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Master memory

Field : Mathematics and computer sciences
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Option : Functional analysis

Theme

On generalized Orlicz spaces

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Introduction

The Orlicz spaces were introduced by Z.W.Birnbaum and W.Orlicz (1931)(see [3]) as a natural generalization of the classical Lebesgue spaces L^p, $1 < p < +\infty$. For this generalization the function x^p entering in the definition of L^p space is replaced by a more general convex function Φ , which is called an N-function and he studied on the Orlicz space associated to N-function. The first detailed study on Orlicz spaces was given by Krasnosel'skii and Rutickii (1961) (see [9]) and they considered the function Φ as an N-function that is based on the integral representation of the real valued convex function.

N-function and Young function are defined on $\mathbb{R}$ and Φ is taken an even function in Krasnosel'skii and Rutuckii (1961), Rao and Ren (1991) (see [17]) respectively. But in this memory we take the domain of the Φ as $[0, +\infty)$ for the convenience with the other definitions. Also, we recall that an N-function Φ is finite real valued convex function defined on $[0, +\infty)$, so this implies that Φ is necessarily continuous.

However, a Young function can have infinite value at a point, and hence may be discontinuous at such a point.

Moreover, recently, in several studies about Orlicz spaces especially on the composition operators (Arora and et al. (2007)(see [1]), Kumar (1997) (see [8]), Raj and Khosla (2009)(see [16])), the function Φ is defined differently from the Young function used in Rao and Ren's works but again they called this new function Φ as a Young function. We know that there are four different type of spaces : classical Lebesgue spaces L^p , Orlicz spaces, variable exponent Lebesgue spaces $L^{p(\cdot)}$ and generalized Orlicz spaces. Naturally, L^p-spaces are Orlicz spaces and $L^{p(\cdot)}$-spaces, and Orlicz and $L^{p(\cdot)}$-spaces are generalized Orlicz spaces. Orlicz spaces and $L^{p(\cdot)}$-spaces have different nature, and neither of them is a subset of the other.

As generalized Orlicz spaces have been an area of growing interest recently. so, the main topic treated in this memory is the representation of some definitions and basic properties of Φ-function and we use them to study some properties of generalized Orlicz spaces as convergence, completeness, separability, uniform convexity, reflexivity and density of smooth functions. The reader can find a lot of information about in the excellent monograph [4] and [6]

Generalized Orlicz spaces has become part of the mainstream research fields in contemporary functional analysis. so, In this memory, we study some definitions and basic properties of Φ-function and we use them to study some properties of generalized Orlicz spaces (also known as Musielak-Orlicz spaces) as convergence, completeness, separability, uniform convexity, reflexivity and density of smooth functions.

Preliminary notions

Preliminary notions

Some results about integration that everyone must know

Theorem

Monotone convergence theorem, Beppo Levi(H. Brezis 2010 [2])
Let (f_n) be a sequence of functions in L^1 that satisfy

(a) $f_1 \leq f_2 \leq ... \leq f_n \leq f_{n+1} \leq ...$ a.e. on Ω,

(b) $\sup_n \int f_n < \infty$.

Then $f_n(x)$ converges a.e. on Ω to a finite limit, which we denote by $f(x)$; the function f belongs to L^1 and $\|f_n - f\|_1 \to 0$.Let (f_n) be a sequence of functions in L^1 that satisfy

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(b) $\sup_n \int f_n < \infty$.

Then $f_n(x)$ converges a.e. on Ω to a finite limit, which we denote by $f(x)$; the function f belongs to L^1 and $\|f_n - f\|_1 \to 0$.

Some results about integration that everyone must know

Theorem

Dominated convergence theorem, Lebesgue(H. Brezis 2010 [2])

Let (f_n) be a sequence of functions in L^1 that satisfy

(a) $f_n(x) \to f(x)$ a.e. on Ω

(b) there is a function $g \in L^1$ such that for all n, $|f_n(x)| \leq g(x)$ a.e. on Ω. Then $f \in L^1$ and $\|f_n - f\|_1 \to 0$.

Fatou's lemma(H. Brezis 2010 [2])

Let (f_n) be a sequence of functions in L^1 that satisfy

(a) for all n, $f_n \geq 0$ a.e.

(b) $\sup_n \int f_n < \infty$.

For almost all $x \in \Omega$ we set $f(x) = \liminf_{n\to\infty} f_n(x) \leq +\infty$. Then $f \in L^1$ and

$$\int f \leq \liminf_{n\to\infty} \int f_n.$$

A basic example is the case in which $\Omega = \mathbb{R}^N$, $\mathcal{M}$ consists of the Lebesgue measurable sets, and μ is the Lebesgue measure on $\mathbb{R}^N$.

Definition (C.P. Niculescu and...(2006))[12]

[Convex Functions]
A function $f : I \to \mathbb{R}$ is called convex if

(1) $$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y)$$

for all points x and y in I and all $\lambda \in [0, 1]$.

Definition(C.P. Niculescu and...(2006))[12]

Let V be a real vector space. We say that function $\|\cdot\|$ from V to $[0, \infty]$ is a quasinorm if :

(a) $\|f\| = 0$ if and only if $f = 0$.
(b) $\|af\| = |a| \, \|f\|$ for all $f \in V$ and $a \in \mathbb{R}$.
(c) There exists $\lambda > 0$ such that $\|f + g\| \leq \lambda(\|f\| + \|g\|)$ for all $f, g \in V$.

- If $\lambda = 1$ in (c), then $\|\cdot\|$ is called a norm. A (quasi)Banach space $(V, \|\cdot\|_V)$ is a (quasi)normed vector space which is complete with respect the (quasi)norm $\|\cdot\|_V$.
- The dual space V^* of a (quasi)Banach space V consists of all bounded, linear functionals $F : V \to \mathbb{R}$. The duality pairing between V^* and V is defined by $\langle F, x\rangle_{V^*,V} = \langle F, x\rangle := F(x)$ for $F \in V, x \in V$. The dual space is equipped with the dual quasinorm $\|f\|_{V^*} := \sup_{\|x\|_V \leq 1} \langle F, x\rangle$, which makes V^* a quasi-Banach space.

- A space is called separable if it contains a dense, countable subset. We denote the bidual space by $V^{**} := (V^{*})$. A quasi-Banach space V is called reflexive if the natural injection $\iota : V \to V^{**}$, given by $\langle \iota x, F \rangle_{V^{**}, V^{*}} := \langle F, x \rangle_{V^{*}, V}$, is surjective. A norm $\|\cdot\|$ on a space V is called uniformly convex if for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for all$x, y \in V$ satisfying $\|x\|, \|y\| \leq 1$ the inequality $\|x - y\| > \varepsilon$ implies $\left\| \frac{x+y}{2} \right\| < 1 - \delta(\varepsilon)$. A quasi-Banach space V is called uniformly convex, if there exists a uniformly convex norm $\|\cdot\|'$, which is equivalent to the original norm of V.

Proposition

(C.P. Niculescu and...(2006))[12] Let V be a Banach space and let $W \subset V$ be closed. Then :

(a) W is a Banach space.

(b) If V is reflexive, then V is reflexive.

(c) If V is separable, then W is separable.

(d) If V is uniformly convex, then V is reflexive.

(e) If V is uniformly convex, then W is uniformly convex.

History of generalized Orlicz spaces

History of generalized Orlicz spaces

Variable exponent Lebesgue spaces appeared in the literature for the first time already in a 1931 article by W. Orlicz [14].Variable exponent spaces have been studied in more than a thousand papers in the past 15 years so we only cite a few monographs on the topic which can be consulted for additional references ([5]).

In 2011, Lars Diening, Petteri Harjulehto, Peter Hästö and Michael Ruzicka [4] have written a book on variable exponent spaces. Recently, in 2019 Petteri Harjulehto and Peter Hasto [6] have presented a systematic treatment of Orlicz and generalized Orlicz spaces (also known as Musielak-Orlicz spaces) in a general framework.

(**Władysław Orlicz**)

Φ-Functions

Equivalent Φ-functions [4] [7]

Definition

A function $h : (0, \infty) \longrightarrow \mathbb{R}$ is almost increasing if there exists a constant $\delta \geq 1$ such that $h(x) \leq \delta h(t)$ for all $0 < x < y$. Almost decreasing is defined analogously. Increasing and decreasing functions are included in the previous definition as the special case $\delta = 1$.

Definition

Let $f : (0, \infty) \longrightarrow \mathbb{R}$ and $p, q > 0$. We say that h satisfies

(2) $$\text{if } \frac{f(y)}{y^p} \text{ is increasing;}$$

(3) $$\text{if } \frac{f(t)}{y^p} \text{ is almost increasing;}$$

(4) $$\text{if } \frac{f(y)}{y^q} \text{ is decreasing;}$$

(5) $$\text{if } \frac{f(y)}{y^q} \text{ is almost decreasing;}$$

Equivalent Φ-functions

We say that f satisfies $(2)_{p>1}$, $(3)_{p>1}$, $(4)_{q<\infty}$ or $(5)_{q<\infty}$ if there exist $p>1$ or $q<\infty$ such that f satisfies (2),(3), (4) or (5), respectively.

Definition

Let $f:[0,\infty)\longrightarrow[0,\infty)$ be increasing with $\lim_{x\longrightarrow 0^+} f(x)=0$, $\lim_{x\longrightarrow\infty} f(x)=\infty$ and $f(0)=0$. Such f is called a Φ-prefunction.
We say that a Φ-prefunction f is a

- (weak) Φ-function, if it satisfies (3) with $p=1$ on $(0,\infty)$;
- (convex) Φ-function if it is left-continuous and convex ;
- (strong) Φ-function if it is continuous in the topology of $[0,\infty]$ and convex.

The sets of weak, convex and strong Φ-function are denoted by Φ_w , Φ_c and Φ_s respectively.

Equivalent Φ-functions

Lemma

If $f \in \Phi_w$ is left-continuous, then $f(x) \leq \liminf_{x_k \to x} f(x_k)$, i.e. f is lower semicontinuous.

Definition

Two functions f and g are equivalent, $f \simeq g$, if there exists $\eta \geq 1$ such that $f(\frac{x}{\eta}) \leq g(x) \leq f(\eta x)$ for all $x \geq 0$.

Lemma

Let $f, g : [0, \infty) \to [0, \infty]$ be increasing with $f \simeq g$.
(a) If f is a Φ-prefunction, then g is a Φ-prefunction.
(b) If f satisfies (3), then g satisfies (3).
(c) If f satisfies (5), then g satisfies (5).

Equivalent Φ-functions

Example [6]

we consider an example shows that (2) without the "almost" is not invariant under equivalence of Φ-functions.
Assume that $f(y) := y^2$ and
$g(y) := y^2 + \max\{y-1, 0\}$ for $y \geq 0$, see Figure 1.
Then $f(y) \leq g(y) \leq f(2y)$ so that $f \simeq g$.
Clearly f satisfies (2) with $p = 2$. Suppose that g satisfies (2) for $p \geq 1$. We write the condition at points $y = 2$ and $x = 3$:

$$\frac{4+1}{2^p} = \frac{g(2)}{2^p} \leq \frac{g(3)}{3^p} = \frac{9+2}{3^p}$$

i.e.$(\frac{3}{2})^p \leq \frac{11}{5}$.This means that $p < 2$, hence g does not satisfy (2) with $p = 2$

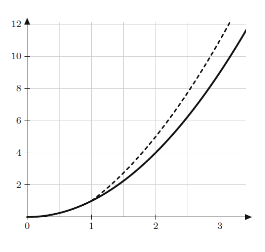

Figure 1 – Functions f(solid) and g (dashed) from Example 18

Lemma

Let f , g : $[0, \infty) \to [0, \infty]$.
(a) If f satisfies (3) with $p = 1$ and $f \approx g$, then $f \simeq g$ and g satisfies (3) with $p = 1$.
(b) If f satisfies $(5)_{q<\infty}$, then $f \simeq g$ implies $f \approx g$.

Upgrading Φ-functions

Lemma

If $f \in \Phi_W$ satisfies (3) with $p \geq 1$, then there exists $g \in \Phi_c$ equivalent to f such that $g^{1/p}$ is convex. In particular, g satisfies (2).

Corollary

If $f \in \Phi_w$, then there exists a constant $\lambda > 0$ such that

$$f\left(\lambda \sum_{k=1}^{\infty} \alpha_k \xi_k\right) \leq \sum_{k=1}^{\infty} f(\alpha_k)\xi_k$$

for all $\alpha_k, \xi_k \geq 0$ with $\sum \xi_k = 1$.

theorem

Every weak Φ-function is equivalent to a strong Φ-function

Upgrading Φ-functions

Definition

We say that a function $f : [0,\infty) \to [0,\infty]$ satisfies Δ_2, or is doubling if there exists a constant $\theta \geq 2$ such that .

$$f(2y) \leq \theta f(y) \quad \text{for all} \quad y \geq 0.$$

Lemma

(a) If $f \in \Phi_W$, then Δ_2 is equivalent to $(5)_{q<\infty}$,
(b) If $f \in \Phi_c$, then Δ_2 is equivalent to $(4)_{q<\infty}$.

Proposition

If $f \in \Phi_w$ satisfies (5), then there exists $g \in \Phi_s$ with $g \approx f$ which is a strictly increasing bijection.

Inverse Φ-Functions

Definition

By $f^{-1} : [0, \infty] \to [0, \infty]$ we denote the left-inverse of

$$f : [0, \infty] \to [0, \infty],$$

$$f^{-1}(\eta) := \inf\{y \geq 0 : f(y) \geq \eta\}.$$

In the Lebesgue case, the inverse of $y \mapsto y^p$ is $y \mapsto y^{\frac{1}{p}}$. As a more general intuition, this means that we flip (mirror) the function over the line $y = x$, and choose the value of the discontinuities so as to make the function left-continuous.

Inverse Φ-Functions

Example[6]

We define $f : [0, \infty] \to [0, \infty]$ by

$$f(y) := \begin{cases} 0 & \text{if } y \in [0, 2] \\ y - 2 & \text{if } y \in (2, 4] \\ 3 & \text{if } y \in (4, 6] \\ y - 3 & \text{if } y \in (6, \infty] \end{cases}$$

see Figure 02 . Then $f \in \Phi_W \backslash \Phi_c$ and the left-inverse is given by

$$f^{-1}(y) := \begin{cases} 0 & \text{if } y = 0 \\ y + 2 & \text{if } y \in (0, 2] \\ 4 & \text{if } y \in (2, 3] \\ y + 3 & \text{if } y \in (3, \infty] \end{cases}$$

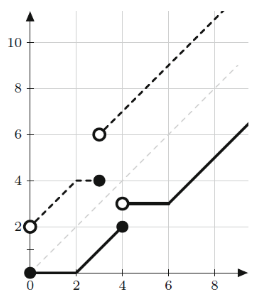

Figure 2 – A weak Φ-function (solid) and its left-inverse (dashed)

Inverse Φ-Functions

With these expressions we can calculate the compositions *fo* f^{-1} and $f^{-1}o\ f$:

$$f(f^{-1}(y)) := \begin{cases} y & \text{if} \quad y \in [0,2] \\ 2 & \text{if} \quad y \in (2,3] \\ y & \text{if} \quad y \in (3,\infty] \end{cases}$$

$$f^{-1}(f(y)) := \begin{cases} 0 & \text{if} \quad y \in [0,2] \\ y & \text{if} \quad y \in (2,4] \\ 4 & \text{if} \quad y \in (4,6] \\ y & \text{if} \quad y \in (6,\infty] \end{cases}$$

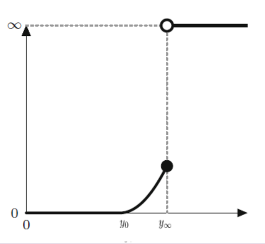

Figure 3 – y_0 **and** y_∞

We next investigate when the composition of f and f^{-1} is the identity. Note that the following result holds only for convex Φ-functions. In the set $\Phi_W \backslash \Phi_c$ the behaviour of the composition is more complicated, as indicated by the previous example.
In the proof we use $y_0 := \sup\{y : f(y) = 0\}$ and $y_\infty := \inf\{y : f(y) = \infty\}$ which are illustrated in figure 03.

Inverse Φ-Functions

Lemma

Let $f \in \Phi_c$, $y_0 := \sup\{y : f(y) = 0\}$ and $y_\infty := \inf\{y : f(y) = \infty\}$. Then

$$f^{-1}(f(y)) = \begin{cases} 0, & y \leq y_0, \\ t, & y_0 < y \leq y_\infty \\ y_\infty, & y > y_\infty. \end{cases} \quad \text{and} \quad f(f^{-1}(\eta)) = \min\{\eta, f(y_\infty)\}.$$

In particular, if $f \in \Phi_s$, then $f(f^{-1}(x)) \equiv x$.
Note that the last property means that f^{-1} is in fact the right-inverse of f in the sense of abstract algebra when $f \in \Phi_s$.

Corollary

Let $f \in \Phi_c$. If $f(x) \in (0, \infty)$, then $f^{-1}(f(x)) = x$.
Indeed, if $f \in \Phi_c$ satisfies $(5)_{q<\infty}$, then f is bijective and f^{-1} is just the regular inverse.

Inverse Φ-Functions

Theorem

Let $f, g \in \Phi_w$. Then $f \simeq g$ if and only if $f^{-1} \approx g^{-1}$.

Proposition

Let $f \in \Phi_w$ and $p, q > 0$. Then
(a) f satisfies (3) if and only if f^{-1} satisfies (5) with $q = \frac{1}{p}$.
(b) f satisfies (5) if and only if f^{-1} satisfies (3) with $p = \frac{1}{q}$.

Lemma

Let $f : [0, \infty] \to [0, \infty], \eta, y \geq 0$ and $\nu > 0$.

(a) Then f^{-1} is increasing, $f^{-1}(0) = 0$, $f^{-1}(f(y)) \leq y$ and $f(f^{-1}(\eta) - \nu) < \eta$ when $f^{-1}(\eta) \geq \nu$.
(b) If f is left-continuous with $f(0) = 0$, then $f(f^{-1}(\eta)) \leq \eta$.
(c) If f is increasing, then f^{-1} is left-continuous, $y \leq f^{-1}(f(y) + \nu)$ and $\eta \leq f(f^{-1}(\eta) + \nu)$.
(d) If f satisfies (3) with $p = \nu$, then $f^{-1}(f(y)) \approx y$, when $f(y) \in (0, \infty)$.
(e) If f with $\lim_{y \to 0^+} f(y) = 0$ satisfies $(5)_{q<\infty}$, then $f(f^{-1}(\eta)) \approx \eta$.

Inverse Φ-Functions

Lemma

Let $f : [0, \infty] \to [0, \infty]$. Then f is increasing and left-continuous if and only if $(f^{-1})^{-1} = f$.

Definition

We say that $\pi : [0, \infty] \to [0, \infty]$ belongs to Φ_w^{-1} if it is increasing, left-continuous, satisfies (5) with $q = 1$, $\pi(y) = 0$ if and only if $y = 0$, and, $\pi(y) = \infty$ if and only if $y = \infty$.

Let us denote be Φ_{w+} the set of left-continuous weak Φ-functions. We next show that Φ_w^{-1} characterizes inverses of Φ_{w+}-functions and that Φ_w^{-1} is an involution in Φ_{w+}.

Proposition

The transformation $f \mapsto f^{-1}$ is a bijection from Φ_{W+} to Φ_W^{-1},

(a) If $f \in \Phi_{W+}$, then $f^{-1} \in \Phi_W^{-1}$ and $(f^{-1})^{-1} = f$.

(b) If $\pi \in \Phi_W^{-1}$, then $\pi^{-1} \in \Phi_{W+}$ and $(\pi^{-1})^{-1} = \pi$.

Conjugate Φ-Functions [4],[6]

Definition

Let $f : [0, \infty) \to [0, \infty]$. We denote by f^* the conjugate function of f which is defined, for $v \geq 0$, by

$$f^*(v) := \sup_{y \geq 0} \Big(yv - f(y) \Big).$$

In the Lebesgue case $y \mapsto \frac{1}{p} y^p$, the conjugate is given by $y \mapsto \frac{1}{p'} y^{p'}$, where p' is the Hölder conjugate exponent. By definition of f^*,

$$yv \leq f(y) + f^*(v) \tag{6}$$

for every $y, v \geq 0$. This is called Young's inequality.

Lemma

If $f \in \Phi_w$, then $f^* \in \Phi_c$.

Conjugate Φ-Functions

Lemma

Let $f, g : [0,\infty) \to [0,\infty]$ and $\delta, \gamma > 0$.

(a) If $f \le g$, then $g^* \le f^*$.

(b) If $g(y) = \delta f(\gamma y)$ for all $y \ge 0$, then $g^*(v) = \delta f^*(\frac{v}{\delta\gamma})$ for all $v \ge 0$.

(c) If $f \simeq g$, then $f^* \simeq g^*$.

Lemma

Let $f \in \Phi_c$ and $\gamma := \lim_{y\to 0^+} \frac{f(y)}{y} = f'(0)$. Then $f^*(x) = 0$ if and only if $x \le \gamma$. Here $f'(0)$ is the right derivative of a convex function at the origin.

Proposition

Let $f \in \Phi_W$. Then $f^{**} \simeq f$ and f^{**} is the greatest convex minorant of f. In particular, if $f \in \Phi_c$, then $f^{**} = f$ and

$$f(y) = \sup_{v\ge 0}(yv - f^*(v)) \quad \text{for all} \quad y \ge 0$$

Conjugate Φ-Functions

Corollary

Let $f, g \in \Phi_c$. Then $f \leq g$ if and only if $g^* \leq f^*$.

Lemma

Let $f \in \Phi_c$ and $\gamma := \lim_{y \to 0^+} \frac{f(y)}{y} = f'(0)$. Then $f^*(x) = 0$ if and only if $x \leq \gamma$. Here $f'(0)$ is the right derivative of a convex function at the origin.

Theorem

If $f \in \Phi_w$, then $f^{-1}(y)(f^*)^{-1}(y) \approx y$.

Proposition

Let $f \in \Phi_W$. Then f satisfies (3) or (5) if and only if f^* satisfies (5) with $q = p'$ or (3) with $p = q'$, respectively.

Definition

We say that $f \in \Phi_w$ satisfies ∇_2, if f^* satisfies Δ_2.

Generalized Φ-Functions

Definition

Let $f : M \times [0, \infty) \to \mathbb{R}$ and $p, q > 0$. We say that f satisfies (3) or (5), if there exists $\delta \geq 1$ such that the function $y \mapsto f(x, y)$ satisfies (3) or (5) with a constant δ, respectively, for μ-almost every $x \in M$. When $\delta = 1$, we use the notation (2) and (4).

Definition

Let (M, Γ, μ) be a σ-finite, complete measure space. A function $f : M \times [0, \infty) \to [0, \infty]$ is said to be a (generalized) Φ-prefunction on (M, Γ, μ) if $x \mapsto f(x, |h(x)|)$ is measurable for every $h \in L^0(M, \mu)$ and $f(x, \cdot)$ is a Φ- prefunction for μ-almost every $x \in M$. We say that the Φ-prefunction f is

- a(generalized weak) Φ-function if f satisfies (3) with $p = 1$;
- a(generalized convex) Φ-function if $f(x, \cdot) \in \Phi_c$ for μ-almost all $x \in M$;
- a(generalized) strong) Φ-function if $f(x, \cdot) \in \Phi_s$ for μ-almost all $x \in M$.

If f is a generalized weak Φ-function on (M, Γ, μ),we write $f \in \Phi_w(M, \mu)$ and similarly we define $f \in \Phi_c(M, \mu)$ and $f \in \Phi_s(M, \mu)$. If Ω is an open subset of $\mathbb{R}^n$ and μ is the n-dimensional Lebesgue measure we omit μ and abbreviate $\Phi_w(\Omega)$, $\Phi_c(\Omega)$ or $\Phi_s(\Omega)$. Or we say that f is a generalized (weak/convex/strong)Φ-function on Ω. Unless there is danger of confusion, we will omit the word "generalized".
Clearly $\Phi_s(M, \mu) \subset \Phi_c(M, \mu) \subset \Phi_w(M, \mu)$. Every Φ-function is a generalized Φ-function if we set $f(x, y) := f(y)$ for $x \in M$ and $y \in [0, \infty)$.

Generalized Φ-Functions

Example

Let $\phi : M \to [1,\infty]$ be a measurable function and define $\phi_\infty := \limsup_{|x|\longrightarrow\infty} \phi(x)$. Let us interpret $y^\infty := \infty\chi_{(1,\infty]}(y)$. Let $\psi : M \to (0,\infty)$ be a measurable function and $1 \leq x < y < \infty$. Let us define, for $y \geq 0$,

$$\begin{aligned} f_1(x,y) &:= y^{\phi(x)}\psi(x) \\ f_2(x,y) &:= y^{\phi(x)}\log(e+y) \\ f_3(x,y) &:= \min\{y^{\phi(x)}, y^{\phi_\infty}\} \\ f_4(x,y) &:= y^{\phi(x)} + \sin(y) \\ g_1(x,y) &:= y^{\phi} + \psi(x)y^t \\ g_2(x,y) &:= (y-1)_+^s + \psi(x)(y-1)_+^t \end{aligned}$$

Observe that

$f_3 \in \Phi_W(M,\mu)\backslash\Phi_c(M,\mu)$ when ϕ is non-constant,

$f_4 \in \Phi_W(M,\mu)\backslash\Phi_c(M,\mu)$ when $\inf_{x\in M}\phi(x) \leq \frac{3}{2}$,

$f_1, f_2 \in \Phi_c(M,\mu)\backslash\Phi_s(M,\mu)$ when $\phi = \infty$ in a set of positive measure, and

$g_1, g_2 \in \Phi_s(M,\mu)$ when $\phi, t \in [1,\infty)$.

Moreover, if $\phi(x) < \infty$ for μ-almost every x, then $f_1, f_2 \in \Phi_s(M,\mu)$.

Measurability

Theorem

Let $f : M \times [0, \infty) \to [0, \infty], x \mapsto f(x, y)$ be measurable for every $y \geq 0$ and $y \mapsto f(x, y)$ be increasing and left-continuous for μ-almost every x. If $h \in L^0(M, \mu)$ is measurable, then $x \mapsto f(x, |h(x)|)$ is measurable.

Definition

We say that $f, g : M \times [0, \infty) \to [0, \infty]$ are equivalent, $f \simeq g$, if there exist $\tau > 1$ such that for all $y \geq 0$ and μ-almost all $x \in M$ we have

$$g(x, \frac{y}{\tau}) \leq f(x, y) \leq g(x, \tau y).$$

Measurability

Lemma

Let $f, g : M \times [0, \infty) \to [0, \infty], f \simeq g$, be increasing with respect to the second variable, and $x \mapsto f(x, |h(x)|)$ and $x \mapsto g(x, |h(x)|)$ be measurable for every measurable f .

(a) If f is a generalized Φ-prefunction, then g is a generalized Φ-prefunction.

(b) If f satisfies (3), then g satisfies (3).

(c) If f satisfies (5), then g satisfies (5).

Lemma

If $f \in \Phi_W(M, \mu)$, then $f^* \in \Phi_c(M, \mu)$.

Lemma

If $f \in \Phi_W(M, \mu)$ satisfies (3) with $p \geq 1$, then there exists $g \in \Phi_c(M, \mu)$ equivalent to f such that $g^{1/p}$ is convex. In particular, g satisfies (2).

Measurability

Theorem

Every weak Φ-function is equivalent to a strong Φ-function

Proposition

If $f \in \Phi_W(M,\mu)$ satisfies $(5)_{q<\infty}$, then there exists $g \in \Phi_s(M,\mu)$ with $g \approx f$ such that $y \mapsto g(x,y)$ is a strictly increasing bijection for μ-almost every $x \in M$.

Lemma

Let $f : M \times [0,\infty) \to [0,\infty]$. If $y \mapsto f(x,y)$ is increasing for μ-almost every x and if $x \mapsto f(x,y)$ is measurable for every $y \geq 0$, then $x \mapsto f^{-1}(x,|h(x)|)$ is measurable for every measurable h .

Measurability

Definition

We say that $\pi : M \times [0,\infty] \to [0,\infty]$ belongs to $\Phi_W^{-1}(M,\mu)$ if it satisfies (3) with $p = 1$, $x \mapsto \xi(x,y)$ is measurable for all y and if for μ-almost every $x \in A$ the function $y \mapsto \pi(x,y)$ is increasing, left-continuous, and $\pi(x,y) = 0$ if and only if $y = 0$ and $\pi(x,y) = \infty$ if and only if $y = \infty$.

Proposition

The transformation $f \mapsto f^{-1}$ is a bijection from Φ_{W_+} to Φ_W^{-1} :

(a) If $f \in \Phi_{W_+}(M,\mu)$, then $f^{-1} \in \Phi_W^{-1}(M,\mu)$ and $(f^{-1})^{-1} = f$.

(b) If $\pi \in \Phi_W^{-1}(M,\mu)$, then $\pi^{-1} \in \Phi_{W_+}(M,\mu))$ and $(\pi^{-1})^{-1} = \pi$.

Weak Equivalence and Weak Doubling

Definition

We say that $f, g : M \times [0, \infty) \to [0, \infty]$ are weakly equivalent, $f \sim g$, if there exist $\tau > 1$ and $h \in L^1(M, \mu)$ such that

$$\varphi(x, y) \leq \psi(x, \tau y) + h(x) \quad \text{and} \quad g(x, y) \leq f(x, \tau y) + h(x)$$

for all $y \geq 0$ and μ-almost all $x \in M$.
An easy calculation shows that $\sim$ is an equivalence relation.It clear from the definitions that $f \simeq g$ implies $f \sim g$ (with h = 0).

Lemma

Let $f, g : M \times [0, \infty) \to [0, \infty]$. If $f \sim g$, then $f^* \sim g^*$.

Weak Equivalence and Weak Doubling

Definition

We say that $f : M \times [0, \infty) \to [0, \infty]$ satisfies the weak doubling condition Δ_2^W if there exist a constant $\theta \geq 2$ and $h \in L^1(M, \mu)$ such that

$$f(x, 2y) \leq \theta f(x, y) + h(x)$$

for μ-almost every $x \in M$ and all $y \geq 0$. We say that f satisfies condition ∇_2^W if f^* satisfies Δ_2^W
If $h \equiv 0$, then we say that the (strong) Δ_2 and ∇_2 conditions hold.

Lemma

Let $f, g : M \times [0, \infty) \to [0, \infty]$ with $f \sim g$.

(a) If f satisfies Δ_2^W , then g satisfies Δ_2^W

(b) If f satisfies ∇_2^W , then g satisfies ∇_2^W

Theorem

If $f \in \Phi_W(M, \mu)$ satisfies Δ_2^W and/or ∇_2^W , then there exists $g \in \Phi_W(M, \mu)$ with $f \sim g$ satisfying Δ_2 and/or ∇_2.

Generalized Orlicz Spaces

Generalized Orlicz Spaces

Modulars [6] [10] [11]

Definition

Let $f \in \Phi_W(M,\mu)$ and let ρ_f be given by

$$\rho_f(h) := \int_M f(x, |h(x)|) d\mu(x)$$

for all $h \in L_0(M,\mu)$. The function ρ_f. is called a modular. The set

$$L^f(M,\mu) := \{h \in L_0(M,\mu) : \rho_f(\lambda h) < \infty \quad \text{for some} \quad \lambda > 0\}$$

is called a generalized Orlicz space. If the set and measure are obvious from the context we abbreviate $L^f(M,\mu) = L^f$.

Lemma

Let $f \in \Phi_W(M,\mu)$.

(a) Then $L^f(M,\mu) = \{h \in L^0(M,\mu) : \lim_{\beta \to 0^+} \rho_f(\beta h) = 0\}$.

(b) If, additionally,f satisfies $(5)_{q<\infty}$, then

$$L^f(M,\mu) = \{h \in L^0(M,\mu) : \rho_f(h) < \infty\}.$$

Modulars

Lemma

Let $f \in \Phi_W(M,\mu)$ and $h_n, h, r \in L^0(M,\mu)$. In (a) and (b), we assume also that f is left-continuous.

(a) If $h_n \to h$ μ-almost everywhere, then $\rho_f(h) \leq \liminf_{n\to\infty} \rho_f(h_n)$.

(b) If $|h_n| \nearrow |h|$ μ-almost everywhere, then $\rho_f(h) = \lim_{n\to\infty} \rho_f(h_n)$

(c) If $h_n \to h$ μ-almost everywhere, $|h_n| \leq |r|$ μ-almost everywhere, and $\rho_f(\beta r) < \infty$ for every $\beta > 0$, then $\lim_{n\to\infty} \rho_f(\beta\,|h - h_k|) = 0$ for every $\beta > 0$.

Lemma

Let $f \in \Phi_W(M,\mu)$ satisfy $(4)_{q<\infty}$. Let $h_i, r_i \in L^f(\mathbb{R}^n)$ for $i = 1, 2, \ldots$ with $(\rho_f(h_i))_{i=1}^{\infty}$ bounded. If $\rho_f(h_i - r_i) \to 0$ as $i \to \infty$, the

$$|\rho_f(h_i) - \rho_f(r_i)| \to 0 \quad \text{as} \quad i \to \infty.$$

Quasinorm and the Unit Ball Property

Definition

Let $f \in \Phi_W(M,\mu)$. For $h \in L^0(M,\mu)$ we denote

$$\|h\|_{L^f(M,\mu)} := \inf\left\{\beta > 0, \rho_f\left(\frac{h}{\beta}\right) \leq 1\right\}.$$

We abbreviate $\|h\|_{L^f(M,\mu)} = \|h\|_f$ if the set and measure are clear from the context.

Remark

Observe that we can write the space L^f with this functional as follows :

$$L^f(M,\mu) = \{h \in L^0(M,\mu) : \|h\|_{L^f(M,\mu)} < \infty\}.$$

As is the case for Lebesgue spaces, we identify functions which coincide μ-almost everywhere, since $\|f\|_f = 0$ only implies that $f = 0$ a.e.

Quasinorm and the Unit Ball Property

Lemma

(a) If $f \in \Phi_W(M,\mu)$, then $\|\cdot\|_f$ is a quasinorm.

(b) If $f \in \Phi_c(M,\mu)$, then $\|\cdot\|_f$ is a norm.

Lemma

Unit Ball Property

Let $f \in \Phi_W(M,\mu)$. Then

$$\|h\|_f < 1 \quad \Rightarrow \quad \rho_f(h) \le 1 \quad \Rightarrow \quad \|h\|_f \le 1$$

If f is left-continuous, then $\rho_f(h) \le 1 \Leftrightarrow \|h\|_f \le 1$.

Quasinorm and the Unit Ball Property

Example

we can show that $\|h\|_f = 1$ does not imply $\rho_f(h) \leq 1$ if the Φ-function is not left-continuous.
Let $f(y) := \infty\chi_{[1,\infty)}(y)$ and $h \equiv 1$. Then $f \in \Phi_W$ and $\rho_f(h) = \infty$. Since $\rho_f(h/\beta) \leq 1$ if and only if $\beta > 1$, we have $\|h\|_f = 1$.

Proposition

Let $f, g \in \Phi_W(M,\mu)$. If $f \simeq g$, then $L^f(M,\mu) = L^g(M,\mu)$ and the norms are comparable.

Corollary

Let $f \in \Phi_W(M,\mu)$. Then

$$\left\|\sum_{j=1}^{\infty} h_i\right\|_f \lesssim \sum_{j=1}^{\infty} \|h_j\|_f$$

Quasinorm and the Unit Ball Property

Theorem

Let $f, g \in \Phi_W(M,\mu)$ and let the measure μ be *atom* $-$ *less*. Then $L^f(M,\mu) \hookrightarrow L^g(M,\mu)$ if and only if there exist $\theta > 0$ and $\varphi \in L^1(M,\mu)$ with $\|\varphi\|_1 \leq 1$ such that

$$g\left(x, \frac{y}{\theta}\right) \leq f(x,y) + \varphi(x)$$

for μ-almost all $x \in M$ and all $y \geq 0$.

Corollary

Let $f, g \in \Phi_W(M,\mu), f \sim g$. Then $L^f(M,\mu) = L^g(M,\mu)$ and the norms are comparable.

Corollary

Let $f \in \Phi_W(M,\mu)$ and $f \in L^f(M,\mu)$ and let a be the constant from (3) with $p = 1$.

(a) If $\|h\|_f < 1$, then $\rho_f(h) \leq \delta \|h\|_f$.

(b) If $\|h\|_f > 1$, then $\|h\|_f \leq \delta \rho_f(h)$.

(c) In any case, $\|h\|_f \leq \delta \rho_f(h) + 1$.

Quasinorm and the Unit Ball Property

Lemma

Let $f \in \Phi_W(M,\mu)$ satisfy (3) and (5), $1 \leq p \leq q < \infty$. Then

$$\min\left\{\left(\frac{1}{\delta}\rho_f(h)\right)^{\frac{1}{p}}, \left(\frac{1}{\delta}\rho_f(h)\right)^{\frac{1}{q}}\right\} \leq \|h\|_f \leq \max\left\{\left(\delta\rho_f(h)\right)^{\frac{1}{p}}, \left(\delta\rho_f(h)\right)^{\frac{1}{q}}\right\}$$

for $h \in L^0(M,\mu)$, where δ is the maximum of the constants from (3) and (5).

Corollary

Let $f \in \Phi_W(M,\mu)$ satisfy (3), $1 \leq p < \infty$.Then

$$\min\left\{\left(\frac{1}{\delta}\rho_f(h)\right)^{\frac{1}{p}}, 1\right\} \leq \|h\|_f \leq \max\left\{\left(\delta\rho_f(\delta)\right)^{\frac{1}{p}}, 1\right\}$$

for $h \in L^0(M,\mu)$, where δ is the constant from (3).

Quasinorm and the Unit Ball Property

Lemma

[Hölder's Inequality]

Let $f \in \Phi_W(M,\mu)$.Then

$$\int_M |\varphi|\,|\psi|\,d\mu \le \|\varphi\|_f\,\|\psi\|_{f^*}$$

for all $\varphi \in L^f(M,\mu)$ and $\psi \in L^{f^*}(M,\mu)$. Moreover, the constant 2 cannot in general be replaced by any smaller number.

Example

We consider an example shows that the extra constant 2 in Hölder's inequality cannot be omitted.

Let $f(y) = \frac{1}{2}y^2$. Then a short calculation gives that

$f^*(y) = \sup_{v\ge 0}(vy - \frac{1}{2}v^2) = \frac{1}{2}y^2$. Let $\varphi \equiv \psi \equiv 1$. Then $\int_0^1 \varphi\psi dt = 1$. On the other hand,

$$\inf\left\{\beta > 0 : \int_0^1 \frac{1}{2}\left(\frac{1}{\beta}\right)^2 dt \le 1\right\} = \frac{1}{\sqrt{2}}$$

and thus $\|\varphi\|_{L^f(0,1)} = \|\psi\|_{L^{f^*}(0,1)} = \frac{1}{\sqrt{2}}$ and $\|\varphi\|_{L^f(0,1)}\,\|\psi\|_{L^{f^*}(0,1)} = \frac{1}{2}$.

Convergence and Completeness

Lemma

Let $f \in \Phi_W(M,\mu)$.Then $\|h_n\|_f \to 0$ as $k \to \infty$ if and only if $\lim_{n\to\infty} \rho_f(\beta h_n) = 0$ for all $\beta > 0$.

Definition

Let $f \in \Phi_W(M,\mu)$ and $h_n, h \in L^f(M)$.We say that h_θ is modular convergent (ρ_f-convergent) to h if $\rho(\beta(h_n - h)) \to 0$ as $n \to \infty$ for some $\beta > 0$.

Lemma

Let $f \in \Phi_W(M,\mu)$.Modular convergence and norm convergence are equivalent if and only if $\rho(h_n) \to 0$ implies $\rho(2h_n) \to 0$.

corollary

Let $f \in \Phi_W$ satisfy $(5)_{q<\infty}$. Then modular convergence and norm convergence are equivalent.

Convergence and Completeness

lemma

Let $f \in \Phi_W(M,\mu)$ and $\mu(M) < \infty$. Then every $\|\cdot\|_f$-Cauchy sequence is also a Cauchy sequence with respect to convergence in measure.

Lemma

Let $f \in \Phi_W(M,\mu)$. Then every $\|\cdot\|_f$-Cauchy sequence $(h_n) \subset L^f$ has a subsequence which converges μ-a.e. to a measurable function h .

Theorem

(a) If $f \in \Phi_W(M,\mu)$, then $L^f(M,\mu)$ is a quasi-Banach space.

(b) If $f \in \Phi_c(M,\mu)$, then $L^f(M,\mu)$ is a Banach space.

Lemma

Let $f \in \Phi_W(M,\mu)$ be left-continuous and $h, h_n \in L^0(M,\mu)$.

(a) If $h_n \to h$ μ-almost everywhere, then $\|h\|_f \leq \liminf_{n\to\infty} \|h_n\|_f$.

(b) If $|h_n| \nearrow |h|$ μ-almost everywhere with $h_n \in L^f(M,\mu)$ and $\sup_n \|h\|_f < \infty$, then $h \in L^f(M,\mu)$ and $\|h_n\|_f \nearrow \|h\|_f$.

Associate Spaces

Definition

Let $f \in \Phi_W(M,\mu)$. Then by $(L^f(M,\mu))^*$ we denote the dual space of $L^f(M,\mu)$. Furthermore, we define $\phi_f : (L^f(M,\mu))^* \to [0,\infty]$ by

$$\phi_f(G) := \sup_{h \in L^f(M,\mu)} \Big(|G(h)| - \rho_f(h) \Big).$$

Remark

Note the difference between the spaces $(L^f(M,\mu))^*$ and $L^{f^*}(M,\mu)$: the former is the dual space of $L^f(M,\mu)$, whereas the latter is the generalized Orlicz space defined by the conjugate modular f^*.

By definition of the functional ϕ_f we have

$$(7) \qquad |G(h)| \leq \rho_f(h) - \phi_f(G)$$

for all $h \in L^f(M,\mu)$ and $G \in (L^f(M,\mu))^*$.This is a generalized version of the classical Young inequality.

The function ϕ_f is actually a semimodular on the dual space. We refer to [4] for details.

In the definition of ϕ_f the supremum is taken over all $L^f(M,\mu)$. However, it is possible to restrict this to the closed unit ball when G is in the unit ball and f is convex.

Associate Spaces

Lemma

Let $f \in \Phi_c(M,\mu)$. If $G \in (L^f(M,\mu))^*$ with $\|G\|_{(L^f)^*} \leq 1$, then

$$\phi_f(G) = \sup_{h \in L^f, \|h\|_f \leq 1} \Big(G\,|(h)| - \rho(h) \Big) = \sup_{h \in L^f, \rho_f \leq 1} \Big(|G(h)| - \rho(h) \Big).$$

Lemma

Let $f \in \Phi_W(M,\mu)$. There exist a sequence of positive functions $h_n \in L^f(M,\mu), n \in \mathbb{N}$, such that $h_n \nearrow 1$ and $\{h_n = 1\} \nearrow M$.

Associate Spaces

Definition

We define the associate space of $L^f(M,\mu)$ as the space $(L^f)'(M,\mu) := \{h \in L^0(M,\mu) : \|h\|_{(L^f)'} < \infty\}$ with the norm

$$\|h\|_{(L^f)'} := \sup_{\|\psi\|_f \leq 1} \int_M h\psi d\mu$$

Theorem

[Norm Conjugate Formula]

If $f \in \Phi_W(M,\mu)$, then

$$(L^f)' = L^{f^*}$$

and the norms are comparable. Moreover, for all $h \in L^0(M,\mu)$

$$\|h\|_f \approx \sup_{\|\psi\|_{f*} \leq 1} \int_M |h\psi| \, d\mu.$$

Separability

Proposition

Let $f \in \Phi_W(M, \mu)$ satisfy the assumption $(5)_{q<\infty}$. Then the sets $S(M, \mu) \cap L^f(M, \mu)$ and $L^\infty(M, \mu) \cap L^f(M, \mu)$ are dense in $L^f(M, \mu)$.

Theorem

Let $f \in \Phi_W(M, \mu)$ satisfy $(5)_{q<\infty}$, and let μ be separable. Then $L^f(M, \mu)$ is separable.

Uniform Convexity and Reflexivity [6]

Definition

We say that $f \in \Phi_c(M, \mu)$ is uniformly convex if for every $\nu > 0$ there exists $d \in (0, 1)$ such that

$$f\left(x, \frac{y+z}{2}\right) \leq (1-d)\frac{f(x,y)+f(x,z)}{2}$$

for μ-almost every $x \in M$ whenever $y, z \geq 0$ and $|z-y| \geq \nu \max\{|z|, |y|\}$

Proposition

The function $f \in \Phi_W(M, \mu)$ is equivalent to a uniformly convex Φ-function if and only it satisfies $(5)_{p>1}$.

Definition

A vector space V is uniformly convex if it has a norm $\|\cdot\|$ such that for every $\nu > 0$ there exists $\delta > 0$ with

$$\|u-v\| \geq \nu \quad \text{or} \quad \|u+v\| \leq 2(1-d)$$

for all $u, v \in X$ with $\|u\| = \|v\| = 1$.

Uniform Convexity and Reflexivity

Lemma

Let $f \in \Phi_c(M,\mu)$ be uniformly convex. Then for every $\nu > 0$ there exists $d_2 > 0$ such that

$$f\left(x, \left|\frac{z+y}{2}\right|\right) \leq (1-d_2)\frac{f(x,|z|)+f(x,|y|)}{2}$$

for all $z, y \in \mathbb{R}$ with $|z-y| > \nu \max\{|z|, |y|\}$ and every $x \in M$.

Lemma

Let $f \in \Phi_c(M,\mu)$ be uniformly convex. Then for every $\nu > 0$ there exists $d > 0$ such that

$$\rho_f\left(\frac{\varphi-\psi}{2}\right) < \nu\frac{\rho_f(\varphi)+\rho_f(\psi)}{2} \quad \text{or} \quad \rho_f\left(\frac{\varphi+\psi}{2}\right) \leq (1-d)\frac{\rho_f(\varphi)+\rho_f(\psi)}{2}$$

for all $\varphi, \psi \in L^0(M,\mu)$.

Uniform Convexity and Reflexivity

Theorem

Let $f \in \Phi_c(M,\mu)$ be uniformly convex and satisfy $(5)_{q<\infty}$.Then $L^f(M,\mu)$ is uniformly convex with norm $\|\cdot\|_f$.
In particular, if f satisfies $(3)_{p>1}$ and $(5)_{q<\infty}$, then $L^f(M,\mu)$ is uniformly convex and reflexive.

Corollary

Let $f \in \Phi_W(M,\mu)$.If f satisfies Δ_2^W and ∇_2^W,then $L^f(M,\mu)$ is uniformly convex and reflexive.

The Weight Condition $(C0)$ and Density of Smooth Functions

Definition

We say that $f \in \Phi_W(M,\mu)$ satisfies $(C0)$,if there exists a constant $\lambda \in (0,1]$ such that $\lambda \leq f^{-1}(x,1) \leq \frac{1}{\lambda}$ for μ-almost every $x \in M$.

Example

Let $f(x,y) = \frac{1}{p(x)} y^{p(x)}$ where $p : M \to [1,\infty)$ is measurable, and $g(x,y) = y^p + \psi(x) y^q$ where $1 \leq p < q < \infty$ and $g : M \to [0,\infty)$ is measurable. Then $f, g \in \Phi_s(M,\mu)$. Since $f^{-1}(x,y) = (p(x)y)^{1/p(x)}$, we see that f satisfies $(C0)$ (without assumptions for p). By Corollary 57, g satisfies $(C0)$ if and only if $g \in L^\infty(M,\mu)$.

Lemma

Let $f \in \Phi_W(M,\mu)$ satisfy $(C0)$. Then there exists $g \in \Phi_s(M,\mu)$ with $f \simeq g$ and $g(x,1) = g^{-1}(x,1) = 1$ for μ-almost every $x \in M$.

The Weight Condition $(C0)$ and Density of Smooth Functions

Corollary

Let $f \in \Phi_W(M,\mu)$. Then f satisfies $(C0)$ if and only if there exists $\lambda \in (0,1]$ such that $f(x,\lambda) \leq 1 \leq f(x,1/\lambda)$ for μ-almost every $x \in M$.

Lemma

If $f \in \Phi_W(M,\mu)$ satisfies $(C0)$, then f^* satisfies $(C0)$

Lemma

Let $f \in \Phi_W(M,\mu)$ satisfy $(C0)$, (3) and (5), $p \in [1,\infty)$ and $q \in [1,\infty]$.Then

$$L^p(M,\mu) \cap L^q(M,\mu) \hookrightarrow L^f(M,\mu) \hookrightarrow L^p(M,\mu) + L^q(M,\mu)$$

and the embedding constants depend only on $(C0)$, (3) and (5).

The Weight Condition $(C0)$ and Density of Smooth Functions

Corollary

Let M have finite measure and let $f \in \Phi_W(M, \mu)$ satisfy $(C0)$ and (3).Then $L^f(M, \mu) \hookrightarrow L^p(M, \mu)$ and there exists λ such that

$$\int_M |h|^p \, d\mu \lesssim \int_M f(x, |h|) d\mu + \mu\Big(\{0 < |h| < \frac{1}{\lambda}\}\Big).$$

Corollary

Let M have finite measure and let $f \in \Phi_W(M, \mu)$ satisfy $(C0)$.Then $L^\infty(M, \mu) \hookrightarrow L^f(M, \mu)$.

Definition

A normed space $(V, \|\cdot\|_V)$ with $V \subset L^0(M, \mu)$ is called a Banach function space, if

(a) $(V, \|\cdot\|_V)$ is circular, solid and satisfies the Fatou property.

(b) If $\mu(A) < \infty$, then $\chi_A \in X$.

(c) If $\mu(A) < \infty$, then $\chi_A \in V'$,i.e. $\int_A |h| \, d\mu \leq c(A) \|f\|_V$ for all $f \in V$.

The Weight Condition $(C0)$ and Density of Smooth Functions

Theorem

Let $f \in \Phi_W(M,\mu)$ satisfy $(C0)$.Then $L^f(M,\mu)$ is a Banach function space.

Lemma

Let $f \in \Phi_W(\Omega)$ satisfy $(5)_{q<\infty}$.Then $L_0^f(\Omega)$ is dense in $L^f(\Omega)$.

Theorem

If $f \in \Phi_W(\Omega)$ satisfies $(C0)$ and $(5)_{q<\infty}$, then $C_0^\infty(\Omega)$ is dense in $L^f(\Omega)$.

Conclusion

Conclusion

In this memory, we presented a study on generalized Orlicz spaces and and their basic properties. This work raises a number of questions that deserve to be addressed. subsequently melted. For example, it would be wise to think in perspective of following :

- Does the next theorem hold for all Φ-prefunctions ?
  Assume that $f, g \in \Phi_W$. Then $f \simeq g$ if and only if $f^{-1} \approx g^{-1}$.
- Is the next lemma true if we assume $(5)_{q<\infty}$ instead of $(4)_{q<\infty}$?
  Let $f \in \Phi_W(M, \mu)$ satisfy $(4)_{q<\infty}$. Let $h_i, r_i \in L^f(\mathbb{R}^n)$ for $i = 1, 2, \ldots$ with $(\rho_f(h_i))_{i=1}^{\infty}$ bounded. If $\rho_f(h_i - r_i) \to 0$ as $i \to \infty$, then

  $$|\rho_f(h_i) - \rho_f(r_i)| \to 0 \quad \text{as} \quad i \to \infty.$$

- Does the next theorem hold without the assumption $(C0)$?
  If $f \in \Phi_W(\Omega)$ satisfies $(C0)$ and $(5)_{q<\infty}$, then $C_0^\infty(\Omega)$ is dense in $L^f(\Omega)$.

For this reason we think that the memory will be useful also for researchers interested in the Orlicz case only.

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