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On the Continuity of the free boundary in a class of two-dimensional elliptic problems with Neuman boundary condition

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Résumé : In this work, we study the continuity of free boundary, in a class of elliptic problems, with Neuman boundary condition, which generalize the work of [5]. We prove that the free boundary is represented locally by a family of continuous functions.

Mots-Clefs : free boundary, continuity, Neuman boundary condition.

1 Statement of the problem and preliminary results

Let Ω be a bounded domain of $\mathbb{R}^2$, with C^1 boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. Let $a = (a_{ij})$ be a two-by-two matrix with for λ and Λ are positive constants;

$$a_{ij} \in L^\infty(\Omega), \qquad |a(x)| \leq \Lambda, \qquad \text{for a.e. } x \in \Omega, \qquad a(x)\xi.\xi \geq \lambda|\xi|^2 \quad \forall \xi \in \mathbb{R}^2, \qquad \text{for a.e. } x \in \Omega,$$

Let $H = (H_1, H_2) \in C^1(\overline{\Omega})$ be a vector function, satisfying for some positive constants $\bar{H} \geq \underline{H}$ and $p > 2$

$$\begin{aligned} &|H_1(x)| \leq \bar{H},\ \underline{H} \leq H_2(x) \leq \bar{H} \qquad \operatorname{div} H(x) \geq 0 \qquad \text{for a.e.}\, x \in \Omega \\ &\operatorname{div} H(x) \in L^p_{loc}(\Omega), \qquad H(x).\nu \neq 0\, \forall x \in \partial\Omega. \end{aligned}$$

Let $\beta(x, v)$ be a nonnegative, continuous function such that $\beta(x, .)$ non-decreasing for a.e. $x \in \Gamma_3$.

We consider the following problem

$$(P) \begin{cases} \text{Find}(u, \chi) \in H^1(\Omega) \times L^\infty(\Omega) \text{ such that } : \\ (i) \quad u \geq 0, \quad 0 \leq \chi \leq 1, \quad u(1-\chi) = 0 \ \text{ a.e. in } \Omega \\ (i) \quad u = 0 \quad \text{ a.e. on } \Gamma_2 \\ (ii) \quad \displaystyle\int_\Omega \big(a(x)\nabla u + \chi H(x)\big).\nabla\xi dx \leq \int_\Gamma \beta(x, \varphi - u)\xi d\sigma(x) \\ \qquad\qquad \forall \xi \in H^1(\Omega), \quad \xi \geq 0 \text{ on } \Gamma_2 \end{cases}$$

Consider the following differential system: $(E(w,h)) \begin{cases} X'(t,w,h) = H(X(t,w,h)) \\ X(0,w,h) = (w,h) \end{cases}$

where: $h \in \pi_{x_2}(\Omega), w \in \pi_{x_1}(\Omega \cap [x_2 = h])$.

This system has a maximal solution $X(.,w,h)$ defined on: $(\alpha_-(w,h),\alpha_+(w,h))$ and consider the mappings:

$$T_h: \begin{array}{ccc} D_h & \longrightarrow & T_h(D_h) \\ (t,w) & \longmapsto & T_h(t,w)=X(t,w) \end{array} \qquad S_h: \begin{array}{ccc} D_h & \longrightarrow & S_h(D_h) \\ (t,w) & \longmapsto & S_h(t,w)=(w,\tau) \end{array}$$

Where: $\tau = L_h(t,w) = \int_{\alpha_-(w)}^{t} |X'(s,w)|ds = \int_{\alpha_-(w)}^{t} |H(X(s,w))|ds$

2 Continuity of the Free Boundary

We define the function Φ_h in $\pi_{x_1}(\Omega\cap[x_2=h])$ by:

$$\Phi_h(w) = \begin{cases} \sup\{\tau : (w,\tau)\in S_h(D_h) : \widetilde{u}(w,\tau)>0\} & : \text{ if this set is not empty} \\ 0 & : \text{ otherwise} \end{cases} \tag{1}$$

Lemma 1 *If we have for some positive number μ,*

$$H.\nu - \beta(x,\varphi(x)) > \mu \quad in T.$$

then we have for $\epsilon>0$ small enough

$$\int_{\mathcal{T}_h(D)} \Big(a(x)\nabla v + \theta.H(x)\Big).\nabla\zeta dx \geq \int_{\Gamma_3} \beta(x,\varphi)\zeta d\sigma(x)$$

$$\forall\zeta\in H^1(\mathcal{T}_h(D)), \quad \zeta\geq 0, \quad \zeta=0 \ on\ \partial\mathcal{T}_h(D)\setminus\Gamma_3$$

The main result is the following theorem:

Theorem 2 *Let $w_0\in\pi_{x_1}\{x_2=h\}$ such that $(w_0,\Phi_h(w_0)\in S_h(D_h)$ and:*

$$\beta(X(\alpha_+(w_0),w_0),\varphi(X(\alpha_+(w_0),w_0))) < \widetilde{H}(w_0,\tau_+(w_0)).\nu(X(\alpha_+(w_0),w_0)) \tag{2}$$

Then Φ_h is contiuous at w_0.

References

[1] M. Chipot and A. Lyaghfouri . *The dam problem with linear Darcy's law and nonlinear leaky boundary conditions.* Advances in Differential Equations, Vol. 3, No. 1: 1–50, 1998.

[2] S. Challal and A. Lyaghfouri. *A Filtration Problem through a Heterogeneous Porous Medium.* Interfaces and Free Boundaries, 6: 55–79, 2004.

[3] S. Challal and A. Lyaghfouri. *On the Continuity of the Free Boundary in Problems of type* $div(a(x)\nabla u) = -(\chi(u)h(x))_{x_1}$. Nonlinear Analysis : Theory, Methods & Applications, Vol. 62, 2: 283–300, 2005.

[4] D. Gilbarg, N.S. Trudinger. *Elliptic Partial Differential Equations of Second Order.* Springer-Verlag, 1983.

[5] A. Saadi. *Coninuity of the free boundary in elliptic problems with Neuman boundary condition.* Electronic Journal of Differential Equations, Vol. 2015, No. 160: 1–16, 2015.