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New Method for Constructing Exact Solutions to Nonlinear PDEs

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Abstract: We propose in this paper a new approach to construct exact solutions of nonlinear PDEs. The method used is called "the travelling profiles method". The travelling profiles method enables us to obtain many exact solutions to large classes of nonlinear PDEs.

Keywords: Nonlinear PDE; exact solutions ; travelling profiles method

1 Introduction

In recent years, a certain number of methods had been developed for seeking exact solutions to nonlinear PDEs, a variety of powerful methods such as the Hirota's bilinear methods [4] based on the Hirota transformation, the truncated Painlevé expansion method [1,12]; the homogeneous balance method [12,13], the special "separation" of the variables [6] were used to investigate nonlinear problems.

In this paper, we present a new approach to find exact solutions to some nonlinear PDEs. The approach presented one will be called "the traveling profiles method" (TPM).

Consider the following equation :

$$\frac{\partial u}{\partial t} = A_x u, \tag{1.1}$$

where $A_x u$ is a linear or nonlinear differential operator.

2 The travelling profiles method (TPM):

The principle of this method is to seek the solution of the problem (1.1) in the form

$$u(x,t) = c(t)\,\psi(\xi) \text{ with } \xi = \frac{x - b(t)}{a(t)},\ a, b, c \in \mathbb{R}, \tag{2.1}$$

where ψ is in L^2, that one will call "the based-profile". The parameters $a(t), b(t), c(t)$ are real valued functions of t.

The coefficients $c(t), a(t), b(t)$ are determined by the solution of minimizition problem:

$$\min_{\dot{c},\dot{a},\dot{b}} \int_{-\infty}^{+\infty} \left| \frac{\partial u}{\partial t} - A_x u \right|^2 dx, \tag{2.2}$$

therefore, we obtain three orthogonality equations which are read

$$\begin{cases} \langle \frac{\partial u}{\partial t} - A_x u, \psi \rangle = 0 \\ \langle \frac{\partial u}{\partial t} - A_x u, \xi \psi'_\xi \rangle = 0 \\ \langle \frac{\partial u}{\partial t} - A_x u, \psi'_\xi \rangle = 0 \end{cases} \tag{2.3}$$

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where $\langle .,. \rangle$ is the inner product in L^2 space.

The PDE (1.1) is then transformed into a set of three coupled ODE's :

$$\begin{array}{l} \frac{\dot{c}}{c}\langle \psi,\psi\rangle - \frac{\dot{a}}{a}\langle \xi\psi'_\xi,\psi\rangle - \frac{\dot{b}}{a}\langle \psi'_\xi,\psi\rangle = \frac{1}{c}\langle A_\xi u,\psi\rangle \\ \frac{\dot{c}}{c}\langle \xi\psi'_\xi,\psi\rangle - \frac{\dot{a}}{a}\langle \xi\psi'_\xi,X\psi'_\xi\rangle - \frac{\dot{b}}{a}\langle \xi\psi'_\xi,\psi'_\xi\rangle = \frac{1}{c}\langle A_\xi u,\xi\psi'_\xi\rangle \\ \frac{\dot{c}}{c}\langle \psi,\psi'_\xi\rangle - \frac{\dot{a}}{a}\langle \xi\psi'_\xi,\psi'_\xi\rangle - \frac{\dot{b}}{a}\langle \psi'_\xi,\psi'_\xi\rangle = \frac{1}{c}\langle A_\xi u,\psi'_\xi\rangle \end{array} \quad (2.4)$$

2.1 A priori estimates of solutions :

Let:

$$V_t = \{\psi,\ \xi\psi'_\xi,\ \psi'_\xi\}$$

the subspace of L^2 generated by associated functions to ψ at the moment t.

From relations (2.3), it is deduced that $\frac{\partial u}{\partial t} - A_x u$ is orthogonal to subspace V_t.

In particular we have $\frac{\partial u}{\partial t}\epsilon V_t$, then $\langle \frac{\partial u}{\partial t} - A_x u, \frac{\partial u}{\partial t}\rangle = 0$, thus if also $A_x u$ belongs to V_t then the method provides us a weakly exact solution, which is written under the form

$$u(x,t) = c(t)\,\psi\left[\frac{x-b(t)}{a(t)}\right]. \quad (2.5)$$

Now we want to establish conditions on the method to find exact solutions to equation (1.1).

2.2 Exact solutions to some nonlinear PDEs

Theorem 1 *For $\psi\epsilon C^2\cap L^2$, the equation (1.1) admits an exact solution in the form $u(x,t) = c(t)\,\psi\left[\frac{x-b(t)}{a(t)}\right]$, if*

1. $A_x u = \frac{c^p}{a^q}A_\xi\psi,$ for $p,q\in\mathbb{R}$,

2. the "based profile" ψ is a solution of the following equation:

$$A_\xi\psi = \alpha\psi + \beta\xi\psi'_\xi + \gamma\psi'_\xi, \qquad \text{where } \alpha,\beta,\gamma\in\mathbb{R}\ \text{, with } \alpha,\beta,\gamma\neq 0, \quad (2.6)$$

in this case, the coefficients $c(t), a(t), b(t)$ are determined by the system:

$$\begin{array}{l} \dot{c} = \frac{c^p}{a^q}\alpha \\ \dot{a} = -\frac{c^{p-1}}{a^{q-1}}\beta \\ \dot{b} = -\frac{c^{p-1}}{a^{q-1}}\gamma \end{array} \quad (2.7)$$

Proof. According to the estimation principle of this method, if $A_x u$ belongs to the subspace V_t, then the function $u(x,t) = c(t)\,\psi(\xi)$ is an exact solution of equation (1.1), in this case the term $A_\xi\psi$ can be expressed as a linear combination of functions $\psi, \xi\psi'_\xi$, and ψ'_ξ, thus $A_x\psi = \alpha\psi + \beta\xi\psi'_\xi + \gamma\psi'_\xi$, for $\alpha,\beta,\gamma\in\mathbb{R}$.

The system (2.7) is obtained as follow: when one replaces $A_\xi(\psi)$ by the combination $\alpha\psi+\beta\xi\psi'_\xi+\gamma\psi'_\xi$ in (2.4), we obtain the system:

$$MX = \frac{c^{p-1}}{a^q}MF \quad (2.8)$$

with

$$M = \begin{pmatrix} \langle\psi,\psi\rangle & \langle\xi\psi'_\xi,\psi\rangle & \langle\psi'_\xi,\psi\rangle \\ \langle\xi\psi'_\xi,\psi\rangle & \langle\xi\psi'_\xi,\xi\psi'_\xi\rangle & \langle\xi\psi'_\xi,\psi'_\xi\rangle \\ \langle\psi'_\xi,\psi\rangle & \langle\xi\psi'_\xi,\psi'_\xi\rangle & \langle\psi'_\xi,\psi'_\xi\rangle \end{pmatrix}, \quad X = \begin{pmatrix} \frac{\dot{c}}{c} \\ -\frac{\dot{a}}{a} \\ -\frac{\dot{b}}{a} \end{pmatrix} \text{ and } F = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

where $\langle .,. \rangle$ is the inner product in L^2.

The matrix in system (2.8) is symmetric and invertible, then (2.8) can be written under the form (2.7). ■

2.2.1 Resolution of the differential system:

We can resolve the system (2.7) as follow: from (2.7) we have

$$\begin{cases} c(t) = K_0\, a(t)^{\frac{-\alpha}{\beta}}, \\ b(t) = \frac{\gamma}{\beta}\, a(t) + K_0' \end{cases}, \text{ with } K_0,\, K_0' \text{ constants,} \tag{2.9}$$

If we replace (2.9) in (2.7), we can deduct finally:

$$\begin{array}{l} a(t) = \left[A(-K_0^{p-1}\beta t + K_1)\right]^{\frac{1}{A}}, \\ c(t) = K_0 \left[A(-K_0^{p-1}\beta t + K_1)\right]^{\frac{-\alpha}{\beta A}} \qquad , \\ b(t) = \frac{\gamma}{\beta}\left[A(-K_0^{p-1}\beta t + K_1)\right]^{\frac{1}{A}} + K_0' \end{array} \tag{2.10}$$

with K_0, K_0', K_1 constants and $A = q + \frac{\alpha}{\beta}(p-1) \neq 0$.

Now, to illustrate the idea of this method we have this example.

2.2.2 Example :

Let the equation

$$\frac{\partial u}{\partial t} = (u^2)_{xx}, \tag{2.11}$$

in this case we have $A_\xi u = \frac{c^2}{a^2} A_\xi \psi$, if we seek an exact solution like $u\,(x,t) = c(t)\psi\left(\frac{x-b(t)}{a(t)}\right)$, then the "based-profile" ψ must verify the following ODE:

$$(\psi^2)_{\xi\xi} = \alpha\psi + \beta\xi\psi_\xi' + \gamma\psi_\xi'. \tag{2.12}$$

If we take for exemple $\alpha = \beta$, and for γ, the equation (2.12) can be written in the form

$$\frac{d}{d\xi}\left[(\psi^2)_\xi - (\beta\xi + \gamma)\right] = 0$$

then we obtain

$$(\psi^2)_\xi - (\beta\xi + \gamma) = k$$

for $k = 0$ we have

$$\psi\,(\xi) = \frac{1}{2}(\frac{\beta}{2}\xi^2 + \gamma\xi + k'), \quad \text{with } k' \text{ constant.} \tag{2.13}$$

Then an exact solution to equation (2.11) takes the form:

$$u\,(x,t) = \frac{1}{2}\left[\frac{\beta}{2}(\frac{x-b(t)}{a(t)})^2 + \gamma(\frac{x-b(t)}{a(t)}) + k'\right], \quad \text{with } k' \text{ constant} \tag{2.14}$$

where $c(t), a(t)$, and $b(t)$ are given by:

$$\begin{array}{l} a(t) = [-\beta K_0 t + K_1]^{\frac{1}{3}} \\ c(t) = K_0\,[-\beta K_0 t + K_1]^{\frac{-1}{3}} \qquad , \\ b(t) = -\frac{\gamma}{\beta}\,[-\beta K_0 t + K_1]^{\frac{-1}{3}} + K_0' \end{array}$$

with K_0, K_0', K_1 constants.

3 Conclusion

A method to construct exact solutions to some PDEs is presented in this paper. This method enables us to obtain exact solutions to large classes of nonlinear PDEs. It gives us the possiblity to obtain very varied choice of classes of exact solutions. The idea of our method (TPM) is well illustrated by an example. This approach is very promising and can also bring new results for other applications in PDEs.

References

[1] Cariello F, Tabor M: Similarity reduction from extended painleve expansion for nonintegral evolution equation. *Physica D.* 53: 59-70(1991)

[2] Ibragimov, N. H.(Editor): CRC Handbook of Lie group Analysis of Differential Equations. *Exact solutions and Conservation Laws, CRSPress , Boca Raton.* (1994)

[3] Huibin L, Kelin W: Exact solutions for two nonlinear equations. *J. Phys. Math. Gen.* 23: 3923-3928(1990)

[4] Hirota R: exact N-soluton solutions of the wave of long in shal low water in nonlinear lattices. *J.Math.Phys.* 14: 810-814(1973)

[5] Galaktionov V.A, Posashkov V.A: New exact solutions of parabolic equation with quadratic nonlinearities, *USSR. Compt. Math. Match. Phys*.29(2):112-119 (1989)

[6] Galaktionov V.A, Posashkov V. A, Svirshchevskii S. R.: Generalized separation of variables for differential equations with polynomial right-hand sides. *Dif. Uravneniya*. 31(2): 253-265(1995)

[7] Otwinowski M: Exact traveling wave solutions in nonlinear equations. *Phys. Lett. A.* 128 (9): 483-487 (1988)

[8] Polyanin. A.D, Zaitsev. V. F: Handbook of Nonlinear Partial Equation. *Chapman&Hall/CRC, Boca Raton.* (2004)

[9] Polyanin. A.D, Zaitsev. V. F: Handbook of Exact Solutions for Ordinary Differential Equations. *CRC Press, Boca Raton, New York.* (1995)

[10] Polyanin, A.D, Alexei I. Zhurov, Andrei V. Vyazmin: Generalized Separation of Variables and Mass Transfer Equations. *J. Non Equilib. Thermodyn*. 25: 251-267(2000)

[11] Rudykh. G. A, Semenov. E. I: On new exact solutions of one-dimensional nonlinear diffusion equation with a source. *Zhurn. Vychisl. Matem. i Matem. Fiziki.* 38(6): 971-977(1998)

[12] Wang M: Exact solutions for a compound KDV equation. *Phys.Lett.A*. 216: 67-75 (1996)

[13] Wang Y C, Wang L X, Zhang W B: Application of the Adomian Decomposition Method to Full Nonlinear Sine-Gordon Equation. *International Journal of Nonlinear Science.* 2(1): 29-38(2006)

[14] Yang L, Zhu Z , Wang Y: Exact solutions of nonlinear equations.*Phys. Lett. A.* 260:55-59 (1999)

[15] Yiqing Li ,Lixin Tian, Yuhai Wu: On the Bifurcation of TravelingWave Solution of Generalized Camassa-Holm Equation, *International Journal of Nonlinear Science*. 6(1):34-45 (2008)

[16] Yuanxi Xie: New Explicit and Exact Solutions to the MKdV Equation.*International Journal of Nonlinear Science.* 6(2):124-128(2008)

[17] Yusufoglu E, Bekir A: On the extended tanh method applications of nonlinear equations. *International Journal of nonlinear Sciences*. 4(1):10-16 (2007)