Surveys in Mathematics and its Applications

ISSN 1842-6298 (electronic), 1843-7265 (print)
Volume 12 (2017), 103 – 115

BOUNDARY VALUE PROBLEM FOR CAPUTO-HADAMARD FRACTIONAL DIFFERENTIAL EQUATIONS

Yacine Arioua and Nouredine Benhamidouche

Abstract. The aim of this work is to study the existence and uniqueness solutions for boundary value problem of nonlinear fractional differential equations with Caputo-Hadamard derivative in bounded domain. We used the standard and Krasnoselskii's fixed point theorems. Some new results of existence and uniqueness solutions for Caputo-Hadamard fractional equations are obtained.

1 Introduction

The origins of fractional calculation go back to the late 17th century. In fact, some mathematicians (L'Hopital, Leibniz(1695)) began to consider how to define the fractional derivative. But it is only during the last three decades that fractional calculation has been the most interesting and the applications of fractional derivatives have become more diversified. There are several definitions of fractional derivatives, the definitions of Riemann-Liouville (1832), Riemann (1849) Caputo (1997), Grunwald-Letnikov (1867) as well as Hadamard (1891, [10]) which are the most used.

The Fractional order differential equations are generalizations of classical integer order differential equations and they are increasingly used to model problems in fluid dynamics, finance, and other areas of application. Recent investigations have shown that sometimes physical systems can be modeled more accurately using fractional derivative formulations, [16]. The reader interested in the subject is refereed to the books (Kilbas et al. 2006 [12]; Klimek 2009; Podlubny 1999 [18]; Samko et al. 1993 [19], Diethelm, 2010, [7]).

In recent years, some authors have investigated the existence and uniqueness of solutions for nonlinear fractional differential equation boundary value problems. For a small sample of such work, we refer the reader to works by Ahmad and Ntouyas [1, 2, 3], Alsaedi et al. [4], Benchohra, Hamani and Ntouyas [5], Mengmeng and

2010 Mathematics Subject Classification: 34A08; 34A37.
Keywords: Caputo-Hadamard derivative; Fractional differential equations; Fixed point theorems.

Jinrong [17].

Alsaedi et al. [4] studied the following boundary value problem of nonlinear Hadamard fractional differential equations:

$$\begin{cases} \mathcal{D}^{\alpha}x(t) = f(t, x(t)), \ 1 < t < e, \ 1 < \alpha \leq 2, \\ x(1) = 0, \ \frac{A}{\Gamma(\gamma)} \int_1^{\eta} \left(\log \frac{\eta}{s}\right)^{\gamma-1} x(s) \frac{ds}{s} + Bx(e) = c, \ \gamma > 0, \ 1 < \eta < e, \end{cases}$$

where $\mathcal{D}^{\alpha}$ is the Hadamard fractional derivative of order α, $f : [1, e] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, and A, B, c are real constants.
Mengmeng LI, and Jinrong Wang [17] studied the existence of local and global solutions to the Hadamard type fractional differential equation

$$\begin{cases} {}^{H}\mathcal{D}^{\alpha}_{a^+}u(t) = f(t, u(t)), \ t \in J, \ 0 < \alpha < 1, \\ {}^{H}\mathcal{D}^{\alpha-1}_{a^+}u(a) = c, \ c \in \mathbb{R}, \end{cases}$$

where $J = [a, a + h]$, $h > 0$ or $[a, +\infty)$ and the symbol ${}^{H}\mathcal{D}^{\alpha}_{a^+}$ is the Hadamard fractional derivative.

Bashir Ahmad and Sotiris K. Ntouyas [1] studied the existence and uniqueness of solutions for the following boundary value problem

$$\begin{cases} \mathcal{D}^{\alpha}u(t) = f(t, u(t)), \ 1 < t < e, \ 1 < \alpha \leq 2, \\ u(1) = 0, \ u(e) = \mathcal{I}^{\beta}u(e) = \frac{A}{\Gamma(\beta)} \int_1^{e} \left(\log \frac{e}{s}\right)^{\beta-1} u(s) \frac{ds}{s}, \ \beta > 0, \end{cases}$$

where $\mathcal{D}^{\alpha}$ is the Hadamard fractional derivative of order α, and $\mathcal{I}^{\beta}$ is the Hadamard fractional integral of order β and $f : [1, e] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function.

In this paper, we discuss the existence of a positive solution to boundary value problem of nonlinear fractional differential equation:

$$ {}^{C}\mathcal{D}^{\alpha}_{1^+}u(t) + f(t, u(t)) = 0, \ 1 < t < e, \ 2 < \alpha \leq 3, \tag{1.1}$$

with fractional boundary conditions:

$$u(1) = u'(1) = 0, \ \left({}^{C}\mathcal{D}^{\alpha-1}_{1^+}u\right)(e) = \left({}^{C}\mathcal{D}^{\alpha-2}_{1^+}u\right)(e) = 0, \tag{1.2}$$

where ${}^{C}\mathcal{D}^{\alpha}_{1^+}$ is the Caputo-Hadamard fractional derivative of order α, and $f : [1, e] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function.

We obtain two different results for the existence and uniqueness of the solution for this boundary value problem.

The major result of this paper is a generalization of the findings in [1], for the existence solution to boundary value problems of fractional integro-differential equation with Hadamard derivative.

2 Preliminaries

At first, we recall some concepts on fractional calculus and present some additional properties that will be used later. For more details, we refer to [12]. We present some basic definitions and results from fractional calculus theory.

Definition 1 (*Hadamard* fractional integral)**.** *(see [12])*
The left-sided fractional integral of order $\alpha > 0$ of a function $y : (a, b) \to \mathbb{R}$ is given by

$$\mathcal{I}_{a^+}^{\alpha} y\,(t) = \frac{1}{\Gamma\,(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} y\,(s)\,\frac{ds}{s}, \tag{2.1}$$

provided the right integral converges.

Definition 2 (*Hadamard* fractional derivative)**.** *(see [12]).*
The left-sided Hadamard fractional derivative of order $\alpha \geq 0$ of a continuous function $y : (a, b) \to \mathbb{R}$ is given by

$$\mathcal{D}_{a^+}^{\alpha} f\,(t) = \delta^n\,\mathcal{I}_{a^+}^{n-\alpha} = \frac{1}{\Gamma\,(n-\alpha)} \left(t\frac{d}{dt}\right)^n \int_a^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} y\,(s)\,\frac{ds}{s}, \tag{2.2}$$

where $n = [\alpha]+1$, and $[\alpha]$ denotes the integer part of the real number α and $\delta = t\frac{d}{dt}$. provided the right integral converges.

There is a recent generalization introduced by Jarad and al in [11], where the authors define the generalization of the Hadamard fractional derivatives and present properties of such derivatives. This new generalization is now known as the Caputo-Hadamard fractional derivatives and is given by the following definition:

Definition 3 (*Caputo-Hadamard* fractional derivative)**.** *(see [11]).*
Let $\alpha \geq 0$, and $n = [\alpha] + 1$. If $y(x) \in AC_{\delta}^{n}[a, b]$, where $0 < a < b < \infty$ and

$$AC_{\delta}^{n}[a, b] = \{g : [a, b] \to C : \delta^{n-1} g(x) \in AC[a, b]\}.$$

The left-sided Caputo-type modification of left- Hadamard fractional derivatives of order α is given by

$$^{C}\mathcal{D}_{a^+}^{\alpha} y\,(t) = \mathcal{D}_{a^+}^{\alpha} \left(y\,(t) - \sum_{k=0}^{n-1} \frac{\delta^k y\,(a)}{k!} \left(\log \frac{t}{a}\right)^k\right) \tag{2.3}$$

Theorem 4. *([11], Theorem 2.1).*
Let $\alpha \geq 0$, and $n = [\alpha] + 1$. If $y(t) \in AC_{\delta}^{n}[a, b]$, where $0 < a < b < \infty$. Then $^{C}\mathcal{D}_{a^+}^{\alpha} f\,(t)$ exist everywhere on $[a, b]$ and
(i) if $\alpha \notin \mathbb{N}_0$, $^{C}\mathcal{D}_{a^+}^{\alpha} f\,(t)$ can be represented by

$$^{C}\mathcal{D}_{a^+}^{\alpha} y\,(t) = \mathcal{I}_{a^+}^{n-\alpha}\,\delta^n y\,(t) = \frac{1}{\Gamma\,(n-\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \delta^n y\,(s)\,\frac{ds}{s}, \tag{2.4}$$

(ii) if $\alpha \in \mathbb{N}_0$, then

$$ {}^{C}\mathcal{D}^{\alpha}_{a+} y(t) = \delta^n y(t). \tag{2.5}$$

In particular

$$ {}^{C}\mathcal{D}^{0}_{a+} y(t) = y(t). \tag{2.6}$$

Caputo-Hadamard fractional derivatives can also be defined on the positive half axis $\mathbb{R}^+$ by replacing a by 0 in formula (2.4) provided that $y(t) \in AC^n_\delta(\mathbb{R})^+$. Thus one has

$$ {}^{C}\mathcal{D}^{\alpha}_{0+} y(t) \frac{1}{\Gamma(n-\alpha)} \int_0^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \delta^n y(s) \frac{ds}{s}. \tag{2.7}$$

Proposition 5. *(see [11, 12]).*
Let $\alpha > 0$, $\beta > 0, n = [\alpha] + 1$, and $a > 0$, then

$$\begin{aligned} &\mathcal{I}^{\alpha}_{a+}\left(\log \tfrac{t}{a}\right)^{\beta-1}(x) = \tfrac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\log \tfrac{x}{a}\right)^{\beta+\alpha-1}, \\ &{}^{C}\mathcal{D}^{\alpha}_{a+}\left(\log \tfrac{t}{a}\right)^{\beta-1}(x) = \tfrac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\log \tfrac{x}{a}\right)^{\beta-\alpha-1}, \quad \beta > n, \\ &{}^{C}\mathcal{D}^{\alpha}_{a+}\left(\log \tfrac{t}{a}\right)^{k} = 0, \quad k = 0, 1, ..., n-1. \end{aligned} \tag{2.8}$$

Theorem 6. *(see [8]).*
Let $u(t) \in AC^n_\delta[a,b]$, $0 < a < b < \infty$ and $\alpha \geq 0, \beta \geq 0$, Then

$$\begin{aligned} &{}^{C}\mathcal{D}^{\alpha}_{a+}\, \mathcal{I}^{\beta}_{a+}\, u(t) = \mathcal{I}^{\beta-\alpha}_{a+} u(t), \\ &{}^{C}\mathcal{D}^{\alpha}_{a+}\, {}^{C}\mathcal{D}^{\beta}_{a+} u(t) = {}^{C}\mathcal{D}^{\beta-\alpha}_{a+}\, u(t). \end{aligned} \tag{2.9}$$

Lemma 7. *(see [11]).*
Let $\alpha \geq 0$, and $n = [\alpha]+1$. If $u(t) \in AC^n_\delta[a,b]$, then the Caputo-Hadamard fractional differential equation

$$ {}^{C}\mathcal{D}^{\alpha}_{a+} u(t) = 0 \tag{2.10}$$

has a solution:

$$ u(t) = \sum_{k=0}^{n-1} c_k \left(\log \frac{t}{a}\right)^k \tag{2.11}$$

and the following formula holds:

$$ \mathcal{I}^{\alpha}_{a+}\, {}^{C}\mathcal{D}^{\alpha}_{a+} u(t) = u(t) + \sum_{k=0}^{n-1} c_k \left(\log \frac{t}{a}\right)^k \tag{2.12}$$

where $c_k \in \mathbb{R}$, $k = 1, 2, ..., n-1$.

Definition 8. *(see [9]). Let (X, d) be a metric space.*
A mapping $F : X \to X$ is Lipschitz continuous if there exists a constant $L > 0$ such that

$$ d(Fx, Fy) \leq Ld(x, y) \text{ for all } x, y \in X. $$

If $0 \leq L < 1$, then F is called a contraction mapping and L is called the contractivity factor of F.

Theorem 9. *(Banach contraction principle, [9]).*
Let (X, d) be a complete metric space and $F : X \to X$ be contractive. Then F has a unique fixed point u, and furthermore, for any $x \in X$, the sequence $(F^k x)_{k\geq 0}$, where $F^k = \underbrace{F \circ F \circ ... \circ F}_{k}$ converges and $\lim\limits_{k\to\infty} F^k x = u$ for each $x \in X$.

Theorem 10. *(Krasnoselskii's fixed point theorem, [15]).*
Let M be a closed, bounded, convex and nonempty subset of a Banach space X. Let A, B be the operators such that
(a) $Ax + By \in M$ whenever $x, y \in M$
(b) A is compact and continuous
(c) B is a contraction mapping.
Then there exists $z \in M$ such that $z = Az + Bz$.

3 Main Results

First, we prove a preparatory lemma for boundary value problem of linear fractional differential equations with Caputo-Hadamard derivative.

Lemma 11. *Let $y(t) \in AC^n_\delta[1, e]$.*
Then the unique solution of the following Caputo-Hadamard fractional differential equation:

$$ {}^C\mathcal{D}^\alpha_{a^+} u\,(t) = y(t), \quad t \in [1, e]\,,\ 2 < \alpha \leq 3, \tag{3.1}$$

with the fractional boundary conditions

$$u\,(1) = u'\,(1) = 0,\ \left({}^C\mathcal{D}^{\alpha-1}_{1^+} u\right)(e) = \left({}^C\mathcal{D}^{\alpha-2}_{1^+} u\right)(e) = 0, \tag{3.2}$$

is given by the integral equation

$$u\,(t) = \int_1^e G\,(t, s)\, y\,(s)\, ds, \tag{3.3}$$

where

$$G(t,s) =$$
$$= \begin{cases} \dfrac{\left(\log\frac{t}{s}\right)^{\alpha-1}}{s\Gamma(\alpha)} + \dfrac{\left[\frac{\Gamma(5-\alpha)(\log t)^2}{2} - \frac{\Gamma(6-\alpha)(\log t)^3}{6}\right]}{s} + \dfrac{\left[\frac{(4-\alpha)\Gamma(6-\alpha)(\log t)^3}{6} - \frac{\Gamma(6-\alpha)(\log t)^2}{2}\right]\left(\log\frac{e}{s}\right)}{s}, & 1 \leq s \leq t \leq e \\ \dfrac{\left[\frac{\Gamma(5-\alpha)}{2}(\log t)^2 - \frac{\Gamma(6-\alpha)}{6}(\log t)^3\right]}{s} + \dfrac{\left[\frac{(4-\alpha)\Gamma(6-\alpha)}{6}(\log t)^3 - \frac{\Gamma(6-\alpha)}{2}(\log t)^2\right]\left(\log\frac{e}{s}\right)}{s}, & 1 \leq t \leq s \leq e \end{cases} \tag{3.4}$$

is called the Green function of boundary value problem (3.1)-(3.2).

Proof. By Lemma 7, (2.12), we can reduce the equation (3.1) to an equivalent integral equation

$$u\,(t) = \mathcal{I}^\alpha_{1^+} y(t) + c_0 + c_1\,(\log t) + c_2\,(\log t)^2 + c_3\,(\log t)^3\,,$$

for some constants $c_0, c_1, c_2, c_3 \in \mathbb{R}$.
As boundary conditions for Problem (3.1), by Theorem 6, (2.9), and Proposition 5 (2.8), we have

$$\begin{aligned}
u(1) &= u'(1) = 0 \text{ implies that } c_0 = c_1 = 0 \\
\left({}^{C}\mathcal{D}_{1^+}^{\alpha-1}u\right)(e) &= \left({}^{C}\mathcal{D}_{1^+}^{\alpha-1}\,\mathcal{I}_{1^+}^{\alpha}y\right)(e) + c_2\left({}^{C}\mathcal{D}_{1^+}^{\alpha-1}(\log t)^2\right)(e) + c_3\left({}^{C}\mathcal{D}_{1^+}^{\alpha-1}(\log t)^3\right)(e) \\
&= \left(\mathcal{I}_{1^+}^{1}y\right)(e) + \frac{2c_2}{\Gamma(4-\alpha)} + \frac{6c_3}{\Gamma(5-\alpha)} = 0, \\
\left({}^{C}\mathcal{D}_{1^+}^{\alpha-2}u\right)(e) &= \left({}^{C}\mathcal{D}_{1^+}^{\alpha-2}\,\mathcal{I}_{1^+}^{\alpha}y\right)(e) + c_2\left({}^{C}\mathcal{D}_{1^+}^{\alpha-2}(\log t)^2\right)(e) + c_3\left({}^{C}\mathcal{D}_{1^+}^{\alpha-2}(\log t)^3\right)(e) \\
&= \left(\mathcal{I}_{1^+}^{2}y\right)(e) + \frac{2c_2}{\Gamma(5-\alpha)} + \frac{6c_3}{\Gamma(6-\alpha)} = 0,
\end{aligned}$$

that is,

$$\begin{aligned}
c_2 &= \frac{\Gamma(5-\alpha)}{2}\left(\mathcal{I}_{1^+}^{1}y\right)(e) - \frac{\Gamma(6-\alpha)}{2}\left(\mathcal{I}_{1^+}^{2}y\right)(e) \\
&= \frac{\Gamma(5-\alpha)}{2}\int_1^e y(s)\frac{ds}{s} - \frac{\Gamma(6-\alpha)}{2}\int_1^e\left(\log\frac{e}{s}\right)y(s)\frac{ds}{s}. \\
c_3 &= \frac{(4-\alpha)\Gamma(6-\alpha)}{6}\left(\mathcal{I}_{1^+}^{2}y\right)(e) - \frac{\Gamma(6-\alpha)}{6}\left(\mathcal{I}_{1^+}^{1}y\right)(e) \\
&= \frac{(4-\alpha)\Gamma(6-\alpha)}{6}\int_1^e\left(\log\frac{e}{s}\right)y(s)\frac{ds}{s} - \frac{\Gamma(6-\alpha)}{6}\int_1^e y(s)\frac{ds}{s}.
\end{aligned}$$

Therefore, the unique solution of the boundary value problem (3.1)-(3.2) is

$$\begin{aligned}
&u(t) = \\
&= \tfrac{1}{\Gamma(\alpha)}\textstyle\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}y(s)\frac{ds}{s} + \left[\frac{\Gamma(5-\alpha)}{2}(\log t)^2 - \frac{\Gamma(6-\alpha)}{6}(\log t)^3\right]\int_1^e y(s)\frac{ds}{s} \\
&+ \left[\tfrac{(4-\alpha)\Gamma(6-\alpha)}{6}(\log t)^3 - \tfrac{\Gamma(6-\alpha)}{2}(\log t)^2\right]\textstyle\int_1^e\left(\log\frac{e}{s}\right)y(s)\frac{ds}{s} \\
&= \textstyle\int_1^t\left[\frac{\left(\log\frac{t}{s}\right)^{\alpha-1}}{s\Gamma(\alpha)} + \frac{\left[\frac{\Gamma(5-\alpha)}{2}(\log t)^2 - \frac{\Gamma(6-\alpha)}{6}(\log t)^3\right]}{s} + \frac{\left[\frac{(4-\alpha)\Gamma(6-\alpha)}{6}(\log t)^3 - \frac{\Gamma(6-\alpha)}{2}(\log t)^2\right]\left(\log\frac{e}{s}\right)}{s}\right]y(s)\,ds \\
&+ \textstyle\int_t^e\left[\frac{\left[\frac{\Gamma(5-\alpha)}{2}(\log t)^2 - \frac{\Gamma(6-\alpha)}{6}(\log t)^3\right]}{s} + \frac{\left[\frac{(4-\alpha)\Gamma(6-\alpha)}{6}(\log t)^3 - \frac{\Gamma(6-\alpha)}{2}(\log t)^2\right]\left(\log\frac{e}{s}\right)}{s}\right]y(s)\,ds \\
&= \textstyle\int_1^e G(t,s)\,y(s)\,ds.
\end{aligned}$$

□

We now turn to the question of existence for boundary value problem (1.1)-(1.2). Let $E = C([1,e],\mathbb{R})$ with $\|u\| = \max_{t\in[1.e]}|u(t)|$ be Banach space. If u is a solution of (1.1)-(1.2) then

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s,u(s)) \frac{ds}{s} \\ &+ \left[\frac{\Gamma(5-\alpha)}{2} (\log t)^2 - \frac{\Gamma(6-\alpha)}{6} (\log t)^3\right] \int_1^e f(s,u(s)) \frac{ds}{s} \\ &+ \left[\frac{(4-\alpha)\Gamma(6-\alpha)}{6} (\log t)^3 - \frac{\Gamma(6-\alpha)}{2} (\log t)^2\right] \int_1^e \left(\log \frac{e}{s}\right) f(s,u(s)) \frac{ds}{s}. \end{aligned}$$

Define a mapping $F : E \to E$ by

$$\begin{aligned} Fu(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s,u(s)) \frac{ds}{s} \\ &+ \left[\frac{\Gamma(5-\alpha)}{2} (\log t)^2 - \frac{\Gamma(6-\alpha)}{6} (\log t)^3\right] \int_1^e f(s,u(s)) \frac{ds}{s} \\ &+ \left[\frac{(4-\alpha)\Gamma(6-\alpha)}{6} (\log t)^3 - \frac{\Gamma(6-\alpha)}{2} (\log t)^2\right] \int_1^e \left(\log \frac{e}{s}\right) f(s,u(s)) \frac{ds}{s}. \end{aligned} \tag{3.5}$$

and the some important constants

$$\omega = \frac{1}{\Gamma(\alpha+1)} + \frac{\Gamma(5-\alpha)}{2} + \frac{(4-\alpha)\Gamma(6-\alpha)}{12}. \tag{3.6}$$

We will prove the first following result via standard fixed point theorems.

Theorem 12. *Assume that $f(t,u)$ is continuous on $[1,e] \times \mathbb{R}$, and there exists a constant $L > 0$ such that:*

$$(H_1) \qquad \forall u, v \in \mathbb{R};\ \ |f(t,u) - f(t,v)| \leq L|u-v|,\ t \in [1,e].$$

If

$$L\omega < 1, \tag{3.7}$$

then the fractional boundary value problem (1.1)-(1.2) has a unique solution in $[1,e]$.

Proof. We will consider the equivalent integral equation problem given by $Fu = u$, where the operator Fu is defined by (3.5). Using the Banach contraction principle, we shall show that Fu has a fixed point.
Fixing $\max\limits_{t\in[1.e]} |f(t,0)| = M$ and choosing $r \geq \frac{M\omega}{1-L\omega}$. We prove that $F(B_r) \subset B_r$, where

$$B_r = \{u \in C(E,\mathbb{R}) /\ \ \|u\| \leq r\},$$

For $u \in B_r$, we have
$\|Fu\| \leq$

$$\leq \max_{t\in[1.e]}\left\{\begin{array}{l}\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}|f(s,u(s))|\frac{ds}{s}+\left[\frac{\Gamma(5-\alpha)}{2}(\log t)^2-\frac{\Gamma(6-\alpha)}{6}(\log t)^3\right]\int_1^e|f(s,u(s))|\frac{ds}{s}\\+\left[\frac{(4-\alpha)\Gamma(6-\alpha)}{6}(\log t)^3-\frac{\Gamma(6-\alpha)}{2}(\log t)^2\right]\int_1^e\left(\log\frac{e}{s}\right)|f(s,u(s))|\frac{ds}{s}\end{array}\right\}$$
$$\leq \max_{t\in[1.e]}\left\{\begin{array}{l}\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}[|f(s,u(s))-f(s,0)|+|f(s,0)|]\frac{ds}{s}\\+\left[\frac{\Gamma(5-\alpha)}{2}(\log t)^2-\frac{\Gamma(6-\alpha)}{6}(\log t)^3\right]\int_1^e[|f(s,u(s))-f(s,0)|+|f(s,0)|]\frac{ds}{s}\\+\left[\frac{(4-\alpha)\Gamma(6-\alpha)}{6}(\log t)^3-\frac{\Gamma(6-\alpha)}{2}(\log t)^2\right]\int_1^e\left(\log\frac{e}{s}\right)[|f(s,u(s))-f(s,0)|+|f(s,0)|]\frac{ds}{s}\end{array}\right\}$$
$$\leq (Lr+M)\max_{t\in[1.e]}\left\{\begin{array}{l}\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{ds}{s}+\left[\frac{\Gamma(5-\alpha)}{2}(\log t)^2-\frac{\Gamma(6-\alpha)}{6}(\log t)^3\right]\int_1^e\frac{ds}{s}\\+\left[\frac{(4-\alpha)\Gamma(6-\alpha)}{6}(\log t)^3-\frac{\Gamma(6-\alpha)}{2}(\log t)^2\right]\int_1^e\left(\log\frac{e}{s}\right)\frac{ds}{s}\end{array}\right\}$$
$$\leq (Lr+M)\max_{t\in[1.e]}\left\{\begin{array}{l}\frac{1}{\Gamma(\alpha+1)}\left[-\left(\log\frac{t}{s}\right)^{\alpha}\right]_1^t+\left[\frac{\Gamma(5-\alpha)}{2}(\log t)^2-\frac{\Gamma(6-\alpha)}{6}(\log t)^3\right][(\log s)]_1^e\\+\left[\frac{(4-\alpha)\Gamma(6-\alpha)}{6}(\log t)^3-\frac{\Gamma(6-\alpha)}{2}(\log t)^2\right]\left[-\frac{1}{2}\left(\log\frac{e}{s}\right)^2\right]_1^e\end{array}\right\}$$
$$\leq (Lr+M)\max_{t\in[1.e]}\left\{\begin{array}{l}\frac{1}{\Gamma(\alpha+1)}(\log t)^{\alpha}+\left[\frac{\Gamma(5-\alpha)}{2}(\log t)^2-\frac{\Gamma(6-\alpha)}{6}(\log t)^3\right]\\+\left[\frac{(4-\alpha)\Gamma(6-\alpha)}{6}(\log t)^3-\frac{\Gamma(6-\alpha)}{2}(\log t)^2\right]\left(\frac{1}{2}\right)\end{array}\right\}$$
$$\leq (Lr+M)\left\{\frac{1}{\Gamma(\alpha+1)}+\frac{\Gamma(5-\alpha)}{2}+\frac{(4-\alpha)\Gamma(6-\alpha)}{12}\right\}$$
$$\leq (Lr+M)\,\omega$$
$$\leq r.$$

which proves that $F(B_r)\subset B_r$.

Now let $u,v\in C([1,e],\mathbb{R})$. Then, for $t\in[1,e]$, we have

$$|(Fu)(t)-(Fv)(t)|\leq$$
$$\leq \max_{t\in[1.e]}\left\{\begin{array}{l}\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}|f(s,u(s))-f(s,v(s))|\frac{ds}{s}+\\\left[\frac{\Gamma(5-\alpha)}{2}(\log t)^2-\frac{\Gamma(6-\alpha)}{6}(\log t)^3\right]\int_1^e|f(s,u(s))-f(s,v(s))|\frac{ds}{s}+\\\left[\frac{(4-\alpha)\Gamma(6-\alpha)(\log t)^3}{6}-\frac{\Gamma(6-\alpha)(\log t)^2}{2}\right]\int_1^e\left(\log\frac{e}{s}\right)|f(s,u(s))-f(s,v(s))|\frac{ds}{s}\end{array}\right\}$$
$$\leq L\|u-v\|\max_{t\in[1.e]}\left\{\begin{array}{l}\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{ds}{s}+\left[\frac{\Gamma(5-\alpha)}{2}(\log t)^2-\frac{\Gamma(6-\alpha)}{6}(\log t)^3\right]\int_1^e\frac{ds}{s}\\+\left[\frac{(4-\alpha)\Gamma(6-\alpha)}{6}(\log t)^3-\frac{\Gamma(6-\alpha)}{2}(\log t)^2\right]\int_1^e\left(\log\frac{e}{s}\right)\frac{ds}{s}\end{array}\right\}$$
$$\leq L\|u-v\|\max_{t\in[1.e]}\left\{\begin{array}{l}\frac{1}{\Gamma(\alpha+1)}(\log t)^{\alpha}+\left[\frac{\Gamma(5-\alpha)}{2}(\log t)^2-\frac{\Gamma(6-\alpha)}{6}(\log t)^3\right]\\+\left[\frac{(4-\alpha)\Gamma(6-\alpha)}{6}(\log t)^3-\frac{\Gamma(6-\alpha)}{2}(\log t)^2\right]\left(\frac{1}{2}\right)\end{array}\right\}$$
$$\leq L\|u-v\|\left\{\frac{1}{\Gamma(\alpha+1)}+\frac{\Gamma(5-\alpha)}{2}+\frac{(4-\alpha)\Gamma(6-\alpha)}{12}\right\}$$
$$\leq L\omega\|u-v\|.$$

Therefore

$$\|Fu-Fv\|\leq L\omega\|u-v\|.$$

From the assumption (3.7) and the preceding estimate, it follows that F is a contraction mapping. Applying theorem 9, the operator F has a fixed point which corresponds to the unique solution of the problem (1.1)-(1.2). This completes the proof. □

We will prove the second result following existence result via Krasnoselskii's fixed point theorem.

Theorem 13. *Let $f : [1, e] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying (H_2). In addition, we assume that:*

$$(H_2) \qquad f(t,u) \leq \mu(t) \quad \forall (t,u) \in [1,e] \times \mathbb{R} \text{ and } \mu \in C([1,e], \mathbb{R}^+).$$

Then the problem (1.1)-(1.2) has at least one solution on $[1, e]$ *if*

$$\frac{\Gamma(5-\alpha)}{2} + \frac{(4-\alpha)\Gamma(6-\alpha)}{12} < 1. \tag{3.8}$$

Proof. We define $\sup\limits_{t\in[1.e]} |\mu(t)| = \|\mu\|$ and choose a suitable constant $\bar{r}$ as

$$\bar{r} \geq \|\mu\| \, \omega$$

where ω is defined by (3.6). We define the operators P and Q on

$$B_{\bar{r}} = \{u \in C([1,e], \mathbb{R}) : \|u\| \leq \bar{r}\}$$

as

$$Pu(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s, u(s)) \frac{ds}{s},$$

$$\begin{aligned} Qu(t) = & \left[\tfrac{\Gamma(5-\alpha)}{2} (\log t)^2 - \tfrac{\Gamma(6-\alpha)}{6} (\log t)^3\right] \int_1^e f(s, u(s)) \tfrac{ds}{s} \\ & + \left[\tfrac{(4-\alpha)\Gamma(6-\alpha)}{6} (\log t)^3 - \tfrac{\Gamma(6-\alpha)}{2} (\log t)^2\right] \int_1^e \left(\log \tfrac{e}{s}\right) f(s, u(s)) \tfrac{ds}{s}. \end{aligned}$$

For $u, v \in B_{\bar{r}}$ we find that

$$\begin{aligned} \|Pu + Qv\| & \leq \|\mu\| \left\{ \begin{array}{l} \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{ds}{s} + \left[\frac{\Gamma(5-\alpha)}{2} (\log t)^2 - \frac{\Gamma(6-\alpha)}{6} (\log t)^3\right] \int_1^e \frac{ds}{s} \\ + \left[\frac{(4-\alpha)\Gamma(6-\alpha)}{6} (\log t)^3 - \frac{\Gamma(6-\alpha)}{2} (\log t)^2\right] \int_1^e \left(\log \frac{e}{s}\right) \frac{ds}{s} \end{array} \right\} \\ & \leq \|\mu\| \, \omega \\ & \leq \bar{r}. \end{aligned}$$

Thus $Pu + Qv \in B_{\bar{r}}$. It follows from the assumption (12 H_1) together with (3.8) that Q is a contraction mapping. Continuity of f implies that the operator P is continuous. Also P is uniformly bounded on $B_{\bar{r}}$ as

$$\|Pu\| \leq \frac{\|\mu\|}{\Gamma(\alpha + 1)}.$$

Now we will prove the compactness of the operator P.
We define $\sup\limits_{t\in[1.e]\times B_{\bar{r}}} |f(t,u)| = \bar{f} < \infty$, $\tau_1, \tau_2 \in [1, e]$ with $\tau_1 < \tau_2$ and consequently,

we have

$$
\begin{aligned}
&\left|(Pu)(\tau_1)-(Pu)(\tau_2)\right| \leq \\
\leq\ & \frac{\bar{f}}{\Gamma(\alpha)}\left|\int_1^{\tau_1}\left(\log\frac{\tau_1}{s}\right)^{\alpha-1}\frac{ds}{s}-\int_1^{\tau_2}\left(\log\frac{\tau_2}{s}\right)^{\alpha-1}\frac{ds}{s}\right| \\
\leq\ & \frac{\bar{f}}{\Gamma(\alpha)}\left|\int_1^{\tau_1}\left[\left(\log\frac{\tau_1}{s}\right)^{\alpha}-\left(\log\frac{\tau_2}{s}\right)^{\alpha}\right]\frac{ds}{s}\right|+\frac{\bar{f}}{\Gamma(\alpha)}\left|\int_{\tau_1}^{\tau_2}\left(\log\frac{\tau_2}{s}\right)^{\alpha}\frac{ds}{s}\right| \\
\leq\ & \frac{\bar{f}}{\Gamma(\alpha+1)}\left[\left|(\log\tau_1)^{\alpha}+\left(\log\frac{\tau_2}{\tau_1}\right)^{\alpha}-(\log\tau_2)^{\alpha}\right|+\left|\left(\log\frac{\tau_2}{\tau_1}\right)^{\alpha}\right|\right],
\end{aligned}
$$

tends to zero as $\tau_1-\tau_2\to 0$. Thus, P is equicontinuous. So P is relatively compact on $B_{\bar{r}}$. Hence, by the Arzela-Ascoli theorem, P is compact on $B_{\bar{r}}$. Thus all the assumptions of Theorem 10 are satisfied. So the conclusion of Theorem 10 implies that the fractional boundary value problem (1.1)-(1.2) has at least one solution on $[1,e]$. The proof is completed. □

Example 14. *Consider the following boundary value problem*

$$
\begin{cases}
{}^{C}\mathcal{D}_{1^+}^{\alpha}u(t)=\frac{\arctan t}{2+|u|}, & t\in[1,e], \\
u(1)=u'(1)=0, & \\
\left({}^{C}\mathcal{D}_{1^+}^{\alpha-1}u\right)(e)=\left({}^{C}\mathcal{D}_{1^+}^{\alpha-2}u\right)(e)=0, &
\end{cases}
\tag{3.9}
$$

Here for $f(t,u(t))=\frac{\arctan t,}{1+|u|}$ *we have*
(12 H_1) $\quad \forall u,v\in\mathbb{R};\ |f(t,u)-f(t,v)|\leq\frac{\pi}{8}|u-v|,\ t\in[1,e]$
If $\alpha=2.75$, *then*

$$
\omega=\frac{1}{\Gamma(3.75)}+\frac{\Gamma(2.25)}{2}+\frac{(1.25)\,\Gamma(3.25)}{12}\simeq 1.05805...
$$

Therefore (12 H_1) is satisfied with $L=\frac{\pi}{8}$, *Further,* $L\omega=\frac{\pi}{8}\times(1.05805)\simeq 0.414952<1$.
Thus, by Theorem 12, the boundary value problem (3.9) *has a unique solution on* $[1,e]$.

Example 15. *Consider the boundary value problem*

$$
\begin{cases}
{}^{C}\mathcal{D}_{1^+}^{\alpha}u(t)=\frac{\log t}{1+u^2}, & t\in[1,e], \\
u(1)=u'(1)=0, & \\
\left({}^{C}\mathcal{D}_{1^+}^{\alpha-1}u\right)(e)=\left({}^{C}\mathcal{D}_{1^+}^{\alpha-2}u\right)(e)=0, &
\end{cases}
\tag{3.10}
$$

Here for $f(t,u(t))=\frac{\log t}{1+u^2}$ *we have*

$$
(13H_2)\qquad |f(t,u)|\leq\mu(t)=\log t,\ \forall(t,u)\in[1,e]\times\mathbb{R}
$$

If $\alpha = 2.75$, *(13 H_2) is satisfied and*

$$\frac{\Gamma(5-\alpha)}{2} + \frac{(4-\alpha)\Gamma(6-\alpha)}{12} \simeq 0.83205 < 1.$$

Thus, by Theorem 13, the boundary value problem (3.10) *has at least one solution on* $[1, e]$.

Acknowledgment. The authors are deeply grateful to the anonymous referees for their kind comments.

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Arioua Yacine
Laboratory for Pure and Applied Mathematics, University of M'sila, Bp 166 M'sila, 28000, Algeria.
e-mail: ariouayacine@ymail.com

Benhamidouche Nouredine
Laboratory for Pure and Applied Mathematics, University of M'sila, Bp 166 M'sila, 28000, Algeria.
e-mail: benhamidouche@yahoo.fr

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