# Existence of general self similar solution to porous medium equation

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**Abstract:** In this paper, we propose a general self similar solutions to the porous medium. We also discuss their existence for some initial data in one dimension. We find for some particular cases an explicit exact solutions. these solutions are called "travelling profile solutions".

**Keywords:** *Porous medium equation - exact solution - general self similar solution.*

## 1 Introduction:

In general, the partial differential equations does not admit an exact solutions, particularly when we imposed initial /boundary conditions. But for some classes of PDEs which enjoy certain symmetries we can find their exact solutions for many particular cases. With some finite or infinite transformations, these partial differential equations becomes invariant and are exactly reduced to ordinary differential equations which can be integrated in a closed form. These solutions are called "self similar solutions".

The porous medium equation it one of class of equations which admit these properties of similarity. This equation is written in the form

$$\left\{ \frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2} \left(u^m\right) , \right. \tag{1.1}$$

where $x \in \mathbb{R}$, $t > 0$, and $m > 1$, is a fixed real number.

Equation (1.1) arises in many other applications, e.g, in the theory of ionized gases at high temperature [17] for values of $m > 1$, and in various models in plasma physics [8] for values of $m < 1$. Of course, for $m = 1$, equation (1.1) is the classical equation of heat conduction. In these lectures we will focus on the case $m > 1$.

A classical "self similarity solutions" from which the known explicit analytical solution of porous media equation may be obtained as special case, takes the form

$$u(x,t) = t^{\alpha} f(\eta), \qquad \eta = xt^{-\beta}, \tag{1.2}$$

where $\alpha$ and $\beta$ (constants) and the profile $f$ are to be determined. Many authors has been study this problem, Aronson [1], Barenblat [4, 5]. In the case of equation (1.1) the similarity exponent $\alpha$ and $\beta$ have to satisfy

$$2\beta + \alpha(m-1) = 1, \tag{1.3}$$

this form is called self-similarity of Type I. We note that such self similar solutions are the guideline for the general PDE theory.
There is another form of self-similarity of Type II with

$$u(x,t) = (T-t)^{-\alpha} f(x(T-t)^{-\beta}), \tag{1.4}$$

for $T > 0$. with the same relation between $\alpha$ and $\beta$ as before.

Also, there is a general self similar solution, where we seek the solution in the form:

$$u = c(t) f(\eta), \quad \eta = \frac{x}{a(t)}, \tag{1.5}$$

where $a(t)$, $c(t)$ and the profile $f$ are to be determined. Gilding and Peletier has studied this form for some particular cases of parameters $a(t)$ and $c(t)$ [11].

In this work we want to find a most general form of self similar solutions to equation (1.1), which are written in the form:

$$u\,(x,t) = c\,(t)\, f\left[\frac{x - b(t)}{a\,(t)}\right], \; a, \; c, \; b \in \mathbb{R}^{+}, \tag{1.6}$$

where $a(t), c(t), b(t)$ and the profile $f$ are to be determined. We prove.also the existence of these solutions under certain conditions.

# 2 General self similar solution to porous medium equation:

In this work we search a general form of self similar solutions, in the form:

$$u(x,t) = c(t)\, f(\eta), \text{ with } \eta = \frac{x - b(t)}{a(t)}, \ a, b, c \in \mathbb{R}^{+}. \tag{2.1}$$

The parameters $a(t), c(t), b(t)$ are real valued functions of $t$.
If we replace this form of solutions in equation (1.1) we find ,

$$\frac{\dot{c}}{c} f - \frac{\dot{a}}{a}\eta f'_{\eta} - \frac{\dot{b}}{b} f'_{\eta} = \frac{c^{m-1}}{a^2} (f^m)''_{\eta\eta}, \tag{2.2}$$

this equation depends of many unknown parameters, our aim is to determine the coefficients $a(t), c(t), b(t)$ and the profile $f$.
In that case, a simple separation of variables argument implies that the following three conditions must hold:

$$\begin{cases} \frac{\dot{c}}{c} &= \frac{c^{m-1}}{a^2}\alpha \\ \frac{\dot{a}}{a} &= -\frac{c^{m-1}}{a^2}\beta \\ \frac{\dot{b}}{a} &= -\frac{c^{m-1}}{a^2}\gamma \end{cases} \tag{2.3}$$

with parameters $\alpha, \beta, \gamma \in \mathbb{R}$, and the profile $f$ must satisfy the equation

$$(f^m)''_{\eta\eta}(\eta) = \alpha f(\eta) + \beta\eta f'_{\eta}(\eta) + \gamma f'_{\eta}(\eta) \tag{2.4}$$

## 2.1 Resolution of the differential system:

At the boundaries, we impose the lateral boundary conditions

$$a(0) = 1, \ c(0) = 1, \ b(0) = 0, \tag{2.5}$$

we can see that from (2.3) we have

$$\begin{cases} c(t) = a(t)^{\frac{-\alpha}{\beta}} \\ b(t) = \frac{\gamma}{\beta} a(t) + K_2 \end{cases}, \tag{2.7}$$

if we replace (2.7) in (2.3) then we deduct

$$\begin{cases} a(t) = (1 - A\beta t)^{\frac{1}{A}}, \\ c(t) = (1 - A\beta t)^{\frac{-\alpha}{\beta A}} \\ b(t) = \frac{\gamma}{\beta}(1 - A\beta t)^{\frac{1}{A}} - \frac{\gamma}{\beta} \end{cases}, \ 0 < t < T, \tag{2.8}$$

for :

$$2\beta + (m-1)\alpha > 0 \tag{2.9}$$

with $A = 2 + \frac{\alpha}{\beta}(m-1)$ and $.\ T = \frac{1}{2\beta+(m-1)\alpha}$,
and

$$\begin{cases} a(t) = \exp(-\beta t) \\ c(t) = \exp(\alpha t) \\ b(t) = \frac{\gamma}{\beta}\exp(-\beta t) - \frac{\gamma}{\beta} \end{cases}, \quad 0 < t < \infty, \tag{2.10}$$

for :

$$2\beta + (m-1)\alpha = 0. \tag{2.11}$$

we have proved the following theorem

**Theorem 1** *The function*

$$u(x,t) = c(t)\, f(\eta), \ \ \textit{with}\ \eta = \frac{x - b(t)}{a(t)}, \ a(t), \ c(t), \ b(t) > 0, \ x \in \mathbb{R}.$$

*is an exact solution of problem (1.1), if the "based profile" f is a solution of following differential equation*

$$(f^m)''_{\eta\eta} = \alpha f + \beta\eta f'_\eta + \gamma f'_\eta, \qquad \textit{where}\ \alpha, \beta, \gamma \in \mathbb{R},$$

*in this case, the coefficients $c(t), a(t)$ and $b(t)$ are determined by the system:*

$$\begin{cases} \frac{\dot{c}}{c} &= \frac{c^{m-1}}{a^2}\alpha \\ \frac{\dot{a}}{a} &= -\frac{c^{m-1}}{a^2}\beta \\ \frac{\dot{b}}{a} &= -\frac{c^{m-1}}{a^2}\gamma \end{cases}.$$

## 3 Existence and uniqueness of the "based profile":

In this section, we discuss the existence and uniqueness of weak solutions with compact support for the boundary value problem

$$(f^m)''_{\eta\eta} = \alpha f + \beta\eta f'_\eta + \gamma f'_\eta, \quad 0 < \eta < \infty, \ \text{where}\ \alpha, \beta, \gamma \in \mathbb{R}, \tag{3.1}$$

with $\eta = \frac{x-b(t)}{a(t)}$, and

$$f(0) = V \quad \text{and} \quad f(\infty) = 0, \tag{3.2}$$

where $V > 0$ are arbitrary real constants. This equation has been investigated in detail in a series papers (Gilding and Peletier, 1976,1977;

Gilding 1980, [11]), with $\gamma = 0$.
Thus the solution $u(x,t)$ satisfy the lateral boundary condition

$$u(b(t),t) = c(t)V, \text{ with } V \in \mathbb{R}^{+}, \tag{3.3}$$

to the porous medium equation (1.1) in the domain $b(t) < x < \infty,\ t > 0$

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2}(u^m), \quad b(t) < x < \infty,\ t > 0. \tag{3.4}$$

Our aim is to generalize the results of [11] for $\gamma \neq 0$, we follow definition.

**Definition :** A function $f$ is a weak solution of (3.1) if it satisfies the following conditions.
**a)** $f$ is bounded, continuous, and nonnegative on $[0,\infty)$.
**b)** $(f^m)(\eta)$ has a continuous derivative with respect to $\eta$ on $(0,\infty)$.
**c)** $f$ satisfies the equation

$$\int_0^\infty \phi' \left\{ (f^m)' - (\beta\eta + \gamma) f \right\} d\xi + (\alpha - \beta) \int_0^\infty \phi f d\eta = 0$$

for all $\phi \in C_0^1(0,\infty)$.

We prove the following theorem

**Theorem 2** *Suppose that $V > 0$. Then the boundary value problem (3.1), (3.2) has a weak solution with compact support if and only if $\beta \leq 0,\ \gamma \leq 0$ and $\alpha - 2\beta > 0$. Furthermore, this weak solution is unique.*

To prove this theorem, we pose the following boundary value problem for (3.1),

$$f(0) = V \tag{3.5}$$

and

$$f(\lambda) = 0, \quad (f^m)'(\lambda) = 0 \tag{3.6}$$

where $\lambda > 0$ is a real number. Using a shooting argument with $\lambda > 0$ as the shooting parameter, we first prove the following theorem for the existence and uniqueness of classical solutions for (3.1) with the boundary conditions (3.5) and (3.6).

**Theorem 3** *Suppose that $V > 0$. Then the boundary value problem (3.1), (3.5), and (3.6) has a unique solution and there exists a unique $\lambda(V) > 0$ such that $f(\eta; \lambda(V))$ is positive on $(0, \lambda)$ if and only if $\beta \leq 0,\ \gamma \leq 0$ and $\alpha - 2\beta > 0$.*

We first determine necessary conditions on the parameters $\alpha$, $\beta$ and $\gamma$ for the existence of a nontrivial weak solution of (3.1) with compact support.

**Lemma 4** *There exists a nontrivial weak solution of (3.1) with a compact support only when $\beta = \gamma = 0$ and $\alpha > 0$ or $\beta < 0$ and $\gamma < 0$.*

**Proof.** Suppose that $f(\eta;\lambda)$ is a nontrivial weak solution of (3.1) with compact support. Then $f > 0$ in $(\lambda - \varepsilon, \lambda)$ and $f = 0$ in $[\lambda, \infty)$ for some $\lambda > 0$ and $\varepsilon > 0$.
It follows that f is a classical solution of (3.1) on $(\lambda - \varepsilon, \lambda)$ and satisfies (3,6) at $\eta = \lambda$; that is, $f(\lambda) = 0$, $(f^m)'(\lambda) = 0$. Integrating (3.1) from $\eta$ to $\lambda$, where $\lambda - \varepsilon < \eta < \lambda$, we get:

$$-(f^m)'(\eta) = -(\beta\eta + \gamma) f(\eta) + (\alpha - \beta) \int_\eta^\lambda f(\xi)\, d\xi. \tag{3.7}$$

The continuity of $f$ and $(f^m)'$ ensures the existence of $\eta_0 \in (\lambda - \varepsilon, \lambda)$ such that $f'(\eta_0) < 0$. This implies that the LHS of (3.7) is positive at $\eta = \eta_0$ , and, therefore, $-(\beta\eta_0 + \gamma)$ and $\alpha - \beta$ cannot both be less than zero. Thus, $\beta = \gamma = 0$ implies that $\alpha > 0$.
Now consider the case $\beta > 0$ et $\gamma > 0$. This requires that $\alpha - \beta > 0$, and hence $\alpha > 0$. We easily check from (3.1) that $f$ cannot have a maximum as long as $f$ is positive. Therefore, $f$ does not assume a maximum at any point in $(\lambda - \varepsilon, \lambda)$, thus, $f'(\eta_0) < 0$ on $(\lambda - \varepsilon, \lambda)$. It follows from (3,7) that

$$-m f^{m-2}(\eta) f'(\eta) + (\beta\eta + \gamma)\eta \le (\alpha - \beta)(\lambda - \eta), \tag{3.8}$$

where we have used the fact that $f(\xi) \le f(\eta)$ for $\xi \in (\eta, \lambda)$, $\lambda - \varepsilon < \eta < \lambda$. As $\eta \to \lambda$ in (3,8), LHS becomes positive, and the RHS tends to zero, a contradiction.
Thus we have shown that $\beta = \gamma = 0$ and $\alpha > 0$ or $\beta < 0$ and $\gamma < 0$ are the only cases for which a nontrivial weak solution of (3.1) exists with a compact support. ■

## 3.1 The case when $\beta = \gamma = 0$ and $\alpha > 0$

With $\beta = \gamma = 0$ and $\alpha > 0$, (3.1) becomes

$$(f^m)'' = \alpha (f^m)^{\frac{1}{m}}, \tag{3.9}$$

Substituting $f^m = g$ in (3.9) and integrating we get

$$\left(g'\right)^2 = \frac{2\alpha m}{m+1} g^{\frac{m+1}{m}},$$

Solving (3.9) for $g$ and using (3.6), we obtain

$$g = \left[ \frac{\alpha (m-1)^2}{2m(m+1)} (\lambda - \eta)^2 \right]^{\frac{m}{m-1}}, \ 0 < \eta < \lambda,$$

thus

$$f(\eta; \lambda) = \left[ \frac{\alpha (m-1)^2}{2m(m+1)} (\lambda - \eta)^2 \right]^{\frac{1}{m-1}}, \ 0 < \eta < \lambda, \tag{3.10}$$

is the unique solution of the problem (3.1) satisfying (3.6).

We observe that

$$f(0; \lambda) = \left[ \frac{\alpha (m-1)^2}{2m(m+1)} \lambda^2 \right]^{\frac{1}{m-1}}.$$

Because $m > 1$, $f(0; \lambda)$ is a continuous function of $\lambda$ with $f(0; 0) = 0$ and $f(0; \infty) = \infty$; furthermore, $f$ is a continuous and monotonically increasing function of a. This implies that, for a given $V > 0$, there exists a unique $\lambda(V)$ such that $f(0; \lambda(V)) = V$. Therefore, $f(\eta; \lambda(V))$ is the unique solution of (3.1) satisfying (3.5) and (3.6). An easy calculation shows that

$$\lambda(V) = \left[ \frac{2m(m+1)}{\alpha (m-1)^2} V^{m-1} \right]^{\frac{1}{2}}.$$

### 3.2 The case when $\beta < 0$ and $\gamma < 0$

We give below an elementary lemma for the case $\beta < 0$ and $\gamma < 0$.

**Lemma 5** *Suppose that* $0 < \mu < \lambda$ *and* $f$ *is a positive solution of (3.1) on* $[\mu, \lambda)$ *satisfying (3.6). Then the following results hold.*
***(i)*** $f'(\eta) < 0$ *on* $[\mu, \lambda)$ *provided that* $\alpha - \beta \geq 0$.
***(ii)*** *Suppose that* $\alpha - \beta < 0$ *and* $f'(\eta_0) = 0$ *for some* $\eta_0 \in [\mu, \lambda)$. *Then* $f$ *has a maximum at* $\eta_0$ *and* $\eta_0 < \frac{\lambda(\alpha-\beta)-\gamma}{\alpha}$.
*Suppose that* $f$ *is a positive solution of (3.1) and (3.6) on* $[0, \lambda)$. *Then*

$$f'(0) < 0, \ \text{for } \alpha - \beta \geq 0.$$

**Proof.**
**(i)** Integration of (3.1) from $\mu < \eta < \lambda$ we obtain

$$-(f^m)'(\eta) = -(\beta\eta + \gamma) f(\eta) + (\alpha - \beta) \int_\eta^\lambda f(\xi)\, d\xi. \tag{3.11}$$

Because $\beta < 0$ and $\gamma < 0$, the RHS of (3.11) is positive when $\alpha - \beta \geq 0$ and hence $(f^m)^{'}(\eta) < 0$. This implies that $f^{'}(\eta) < 0$ on $[\mu, \lambda)$.
**(ii)** if $\alpha - \beta < 0$ then $\alpha < 0$ (because $\beta < 0$), by (3.1), $f^{''}(\eta_0) < 0$ when $f^{'}(\eta_0) = 0$, thus $f$ has a maximum at $\eta = \eta_0$ and is strictly decreasing on $(\eta_0, \lambda)$; that is, $f^{'}(\eta) < 0$ on $(\eta_0, \lambda)$. Putting $\eta = \eta_0$ in (3.11), we have:

$$0 = -(\beta\eta_0 + \gamma) f(\eta_0) + (\alpha - \beta) \int_{\eta_0}^{\lambda} f(\xi)\, d\xi > -(\beta\eta_0 + \gamma) f(\eta_0) + (\alpha - \beta)(\lambda - \eta_0) f(\eta_0),$$

therefore,

$$-(\beta\eta_0 + \gamma) + (\alpha - \beta)(\lambda - \eta_0) < 0 \text{ or } \eta_0 < \frac{\lambda(\alpha - \beta) - \gamma}{\alpha}$$

With $\eta = 0$, (3.11) becomes

$$-(f^m)^{'}(0) = -\gamma f(0) + (\alpha - \beta) \int_{\eta}^{\lambda} f(\xi)\, d\xi. \tag{3.12}$$

The result for $f^{'}(0)$ follows immediately from (3.12). ■

In the next lemma, we prove the local existence and uniqueness of a solution of (3.1) satisfying (3.6). This is accomplished by formulating an equivalent integral equation following the work of Atkinson and Peletier [4].

**Lemma 6** *Suppose that $\beta < 0$, $\gamma < 0$ and $\alpha$ is any real number. Then, for any $\lambda > 0$, equation (3.1) with initial condition (3.6) at $\eta = \lambda$, has a unique positive solution in a neighborhood $(\lambda - \varepsilon, \lambda)$ of $\lambda$, here, $\varepsilon > 0$ is a constant.*

**Proof.** Suppose that $f$ is a positive solution in a left neighborhood of $\eta = \lambda$. By lemma 5, $f^{'}(\eta) < 0$ for $\eta \in (\lambda - \varepsilon, \lambda)$ for some $\varepsilon > 0$.
Let $\eta = G(f)$ where $G$ is the inverse of $f$ on $(\lambda - \varepsilon, \lambda)$. Rewriting (3.11), we have:

$$(f^m)^{'}(\eta) = (\alpha\eta + \gamma) f(\eta) + (\alpha - \beta) \int_{\eta}^{\lambda} \xi f^{'}(\xi)\, d\xi. \tag{3.13}$$

With $G(f) = \eta$ in (3.13) we have:

$$\frac{dG}{df} = \frac{m f^{m-1}}{(\alpha G + \gamma) f - (\alpha - \beta) \int_0^f G(\varphi) d\varphi}, \tag{3.14}$$

equation (3.14) is an integro-differential equation for $G = G(f)$. Integrating (3.14) from 0 to $f$ , we obtain

$$G(f) - \lambda = m \int_0^f \frac{\phi^{m-1} d\phi}{(\alpha G + \gamma)\, \phi - (\alpha - \beta) \int_0^\phi G(\psi) d\psi}. \tag{3.15}$$

Let

$$H(f) = 1 - \lambda^{-1} G(f), \tag{2.16}$$

Then, equation (3.15) becomes

$$H(f) = \frac{m}{\lambda^2} \int_0^f \frac{\phi^{m-1} d\phi}{(-\beta - \gamma)\, \phi + \alpha \phi H(\phi) - (\alpha - \beta) \int_0^\phi H(\psi) d\psi}. \tag{3.17}$$

By using the Banach–Cacciopoli contraction mapping principle (see Hartman [13]), we now show that equation (3.17) admits a unique positive solution in a right neighborhood of $f = 0$. Let $X$ be the set of all bounded functions $H(f)$ on $[0, h], h > 0$, satisfying

$$0 \leq H(f) \leq \rho = \frac{|\beta + \gamma|}{2(|\alpha| + |\alpha - \beta|)}. \tag{3.18}$$

Let $\|..\|$ be the sup norm defined on $X$. Then $X$ is a complete metric space.

$$M\,(H)\,(f) = \frac{m}{\lambda^2} \int_0^f \frac{\phi^{m-1} d\phi}{-\,(\beta + \gamma)\, \phi + \alpha \phi H(\phi) - (\alpha - \beta) \int_0^\phi H(\psi) d\psi}, \; H(f) \in X. \tag{3.19}$$

First we show that $M$ maps $X$ into $X$ over $[0, h_0]$, $h \leq h_0$. Let $H \in X$. Clearly,

$$\begin{aligned} -\,(\beta + \gamma)\, \phi + \alpha \phi H(\phi) - (\alpha - \beta) \int_0^\phi H(\psi) &\geq -\,(\beta + \gamma)\, \phi - |\alpha|\, \phi H(\phi) - |\alpha - \beta|\, \|H\|\, \phi \\ &\geq -\,(\beta + \gamma)\, \phi - (|\alpha| + |\alpha - \beta|)\, \|H\|\, \phi \\ &\geq \frac{-\,(\beta + \gamma)\, \phi}{2}, \end{aligned} \tag{3.20}$$

where we have used (3.18). Therefore, from (3.19), we have

$$\begin{aligned} M\,(H)\,(f) &\leq \frac{2m}{-\,(\beta + \gamma)\, \lambda^2} \int_0^f \phi^{m-2} d\phi \\ &= \frac{2m f^{m-2}}{-\,(\beta + \gamma)\, \lambda^2\, (m - 1)} \\ &\leq \frac{2m h^{m-2}}{-\,(\beta + \gamma)\, \lambda^2\, (m - 1)}. \end{aligned} \tag{3.21}$$

Thus, $M(H)$ is well defined on $X$ and $M(H):[0,h]\to\mathbb{R}$ is nonnegative and continuous. The RHS of (3.21) suggests that we may find $h_0,\ h\le h_0$ such that $\|M(H)\|\le\rho,\ H\in X$. Thus $M$ maps $X$ into $X$ for $h\le h_0$. In the next step, we show that $M$ is a contraction map on $X$. Let $H_1,H_2\in X$, and $h\le h_0$.
Then

$$\begin{aligned}\|M(H_1)-M(H_2)\| &\le \frac{4m}{(\beta+\gamma)^2\lambda^2}\int_0^f \phi^{m-3}\left(|\alpha|\,\phi\,\|H_1-H_2\|+|\alpha-\beta|\int_0^\phi \|H_1-H_2\|\,d\psi\right)d\phi\\ &\le \frac{4m}{(m-1)(\beta+\gamma)^2\lambda^2}\left(|\alpha|+|\alpha-\beta|\right)h^{m-1}\|H_1-H_2\|.\end{aligned}$$

Therefore, there exists $h_1\in(0,h_0]$ such that if $h\le h_1$, $M$ is a contraction on $X$. By the Banach–Cacciopoli contraction principle, $M$ has a unique fixed point in $X$ and hence equation (3.17) has a unique solution. This, in turn, implies that there exists a unique positive solution of (3.1), (3.6) in an interval $(\lambda-\varepsilon,\lambda)$ for some $\varepsilon>0$. ■

In the next lemma, we prove that a positive solution $f(\eta;\lambda)$ of (3.1) and (3.6) cannot be unbounded.

**Lemma 7** *Suppose that $\beta<0,\ \gamma<0$ and $\mu\in[0,\lambda)$. Furthermore, let $f$ be a positive solution of (3.1) and (3.6) on $(\mu,\lambda)$. Then $f$ is bounded on $(\mu,\lambda)$ and*

$$\sup f(\eta)\le\left[\frac{(m-1)\lambda}{2m}\max\left\{-(\beta\lambda+2\gamma),\left[(\alpha-2\beta)\lambda-2\gamma\right]\right\}\right]^{\frac{1}{m-1}}$$

**Proof.** We prove this lemma for the following two cases: (i) $\alpha-\beta\ge0$, (ii) $\alpha-\beta<0$.
Case (i). $\alpha-\beta\ge0$.
Because, for this case, $f'(\eta)<0$ on $(\mu,\lambda)$ by Lemma 5, $f(\eta)\ge f(\xi),\ \xi\in(\eta,\lambda)$. By (3.11),

$$-(f^m)'(\eta)\le-(\beta\eta+\gamma)f(\eta)+(\alpha-\beta)f(\eta)(\lambda-\eta),\ \ \mu\le\eta<\lambda,$$

or

$$-mf^{m-2}f'\le-\alpha\eta-\gamma+\lambda(\alpha-\beta)\le-\lambda\beta-\gamma+\alpha(\lambda-\eta),\ \ \mu\le\eta<\lambda. \tag{3.22}$$

Integrating (3.22) from $\eta$ to $\lambda$ gives

$$\frac{m}{m-1}f^{m-1}(\eta)\le\left[-\lambda\beta-\gamma+\frac{1}{2}\alpha(\lambda-\eta)\right](\lambda-\eta),\ \ \mu\le\eta\le\lambda. \tag{3.23}$$

Thus,

$$\frac{m}{m-1}\sup_{(\mu,\lambda)} f^{m-1}(\eta) \le \frac{1}{2}\left[(\alpha-2\beta)\lambda-2\gamma\right]\lambda \tag{3.24}$$

Case (ii). $\alpha-\beta<0$.
By equation (3.11),

$$-\left(f^{m}\right)'(\eta) \le -(\beta\eta+\gamma)\, f(\eta)\,,\ \ \mu\le\eta<\lambda,$$

or

$$-mf^{m-2}f' \le -(\beta\eta+\gamma)\,,\ \ \mu\le\eta<\lambda. \tag{3.25}$$

Integrating (3.25) from $\eta$ to $\lambda$, we have

$$\frac{m}{m-1}f^{m-1}(\eta) \le -\left[\frac{\beta}{2}\left(\lambda^2-\eta^2\right)+\gamma(\lambda-\eta)\right],\ \ \mu\le\eta\le\lambda. \tag{3.26}$$

This, in turn, implies that

$$\frac{m}{m-1}\sup_{(\mu,\lambda)} f^{m-1}(\eta) \le -\frac{\lambda}{2}(\beta\lambda+2\gamma)\,. \tag{3.27}$$

Observe that the bounds in (3.24) and (3.27) are independent of $\mu$ and, therefore, $f(\eta)$ cannot be unbounded as $\eta$ decreases from $\eta=\lambda$. ■

**Lemma 8** *Suppose that $f$ is a positive solution of (3.1) and (3.6) in a left neighborhood of $\eta=\lambda$, and $\beta<0$, $\gamma<0$. Then $f(\eta)>0$ on $[0,\lambda)$ when $\alpha-2\beta>0$.*

**Proof.** Integrating (3,11) from $\eta$ to $\lambda$ we have

$$f^{m}(\eta) = -(\beta\eta+\gamma)\int_{\eta}^{\lambda} f(\xi)\,d\xi + (\alpha-2\beta)\int_{\eta}^{\lambda}(\xi-\eta)\,f(\xi)\,d\xi. \tag{3.28}$$

It is easy to see from (3.28) that, if $\alpha-2\beta>0$, then $f(\eta)>0$ on $(0,\lambda)$. ■

**Prove of Theorem 3:** Now we proceed to prove Theorem 3. We have already proved in Lemma 6 the local existence of a solution about $\eta=\lambda$ for (3.1) and (3.6). This unique local solution may be extended back to $\eta=0$ as a positive solution with $f(0)>0$ if and only if when $\alpha-2\beta>0$ (see Lemma 8). Now if we can prove that there exists $\lambda(V)$ such that $f(0;\lambda(V))=V$, then Theorem 3 is proved. To that end, we use the following result due to Barenblatt (see [4]),. Suppose that $f(\eta;\lambda)$ is a solution of (3.1) and (3.6) on $(0,\lambda)$; then $\omega^{-\frac{2}{m-1}}f(\omega\eta;\omega\lambda)$ is a solution of (3.1) and (3.6) on $(0,\omega\lambda)$ for any $\omega>0$. Let $\omega=\lambda^{-1}$. Then,

$$f(0;\lambda)=\lambda^{\frac{2}{m-1}}f(0;1)=V. \tag{3.31}$$

Because $f(0;1)>0$ for $\alpha-2\beta>0,\ \beta<0,\ \gamma<0$, we get a unique root $\lambda=\lambda(V)$ of (3.31). Thus, $f(\eta;\lambda(V))$ is the unique solution of (3.1), (3.5), and (3.6).

Theorem 3 follows if we add that, for $\beta=\gamma=0$, we have already constructed the explicit solution (3.10)

$$f(\eta;\lambda)=\left[\frac{\alpha\,(m-1)^2}{2m\,(m+1)}\,(\lambda-\eta)^2\right]^{\frac{1}{m-1}},\ 0<\eta<\lambda.$$

**Prove of Theorem 2:**

We observe that

$$f(\eta)=\begin{cases} f(\eta;\lambda), & 0<\eta<\lambda \\ 0, & \lambda<\eta<\infty \end{cases}, \tag{3.32}$$

is a weak solution of (3.1) and (3.6). Now we must show that, given $V>0$, (3.32) is the only solution of (3.1), (3.5), and (3.6) with compact support.

Suppose that $f(\eta)$ is a weak solution of the problem (3.1) and (3.2) with compact support. By Lemma 8, this is possible only if $\alpha-2\beta>0$. Moreover,

$$f(\eta)\begin{cases} >0, & \text{on }\ \eta\in[0,\lambda) \\ =0, & \text{on }\eta\in[\lambda,\infty)\,,\ \lambda>0 \end{cases}$$

By Theorem 3, this is also the unique solution. Thus, we have proved Theorem 2.

We conclude with a discussion of the implications of Theorems 2 and 3 for general form of self similar solutions to equation (1.1).

**Theorem 9** *If $\beta<0,\ \gamma<0$ and $\alpha\geq\frac{2\beta}{1-m}$, the problem (3.4), (3.3) has a weak solution with compact support in the form*

$$u\,(x,t)=c\,(t)\,f\,(\eta)\,,\ \textit{with}\ \eta=\frac{x-b(t)}{a\,(t)},$$

*with the "based profile" $f$ is a solution of following differential equation*

$$(f^m)''_{\eta\eta}=\alpha f+\beta\eta f'_\eta+\gamma f'_\eta,\ \ 0<\eta<\infty.$$

*and the coefficients $c\,(t)\,,a\,(t)$ and $b(t)$ are given by*

*1)*

$$\begin{cases} a(t)=(1-A\beta t)^{\frac{1}{A}} \\ c(t)=(1-A\beta t)^{\frac{-\alpha}{\beta A}} \\ b(t)=\frac{\gamma}{\beta}\,(1-A\beta t)^{\frac{1}{A}}-\frac{\gamma}{\beta} \end{cases},\ 0<t<T.$$

*If* $\alpha > \frac{2\beta}{1-m}$, *with* $A = 2 + \frac{\alpha}{\beta}(m-1)$ *and* $T = \frac{1}{2\beta+(m-1)\alpha}$,
*and by*
*2)*

$$\begin{cases} a(t) = \exp(-\beta t) \\ c(t) = \exp(\alpha t) \\ b(t) = \frac{\gamma}{\beta}\exp(-\beta t) - \frac{\gamma}{\beta} \end{cases}, \quad 0 < t < \infty.$$

*If* $\alpha = \frac{2\beta}{1-m}$.

**Proof.** We have already proved in theorem 3 the existence of "based profile" $f$ with compact support if and only if $\beta < 0,\ \gamma < 0$ and $\alpha - 2\beta > 0$.
The coefficients $c(t), a(t)$ and $b(t)$ are given by (2.8)

$$\begin{cases} a(t) = (1 - A\beta t)^{\frac{1}{A}} \\ c(t) = (1 - A\beta t)^{\frac{-\alpha}{\beta A}} \\ b(t) = \frac{\gamma}{\beta}(1 - A\beta t)^{\frac{1}{A}} - \frac{\gamma}{\beta} \end{cases}, \quad 0 < t < T$$

with $A = 2 + \frac{\alpha}{\beta}(m-1)$ and $T = \frac{1}{2\beta+(m-1)\alpha}$.
if $2\beta + (m-1)\alpha > 0$, ie $\alpha > \frac{2\beta}{1-m}$.
Clearly the coefficients $c(t), a(t)$ and $b(t)$ are defined if $1 - A\beta t > 0$ this implies

$$t < \frac{1}{A\beta} = \frac{1}{2\beta + (m-1)\alpha} = T.$$

We see that the solution $u(x,t)$ blows up at $t = T$. and $T = \frac{1}{2\beta+(m-1)\alpha}$, is the blow-up time, such that the solution is well defined for all $0 < t < T$, while $u(x,t) \to \infty$ as $t = T$.
and (2.10)

$$\begin{cases} a(t) = \exp(-\beta t) \\ c(t) = \exp(\alpha t) \\ b(t) = \frac{\gamma}{\beta}\exp(-\beta t) - \frac{\gamma}{\beta} \end{cases}, \quad 0 < t < \infty.$$

if $2\beta + (m-1)\alpha = 0$, ie $\alpha = \frac{2\beta}{1-m}$.
Finally, we prove the solution $u(x,t) = c(t) f\left[\frac{x-b(t)}{a(t)}\right]$ exist when $\beta < 0,\ \gamma < 0$ and $\alpha \geq \frac{2\beta}{1-m}$. ■

## Conclusion

In this work we have find a new solution of porous media equation in a general form of self similar solutions. We discuss their Existence and uniqueness of the "based profile".

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