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# *Master memory*

**Field** : Mathematics and Computer science
**Branch** : Mathematics
**Option** : Functional analysis

# Theme

## *On generalized Orlicz spaces*

**Presented by :**

Racha BOUSBAA

**In front of the jury composed of :**

| | | | |
|---|---|---|---|
| Bachir GAGUI | M.C.A, | University of M'sila | **President** |
| Hellal ABDELAZIZ | M.A.A, | University of M'sila | **Supervisor** |
| Noureddine DECHOUCHA | M.A.A, | University of M'sila | **Co-Supervisor** |
| Khelifa CHADI | M.A.A, | University of M'sila | **Examiner** |

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# Contents

# Dedication

I'm dedicating this memory to beloved people who have meant and continue to mean so much to me. Although they are no longer of this world, their memories continue to regulate my life.

First and foremost, to my father **Farid Bousbaa** whose love for me knew no bounds and, who taught me the value of hard work. You are the most mentor and best friend could ever have. I appreciate you more than words can ever say,thank you Next, my mother **Fatima El zahra Ferroudj** who raised me, loved me, you have always been a source of inspiration and happiness to me thanks for every things.

I also want to remember my soulmate and my sister **Achwak Bousba** i hope nothing will separate us.

Last but not least I'm dedicating this to my fiance **Ali Tahri** thanks for your support.

In the end i thank all my brothers and sisters.

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# Abstract

Recently, generalized Orlicz spaces has become part of the mainstream research fields in contemporary functional analysis. so, In this memory, we study some definitions and basic properties of $\Phi$-function and we use them to study some properties of generalized Orlicz spaces (also known as Musielak-Orlicz spaces) as convergence, completeness, separability, uniform convexity, reflexivity and density of smooth functions.

**keywords**: $\Phi$-function, Variable exponent Lebesgue spaces $L^{p(\cdot)}$, Orlicz spaces, Generalized Orlicz spaces.

## Résumé

Récemment, les espaces d'Orlicz généralisés sont devenus une partie dans les domaines principaux de recherche en analyse fonctionnelle moderne. Alors, dans cette mémoire, nous étudions quelques définitions et propriétès de base de la fonction $\Phi$ et nous utilisons pour étudier certaines propriétés des espaces d'Orlicz généralisés ( aussi connu comme les espaces de Musielak-Orlicz ) comme convergence, complétude, séparabilité, convexité uniforme, réflexivité et la densité des fonctions indéfiniment dérivables à support compact.

**mots-clés**: $\Phi$-fonction, Espaces de Lebesgue à exposant variable $L^{p(\cdot)}$, Espaces d'Orlicz, Espaces d'Orlicz généralisés.

# List of Symbols

In what follows, we will use the following notations.

| | |
|---|---|
| $\mathbb{R}^n$ | Euclidean, $n$-dimensional space, |
| $x$ | Vecteur de $\mathbb{R}^n$, $x = (x_1, x_2, ..., x_n)$, $x_i \in \mathbb{R}$, $1 \leq i \leq n$ |
| $d\mu$ | or $dx$ Lebesgue measure $N$-dimensional |
| $\|M\|$ | or mes $(M)$ Measure of Lebesgue of a set $M$ |
| $\|\Omega\|$ | Measure of the set $\Omega$ |
| $p.p.$ | almost everywhere, |
| $p^{'}$ | $= \frac{p}{p-1}$, The Hölder conjugate exponent of $p$, |
| $\nearrow$ | Limit of an increasing sequence, |
| $\chi_A$ | Characteristic function of the set $A$, |
| $\Omega$ | Open set in $\mathbb{R}^N$ |
| $\partial\Omega$ | The border of $\Omega$ |
| $B$ | Open ball, |
| $B(x, r)$ | Open ball with center $x$ and radius $r > 0$, |
| $(M, \mu)$ | Measure space, |
| $C^\infty$ | Smooth functions, i.e. it consists of functions which are continuously differentiable arbitrarily many times. |
| $C_0^\infty \subset C^\infty$ | Smooth functions with compact support, |
| $\delta$ | Usually the constant from the $(2.2)_{p>1}$ and $(2.4)_{q<\infty}$ conditions, |
| $f_B^+$ | Essential supremum of $f$ in $B$, |
| $f_B^-$ | Essential infimum of $f$ in $B$, |
| $f, g$ | (Generalized) $\Phi$-function, |
| $f^*$ | Conjugate $\Phi$-function of $f$ with $f^{**} \neq f$, |
| $\Phi_c$ | Set of convex $\Phi$-functions, |
| $\Phi_c(M, \mu)$ | Set of generalized convex $\Phi$-functions, |

| | |
|---|---|
| $f^{-1}$ | Left-inverse of $f$, |
| $\Phi_s$ | Set of strong $\Phi$-functions, |
| $\Phi_s(M,\mu)$ | Set of generalized strong $\Phi$-functions, |
| $\Phi_W$ | Set of weak $\Phi$-functions,, |
| $\Phi_W(M,\mu)$ | Set of generalized weak $\Phi$-functions, |
| $\Phi_{W^+}$ | Set of left-continuous weak $\Phi$-function, |
| $\Delta_2$ | The doubling function, |
| $(M,\Gamma,\mu)$ | a $\sigma$-finite, complete measure space, |
| $\simeq$ | is similar or equal to, |
| $\rho_f$ | Modular function, |
| $\rho_f$ | $:= \int_M f(x, \lvert h(x)\rvert) d\mu(x)$ |
| $L^f$ | Generalized Orlicz space, |
| $L^0(M,\mu)$ | Set of measurable functions, |
| $(L^f)'$ | Associate space of $L^f$, |
| $(L^f)^*$ | Dual space of $L^f$, |
| $L^f_0$ | Generalized Orlicz functions with compact support, |
| $L^0$ | Set of measurable functions, |
| $L^s$ | Classical Lebesgue space, |
| $S$ | Simple functions, |
| $\lvert\cdot\rvert_f$ | Norm with respect to $f$, |
| $\lVert h\rVert_{L^f(M,\mu)}$ | $= \inf\{\beta > 0 : \rho_f\left(\frac{h}{\beta}\right) \leq 1\}$ |
| $\lVert\cdot\rVert_\varphi$ | Luxemburg norm. |
| $\Delta_2^W$ | The weak doubling condition, |
| $\nabla_2^W$ | The weak doubling condition with $f^*$, |
| $y_0$ | $\sup\{y : \quad f(y) = 0\}$, |
| $y_\infty$ | $\inf\{y : \quad f(y) = \infty\}$, |
| $[0,\infty]$ | Compactification of $[0,\infty)$, |
| $(L^f(M,\mu))'$ | $:= \{h \in L^0(M,\mu) : \lVert h\rVert_{(L^f)'} < \infty\}$ |
| $\lVert h\rVert_{(L^f)'}$ | $= \sup_{\lVert r\rVert_\varphi \leq 1} \int_M hr d\mu$ |
| $\lVert h\rVert_{V^*}$ | $= \sup_{\lVert h\rVert_V} \lvert G(h)\rvert$ |
| $V^*$ | The dual space of $V$ |
| $\phi_h$ | $= \sup_{h\in L^f(M,\mu)} \Big( \lvert G(h)\rvert - \rho_f(h) \Big)$ |

# Introduction

The Orlicz spaces were introduced by Z.W.Birnbaum and W.Orlicz (1931)(see [4]) as a natural generalization of the classical Lebesgue spaces $L^p$, $1 < p < +\infty$. For this generalization the function $x^p$ entering in the definition of $L^p$ space is replaced by a more general convex function $\Phi$ , which is called an N-function and he studied on the Orlicz space associated to N-function. The first detailed study on Orlicz spaces was given by Krasnosel'skii and Rutickii (1961) ( see [18] ) and they considered the function $\Phi$ as an N-function that is based on the integral representation of the real valued convex function.

N-function and Young function are defined on $\mathbb{R}$ and $\Phi$ is taken an even function in Krasnosel'skii and Rutuckii (1961), Rao and Ren (1991) ( see [27] ) respectively. But in this memory we take the domain of the $\Phi$ as $[0, +\infty)$ for the convenience with the other definitions. Also, we recall that an N-function $\Phi$ is finite real valued convex function defined on $[0, +\infty)$ , so this implies that $\Phi$ is necessarily continuous. However, a Young function can have infinite value at a point, and hence may be discontinuous at such a point. Moreover, recently, in several studies about Orlicz spaces especially on the composition operators (Arora and et al. (2007)( see [2] ), Kumar (1997) ( see [16] ), Raj and Khosla (2009)( see [25] )), the function $\Phi$ is defined differently from the Young function used in Rao and Ren's works but again they called this new function $\Phi$ as a Young function.

We know that there are four different type of spaces: classical Lebesgue spaces $L^p$ , Orlicz spaces, variable exponent Lebesgue spaces $L^{p(\cdot)}$ and generalized Orlicz spaces. Naturally, $L^p$-spaces are Orlicz spaces and $L^{p(\cdot)}$-spaces, and Orlicz and $L^{p(\cdot)}$-spaces are generalized Orlicz spaces. Orlicz spaces and $L^{p(\cdot)}$-spaces have different nature, and neither of them is a subset of the other.

As generalized Orlicz spaces have been an area of growing interest recently. so, the main topic treated in this memory is the representation of some definitions and basic properties of $\Phi$-function and we use them to study some properties of generalized Orlicz spaces as convergence, completeness, separability, uniform convexity, reflexivity and density of smooth functions. The reader can find a lot of information about in the excellent monograph [10] and [14].

The memory consists of three chapters. In the preliminaries (Chapter 1) we establish the notation of the memory. We introduce some important results on some definitions, examples of Banach spaces and some results about Integration that everyone must know. Also, we recall the most important results about the history of Non-standard growth phenomena.

In Chapter 2 of this memory as indicated in [10] and [14], we introduce the study of some definitions and basic properties of $\Phi$-function. so, we start by studying the properties of almost increasing and almost decreasing, which will be used throughout the memory and we consider two notions of the equivalence of $\Phi$-functions and prove relations between them. As $\Phi$-functions have much better invariance properties than convex or strong F-functions. However, in many cases it is nicer to work with $\Phi$-functions of the latter classes. This can often be achieved by upgrading the F-function that we obtain from some process. The tools for doing so are developed in this chapter. An alternative approach to upgrading is to use the conjugate function. Since the weak $\Phi$-functions are not bijections, they are not strictly speaking invertible. However, we can define a left-continuous function with many properties of the inverse, which we call for simplicity left-inverse. Note that this is not the left-inverse in the sense of abstract algebra. For the elegance of our study, we extend in this chapter all $\Phi$-functions to the interval $[0, \infty]$ by $\phi(\infty) := \infty$. Finally, we study the generalize $\Phi$-functions in such a way that they may depend on the space variable.

In the last chapter (Chapter 3) we use the properties of $\Phi$-functions to study and derive results for function spaces defined by means of $\Phi$-functions. we first study and define the spaces, then see that they are quasi-Banach spaces. after that, we study associate spaces, separability and uniform convexity require some restrictions on the $\Phi$-function. Finally, we will study the density of smooth functions.

# Chapter 1

# Preliminary notions

In this chapter we present the concepts and results used throughout the memory on some definitions, examples of Banach spaces and some results about Integration that everyone must know. see, e.g. the monographs [1], [3],[13] and [24].

## 1.1 Introducton to Banach spaces

**Definition 1.1.1.** Let $V$ be a $\mathbb{K}$-vector space. A functional $\varphi : V \to [0; \infty)$ is called a seminorm, if

**(a)** $\varphi(\lambda x) = |\lambda| \varphi(x), \forall \lambda \in \mathbb{K}; x \in V$,

**(b)** $\varphi(x + y) \leq \varphi(x) + \varphi(y), \forall x, y \in V$.

**Definition 1.1.2.** Let $\varphi$ be a seminorm such that $\varphi(x) = 0 \Rightarrow x = 0$. Then, $\varphi$ is a norm (denoted by $\|\cdot\|$).

**Definition 1.1.3.** A pair $(V, \|\cdot\|)$ is called a normed linear space.

**Lemma 1.1.4.** *Each normed space $(V, \|\cdot\|)$ is a metric space $(V, d)$ with a metric given by $d(x, y) = \|x - y\|$.*

**Definition 1.1.5.** A sequence $\{x_n\}_{n \in N}$ in a normed space $(V, \|\cdot\|)$ is called a Cauchy sequence, if

$$\forall \nu > 0 \quad \exists N(\nu) \in \mathbb{N} \quad \forall n, m \geq N(\nu) \quad \Rightarrow \quad \|x_n - x_m\| \leq \nu.$$

**Definition 1.1.6.** A sequence $\{x_n\}_{n\in N}$ converges to $x$ (which is denoted by $lim_{n\to\infty}x_n = x$), if

$$\forall \nu > 0 \quad \exists N(\nu) \in \mathbb{N} \quad \forall n \geq N(\nu) \quad \Rightarrow \quad \|x_n - x\| < \nu.$$

**Definition 1.1.7.** If every Cauchy sequence $\{x_n\}_{n\in N}$ converges in $V$, then $(V, \|\cdot\|)$ is called a complete space.

**Definition 1.1.8.** A normed linear space $(V, \|\cdot\|)$ which is complete is called a Banach space.

**Lemma 1.1.9.** *Let $(V, \|\cdot\|)$ be a Banach space and $U$ be a closed linear subspace of $V$. Then, $(U, \|\cdot\|)$ is a Banach space as well.*

# Examples of Banach spaces

**Example 1.1.10.** Let $\mathbf{B}(T)$ be a space consisting of all bounded maps $f : T \to \mathbb{K}$. For each $f \in \mathbf{B}(T)$ we set

$$\|f\|_\infty = \sup_{t\in T} |f(t)| \,.$$

Then, $(\mathbf{B}(T); \|\cdot\|_\infty)$ is a Banach space. To prove the assertion we need to show that

**(a)** $\|\cdot\|_\infty$ is a norm,

**(b)** each Cauchy sequence converges to an element from $\mathbf{B}(T)$.

Concerning claim (a), let $\lambda \in \mathbb{K}$ and $f \in \mathbf{B}(T)$. Then,

$$\|\lambda f\|_\infty = \sup_{t\in T} |\lambda f(t)| = |\lambda| \sup_{t\in T} |f(t)| = \lambda \|f\|_\infty \tag{1.1}$$

Let $t_0 \in T$ and $f, g \in \mathbf{B}(T)$. Then,

$$|f(t_0) + g(t_0)| \leq |f(t_0)| + |g(t_0)| \leq \sup_{t\in T} |f(t)| + \sup_{t\in T} |g(t)| = \|f\|_\infty + \|g\|_\infty \,.$$

The right hand side of the inequality above is independent on $t$. Therefore, taking supremum of both sides of the inequality over $t \in T$ yields

$$\|f+g\|_{\infty} \leq \|f\|_{\infty} + \|g\|_{\infty}. \tag{1.2}$$

Finally, let $f$ be such that $\|f\|_{\infty} = 0$, which is equivalent to $\sup_{t\in T} |f(t)| = 0$. This implies that $|f(t)| = 0$ for each $t$. Thus, $f = 0$.

Now, we shall prove the statement (b). Let $\{f_n\}$ be a Cauchy sequence. Then, for every $\nu > 0$ there exists $N = N(\nu)$ such that for all $n, m \geq N$ it holds that $\|f_n - g_m\|_{\infty} < \nu$. In particular,

$$|f_n(t) - g_m(t)| < \nu, \quad \forall t \in T.$$

Thus, for any $t \in T$ the sequence $\{f_n(t)\}_{n\in N}$ converges to some $f(t)$, due to the completeness of $\mathbb{K}$ (real and complex numbers are complete spaces). Define a candidate for a limit of the sequence $\{f_n\}_{n\in N}$, that is, $f : T \to \mathbb{K}$ as

$$f(t) = \lim_{n\to\infty} f_n(t).$$

If follows from the statement above that there exists $N_0 = N_0(\nu, t)$ such that

$$|f_n(t) - f(t)| < \nu, \quad \forall n \geq N_0. \tag{1.3}$$

Without loss of generality we can assume that $N_0(\nu, t) \geq N(\nu)$ for each $t \in T$. Then, for $n \geq N$ it holds that

$$\begin{aligned} |f_n(t) - f(t)| &\leq \left|f_n(t) - f_{N_0(\nu,t)}(t)\right| + \left|f_{N_0(\nu,t)}(t) - f(t)\right| \\ &\leq \left\|f_n - f_{N_0(\nu,t)}\right\|_{\infty} + \nu < 2\nu, \end{aligned}$$

where we used the fact that $\{f_n\}_{n\in N}$ is a Cauchy sequence and (1.3). Moreover, for each $t \in T$ and $N = N(\nu)$ we have

$$|f(t)| \leq |f_N(t)| + |f_N(t) - f(t)| \leq \|f_N\|_{\infty} + 2\nu,$$

which implies $\|f\|_{\infty} \leq \|f_N\|_{\infty} + 2\nu$, and so forth $f \in \mathbf{B}(T)$.

**Example 1.1.11.** Let $T$ be a metric space and $\mathbf{C}_b(T)$ be a space of bounded continuous functions on $T$. Then, $(\mathbf{C}_b; \|\cdot\|_\infty)$ is a Banach space.

To prove the assertion, it is sufficient to show that $\mathbf{C}_b(T)$ is a closed subspace of $\mathbf{B}(T)$ (due to the Lemma 1.9), that is, to show that every sequence in $\mathbf{C}_b(T)$ which converges in $\mathbf{B}(T)$ converges to a point from $\mathbf{C}_b(T)$. Let $\{x_n\}_{n\in N}$ be a sequence of bounded, continuous functions convergent to $x \in \mathbf{B}(T)$. We need to show that $x$ is a continuous function. For any $\nu > 0$ there exists $N = N(\nu) \in \mathbb{N}$, such that $\|x_N - x\|_\infty < \nu/3$, since the sequence is convergent. Now, let $t_0 \in T$. By the continuity of $x_N$, there exists $\delta = \delta(\nu, t_0) > 0$ such that

$$d(t, t_0) < \delta \quad \Rightarrow \quad |x_N(t) - x_N(t_0)| < \nu/3.$$

Therefore, for all t such that $d(t, t_0) < \delta$ it holds that

$$\begin{aligned} |x(t) - x(t_0)| &\le |x(t) - x_N(t)| + |x_N(t) - x_N(t_0)| + |x_N(t_0) - x_N(t_0)| \\ &\le 2\|x - x_N\|_\infty + |x_N(t) - x_N(t_0)| \le 2\nu/3 + \nu/3 = \nu, \end{aligned}$$

which ends the proof.

**Example 1.1.12.** A space of continuous functions vanishing at infinity

$$\mathbf{C}_0(\mathbb{R}^n) = \left\{ f \in \mathbf{C}_b(\mathbb{R}^n) : \lim_{|t|\to\infty} |f(t)| = 0 \right\}$$

with a $\|\cdot\|_\infty$ norm is a Banach space.

**Example 1.1.13.** The following spaces

$$\begin{aligned} c_0 &= \left\{ \{t_n\}_{n\in\mathbb{N}} : t_n \in \mathbb{K}, \lim_{n\to\infty} t_n = 0 \right\}, \\ c &= \left\{ \{t_n\}_{n\in\mathbb{N}} : t_n \in \mathbb{K}, \lim_{n\to\infty} t_n \quad \text{exists} \right\} \end{aligned}$$

with a $\|\cdot\|_\infty$ norm are Banach spaces.

**Remark 1.1.14.** $\mathbf{B}(\mathbb{N})$ is often denoted by $l^\infty$.

## $l^p$ spaces

**Definition 1.1.15.** Let $1 \leq p < \infty$. We define

$$l^p = \left\{ f \in l^\infty : \sum_{n=1}^{\infty} |f_n|^p < \infty \right\} \quad \text{and} \quad \|f\|_p = \sqrt[p]{\sum_{n=1}^{\infty} |f_n|^p}.$$

**Lemma 1.1.16 (Hölder inequality for sequences).** *Let $1 \leq p, q \leq \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, for $f \in l^p$ and $g \in l^q$ it holds that*

**(a)** $f \cdot g \in l^1$,

**(b)** $\|f \cdot g\|_1 \leq \|f\|_p \cdot \|g\|_q$.

**Lemma 1.1.17 (Minkowski inequality for sequences).** *Let $1 \leq p \leq \infty$ and $f, g \in l^p$. Then*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

**Example 1.1.18.** Spaces $(l^p, \|\cdot\|_p)$ are Banach spaces for $1 \leq p \leq \infty$.

Space $(l^\infty, \|\cdot\|_\infty)$ coincides with $(\mathbf{B}(\mathbb{N}), \|\cdot\|_\infty)$, therefore we assume that $p < \infty$. Similarly as in the Example 1, to prove the assertion we need to show that

**(a)** $\|\cdot\|_\infty$ is a norm,

**(b)** each Cauchy sequence converges to an element from $l^p$.

Claim (a) is straightforward, when one uses the Minkowski inequality for the proof of the triangle inequality. For the proof of completness, consider a Cauchy sequence $\{f^n\}_{n\in\mathbb{N}}$. Each element $f^n \in l^p$ is a sequence given by $f^n = (f_1^n, f_2^n, ......)$. Note that $l^p \subset l^\infty$, what holds due to the following estimate

$$\|f\|_p = \sqrt[p]{\sum_{k=1}^{\infty} |f_k|^p} \geq \sqrt[p]{\sup_{k\in\mathbb{K}} |f_k|^p} = \sup_{k\in\mathbb{K}} |f_k| = \|f\|_\infty,$$

for $f = (f_1; f_2; ....)$. Thus, if we consider the sequence $\{f^n\}_{n\in\mathbb{N}}$ as a sequence of elements of $l^\infty$ space, we conclude that there exists exactly one $f \in l^\infty$ such that

$\lim_{n\to\infty} \|f^n - f\|_\infty = 0$ (this follows from the completeness of $(l^\infty, \|\cdot\|_\infty)$).

In particular,

$$\lim_{n\to\infty} f_k^n = f_k \quad \text{for each} \quad k \in \mathbb{N}. \tag{1.4}$$

We shall show that $f = \{f_k\}_{k\in\mathbb{N}}$ is an element of $l^p$ space and that $\{f_n\}_{n\in\mathbb{N}}$ converges to $f$ in $l^p$. For any $\nu > 0$ there exists $N = N(\nu)$, such that for all $n, m \in N$ it holds that

$$\|f^n - f^m\|_p < \nu.$$

In particular, for every $K \in \mathbb{N}$

$$\sqrt[p]{\sum_{k=1}^{K} |f_k^n - f_k^m|^p} \leq \|f^n - f^m\|_p < \nu.$$

Using (1.4) and passing to the limit with $f_k^m$ we obtain

$$\sqrt[p]{\sum_{k=1}^{K} |f_k^n - f_k|^p} < \nu.$$

Since the estimate is valid for all $K$ and the right hand side of the inequality is independent on $K$, it holds also that

$$\sqrt[p]{\sum_{k=1}^{\infty} |f_k^n - f_k|^p} < \nu.$$

Therefore $\|f^n - f\|_p < \nu$, which proves that $x$ is a limit of the sequence in $l^p$. Moreover,

$$\|f\|_p \leq \left\|f - f^N\right\|_p + \left\|f^N\right\|_p \leq \nu + \left\|f^N\right\|_p < +\infty.$$

## 1.2 Minkowski functional

**Definition 1.2.1.** Set $A$ is called an absorbing set if for each $x \in V$ there exists $t \in \mathbb{K}$, such that $t \cdot x \in A$.

**Definition 1.2.2.** Set $A$ is called a balanced set if $x \in A \Rightarrow -x \in A$.

**Definition 1.2.3.** Let $A$ be a convex, absorbing and balanced set. A functional $\mu_A : V \to [0, \infty)$ defined by

$$\mu_A(x) = \inf\left\{t \in (0, \infty) : \frac{x}{t} \in A\right\} \tag{1.5}$$

is called Minkowski functional.

**Lemma 1.2.4.** *Minkowski functional generates a seminorm on $V$. If additionally $A$ is bounded in each direction, that is, for each $x \in V$ a set $(A \cap lin\{x\})$ is a bounded set, then it is a norm.*

*Proof.* We shall concentrate on the essential part of the proof, that is, showing that $\mu_A$ fulfills the triangle inequality. Fix $\nu > 0$ and let $t = \mu_A(x) + \nu, s = \mu_A(y) + \nu$. Then, $t^{-1}x; s^{-1}y \in A$, what follows from the definition of the Minkowski functional. Set $A$ is convex, therefore

$$\frac{t}{t+s} \cdot \frac{x}{t} + \frac{s}{t+s} \cdot \frac{y}{s} = \frac{x+y}{t+s} \in A,$$

which implies that

$$\mu_A(x+y) = \inf\left\{z \in (0, \infty) : \frac{x+y}{z} \in A\right\} \le t + s = \mu_A(x) + \mu_A(y) + 2\nu.$$

Due to a freedom in the choice of $\nu$ the assertion is proved. □

## Examples of normed spaces with a norm introduced by the Minkowski functional

Let $F = F(\Omega)$ be a space of real valued, Lebesgue measurable functions on $\Omega$.

**Example 1.2.5.** Orlicz spaces $L_\rho(\Omega)$.

Let $\rho$ be a non-negative convex function on $[0; \infty)$, such that

$$\rho(0) = 0 \quad \text{and} \quad \lim_{t\to\infty} \rho'(t) = \infty.$$

Define a set

$$A = \Big\{ f \in F : \int_\Omega \rho(|f(x)|)dx \le 1 \Big\}. \tag{1.6}$$

Orlicz space $L_\rho(\Omega)$ is the smallest linear space containing $A$. It can be checked that Minkowski functional $\mu_A$ defines a norm on $L_\rho(\Omega)$.

**Example 1.2.6.** Lebesgue spaces

$$L^p(\Omega) = \Big\{ f \in F : \int_\Omega |f(x)|^p \, dx < \infty \Big\}.$$

The most important class of Orlicz spaces arises when we set $\rho(x) = x^p$, where $1 < p < \infty$. In this case we obtain Lebesgue spaces $L^p(\Omega)$. Analogously as in the example above,

$$A = \Big\{ f \in F : \int_\Omega |f(x)|^p \, dx \le 1 \Big\}.$$

It turns out that Minkowski functional $\mu_A$ is given by the following formula

$$\mu_A(f) = \sqrt[p]{\int_\Omega |f(x)|^p \, dx}.$$

Note that $A$ is a convex, absorbing and balanced set, therefore $\mu_A$ is a seminorm on $L^p(\Omega)$. Moreover, if $\mu_A(f) = 0$, then $\displaystyle\int_\Omega |f(x)|^p \, dx = 0$, which implies $f = 0$ a.e. Thus, $\mu_A$ defines a norm on $L^p(\Omega)$.

**Example 1.2.7.** Generalized Lebesgue spaces

$$L^{p(\cdot)}(\Omega) = \Big\{ f \in F : \int_\Omega |f(x)|^{p(x)} \, dx < \infty \Big\}.$$

The next important class of Orlicz spaces is created when one sets $\rho(x) = x^{p(x)}$, where $p(x)$ fulfills $1 < p_1 \le p(x) \le p_2 < \infty$ for some $p_1, p_2$. In this case we obtain generalized Lebesgue spaces $L^{p(\cdot)}(\Omega)$. Similarly as before,

$$A = \Big\{ f \in F : \int_\Omega |f(x)|^{p(x)} \, dx \le 1 \Big\}.$$

## 1.3 $L^p$ spaces

**Definition 1.3.1.** Let $1 \leq p < \infty$. The space $L^p(\Omega)$ consists of equivalence classes of Lebesgue measurable functions $f : \Omega \to \mathbb{R}$ such that

$$\int_\Omega |f(x)|\, dx < \infty,$$

where two measurable functions are equivalent if they are equal a.e. The $L^p$ norm of $f \in L^p(\Omega)$ is defined by

$$\|\cdot\|_{L^p} = \sqrt[p]{\int_\Omega |f(x)|^p\, dx}.$$

For $p = +\infty$ the definition is slightly different. We say that a function $f$ is essentially bounded, if

$$ess \sup |f| = \inf_{N/|N|=0} \sup_{(\Omega \;\; N)} |f(x)| < \infty.$$

The space $L^\infty(\Omega)$ consists of equivalence classes (two functions are equivalent if they are equal a.e.) of measurable, essentially bounded functions $f : \Omega \to \mathbb{R}$ with a norm

$$\|\cdot\|_{L^\infty} = ess \sup |f|\,.$$

**Remark 1.3.2.** The reason to regard functions that are equal a.e. as equivalent is so that $\|f\|_{L^p} = 0$ implies that $f = 0$ and thus $\|\cdot\|_{L^p}$ is a norm. For example, the characteristic function of the rational numbers $\mathbb{Q}$ is equivalent to 0 in $L^p(\mathbb{R})$, for $1 \leq p \leq \infty$.

**Lemma 1.3.3 (Hölder inequality for integrals).** *Let $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $f \in L^p(\Omega), g \in L^q(\Omega)$. Then, $f \cdot g \in L^1(\Omega)$ and*

$$\|f \cdot g\|_{L^1} \leq \|f\|_{L^p} \cdot \|g\|_{L^q}\,.$$

**Theorem 1.3.4.** *Orlicz spaces $L^\rho(\Omega)$, Lebesgue spaces $L^p(\Omega)$, and generalized Lebesgue spaces $L^{p(\cdot)}(\Omega)$, are Banach spaces.*

*We prove the theorem only for the Lebesgue spaces. In the proof we shall use the following lemma.*

**Lemma 1.3.5.** *For any normed space $(V, \|\cdot\|)$ the following conditions are equivalent*

**(a)** $(V, \|\cdot\|)$ *is a complete space.*

**(b)** *If $\{x_n\}_{n\in N}$ is a sequence in $V$, such that $\sum_{n=1}^{+\infty} \|x_n\| < +\infty$, then there exists $x \in V$ such that*

$$\lim_{N\to+\infty} \left\| \sum_{n=1}^{N} x_n - x \right\| = 0.$$

*The condition (b) simply states that any absolutely convergent series is convergent.*

*Proof.* **(a) $\Rightarrow$ (b)** The implication follows from the fact that $S_N = \sum_{n=1}^{N} x_n$ is a Cauchy sequence.

**(b) $\Rightarrow$ (a)** Let $\{x_n\}_{n\in N}$ be a Cauchy sequence. For each $k \in \mathbb{N}$ there exists $N_k$, such that

$$\|x_m - x_n\| < 2^{-k}, \quad \forall n, m \in N_k.$$

We choose a subsequence $\{x_{n_k}\}_{k\in N}$ such that

$$\left\|x_{n_{k+1}} - x_{n_k}\right\| < 2^{-k}, \quad \forall k \in \mathbb{N}.$$

and denote $y_1 = x_{n_1}, y_k = (x_{n_{k+1}} - x_{n_k})$ for $k > 1$. Therefore $\sum_{i=1}^{+\infty} \|y_i\| < +\infty$. From assumptions it follows that there exists $y \in V$, such that

$$\lim_{N\to+\infty} \left\| \sum_{n=1}^{N} y_n - y \right\| = \lim_{N\to+\infty} \left\|x_{n_{N+1}} - y\right\| = 0.$$

Therefore, the subsequence $\{x_{n_k}\}_{k\in N}$ converges in $X$. A Cauchy sequence, which has a convergent subsequence, converges as well, which ends the proof. $\square$

*Proof.* Checking that $\|\cdot\|_{L^p}$ is a norm, when one uses Minkowski inequality, is straightforward. Note that we have proved the assertion in an alternative way for $1 < p < +\infty$ in the Example 1.2.6.

For the proof of completeness we shall use claim (b) from Lemma 1.1.9. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in $L^p(\Omega)$ such that $\rho = \sum_{n=1}^{+\infty} \|f_n\|_{L^p} < +\infty$. We need to construct a function $f \in L^p(\Omega)$, such that $\lim_{N\to+\infty} \left\|\sum_{n=1}^{N} f_n - f\right\|_{L^p} < +\infty$. Define $\widehat{g_n}, \widehat{g} : \Omega \to \mathbb{R}$ as following

$$\widehat{g_n}(x) = \sum_{i=1}^{n} |f_i(x)| \quad \text{and} \quad \widehat{g}(x) = \sum_{n=1}^{+\infty} |f_n(x)| \,.$$

From Minkowski inequality we obtain

$$\|\widehat{g_n}\|_{L^p} = \left\|\sum_{i=1}^{n} |f_i|\right\|_{L^p} \le \sum_{i=1}^{n} \|f_i\|_{L^p} \le \sum_{n=1}^{+\infty} \|f_n\|_{L^p} = M < +\infty,$$

By construction, $\widehat{g_n}$ converges monotonically to $\widehat{g}$. Therefore, from the monotone convergence theorem and the inequality above it follows that

$$\int_\Omega (\widehat{g}(x))^p dx = \int_\Omega \lim_{n\to\infty} (\widehat{g_n}(x))^p dx = \lim_{n\to\infty} \int_\Omega (\widehat{g_n}(x))^p dx < M^p.$$

which implies that $\widehat{g} \in L^p(\Omega)$ and in particular $\widehat{g}$ is finite a.e. From the latter fact we conclude that

$$f(x) := \sum_{n=1}^{+\infty} f_n(x)$$

is finite a.e. and $f \in L^p(\Omega)$ with $\|f\|_{L^p} \le \|\widehat{g}\|_{L^p}$ . Note that

$$0 \le \left|f(x) - \sum_{i=1}^{n} f_i(x)\right|^p = \left|\sum_{i=n+1}^{+\infty} f_i(x)\right|^p = \Big(\sum_{i=n+1}^{+\infty} |f_i(x)|\Big)^p \le (\widehat{g}(x))^p < M^p.$$

Thus, by the Lebesgue dominated convergence theorem

$$\lim_{n\to\infty} \int_\Omega \left|f(x) - \sum_{i=1}^{n} f_i(x)\right|^p dx = 0.$$

which ends the proof due to the Lemma 1.3 $\square$

## 1.4 Convex Functions

**Definition 1.4.1.** [22]

A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called convex if

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y)$$

for all points $x$ and $y$ in $I$ and all $\lambda \in [0,1]$.

**Remark 1.4.2.**

Suppose that $f : [0,\infty) \to [0,\infty]$ is convex and $f(0) = 0$. Choosing $x = 0$ in the previous definition and $\lambda = \beta$ or $\lambda = \frac{1}{\beta}$, we find

$$f(\beta y) \leq \beta f(y), \qquad \text{for } y \in [0,1],$$

$$f(\beta y) \geq \beta f(y), \qquad \text{for } y \geq 1.$$

**Definition 1.4.3.** [1]

Let $V$ be a real vector space. We say that function $\|\cdot\|$ from $V$ to $[0,\infty]$ is a quasinorm if:

**(a)** $\|f\| = 0$ if and only if $f = 0$.

**(b)** $\|af\| = |a|\,\|f\|$ for all $f \in V$ and $a \in \mathbb{R}$.

**(c)** There exists $\lambda > 0$ such that $\|f + g\| \leq \lambda(\|f\| + \|g\|)$ for all $f, g \in V$.

- If $\lambda = 1$ in (c), then $\|\cdot\|$ is called a norm. A (quasi)Banach space $(V, \|\cdot\|_V)$ is a (quasi)normed vector space which is complete with respect the (quasi)norm $\|\cdot\|_V$.

- The dual space $V^*$ of a (quasi)Banach space $V$ consists of all bounded, linear functionals $F : V \to \mathbb{R}$. The duality pairing between $V^*$ and $V$ is defined by $\langle F, x\rangle_{V^*,V} = \langle F, x\rangle := F(x)$ for $F \in V, x \in V$. The dual space is equipped with the dual quasinorm $\|f\|_{V^*} := \sup_{\|x\|_V \le 1}\langle F, x\rangle$, which makes $V^*$ a quasi-Banach space.

**Remark 1.4.4.** [1]

- A space is called separable if it contains a dense, countable subset. We denote the bidual space by $V^{**} := (V^*)$. A quasi-Banach space $V$ is called reflexive if the natural injection $\iota : V \to V^{**}$, given by $\langle \iota x, F\rangle_{V^{**},V^*} := \langle F, x\rangle_{V^*,V}$, is surjective. A norm $\|\cdot\|$ on a space $V$ is called uniformly convex if for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for all$x, y \in V$ satisfying $\|x\|, \|y\| \le 1$ the inequality $\|x - y\| > \varepsilon$ implies $\left\|\frac{x+y}{2}\right\| < 1 - \delta(\varepsilon)$.

- A quasi-Banach space $V$ is called uniformly convex, if there exists a uniformly convex norm $\|\cdot\|'$, which is equivalent to the original norm of $V$.

**Proposition 1.4.5.** *[1]*

*Let $V$ be a Banach space and let $W \subset V$ be closed. Then:*

**(a)** *$W$ is a Banach space.*

**(b)** *If $V$ is reflexive, then $V$ is reflexive.*

**(c)** *If $V$ is separable, then $W$ is separable.*

**(d)** *If $V$ is uniformly convex, then $V$ is reflexive.*

**(e)** *If $V$ is uniformly convex, then $W$ is uniformly convex.*

**Remark 1.4.6.**

We say that a (quasi-)Banach space $V$ is continuously embedded into a (quasi-)Banach space $W$ , $V \hookrightarrow W$ , if $V \subset W$ and there exists a constant $C > 0$ such that $\|x\|_W \leq C\|x\|_V$ for all $x \in V$.

The embedding of $V$ into $W$ is called compact, $V \hookrightarrow\hookrightarrow W$, if $V \hookrightarrow W$ and bounded sets in $V$ are precompact in $W$.

A sequence $\{x_k\}_{k\in\mathbb{N}} \subset V$ is called (strongly) convergent to $x \in V$, if $\lim_{k\to+\infty} \|x_k - x\|_V = 0$. It is called weakly convergent if $\lim_{k\to+\infty}\langle F, x_k\rangle = 0$ for all $F \in V^*$.

Let $(V, \|\cdot\|_V)$ be a Banach space and $A \subset V$ a set. The closure of $A$ with respect to the norm $\|\cdot\|_V$, $\overline{A}^{\|\cdot\|_V}$ is the smallest closed set that contains $A$.

## 1.5 Some results about Integration

**Theorem 1.5.1 (monotone convergence theorem, Beppo Levi).** *[3]*

*Let $(h_n)$ be a sequence of functions in $L^1$ that satisfy*

**(a)** $h_1 \leq h_2 \leq ... \leq h_n \leq h_{n+1} \leq ...$ *a.e. on* $\Omega$,

**(b)** $\sup_n \displaystyle\int h_n < \infty$.

*Then $h_n(x)$ converges a.e. on $\Omega$ to a finite limit, which we denote by $h(x)$; the function $h$ belongs to $L^1$ and $\|h_n - h\|_1 \to 0$.*

**Theorem 1.5.2 (dominated convergence theorem, Lebesgue).** *[3]*

*Let $(h_n)$ be a sequence of functions in $L^1$ that satisfy*

**(a)** $h_n(x) \to h(x)$ *a.e. on* $\Omega$

**(b)** *there is a function $\psi \in L^1$ such that for all $n$, $|h_n(x)| \leq \psi(x)$ a.e. on $\Omega$. Then $h \in L^1$ and $\|h_n - h\|_1 \to 0$.*

**Lemma 1.5.3 (Fatou's lemma).** *[3]*

*Let $(h_n)$ be a sequence of functions in $L^1$ that satisfy*

**(a)** *for all $n$, $h_n \geq 0$ a.e.*

**(b)** $\sup_n \int h_n < \infty$.

*For almost all $x \in \Omega$ we set $h(x) = \liminf_{n\to\infty} h_n(x) \leq +\infty$. Then $h \in L^1$ and*

$$\int h \leq \liminf_{n\to\infty} \int h_n.$$

*A basic example is the case in which $\Omega = \mathbb{R}^N$, $\mathcal{M}$ consists of the Lebesgue measurable sets, and $\mu$ is the Lebesgue measure on $\mathbb{R}^N$.*

## 1.6 History of generalized Orlicz spaces

Variable exponent Lebesgue spaces appeared in the literature for the first time already in a 1931 article by W. Orlicz [23]. In this article the following question is considered: let $(p_i)$, with $p_i > 1$, and $(f_i), f_i \geq 0$ be real-valued sequences. What is the necessary and sufficient condition on $(g_i)$ for $\sum_i f_i g_i$ to converge whenever $\sum_i f_i^{pi}$ converges ? It turns out that the answer is that $\sum_i (\beta g_i)^{p'_i}$ should converge for some $\beta > 0$ and $p'_i = \frac{p_i}{p_i - 1}$. This is essentially Hölder's inequality in the space $\ell^{p(\cdot)}$. Orlicz also considered the variable exponent function space $L^{p(\cdot)}$ on the real line, and proved the Hölder inequality in this setting.

Variable exponent spaces have been studied in more than a thousand papers in the past 15 years so we only cite a few monographs on the topic which can be consulted for additional references ([7], [8], [11], [26]).

After this one paper [23], Orlicz abandoned the study of variable exponent spaces, to concentrate on the theory of the function spaces that now bear his name (but see also [20]). In the theory of Orlicz spaces, one defines the space $L^f$ in an open set

$\Omega \cap \mathbb{R}^n$ to consist of those measurable functions $\psi : \Omega \to \mathbb{R}$ for which

$$\rho(\beta\psi) = \int_{\Omega} f(\beta |\psi(x)|)dx < \infty$$

for some $\beta > 0$ ($f$ has to satisfy certain conditions, see Definition 2.1.4). Abstracting certain central properties of $\varrho$,one is led to a more general class of so-called modular function spaces which were first systematically studied by H. Nakano [21].

Following the work of Nakano, modular spaces were investigated by several people, most importantly by the groups at Sapporo (Japan), Voronezh (USSR),and Leiden (Netherlands). Somewhat later, a more explicit version of these spaces, modular function spaces, were investigated by Polish mathematicians, for instance by H.Hudzik, A. Kaminska and J.Musielak.

For a comprehensive presentation of modular function spaces and generalized Orlicz spaces, see the monograph [19] by Musielak.

Harmonic analysis in generalized Orlicz spaces has only recently been studied. In 2005, Lars Diening [9] investigated the boundedness of the maximal operator on $L^f$ and gave abstract conditions on the $\Phi$-function for the boundedness to hold. In the variable exponent case it led to the result that the maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ if and only if it is bounded on $L^{p'(\cdot)}(\mathbb{R}^n)$(provided $1 < p^- \leq p^+ < \infty$). Unfortunately,this result has still not been successfully extended to generalized Orlicz spaces.

In 2011, Lars Diening, Petteri Harjulehto, Peter Hästö and Michael Ruzicka [10] have written a book on variable exponent spaces. Recently, in 2019 Petteri Harjulehto and Peter Hasto [14] have presented a systematic treatment of Orlicz and generalized Orlicz spaces (also known as Musielak-Orlicz spaces) in a general framework.

# Chapter 2

# Φ-Functions

In this chapter, we will study some definitions and basic properties of Φ-function as almost increasing, almost decreasing, equivalent Φ -Functions, upgrading Φ-functions, inverse Φ-Functions, conjugate Φ-Functions and generalized Φ-Functions. this chapter is based on the monographs [10, 14, 18, 27].

## 2.1 Equivalent Φ-Functions

We begin by studying the properties of almost increasing and almost decreasing, which will be used throughout the memory. Finally, we consider two notions of the equivalence of Φ-functions and study relations between them.

**Definition 2.1.1.** A function $h : (0, \infty) \longrightarrow \mathbb{R}$ is almost increasing if there exists a constant $\delta \geq 1$ such that $h(x) \leq \delta h(y)$ for all $0 < x < y$. Almost decreasing is defined analogously.

Increasing and decreasing functions are included in the previous definition as the special case $a = 1$.

**Definition 2.1.2.** Let $f : (0, \infty) \longrightarrow \mathbb{R}$ and $p, q > 0$. We say that $f$ satisfies

$$\frac{f(x)}{x^p} \text{ is increasing;} \tag{2.1}$$

$$\frac{f(x)}{x^p} \text{ is almost increasing;} \tag{2.2}$$

$$\frac{f(x)}{x^q} \text{ is decreasing;} \tag{2.3}$$

$$\frac{f(x)}{x^q} \text{ is almost decreasing;} \tag{2.4}$$

We say that $f$ satisfies $(2.1)_{p>1}$, $(2.2)_{p>1}$, $(2.3)_{q<\infty}$ or $(2.4)_{q<\infty}$ if there exist $p > 1$ or $q < \infty$ such that $f$ satisfies (2.1),(2.2), (2.3) or (2.4), respectively.

**Remark 2.1.3.** If $f$ satisfies (2.2) in respect to $p_1$. Then it satisfies (2.2) in respect to $p_2$ for $p_2 < p_1$ and it does not satisfy (2.4) for $q < p_1$. Likewise, if $f$ satisfies (2.4) in respect of $q_1$ then it satisfies (2.4) in respect to $q_2$ for $q_2 < q_1$ and it does not satisfy (2.2) for $p > q_1$.

**Definition 2.1.4.** Let $f : [0, \infty) \longrightarrow [0, \infty]$ be increasing with $\lim_{x\longrightarrow 0^+} f(x) = 0$, $\lim_{x\longrightarrow\infty} f(x) = \infty$ and $f(0) = 0$. Such $f$ is called a $\Phi$-prefunction.

We say that a $\Phi$-prefunction $f$ is a

- (weak) $\Phi$-function, if it satisfies (2.2) with $p = 1$ on $(0, \infty)$;
- convex $\Phi$-function if it is left-continuous and convex;
- strong $\Phi$-function if it is continuous in the topology of $[0, \infty]$ and convex.

The sets of weak, convex and strong $\Phi$-function are denoted by $\Phi_W$ , $\Phi_c$ and $\Phi_s$ respectively.

**Remark 2.1.5.** Note that when we speak about $\Phi$-functions, we mean the weak $\Phi$-functions of the previous definition.

Continuity in the topology of $C([0, \infty); [0, \infty])$ means that

$$\lim_{x\longrightarrow y} h(x) = h(y)$$

for every point $y \in [0, \infty)$ regadless of whether $h(y)$ is finite or infinite.

If $f$ is convex and $f(0) = 0$, then we obtain for $0 < x < y$ that

$$f(x) = f\Big(\frac{x}{y}y + 0\Big) \leq \frac{x}{y}f(y) + \Big(1 - \frac{x}{y}\Big)f(0) = \frac{x}{y}f(y), \tag{2.5}$$

i.e. (2.1) with $p = 1$, holds. Therefore, it follows from the definition that $\Phi_s \subset \Phi_c \subset \Phi_W$.

As a convex function, $f$ is continuous in $[0, \infty)$ if and only if it is finite on $[0, \infty)$. Let us show that if $f \in \Phi_c$ satisfies $(2.4)_{q<\infty}$, then $f \in \Phi_s$. Since $\lim\limits_{x \longrightarrow 0^+} f(x) = 0$, we can find $x > 0$ with $f(x) < \infty$. By $(2.4)_{q<\infty}$ we obtain $f(y) \leq \delta \frac{y^q}{x^q} f(x)$ for $y > x$. Hence $f < \infty$ in $[0, \infty)$ and thus it is continuous.

**Example 2.1.6.** [14]

For $x \geq 0$, we define (see Fig. 2.1.6)

$$\begin{aligned} f^p(x) &:= \frac{1}{p} x^p, \qquad p \in (0, \infty), \\ f_{\max}(x) &:= (\max\{0, x-1\})^2, \\ f_{\sin}(x) &:= x + \sin(x), \\ f_{\exp}(x) &:= \exp(x) - 1, \\ f^\infty(x) &:= \infty \chi_{(1,\infty)}(x), \\ f^{\infty,2}(x) &:= f^\infty(x) + \frac{2x-1}{1-x} \chi_{(1/2,1)}(x). \end{aligned}$$

Observe that $f^p \in \Phi_s$ if and only if $p \geq 1$. Furthermore, $f_{\max}$ ,$f_{\exp}$ , $f^{\infty,2} \in \Phi_s$ ,$f^\infty \in \Phi_c \backslash \Phi_s$ and $f_{\sin} \in \Phi_W \backslash \Phi_c$.

- Observe that $f^1 \simeq f_{\sin}$ and $f^\infty \simeq f^{\infty,2}$ but $f^\infty \not\approx f^{\infty,2}$, as $f \approx g$ means that $C_1 f(t) \leq g(t) \leq C_2 f(t)$ for all relevant values of $t$.

  Therefore, neither $\Phi_c$ nor $\Phi_s$ is invariant under equivalence of $\Phi$-functions.

  For the equivalence $\simeq$ see Definition (2.1.9)

- Observe that $f^p \to f^\infty$ and $\frac{1}{p} f^{\infty,2} \to f^\infty$ point-wise as $p \to \infty$. Therefore, $\Phi_s$ is not invariant under point-wise limits of $\Phi$-functions.

- Note that $\min\{f^1, f^2\} \in \Phi_W \backslash \Phi_c$, so $\Phi_c$ is not preserved under point-wise minimum.

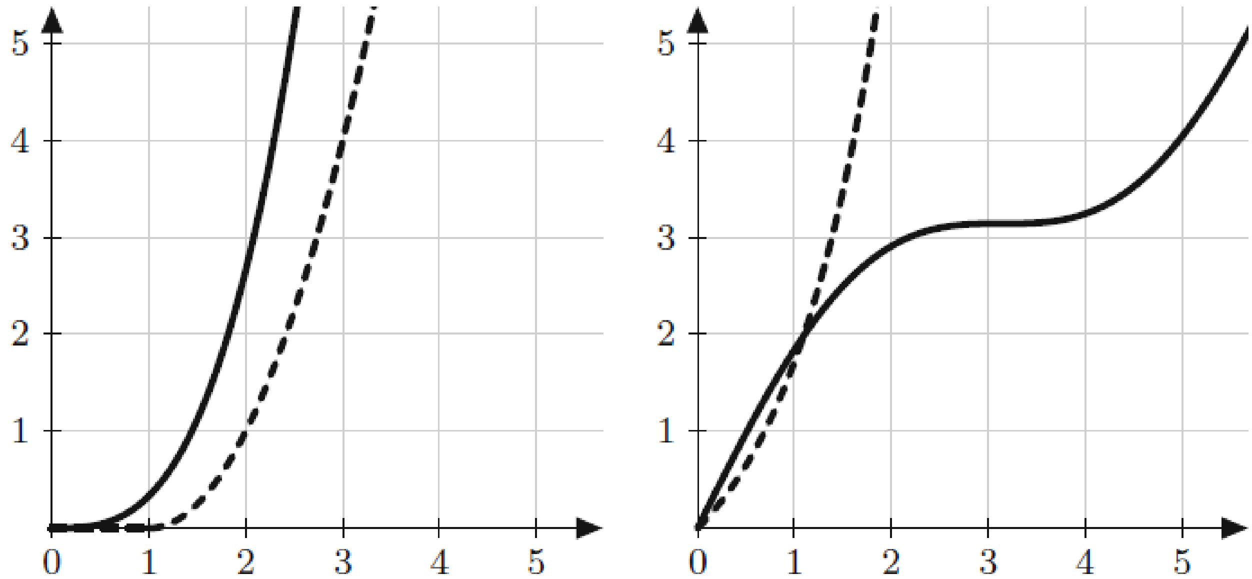


Figure 2.1: **Functions from Example 2.1.6.Left:** $f^3$ **(solid) and** $f_{\max}$ **(dashed). Right:** $f_{\sin}$ **(solid) and** $f_{\exp}$ **(dashed)**

**Lemma 2.1.7.** *If $f \in \Phi_W$ is left-continuous, then $f(x) \leq \liminf_{x_k \to x} f(x_k)$, i.e. $f$ is lower semicontinuous.*

*Proof.* Suppose that $x_k \longrightarrow x$ and $x'_k := \min\{x, x_k\}$. we have $x'_k \to x^-$, since $f$ is left-continuous and increasing, we get $f(x) = \lim_{x_k \to x} f(x'_k) \leq \liminf_{x_k \to x} f(x_k)$ $\square$

**Remark 2.1.8.** [14]

Assume that $D := (1, \infty)$, $f^\infty(x) := \infty\chi_{(1,\infty)}(x)$, since $f$ is not continuous, we have $0 = f^\infty(1) < \inf_{x \in A} f^\infty(x) = \infty$.

but, in general, if $f$ a $\Phi$-prefunction then

$$f(\inf D) \leq \inf_{x \in D} f(x)$$

for every non-empty set $D \subset [0, \infty)$

However although, if $f$ a $\Phi$-prefunction then, for every $\sigma > 1$ we have

$$\inf f(D) \leq f(\sigma \inf D)$$

In fact, if $\inf D = 0$, then $\inf f(D) \leq f(0) = 0$ as well as $\lim_{x \to 0^+} f(x) = 0$,

If $\inf D > 0$, then $\inf D < \sigma \inf D$, by characterizations of the infimum, there exists $x \in D$ such that $x < \sigma \inf D$.

Consequently, using the monotonicity of $f$, we get $\inf f(D) \leq f(\sigma \inf D)$.

Moreover, if $f$ satisfy (2.2) with $p = 1$ and constant $\delta$. we can show that

- $f(\beta x) \leq \delta \beta f(x)$ for all $\beta \in [0, 1]$ and $x \geq 0$; and

- $f(\gamma x) \geq \frac{\gamma}{\delta} f(x)$ for all $\gamma \in [1, \infty)$ and $x \geq 0$

  For $x = 0$, previous inequalities are true,

  For $x > 0$, since $f$ satisfies (2.2) with $p = 1$ and $\beta x \leq x \leq \gamma x$, we deduce that

$$\frac{1}{\delta} \frac{f(\beta x)}{\beta x} \leq \frac{f(x)}{x} \leq \delta \frac{f(\gamma x)}{\gamma x}$$

  Then previous inequalities are true.

**Definition 2.1.9.** Two functions $f$ and $g$ are equivalent, $f \simeq g$, if there exists $\eta \geq 1$ such that $f(\frac{x}{\eta}) \leq g(x) \leq f(\eta x)$ for all $x \geq 0$.

**Remark 2.1.10.** Observe that $\simeq$ is an equivalence relation in a set of functions from $[0, \infty)$ to $[0, \infty]$ :

- $f \simeq f$ for every $f$. (reflexivity);

- $f \simeq g$ implies $g \simeq f$ (symmetry);

- $f_1 \simeq f_2$ and $f_2 \simeq f_3$ imply $f_1 \simeq f_3$ (transitivity).

**Lemma 2.1.11.** *Let $f, g\ : [0, \infty) \to [0, \infty]$ be increasing with $f \simeq g$.*

*(a) If $f$ is a $\Phi$-prefunction, then $g$ is a $\Phi$-prefunction.*

*(b) If $f$ satisfies (2.2), then $g$ satisfies (2.2).*

*(c) If $f$ satisfies (2.4), then $g$ satisfies (2.4).*

*Proof.* For (a), suppose that $f$ is a $\Phi$-prefunction. Let $\eta \geq 1$ be such that $f(\frac{x}{\eta}) \leq g(x) \leq f(\eta x)$. we find,

$$0 \leq \lim_{x \to 0^+} g(x) \leq \lim_{x \to 0^+} f(\eta x) = \lim_{x \to 0^+} g(x) = 0$$

and

$$\lim_{x \to \infty} g(x) \geq \lim_{x \to \infty} f(\frac{x}{\eta}) = \lim_{x \to \infty} f(x) = \infty$$

when $x = 0$, we get $g(0) = 0$.

For (b), Let $0 < x < y$ and assume first that $\eta^2 x < y$, where $\eta$ is the constant from the equivalence. By (2.2) of $f$ with constant $\delta$, we get

$$\frac{g(x)}{x^p} \leq \frac{f(\eta x)}{x^p} = \eta^p \frac{f(\eta x)}{(\eta x)^p} \leq \delta \eta^p \frac{f(\frac{y}{\eta})}{(\frac{y}{\eta})^p} \leq \delta \eta^{2p} \frac{g(y)}{y^p}$$

Assume then that $y \in (x, \eta^2 x]$. since $g$ is increasing, we get

$$\frac{g(x)}{x^p} \leq \frac{g(y)}{x^p} = \frac{y^p}{x^p} \frac{g(y)}{y^p} \leq \eta^{2p} \frac{g(y)}{y^p}$$

we have shown that $g$ satisfies (2.2) with constant $\delta \eta^{2p}$.

The proof of (c) is similar to (b). $\square$

**Example 2.1.12.** [14]

we consider an example shows that (2.1) without the "almost" is not invariant under equivalence of $\Phi$-functions.

Assume that $f(y) := y^2$ and $g(y) := y^2 + \max\{y - 1, 0\}$ for $y \geq 0$. Then $f(y) \leq g(y) \leq f(2y)$ so that $f \simeq g$. (see Fig.2.1.12)

Clearly $f$ satisfies (2.1) with $p = 2$. Suppose that $g$ satisfies (2.1) for $p \geq 1$. we write the condition at points $y = 2$ and $x = 3$:

$$\frac{4+1}{2^p} = \frac{g(2)}{2^p} \leq \frac{g(3)}{3^p} = \frac{9+2}{3^p}$$

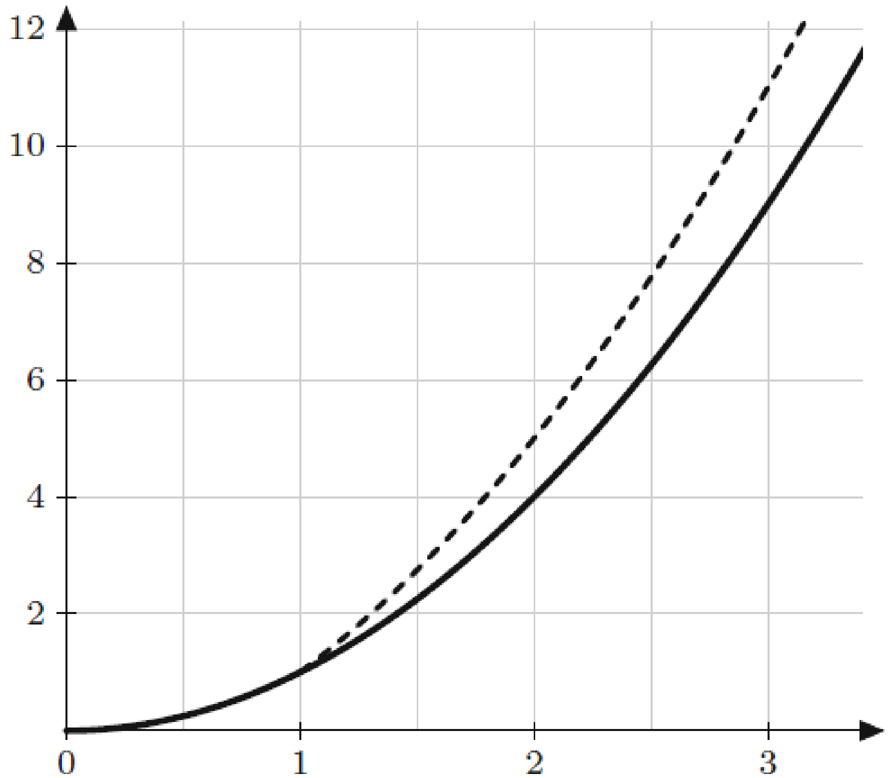


Figure 2.2: **Functions $f$ (solid) and $g$ (dashed) from Example 2.1.12**

i.e.$(\frac{3}{2})^p \leq \frac{11}{5}$. this means that $p < 2$, hence $g$ does not satisfy (2.1) with $p = 2$.

**Lemma 2.1.13.** *Let $f$ , $g : [0, \infty) \to [0, \infty]$.*

*(a) If $f$ satisfies* (2.2) *with $p = 1$ and $f \approx g$ , then $f \simeq g$ and $g$ satisfies* (2.2) *with $p = 1$.*

*(b) If $f$ satisfies* $(2.4)_{q<\infty}$*, then $f \simeq g$ implies $f \approx g$.*

*Proof.* Suppose that $f \approx g$ with constant $\eta \geq 1$. Then $g(y) \leq \eta f(y) \leq f(\delta\eta y)$ by (2.2) with $p = 1$. The lower bound is similar, and thus $f(y) \simeq g(y)$ . As (2.2) with $p = 1$ gives as follows :

$$\frac{g(x)}{x} \leq \eta\frac{f(x)}{x} \leq \delta\eta\frac{f(x)}{x} \leq \delta\eta^2\frac{g(y)}{y}$$

for $0 < x < y$.

Note that here we do not need the function to be increasing, in contrast to Lemma (2.1.11).

Assume next that $f \simeq g$and $f$ satisfies (2.4). Then $g(y) \leq f(\eta y) \leq \delta \eta^q f(y)$ by (2.4). The lower bound is similar, and thus $f(y) \approx g(y)$. $\square$

## 2.2 Upgrading Φ-Functions

Recall that, we can always estimate a weak Φ-functions as follows:

$$f(x+y) \leq f(2\max\{x,y\}) \leq f(2x) + f(2y)$$

So, we will show some convexity-type properties.

**Lemma 2.2.1.** *If $f \in \Phi_W$ satisfies* (2.2) *with $p \geq 1$, then there exists $g \in \Phi_c$ equivalent to $f$ such that $g^{1/p}$ is convex. In particular, $g$ satisfies* (2.1).

*Proof.* see [14]. $\square$

**Corollary 2.2.2.** *If $f \in \Phi_W$, then there exists a constant $\lambda > 0$ such that*

$$f\Big(\lambda \sum_{k=1}^{\infty} \alpha_k \xi_k\Big) \leq \sum_{k=1}^{\infty} f(\alpha_k)\xi_k$$

*for all $\alpha_k, \xi_k \geq 0$ with $\sum \xi_k = 1$.*

*Proof.* Assume first that $g \in \Phi_c$. denote $\xi'_{m+1} := \sum_{k=m+1}^{\infty} \xi_\omega$ and $\alpha'_{m+1} = 0$. Then by convexity

$$g\Big(\sum_{k=1}^{m} \alpha_k \xi_k\Big) = f\Big(\sum_{k=1}^{m} \alpha_k \xi_k + \alpha'_{m+1}\xi'_{m+1}\Big) \leq \sum_{k=1}^{m} g(\alpha_k)\xi_k + g(\alpha'_{m+1})\xi'_{m+1} \leq \sum_{k=1}^{\infty} g(\alpha_k)\xi_k.$$

The inequality follows with $\lambda = 1$ by left-continuity as $m \to \infty$.

Let then $f \in \Phi_W$. By Lemma (2.2.1), there exists $g \in \Phi_c$ such that $f \simeq g$ with constant $\eta \geq 1$. Choose $\lambda := \eta^{-2}$. Then

$$f\Big(\lambda \sum_{k=1}^{\infty} \alpha_k \xi_k\Big) \leq g\Big(\frac{1}{\eta} \sum_{k=1}^{\infty} \alpha_k \xi_k\Big) \leq \sum_{k=1}^{\infty} g\Big(\frac{1}{\eta}\alpha_k\Big)\xi_k \leq \sum_{k=1}^{\infty} f(\alpha_k)\xi_k$$

. $\square$

**Theorem 2.2.3.** *Every weak $\Phi$-function is equivalent to a strong $\Phi$-function*

*Proof.*

Assume that $f \in \Phi_W$ satisfies (2.2) with $p = 1$. By Lemma (2.2.1) there exists $f_c \in \Phi_c$ with $f_c \simeq f$. since $f_c$ convex then $f_c$ is continuous in the set $\{f_c < \infty\}$.

If $\{f_c < \infty\} = [0, \infty)$ we are done.

Otherwise, denote $y_\infty := \inf\{y : f_c(y) = \infty\} \in (0, \infty)$ and define

$$f_x(y) := f_c(y) + \frac{2y - y_\infty}{y_\infty - y}\chi_{(\frac{1}{2}y_\infty, y_\infty)}(y) + \infty\chi_{[y_\infty,\infty)}(y)$$

As the sum of three convex functions, $f_x$ is convex. Furthermore, $f_x = f_c$ in $[0, \frac{1}{2}y_\infty] \bigcup [y_\infty, \infty)$. Hence $f_x(\frac{y}{2}) \leq f_c(y) \leq f_x(y)$, so that $f_x \simeq f_c \simeq f$. since $f_x$ is increasing we obtain by Lemma (2.1.11)(a) that $f_x$ is a $\Phi$-prefunction. Since also $\lim_{y \to y_\infty} f_x(y) = \infty$ , we conclude that $f_x \in \Phi_s$. □

**Remark 2.2.4.** Observe that theorem 2.2.3 is not true with the $\approx$-equivalence. Indeed, from Example (2.1.6), if $g \approx f^\infty$, then necessarily $g = f^\infty$.

**Definition 2.2.5.** We say that a function $f : [0, \infty) \to [0, \infty]$ satisfies $\Delta_2$, or is doubling if there exists a constant $\theta \geq 2$ such that .

$$f(2y) \leq \theta f(y) \quad \text{for all} \quad y \geq 0.$$

Next we show that $(2.4)_{q<\infty}$ is a quantitative version of doubling.

**Lemma 2.2.6.** .

*(a) If $f \in \Phi_W$, then $\Delta_2$ is equivalent to $(2.4)_{q<\infty}$,*

*(b) If $f \in \Phi_c$, then $\Delta_2$ is equivalent to $(2.3)_{q<\infty}$.*

*Proof.* Assume that $f$ satisfies $\Delta_2$ and let $0 < x < y$. Choose an integer $n \geq 1$ such that $2^{n-1}x < y \leq 2^n x$. we have

$$f(y) \leq f(2^n x) \leq \theta f(2^{n-1}x) \leq ... \leq \theta^n f(x)$$

We define $q := \log_2(\theta) \geq 1$. Then the previous inequality and $y > 2^{n-1}x$ yield that

$$\frac{f(y)}{y^q} \leq \theta^n \frac{f(x)}{y^q} \leq \theta^n \frac{f(x)}{2^{q(n-1)}x^q} = \theta \frac{f(x)}{x^q}$$

Thus $f$ satisfies (2.4).

Assume then $(2.4)_{q<\infty}$ holds. Then there exists $q > 1$ such that

$$\frac{f(2y)}{(2y)^q} \leq \delta \frac{f(y)}{y^q}$$

so $f(2y) \leq \delta 2^q f(y)$. Thus $(2.4)_{q<\infty}$ implies $\Delta_2$ with $\theta = \delta 2^q$. Hence (a) is proved.

For (b) we are left to show that convexity and $\Delta_2$ yield $(2.3)_{q<\infty}$ for some $q \geq 1$.

Let $y \geq 2x$ and $q$ be the exponent defined in (a). By case (a),

$$\frac{f(y)}{f(x)} \leq \theta \Big(\frac{y}{x}\Big)^q \leq \Big(\frac{y}{x}\Big)^{2q}$$

and $(2.3)_{q<\infty}$ holds for $y \geq 2s$ with any exponent that is at least $2q$. Suppose next that $y \in (x, 2x)$. Choose $\epsilon := \frac{y}{x} - 1 \in (0,1)$ and note that $y = (1-\epsilon)x + \epsilon 2x$. By convexity and $\Delta_2$, we find that

$$f(y) \leq (1-\epsilon)f(x) + \epsilon f(2x) \leq (1 - \epsilon + \theta\epsilon)f(x),$$

. Therefore by the generalized Bernoulli inequality in the second step we obtain

$$\frac{f(y)}{f(x)} \leq 1 + (\theta - 1)\epsilon \leq (1+\epsilon)^{\theta-1} = (\frac{y}{x})^{\theta-1},$$

Combining the two cases, we see that (2.3) with $q = q2 := \max\{2q, \theta - 1\}$ holds. □

**Proposition 2.2.7.** *If $f \in \Phi_W$ satisfies $(2.4)_{q<\infty}$, then there exists $g \in \Phi_s$ with $g \approx f$ which is a strictly increasing bijection.*

*Proof.* see [14]. □

## 2.3 Inverse Φ-Functions

we will extend in this section all Φ-functions to the interval $[0,\infty]$ by $f(\infty) := \infty$ and define a left-continuous function with many properties of the inverse, which we call for simplicity left-inverse because weak Φ-functions are not bijections, they are not strictly speaking invertible.

**Definition 2.3.1.** By $f^{-1} : [0,\infty] \to [0,\infty]$, we denote the left-inverse of

$$f : [0,\infty] \to [0,\infty],$$

$$f^{-1}(\eta) := \inf\{y \geq 0 : f(y) \geq \eta\}.$$

**Example 2.3.2.**

We define $f : [0,\infty] \to [0,\infty]$ by

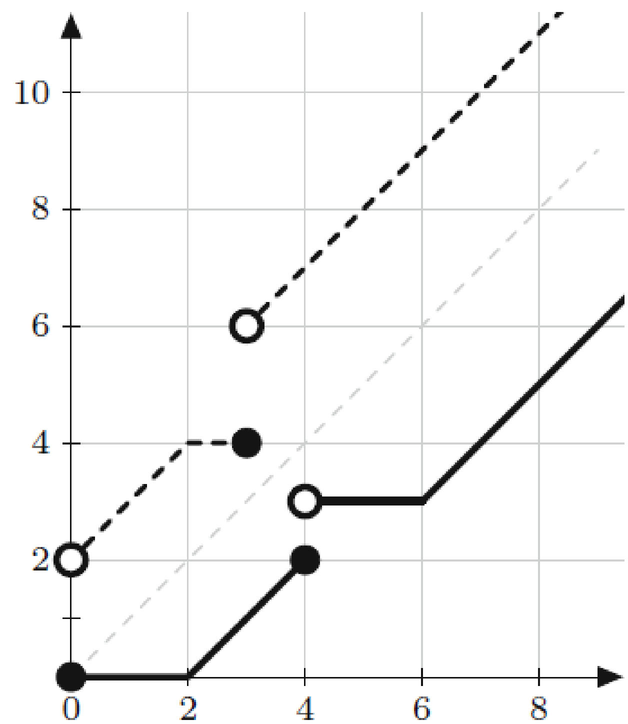


Figure 2.3: **A weak Φ-function (solid) and its left-inverse (dashed)**

$$f(y) := \begin{cases} 0 & \text{if} \quad y \in [0,2] \\ y-2 & \text{if} \quad y \in (2,4] \\ 3 & \text{if} \quad y \in (4,6] \\ y-3 & \text{if} \quad y \in (6,\infty] \end{cases}$$

see Fig. 2.3. Then $f \in \Phi_W \backslash \Phi_c$ and the left-inverse is given by

$$f^{-1}(y) := \begin{cases} 0 & \text{if} \quad y = 0 \\ y+2 & \text{if} \quad y \in (0,2] \\ 4 & \text{if} \quad y \in (2,3] \\ y+3 & \text{if} \quad y \in (3,\infty] \end{cases}$$

With these expressions we can calculate the compositions $f o\, f^{-1}$ and $f^{-1} o\, f$:

$$f(f^{-1}(y)) := \begin{cases} y & \text{if} \quad y \in [0,2] \\ 2 & \text{if} \quad y \in (2,3] \\ y & \text{if} \quad y \in (3,\infty] \end{cases} \qquad f^{-1}(f(y)) := \begin{cases} 0 & \text{if} \quad y \in [0,2] \\ y & \text{if} \quad y \in (2,4] \\ 4 & \text{if} \quad y \in (4,6] \\ y & \text{if} \quad y \in (6,\infty] \end{cases}$$

see Fig. 2.4.

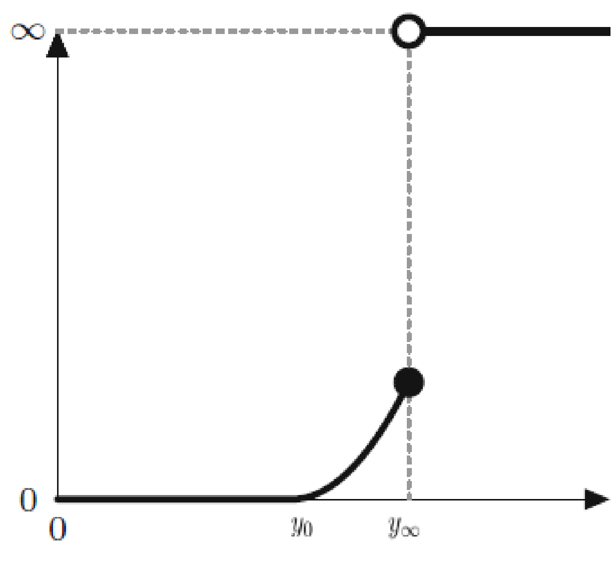


Figure 2.4: $y_0$ **and** $y_\infty$

Note that the following result holds only for convex $\Phi$-functions.

**Lemma 2.3.3.** *Let $f \in \Phi_c$, $y_0 := \sup\{y : f(y) = 0\}$ and $y_\infty := \inf\{y : f(y) = \infty\}$. Then*

$$f^{-1}(f(y)) = \begin{cases} 0, & y \leq y_0, \\ t, & y_0 < y \leq y_\infty \\ y_\infty, & y > y_\infty. \end{cases} \quad \text{and} \quad f(f^{-1}(\eta)) = \min\{\eta, f(y_\infty)\}.$$

*In particular, if $f \in \Phi_s$, then $f(f^{-1}(x)) \equiv x$.*

*Proof.* see [14]. □

**Remark 2.3.4.** In the proof ( see [14]) we used the fact that every convex $\Phi$-function is strictly increasing on $f^{-1}(0, \infty) = (y_0, y_\infty)$. This also yields the following result.

**Corollary 2.3.5.** *Let $f \in \Phi_c$. If $f(x) \in (0, \infty)$, then $f^{-1}(f(x)) = x$.*

*Indeed, if $f \in \Phi_c$ satisfies $(2.4)_{q<\infty}$, then $f$ is bijective and $f^{-1}$ is just the regular inverse.*

**Theorem 2.3.6.** *Let $f, g \in \Phi_W$. Then $f \simeq g$ if and only if $f^{-1} \approx g^{-1}$.*

*Proof.* Suppose first that $f \simeq g$, i.e. $f\left(\frac{y}{\eta}\right) \leq g(y) \leq f\left(\eta y\right)$ for all $y \geq 0$. Then

$$g^{-1}(\eta) = \inf\{y \geq 0 : g(y) \geq \eta\} \geq \inf\{y \geq 0 : f(\eta y) \geq \eta\} = \frac{1}{\eta} f^{-1}(\eta)$$

and

$$g^{-1}(\eta) = \inf\{y \geq 0 : g(y) \geq \eta\} \leq \inf\{y \geq 0 : f\left(\frac{y}{\eta}\right) \geq \eta\} = \eta f^{-1}(\eta)$$

Thus $f \simeq g$ implies $f^{-1} \approx g^{-1}$.

Suppose then that $f^{-1} \approx g^{-1}$. By Theorem (2.2.3) there exist $f_x, g_x \in \Phi_s$ such that $f_x \simeq f$ and $g_x \simeq g$. By the first part of the proof, $f_x^{-1} \approx f^{-1}$ and $g_x^{-1} \approx g^{-1}$ so that $f_x^{-1} \approx g_x^{-1}$ by transitivity of $\approx$. If we show that this implies $f_x \simeq g_x$, then the claim follows, since " $\simeq$ " is an equivalence relation.

Let $y_0 := \sup\{y : f_x(y) = 0\}$ and $y_\infty := \inf\{y : f_x(y) = \infty\}$. Let us first assume that $y \in (y_0, y_\infty)$. We obtain by Corollary (2.3.5) and $f_x^{-1} \approx g_x^{-1}$ that

$$\frac{1}{\eta}y = \frac{1}{\eta}f_x^{-1}(f_x(y)) \leq g_x^{-1}(f_x(y)) \leq \eta f_x^{-1}(f_x(y)) = \eta y.$$

Then we take $g_s$ of both sides and use Lemma (2.3.3) to obtain that $g_x(\frac{1}{\eta}y) \leq g_x(g_x^{-1}(f_x(y))) = f_x(y) \leq g_x(\eta y)$. We have shown the claim for $y \in (y_0, y_\infty)$. By continuity, $g_x(\frac{1}{\eta}y_0) = \lim_{y \to y_0^+} g_x(\frac{1}{\eta}y) \leq \lim_{y \to y_0^+} f_x(y) = f_x(y_0) = 0$ and hence $g_x(\frac{1}{\eta}y) = 0$ for $y \in (0, y_0]$. The inequality $f(y) \leq g(\eta y)$ is clear, since $f(y) = 0$ when $y \leq y_0$. Similarly, we prove that $g_x(\frac{1}{\eta}y) \leq \infty \leq g_x(\eta y)$ when $y \geq y_\infty$. □

**Proposition 2.3.7.** *Let $f \in \Phi_W$ and $p, q > 0$. Then*

*(a) $f$ satisfies* (2.2) *if and only if $f^{-1}$ satisfies* (2.4) *with $q = \frac{1}{p}$.*

*(b) $f$ satisfies* (2.4) *if and only if $f^{-1}$ satisfies* (2.2) *with $p = \frac{1}{q}$ .*

*Proof.* Suppose first that $f \in \Phi_s$. Let $\eta \in (0, \infty)$ and $y_\infty := \inf\{y : f(y) = \infty\}$. Since $f$ is surjection, there exists $y \in (0, y_\infty)$ such that $f(y) = \eta$.

We obtain by Corollary 2.3.5 that

$$\frac{f^{-1}(\eta)}{\eta^{\frac{1}{p}}} = \frac{f^{-1}(f(y))}{f(y)^{\frac{1}{p}}} = \Big(\frac{f(y)}{y^p}\Big)^{\frac{-1}{p}}$$

Hence the fraction on the left-hand side is almost decreasing if and only if the fraction on the right-hand side is almost increasing, and vice versa.

This proves the claim for $f$ restricted to $(y_0, y_\infty]$. If $y \leq y_0$ or $y > y_\infty$, then $f(y) = 0$ or $f(y) = \infty$, and the claim is vacuous.

Consider then the general case $f \in \Phi_W$. By Theorem 2.2.3 there exists $g \in \Phi_s$ with $g \simeq f$. By Theorem 2.3.6, $g^{-1} \approx f^{-1}$, so the claim follows from the first part of the proof, by Lemma 2.1.11. □

**Lemma 2.3.8.** *Let $f, g : [0, \infty] \to [0, \infty]$ be increasing. Then the following implications hold:*

$$f \leq g \quad \Rightarrow \quad g^{-1} \leq f^{-1}$$

$$f^{-1} < g^{-1} \quad \Rightarrow \quad g < f$$

*Proof.* Suppose that $f \leq g$ and $\eta \geq 0$. Then

$$\{y \geq 0 : f(y) \geq \eta\} \subset \{y \geq 0 : g(y) \geq \eta\}$$

and so $f^{-1}(\eta) \leq g^{-1}(\eta)$. □

**Lemma 2.3.9.** *Let $f : [0, \infty] \to [0, \infty], \eta, y \geq 0$ and $\nu > 0$.*

**(a)** *Then $f^{-1}$ is increasing, $f^{-1}(0) = 0$, $f^{-1}(f(y)) \leq y$ and $f(f^{-1}(\eta) - \nu) < \eta$ when $f^{-1}(\eta) \geq \nu$.*

**(b)** *If $f$ is left-continuous with $f(0) = 0$, then $f(f^{-1}(\eta)) \leq \eta$.*

**(c)** *If $f$ is increasing, then $f^{-1}$ is left-continuous, $y \leq f^{-1}(f(y) + \nu)$ and $\eta \leq f(f^{-1}(\eta) + \nu)$.*

**(d)** *If $f$ satisfies (2.2) with $p = \nu$, then $f^{-1}(f(y)) \approx y$, when $f(y) \in (0, \infty)$.*

**(e)** *If $f$ with $\lim_{y \to 0^+} f(y) = 0$ satisfies $(2.4)_{q<\infty}$, then $f(f^{-1}(\eta)) \approx \eta$.*

*Proof.* see [14] □

**Example 2.3.10.** we consider an example shows that the left-continuity is crucial in Lemma 2.3.9(b).If

$$f(y) := \begin{cases} 2y & \text{if } y \in [0, 1); \\ y + 2 & \text{if } y \in [1, \infty], \end{cases}$$

then

$$f^{-1}(y) := \begin{cases} \frac{1}{2}y & \text{if } y \in [0, 2) \\ 1 & \text{if } y \in [2, 3) \\ y - 2 & \text{if } y \in [3, \infty], \end{cases}$$

and thus $f(f^{-1}(2)) = f(1) = 3$.

Let $f^{\infty}(y) := \infty\chi_{(1,\infty)}(y)$. Then $(f^{\infty})^{-1} = \chi_{(0,\infty]}$ and thus

$$(f^{\infty})^{-1}(f^{\infty}(\frac{1}{2})) = (f^{\infty})^{-1}(0) = 0 < \frac{1}{2}$$

and $f^{\infty}((f^{\infty})^{-1}(1)) = f^{\infty}(1) = 0 < 1$.

**Lemma 2.3.11.** *Let $f : [0,\infty] \to [0,\infty]$. Then $f$ is increasing and left-continuous if and only if $(f^{-1})^{-1} = f$.*

*Proof.* Suppose first that $f$ is increasing and left-continuous. Let $0 < \nu < y$. Lemma 2.3.9(a) yields that $f^{-1}(\eta) \leq f^{-1}(f(y-\nu)) \leq y - \nu$ for $0 \leq \eta \leq f(y-\nu)$. so we obtain that

$$(f^{-1})^{-1}(y) = \inf\{\eta \geq 0 : f^{-1}(\eta) \geq t\} \geq f(y-\nu)$$

Since $f$ is left-continuous, this yields $(f^{-1})^{-1}(y) \geq f(y)$ as $\nu \to 0^+$.
If $y = 0$, the inequality also holds.
By Lemma 2.3.9(c), $f^{-1}(f(y)+\nu) \geq y$ and hence $(f^{-1})^{-1}(y) \leq f(y) + \nu$. while $\nu \to 0^+$ we obtain $(f^{-1})^{-1}(y) \leq f(y)$. hence $(f^{-1})^{-1} = f$.

Assume then conversely that $(f^{-1})^{-1} = f$. By Lemma 2.3.9(a), $f^{-1}$ is increasing. Hence Lemma 2.3.9(c) implies that $(f^{-1})^{-1}$ is left-continuous. Furthermore, it is increasing by Lemma 2.3.9(a). Since $f = (f^{-1})^{-1}$, also $f$ is increasing and left-continuous. □

**Definition 2.3.12.** We say that $\pi : [0,\infty] \to [0,\infty]$ belongs to $\Phi_W^{-1}$ if it is increasing, left-continuous, satisfies (2.4) with $q = 1$, $\pi(y) = 0$ if and only if $y = 0$, and, $\pi(y) = \infty$ if and only if $y = \infty$.

**Proposition 2.3.13.** *The transformation $f \mapsto f^{-1}$ is a bijection from $\Phi_{W+}$ to $\Phi_W^{-1}$,*

**(a)** *If $f \in \Phi_{W+}$, then $f^{-1} \in \Phi_W^{-1}$ and $(f^{-1})^{-1} = f$.*

**(b)** *If $\pi \in \Phi_W^{-1}$ , then $\pi^{-1} \in \Phi_{W+}$ and $(\pi^{-1})^{-1} = \pi$.*

*Proof.* see [14].
such as $\Phi_{W+}$ the set of left-continuous weak $\Phi$-functions and $\Phi_W^{-1}$ characterizes inverses of $\Phi_{W+}$-functions. □

## 2.4 Conjugate Φ-Functions

This section is based on [10],[14]

**Definition 2.4.1.** Let $f : [0, \infty) \to [0, \infty]$. We denote by $f^*$ the conjugate function of $f$ which is defined, for $v \geq 0$, by

$$f^*(v) := \sup_{y \geq 0} \Big( yv - f(y) \Big).$$

In the Lebesgue case $y \mapsto \frac{1}{p} y^p$, the conjugate is given by $y \mapsto \frac{1}{p'} y^{p'}$, where $p'$ is the Hölder conjugate exponent. By definition of $f^*$,

$$yv \leq f(y) + f^*(v) \tag{2.6}$$

for every $y, v \geq 0$. This is called Young's inequality.

**Lemma 2.4.2.** *If $f \in \Phi_W$, then $f^* \in \Phi_c$.*

*Proof.* For $v = 0$, we have $f^*(0) = \sup_{y \geq 0} \Big( - f(y) \Big) = f(0) = 0$.

$f^*$ is increasing ?

Let $v < w$, we have

$$f^*(v) = \sup_{y \geq 0} \Big( yv - f(y) \Big) \leq \sup_{y \geq 0} \Big( y(w - v) + yv - f(y) \Big) = f^*(w)$$

So $f^*(y) \geq f^*(0) = 0$ for all $y > 0$. Since $\lim_{y \to 0} f(y) = 0$, there exists $y_1 > 0$ with $f(y_1) < \infty$. Then $f^*(v) \geq vy_1 - f(y_1)$, so that $\lim_{v \to \infty} f^*(v) = \infty$. In addition, there exists $y_2 > 0$ with $f(y_2) > 0$. from (2.2) with $p = 1$ we get $f(y) \geq \frac{f(y_2)}{\delta y_2} y$ when $y \geq y_2$. Hence

$$f^*(v) \leq \max \Big\{ y_2 v, \sup_{y > y_2} (yv - \frac{f(y_2) y}{\delta y_2}) \Big\}$$

When $v < \frac{f(y_2)}{\delta y_2}$ we find $\lim_{v \to 0^+} f^*(v) = 0$. Hence $f^*$ is a Φ-prefunction.

$f^*$ is convex ?. let $\sigma \in (0,1)$ and $v, w \geq 0$. we obtain

$$\begin{aligned} f^*(\sigma v + (1-\sigma)w) &= \sup_{y\geq 0}\Big(y(\sigma v + (1-\sigma)w) - f(y)\Big) \\ &= \sup_{y\geq 0}\Big(\sigma(yv - f(y)) + (1-\sigma)(yw - f(y))\Big) \\ &\leq \sigma \sup_{y\geq 0}(yv - f(y)) + (1-\sigma)\sup_{y\geq 0}(yw - f(y)) \\ &= \sigma f^*(v) + (1-\sigma) f^*(w) \end{aligned}$$

Let $v_\infty^* := \inf\{v > 0 : f^*(v) = \infty\}$. Convexity implies continuity in $[0, v_\infty^*)$ and continuity is clear in $(v_\infty^*, \infty)$. So we are left to show left-continuity at $v_\infty^*$.

For every $\beta \in (0,1)$ we have $f^*(\beta v_\infty^*) \leq f^*(v_\infty^*)$ and hence $\limsup_{\beta\to 1^-} f^*(\beta v_\infty^*) \leq f^*(v_\infty^*)$. Let $k < f^*(v_\infty^*)$ and $y_1$ be such that $y_1 v_\infty^* - f(y_1) \geq k$. Then

$$\lim\inf_{\beta\to 1^-} f^*(\beta v_\infty^*) \geq \lim\inf_{\beta\to 1^-}\Big(y_1\beta v_\infty^* - f(y_1)\Big) = y_1 v_\infty^* - f(y_1) \geq k$$

when $k \to f^*(v_\infty^*)^-$, we find $\liminf_{\beta\to 1^-} f^*(\beta v_\infty^*) \geq f^*(v_\infty^*)$ and thus $f^*$ is left-continuous. □

**Remark 2.4.3.** Observe that $f^*$ is convex and left-continuous even if $f$ is not. The previous lemma does not extend to strong $\Phi$-functions:

Indeed, if $f(y) := y$, then $f \in \Phi_s$ but $f^*(y) = \infty\chi_{(1,\infty)}(y)$ belongs to $\Phi_c \setminus \Phi_s$.

**Lemma 2.4.4.** *Let $f, g : [0,\infty) \to [0,\infty]$ and $\delta, \gamma > 0$.*

**(a)** *If $f \leq g$, then $g^* \leq f^*$.*

**(b)** *If $g(y) = \delta f(\gamma y)$ for all $y \geq 0$, then $g^*(v) = \delta f^*(\frac{v}{\delta\gamma})$ for all $v \geq 0$.*

**(c)** *If $f \simeq g$, then $f^* \simeq g^*$.*

*Proof.* For (a). letting $f(y) \leq g(y)$ for all $y \geq 0$. we obtain

$$g^*(v) = \sup_{y\geq 0}\Big(yv - g(y)\Big) \leq \sup_{y\geq 0}\Big(yv - f(y)\Big) = f^*(v)$$

for all $v \geq 0$.

For (b), assume that $\delta, \gamma > 0$ and $g(y) = \delta f(\gamma y)$ for all $y \geq 0$. we find

$$\begin{aligned} g^*(v) &= \sup_{y\geq 0}\Big(yv - g(y)\Big) = \sup_{y\geq 0}\Big(yv - \delta f(\gamma y)\Big) \\ &= \sup_{y\geq 0}\delta\Big(\gamma y\frac{v}{\delta\gamma} - f(\gamma y)\Big) = \delta f^*\Big(\frac{v}{\delta\gamma}\Big) \end{aligned}$$

for all $v \geq 0$.

For (c), suppose that $f(\frac{y}{\tau}) \leq g(y) \leq f(\tau y)$. using (a), we get

$$f^*(\tau y) \leq g^*(y) \leq f^*(\frac{y}{\tau})$$

by (b), we obtain

$$f^*(\tau y) = (f(\frac{y}{\tau}))^* \leq g^*(y) \leq (f(\tau y))^* = f^*(\frac{y}{\tau})$$

□

**Remark 2.4.5.** In (c) of the previous lemma is false for $\approx$.

Indeed, we consider $f(y) = y$ and $g(y) = 2y$. Then $f \approx g$ and $f \simeq g$. However, $f^* = \infty\chi_{(1,\infty)}$ and $g^* = \infty\chi_{(2,\infty)}$, so that $f^* \not\approx g^*$.

If $f \in \Phi_W \setminus \Phi_c$ then $f^{**} \neq f$, with $f^{**} = (f^*)^*$.

**Proposition 2.4.6.**

*Let $f \in \Phi_W$. Then $f^{**} \simeq f$ and $f^{**}$ is the greatest convex minorant of $f$.*

*In particular, if $f \in \Phi_c$, then $f^{**} = f$ and*

$$f(y) = \sup_{v\geq 0}(yv - f^*(v)) \quad \textit{for all} \quad y \geq 0$$

*Proof.* Let us first assume that $f \in \Phi_c$ and prove the latter part of the proposition. By Lemma 2.4.2 we have $f^{**} \in \Phi_c$. By definition of $f^{**}$ and Young's inequality (2.4.1) we obtain

$$f^{**}(y) = \sup_{v\geq 0}(yv - f^*(v)) \leq \sup_{v\geq 0}(f(y) + f^*(v) - f^*(v)) = f(y), \tag{2.7}$$

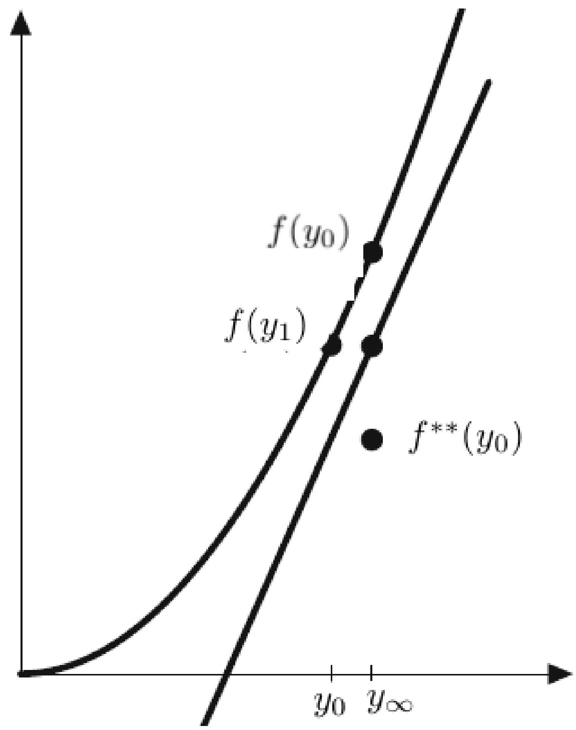


Figure 2.5: **Sketch of the case** $f^{**}(y_0) < f(y_0)$

It remains to show $f^{**}(y) \geq f(y)$. We prove this by contradiction. Assume to the contrary that there exists $y_0 \geq 0$ with $f^{**}(y_0) < f(y_0)$, see Fig. 2.6.

Suppose first that $f(y_0) < \infty$. Since $f$ is left-continuous, there exists $y_1 < y_0$ with $f(y_1) > \frac{1}{2}(f(y_0) + f^{**}(y_0))$. Let $\rho := \frac{f(y_0)-f(y_1)}{y_0-y_1}$. Since $f$ is increasing, it follows by convexity that

$$f(y) \geq \rho(y - y_0) + \frac{1}{2}(f(y_0) + f^{**}(y_0)).$$

Therefore, by Young's inequality for $f^*$,

$$\begin{aligned} f^*(\rho) = \sup_{y\geq 0}(\rho y - f(y)) &\leq \rho y_0 - \frac{1}{2}(f(y_0) + f^{**}(y_0) \\ &\leq f^*(\rho) + f^{**}(y_0) - \frac{1}{2}(f(y_0) + f^{**}(y_0)) < f^*(\rho) \end{aligned}$$

a contradiction.

The case $f(y_0) = \infty$ is handled similarly, with the estimate $f(y) \geq \rho(y - y_\infty) + 2f^{**}(y_0)$ for suitably big $\rho$.

We next consider the general case $f \in \Phi_W$. By Theorem 2.2.3, there exists $g \in \Phi_s$ with $f \simeq g$. Using Lemma 2.4.4(c) twice we obtain $f^{**} \simeq g^{**}$.

By the first part of the proof, $g = g^{**}$ and thus $f^{**} \simeq g \simeq f$.

We already know by (2.7) that $f^{**}$ is a convex minorant of $f$. Suppose that $g$ is also a convex minorant of $f$. By taking $\max\{g, 0\}$ we may assume that $g$ is non-negative. By Lemma 2.4.4(a), used twice, we have $g^{**} \leq f^{**}$. But since $g$ is convex, the first part of the proof implies that $g^{**} = g$ so that $g \leq f^{**}$. Hence $f^{**}$ is the greatest convex minorant. $\square$

**Corollary 2.4.7.** *Let $f, g \in \Phi_c$. Then $f \leq g$ if and only if $g^* \leq f^*$.*

*Proof.* Lemma 2.4.4(a) yields the implication from $\Phi$-functions to conjugate functions. The reverse implication follows using Lemma 2.4.4(a) for $f^*$ and $g^*$ and Proposition 2.4.6 that gives $f^{**} = f$ and $g^{**} = g$. $\square$

**Lemma 2.4.8.** *Let $f \in \Phi_c$ and $\gamma := \lim_{y \to 0^+} \frac{f(y)}{y} = f'(0)$. Then $f^*(x) = 0$ if and only if $x \leq \gamma$.*

*Here $f'(0)$ is the right derivative of a convex function at the origin.*

*Proof.* Since $f$ is convex, it satisfies (2.1) with $p = 1$. In particular, $\frac{f(y)}{y} \geq f'(0) = \gamma$. We observe that

$$f^*(x) = \sup_{y \geq 0} \Big( x - \frac{f(y)}{y} \Big).$$

If $x \leq \gamma$, then the parenthesis is non-positive, so $f^*(x) = 0$. If $x > \gamma$, then the parenthesis is positive for some sufficiently small $y > 0$, and so $f^*(x) > 0$. $\square$

The following theorem gives a simple formula for approximating the inverse of $f^*$. This results was previously shown for N-functions in [[10], Lemma 2.6.11] and for generalized $\Phi$-functions in [[6], Lemma 2.3] but the later proof includes a mistake.

**Theorem 2.4.9.** *If $f \in \Phi_W$, then $f^{-1}(y)(f^*)^{-1}(y) \approx y$.*

*Proof.* see [14]. $\square$

**Proposition 2.4.10.** *Let $f \in \Phi_W$. Then $f$ satisfies (2.2) or (2.4) if and only if $f^*$ satisfies (2.4) with $q = p'$ or (2.2) with $p = q'$ , respectively.*

*Proof.* We start with the special cases (2.1) and (2.3). We have that $f$ satisfies (2.1) if and only if $\frac{f(y^{1/p})}{y}$ is increasing, similarly for $f^*$ and (2.3). From the definition of the conjugate function,

$$\frac{f^*(x^{\frac{1}{p}})}{x} = \frac{1}{x} \sup_{y \geq 0} \Big( y x^{\frac{1}{p}} - f(y) \Big) = \sup_{w \geq 0} w \Big( w^{-\frac{1}{p}} - \frac{f((xw)^{\frac{1}{p'}})}{xw} \Big)$$

where we used the change of variables $y =: (xw)^{\frac{1}{p'}}$. From this expression, we see that $f^*$ satisfies (2.3) and (2.1) if satisfies (2.1) with $p = q'$ and (2.3) with $q = p'$, respectively. For the opposite implication, we use $(f^*)^* \simeq f$ from Proposition 2.4.6.

Suppose now that $f$ satisfies (2.2). Then $g(x) := x^p \inf_{y \geq x} y^{-p} f(y)$ satisfies (2.1). A short calculation shows that $f \approx g$ : by $(2.2)_{p>1}$ we obtain

$$\frac{1}{\delta} f(x) = \frac{1}{\delta} x^p \frac{f(x)}{x^p} \leq g(x) \leq x^p \frac{f(x)}{x^p} = f(x)$$

By the above argument, $g^*$ satisfies (2.3) and by Lemma 2.4.4(c),$f^* \simeq g^*$. For (2.4), we can argue in the same way with the auxiliary function $g(x) := x^q \sup_{y \leq x} y^{-q} f(y)$.

□

**Definition 2.4.11.** We say that $f \in \Phi_W$ satisfies $\nabla_2$, if $f^*$ satisfies $\Delta_2$.

We can now connect this concept from the theory of Orlicz spaces to the assumptions as Proposition 2.4.10 and Lemma 2.2.6 yield the following result.

**Corollary 2.4.12.** *A function $f \in \Phi_W$ satisfies $\nabla_2$ if and only if it satisfies* (2.2).

## 2.5 Generalized Φ-Functions

we will generalize Φ-functions in such a way that they may depend on the space variable. Let $(M, \Gamma, \mu)$ be a $\sigma$-finite, complete measure space. In what follows we always make the natural assumption that the measure $\mu$ is not identically zero.

**Definition 2.5.1.** Let $f : M \times [0, \infty) \to \mathbb{R}$ and $p, q > 0$. We say that f satisfies (2.2) or (2.4), if there exists $\delta \geq 1$ such that the function $y \mapsto f(x, y)$ satisfies (2.2) or (2.4) with a constant $\delta$, respectively, for $\mu$-almost every $x \in M$. When $\delta = 1$, we use the notation (2.1) and (2.3).

**Remark 2.5.2.** Observe that in the almost increasing and decreasing conditions we require that the same constant applies to almost every point. Furthermore, if we define $f(x,y) = g(y)$ for every $x$, then $f$ satisfies (2.2) in the sense of the previous definition if and only if $g$ satisfies (2.2) in the sense of Definition 2.1.2. The same applies to the other terms. Therefore, there is no need to distinguish between the conditions based on whether there is an $x$-dependence of the function or not.

**Definition 2.5.3.** Let $(M, \Gamma, \mu)$ be a $\sigma$-finite, complete measure space. A function $f : M \times [0, \infty) \to [0, \infty]$ is said to be a (generalized) $\Phi$-prefunction on $(M, \Gamma, \mu)$ if $x \mapsto f(x, |f(x)|)$ is measurable for every $f \in L^0(M, \mu)$ and $f(x, \cdot)$ is a $\Phi$- prefunction for $\mu$-almost every $x \in M$. We say that the $\Phi$-prefunction $f$ is

- a(generalized weak) $\Phi$-function if $f$ satisfies (2.2) with $p = 1$;
- a(generalized) convex $\Phi$-function if $f(x, \cdot) \in \Phi_c$ for $\mu$-almost all $x \in A$;
- a(generalized) strong $\Phi$-function if $f(x, \cdot) \in \Phi_s$ for $\mu$-almost all $x \in A$.

If $f$ is a generalized weak $\Phi$-function on $(M, \Gamma, \mu)$,we write $f \in \Phi_W(M, \mu)$ and similarly we define $f \in \Phi_c(M, \mu)$ and $f \in \Phi_s(M, \mu)$. If $\Omega$ is an open subset of $\mathbb{R}^n$ and $\mu$ is the $n$-dimensional Lebesgue measure we omit $\mu$ and abbreviate $\Phi_W(\Omega)$, $\Phi_c(\Omega)$ or $\Phi_s(\Omega)$. Or we say that $f$ is a generalized (weak/convex/strong)$\Phi$-function on $\Omega$. Unless there is danger of confusion, we will omit the word "generalized".

Clearly $\Phi_s(M, \mu) \subset \Phi_c(M, \mu) \subset \Phi_W(M, \mu)$. Every $\Phi$-function is a generalized $\Phi$-function if we set $f(x, y) := f(y)$ for $x \in M$ and $y \in [0, \infty)$. Next we give some examples of non-trivial generalized $\Phi$-functions.

**Example 2.5.4.** Let $\phi : M \to [1, \infty]$ be a measurable function and define $\phi_\infty := \limsup_{|x| \longrightarrow \infty} \phi(x)$. Let us interpret $y^\infty := \infty \chi_{(1,\infty]}(y)$. Let $\psi : M \to (0, \infty)$ be a measurable function and $1 \leq x < y < \infty$. Let us define, for $y \geq 0$,

$$
\begin{aligned}
f_1(x,y) &:= y^{\phi(x)}\psi(x) \\
f_2(x,y) &:= y^{\phi(x)}\log(e+y) \\
f_3(x,y) &:= \min\{y^{\phi(x)}, y^{\phi_\infty}\} \\
f_4(x,y) &:= y^{\phi(x)} + \sin(y) \\
g_1(x,y) &:= y^{\phi} + \psi(x)y^{t} \\
g_2(x,y) &:= (y-1)_+^s + \psi(x)(y-1)_+^t
\end{aligned}
$$

Observe that

$f_3 \in \Phi_W(M,\mu)\backslash\Phi_c(M,\mu)$ when $\phi$ is non-constant,

$f_4 \in \Phi_W(M,\mu)\backslash\Phi_c(M,\mu)$ when $\inf_{x\in M}\phi(x) \leq \frac{3}{2}$,

$f_1, f_2 \in \Phi_c(M,\mu)\backslash\Phi_s(M,\mu)$ when $\phi = \infty$ in a set of positive measure, and

$g_1, g_2 \in \Phi_s(M,\mu)$ when $\phi, t \in [1,\infty)$.

Moreover, if $\phi(x) < \infty$ for $\mu$-almost every $x$, then $f_1, f_2 \in \Phi_s(M,\mu)$.

## Measurability

Observe that in the definition of generalized $\Phi$-functions we have directly assumed that $x \mapsto f(x,|h(x)|)$ is measurable. If $f$ is left-continuous, then this assumption can be replaced with the conditions from the next theorem.

**Theorem 2.5.5.** *Let $f : M \times [0,\infty) \to [0,\infty]$, $x \mapsto f(x,y)$ be measurable for every $y \geq 0$ and $y \mapsto f(x,y)$ be increasing and left-continuous for $\mu$-almost every $x$. If $h \in L^0(M,\mu)$ is measurable, then $x \mapsto f(x,|h(x)|)$ is measurable.*

*Proof.* We have to show that $D_\alpha := \{x \in M : f(x,|h(x)|) > \alpha\}$ is measurable for every $\alpha \in \mathbb{R}$. Let us write $T_\alpha(y) := \{x \in M : f(x,t) > \alpha\} \cap \{x \in M : |f(x)| \geq y\}$, for $y \geq 0$. Then for each $y$ we have $T_\alpha(y) \subset D_\alpha$ since $f$ is increasing. Assume then that $x \in D_\alpha$. Let $(y_i)$ be a sequence of non-negative rational numbers converging to $|f(x)|$ from below. By the left-continuity of $f$, we have $\lim_{i\to\infty} f(x, y_i) = f(x, |h(x)|)$. Thus there exists $i_0$ such that $f(x, y_{i_0}) > \alpha$ and $0 \leq y_{i_0} \leq |f(x)|$. This yields that $x \in T_\alpha(y_{i_0})$. We have shown$D_\alpha = \bigcup_{y\in\mathbb{Q}\cap[0,\infty)} T_\alpha(y)$. Since each $T_\alpha(y)$ is measurable by assumption and the union is countable, the set $D_\alpha$ is measurable. □

The next example shows that $x \mapsto f(x, |h(x)|)$ need not to be measurable if we omit left-continuity of $f$.

**Example 2.5.6.** Consider the Lebesgue measure on $[1, 2]$ and let $T \subset [1, 2]$ be a non-measurable set. We define $f : [1, 2] \times [0, \infty) \to [0, \infty]$ by

$$f(x, y) := \chi_T(y)\chi_{\{x\}}(y) + \infty\chi_{(x,\infty)}(y).$$

For constant $y \geq 0, x \mapsto f(x, y)$ is decreasing and hence measurable.For each $x \in [1, 2], y \mapsto f(x, y)$ belongs to $\Phi_W$, but it is left-continuous only when $\chi_T(x) = 0$ i.e. when $x \notin T$. Let $h : [1, 2] \to \mathbb{R}, h(x) := x$. Then $f$ is continuous, and hence measurable. But $f(x, |h(x)|) = f(x, x) = \chi_T(x)$ is not a measurable function.

Properties of $\Phi$-functions are generalized point-wise uniformly to the generalized $\Phi$-function case. For instance we define equivalence as follows.

**Definition 2.5.7.** We say that $f, g : M \times [0, \infty) \to [0, \infty]$ are equivalent, $f \simeq g$, if there exist $\tau > 1$ such that for all $y \geq 0$ and $\mu$-almost all $x \in M$ we have

$$g(x, \frac{y}{\tau}) \leq f(x, y) \leq g(x, \tau y).$$

**Lemma 2.5.8.** *Let $f, g : M \times [0, \infty) \to [0, \infty], f \simeq g$, be increasing with respect to the second variable, and $x \mapsto f(x, |h(x)|)$ and $x \mapsto g(x, |h(x)|)$ be measurable for every measurable $f$ .*

**(a)** *If $f$ is a generalized $\Phi$-prefunction, then $g$ is a generalized $\Phi$-prefunction.*

**(b)** *If $f$ satisfies (2.2), then $g$ satisfies (2.2).*

**(c)** *If $f$ satisfies (2.4), then $g$ satisfies (2.4).*

Let us here show how the upgrading results can be conveniently obtained by means of the conjugate function. We first show that $f^*$ is measurable, i.e. we generalize Lemma 2.4.2.

**Lemma 2.5.9.** *If $f \in \Phi_W(M, \mu)$, then $f^* \in \Phi_c(M, \mu)$.*

*Proof.* By Lemma 2.4.2 and Theorem 2.5.5, it is enough to show that $x \mapsto f^*(x, y)$ is measurable for every $y \geq 0$. We first show that

$$\sup_{v \geq 0}(vy - f(x, v)) = \sup_{v \in \mathbb{Q} \cap [0,\infty)} (vy - f(x, v))$$

The inequality "$\geq$" is obvious. Suppose that $v \in (0, \infty) \setminus \mathbb{Q}$ and let $v_j \in (v - \frac{1}{j}, v) \cap \mathbb{Q}$. Since $f$ is increasing, we obtain $vy - f(x, v) \leq vy - f(x, v_j) \leq v_j y - f(x, v_j) + \frac{y}{j}$. When $j \to \infty$ Ą, we obtain the inequality "$\leq$".

Let $c \geq 0$. Then $f^*(x, y) \leq c$ if and only if $vy - (x, v) \leq c$ for all $v \in \mathbb{Q} \cap [0, \infty)$. Thus

$$\{x : f^*(x, y) \leq c\} = \bigcap_{v \in \mathbb{Q} \cap [0,\infty)} \{x : vy - f(x, v) \leq c\}$$

is measurable as a countable intersection of measurable sets, and hence $x \mapsto f^*(x, y)$ is measurable. $\square$

Let us then consider Lemma 2.2.1 and Theorem 2.2.3 which show that every weak $\Phi$-function is equivalent to a strong $\Phi$-function.

**Lemma 2.5.10.** *If $f \in \Phi_W(M, \mu)$ satisfies* (2.2) *with $p \geq 1$, then there exists $g \in \Phi_c(M, \mu)$ equivalent to $f$ such that $g^{1/p}$ is convex. In particular, $g$ satisfies* (2.1).

*Proof.* We first observe that $\pi := f^{\frac{1}{p}} \in \Phi_W(M,\mu)$. It follows by Lemma 2.5.9 that $\pi^{**} \in \Phi_c(M,\mu)$ and by Proposition 2.4.6 that $\pi \simeq \pi^{**}$. Then $g := (\pi^{**})^p$ is the required convex $\Phi$-function. □

**Theorem 2.5.11.** *Every weak $\Phi$-function is equivalent to a strong $\Phi$-function*

*Proof.* Let $f_c$ be from Lemma 2.5.10 . By the proof of Theorem 2.2.3 we need only to show that the functions in the proof satisfy the measurability property. There we defined $y_\infty(x) := \inf\{y : f_c(x,y) = \infty\} \in (0,\infty)$ and

$$f_s(x,y) := f_c(x,y) + \frac{2y - y_\infty(x)}{y_\infty(x) - y}\chi_{(\frac{1}{2}y_\infty(x), y_\infty(x))}(y) + \infty\chi_{[y_\infty(x),\infty)}(y).$$

Then $f_s$ is left-continuous and hence by Theorem 2.5.5 we need to show that $x \mapsto f_s(x,y)$ is measurable for every $y \geq 0$.

This is clear if $x \mapsto y_\infty(x)$ is measurable. Let $e \geq 0$. Then $\inf\{y : f_c(x,y) = \infty\} \leq e$ if and only if $f_c(x, e+\epsilon) = \infty$ for all $\epsilon > 0$. The later implies that $f_c(x, e+\kappa) = \infty$ for all $\kappa \in \mathbb{Q} \cap (0,\infty)$. We obtain that

$$\{x : y_\infty(x) \leq e\} = \bigcap_{\kappa \in \mathbb{Q}\cap(0,\infty)} \{x : f(x, e+\kappa) = \infty\}$$

is measurable as a countable intersection of measurable sets, and hence $y_\infty$ is measurable. □

The proof of following proposition is the same as the proof of Proposition 2.2.7 except that it is based on Lemma 2.5.10, not its preliminary version Lemma 2.2.1.

**Proposition 2.5.12.** *If $f \in \Phi_W(M,\mu)$ satisfies $(2.4)_{q<\infty}$, then there exists $g \in \Phi_s(M,\mu)$ with $g \approx f$ such that $y \mapsto g(x,y)$ is a strictly increasing bijection for $\mu$-almost every $x \in M$.*

*Then we consider the inverse of a generalized $\Phi$-function. For that we prove an extra lemma.*

**Lemma 2.5.13.** *Let $f : M \times [0,\infty) \to [0,\infty]$. If $y \mapsto f(x,y)$ is increasing for $\mu$-almost every $x$ and if $x \mapsto f(x,y)$ is measurable for every $y \geq 0$, then $x \mapsto f^{-1}(x,|f(x)|)$ is measurable for every measurable $f$ .*

*Proof.* By Lemma 2.3.9(c), $f^{-1}$ is left-continuous and hence by Theorem 2.5.5 we need to show that $x \mapsto f^{-1}(x,\eta)$ is measurable for every $\eta \geq 0$.

Let $b, \eta \geq 0$. Then $f^{-1}(x,\eta) = \inf\{y : f(x,y) \geq \eta\} > b$ if and only if there exists $\nu > 0$ such that $f(x, b+\nu) < \eta$. Since $f$ is increasing with respect to the second variable, the later implies that $f(x, b+\kappa) < \eta$ for all $\kappa \in \mathbb{Q} \cap (0,\nu]$. Thus

$$\{x : f^{-1}(x,\eta) > b\} = \bigcup_{\kappa \in \mathbb{Q}\cap(0,\infty)} \{x : f(x, b+\kappa) < \eta\}$$

is measurable as a countable union of measurable sets, and hence $x \mapsto f^{-1}(x,\eta)$ is measurable.

Next we generalize Definition 2.3.12 to $\Phi_W(M,\mu)$-functions □

**Definition 2.5.14.** We say that $\pi : M \times [0,\infty] \to [0,\infty]$ belongs to $\Phi_W^{-1}(M,\mu)$ if it satisfies (2.2) with $p = 1$, $x \mapsto \xi(x,y)$ is measurable for all $y$ and if for $\mu$-almost every $x \in A$ the function $y \mapsto \pi(x,y)$ is increasing, left-continuous, and $\pi(x,y) = 0$ if and only if $y = 0$ and $\pi(x,y) = \infty$ if and only if $y = \infty$.

We denote $\Phi_W + (M,\mu)$ the set of left-continuous generalized weak $\Phi$-functions.

**Proposition 2.5.15.** *The transformation $f \mapsto f^{-1}$ is a bijection from $\Phi_{W+}$ to $\Phi_W^{-1}$ :*

**(a)** *If $f \in \Phi_{W_+}(M,\mu)$, then $f^{-1} \in \Phi_W^{-1}(M,\mu)$ and $(f^{-1})^{-1} = f$.*

**(b)** *If $\pi \in \Phi_W^{-1}(M,\mu)$, then $\pi^{-1} \in \Phi_{W_+}(M,\mu))$ and $(\pi^{-1}){-1} = \pi$.*

## Weak equivalence and weak Doubling

We can also define some properties which are properly generalized in the sense that they have no analogue in the case that does not depend on the space variable.

**Definition 2.5.16.** We say that $f, g : M \times [0, \infty) \to [0, \infty]$ are weakly equivalent, $f \sim g$, if there exist $\tau > 1$ and $g \in L^1(M, \mu)$ such that

$$f(x, y) \leq g(x, \tau y) + g(x) \quad \text{and} \quad g(x, y) \leq f(x, \tau y) + g(x)$$

for all $y \geq 0$ and $\mu$-almost all $x \in M$.

An easy calculation shows that $\sim$ is an equivalence relation.It clear from the definitions that $f \simeq g$ implies $f \sim g$ (with $g = 0$). Later in Theorem 3.2.9 we show that $f \sim g$ if and only if $L^f(M, \mu) = L^g(M, \mu)$. Also weak equivalence is preserved under conjugation:

**Lemma 2.5.17.** *Let $f, g : M \times [0, \infty) \to [0, \infty]$. If $f \sim g$ , then $f^* \sim g^*$.*

*Proof.* Let $f \sim g$. Then we obtain

$$\begin{aligned} f^*(x, \tau v) &= \sup_{y \geq 0}(y\tau y - f(x, y)) \geq \sup_{y \geq 0}(y\tau v - g(x, \tau y) - g(x)) \\ &= \sup_{y \geq 0}(y\tau v - g(x, \tau y)) - g(x) = g^*(x, \tau v) - g(x) \end{aligned}$$

and similarly $f^*(x, v) \leq g^*(x, \tau v) + g(x)$. $\square$

**Definition 2.5.18.** We say that $f : M \times [0, \infty) \to [0, \infty]$ satisfies the weak doubling condition $\Delta_2^W$ if there exist a constant $\theta \geq 2$ and $\psi \in L^1(M, \mu)$ such that

$$f(x, 2y) \leq \theta f(x, y) + \psi(x)$$

for $\mu$-almost every $x \in M$ and all $y \geq 0$. We say that $f$ satisfies condition $\nabla_2^W$ if $f^*$ satisfies $\Delta_2^W$

If $\psi \equiv 0$, then we say that the (strong) $\Delta_2$ and $\nabla_2$ conditions hold.

**Remark 2.5.19.** Note that the $\Delta_2$ and $\nabla_2$-conditions for $x$-independent $\Phi$-prefunctions have been defined in Definition 2.2.5. By writing $f(x, y) := f(y)$ we see that the definitions are equivalent. Since the constant $\theta$ in Definition 2.5.22 is the same for $\mu$-almost every $x \in M$, we see by Lemma 2.2.6 that $\Delta_2$ is equivalent to $(2.4)_{q<\infty}$ and by Corollary 2.4.12 that $\nabla_2$ is equivalent to $(2.2)_{p>1}$.

**Lemma 2.5.20.** *Let $f, g : M \times [0,\infty) \to [0,\infty]$ with $f \sim g$.*

**(a)** *If $f$ satisfies $\Delta_2^W$ , then $g$ satisfies $\Delta_2^W$*

**(b)** *If $f$ satisfies $\nabla_2^W$ , then $g$ satisfies $\nabla_2^W$*

*Proof.* (a) Choose an integer $n \geq 1$ such that $2^{n-1} < 2\tau^2 \leq 2^n$. Then, by iterating the $\Delta_2^W$ assumption, we conclude that

$$f(x, 2\tau y) \leq f(x, 2^n \frac{y}{\tau}) \leq \theta f(x, 2^{n-1}\frac{y}{\tau}) + \psi(x) \leq ... \lesssim f(x, y\tau) + \psi(x).$$

Denote by $\psi_2$ the function from $\sim$ . We find that

$$\begin{aligned} g(x, 2y) &\leq f(x, 2\tau y) + \psi_2(x) \lesssim f(x, \frac{y}{\tau}) + \psi(x) + \psi_2(x) \\ &\leq g(x, y) + \psi(x) + 2\psi_2(x). \end{aligned}$$

(b) Since $f \sim g$, Lemma 2.5.17 yields $f^* \sim g^*$. Since $f^*$ satisfies $\Delta_2^W$ so does $g^*$ by (a). This means that $g$ satisfies $\nabla_2^W$ . □

Next we show that weak doubling can be upgraded to strong doubling via weak equivalence of Φ-functions.

**Theorem 2.5.21.** *If $f \in \Phi_W(M, \mu)$ satisfies $\Delta_2^W$ and/or $\nabla_2^W$ , then there exists $g \in \Phi_W(M, \mu)$ with $f \sim g$ satisfying $\Delta_2$ and/or $\nabla_2$.*

*Proof.* By Theorem 2.5.11 and Lemmas 2.5.17 and 2.5.20, we may assume without loss of generality that $f \in \Phi_s(M, \mu)$.By the assumptions,

$$f(x, 2y) \leq \theta f(x, y) + \psi(x) \quad \text{and/or} \quad f^*(x, 2y) \leq \theta f^*(x, y) + \psi(x)$$

for some $\theta > 2, \psi \in L^1, y \geq 0$ and $\mu$-almost all $x \in M$. Using $f = f^{**}$ (Proposition 2.4.6), the definition of the conjugate Φ-function and Lemma 2.4.4(b), we rewrite the second inequality as

$$\begin{aligned}
f(x,2y) &= \sup_{v\geq 0}\Big(2yv - f^*(x,v)\Big) \\
&\leq \sup_{v\geq 0}\Big(2yv - \frac{1}{\theta}(f^*(x,2v)-\psi(x))\Big) \\
&= \sup_{v\geq 0}\Big(2yv - \frac{1}{\theta}f^*(x,2v)\Big) + \frac{1}{\theta}\psi(x) \\
&= \frac{1}{\theta}f^{**}(x,\theta y) + \frac{1}{\theta}\psi(x) = \frac{1}{\theta}f(x,\theta y) + \frac{1}{\theta}\psi(x).
\end{aligned}$$

Define $y_x := f^{-1}(x,\psi(x))$ and suppose that $y > y_x$ so that $\psi(x) \leq f(x,y)$ by Lemma 2.3.3. By (2.1) with $p = 1$, we conclude that $\theta\psi(x) \leq \theta f(x,y) \leq f(x,\theta y)$. Hence in the case $y > y_x$ we have

$$f(x,2y) \leq (\theta+1)f(x,y) \quad \text{and/or} \quad f(x,2y) \leq \frac{\theta+1}{\theta^2}f(x,\theta y).$$

Let $q := \log_2(\theta+1)$ and $p := \frac{\log(\theta^2/(\theta+1))}{\log(\theta/2)}$.Note that $p > 1$ since $\frac{\theta^2}{\theta+1} > \frac{\theta}{2}$ and $\theta > 2$. Divide the first inequality by $(2y)^q$ and the second one by $(2y)^p$:

$$\begin{aligned}
\frac{f(x,2y)}{(2y)^q} &\leq \frac{\theta+1}{2^q}\frac{f(x,y)}{y^q} = \frac{f(x,y)}{y^q} \quad \text{and/or} \\
\frac{f(x,2y)}{(2y)^p} &\leq \frac{(\theta+1)\theta^p}{\theta^2 2^p}\frac{f(x,\theta y)}{\theta y^p} = \frac{f(x,\theta y)}{\theta y^p}.
\end{aligned}$$

Let $l > y \geq y_x$.Then there exists $n \in \mathbb{N}$ such that $2^n y < l \leq 2^{n+1}y$, Hence

$$\frac{f(x,l)}{l^q} \leq \frac{f(x,2^{n+1}y)}{(2^n y)^q} = 2^q\frac{f(x,2^{n+1}y)}{(2^{n+1}y)^q} \leq 2^q\frac{f(x,2^n y)}{(2^n y)^q} \leq ... \leq 2^q\frac{f(x,y)}{y^q},$$

so $f$ satisfies (2.4) for $y \geq y_x$. Similarly, we find that $f$ satisfies (2.2) for $y \geq y_x$. Define

$$g(x,y) := \begin{cases} f(x,y), & \text{for } y \geq y_x; \\ c_x y^p & \text{otherwise,} \end{cases}$$

where $c_x$ is chosen so that the $g$ is continuous at $y_x$.Then $g$ satisfies (2.2) and/or (2.4) on $[0,y_x]$ and $[y_x,\infty)$, hence on the whole real axis.

Furthermore, $f(x,y) = g(x,y)$ when $y \geq y_x$ ,and so it follows that $|f(x,y)-g(x,y)| \leq f(x,y_x) \leq \psi(x)$ (Lemma 2.3.3). Since $\psi \in L^1$,this means that $f \sim g$, so $g$ is the required function.

Finally we show that $g \in \Phi_w(M, \mu)$. The function $x \mapsto c_x = \frac{f(x, y_x)}{y_x^p}$ is measurable since $y_x = f^{-1}(x, \psi(x))$ is measurable (Lemma 2.5.13), thus we obtain that $x \mapsto g(x, y)$ is measurable. It is clear that $y \mapsto g(x, y)$ is a left-continuous $\Phi$-prefunction for $\mu$-almost every $x$ and hence the measurability property follows from Theorem 2.5.5. Since $g$ satisfies (2.2) with $p = 1$ on $[0, y_x]$ and $[y_x, \infty)$ for $\mu$-almost every $x$, it satisfies (2.2) with $p = 1$. $\square$

# Chapter 3

# Generalized Orlicz Spaces

In the previous chapter, we studied the properties of $\Phi$-functions. In this chapter, we use them to study and derive results for function spaces defined by means of $\Phi$-functions.

## 3.1 Modulars

see, e.g., the monographs [5, 14, 17, 19].

**Definition 3.1.1.** Let $f \in \Phi_W(M, \mu)$ and let $\rho_f$ be given by

$$\rho_f(h) := \int_A f(x, |h(x)|) d\mu(x), \qquad \text{for all } h \in L^0(M, \mu).$$

The function $\rho_f$ is called a modular.

$$\text{The set} \quad L^f(M, \mu) := \{h \in L^0(A, \mu) : \rho_f(\lambda h) < \infty \quad \text{for some} \quad \lambda > 0\}$$

is called a generalized Orlicz space. we denote $L^f(M, \mu) = L^f$.

**Remark 3.1.2.**

Generalized Orlicz spaces are also called Musielak-Orlicz spaces,

If $\lim_{\beta \to 1^-} \rho(\beta h) = \rho(h)$ then $\rho$ is left-continuous.

If $f \in \Phi_c(M, \mu)$ is strictly increasing, then $\rho_f$ to be called a modular.

**Example 3.1.3.** As in example 2.1.6, we consider the following $\Phi$-functions:

$$\begin{aligned}
f^p(y) &:= \frac{1}{p}y^p, p \in (0,\infty) \\
f_{\max}(y) &:= (\max\{0, y-1\})^2, \\
f_{\sin}(y) &:= y + \sin(y), \\
f_{\exp}(y) &:= \exp(y) - 1, \\
f^{\infty}(y) &:= \infty\chi_{(1,\infty)}(y), \\
f^{\infty,2}(y) &:= f^{\infty}(y) + \frac{2y-1}{1-y}\chi_{(1/2,1)}(y).
\end{aligned}$$

which generate Orlicz spaces

$$L^{f^p} = L^p, \quad L^{f_{\max}} = L^2 + L^{\infty}, \quad L^{f_{\sin}} = L^1, \quad L^{f_{\exp}} = \exp L, \quad L^{f^{\infty}} = L^{f^{\infty,2}} = L^{\infty}.$$

**Lemma 3.1.4.** *Let $f \in \Phi_W(M,\mu)$.*

**(a)** *Then $L^f(M,\mu) = \{h \in L^0(M,\mu) : \lim_{\beta\to 0^+} \rho_f(\beta h) = 0\}$.*

**(b)** *If, additionally,$f$ satisfies $(2.4)_{q<\infty}$, then*

$$L^f(M,\mu) = \{h \in L^0(M,\mu) : \rho_f(h) < \infty\}.$$

*Proof.* **(a)** First inclusion, suppose there exists $\beta > 0$ such that $\rho_f(\beta f) < \infty$. by (2.2) with $p = 1$ we get

$$f(x, y\beta\,|h(x)|) \leq \delta y f(x, \beta\,|h(x)|)$$

for $y \in (0,1)$ and $\mu$-almost all $x \in M$. This gives that

$$\int_M f(x, y\beta\,|h(x)|)d\mu(x) \leq \delta y \int_M f(x,\beta\,|h(x)|)d\mu(x)$$

and hence $\lim_{\beta\to 0^+} \rho_f(\beta h) = 0$.

Second inclusion, if $\lim_{\beta\to 0^+} \rho_f(\beta h) = 0$, then there exists $\beta > 0$ such that $\rho_f(\beta h) < \infty$. Hence

$$\{h \in L^0(M,\mu) : \lim_{\beta\to 0^+} \rho_f(\beta h) = 0\} \subset L^f(M,\mu).$$

**(b)** For the first direction, suppose there exists $\beta \in (0,1)$ such that $\rho_f(\beta h) < \infty$ (the case $\beta \geq 1$ is clear). Then (2.4) gives

$$\rho_f(h) \leq \int_M \delta\beta^{-q} f(x, \beta\,|h|)d\mu(x) = \delta\beta^{-q}\rho_f(\beta h) < \infty$$

The inclusion $\{h \in L^0(M,\mu) : \rho_f(h) < \infty\} \subset L^f(M,\mu)$ is obvious.

$\square$

We will show some properties are called Fatou's lemma, monotone convergence and dominated convergence for the modular, respectively.

**Lemma 3.1.5.** *Let $f \in \Phi_W(M,\mu)$ and $h_n, h, r \in L^0(M,\mu)$. In (a) and (b), we assume also that $f$ is left-continuous.*

**(a)** *If $h_n \to h$ $\mu$-almost everywhere, then $\rho_f(h) \leq \liminf_{n\to\infty} \rho_f(h_n)$.*

**(b)** *If $|h_n| \nearrow |h|$ $\mu$-almost everywhere, then $\rho_f(h) = \lim_{n\to\infty} \rho_f(h_n)$*

**(c)** *If $h_n \to h$ $\mu$-almost everywhere, $|h_n| \leq |r|$ $\mu$-almost everywhere, and $\rho_f(\beta r) < \infty$ for every $\beta > 0$, then $\lim_{n\to\infty} \rho_f(\beta\,|h - h_k|) = 0$ for every $\beta > 0$.*

*Proof.* To prove (a). Using Lemma 2.1.7 and $f$ is left-continuous, then the mapping $f(x,\cdot)$ is lower semicontinuous. we can use Fatou's lemma to conclude that

$$\begin{aligned}\rho_f(h) &= \int_M f(x, \lim_{n\to\infty} |h_n|)d\mu \leq \int_M \liminf_{n\to\infty} f(x,|h_n|)d\mu \\ &= \liminf_{n\to\infty} \int_M f(x,|h_n|)d\mu = \liminf_{n\to\infty} \rho_f(h_n)\end{aligned}$$

To prove (b), Assume that $|h_n| \nearrow |h|$, if $\rho_f(h) = \infty$, then by (a) we get $lim_{n\to\infty}\rho_f(h_n) = \infty$, So if $\rho_f(h) < \infty$. by the left-continuity and monotonicity of $f(x,\cdot)$, we have $0 \leq f(x,|h_n|) \nearrow f(x,|h|)$ $\mu$-almost everywhere. hence, monotone convergence implies

$$\begin{aligned}\rho_f(h) &= \int_M f(x, \lim_{n\to\infty} |h_n|)d\mu = \int_M \lim_{n\to\infty} f(x,|h_n|)d\mu \\ &= \lim_{n\to\infty} \int_M f(x,|h_n|)d\mu = \lim_{n\to\infty} \rho_f(h_n)\end{aligned}$$

To prove (c), suppose that $h_n \to h$ $\mu$-almost everywhere, $|h_n| \leq |r|$, and $\rho_f(\beta r) < \infty$ for every $\beta > 0$. then $|h_n - h| \to 0$ $\mu$-almost everywhere, $|h| \leq |r|$ and $\lambda |h_n - h| \leq 2\beta |r|$. since $\rho_f(2\beta r) < \infty$, we can use dominated convergence to conclude that

$$\lim_{n\to\infty} \rho_f(\beta |h - h_n|) = \int_M f(x, \lim_{n\to\infty} \beta |h - h_n|) d\mu = 0$$

$\square$

**Lemma 3.1.6.** *Let $f \in \Phi_W(M, \mu)$ satisfy $(2.3)_{q<\infty}$. Let $h_i, r_i \in L^f(\mathbb{R}^n)$ for $i = 1, 2, ...$ with $(\rho_f(h_i))_{i=1}^{\infty}$ bounded. If $\rho_f(h_i - r_i) \to 0$ as $i \to \infty$, then*

$$|\rho_f(h_i) - \rho_f(r_i)| \to 0 \quad as \quad i \to \infty.$$

*Proof.* Since $f$ is increasing and satisfies (2.3), this yields

$$\begin{aligned} f(x, r_i) &\leq f(x, |r_i - h_i| + |h_i|) \leq f(x, 2|r_i - h_i|) + f(x, 2|h_i|) \\ &\leq 2^q f(x, |r_i - h_i|) + 2^q f(x, |h_i|) \end{aligned}$$

and hence $(\rho_f(h_i))_{i=1}^{\infty}$ is bounded. choosing $C > 0$ such that $\rho_f(h_i) \leq C$ and $\rho_f(r_i) \leq C$. let $\beta > 0$ and note that $|h_i| \leq |h_i - r_i| + |r_i|$.
if $|h_i - r_i| \leq \beta |r_i|$, then by (2.3) we find

$$f(x, |h_i|) \leq f(x, (1 + \beta) |r_i|) \leq (1 + \beta)^q f(x, |r_i|).$$

If, on the other hand, $|h_i - r_i| > \beta |r_i|$, then we estimate by (2.3)

$$f(x, |h_i|) \leq f(x, (1 + \frac{1}{\beta}) |h_i - r_i|) \leq (1 + \frac{1}{\beta})^q f(x, |h_i - r_i|)$$

we integrate over $x \in M$, we obtain

$$\begin{aligned} \rho_f(h_i) - \rho_f(r_i) &= \int_M f(x, |h_i - r_i + r_i|) - f(x, |r_i|) d\mu(x) \\ &\leq (1 + \frac{1}{\beta})^q \rho_f(h_i - r_i) + ((1 + \beta)^q - 1) \rho_f(r_i). \end{aligned}$$

By combining the inequalities gives,

$$|\rho_f(h_i) - \rho_f(r_i)| \leq (1 + \frac{1}{\beta})^q \rho_f(h_i - r_i) + ((1+\beta)^q - 1)(\rho_f(h_i) - \rho_f(r_i)).$$

Let $\nu > 0$ be given. since $\rho_f(f_j) - \rho_f(r_i) \leq 2C$, we can choose $\beta$ so small that

$$((1+\beta)^q - 1)(\rho_f(h_i) - \rho_f(r_i)) \leq \frac{\nu}{2}.$$

we can then choose $i_0$ so large that

$$(1 + \frac{1}{\beta})^q \rho_f(h_i - r_i) \leq \frac{\nu}{2}.$$

when $i \geq i_0$ and it follows that $|\rho_f(h_i) - \rho_f(r_i)| \leq \nu$. □

## 3.2 Quasinorm and the unit ball property

**Definition 3.2.1.** Let $f \in \Phi_W(M, \mu)$. for $h \in L^0(M, \mu)$, we denote

$$\|h\|_{L^f(M,\mu)} := \inf \left\{ \beta > 0, \rho_f\left(\frac{h}{\beta}\right) \leq 1 \right\}.$$

we abbreviate $\|h\|_{L^f(M,\mu)} = \|h\|_f$.

**Remark 3.2.2.** $L^f$ can appear as follows:

$$L^f(M, \mu) = \{h \in L^0(M, \mu) : \|h\|_{L^f(M,\mu)} < \infty\}.$$

**Lemma 3.2.3.**

**(a)** *If $f \in \Phi_W(M, \mu)$, then $\|\cdot\|_f$ is a quasinorm.*

**(b)** *If $f \in \Phi_c(M, \mu)$, then $\|\cdot\|_f$ is a norm.*

*Proof.* To prove (a), suppose first that $f \in \Phi_W(M, \mu)$. if $h = 0$ a.e., then $\|h\|_f = 0$. If $\|h\|_f = 0$, then $\rho_f\left(\frac{h}{\beta}\right) \leq 1$ for all $\beta > 0$. when $h(x) \neq 0$, we have $\frac{|h(x)|}{\beta} \to \infty$ when $\beta \to 0^+$. since $\lim_{y\to\infty} f(x, y) = \infty$ for $\mu$-almost every $x$, we obtain that $h(x) = 0$ for $\mu$- almost every $x \in M$.

Let $f \in L^f(M,\mu)$ and $e \in \mathbb{R}$. by definition, $\rho_f(h) = \rho_f(|h|)$. with the change of variables $\beta' := \beta/|e|$, we find

$$\begin{aligned}
\|eh\|_f &= \inf\left\{\beta > 0, \rho_f\Big(\frac{eh}{\beta}\Big) \le 1\right\} = \inf\left\{\beta > 0, \rho_f\Big(\frac{h}{\beta/|e|}\Big) \le 1\right\} \\
&= |e| \inf\left\{\beta' > 0, \rho_f\Big(\frac{h}{\beta'}\Big) \le 1\right\} = |e|\,\|h\|_f\,.
\end{aligned}$$

Hence $\|\cdot\|_f$ is homogeneous.

Let $h, r \in L^f(M,\mu)$ and $v > \|h\|_f$. and $w > \|r\|_f$. then $\rho_f(h/v) \le 1$ and $\rho_f(r/w) \le 1$ by the definition of the norm. using (2.2) with $p = 1$, we get

$$f\Big(x, \frac{|h|}{2\delta v}\Big) \le \frac{1}{2} f\Big(x, \frac{|h|}{v}\Big) \quad \text{and} \quad f\Big(x, \frac{|r|}{2\delta w}\Big) \le \frac{1}{2} f\Big(x, \frac{|r|}{w}\Big).$$

Thus we obtain that

$$\begin{aligned}
\int_M f\Big(x, \frac{|h+r|}{4\delta(v+w)}\Big) d\mu &\le \int_M f\Big(x, \frac{|2h|}{4\delta v}\Big) + f\Big(x, \frac{|2r|}{4\delta w}\Big) d\mu \\
&\le \frac{1}{2}\int_M f\Big(x, \frac{|h|}{v}\Big) + f\Big(x, \frac{|r|}{w}\Big) d\mu \\
&\le \frac{1}{2} + \frac{1}{2} = 1
\end{aligned}$$

and hence $\|h+r\|_f \le 4\delta v + 4\delta w$ which yields that $\|h+r\|_f \le 4\delta\,\|h\|_f + 4\delta\,\|r\|_f$. This completes the proof of (a).

To prove (b), suppose that $f$ is convex. Let $v > \|h\|_f$ and $w > \|r\|_f$. using the convexity of $f$, we obtain

$$\begin{aligned}
\int_M f\Big(x, \frac{|h+r|}{v+w}\Big) d\mu &\le \int_M f\Big(x, \frac{v}{v+w}\frac{|h|}{v} + \frac{w}{v+w}\frac{|r|}{w}\Big) d\mu \\
&\le \int_M \frac{v}{v+w} f\Big(x, \frac{|h|}{v}\Big) + \frac{w}{v+w} f\Big(x, \frac{|r|}{w}\Big) d\mu \\
&\le \frac{v}{v+w} + \frac{w}{v+w} = 1
\end{aligned}$$

Thus $\|h+r\|_f \le v+w$, which yields $\|h+r\|_f \le \|h\|_f + \|r\|_f$, as required for (b). $\square$

We will show a fundamental relation between the norm and the modular.

**Lemma 3.2.4 (Unit ball property).** *Let $f \in \Phi_W(M,\mu)$. Then*

$$\|h\|_f < 1 \quad \Rightarrow \quad \rho_f(h) \le 1 \quad \Rightarrow \quad \|h\|_f \le 1$$

*If $f$ is left-continuous, then $\rho_f(h) \le 1 \Leftrightarrow \|h\|_f \le 1$.*

*Proof.* Assume that $\rho_f(h) \le 1$, by definition of $\|\cdot\|_f$, we get $\|h\|_f \le 1$. Furthermore, if $\|h\|_f < 1$, then $\rho_f(h/\beta) \le 1$ for some $\beta < 1$, since $\rho$ is increasing, it follows that $\rho_f(h) \le 1$.

If $\|h\|_f \le 1$, then $\rho_f(h/\beta) \le 1$ for all $\beta > 1$. when $\rho$ is left-continuous it follows that $\rho_f(h) \le 1$. □

**Example 3.2.5.** Let $f(y) := \infty\chi_{[1,\infty)}(y)$ and $h \equiv 1$, then $f \in \Phi_W$ and $\rho_f(h) = \infty$, since $\rho_f(h/\beta) \le 1$ if and only if $\beta > 1$, we have $\|h\|_f = 1$.
we have shown that if the $\Phi$-function is not left-continuous, then $\|h\|_f = 1$ does not imply $\rho_f(h) \le 1$.

**Proposition 3.2.6.** *Let $f, g \in \Phi_W(M,\mu)$. If $f \simeq g$, then $L^f(M,\mu) = L^g(M,\mu)$ and the norms are comparable.*

*Proof.* Assume that $g(x, \frac{y}{\tau}) \le f(x,y) \le g(x, \tau y)$ and $h \in L^f(M,\mu)$, then there exists $\beta > 0$ such that $\rho_g(\frac{\beta}{\tau}h) \le \rho_f(\beta h) < \infty$. thus $h \in L^g(M,\mu)$. the other direction is similar and hence $L^f(M,\mu) = L^g(M,\mu)$ as sets.
Let $\nu > 0$ and $\beta = \|h\|_f + \nu$. then

$$\rho_g\Big(\frac{h}{\tau\beta}\Big) \le \rho_f\Big(\frac{h}{\beta}\Big) \le 1$$

and hence $\|f\|_g \le \tau\beta = \tau(\|h\|_f + \nu)$. Letting $\nu \to 0^+$ we obtain that $\|h\|_g \le \tau \|h\|_f$. the other direction is similar and so the norms are comparable. □

**Corollary 3.2.7.** *Let $f \in \Phi_W(M,\mu)$. Then*

$$\left\| \sum_{j=1}^{\infty} h_i \right\|_f \lesssim \sum_{j=1}^{\infty} \|h_j\|_f$$

*Proof.* see [14]. $\square$

**Remark 3.2.8.** Recall that, a measure $\mu$ is called $atom-less$ if for any measurable set $A$ with $\mu(A) > 0$ there exists a measurable subset $A'$ of $A$ such that $\mu(A) > \mu(A') > 0$.

**Theorem 3.2.9.** *Let $f, g \in \Phi_W(M, \mu)$ and let the measure $\mu$ be $atom - less$. Then $L^f(M, \mu) \hookrightarrow L^g(M, \mu)$ if and only if there exist $\theta > 0$ and $\varphi \in L^1(M, \mu)$ with $\|\varphi\|_1 \leq 1$ such that*

$$g\Big(x, \frac{y}{\theta}\Big) \leq f(x, y) + \varphi(x)$$

*for $\mu$-almost all $x \in M$ and all $y \geq 0$.*

*Proof.* see [14]. $\square$

**Corollary 3.2.10.** *Let $f, g \in \Phi_W(M, \mu), f \sim g$. Then $L^f(M, \mu) = L^g(M, \mu)$ and the norms are comparable.*

**Corollary 3.2.11.** *Let $f \in \Phi_W(M, \mu)$ and $h \in L^f(M, \mu)$ and let $\delta$ be the constant from* (2.2) *with $p = 1$.*

**(a)** *If $\|h\|_f < 1$, then $\rho_f(h) \leq \delta \|h\|_f$.*

**(b)** *If $\|h\|_f > 1$, then $\|h\|_f \leq \delta \rho_f(h)$.*

**(c)** *In any case, $\|h\|_f \leq \delta \rho_f(h) + 1$.*

*Proof.* For (a), if $h = 0$ is trivial case, otherwise suppose that $0 < \|h\|_f < 1$. Let $\beta > 1$ be so small that $\beta \|h\|_f < 1$. By unit ball property (Lemma 3.2.4) and $\left\|\frac{h}{\beta\|h\|_f}\right\|_f < 1$, it follows that $\rho_f(\frac{h}{\beta\|h\|_f}) \leq 1$. since $\beta \|h\|_f \leq 1$, by (2.2) with $p = 1$ we have

$$\frac{1}{\delta\beta \|h\|_f}\rho_f(h) \leq \rho_f\Big(\frac{h}{\beta \|h\|_f}\Big) \leq 1$$

as $\beta \to 1^+$ we find that $\rho_f(h) \leq \delta \|h\|_f$.

For (b), suppose that $\|h\|_f > 1$. then $\rho_f(\frac{h}{\beta}) > 1$ for $1 < \beta < \|h\|_f$ and by (2.2) with $p = 1$ we obtain

$$\frac{\delta}{\beta}\rho_f(h) \geq \rho_f(\frac{h}{\beta}) > 1$$

as $\beta \to \|h\|_f^-$ we get that $\delta\rho_f(h) \geq \|h\|_f$.

For (c), using (b), we get the claim (c). $\square$

**Remark 3.2.12.** we consider $h \equiv 1$, $f(x,y) := \infty\chi_{(1,\infty)}(y)$ and $g(x,y) := \infty\chi_{[1,\infty)}(y)$. then $f$ and $g$ are $\Phi$-functions and $\|h\|_f = \|h\|_g = 1$ but $\rho_f(h) = 0$ and $\rho_g(h) = \infty$. so we have shown that in Corollary 3.2.11(a) and (b) the case $\|h\|_f = 1$ is excluded.

**Lemma 3.2.13.** *Let $f \in \Phi_W(M,\mu)$ satisfy* (2.2) *and* (2.4), $1 \leq p \leq q < \infty$. *Then*

$$\min\Big\{(\frac{1}{\delta}\rho_f(h))^{\frac{1}{p}}, (\frac{1}{\delta}\rho_f(h))^{\frac{1}{q}}\Big\} \leq \|h\|_f \leq \max\Big\{(\delta\rho_f(\delta))^{\frac{1}{p}}, (\delta\rho_f(h))^{\frac{1}{q}}\Big\}$$

*for $h \in L^0(M,\mu)$, where $\delta$ is the maximum of the constants from* (2.2) *and* (2.4).

*Proof.* we begin with the proof of the first inequality. let $v \in (0, \rho_f(h))$ and assume first that $\frac{v}{\delta} \leq 1$. then (2.2) gives that

$$f\Big(x, \frac{|h(x)|}{(v/\delta)^{\frac{1}{p}}}\Big) \geq \frac{\delta}{\delta v} f(x, |h(x)|) = \frac{1}{v} f(x, |h(x)|).$$

Integrating over $M$, we find that $\rho_f(h/(v/\delta)^{1/p}) > 1$, which yields $\|h\|_f \geq (v/\delta)^{\frac{1}{p}}$.
If $\frac{v}{\delta} > 1$, we similarly use (2.4) to conclude that $\|h\|_f \geq (\frac{v}{\delta})^{\frac{1}{q}}$.
The first inequality follows as $v \to \rho_f(h)^-$.

For the second inequality. let $v > \rho_f(h)$ and assume first that $\delta v \leq 1$. then (2.4) gives that

$$f\Big(x, \frac{|h(x)|}{(\delta v)^{\frac{1}{q}}}\Big) \leq \frac{\delta}{\delta v} f(x, |h(x)|) = \frac{1}{v} f(x, |h(x)|).$$

Integrating over $M$, we find that $\rho_f(h/(\delta v)^{1/q}) \le 1$, which yields $\|h\|_f \le (\delta v)^{\frac{1}{q}}$ .

If $\delta v > 1$, we similarly use (2.2) to conclude that $\|h\|_f \le (\delta v)^{\frac{1}{p}}$.

The second inequality follows as $v \to \rho_f(h)^+$. $\square$

If $q = \infty$, we get the following corollary.

**Corollary 3.2.14.** *Let $f \in \Phi_W(M,\mu)$ satisfy* (2.2), $1 \le p < \infty$.*Then*

$$\min\left\{(\frac{1}{\delta}\rho_f(h))^{\frac{1}{p}}, 1\right\} \le \|h\|_f \le \max\left\{(\delta\rho_f(\delta))^{\frac{1}{p}}, 1\right\}$$

*for $h \in L^0(M,\mu)$, where $\delta$ is the constant from* (2.2).

Let us next show the generalization of the classical Hölder inequality $\int |\varphi|\,|\psi|\,d\mu \le \|\varphi\|_p \|\psi\|_{p'}$ to generalized Orlicz spaces.

**Lemma 3.2.15 (Hölder's Inequality).** *Let $f \in \Phi_W(M,\mu)$.Then*

$$\int_M |\varphi|\,|\psi|\,d\mu \le 2\,\|\varphi\|_f\,\|\psi\|_{f^*}$$

*for all $\varphi \in L^f(M,\mu)$ and $\psi \in L^{f^*}(M,\mu)$. Moreover, the constant* 2 *cannot in general be replaced by any smaller number.*

*Proof.* Let $\varphi \in L^f$ and $\psi \in L^{f^*}$ with $v > \|\varphi\|_f$ and $w > \|\psi\|_{f^*}$. By the unit ball property, $\rho_f(\varphi/v) \le 1$ and $\rho_{f^*}(\psi/w) \le 1$. thus, using Young's inequality (2.6), we obtain

$$\int_M \frac{|\varphi|}{v}\frac{|\psi|}{w}d\mu \le \int_M f\Big(x, \frac{|\varphi|}{v}\Big) + f^*\Big(x, \frac{|\psi|}{w}\Big)d\mu = \rho_f\Big(\frac{\varphi}{v}\Big) + \rho_{f^*}\Big(\frac{\psi}{w}\Big) \le 2.$$

Multiplying by $vw$, we get the inequality as $v \to \|\varphi\|_f^+$ and $w \to \|\psi\|_{f^*}^+$.

**Example 3.2.16.** Suppose that $f(y) = \frac{1}{2}y^2$. then $f^*(y) = \sup_{v\ge 0}(vy - \frac{1}{2}v^2) = \frac{1}{2}y^2$. let $\varphi \equiv \psi \equiv 1$. then $\int_0^1 \varphi\psi dt = 1$. otherwise,

$$\inf\left\{\beta > 0 : \int_0^1 \frac{1}{2}\Big(\frac{1}{\beta}\Big)^2 dt \le 1\right\} = \frac{1}{\sqrt{2}}$$

and thus $\|\varphi\|_{L^f(0,1)} = \|\psi\|_{L^{f^*}(0,1)} = \frac{1}{\sqrt{2}}$ and $\|\varphi\|_{L^f(0,1)}\,\|\psi\|_{L^{f^*}(0,1)} = \frac{1}{2}$.

so we have shown that the extra constant 2 in Hölder's inequality cannot be omitted.

□

## 3.3 Convergence and completeness

**Lemma 3.3.1.** *Let $f \in \Phi_W(M,\mu)$.Then $\|h_n\|_f \to 0$ as $n \to \infty$ if and only if $\lim_{n\to\infty} \rho_f(\beta h_n) = 0$ for all $\beta > 0$.*

*Proof.* Assume first that$\rho_f(\beta h_n) \to 0$ for all $\beta > 0$. then $\rho_f(\beta h_n) \leq 1$ for large $k$. by unit ball property (Lemma 3.2.4), $\|h_n\|_f \leq 1/\beta$ for the same $n$. since $\beta > 0$ was arbitrary, we get $\|h_n\|_f \to 0$.

Assume now that $\|h_n\|_f \to 0$. Let $\theta > 1$ and $\beta > 0$. then $\|\theta\beta h_n\|_f < 1$ for large $n$. thus $\rho_f(\theta\beta h_n) \leq 1$ for large $n$, by unit ball property (Lemma 3.2.4). Hence by (2.2) with $p = 1$

$$\begin{aligned}\rho_f(\beta h_n) &= \int_M f(x, \beta\,|h_n|)d\mu \leq \int_M \frac{\delta}{\theta} f(x, \theta\,|\beta h_n|)d\mu \\ &= \frac{\delta}{\theta}\rho_f(\theta\beta h_n) \leq \frac{\delta}{\theta}\end{aligned}$$

for all $\theta > 1$ and all large $n$. this implies $\rho_f(\beta h_n) \to 0$. □

**Definition 3.3.2.** Let $f \in \Phi_W(M,\mu)$ and $h_n, h \in L^f(M)$. We say that $h_n$ is modular convergent ($\rho_f$-convergent) to $f$ if $\rho(\beta(h_n - h)) \to 0$ as $n \to \infty$ for some $\beta > 0$.

**Remark 3.3.3.** Lemma 3.3.1 gives that for norm convergence we have $\lim_{n\to\infty} \rho(\beta(y_n - y)) = 0$ for all $\beta > 0$, while for modular convergence this only has to hold for some $\beta > 0$. so we have shown that modular convergence is weaker than norm convergence.

**Lemma 3.3.4.** *Let $f \in \Phi_W(M,\mu)$. Modular convergence and norm convergence are equivalent if and only if $\rho(h_n) \to 0$ implies $\rho(2h_n) \to 0$.*

*Proof.* Assume first that $\rho_f(h_n) \to 0$ implies $\rho_f(2h_n) \to 0$. Let $h_n \in L^f$ with $\rho_f(\beta_0 h_n) \to 0$ for some $\beta_0 > 0$. We have to show that $\rho_f(\beta h_n) \to 0$ for all $\beta > 0$. For fixed $\beta > 0$ choose $m \in \mathbb{N}$ such that $2^m\beta_0 \geq \beta$. then by repeated application of the assumption we get $\lim_{k\to\infty} \rho(2^m\beta_0 h_n) = 0$. since $f$ is increasing we obtain $0 \leq \lim_{k\to\infty} \rho_f(\beta h_n) \leq \lim_{k\to\infty} \rho_f(2^m\beta_0 h_n) = 0$. by Lemma 3.3.1 we get $h_n \to 0$.

Assume then modular convergence and norm convergence be equivalent and let $\rho(h_n) \to 0$ with $h_n \in L^f$. then $h_n \to 0$ (norm convergence) and by Lemma 3.3.1 it follows that $\rho(2h_n) \to 0$. □

**Corollary 3.3.5.** *Let $f \in \Phi_W$ satisfy $(2.4)_{q<\infty}$. Then modular convergence and norm convergence are equivalent.*

**Lemma 3.3.6.** *Let $f \in \Phi_W(M,\mu)$ and $\mu(M) < \infty$. Then every $\|\cdot\|_f$-Cauchy sequence is also a Cauchy sequence with respect to convergence in measure.*

*Proof.* Fix $\nu > 0$ and let $E_y := \{x \in M : f(x,y) = 0\}$ for $y > 0$. Then $E_y$ is measurable. For $\mu$-almost all $x \in M$ the function $y \mapsto f(x,y)$ is increasing, so $E_y \subset E_z \cup G$ for all $y > z$ with $\mu(G) = 0$ and $G$ independent of $z$ and $y$. Since $\lim_{y\to\infty} f(x,y) = \infty$ for $\mu$-almost every $x \in M$ and $\mu(M) < \infty$, we obtain that $\lim_{n\to\infty} \mu(E_n) = 0$. Thus, there exists $\theta \in \mathbb{N}$ such that $\mu(E_n) < \nu$.

For a $\mu$-measurable set $F \subset M$ define

$$\nu_n(F) := \rho_f(\theta\chi_F) = \int_F f(x,\theta)d\mu$$

If $F$ is $\mu$-measurable with $\nu_\theta(F) = 0$, then $f(x,\theta) = 0$ for $\mu$-almost every $x \in F$. Thus $\mu(F \backslash E_\theta) = 0$ by the definition of $E_\theta$. Hence, $F$ is a $\mu|_{M\backslash E_\theta}$-null set, which means that the measure $\mu|_{M\backslash E_\theta}$ is absolutely continuous with respect to $\nu_\theta$.

Since $\mu(M \setminus E_\theta) \leq \mu(M) < \infty$ and $\mu|_{M\backslash E_\theta}$ is absolutely continuous with respect to $\nu_\theta$, there exists $\alpha \in (0,1)$ such that $\nu_\theta(F) \leq \alpha$ implies $\mu(F \setminus E_\theta) \leq \nu$.

Since $h_n$ is a $\|\cdot\|_f$-Cauchy sequence, there exists $n_0 \in \mathbb{N}$ such that $\left\|\frac{\theta\delta}{\alpha\nu}(h_m - h_n)\right\|_f <$

1 for all $m, n \geq n_0$, with $\delta$ from (2.2) with $p = 1$. Assume in the following that $m, k \geq k_0$. Then (2.2) with $p = 1$ and the unit ball property (Lemma 3.2.4) imply

$$\rho_f\Big(\frac{\theta}{\nu}(h_m - h_n)\Big) \leq \alpha\rho_f\Big(\frac{\theta\delta}{\nu\alpha}(h_m - h_n)\Big) \leq \alpha$$

Let us write $F_{m,n,\nu} := \{x \in M : |h_m(x) - h_n(x)| \geq \nu\}$. Then

$$\nu_\theta(F_{m,n,\nu}) = \int_{F_{m,n,\nu}} f(x,\theta)d\mu(x) \leq \rho_f\Big(\frac{\theta}{\nu}(h_m - h_n)\Big) \leq \alpha.$$

By the choice of $\alpha$, this implies that $\mu(F_{m,n,\nu} \setminus E_\theta) \leq \nu$. With $\mu(E_\theta) < \nu$ we have $\mu(F_{m,n,\nu}) \leq 2\nu$. Since $\nu > 0$ was arbitrary, this proves that $h_n$ is a Cauchy sequence with respect to convergence in measure. □

**Lemma 3.3.7.** *Let $f \in \Phi_W(M,\mu)$. Then every $\|\cdot\|_f$-Cauchy sequence $(h_n) \subset L^f$ has a subsequence which converges $\mu$-a.e. to a measurable function $h$ .*

*Proof.* Recall that $\mu$ is $\sigma$-finite. Let $M := \bigcup_{i=1}^\infty M_i$ with $M_i$ pairwise disjoint and $\mu(M_i) < \infty$ for all $i \in N$. Then, by Lemma 3.3.6, $(h_n)$ is a Cauchy sequence with respect to convergence in measure on $M_1$. Therefore there exists a measurable function $f : M_1 \to \mathbb{R}$ and a subsequence of $(h_n)$ which converges to $h$ $\mu$-almost everywhere. Repeating this argument for every $M_i$ and passing to the diagonal sequence, we get a subsequence $(h_{n_j})$ and a $\mu$-measurable function $h : M \to \mathbb{R}$ such that $h_{n_j} \to f$ $\mu$-almost everywhere. □

Now, we study the completeness of $L^f$.

**Theorem 3.3.8.**

**(a)** *If $f \in \Phi_W(M,\mu)$, then $L^f(M,\mu)$ is a quasi-Banach space.*

**(b)** *If $f \in \Phi_c(M,\mu)$, then $L^f(M,\mu)$ is a Banach space.*

*Proof.* By Lemma 3.2.3, $\|\cdot\|_f$ is a quasinorm if $f \in \Phi_W(M,\mu)$ and a norm if $f \in \Phi_c(M,\mu)$. It remains to prove completeness.

Let $(h_n)$ be a Cauchy sequence. By Lemma 3.3.7, there exists a subsequence $h_{n_i}$ and a $\mu$-measurable function $f : M \to \mathbb{R}$ such that $h_{n_i} \to f$ for $\mu$-almost every $x \in M$. This implies $f(x, c\,|h_{n_i}(x) - h(x)|) \to 0$ $\mu$-almost everywhere for every $c > 0$. Let $\beta > 0$ and $0 < \nu < 1$. Since $(h_n)$ is a Cauchy sequence, there exists $N = N(\beta,\nu) \in \mathbb{N}$ such that $\|\beta(h_m - h_n)\|_f < \nu/\delta$, for all $m, n \geq N$, with a from (2.2) with $p = 1$. By Corollary 3.2.11(a) this implies $\rho_f(\beta(h_m - h_n)) \leq \nu$ for all $m, n \geq N$.

Since $f$ is increasing, we obtain

$$f(x, \lim_{i\to\infty} \frac{\beta}{2}\,|h_m - h_{n_i}|) \leq \liminf_{i\to\infty} f(x, \beta\,|h_m - h_{n_i}|)$$

. Hence Fatou's lemma yields that

$$\begin{aligned}
\rho_f(\frac{\beta}{2}(h_m - h_n)) &= \int_M f(x, \lim_{i\to\infty}\frac{\beta}{2}\,|h_m - h_{n_i}|)d\mu \\
&\leq \int_M \liminf_{i\to\infty} f(x, \beta\,|h_m - h_{n_i}|)d\mu \\
&\leq \liminf_{i\to\infty} \int_M f(x, \beta\,|h_m - h_{n_i}|)d\mu \leq \nu
\end{aligned}$$

Thus $\rho_f(\frac{\beta}{2}(h_m - h_n)) \to 0$ for $m \to \infty$ and every $\beta > 0$, so that $\|h_m - h\|_f \to 0$ by Lemma 3.3.1. therefore every Cauchy sequence converges in $L^f$. ☐

**Remark 3.3.9.** Let $f \in \Phi_W(M,\mu)$. Then $L^f(M,\mu)$ is circular, i.e.

$$\|h\|_f = \||h|\|_f \quad \text{for all} \quad h \in L^f. \tag{3.1}$$

If $\varphi \in L^f$, $\psi \in L^0(M,\mu)$, and $0 \leq |\psi| \leq |\varphi|$ $\mu$-almost everywhere, then $L^f(M,\mu)$ is solid, i.e.

$$\psi \in L^f \quad \text{and} \quad \|\psi\|_f \leq \|\varphi\|_f\,. \tag{3.2}$$

**Lemma 3.3.10.** *Let $f \in \Phi_W(M,\mu)$ be left-continuous and $h, h_n \in L^0(M,\mu)$.*

**(a)** *If $h_n \to h$ $\mu$-almost everywhere, then $\|h\|_f \le \liminf_{n\to\infty} \|h_n\|_f$.*

**(b)** *If $|h_n| \nearrow |f|$ $\mu$-almost everywhere with $h_n \in L^f(M,\mu)$ and $\sup_n \|h\|_f < \infty$, then $h \in L^f(M,\mu)$ and $\|h_n\|_f \nearrow \|h\|_f$.*

*Proof.* For (a) let If $h_n \to f$ $\mu$-almost everywhere. There is nothing to prove for $\liminf_{n\to\infty} \|h_n\|_f = \infty$. Otherwise, let $\beta > \liminf_{n\to\infty} \|h_n\|_f$. Then $\|h_n\|_f < \beta$ for some large $n$. Thus by the unit ball property (Lemma 3.2.4), $\rho_f(h_n/\beta) \le 1$ for large $n$. Now Fatou's lemma for the modular (Lemma 3.1.5) implies $\rho_f(h/\beta) \le 1$. So $\|h\|_f \le \beta$ again by the unit ball property. Thus we have $\|h\|_f \le \liminf_{n\to\infty} \|h_n\|_f$.

It remains to prove (b). So let $|h_n| \nearrow |h|$ $\mu$-almost everywhere with $\sup_n \|h\|_f < \infty$. By (a) we obtain $\|h\|_f \le \liminf_{n\to\infty} \|h_n\|_f \le \sup_n \|h_n\|_f < \infty$, which also proves $f \in L^f$. On the other hand, $|h_n| \nearrow |f|$ and solidity (3.2) implies that $\|h_n\|_f \nearrow \limsup_{n\to\infty} \|h_n\|_f \le \|h\|_f$. It follows that $\lim_{n\to\infty} \|h_n\|_f = \|f\|_f$ and $\|h_n\|_f \nearrow \|f\|_f$. □

## 3.4 Associate spaces

First, recall that the dual space $V^*$ of a normed space $V$ consists of all bounded linear functions from $V$ to $\mathbb{R}$. Equipped with the norm

$$\|h\|_{V^*} := \sup_{\|h\|_V \le 1} |G(h)|\,,$$

$V^*$ is a Banach space, see for example [24].

second, we will show that the second associate space is always isomorphic to the space itself, whereas the second dual space is only isomorphic under certain additional conditions.

**Definition 3.4.1.** Let $f \in \Phi_W(M,\mu)$. Then by $(L^f(M,\mu))^*$ we denote the dual space of $L^f(M,\mu)$. Furthermore, we define $\phi_f : (L^f(M,\mu))^* \to [0,\infty]$ by

$$\phi_f(G) := \sup_{h \in L^f(M,\mu)} \Big( |G(h)| - \rho_f(h) \Big).$$

**Remark 3.4.2.** Note the difference between the spaces $(L^f(M,\mu))^*$ and $L^{f^*}(M,\mu)$: the former is the dual space of $L^f(M,\mu)$, whereas the latter is the generalized Orlicz space defined by the conjugate modular $f^*$.

By definition of the functional $\phi_f$ we have

$$|G(h)| \leq \rho_f(h) - \phi_f(G) \tag{3.3}$$

for all $h \in L^f(M,\mu)$ and $G \in (L^f(M,\mu))^*$.This is a generalized version of the classical Young inequality.

The function $\phi_f$ is actually a semimodular on the dual space. We refer to [10] for details.

In the definition of $\phi_f$ the supremum is taken over all $L^f(M,\mu)$. However, it is possible to restrict this to the closed unit ball when $G$ is in the unit ball and $f$ is convex.

**Lemma 3.4.3.** *Let $f \in \Phi_c(M,\mu)$. If $G \in (L^f(M,\mu))^*$ with $\|G\|_{(L^f)^*} \leq 1$, then*

$$\phi_f(G) = \sup_{h\in L^f, \|h\|_f \leq 1} \Big( |G(h)| - \rho(h) \Big) = \sup_{h \in L^f, \rho_f(h) \leq 1} \Big( |G(h)| - \rho(h) \Big).$$

*Proof.* The equivalence of the suprema follows from the unit ball property (Lemma 3.2.4). Let $\|G\|_{(L^f)^*} \leq 1$. By the definition of the dual norm we have

$$\begin{aligned} \sup_{\|h\|_f > 1} \Big( |G(h)| - \rho(f) \Big) &\leq \sup_{\|h\|_f > 1} \Big( \|G\|_{(L^f)^*} \|h\|_f - \rho_f(h) \Big) \\ &\leq \sup_{\|h\|_f > 1} \Big( \|h\|_f - \rho_f(h) \Big). \end{aligned}$$

If $\|h\|_f > 1$, then $\rho_f(h) \geq \|h\|_f$ by Corollary 3.2.11, and so the right-hand side of the previous inequality is non-positive. Since $\rho^*$ is defined as a supremum, and is always non-negative, we see that f with $\|h\|_f > 1$ does not affect the supremum, and so the claim follows. □

The next lemma shows that we can approximate the function 1 with a monotonically increasing sequence of functions in the generalized Orlicz space. This will allow us to generalize several results from ( [10], Chapter 2 ) without the extraneous assumption $L^\infty \subset L^f$ that was used there.

**Lemma 3.4.4.** *Let $f \in \Phi_W(M,\mu)$. There exist a sequence of positive functions $\varphi_n \in L^f(M,\mu), n \in \mathbb{N}$, such that $\varphi_n \nearrow 1$ and $\{\varphi_n = 1\} \nearrow M$.*

*Proof.* We set $\varphi(x) := f^{-1}(x,1)$. Then $\varphi$ is measurable by Lemma 2.5.13 and $f(x,\varphi(x)) \leq 1$ by Lemma 2.3.9(b). Let us define $\varphi_n := \min\{n\varphi\chi_{B(0,n)\cap M}, 1\}$. Then

$$\rho_f(\frac{1}{n}\varphi_n) \leq \int_{B(0,n)\cap M} f(x, \min\{\varphi, 1/n\})dx \leq |B(0,n)| < \infty,$$

so that $\varphi_n \in L^f(M)$.By $\lim_{y\to 0^+} f(x,y) = 0$ we have $\varphi > 0$. It follows that $n\varphi\chi_{B(0,n)\cap M} \nearrow \infty$ for $\mu$-almost every $x \in M$, and so $\{\varphi_n = 1\} \nearrow M \setminus F, \mu(F) = 0$. By modifying $h_k$ in a set of measure zero, we obtain the claim. $\square$

**Definition 3.4.5.** We define the associate space of $L^f(M,\mu)$ as the space $(L^f)'(M,\mu) := \{h \in L^0(M,\mu) : \|h\|_{(L^f)'} < \infty\}$ with the norm

$$\|h\|_{(L^f)'} := \sup_{\|\psi\|_f \leq 1} \int_M h\psi d\mu$$

If $\psi \in (L^f)'$ and $h \in L^f$, then $h\psi \in L^1$ by the definition of the associate space. In particular, the integral $\int_M h\psi d\mu$ is well defined and

$$\left|\int_M h\psi d\mu\right| \leq \|\psi\|_{(L^f)'} \|h\|_{L^f}.$$

By $J_h$ we denote the functional $\psi \mapsto \int_M h\psi d\mu$. Clearly $J_h \in (L^f)^*$ when $h \in (L^f)'$ so $J_. : (L^f)' \to (L^f)^*$. The next result shows that the associate space of $L^f$ is always given by $L^{f^*}$. In this sense the associate space is much nicer than the general dual space.

**Theorem 3.4.6 (Norm conjugate formula).** *If $f \in \Phi_W(M,\mu)$, then*

$$(L^f)' = L^{f^*}$$

*and the norms are comparable. Moreover, for all $h \in L^0(M,\mu)$*

$$\|h\|_f \approx \sup_{\|\psi\|_{f^*} \leq 1} \int_M |h\psi| \, d\mu.$$

*Proof.* By Theorem 2.5.11 there exists $g \in \Phi_s(M,\mu)$ such that $f \simeq g$. Then $L^f = L^g$ and $\|f\|_f \approx \|f\|_g$ by Proposition 3.2.6.

Let $h \in (L^g)'$ with $\|h\|_{(L^g)'} \leq 1$ and $\nu > 0$. Let $\{\varepsilon_1, \varepsilon_2, ...\}$ be an enumeration of non-negative rational numbers with $\varepsilon_1 = 0$. For $n \in \mathbb{N}$ and $x \in M$ define

$$s_n(x) := \max_{i \in \{1,...n\}} \{\varepsilon_i \, |h(x)| - g(x, \varepsilon_i)\}.$$

The special choice $\varepsilon_1 = 0$ implies that $s_n(x) \geq 0$ for all $x \geq 0$. Since $\mathbb{Q}$ is dense in $[0, \infty)$ and $g(x, \cdot)$ is left-continuous, $s_n(x) \nearrow g^*(x, |h(x)|)$ for $\mu$-almost every $x \in M$ as $n \to \infty$.

Since $h$ and $g(\cdot, y)$ are measurable functions, the sets

$$T_{j,n} := \left\{ x \in M : \varepsilon_j \, |h(x)| - g(x, \varepsilon_j) = \max_{i=1,...n} (\varepsilon_i \, |h(x)| - g(x, \varepsilon_i)) \right\}$$

are measurable.

Let $P_{j,n} := T_{j,n} \setminus (T_{1,n} \cup ... \cup T_{j-1,n})$ and define

$$\zeta_n := \sum_{j=1}^{n} \varepsilon_j \chi_{P_{j,n}}.$$

Then $\zeta_n$ is measurable and bounded and

$$s_n(x) = \zeta_n(x) \, |h(x)| - g(x, \zeta_k(x))$$

for all $x \in M$.

Let $\psi_n \in L^g(M,\mu)$ be as in Lemma 3.4.4. Since $\zeta_n$ is bounded and $\psi_n \in L^g(M,\mu)$, it follows that $r := sgn(h)\psi_n\zeta_n \in L^g(M,\mu)$.

Since $\phi_g$ is defined in Definition 3.4.1 as a supremum over functions in $L^g$, we get a lower bound by using the particular function $r\chi_T$. Thus

$$\begin{aligned}
\phi_g(J_h) &\geq \left|J_h(r\chi_{\{\psi_n=1\}})\right| - \rho_g(r\chi_{\{\psi_n=1\}}) = \int_{\{\psi_n=1\}} hr - g(x,|r|)d\mu \\
&\geq \int_{\{\psi_n=1\}} \zeta_n\,|h| - g(x,|\zeta_n|)d\mu = \int_M s_n\chi_{\{\psi_n=1\}}d\mu.
\end{aligned}$$

Since $s_n\chi_{\{\psi_n=1\}} \nearrow g^*(x,|h|)$ $\mu$-almost everywhere, it follows by monotone convergence that $\phi_g(J_h) \geq \rho_{g^*}(h)$. From the definitions of $\phi_g$ and $\rho_{g^*}$ we conclude by Young's inequality (2.6) that

$$\phi_g(J_h) = \sup_{\varphi\in L^g} \int_M h\varphi - g(x,\varphi)d\mu \leq \sup_{\varphi\in L^g} \int_M g^*(x,h)d\mu = \rho_{g^*}(h).$$

Hence $\phi_g(J_h) = \rho_{g^*}(h)$.

Recall that we are assuming $\|h\|_{(L^g)'} \leq 1$ and denote $G := \{\varphi \in L^g : \|\varphi\|_g \leq 1\}$. Then Lemma 3.4.3 and the definition of the associate space yield

$$\phi_g(J_h) = \sup_{\varphi\in G}\Big(|J_h(\varphi)| - \rho_g(\varphi)\Big) \leq \sup_{\varphi\in G}\Big(\|\varphi\|_g - \rho_g(\varphi)\Big) \leq \sup_{\varphi\in G}\|\varphi\|_g \leq 1.$$

Hence also $\rho_{g^*}(h) = \phi_g(J_h) \leq 1$ and it follows from the unit-ball property that $\|h\|_{g^*} \leq 1$. By a scaling argument, we obtain $\|h\|_{g^*} \leq \|h\|_{(L^g)'}$.
Hölder's inequality (Lemma 3.2.15) implies that $\|h\|_{(L^g)'} \leq 2\,\|h\|_{g^*}$. In view of the previous paragraph, $\|h\|_{(L^g)'} \approx \|h\|_{g^*}$.

Taking into account that $g^{**} \simeq g$ (Proposition 2.4.6), we have shown that $L^f = L^g = (L^{g^*})'$. By the definition of the associate space norm, this means that

$$\|h\|_f \approx \|h\|_g \approx \sup_{\|\varphi\|_{g^*}\leq 1} \int_M |h|\,|\varphi|\,d\mu$$

for $h \in L^g$. By Lemma 2.4.4, $g^* \simeq f^*$ and hence $\|\varphi\|_{g^*} \approx \|\varphi\|_{f^*}$ (Proposition 2.4.4). By $\frac{1}{\tau}\|\varphi\|_{g^*} \leq \|\varphi\|_{f^*} \leq \tau\,\|\varphi\|_{g^*}$ we obtain

$$\sup_{\|\varphi\|_{g^*}\leq 1} \int_M |h|\,|\varphi|\,d\mu \geq \sup_{\tau\|\varphi\|_{g^*}\leq 1} \int_M |h|\,|\varphi|\,d\mu \geq \frac{1}{\tau} \sup_{\|\tau\varphi\|_{g^*}\leq 1} \int_M |h|\,|\tau\varphi|\,d\mu$$

and similarly for the other direction. Thus the claim is proved in the case $f \in L^f$.

In the case $h \in L^0 \setminus L^g$, we can approximate $\psi_n \min\{|h|, n\} \nearrow |h|$ as before. Since $\psi_n \min\{|h|, n\} \in L^g$, the previous result implies that the formula holds, in the form $\infty = \infty$, when $h \in L^0 \setminus L^g$. $\square$

## 3.5 Separability

Recall that a (quasi-)Banach space is separable if it contains a dense, countable subset so, we will study separability and other density results.

**Remark 3.5.1.** We say that a function is simple if it is a linear combination of characteristic functions of measurable sets, $\sum_{j=1}^n s_j \chi_{F_j}(x)$ with $\mu(F_1), ..., \mu(F_n) < \infty$ and $s_1, ..., s_n \in \mathbb{R}$.
We denote the set of simple functions by $S(M, \mu)$, or, when $M$ and $\mu$ are clear, by $S$.

**Proposition 3.5.2.** *Let $f \in \Phi_W(M, \mu)$ satisfy the assumption $(2.4)_{q<\infty}$. Then the sets $S(M, \mu) \cap L^f(M, \mu)$ and $L^\infty(M, \mu) \cap L^f(M, \mu)$ are dense in $L^f(M, \mu)$.*

*Proof.* Let $h \in L^f(M, \mu)$ with $h \geq 0$. Since $h$ is measurable, there exist $\psi_n := \sum_{j=1}^n s_j \chi_{F_j}(x)$ with measurable sets $F_j$ and $0 \leq h_n \nearrow h$ $\mu$-almost everywhere. Note that it does not necessary hold that $\mu(F_j) < \infty$. Since $\mu$ is $\sigma$-finite, there exist sets $(M_j)$ such that $M = \bigcup_{j=1}^\infty M_j$ and $\mu(M_j) < \infty$ for every $j$.We define $\widetilde{h}_n := \sum_{j=1}^n s_j \chi_{F_j}(x) \chi_{\cup_{i=1}^n} M_i(x)$. Then $\widetilde{h}_n \in S$ and $0 \leq \widetilde{h}_n \nearrow h$ $\mu$-almost everywhere.
Since $0 \leq \widetilde{h}_n \leq h$ we find that $\widetilde{h}_n \in L^f(M, \mu)$. Since $f$ satisfies $(2.4)_{q<\infty}$, norm and modular convergence are equivalent by Corollary 3.3.5. Let $\beta > 0$ be such that $\rho_f(\beta h) < \infty$.Then $\beta \left| h - \widetilde{h}_n \right| \leq \beta |h|$ and hence by dominated convergence
$\rho_f\left(\beta \left| h - \widetilde{h}_n \right|\right) \to 0$ as $n \to \infty$. Since norm and modular convergence are equivalent this yields that $\widetilde{h}_n \to h$ in $L^f(M, \mu)$. Thus, $h$ is in the closure of $S \cap L^f(M, \mu)$. If we drop the assumption $h \geq 0$, then we obtain the same result by considering the positive and negative parts of $h$ separately.

Since every simple function is bounded, it follows that the larger set $L^\infty \cap L^f$ is also dense in $L^f$. $\square$

We say that a measure $\mu$ is separable if there exists a sequence $(F_n) \subset \Gamma$ with the following properties:

**(a)** $\mu(F_n) < \infty$ for all $n \in \mathbb{N}$,

**(b)** for every $F \in \Gamma$ with $\mu(F) < \infty$ and every $\nu > 0$ there exists an index $n$ such that $\mu(F \triangle F_n) < \nu$, where $\triangle$ denotes the symmetric difference defined as $F \triangle F_n := (F \setminus F_n) \cup (F_n \setminus F)$.

For instance the Lebesgue measure on $\mathbb{R}^n$ and the counting measure on $\mathbb{Z}^n$ are separable. Under $(2.4)_{q<\infty}$, the separability of the measure implies separability of the space. Since $L^\infty$ is not separable, the assumption $(2.4)_{q<\infty}$ is reasonable.

**Theorem 3.5.3.** *Let $f \in \Phi_W(M,\mu)$ satisfy $(2.4)_{q<\infty}$, and let $\mu$ be separable. Then $L^f(M,\mu)$ is separable.*

*Proof.* Let $S_0$ be the set of all simple functions of the form $\sum_{j=1}^n \alpha_j \chi_{F_j}$ with $\alpha_j \in \mathbb{Q}$ and $F_j$ is as in the definition of a separable measure, so that $S_0$ is countable.

By Proposition 3.5.2 it suffices to prove that $S_0$ is dense in $S$.
Let $h \in S \cap L^f$ be non-negative. Then we can write $h$ in the form $\sum_{j=1}^n \gamma_j \chi_{G_j}$ with $\gamma_j \in (0,\infty), G_j \in \Gamma$ pairwise disjoint and $\mu(G_j) < \infty$ for all $j$. Let $\psi_n$ be as in Lemma 3.4.4.

Fix $\nu \in (0,1)$. Let $\beta \in (0,1]$ be such that $\rho_f(\beta h) < \infty$. By $(2.4)_{q<\infty}$, we obtain

$$\rho_f(6h\chi_E) = \int_E f(x,6h)d\mu \le \frac{\delta 6^q}{\beta^q} \le \int_E f(x,\beta h)d\mu$$

and similarly $\rho_f(h) < \infty$. By the absolute continuity of the integral we may choose $\epsilon_1 > 0$ such that

$$\rho_f(6h\chi_E) < \nu$$

for every measurable set $E$ with $\mu(E) < n\epsilon_1$.

Next choose $j_0 \in \mathbb{N}$ such that $\mu(\bigcup G_j \setminus \{\psi_{j_0} = 1\}) < \frac{1}{2}\epsilon_1$. By (2.4) and absolute continuity of the integral, we can choose $\epsilon_2 > 0$ such that

$$\rho_f(6\gamma\psi_{j_0}\chi_E) < \nu$$

for every measurable set $E$ with $\mu(E) < n\epsilon_2$, where $\gamma := \max\{\gamma_j\}$. Then choose rational numbers $\alpha_1, ..., \alpha_n \in (0, \infty)$ such that $|\gamma_j - \alpha_j| < \nu\gamma_j$ for $j = 1, ..., n$.

Furthermore, for each $j$ we find $l_j$ such that $\mu(G_j \triangle F_{l_j}) < \min\{\frac{1}{2}\epsilon_1, \epsilon_2\}$.

Let $\varphi := \psi_{j_0} \sum_{j=1}^n \alpha_j \chi_{F_{l_j}}$. Then

$$\begin{aligned}
|h - \varphi| &= \left|\sum_{j=1}^n (\gamma_j - \alpha_j)\chi_{G_j}\right| + \left|\sum_{j=1}^n \alpha_j(\chi_{G_j} - \psi_{j_0}\chi_{F_{l_j}})\right| \\
&\leq \left|\sum_{j=1}^n |\gamma_j - \alpha_j|\,\chi_{G_j}\right| + \sum_{j=1}^n (\alpha_j \chi_{G_j \setminus (F_{l_j} \cap \psi_{j_0} = 1\})} + \psi_{j_0}\alpha_j\chi_{F_{l_j} \setminus G_j}) \\
&\leq \nu h + 2\sum_{j=1}^n (\gamma_j \chi_{G_j \setminus (F_{l_j} \cap \psi_{j_0} = 1\})} + \gamma\psi_{j_0}\chi_{F_{l_j} \setminus G_j}).
\end{aligned}$$

Denote $E := \bigcup_j G_j \setminus (F_{l_j} \cap \psi_{j_0} = 1\})$ and $E' := \bigcup_j F_{l_j} \setminus G_j$. Then $\mu(E) \leq \sum_{j=1}^n (\mu(G_j \setminus F_{l_j}) + \mu(G_j \setminus \psi_{j_0} = 1\})) \leq \frac{n}{2}\epsilon_1 + \frac{n}{2}\epsilon_1 = n\epsilon_1$ and $\mu(E') \leq n\epsilon_2$. Taking $f$ of both sides of the previous estimate for $|h - \varphi|$, and integrating over $M$, we find by $(2.4)_{q<\infty}$ that

$$\begin{aligned}
\rho_f(h - \varphi) &\leq \rho_f(\nu h + 2h\chi_E + 2\gamma\psi_{j_0}\chi_{E'}) \\
&\leq \rho_f(3\nu h) + \rho_f(6h\chi_E) + \rho_f(6\gamma\psi_{j_0}\chi_{E'})) \\
&\lesssim \nu\rho_f(h) + 2\nu.
\end{aligned}$$

It follows that $\rho_f(h - \varphi) \to 0$ as $\nu \to 0^+$. Since norm and modular convergence are equivalent (Corollary 3.3.5), this implies the claim. $\square$

## 3.6 Uniform convexity and reflexivity

The section is based on [14] and [15] so, we will study the reflexivity of $L^f$ by means of uniform convexity, since it is well known that the latter implies the former.

**Definition 3.6.1.** We say that $f \in \Phi_c(M, \mu)$ is uniformly convex if for every $\nu > 0$ there exists $d \in (0,1)$ such that

$$f\Big(x, \frac{y+z}{2}\Big) \leq (1-d)\frac{f(x,z)+f(x,y)}{2}$$

for $\mu$-almost every $x \in M$ whenever $y, z \geq 0$ and $|z-y| \geq \nu \max\{|z|, |y|\}$.

Uniformly convex $\Phi$-functions can be very neatly described in terms of equivalent $\Phi$-functions and $(2.2)_{p>1}$.

**Proposition 3.6.2.** *The function $f \in \Phi_W(M, \mu)$ is equivalent to a uniformly convex $\Phi$-function if and only it satisfies $(2.2)_{p>1}$.*

*Proof.* Assume first that $f$ satisfies (2.2) with $p > 1$. By Lemma 2.5.10, there exists $g \in \Phi_c(M, \mu)$ such that $f \simeq g$ and $g^{\frac{1}{p}}$ is convex. The claim follows once we show that $g$ is uniformly convex. Let $\nu \in (0,1)$ and $z - y \geq \nu z$, with $z > y > 0$.
Since $g^{\frac{1}{p}}$ is convex,

$$f\Big(x, \frac{z+y}{2}\Big)^{\frac{1}{p}} \leq \frac{f(x,z)^{\frac{1}{p}} + g(x,y)^{\frac{1}{p}}}{2}$$

Since $y \leq (1-\nu)z$ and $g$ is convex, we find that

$$g(x,y) \leq g(x,(1-\nu)z) \leq (1-\nu)g(x,z)$$

. Therefore, $g(x,y)^{\frac{1}{p}} \leq (1-\nu')g(x,z)^{\frac{1}{p}}$ for some $\nu' > 0$ depending only on $\nu$ and $p$. Since $y \mapsto y^p$ is uniformly convex, we obtain that

$$\Big(\frac{g(x,z)^{\frac{1}{p}} + g(x,y)^{\frac{1}{p}}}{2}\Big)^p \leq (1-d)\frac{f(x,z)+f(x,y)}{2}$$

Combined with the previous estimate, this shows that $g$ is uniformly convex.

Assume now conversely, that $f \simeq g$ and $g$ is uniformly convex. Choose $\nu = \frac{1}{2}$ and $y = 0$ in the definition of uniform convexity:

$$g(x, \frac{z}{2}) \leq \frac{1}{2}(1-d)g(x, z).$$

Divide this equation with $(z/2)^p$ where $p > 1$ is given by $2^{p-1}(1-d) = 1$:

$$\frac{g(x, \frac{z}{2})}{(z/2)^p} \leq 2^{p-1}(1-d)\frac{g(x, z)}{s^p} = \frac{g(x, z)}{z^p}$$

The previous inequality holds for every $z > 0$. If $0 < y < z$, then we can choose $n \in \mathbb{N}$ such that $2^n y \leq z < 2^{n+1} y$. Then by the previous inequality and monotonicity of $g$,

$$\frac{g(x, y)}{y^p} \leq \frac{g(x, 2y)}{(2y)^p} \leq ... \leq \frac{g(x, 2^n y)}{(2^n y)^p} \leq 2^p \frac{g(x, z)}{z^p}.$$

Hence $g$ satisfies (2.2) with $p > 1$. Since this property is invariant under equivalence (Lemma 2.1.11), it holds for $f$ as well. $\square$

**Definition 3.6.3.** A vector space $V$ is uniformly convex if it has a norm $\|\cdot\|$ such that for every $\nu > 0$ there exists $d > 0$ with

$$\|u - v\| \geq \nu \quad \text{or} \quad \|u + v\| \leq 2(1 - d)$$

for all $u, v \in V$ with $\|u\| = \|v\| = 1$.

**Remark 3.6.4.** In the Orlicz case, it is well known that the space $L^f$ is reflexive and uniformly convex if and only if $f$ and $f^*$ are doubling.

**Lemma 3.6.5.** *Let $f \in \Phi_c(M, \mu)$ be uniformly convex. Then for every $\nu > 0$ there exists $d_2 > 0$ such that*

$$f\Big(x, \Big|\frac{z + y}{2}\Big|\Big) \leq (1 - d_2)\frac{f(x, |z|) + f(x, |y|)}{2}$$

*for all $z, y \in \mathbb{R}$ with $|z - y| > \nu \max\{|z|, |y|\}$ and every $x \in M$.*

*Proof.* Fix $\nu \in (0,1)$ and let $d > 0$ be as in Definition 3.6.1. Let $|z-y| > \nu \max\{|z|,|y|\}$. If $||z|-|y|| > \nu \max\{|z|,|y|\}$, then the claim follows by uniform convexity of $f$, $|z+y| \le |z|+|y|$ and the choice $d_2 := d$. So assume in the following $||z|-|y|| \le \nu \max\{|z|,|y|\}$. Since $|z-y| > \nu \max\{|z|,|y|\}$, it follows that $z$ and $y$ have opposite signs, and that

$$\left|\frac{z+y}{2}\right| = \left|\frac{|z|+|y|}{2}\right| \le \frac{\nu}{2}\max\{|z|,|y|\}.$$

Then it follows from convexity that

$$f\Big(x, \left|\frac{z+y}{2}\right|\Big) \le \frac{\nu}{2} f(x, \max\{|z|,|y|\}) \le \nu \frac{f(x,|z|)+f(x,|y|)}{2}.$$

Therefore the claim holds with $d_2 := \min\{d, 1-\nu\}$. $\square$

**Lemma 3.6.6.** *Let $f \in \Phi_c(M,\mu)$ be uniformly convex. Then for every $\nu > 0$ there exists $d > 0$ such that*

$$\rho_f\Big(\frac{\varphi-\psi}{2}\Big) < \nu \frac{\rho_f(\varphi)+\rho_f(\psi)}{2} \quad \text{or} \quad \rho_f\Big(\frac{\varphi+\psi}{2}\Big) \le (1-d)\frac{\rho_f(\varphi)+\rho_f(\psi)}{2}$$

*for all $\varphi, \psi \in L^0(M,\mu)$.*

*Proof.* Fix $\nu > 0$. Let $d_2 > 0$ be as in Lemma 3.6.5 for $\nu/4$. There is nothing to show if $\rho_f(\varphi) = \infty$ or $\rho_f(\psi) = \infty$. So in the following let $\rho_f(\varphi), \rho_f(\psi) < \infty$, which imply by convexity that $\rho_f\big(\frac{\varphi+\psi}{2}\big), \rho_f\big(\frac{\varphi-\psi}{2}\big) < \infty$.

Assume that $\rho_f\big(\frac{\varphi-\psi}{2}\big) \ge \nu \frac{\rho_f(\varphi)+\rho_f(\psi)}{2}$. We show that the second inequality in the statement of the lemma holds with $d = \frac{d_2 \nu}{2}$. Define

$$F := \Big\{x \in M : |\varphi(x)-\psi(x)| > \frac{\nu}{2}\max\{|\varphi(x)|,|\psi(x)|\}\Big\}.$$

By (2.1) with $p = 1$, for $\mu$-almost all $x \in M \setminus F$, we have

$$\begin{aligned} f\Big(x, \frac{|\varphi(x)-\psi(x)|}{2}\Big) &\le \frac{\nu}{4} f(x, \max\{|\varphi(x)|,|\psi(x)|\}) \\ &\le \frac{\nu}{2}\frac{f(x,|\varphi(x)|)+f(x,|\psi(x)|)}{2}. \end{aligned}$$

It follows that

$$\rho_f\Big(\chi_{M\setminus F}\frac{\varphi-\psi}{2}\Big)\le\frac{\nu}{2}\frac{\rho_f(\chi_{M\setminus F}\varphi)+\rho_f(\chi_{M\setminus F}\psi)}{2}\le\frac{\nu}{2}\frac{\rho_f(\varphi)+\rho_f(\psi)}{2}.$$

This and $\rho_f\Big(\frac{\varphi-\psi}{2}\Big)\ge\nu\frac{\rho_f(\varphi)+\rho_f(\psi)}{2}$ imply

$$\rho_f\Big(\chi_F\frac{\varphi-\psi}{2}\Big)=\rho_f\Big(\frac{\varphi-\psi}{2}\Big)-\rho_f\Big(\chi_{M\setminus F}\frac{\varphi-\psi}{2}\Big)\ge\frac{\nu}{2}\frac{\rho_f(\varphi)+\rho_f(\psi)}{2}. \qquad (3.4)$$

On the other hand it follows by the definition of $F$ and the choice of $d_2$ in Lemma 3.6.5 that

$$\rho_f\Big(\chi_F\frac{\varphi+\psi}{2}\Big)\le(1-d_2)\frac{\rho_f(\chi_F\varphi)+\rho_f(\chi_F\psi)}{2}. \qquad (3.5)$$

Since $\frac{1}{2}(f(x,\varphi)+f(x,\psi))-f(x,\frac{\varphi+\psi}{2})\ge 0$ on $M\setminus F$ (by convexity), we obtain

$$\frac{\rho_f(\varphi)+\rho_f(\psi)}{2}-\rho_f\Big(\frac{\varphi+\psi}{2}\Big)\ge\frac{\rho_f(\chi_F\varphi)+\rho_f(\chi_F\psi)}{2}-\rho_f\Big(\chi_F\frac{\varphi+\psi}{2}\Big).$$

This, (3.5), convexity and (3.4) imply

$$\begin{aligned}\frac{\rho_f(\varphi)+\rho_f(\psi)}{2}-\rho_f\Big(\frac{\varphi+\psi}{2}\Big) &\ge d_2\frac{\rho_f(\chi_F\varphi)+\rho_f(\chi_F\psi)}{2}\\ &\ge d_2\rho_f\Big(\chi_F\frac{\varphi-\psi}{2}\Big)\ge\frac{d_2\nu}{2}\frac{\rho_f(\varphi)+\rho_f(\psi)}{2}.\end{aligned}$$

□

**Theorem 3.6.7.** *Let $f\in\Phi_c(M,\mu)$ be uniformly convex and satisfy $(2.4)_{q<\infty}$. Then $L^f(M,\mu)$ is uniformly convex with norm $\|\cdot\|_f$.*

*In particular, if $f$ satisfies $(2.2)_{p>1}$ and $(2.4)_{q<\infty}$, then $L^f(M,\mu)$ is uniformly convex and reflexive.*

*Proof.* Fix $\nu>0$. Let $\varphi,\psi\in L^f(M,\mu)$ with $\|\varphi\|_f,\|\psi\|_f\le 1$ and $\|\varphi-\psi\|_f>\nu$. Then $\big\|\frac{\varphi-\psi}{2}\big\|_f>\frac{\nu}{2}$ and by Lemma 3.2.13 there exists $\beta=\beta(\nu)>0$ such that $\rho_f(\frac{\varphi-\psi}{2})>\beta$. By the unit ball property (Lemma 3.2.4) we have $\rho_f(\varphi),\rho_f(\psi)\le 1$, so $\rho_f(\frac{\varphi-\psi}{2})>\beta\frac{\rho_f(\varphi)+\rho_f(\psi)}{2}$. By Lemma 3.6.6, there exists $\gamma=\gamma(\alpha)>0$ such that $\rho_f(\frac{\varphi-\psi}{2})\le(1-\gamma)\frac{\rho_f(\varphi)+\rho_f(\psi\psi)}{2}\le 1-\gamma$. Since $f$ is a convex $\Phi$-function, it satisfies

$(2.1)_{p>1}$ and by Lemma 2.2.6 $(2.4)_{q<\infty}$ implies $(2.3)_{q<\infty}$. Now Lemma 3.2.13 implies the existence of $d = d(\gamma) > 0$ with $\left\|\frac{\varphi-\psi}{2}\right\|_f \leq 1-d$. This proves the uniform convexity of the norm $\|\cdot\|_f$. □

**Remark 3.6.8.** If $f$ satisfies$(2.2)_{p>1}$ and $(2.4)_{q<\infty}$, then it is equivalent to some $g \in \Phi_c(M,\mu)$ which is uniformly convex and satisfies $(2.4)_{q<\infty}$, by Proposition 3.6.2. Hence by the first part $L^g$ is uniformly convex and by Proposition 1.3.5 ( [?] )it is reflexive. Since $L^f = L^g$ by Proposition 3.2.6, the same holds for $L^f$.

The conditions $(2.2)_{p>1}$ and $(2.4)_{q<\infty}$ can be generalized further.

**Corollary 3.6.9.** *Let $f \in \Phi_W(M,\mu)$. If $f$ satisfies $\Delta_2^W$ and $\nabla_2^W$, then $L^f(M,\mu)$ is uniformly convex and reflexive.*

*Proof.* By Theorem 2.5.21, Lemma 2.2.6 and Corollary 2.4.12, there exists $g \in \Phi_W(M,\mu)$ which satisfies $(2.4)_{q<\infty}$, $(2.2)_{p>1}$ and $f \sim g$. Hence by Theorem 3.6.7, $L^g$ is uniformly convex and reflexive. Since $f \sim g$, Corollary 3.2.10 and Proposition 3.2.6 imply that $L^f = L^g$, and hence we have proved that $L^f$ is uniformly convex and reflexive. □

## 3.7 Density of smooth functions and the weight condition $(C0)$

**Definition 3.7.1.** We say that $f \in \Phi_W(M,\mu)$ satisfies $(C0)$, if there exists a constant $\lambda \in (0,1]$ such that $\lambda \leq f^{-1}(x,1) \leq \frac{1}{\lambda}$ for $\mu$-almost every $x \in M$.

Equivalently, this means that there exists $\lambda \in (0,1]$ such that $f(x,\lambda) \leq 1 \leq f(x,1/\lambda)$ for $\mu$-almost every $x \in M$ (cf. Corollary 3.7.5).

**Example 3.7.2.** Let $f(x,y) = \frac{1}{p(x)} y^{p(x)}$ where $p : M \to [1,\infty)$ is measurable, and $g(x,y) = y^p + \psi(x)y^q$ where $1 \leq p < q < \infty$ and $\psi : M \to [0,\infty)$ is measurable. Then $f, g \in \Phi_s(M,\mu)$. Since $f^{-1}(x,y) = (p(x)y)^{1/p(x)}$, we see that $f$ satisfies $(C0)$

(without assumptions for $p$). By Corollary 3.7.5, $g$ satisfies $(C0)$ if and only if $\psi \in L^\infty(M,\mu)$.

**Remark 3.7.3.** By Theorem 2.3.6 we have $f \simeq g$ if and only if $f^{-1} \approx g^{-1}$ and thus $(C0)$ is invariant under equivalence of weak $\Phi$-functions.

Note that if $f$ satisfies $(C0)$, then it is not necessary that $f(x,1) \approx 1$. For instance, for $f_\infty(y) = \infty\chi_{(1,\infty)}$, we have $f_\infty^{-1}(x,1) = 1$ whereas $f_\infty$ only takes values 0 and $\infty$. However there exists an equivalent weak $\Phi$-function for which also $f(x,1)$ is controlled.

**Lemma 3.7.4.** *Let $f \in \Phi_W(M,\mu)$ satisfy $(C0)$. Then there exists $g \in \Phi_s(M,\mu)$ with $f \simeq g$ and $g(x,1) = g^{-1}(x,1) = 1$ for $\mu$-almost every $x \in M$.*

*Proof.* By Theorem 2.5.11 there exists $g_1 \in \Phi_s(M,\mu)$ with $f \simeq g_1$. Since $f$ satisfies $(C0)$ so does $g_1$. We set

$$g_2(x,t) := g_1(x, g_1^{-1}(x,1)t).$$

By Lemma 2.5.13, $x \mapsto g_1^{-1}(x,y)$ is measurable. Thus $x \mapsto g_2(x,y)$ is measurable for fixed $y$ by the definition of generalized $\Phi$-prefunction. Then $f_2$ satisfies the measurability condition of $\Phi_s(M,\mu)$ by Theorem 2.5.5.

We show that $g_2 \in \Phi_s(M,\mu)$. The function $g_2$ is increasing since $g_1$ is increasing. By Lemma 2.5.8, $g_2$ is a $\Phi$-prefunction. Since $y \mapsto g_1(x,y)$ is convex we obtain that

$$\begin{aligned} g_2(x, ay + (1-a)z) &= g_1(x, ag_1^{-1}(x,1)y + (1-a)g_1^{-1}(x,1)z) \\ &= ag_1(x, g_1^{-1}(x,1)y) + (1-a)g_1(x, g_1^{-1}(x,1)z) \\ &= ag_2(x,y) + (1-a)g_2(x,y) \end{aligned}$$

for every $a \in [0,1]$ and $y, z \geq 0$. Since $y \mapsto g_1(x,y)$ is continuous into the compactification $[0,\infty]$ for $\mu$-almost every $x$ and $g_1^{-1}(x,1)$ is independent of $y$, we obtain that

$y \mapsto g_2(x, y)$ is continuous for $\mu$-almost every $x$.

Since $g_1$ satisfies $(C0)$, we have $g_1 \simeq g_2$. By Lemma 2.3.3,

$$g_2(x, 1) = g_1(x, g_1^{-1}(x, 1)) = 1$$

for $\mu$-almost every $x \in M$. By Corollary 2.3.5, this implies $g_2^{-1}(x, 1) = 1$ for $\mu$-almost every $x \in M$. □

**Corollary 3.7.5.** *Let $f \in \Phi_W(M, \mu)$ . Then $f$ satisfies $(C0)$ if and only if there exists $\lambda \in (0, 1]$ such that $f(x, \lambda) \le 1 \le f(x, 1/\lambda)$ for $\mu$-almost every $x \in M$.*

*Proof.* Assume first that $(C0)$ holds. By Lemma 3.7.4, there exists $g \in \Phi_s(M, \mu)$ with $g(x, 1) = 1$ and $g \simeq f$. This implies the inequality.

Assume then that the inequality holds. By the definition of $f^{-1}$, the inequality $f(x, \frac{1}{\lambda}) \ge 1$ yields $f^{-1}(x, 1) \le \frac{1}{\lambda}$. By (2.2) with $p = 1$ and $f(x, \lambda) \le 1$, we obtain

$$\frac{f(x, \lambda/(2\delta))}{\lambda/(2\delta)} \le \delta \frac{f(x, \lambda)}{\lambda} \le \frac{\delta}{\lambda}$$

so that $f(x, \frac{\lambda}{2\delta}) \le \frac{1}{2}$. This yields that $f^{-1}(x, 1) \ge \frac{\lambda}{2\delta}$. □

**Corollary 3.7.6.** *Let $f \in \Phi_W(M, \mu)$ .If there exists $T > 0$ such that $f(x, T) \approx 1$, then $f$ satisfies $(C0)$.*

*Proof.* Let $s \le f(x, T) \le t$. We may assume that $s \in (0, 1]$ and $t \ge 1$. By (2.2) with $p = 1$ we obtain

$$\frac{f(x, T/(t\delta))}{T/(t\delta)} \le \delta \frac{f(x, T)}{T} \le \frac{t\delta}{T} \quad \text{and} \quad \frac{s}{T} \le \frac{f(x, T)}{T} \le \delta \frac{f(x, \delta T/s)}{\delta T/s}$$

Thus $f(x, T/(t\delta)) \le 1$ and $f(x, \delta T/s) \ge 1$ and the claim follows from Corollary 3.7.5. □

**Lemma 3.7.7.** *If $f \in \Phi_W(M, \mu)$ satisfies $(C0)$, then $f^*$ satisfies $(C0)$.*

*Proof.* By Theorem 2.4.9 and $(C0)$ of $f$ we obtain

$$(f^*)^{-1}(x,1) \approx \frac{1}{f^{-1}(x,1)} \leq \frac{1}{\lambda} \quad \text{and} \quad (f^*)^{-1}(x,1) \approx \frac{1}{f^{-1}(x,1)} \geq \lambda$$

for $\mu$-almost every $x \in M$. $\square$

**Remark 3.7.8.** We next characterize the embeddings of the sum and the intersection of generalized Orlicz spaces.

Let us introduce the usual notation. Recall that for two normed spaces $V$ and $W$ ( which are both subsets of a vector spaces $Z$) we equip the intersection $V \cap W$ and the sum $V + W := \{\varphi + \psi : \varphi \in V, \psi \in W\}$ with the norms

$$\|\phi\|_{V\cap W} := \max\{\|\phi\|_V, \|\phi\|_W\} \quad \text{and} \quad \|\phi\|_{V+W} := \inf_{\phi=\varphi+\psi, \varphi\in V, \psi\in W}(\|\varphi\|_V + \|\psi\|_W).$$

In the next lemma we use the convention that every $f \in \Phi_W(M,\mu)$ satisfies (2.4) with $q = \infty$ and constant 1. Hence we always have $L^1 \cap L^\infty \hookrightarrow L^f \hookrightarrow L^1 + L^\infty$. Notice that the first embedding requires only $f(x, \frac{1}{\lambda}) \geq 1$ whereas the second requires only $f(x,\lambda) \leq 1$.

**Lemma 3.7.9.** *Let $f \in \Phi_W(M,\mu)$ satisfy $(C0)$, (2.2) and (2.2), $p \in [1,\infty)$ and $q \in [1,\infty]$. Then*

$$L^p(M,\mu) \cap L^q(M,\mu) \hookrightarrow L^f(M,\mu) \hookrightarrow L^p(M,\mu) + L^q(M,\mu)$$

*and the embedding constants depend only on $(C0)$, (2.2) and (2.4).*

*Proof.* Let $\lambda \in (0,1]$ be the constant from Corollary 3.7.5 and a be the maximum of the constant from $(2.4)_{q<\infty}$ and $(2.2)_{p>1}$.

Let us first study $L^f(M,\mu) \hookrightarrow L^p(M,\mu) + L^q(M,\mu)$. Let $h \in L^f$ with $\|h\|_f < 1$ so that $\rho_f(h) \leq 1$ by the unit ball property (Lemma 3.2.4). We may assume that $h \geq 0$ since otherwise we may study $|h|$. We assume that $p, q \in [1,\infty)$. The cases $q = \infty$ follows by simple modifications.

Define $h_1 := h\chi_{\{0\le h\le \frac{1}{\lambda}\}}$ and $h_2 := h\chi_{\{h>\frac{1}{\lambda}\}}$. By Corollary 3.7.5, (2.2) and (2.4), we have

$$\lambda^p \le \frac{f(x,1/\lambda)}{1/\lambda^p} \le \delta\frac{f(x,y)}{y^p} \quad \text{and} \quad \lambda^q \le \frac{f(x,1/\lambda)}{1/\lambda^q} \le \delta\frac{f(x,z)}{z^p}$$

for $z \le \frac{1}{\lambda} \le y$. Using these we obtain that

$$\frac{\lambda^p}{\delta}\int_M h_1^p dx \le \int_M f(x,h_1)dx \le 1 \quad \text{and} \quad \frac{\lambda^q}{\delta}\int_M h_2^q dx \le \int_M f(x,h_2)dx \le 1.$$

Thus we have $\|h\|_{L^p+L^q} \le \frac{\delta^{1/p}}{\lambda} + \frac{\delta^{1/g}}{\lambda}$ and claims follows by the scaling argument,i.e. by using this result for $h/(\|h\|_f + \nu)$ and then letting $\nu \to 0^+$.

Then we consider the embedding $L^p(M,\mu)\cap L^q(M,\mu) \hookrightarrow L^f(M,\mu)$ and assume that $\|h\|_{L^p\cap L^q} \le \frac{1}{\delta}\min\{\lambda^p,\lambda^q\}$.

Define $h_1 := h\chi_{\{0\le h\le\lambda\}}$ and $h_2 := h\chi_{\{h>\lambda\}}$. By Corollary 3.7.5, (2.2) and (2.4) we have

$$\frac{f(x,y)}{y^p} \le \delta\frac{f(x,\lambda)}{\lambda^p} \le \frac{\delta}{\lambda^p} \quad \text{and} \quad \frac{f(x,z)}{z^q} \le \delta\frac{f(x,\lambda)}{\lambda^q} \le \frac{\delta}{\lambda^q}$$

for $y \le \beta \le z$. Using these we obtain that

$$\int_M f(x,h_1)dx \le \frac{\delta}{\lambda^p}\int_M h_1^p dx \le 1 \quad \text{and} \quad \int_M f(x,h_2)dx \le \frac{\delta}{\lambda^q}\int_M h_2^q dx \le 1$$

Thus we have $\|h\|_f \le 1$ and claims follows by the scaling argument. □

Next we give an example which shows that assumption $(C0)$ is not redundant in Lemma 3.7.9.

**Example 3.7.10.** Let $f(x,y) = y^2|x|^2$. Then $f \in \Phi_s(\mathbb{R})$ satisfies (2.1) with $p = 2$ and (2.3) $q = 2$ but not $(C0)$.

First we show that $L^f(\mathbb{R}) \hookrightarrow L^2(\mathbb{R}) + L^\infty(\mathbb{R})$ does not hold.

For that let $h(x) := \frac{1}{\sqrt{|x|}}\chi_{(-1,1)}$. Then

$$\int_{\mathbb{R}} f(x,h)dx = \int_{(-1,1)} \frac{1}{|x|}|x|^2\,dx \approx 1$$

and thus $h \in L^f(\mathbb{R})$. Let $h_1 \in L^2(\mathbb{R})$ and $h^2 \in L^\infty(\mathbb{R})$ be such that $h = h_1 + h_2$. Then we find $t > 0$ such that $h(x) = h_1(x)$ for all $x \in (-t, t)$ and obtain

$$\int_{(-t,t)} h_1^2 dx = \int_{(-t,t)} \frac{1}{|x|} dx = \infty$$

and thus such a decomposition does not exist.

Next we show that $L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \hookrightarrow L^f(\mathbb{R})$ does not hold.

Let $\psi(x) := \min\{1, |x|^{-5/4}\}$. A short calculation shows that $\psi \in L^1(\mathbb{R})$. Since $0 < \psi \leq 1$ this yields that $\psi \in L^2(\mathbb{R})$. On the other hand for every $\lambda > 0$ we have

$$\int_{\mathbb{R}} f(x, \lambda\psi) dx \geq \int_{\mathbb{R}\setminus(-1,1)} (\lambda\, |x|^{-5/4})^2\, |x|^2\, dx \approx \lambda^2 \int_1^\infty |x|^{\frac{-1}{2}}\, dx = \infty$$

and thus $\psi \notin L^f(\mathbb{R})$.

When the set $M$ has finite measure, the previous result simplifies and we get the following corollaries.

**Corollary 3.7.11.** *Let $M$ have finite measure and let $f \in \Phi_W(M, \mu)$ satisfy $(C0)$ and* (2.2)*. Then $L^f(M, \mu) \hookrightarrow L^p(M, \mu)$ and there exists $\lambda$ such that*

$$\int_M |h|^p\, d\mu \lesssim \int_M f(x, |h|) d\mu + \mu\Big(\{0 < |h| < \frac{1}{\lambda}\}\Big).$$

*Proof.* Let $\lambda \in (0, 1]$ be from Corollary 3.7.5. Then by (2.2) and $(C0)$

$$\delta \frac{f(x, y)}{y^p} \geq \frac{f(x, 1/\lambda)}{1/\lambda^p} \geq \lambda^p$$

for all $y \geq \frac{1}{\lambda}$,so that $\delta f(x, y) \geq \lambda^p y^p$.Thus

$$\lambda^p y^p \leq \delta f(x, y) + \chi_{\{0<y<\frac{1}{\lambda}\}},$$

which yields the claim for the modulars when we set $y := |f(x)|$ and integrate over $x \in M$.

The embedding follows from Lemma 3.7.9 since $L^p(M, \mu) + L^\infty(M, \mu) = L^p(M, \mu)$. Similarly, since $L^\infty(M, \mu) = L^p(M, \mu) \cap L^\infty(M, \mu)$ when $\mu(M) < \infty$, Lemma 3.7.9 also implies the following result. ∎

**Corollary 3.7.12.** *Let $M$ have finite measure and let $f \in \Phi_W(M,\mu)$ satisfy $(C0)$. Then $L^\infty(M,\mu) \hookrightarrow L^f(M,\mu)$.*

The next example shows that the previous result need not hold if $f$ does not satisfy $(C0)$.

**Example 3.7.13.** Let $(0,1) \subset \mathbb{R}$ and $f(x,y) := \frac{y}{|x|}$. Then $f \in \Phi_s(0,1)$ and $f$ does not satisfy $(C0)$. Let $h \equiv 2 \in L^\infty(M,\mu)$. We obtain

$$\int_0^1 f(x, \beta\,|h|)dx \leq \int_0^1 \frac{2\beta}{x} dx = \infty$$

for all $\beta > 0$ and hence $h \notin L^f(0,1)$.

Next we show that $L^f(M,\mu)$ is a Banach function space provided that $f$ satisfies (C0).

**Definition 3.7.14.** A normed space $(V, \|\cdot\|_V)$ with $V \subset L^0(M,\mu)$ is called a Banach function space, if

**(a)** $(V, \|\cdot\|_V)$ is circular, solid and satisfies the Fatou property ( see remark 3.3.9).

**(b)** If $\mu(A) < \infty$, then $\chi_A \in V$.

**(c)** If $\mu(A) < \infty$, then $\chi_A \in V'$, i.e. $\displaystyle\int_A |h|\,d\mu \leq c(A)\,\|h\|_V$ for all $h \in V$.

We have seen that the properties in (a) always hold. The next theorem shows that $(C0)$ implies the other two.

**Theorem 3.7.15.** *Let $f \in \Phi_W(M,\mu)$ satisfy $(C0)$. Then $L^f(M,\mu)$ is a Banach function space.*

*Proof.* Circularity and solidity hold by (3.1) and (3.2). The Fatou property holds by Lemma 3.3.10. So we only check (b) and (c).

For (b), let $\mu(A) < \infty$. By Corollary 3.7.5 there exists $\lambda > 0$ such that $f(x, \lambda) \leq 1$ and hence

$$\int_M f(x, \lambda\chi_A) d\mu = \int_A f(x, \lambda) d\mu \leq \mu(A)$$

so that $\chi_A \in L^f(M, \mu)$. By Theorem 3.4.6, $(L^f)' = L^{f^*}$, and by Lemma 3.7.7, $f^*$ satisfies $(C0)$. Therefore (c) follows from (b) of $f^*$. $\square$

**Remark 3.7.16.** At the end of this section we give some basic density results in $\Omega \subset \mathbb{R}^n$ with the Lebesgue measure.

Note that the assumption $(2.4)_{q<\infty}$ is not redundant, since the results do not hold in $L^\infty$. Let us denote by $L_0^f(\Omega)$ the set of functions from $L^f(\Omega)$ whose support is compactly in $\Omega$.

**Lemma 3.7.17.** *Let $f \in \Phi_W(\Omega)$ satisfy $(2.4)_{q<\infty}$. Then $L_0^f(\Omega)$ is dense in $L^f(\Omega)$.*

*Proof.* Let $h \in L^f(\Omega)$ and let $\beta > 0$ be such that $\displaystyle\int_\Omega f(x, \beta h) dx < \infty$.
Define $h_j := h\chi_{B(0,j)}$. Then

$$\int_\Omega f(x, \beta\,|h - h_j|) dx = \int_{\Omega \setminus B(0,j)} f(x, \beta\,|h|) dx \to 0$$

as $j \to \infty$ by the absolute continuity of the integral. Hence $(h_i)$ is modular convergent to $h$ and thus the claim follows by Corollary 3.3.5. $\square$

If we also have $(C0)$, then a stronger result holds.

**Theorem 3.7.18.** *If $f \in \Phi_W(\Omega)$ satisfies $(C0)$ and $(2.4)_{q<\infty}$, then $C_0^\infty(\Omega)$ is dense in $L^f(\Omega)$.*

*Proof.* Let $f$ satisfy (2.4) and note that simple functions are dense in $L^f(\Omega)$ by Proposition 3.5.2. Since every simple function belongs to $L^1(\Omega) \cap L^q(\Omega)$, it can be approximated by a sequence of $C_0^\infty(\Omega)$ functions in the same space.
By Lemma 3.7.9, $L^1(\Omega) \cap L^q(\Omega) \hookrightarrow L^f(\Omega)$ so the claim follows. $\square$

# Conclusion

In this memory, we presented a study on generalized Orlicz spaces and and their basic properties. This work raises a number of questions that deserve to be addressed. subsequently melted. For example, it would be wise to think in perspective of following:

- Does the next theorem hold for all $\Phi$-prefunctions ?

  Assume that $f, g \in \Phi_W$. Then $f \simeq g$ if and only if $f^{-1} \approx g^{-1}$.

- Is the next lemma true if we assume $(2.4)_{q<\infty}$ instead of $(2.3)_{q<\infty}$ ?

  Let $f \in \Phi_W(M, \mu)$ satisfy $(2.3)_{q<\infty}$. Let $h_i, r_i \in L^f(\mathbb{R}^n)$ for $i = 1, 2, ...$ with $(\rho_f(h_i))_{i=1}^{\infty}$ bounded. If $\rho_f(h_i - r_i) \to 0$ as $i \to \infty$, then

$$|\rho_f(h_i) - \rho_f(r_i)| \to 0 \quad \text{as} \quad i \to \infty.$$

- Does the next theorem hold without the assumption $(C0)$ ?

  If $f \in \Phi_W(\Omega)$ satisfies $(C0)$ and $(2.4)_{q<\infty}$, then $C_0^\infty(\Omega)$ is dense in $L^f(\Omega)$.

For this reason we think that the memory will be useful also for researchers interested in the Orlicz case only.

# Bibliography


[1] R. Adams. *Sobolev Spaces*. Pure and Applied Mathematics, Vol. **65** (Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975)

[2] S.C. Arora, G. Datt and S. Verma; *Multiplication and Composition Operators on Orlicz-Lorentz Spaces.* Int. Journal of Math. Analysis, Vol. 1, no. **25**, (2007), 1227-1234.

[3] H. Brezis; *Functional Analysis , Sobolev spaces and partial Differential Equations*, Springer, New York, Dordrecht Heidelberg, London, (2010).

[4] Z. W. Birnbaum, and W. Orlicz; *Über die Verallgemeinerung des Begriffes der zueinander Konjugierten Potenzen.* Studia Mathematica, **3** (1931), 1-67.

[5] V.V. Chistyakov, *Metric Modular Spaces.* Theory and Applications. Springer Briefs in Mathematics, Springer, Cham, (2015).

[6] U. Cruz and P. Hästö; *Extrapolation and interpolation in generalized Orlicz spaces.* Trans. Am. Math. Soc. **370**(6), (2018), 4323-4349.

[7] R.E. Castillo and H. Rafeiro; *An Introductory Course in Lebesgue Spaces.* CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC ( Springer, Cham, 2016)

[8] D. Cruz-Uribe and A. Fiorenza; *Variable Lebesgue Spaces: Foundations and Harmonic Analysis*(Birkhäuser, Basel, 2013)

[9] L. Diening; *Maximal function on Orlicz-Musielak spaces and generalized Lebesgue spaces.* Bull. Sci. Math. **129**,(2005), 657-700.

[10] L. Diening, P. Harjulehto, P. Hästö and M.Ruzicka; *Lebesgue and Sobolev spaces with Variable Exponents.* volume **2017** of Lecture Notes in Mathematics, Springer, Heidelberg, (2011).

[11] D.E. Edmunds and J. Lang, O. Méndez; *Differential Operators on Spaces of Variable Integrability.* World Scientific Publishing, Hackensack, NJ, (2014).

[12] L. Esposito, F. Leonetti and G. Mingione; *Sharp regularity for functionals with (p, q) growth.* J. Differ. Equ. **204**(1),(2004),5-55.

[13] A. I. Ghehioueche; *On Lebesgue and Sobolev Spaces with Variable Exponents.* Master memory, Functional Anal, Dep.Maths, M'sila University, Id:**475**, (2019), 1-50.

[14] P. Harjulehto, P. Hästö; *Orlicz Spaces and Generalized Orlicz Spaces.* Lecture Notes in Mathematics **2236**, Springer Nature Switzerland AG. (2019).

[15] P. Harjulehto and P. Hästö; *Uniform convexity and associate spaces.* Czech. Math. J. **68**(143)(4), (2018), 1011-1020.

[16] R. Kumar; *Composition operators on Orlicz spaces.* Integral Equations and operators theory, Vol. 29, (1997), 17-22.

[17] M.A. Khamsi and W.M. Kozlowski; *Fixed Point Theory in Modular Function Spaces.* (Birkhäuser/Springer, Cham, 2015). With a foreword by W. A. Kirk.

[18] M. A. Krasnosel'skii and Ja. B. Rutickii; *Convex Functions and Orlicz Spaces.* Noordhoff, Ltd.; 1st edition, (1961).

[19] J. Musielak; *Orlicz spaces and modular spaces.* Lecture Notes in Mathematics, **1034**. Springer, Berlin, (1983).

[20] J. Musielak and W. Orlicz; *On modular spaces.* Studia Math. **18**,(1959),49-65.

[21] H. Nakano; *Modulared Semi-Ordered Linear Spaces.* Maruzen Co. Ltd., Tokyo,(1950).

[22] C.P. Niculescu and L.-E. Persson; *Convex Functions and Their Applications* volume **23** of CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC ,Springer, New York.A contemporary approach.(2006)

[23] W. Orlicz; *Über konjugierte Exponentenfolgen.* Studia Math.**3**,(1931),200-211.

[24] W. Rudin; *Functional Analysis, 2nd edn.* (McGraw-Hill Book Co., New York, 1991)

[25] K. Raj and V. Khosla; *Weighted Composition Operators between Spaces of Orlicz-Functions.* International Journal of Algebra, Vol. 3, no.**7**, (2009), 315–324.

[26] V.D. Rãdulescu and D.D. Repovs; *Partial Differential Equations with Variable Exponents.* Monographs and Research Notes in Mathematics (CRC Press, Boca Raton, FL, 2015). Variational methods and qualitative analysis.

[27] M.M. Rao and Z.D. Ren; *Theory of Orlicz spaces.* volume **146** of Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker Inc., New York, (1991).