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Theme

Finite Differences of Fractional Partial Differential Equations

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up to university professors.

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Dedicating

To the one who drenched the empty cup to give me a drop of love

To whom I tried to give us a moment of happiness To those who harvest thorns from my path to pave the way for me to learn

To the big heart (my dear father)

To that which lit life with its light

To the symbol of love and healing balm (my beloved mother)

To the pure, tender hearts and innocent souls to the winds of my life (my brothers)

Now the sails are opened and the anchor is raised, and the ship is launched into a wide sea

Darkness is the sea of life, and in this darkness only the lamp of memories, memories

Far brotherhood to those I loved and loved (my girlfriends)

To those I did not write in my diary while they are present in my memory To them all I dedicate this Humble work.

Contents

Introduction

The theory of fractional derivation is an almost old subject than classical calculus such as we know it today, its origins date back to the end of the 17^{th} century, the time where Newton and Leibniz developed the foundations of differential and integral calculus. In particular, Leibniz introduced the symbol $\frac{d^n f}{dt^n}$ for denotes the n^{th} derivative of a function f. When he announced in a letter to the Hospital (apparently with the implicit assumption that $2 \in N$), The hospital replied: What does $\frac{d^n f}{dt^n}$ mean? if $n = 1/2$?.

This letter from Hpital, written in 1695, is now accepted as the first incident of what we call the fractional shunt, and the fact that Hpital asked for $n = 1/2$, i.e. a fraction (rational number) actually gave rise to the name of this part mathematics.

Fractional differential equations, also known as extraordinary differential equations, are a generalization of differential equations through the application of fractional calculus.

However, analytic solutions of the fractional partial differential equations either do not exist or involve special functions, such as the Fox (H-function) function (Mathai and Saxena 1978) and the Mittag-Leffter function (Podlubny 1998) which are diffcult to evaluate.

Consequently, numerical techniques are required to find the solution of fractional partial differential equations.

This thesis is broken down into three chapters as follows: in the first chapter, which is divided into four sections. In the first section,we will be devoted to a some preliminary concepts will be introduced as the Euler Gamma function, Beta and Mittag-Leffler functions. In the second section, we are interested in elementary defenitions and basic notions relating to fractional calculus: the fractional integrals, Riemann-Liouville, Caputo and Grnwald-Letnikov fractional Derivatives, we also talked about some of their properties and the relationship between them. In the last section of this chapter, will be devoted to Partial Fractional Derivatives.

In the second chapter, we will have to study the numerical ways of Approximations to Riemann-Liouville Derivatives using serval ways of them the Grünwald-Letnikov, L1, L2 and L2C approximation.

In the third chapter of our work, which is divided into three sections. In the first section, we investigate the finite difference methods for the time-fractional equation in one spatial dimension. In the second section, we construct the finite difference methods for the space-fractional equations in one spatial dimension. In the last section of this chapter, we derive the finite difference methods for time-space fractional equations in one space dimension .

CHAPTER 1

BASIC CONCEPTS AND ELEMENTS OF FRACTIONAL CALCULUS

This chapter will be devoted to the primary definitions and basic concepts related to fractional calculus such as the Euler gamma, Beta and Mittag-Leffler functions. In addition to that, it will also present other elements of functional analysis, such as the fractional integration, fractional derivation, relative definitions of operators of fractional order, among others, which will all be at the core of this work.

1.1 Historical Overview

Fractional calculus is not a new topic, in reality it has almost the same history as that of classical calculus. It can be dated back to the Leibniz letter to L'Hpital, dated 30 September 1695, in which the meaning of the one-half order derivative was first discussed with some remarks about its possibility.

Following L'Hspital's and Leibniz's first inquisition, fractional calculus was primarily a study reserved for the best minds in mathematics. Fourier, Euler and Laplace are among the many who tackled the fractional calculus and the mathematical consequences [25]. Several mathematicians used their own notation and methodology to introduce definitions that fit the concept of an integral or derivative non-integral order. The most famous of these definitions of fractional calculus are Riemann-Liouville, Caputo and Grnwald-Letnikov definitions.

Most of the mathematical theory applicable to the study of fractional calculus was developed prior to the turn of the twentieth century. However, it is only during the last century that the most intriguing advances in engineering and scientific application have been achieved. Mathematics had in some cases to change in order to meet the requirements of physical reality.

1.2 Special Functions of Fractional Calculus

In this section, we present the Euler gamma, Beta and Mittag-Leffler functions. These functions play an important role in the theory of fractional calculus.

1.2.1 Euler Gamma Function

The simplest interpretation of the Euler gamma function is simply the generalization of the factorial for all real numbers. The definition of the Euler gamma function is given as follows:

Definition 1. *The Euler gamma function is defined by the so-called Euler integral of the second kind and is given by:*

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt, \tag{1.1}$$

where $t^{z-1} = e^{(z-1)ln(t)}$. *This integral is converge for all complex* $z, (Re(z) > 0)$, *with* $\Gamma(1) = 1$, $\Gamma(0^+) = +\infty$, $\Gamma(z)$ *is a monotonous and strictly decreasing function for* $0 < z \leq 1$.

Property 1. *An important property of the Euler gamma function* $\Gamma(z)$ *is the following recurrence relation:*

$$\Gamma(z+1) = z\Gamma(z), Re(z) > 0. \tag{1.2}$$

When we can demonstrate it by an integration by parts, as follows:

$$\Gamma(z+1) = \int_0^{+\infty} t^z e^{-t} dt = [-t^z e^{-t}]_0^{+\infty} + z \int_0^{+\infty} t^{z-1} e^{-t} dt = z\Gamma(z). \tag{1.3}$$

Property 2. *The Euler gamma function can be extended to the half-plane* $Re(z) \leq 0$ *by:*

$$\Gamma(z) = \frac{\Gamma(z+n)}{(z)_n}, Re(z) > 0, \text{ for } n \in \mathbb{N}^*, Re(z) \notin \mathbb{Z}_0^- = \{..., -3, -2, -1, 0\}. \tag{1.4}$$

Here $(z)_n$ *is the Pochhammer symbol defined for complex* $z \in \mathbb{C}$ *and non-negative integer* $n \in \mathbb{N}$ *by:*

$$\begin{cases} (z)_0 = 1, \\ (z)_n = z(z+1)...(z+n-1), n \in \mathbb{N}^*. \end{cases} \tag{1.5}$$

For a better understanding, the graph of the Euler gamma function $y = \Gamma(x)$ *for real values of* x *is given in (1.1). (1.1) demonstrates the Euler gamma function at and around zero. Note that at negative integer values, the Euler gamma function goes to infinity.*

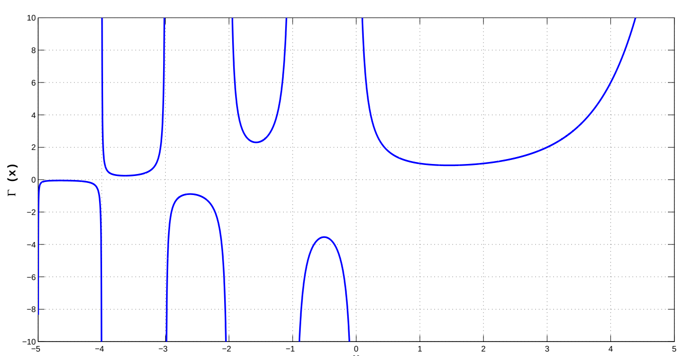

Figure 1.1: Graphe of the Euler gamma function

Property 3. *1. The Euler gamma function generalizes the factorial:*

$$\Gamma(n+1) = n!, \forall n \in \mathbb{N}. \tag{1.6}$$

2.

$$\Gamma(n+\frac{1}{2}) = \frac{2n!\sqrt{\pi}}{4^n n!}. \tag{1.7}$$

1.2.2 Beta Function

Beta function, also known as the Euler integral of the first kind, is an important relationship in fractional calculus. In many cases it is more convenient to use beta function instead of a certain combination of values of the gamma function.

Definition 2. *The Beta function is a type of Euler integral defined by:*

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt, (p, q \in \mathbb{C}, Re(p) > 0, Re(q) > 0), \tag{1.8}$$

where p and q are complex numbers, with $Re(p) > 0$ and $Re(q) > 0$.

Property 4. *For all $p, q \in \mathbb{C}$, with $Re(p) > 0, Re(q) > 0$, we have:*

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}. \tag{1.9}$$

Proof. Let $D = (0, +\infty) \times (0, +\infty)$, we have

$$\Gamma(p)\Gamma(q) = \left(\int_0^{+\infty} x^{p-1}e^{-x}dx\right)\left(\int_0^{+\infty} y^{q-1}e^{-y}dy\right) = \int\int_D x^{p-1}y^{q-1}e^{-(x+y)}dxdy,$$

we put $u = x+y$ and $v = \frac{x}{x+y}$ then we have:

$$\frac{\partial(x,y)}{\partial(x,y)} = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix} = -uv - u(1-v) = -u,$$

and, $D^{'} = \{(u,v)/u \geq 0, 0 \leq v \leq 1\}$ then

$$\begin{aligned}
\int\int_D x^{p-1}y^{p-1}e^{-(x+y)}dxdy &= \int\int_{D'} (uv)^{p-1}(u(1-v))^{q-1}e^{-u}|-u|dudv \\
&= \int\int_{D'} u^{p+q-1}v^{p-1}(1-v)^{q-1}e^{-u}dudv \\
&= \int_0^{+\infty}\int_0^1 u^{p+q-1}v^{p-1}(1-v)^{q-1}e^{-u}dudv \\
&= \left(\int_0^{+\infty} u^{p+q-1}e^{-u}du\right)\left(\int_0^1 v^{p-1}(1-v)^{q-1}dv\right) \\
&= \Gamma(p+q)B(p,q),
\end{aligned}$$

then

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

Equation (1.9) provides the analytical continuation of the Beta function to the entire complex plane via the analytical continuation of the Euler gamma function. It should also be mentioned that Beta function is symmetric, i.e.,

$$B(p,q) = B(q,p), (\forall p, q \in \mathbb{C}, Re(p) > 0, Re(q) > 0). \tag{1.10}$$

1.2.3 Mittag-Leffler Function

The Mittag-Leffler function is an important function that is widely used in the field of fractional calculus. Just as the exponential naturally arises out of the solution to integer order differential equations, the Mittag-Leffler function plays an analogous role in the solution of non-integer order differential equations. The generalization of the single-parameter exponential function has been introduced by G.M. Mittag-Leffler [23] and is designated by the following definition:

Definition 3. *The standard definition of the Mittag-Leffler function is given by*

$$E_\alpha(t) = \sum_{k=0}^{+\infty} \frac{t^k}{\Gamma(\alpha k + 1)}, \alpha > 0, \tag{1.11}$$

it is also common to represent the Mittag-Leffler function in two arguments, α and β. Such that

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{+\infty} \frac{t^k}{\Gamma(\alpha k + \beta)}, \alpha, \beta > 0. \tag{1.12}$$

The last relation is the more generalized form of the function. For $\beta = 1$; we find the relationship (1.11).

Example 1. *From the relation (1.12); we find that*

$$E_{1,1}(t) = E_1(t) = \sum_{k=0}^{+\infty} \frac{t^k}{\Gamma(k+1)} = \sum_{k=0}^{+\infty} \frac{t^k}{k!} = e^t.$$

$$E_{1,2}(t) = \sum_{k=0}^{+\infty} \frac{t^k}{\Gamma(k+2)} = \sum_{k=0}^{+\infty} \frac{t^k}{(k+1)!} = \frac{1}{t} \sum_{k=0}^{+\infty} \frac{t^{k+1}}{(k+1)!} = \frac{1}{t}(e^t - 1).$$

1.3 Basic Fractional Integrals and Derivatives

In this section, we first introduce fractional calculus (i.e., fractional integration and fractional differentiation). Generally speaking, the fractional integral mainly means (fractional) Riemann-Liouville integral. The fractional derivatives consist of at least six kinds of definitions, but they are not equivalent. Here, we present the most frequently used fractional integral and derivatives. Then we study their important properties, some of which are easily confused. Following that, we observe that - only - certain properties of classical derivatives can be generalized to the fractional case.

1.3.1 Fractional Integrals

It is known that calculus means integration and differentiation. Fractional calculus, as its name suggests, refers to fractional integration and fractional differentiation. Fractional integration often means Riemann-Liouville integral. But for fractional differentiation, there are several kinds of fractional derivatives.

1.3.1.1 The Riemann-Liouville Fractional Integrals

Definition 4. ***(The Left and Right Riemann-Liouville fractional integral [27])*** *The left and Right fractional integral (or the left and Right Riemann-Liouville integral) with order $\alpha > 0$ of the given function $f(t), t \in (a, b)$ are defined as*

$$I^{\alpha}_{a,t} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \tag{1.13}$$

and

$$I^{\alpha}_{t,b} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s) ds, \tag{1.14}$$

where $\Gamma(.)$ is the Euler's gamma function.

It is natural to extend the equation (1.13) to the axes $\mathbb{R}$ and $\mathbb{R}^+$. Let us note these operators I^{α}_{+} and $I^{\alpha}_{0^+}$ respectively.

Definition 5. *1. The left Riemann-Liouville integral with order $\alpha > 0$ of a continuous function $f : \mathbb{R} \to \mathbb{R}$ is defined as*

$$I_{+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} (t-s)^{\alpha-1} f(s) ds, \forall t \in \mathbb{R}. \quad (1.15)$$

2. The left Riemann-Liouville integral with order $\alpha > 0$ of a continuous function $f : \mathbb{R}^{+} \to \mathbb{R}$ is defined as

$$I_{0^{+}}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} f(s) ds, \forall t \in \mathbb{R}^{+}. \quad (1.16)$$

1.3.2 Fractional Derivatives

As previously mentioned in the previous section, there are several kinds of fractional derivatives, but they are not equivalent. In the following, the most frequently used fractional derivatives are introduced.

1.3.2.1 The Riemann-Liouville Fractional Derivatives

Definition 6. ***(Riemann-Liouville fractional derivatives [27])*** *The left and right Riemann-Liouville derivatives with order $\alpha > 0$ of the given function $f(t), t \in (a, b)$ are defined as*

$${}^{RL}D_{a,t}^{\alpha} f(t) = \frac{d^m}{dt^m}[I_{a,t}^{m-\alpha} f(t)] = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_{a}^{t} (t-s)^{m-\alpha-1} f(s) ds, \quad (1.17)$$

and

$${}^{RL}D_{t,b}^{\alpha} f(t) = (-1)^m \frac{d^m}{dt^m}[I_{t,b}^{m-\alpha} f(t)] = \frac{(-1)^m}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_{t}^{b} (s-t)^{m-\alpha-1} f(s) ds, \quad (1.18)$$

respectively, where m is a positive integer satisfying $m - 1 \leq \alpha < m$.

1.3.2.2 The Caputo Fractional Derivatives

Definition 7. ***(Caputo fractional derivatives [27])*** *The left and right Caputo derivatives with order $\alpha > 0$ of the given function $f(t), t \in (a, b)$ are defined as*

$${}^{C}D_{a,t}^{\alpha} f(t) = I_{a,t}^{m-\alpha} \left[f^{(m)}(t) \right] = \frac{1}{\Gamma(m-\alpha)} \int_{a}^{t} (t-s)^{m-\alpha-1} f^{(m)}(s) ds, \quad (1.19)$$

and

$${}^{C}D_{t,b}^{\alpha} f(t) = (-1)^m I_{a,t}^{m-\alpha} \left[f^{(m)}(t) \right] = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_{t}^{b} (s-t)^{m-\alpha-1} f^{(m)}(s) ds, \quad (1.20)$$

respectively, where m is a positive integer satisfying $m-1<\alpha\leq m$.

Property 5. ***([27])*** *Let* $\alpha \geq 0$ *and* $m-1<\alpha<m$ *if* $f\in C^m([a,b];R)$ *The Riemann-Liouville derivative and Caputo derivative of* $f(t)$ *have following relation*

$$^{C}D^{\alpha}_{a,t}f(t) = ^{RL}D^{\alpha}_{a,t}\left[f(t)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(t-a)^k\right]. \tag{1.21}$$

Proof. By definition

$$\begin{aligned}
^{RL}D^{\alpha}_{a,t}\left[f(t)-\sum_{k=0}^{m-1}\frac{f^{(k)}(a)}{k!}(t-a)^k\right] &= \left(\frac{d}{dt}\right)^m I^{m-\alpha}_{a,t}\left[f(t)-\sum_{k=0}^{m-1}\frac{f^{(k)}(a)}{k!}(t-a)^k\right]\\
&= \left(\frac{d}{dt}\right)^m \int_a^t \frac{(t-s)^{m-\alpha-1}}{\Gamma(m-\alpha)}\left[f(s)-\sum_{k=0}^{m-1}\frac{f^{(k)}(a)}{k!}(s-a)^k\right]ds
\end{aligned}$$

By part integration

$$g(s)=\left[f(s)-\sum_{k=0}^{m-1}\frac{f^{(k)}(a)}{k!}(s-a)^k\right]\longrightarrow \frac{d}{ds}\left[f(s)-\sum_{k=0}^{m-1}\frac{f^{(k)}(a)}{k!}(s-a)^k\right]$$

$$\frac{(t-s)^{m-\alpha-1}}{\Gamma(m-\alpha)}\longrightarrow -\frac{(t-s)^{m-\alpha}}{\Gamma(m-\alpha+1)}$$

We find

$$\begin{aligned}
I^{m-\alpha}_{a,t}[g(t)] &= \int_a^t \frac{(t-s)^{m-\alpha-1}}{\Gamma(n-\alpha)}\left[f(s)-\sum_{k=0}^{m-1}\frac{f^{(k)}(a)}{k!}(s-a)^k\right]ds\\
&= \left[\frac{-(t-s)^{m-\alpha}}{\Gamma(m-\alpha-1)}g(s)\right]_a^t+\int_a^t\frac{(t-s)^{m-\alpha}}{\Gamma(m-\alpha+1)}\frac{d}{dt}g(s)ds
\end{aligned}$$

Where

$$I^{m-\alpha}_{a,t}[g(t)] = I^{m-\alpha+1}_{a,t}\frac{d}{dt}g(t)$$

Same way for n-times

$$
\begin{aligned}
I_{a,t}^{m-\alpha}[g(t)] &= I_{a,t}^{m-\alpha+m}\frac{d^m}{dt^m}g(t)\\
&= I_{a,t}^{m}I_{a,t}^{m-\alpha}\frac{d^m}{dt^m}g(t)\\
&= I_{a,t}^{m}I_{a,t}^{m-\alpha}\frac{d^m}{dt^m}\left[f(t)-\sum_{k=0}^{m-1}\frac{f^{(k)}(a)}{k!}(t-a)^k\right]\\
&= I_{a,t}^{m}I_{a,t}^{m-\alpha}\frac{d^m}{dt^m}f(t), \frac{d^m}{dt^m}\left[\sum_{k=0}^{m-1}\frac{f^{(k)}(a)}{k!}(t-a)^k\right]=0,
\end{aligned}
$$

then

$$
\begin{aligned}
D_{a,t}^{\alpha}\left[f(t)-\sum_{k=0}^{m-1}\frac{f^{(k)}(a)}{k!}(t-a)^k\right] &= \left(\frac{d}{dt}\right)^m I_{a,t}^{m-\alpha}\left[f(t)-\sum_{k=0}^{m-1}\frac{f^{(k)}(a)}{k!}(t-a)^k\right]\\
&= \left(\frac{d}{dt}\right)^m I_{a,t}^{m-\alpha}\left[g(t)\right]\\
&= \left(\frac{d}{dt}\right)^m I_{a,t}^{m}I_{a,t}^{m-\alpha}\frac{d^m}{dt^m}f(t)\\
&= I_{a,t}^{m-\alpha}\frac{d^m}{dt^m}f(t)\\
&= {}^{C}D_{a,t}^{\alpha}f(t).
\end{aligned}
$$

1.3.2.3 The Grünwald-Letnikov Fractional Derivatives

Definition 8. ***(Grünwald-Letnikov fractional derivatives [27])*** *The left and right Grünwald-Letnikov derivatives with order $\alpha > 0$ of the given function $f(t), t \in (a,b)$ are defined as*

$$
{}^{GL}D_{a,t}^{\alpha}f(t) = \lim_{h\longrightarrow 0} h^{-\alpha}\sum_{k=0}^{N}\omega_k^{\alpha}f(t-kh), \tag{1.22}
$$

and

$$
{}^{GL}D_{t,b}^{\alpha}f(t) = \lim_{h\longrightarrow 0} h^{-\alpha}\sum_{k=0}^{N}\omega_k^{\alpha}f(t+kh), \tag{1.23}
$$

respectively, where $h = \frac{t-a}{N}$ and

$$
\omega_k^{\alpha} = (-1)^k\frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)} = \frac{\Gamma(k-\alpha)}{\Gamma(k+1)\Gamma(-\alpha)}. \tag{1.24}
$$

Property 6. *let $\alpha \geq 0$ and $m-1 < \alpha < m$ if $f \in C^m([a,b];R)$ The Riemann-Liouville derivative and the Grünwald-Letnikov derivative of $f(t)$ have following relation*

$$
{}^{RL}D_{a,t}^{\alpha}f(t) ={}^{GL} D_{a,t}^{\alpha}f(t), {}^{RL}D_{t,b}^{\alpha}f(t) ={}^{GL} D_{t,b}^{\alpha}f(t). \tag{1.25}
$$

1.3.3 Fractional operators properties

Let us turn our attention to the properties of fractional-order integration and differentiation, which are most frequently used in applications.

Property 7. ***([27])*** *The fractional integrals and derivatives are linear operators*

$$D^{\alpha}(\lambda f(t) + \mu g(t)) = \lambda D^{\alpha} f(t) + \mu D^{\alpha} g(t). \tag{1.26}$$

We prove property (7) for the Riemann-Liouville fractional Derivatives, and in similar way we also can prove it for the other operators. So, we have

$$\begin{aligned} D^{\alpha}_{a,t}(\lambda f(t) + \mu g(t)) &= \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_a^t (t-s)^{m-\alpha-1} (\lambda f(s) + \mu g(s)) ds \\ &= \frac{\lambda}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_a^t (t-s)^{m-\alpha-1} f(s) ds + \frac{\mu}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_a^t (t-s)^{m-\alpha-1} g(s) ds \\ &= \lambda D^{\alpha}_{a,t} f(t) + \mu D^{\alpha}_{a,t} g(t). \end{aligned}$$

Remark[1]

We have remark the absence of generalization for the derivative of the product and of the composition of two functions.these characteristics of the classical derivative actually go badly to the fractional. for all the definition used and even with restrictions on the functions:

$$\begin{aligned} D^{\alpha}(fg) &\neq (D^{\alpha} f)g + (D^{\alpha} g)f \\ D^{\alpha}(f/g) &\neq \frac{(D^{\alpha} f)g - (D^{\alpha} g)f}{g^2} \\ D^{\alpha}(f \circ g) &\neq (D^{\alpha} f)(g)(g)'. \end{aligned}$$

1.4 Partial Fractional Derivatives

Similar to the classical partial derivatives, we can also define the partial fractional derivatives [29]. For example,let $0 < \alpha_1, \alpha_2 < 1$, the partial fractional derivative ${}^{RL}D^{\alpha_1+\alpha_2}_{x^{\alpha_1} y^{\alpha_2}} u(x, y)$ is defined by

$$\begin{aligned} {}^{RL}D^{\alpha_1+\alpha_2}_{x^{\alpha_1}y^{\alpha_2}}u(x,y) &= {}^{RL}D^{\alpha_2}_{0,y}\left[{}^{RL}D^{\alpha_2}_{0,x}u(x,y)\right] \\ &= {}^{RL}D^{\alpha_2}_{0,y}\left[\frac{1}{\Gamma(1-\alpha_1)}\frac{d}{dx}\int_0^x (x-s)^{-\alpha_1}u(s,y)ds\right] \\ &= \frac{1}{\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)}\frac{d^2}{dx^2}\int_0^x\int_0^y (x-s)^{-\alpha_1}(y-\tau)^{-\alpha_2}u(s,\tau)d\tau ds, \end{aligned}$$

then

$$ {}^{RL}D^{\alpha_1+\alpha_2}_{x^{\alpha_1}y^{\alpha_2}}u(x,y) = {}^{RL}D^{\alpha_2+\alpha_1}_{y^{\alpha_2}x^{\alpha_1}}u(x,y). \tag{1.27}$$

Definition 9. ***(The partial fractional derivative operator [29])*** *The partial fractional derivative operator* ${}^{RL}D^{\alpha_1+\alpha_2}_{x^{\alpha_1}y^{\alpha_2}}u(x,y)$ *with order* $\alpha_1+\alpha_2$ *is defined by*

$$ {}^{RL}D^{\alpha_1+\alpha_2}_{x^{\alpha_1}y^{\alpha_2}}u(x,y) = \frac{1}{\Gamma(m-\alpha_1)\Gamma(n-\alpha_2)}\frac{d^{m+n}}{dx^m dy^n}\int_0^x\int_0^y (x-s)^{m-\alpha_1-1}(y-\tau)^{n-\alpha_2-1}u(s,\tau)d\tau ds,$$

where $m-1<\alpha_1<m, n-1<\alpha_2<n, m, n$ *are positive integers.*

Definition 10. *[29] The partial fractional derivative operator* ${}^{RL}D^{\alpha_1+\alpha_2+...+\alpha_l}_{x_1^{\alpha_1}x_2^{\alpha_2}...x_l^{\alpha_l}}$ *with order* $(\alpha_1+\alpha_2+ ... +\alpha_l)$ *is defined by*

$$ {}^{RL}D^{\alpha_1+\alpha_2+...+\alpha_l}_{x_1^{\alpha_1}x_2^{\alpha_2}...x_l^{\alpha_l}}u(x_1,...,x_l) = \frac{1}{\prod_{k=1}^{l}\Gamma(m_k-\alpha_k)}\frac{\partial^{m_1+m_2+...+m_l}}{\partial x_1^{m_1}\partial x_2^{m_2}...\partial x_l^{m_l}}\int_0^{x_1}...\int_0^{x_l}$$
$$(x_l-\epsilon_l)^{m_l-\alpha_l-1}...(x_1-\epsilon_1)^{m_1-\alpha_1-1}u(\epsilon_1,...,\epsilon_l)d\epsilon_1...d\epsilon_l,$$

where $m_k-1<\alpha_k<m_k(k=1,2,,l), m_k$ *are positive integers.*

Definition 11. *[29] The partial fractional derivative operator* ${}^{C}D^{\alpha_1+\alpha_2+...+\alpha_l}_{x_1^{\alpha_1}x_2^{\alpha_2}...x_l^{\alpha_l}}$ *with order* $(\alpha_1+\alpha_2+...+\alpha_l)$ *is defined by*

$$ {}^{C}D^{\alpha_1+\alpha_2+...+\alpha_l}_{x_1^{\alpha_1}x_2^{\alpha_2}...x_l^{\alpha_l}}u(x_1,...,x_l) = \frac{1}{\prod_{k=1}^{l}\Gamma(m_k-\alpha_k)}\int_0^{x_1}...\int_0^{x_l}$$
$$(x_l-\epsilon_l)^{m_l-\alpha_l-1}...(x_1-\epsilon_1)^{m_1-\alpha_1-1}\frac{\partial^{m_1+m_2+...+m_l}}{\partial x_1^{m_1}\partial x_2^{m_2}...\partial x_l^{m_l}}u(\epsilon_1,...,\epsilon_l)d\epsilon_1...d\epsilon_l,$$

where $m_k-1<\alpha_k<m_k(k=1,2,,l), m_k$ *are positive integers.*

Fractional-order's partial differential equations (FPDEs) are generalizations of classical partial differential equations. They have been of considerable interest to the recent literature. A considerable attention has been especially devoted to these topics in the fields of visco-elasticity materials, electrochemical processes, dielectric polarization, among others. Increasingly, these models are used in applications such as fluid flow and finance. The solutions of FPDEs play an important role in the proper understanding of qualitative features of many phenomena and processes in various areas of natural sciences. Furthermore, simple solutions are often used in teaching many courses as specific examples which illustrate basic tenets of a theory that admits mathematical formulation.

Definition 12. *(**Fractional Differential Equation [12]**) A fractional differential equation (FDE) is a relationship of the type*

$$F(\eta, u(\eta), D^{\alpha_1}u(\eta), D^{\alpha_2}u(\eta), ...) = 0, \alpha_1, \alpha_2, ... > 0, \tag{1.28}$$

between the variable $\eta \in \mathbb{R}$, and the fractional derivatives of orders $\alpha_1, \alpha_2, ...$ of the unknown function u at the point η. Here $D^{\alpha}u$ presents a fractional differential operator of order $\alpha > 0$.

Definition 13. *A fractional-orders's partial differential equation (FPDE) for the function u is a relationship between u; the independent variables $(\eta_1, \eta_2, ..., \eta_n) \in \mathbb{R}^n$ and one or more fractional derivatives $D^{\alpha_1}_{\eta_1}u, D^{\alpha_2}_{\eta_2}u, ..., D^{\alpha_3}_{\eta_3}D^{\alpha_4}_{\eta_4}u, ...$, that we can write in the form:*

$$F(u, \eta_1, ..., D^{\alpha_1}_{\eta_1}u, D^{\alpha_2}_{\eta_2}u, D^{\alpha_3}_{\eta_3}, D^{\alpha_4}_{\eta_4}u, ...) = 0, \alpha_1, \alpha_2, ... > 0, \tag{1.29}$$

the symbol $D^{\alpha}_{\eta_i}$ presents a fractional differential operator of order α at η_i, $i = 1, 2, ..., n$.

CHAPTER 2

FINITE DIFFERENCE APPROXIMATIONS FOR FRACTIONAL DERIVATIVES

In this chapter, we consider approximation methods to evaluate fractional derivative numerically. In particular we look for approximations for the Riemann-Liouville fractional derivative. The fractional derivative of $f(t)$ in the definition of Riemann-Liouville fractional derivative, Equation (1.17), depends upon $f(t)$ at the times $[0;T]$, which means that the fractional derivative of function $f(t)$ depends on the historical behaviour of $f(t)$ [27]. One of the main approximations of the Riemann-Liouville fractional derivative is the Grnwald-Letnikov approximation based upon the Grnwald-Letnikov definition, given by Equation (1.22) in Chapter 1. Another methods to approximate the fractional derivatives L1,L2 and L2C schemes. Note that, the Riemann-Liouville derivative and the Caputo derivative have the relation equation(1.21), almost all the numerical methods for the RiemannLiouville derivative can be theoretically extended to the Caputo derivative if $f \in C^m([a,b];R)$ conditions.

2.1 The Grünwald-Letnikov Approximation

As mentioned above, one method to approximate the fractional derivatives numerically is the Grünwald-Letnikov approximation. Let $t \in [0, T]$, $\Delta t = T/N$ so that $t_n = n\Delta t$, for $0 \leq n \leq N$. If $f(t) \in C([0, T])$, the Grünwald-Letnikov derivative is equivalent to the Riemann-Liouville derivative (property 6). Using The left Grünwald-Letnikov derivatives, the left Riemann-Liouville derivative can be approximated with first order accuracy by [27, 17, 33]:

$$\left[{}^{RL}D_{0,t}^{\alpha} f(t)\right]_{t=t_n} \approx \frac{1}{\Delta t^{\alpha}} \sum_{k=0}^{n} \omega_k^{\alpha} f(t_{n-k}) + O(\Delta t). \tag{2.1}$$

The Grünwald-Letnikov weights $\omega_k^{\alpha} = (-1)^k \binom{\alpha}{k}$, with $k \geq 0$, are the coefficients of the power series of the generating function $(1 - z)^{\alpha} = \sum_{k=0}^{\infty} \omega_k^{\alpha} z^k$. These weights satisfy the recursive formula:

$$\omega_0^{\alpha} = 1; \omega_k^{\alpha} = (1 - \frac{\alpha + 1}{k})\omega_{k-1}^{\alpha}. \tag{2.2}$$

However, other formulas for the calculation of these weights exit, leading to higher order approximations [27, 17].

The estimate of the accuracy of the Grünwald-Letnikov scheme, using the weights ω_k^{α} given by Equation (2.2), was tested on the functions $f(t) = \sin(t)$, $t \in [0; \pi]$,and $f(t) = t^2$, $t \in [0; 1]$.

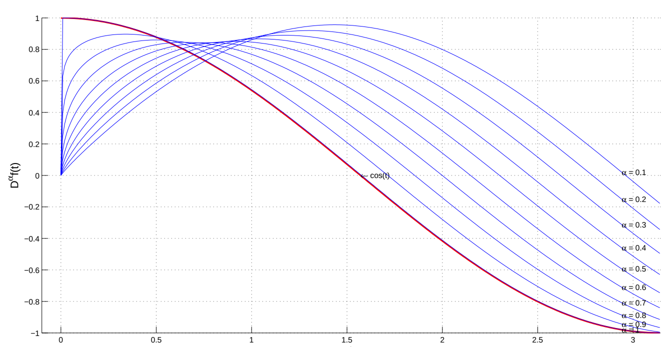

Figure 2.1: Riemann-Liouville derivative with different values of order $0 < \alpha \leq 1$ of $sin(t)$

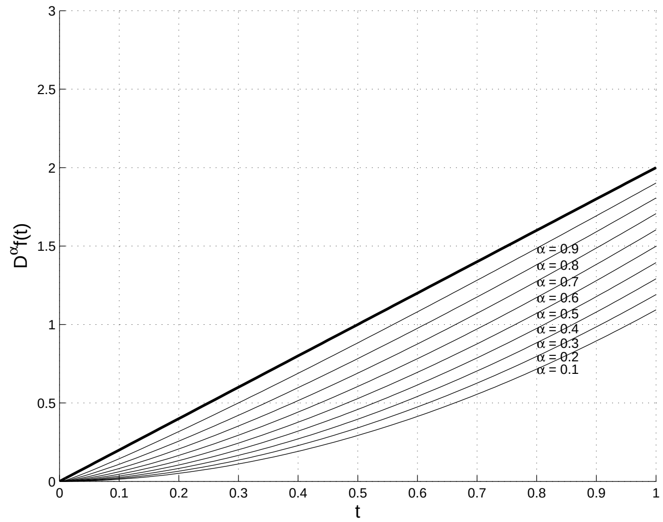

Figure 2.2: Riemann-Liouville derivative with different values of order $0 < \alpha \leq 1$ of t^2

2.2 L1 Approximation

The L1 method is another popular choice for the approximation of fractional derivatives. This approximation is found in many unconditionally stable schemes and is suitable for ($0 < \alpha < 1$) [14, 16, 26, 30]. Nevertheless, similar methods exist for ($1 < \alpha < 2$).

The $L1$ scheme was originally developed by Oldham and Spanier [26]. In this method the function $f(t)$ is defined as a piecewise linear, and the Riemann-Liouville derivative given in Equation (1.17) with $m = 1$ is written as

$$^{RL}D^{\alpha}_{a,t} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-s)^{-\alpha} f(s) ds. \tag{2.3}$$

The L1 approximation scheme is found as

$$^{RL}D^{\alpha}_{0,t} f(t) = ^{C} D^{\alpha}_{0,t} f(t) + \frac{f(0)}{\Gamma(1-\alpha)} t^{-\alpha}. \tag{2.4}$$

Letting $t = t_n = n\Delta t$ and $0 < \alpha < 1$, one gets

$$\begin{aligned}\left[{}^{c}D_{0,t}^{\alpha} f(t)\right]_{t=t_n} &= \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t_n} (t_n - s)^{-\alpha} f'(s) ds \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_n - s)^{-\alpha} f'(s) ds \\ &\approx \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_n - s)^{-\alpha} \frac{f(t_{k+1}) - f(t_k)}{\Delta t} ds + \hat{R}^n \\ &= \sum_{k=0}^{n-1} (f(t_{k+1}) - f(t_k)) \left[\frac{-(t_n - s)^{1-\alpha}}{\Delta t \Gamma(2-\alpha)}\right]_{t_k}^{t_{k+1}} + \hat{R}^n \\ &= \sum_{k=0}^{n-1} (f(t_{k+1}) - f(t_k)) \left[\frac{\Delta t^{-\alpha}((n-k)^{1-\alpha} - (n-k-1)^{1-\alpha})}{\Gamma(2-\alpha)}\right] + \hat{R}^n \\ &= \sum_{k=0}^{n-1} b_{n-k-1}(f(t_{k+1}) - f(t_k)) + \hat{R}^n,\end{aligned}$$

where

$$b_k = \frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)} \left[(k+1)^{1-\alpha} - k^{1-\alpha}\right]. \tag{2.5}$$

In each interval, $k\Delta t \leq s \leq (k+1)\Delta t$ the derivative is then assumed to be constant and is approximated by a first order finite difference approximation. The approximation is then given as

$$\left[{}^{RL}D_{0,t}^{\alpha} f(t)\right]_{t=t_n} \approx \frac{f(0) t_n^{-\alpha}}{\Gamma(1-\alpha)} + \sum_{k=0}^{n-1} b_{n-k-1}(f(t_{k+1}) - f(t_k)) + \hat{R}^n. \tag{2.6}$$

We tested by the functions $f(t) = \sin(t)$, $t \in [0; \pi]$,and $f(t) = e^t$, $t \in [0; 1]$.

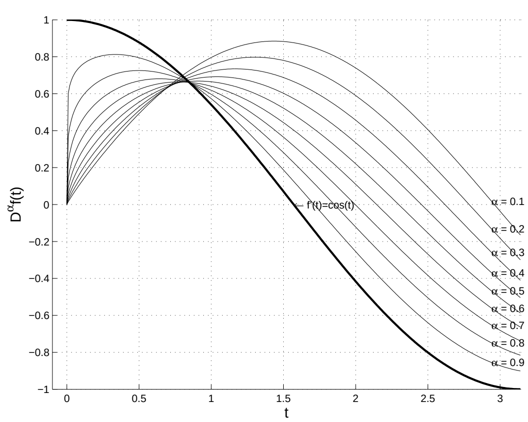

Figure 2.3: Riemann-Liouville derivative with different values of order $0 < \alpha \leq 1$ of $sin(t)$

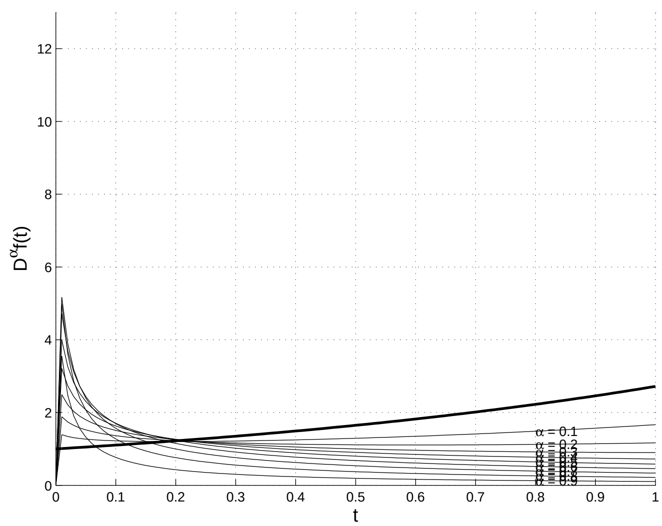

Figure 2.4: Riemann-Liouville derivative with different values of order $0 < \alpha \leq 1$ of e^t

Theorem 1. *Let* $0 < \alpha < 1$ *and* $f(t) \in C^2([0,T])$*. Then*

$$\left|\hat{R}^n\right| = \left|\int_{t_0}^{t_n}(s_n - s)^{-\alpha} f'(s)ds + \sum_{k=0}^{n-1} a_{k+1}^n (f(s_{k+1}) - f(s_k))\right| \leq C(\tau_{max})^{2-\alpha} \max_{0\leq t\leq T} |f''(t)|,$$

where C *is only dependent on* α *and* τ_{max}/τ_{min}*.*

2.3 L2 and L2C methods

The L2 method and its variant L2C method [26, 18] are used to discretise the Riemann-Liouville derivative of order $\alpha (1 < \alpha < 2)$, which can be obtained in a similar way to that of the L1 method.

The L2 approximation scheme is found as

$$^{RL}D_{0,t}^{\alpha} f(t) =^{C} D_{0,t}^{\alpha} f(t) + \frac{f(0)}{\Gamma(1-\alpha)} t^{-\alpha} + \frac{f'(0)}{\Gamma(2-\alpha)} t^{1-\alpha}. \tag{2.7}$$

For the Caputo derivative with order $1 < \alpha < 2$,we have

$$\begin{aligned} \left[{}^{C}D_{0,t}^{\alpha} f(t)\right]_{t=t_n} &= \frac{1}{\Gamma(2-\alpha)} \int_{t_0}^{t_n} (t_n - s)^{1-\alpha} f''(s)ds \\ &= \frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_n - s)^{1-\alpha} f''(s)ds \\ &= \frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} s^{1-\alpha} f''(t_n - s)ds, \end{aligned}$$

where in each interval $k\Delta t \le s \le (k+1)\Delta t$, we have

$$\begin{aligned}\left[{}^{c}D_{0,t}^{\alpha} f(t)\right]_{t=t_n} &\approx \sum_{k=0}^{n-1} \frac{1}{\Gamma(2-\alpha)\Delta t^2} \left(f(t_{n-k-1}) - 2f(t_{n-k}) + f(t_{n-k+1})\right) \int_{t_k}^{t_{k+1}} s^{1-\alpha} ds + O(\Delta t^{3-\alpha}) \\ &= \sum_{k=-1}^{n} W_k f(t_{n-k}).\end{aligned}$$

The L2 approximation scheme is given

$$\left[{}^{RL}D_{0,t}^{\alpha} f(t)\right]_{t=t_n} = \sum_{k=-1}^{n} W_k f(t_{n-k}) + \frac{f(0)}{\Gamma(1-\alpha)} t^{-\alpha} + \frac{f'(0)}{\Gamma(2-\alpha)} t^{1-\alpha}, \tag{2.8}$$

where W_k is defined by

$$W_k = \frac{\Delta t^{-\alpha}}{\Gamma(3-\alpha)} \begin{cases} 1, & k=-1, \\ 2^{2-\alpha} - 3, & k=0, \\ (k+2)^{2-\alpha} - 3(k+1)^{2-\alpha} + 3k^{2-\alpha} - (k-1)^{2-\alpha}, & 1 \le k \le n-2, \\ -2n^{2-\alpha} + 3(n-1)^{2-\alpha} - (n-2)^{2-\alpha}, & k = n-1, \\ n^{2-\alpha} - (n-1)^{2-\alpha}, & k=n. \end{cases} \tag{2.9}$$

On the other hand, we have ${}^{C}D_{0,t}^{\alpha} f(t) ={}^{C} D_{0,t}^{\alpha-1} f(t)$. The L1 method can be used to discretise the $(\alpha-1)$-order Caputo derivative of $f(t)$.

$$\left[{}^{c}D_{0,t}^{\alpha} f(t)\right]_{t=t_{n+1/2}} = a_0 f'(t_{n+1/2}) - \sum_{j=1}^{n} a_{n-j+1} f'(t_{j-1/2}) - (a_n - B_n) f'(t_{1/2}) - B_n f'(t_0), \tag{2.10}$$

where a_n and B_n are defined by

$$a_n = \frac{\Delta t^{1-\alpha}}{\Gamma(3-\alpha)}[(n+1)^{2-\alpha} - n^{2-\alpha}], B_n = \frac{2\Delta t^{1-\alpha}}{\Gamma(3-\alpha)}[(n+1/2)^{2-\alpha} - n^{2-\alpha}]. \tag{2.11}$$

Where in each interval $k\Delta t \le s \le (k+1)\Delta t$.

$$\int_{t_k}^{t_{k+1}} s^{1-\alpha} f''(t_n - s) ds \approx \frac{f(t_n - t_{k+2}) - f(t_n - t_{k+1}) + f(t_n - t_{k-1}) f(t_n - t_k)}{2\Delta t^2} \int_{t_k}^{t_{k+1}} s^{1-\alpha} ds, \tag{2.12}$$

so, the approximation of L2C method for Riemann-Liouville derivative is given as

$$\left[{}^{RL}D_{0,t}^{\alpha} f(t)\right]_{t=t_n} \approx \sum_{k=-1}^{n+1} \hat{W}_k f(t_{n-k}) + \frac{f(0) t_n^{-\alpha}}{\Gamma(1-\alpha)} + \frac{f'(0) t_n^{1-\alpha}}{\Gamma(2-\alpha)}, \tag{2.13}$$

where $\hat{W}_k$ is defined as

$$\hat{W}_k = \frac{\Delta t^{-\alpha}}{2\Gamma(3-\alpha)} \begin{cases} 1, & k=-1, \\ 2^{2-\alpha}-2, & k=0, \\ 3^{2-\alpha}-2^{2-\alpha}, & k=1, \\ (k+2)^{2-\alpha}-2(k+1)^{2-\alpha}+2(k+1)^{2-\alpha}-(k-2)^{2-\alpha}, & 2\leq k\leq n-2, \\ -n^{2-\alpha}-(n-3)^{2-\alpha}+2(n-2)^{2-\alpha}, & k=n-1, \\ -n^{2-\alpha}+2(n-1)^{2-\alpha}-(n-2)^{2-\alpha}, & k=n, \\ n^{2-\alpha}-(n-1)^{2-\alpha}, & k=n+1. \end{cases} \tag{2.14}$$

The accuracy of the L2 and L2C methods depends on α. If $\alpha = 1$, the L2 and L2C methods reduce to the backward difference method and the central difference method for the first order derivative, respectively. If $\alpha = 2$, the L2 method reduces to the central difference method for the second order derivative, and the L2C method reduces to

$$\frac{d^2 f(t_k)}{dt^2} \approx \frac{f(t_{k+2}) + f(t_k) - f(t_{k-1}) - f(t_{k+1})}{2\Delta t^2}. \tag{2.15}$$

With accuracy of order 1. In fact, the L2 method converges with order $O(\Delta t^{3-\alpha})$. Experiments show that the L2 method is more accurate than the L2C method for $1 < \alpha < 1.5$, while the reverse happens for $1.5 < \alpha < 2$. Near $\alpha = 1.5$, the two methods have almost similar results ([18]).

2.4 Finite difference approximations of integer order derivatives

In addition to fractional derivatives, integer order derivative will also need to be approximated throughout the coming chapter. First order time derivatives will appear in space fractional PDEs and are approximated either by central or backward difference operators. On the other hand, second order difference operators are used to approximate second order derivatives in time fractional PDEs. Hence, this section introduces the operators used to approximate integer order derivatives, common to several of the coming schemes.

Denote by $I = (a, b)$. Let be Δt the time step size and N a positive integer with $\Delta t = T/N$ and $t_n = n\Delta t$ for $n = 0, 1, \ldots, N$. Denote by $t_{n+\frac{1}{2}} = \frac{t_n + t_{n+1}}{2}$ for $n = 0, 1, \ldots, n-1$. We can define the space step size $\Delta x = (b-a)/M$, M is a positive integer. The space grid point x_i is given by $x_i = a + i\Delta x, i = 0, 1, \ldots, M$. Let $x_{i+\frac{1}{2}} = \frac{x_i + x_{i+1}}{2}$. For the function $u(x,t) \in C(I \times [0,T])$, denote by $u_n = u_n(.) = u(., t_n)$ and $u_i^n = u(x_i, t_n)$.

Next, we introduce the following notations that will be used in the description of the numerical schemes. At time level $n + 1/2$ it holds that

$$[\frac{\partial u}{\partial t}]_{x_i, t_{n+\frac{1}{2}}} = \delta_t u_i^{n+\frac{1}{2}} + O(\Delta t^2), \text{ where } \delta_t u_i^{n+\frac{1}{2}} = \frac{u_i^{n+1} - u_i^n}{\Delta t}.$$

which holds a similar result, apart from the truncation error, to the approximation of the first time derivative at time $t = (n+1)\Delta t$ with backward differences

$$[\frac{\partial u}{\partial t}]_{x_i, t_{n+1}} = \delta_t u_i^{n+1} + O(\Delta t), \text{ where } \delta_t u_i^{n+1} = \frac{u_i^{n+1} - u_i^n}{\Delta t}.$$

The first order space derivative can be approximated at $x = (i + \frac{1}{2})\Delta x$ with

$$[\frac{\partial u}{\partial x}]_{x_{i+\frac{1}{2}}, t_n} = \delta_t u_{i+\frac{1}{2}}^n + O(\Delta x^2), \text{ where } \delta_t u_{i+\frac{1}{2}}^n = \frac{u_{i+1}^n - u_i^n}{\Delta x}.$$

The second order space derivative can be approximated at $x = i\Delta x$ with

$$[\frac{\partial^2 u}{\partial x^2}]_{x_i, t_n} = \delta_t^2 u_i^n + O(\Delta x^2), \text{ where } \delta_t^2 u_i^n = \frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{\Delta t}.$$

CHAPTER 3

FINITE DIFFERENCE METHODS FOR FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

MMany researchers have investigated ways of finding the solution of fractional partial differential equations (FPDEs) such as analytical solutions [11, 15, 22, 19, 32] and numerical solutions [14, 3, 7, 6, 24, 5]. Some analytic solutions are known but they are difficult to evaluate. Most fractional partial differential equations do not have exact solutions and so consequently numerical techniques must be used to obtain their approximate solutions.

This chapter is divided into three sections. In the first section, we investigate the finite difference methods for the time-fractional equation in one spatial dimension. In the second section, we construct the finite difference methods for the space-fractional equations in one spatial dimension. In the last section of this chapter,we derive the finite difference methods for time-space fractional equations in one space dimension .

3.1 One-Dimensional Time-Fractional Equations

Denote by $I = (a, b)$. Let be Δt the time step size and N a positive integer with $\Delta t = T/N$ and $t_n = n\Delta t$ for $n = 0, 1, \dots, N$. Denote by $t_{n+\frac{1}{2}} = \frac{t_n + t_{n+1}}{2}$ for $n = 0, 1, \dots, N-1$. We can define the space step size $\Delta x = (b-a)/M$, M is a positive integer. The space grid point x_i is given by $x_i = a + i\Delta x, i = 0, 1, \dots, M$. Let $x_{i+\frac{1}{2}} = \frac{x_i + x_{i+1}}{2}$. For the function $u(x,t) \in C(I \times [0,T])$, denote by $u_n = u_n(.) = u(., t_n)$ and $u_i^n = u(x_i, t_n)$.

Next, we introduce the following notations that will be used in the description of the numerical schemes.

$$\delta_x u^n_{i+\frac{1}{2}} = \frac{u^n_{i+1} - u^n_i}{\Delta x}, \delta_x^2 u_i^n = \frac{u^n_{i+1} - 2u^n_i + u^n_{i-1}}{\Delta x^2},$$
$$\delta_t u_i^{n+\frac{1}{2}} = \frac{u_i^{n+1} - u_i^n}{\Delta t}.$$

3.1.1 Caputo Type Subdiffusion Equations

We consider the following Caputo type time-fractional diffusion equation [9, 22]

$$\begin{cases} {}^C D^\alpha_{0,t} u = K_\alpha \partial_x^2 u + g(x,t), & (x,t) \in [a,b] \times [0,T], \\ u(x,0) = \phi_0(x), & a \le x \le b, \\ u(a,t) = u_a(t), u(b,t) = u_b(t), & 0 \le t \le T, \end{cases} \tag{3.1}$$

where $g(x,t) = D^{\alpha-1}_{0,t} f(x,t)$.

If $0 < \alpha < 1$ and the solution $u(x,t) \in C^m([0,T]; R)$, then ${}^C D^\alpha_{0,t} u(x,t) ={}^{RL} D^\alpha_{0,t}(u(x,t) - u(x,0))$. Hence, a natural way to discretize the Caputo derivative in (3.1) is to use the GrnwaldLetnikov approximation, or the L1 method, ...etc., and the space is discretized by the classical methods such as the central difference method or the compact difference method ([8],[9],[13],[37]).

3.1.1.1 Explicit Euler Type Methods

The explicit method is particularly of interest because of its simplicity, easy implementation,and low cost in real computation. Like the explicit Euler method for the heat equation ($\alpha = 1$ in (3.1)), we can present the corresponding explicit method for the fractional sub-diffusion equation (3.1), which can be seen as an extension of the forward Euler method.

Letting $(x,t) = (x_i, t_n)$ in (3.1) leads to

$$^C D_{0,t}^{\alpha} u(x_i, t_n) = K_\alpha \partial_x^2 u(x_i, t_n) + g(x_i, t_n) = K_\alpha \partial_x^2 u(x_i, t_{n-1}) + g(x_i, t_n) + O(\Delta t). \quad (3.2)$$

The Caputo derivative in (3.2) can be discretized by the known methods, i.e., the Grnwald-Letnikov formula or the L1 method, etc.; the space direction is discretized by the central difference method. We have

$$\delta_t^{(\alpha)} u_i^n = K_\alpha \delta_x^2 u_i^{n-1} + g_i^n + O(\Delta t + \Delta x^2), \quad (3.3)$$

Where $\delta_t^{(\alpha)}$ is the approximate operator in time that is to be defined.

The Caputo derivative is discretized by the Grünwald-Letnikov formula and the space direction is discretized by the central difference method, one can get the finite difference method for (3.1):

$$\begin{cases} \delta_t^{(\alpha)}(u_i^n - u_i^0) = K_\alpha \delta_x^2 u_i^{n-1} + g_i^n, & i = 1, 2, \ldots, N-1, \\ u_i^0 = \phi_0(x_i), & i = 0, 1, \ldots, N, \\ u_0^n = U_a(t), u_N^n = U_b(t), & \end{cases} \quad (3.4)$$

where

$$\delta_t^{(\alpha)}(u_i^n - u_i^0) = \frac{1}{\Delta t^\alpha} \sum_{k=0}^{n} \omega_{n-k}^\alpha (u_i^k - u_i^0), \omega_k^\alpha = \frac{\Gamma(k-\alpha)}{\Gamma(k+1)\Gamma(-\alpha)}. \quad (3.5)$$

The Caputo derivative is discretized by the L1 method with the space direction approximated by the central difference scheme; we can derive the method for (3.1):

$$\begin{cases} \delta_t^{(\alpha)} u_i^n = K_\alpha \delta_x^2 u_i^{n-1} + g_i^n, & i = 1, 2, \ldots, N-1, \\ u_i^0 = \phi_0(x_i), & i = 0, 1, \ldots, N, \\ u_0^n = U_a(t), u_N^n = U_b(t), & \end{cases} \quad (3.6)$$

where

$$\delta_t^{(\alpha)} u_i^n = \frac{1}{\Delta t^\alpha} \sum_{k=0}^{n-1} b_{n-k-1}^\alpha (u_i^{k+1} - u_i^k) = \frac{1}{\Delta t^\alpha} \left[b_0 u_i^n \sum_{k=0}^{n-1} (b_{n-k-1}^\alpha - b_{n-k}^\alpha) u_i^k - b_n u_i^0 \right], \quad (3.7)$$

$$b_k^\alpha = \frac{1}{\Gamma(2-\alpha)} [(k+1)^{1-\alpha} - k^{1-\alpha}], \quad (3.8)$$

and

$$\delta_x^2 u_i^{n-1} = \frac{u_{i+1}^{n-1} - 2u_i^{n-1} + u_{i-1}^{n-1}}{\Delta x^2}. \quad (3.9)$$

Now, we discuss the stability of the two methods (3.4) and (3.6).

Theorem 2. *Suppose that $u_i^n (i = 1, 2, \ldots, N-1)$ is the solution to (3.4). Let $\mu = K_\alpha \Delta t^\alpha / \Delta x^2$. if $\mu \le \alpha/2$, then the method (3.4) is stable.*

Proof. Suppose that $u_i^n (i = 1, 2, \ldots, N-1)$ and $g_i^n (i = 1, 2, \ldots, N-1)$ have perturbations $\tilde{u}_i^n (i = 1, 2, \ldots, N-1)$ and $\tilde{g}_i^n (i = 1, 2, \ldots, N-1)$. Denote by $\tilde{\mathbf{u}}^n = (\tilde{u}_1^n, \tilde{u}_2^n, \ldots, \tilde{u}_{N-1}^n)^T, \tilde{\mathbf{g}}^n = (\tilde{g}_1^n, \tilde{g}_2^n, \ldots, \tilde{g}_{N-1}^n)^T$,and

$$A = \begin{bmatrix} \omega_1^\alpha + 2\mu & -\mu & 0 & \ldots & 0 & 0 \\ -\mu & \omega_1^\alpha + 2\mu & -\mu & \ldots & 0 & 0 \\ 0 & -\mu & \omega_1^\alpha + 2\mu & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \omega_1^\alpha + 2\mu & -\mu \\ 0 & 0 & 0 & \ldots & -\mu & \omega_1^\alpha + 2\mu \end{bmatrix}_{(N-1)\times(N-1)}, \tag{3.10}$$

$$B = \begin{bmatrix} 1-2\mu & \mu & 0 & \ldots & 0 & 0 \\ \mu & 1-2\mu & \mu & \ldots & 0 & 0 \\ 0 & \mu & 1-2\mu & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1-2\mu & \mu \\ 0 & 0 & 0 & \ldots & \mu & 1-2\mu \end{bmatrix}_{(N-1)\times(N-1)}. \tag{3.11}$$

Expand the equation (3.4) in the following form

$$\sum_{k=0}^{n} \omega_{n-k}^\alpha (\tilde{u}_i^k - \tilde{u}_i^0) = \mu \tilde{u}_i^{n-1} (\tilde{u}_{i+1}^{n-1} - 2\tilde{u}_i^{n-1} + \tilde{u}_{i-1}^{n-1}) + \Delta t^\alpha \tilde{g}_i^n. \tag{3.12}$$

Then the matrix representation of the perturbation equation (3.12) can be expressed as

$$\begin{cases} \tilde{\mathbf{u}}^1 = B\tilde{\mathbf{u}}^0 + \Delta t^\alpha \tilde{\mathbf{g}}^1, & n = 1, \\ \tilde{\mathbf{u}}^n = -A\tilde{\mathbf{u}}^{n-1} - \sum_{k=1}^{n-2} \omega_{n-k}^\alpha \tilde{\mathbf{u}}^k + \sum_{k=1}^{n} \omega_{n-k}^\alpha \tilde{\mathbf{u}}^0 + \Delta t^\alpha \tilde{\mathbf{g}}^1, & n > 1. \end{cases} \tag{3.13}$$

Since $2\mu \le \alpha = -\omega_1^\alpha$, it is easy to obtain $||A|| \le -\omega_1^\alpha$ and $||B|| \le 1$. Here $||A||$ denotes the spectral norm (or 2-norm) of the matrix A, which is equal to the absolute largest eigenvalue of A when A is symmetric.

We also denote it by

$$b_{n-k} = \sum_{k=0}^{n-1} \omega_k^\alpha = \frac{\Gamma(n-\alpha)}{\Gamma(1-\alpha)\Gamma(n)} = \frac{n^{-\alpha}}{\Gamma(1-\alpha)} + O(n^{-1-\alpha}). \tag{3.14}$$

Then one can easily prove that $\Delta t^\alpha \le C b_{n-1}$, C is a positive constant only dependent on α and T. Next, we prove that

$$||\tilde{\mathbf{u}}^n|| \le ||\tilde{\mathbf{u}}^0|| + C \max_{1\le n\le n_T} ||\tilde{\mathbf{g}}^n|| = E, \tag{3.15}$$

where $||.||$ is the discrete L^2 norm for the vector, which is defined by

$$||\mathbf{u}|| = \left(\sum_{i=1}^{n-1} u_i^2\right)^{\frac{1}{2}}, \mathbf{u} = (u_1, u_2, \ldots, u_{N-1})^T \in R^{N-1}. \tag{3.16}$$

We use the mathematical induction method to prove (3.16). For $n = 1$, we have from (3.15)

$$||\tilde{\mathbf{u}}^1|| = ||B\tilde{\mathbf{u}}^0 + \Delta t^\gamma \tilde{\mathbf{g}}^1|| \leq ||B|| ||\tilde{\mathbf{u}}^0|| + Cb_0||\tilde{\mathbf{g}}^1|| \leq ||\tilde{\mathbf{u}}^0|| + C||\tilde{\mathbf{g}}^1|| \leq E.$$

Suppose that $||\tilde{\mathbf{u}}^n|| \leq E; n = 1, 2, \ldots, m-1$. For $n = m$, one has from (3.15)

$$\begin{aligned}
||\tilde{\mathbf{u}}^m|| &\leq ||A\tilde{\mathbf{u}}^{m-1}|| - \sum_{k=1}^{n-2} \omega_{n-k}^\alpha ||\tilde{\mathbf{u}}^k|| + \sum_{k=1}^{n} \omega_{n-k}^\alpha ||\tilde{\mathbf{u}}^0|| + \Delta t^\alpha ||\tilde{\mathbf{g}}^m|| \\
&\leq ||A||E - \sum_{k=1}^{n-2} \omega_{n-k}^\alpha E + b_{m-1}||\tilde{\mathbf{u}}^0|| + \Delta t^\alpha ||\tilde{\mathbf{g}}^m|| \\
&\leq -\omega_1^\alpha E - \sum_{k=1}^{n-2} \omega_{n-k}^\alpha E + b_{m-1}E \\
&= b_0 E - \sum_{k=1}^{n-1} \omega_1^\alpha E + b_{m-1}E = E.
\end{aligned}$$

We can similarly prove that the explicit method (3.6) is conditionally stable and convergent with order $O(\Delta t + \Delta x^2)$ if $\frac{K_\alpha \Delta t^\alpha}{\Delta x^2} \leq \frac{b_0^\alpha - b_1^\alpha}{2} = \frac{1-2^{-\alpha}}{\Gamma(2-\alpha)}$.The convergence of the explicit method (3.4) was also proved by Gorenflo and AbdelRehim ([10]) in the FourierLaplace domain.

3.1.1.2 Implicit Euler Type Methods

Next, we introduce the typical implicit methods. Let $(x, t) = (x_i, t_n)$ in (3.1). Then we have

$${}^C D_{0,t}^\alpha u(x_i, t_n) = K_\alpha \partial_x^2 u(x_i, t_n) + g(x_i, t_n). \tag{3.17}$$

The time derivative in (3.17) is discretized by the Grünwald-Letnikov formula,and the space derivative is discretized by the central difference method. We can obtain the following finite difference method for (3.1), which is given by:

$$\begin{cases} \delta_t^\alpha (u_i^n - u_i^0) = K_\alpha \delta_x^2 u_i^n + g_i^n, & i = 1, 2, \ldots, N-1, \\ u_i^0 = \phi_0(x_i), & i = 0, 1, \ldots, N, \\ u_0^n = U_a(t), u_N^n = U_b(t), \end{cases} \tag{3.18}$$

where $\delta_t^{(\alpha)}(u_i^n - u_i^0)$ is defined by (3.5).

The L1 method can be used to discretize the Caputo derivative in (3.1) or (3.18),and the space derivative is discretized by the central difference. The corresponding method is given by:

$$\begin{cases} \delta_t^{(\alpha)} u_i^n = K_\alpha \delta_x^2 u_i^n + g_i^n, & i = 1, 2, \ldots, N-1, \\ u_i^0 = \phi_0(x_i), & i = 0, 1, \ldots, N, \\ u_0^n = U_a(t), u_N^n = U_b(t), \end{cases} \tag{3.19}$$

where $\delta_t^{(\alpha)} u_i^n$ is defined by (3.7) and b_k^α is defined by (3.8).

It is easy to prove that the two difference methods (3.19) and (3.7) are unconditionally stable using the Fourier method , and are convergent of order $O(\Delta t + \Delta x^2)$ and $O(\Delta t^{2-\alpha} + \Delta x^2)$, Next, we just give the stability and convergence analysis for (3.7); the stability and convergence for (3.19) is very similar.

Theorem 3. *The finite difference method (3.7) is unconditionally stable.*

Proof. We use the Fourier method. Suppose that $g_i^n = 0$ and $u_i^n = \rho_n e^{i\sigma j \Delta x} (j^2 = -1)$. Inserting u_i^n into (3.7) yields

$$(b_0^\alpha + 4\mu^*)\rho_n = \sum_{k=1}^{n-1} (b_{n-k-1}^\alpha - b_{n-k}^\alpha)\rho_k + b_n^\alpha \rho_0, \tag{3.20}$$

where $\mu^* = \frac{\Delta t^\alpha}{\Delta x^2} K_\alpha \sin^2(\frac{\sigma \Delta x}{2})$.

Next, we use the mathematical induction method to prove that $|\rho_n| \leq |\rho_0|$. It is easy to verify that $0 \leq b_{k+1}^\alpha \leq b_k^\alpha, k = 0, 1, \cdots$.

If $n = 1$, We can get

$$|\rho_1| \leq \frac{b_1^\alpha}{b_0^\alpha + 4\mu^*} |\rho_0| \leq |\rho_0|. \tag{3.21}$$

Suppose that $|\rho_k| \leq |\rho_0|, n = 1, 2, \ldots, m-1$. For $n = m$, we have

$$\begin{aligned} |\rho_m| &\leq \frac{1}{b_0^\alpha + 4\mu \sin^2(\frac{\sigma \Delta x}{2})} \sum_{k=1}^{m-1} (b_{m-k-1}^\alpha - b_{m-k}^\alpha)|\rho_k| + \frac{b_{m-k}^\alpha}{b_0^\alpha + 4\mu^*} |\rho_0| \\ &\leq \frac{1}{b_0^\alpha + 4\mu} \left(\sum_{k=1}^{m-1} (b_{m-k-1}^\alpha - b_{m-k}^\alpha) + b_{m-1}^\alpha \right) |\rho_0| \\ &\leq \frac{b_0^\alpha}{b_0^\alpha + 4\mu} |\rho_0| \leq |\rho_0|. \end{aligned}$$

Therefore,$|\rho_m| \leq |\rho_0|$, so that $|\rho_n| \leq |\rho_0|$ for all $0 \leq n \leq n_T$.

Lemma 1. *Let* $\boldsymbol{u}^k = (u_0^k, u_1^k, \ldots, u_N^k)$ *and* $\boldsymbol{g}^k = (g_0^k, g_1^k, \ldots, g_N^k)$*. The series* b_k *satisfies* $b_0 > 0, \sum_{k=1}^{\infty} |b_k| \leq b_0, b_k = O(k^{-\gamma}), B_n = O(n^{-\gamma})$*, and* $\Delta t^\gamma \leq C b_n, C$ *is independent of* n *and* Δt*. If*

$$b_0(\boldsymbol{u}^n, \boldsymbol{u}^n)_N \leq \sum_{k=1}^{n-1} b_{n-k}(\boldsymbol{u}^k, \boldsymbol{u}^n)_N + B_n(\boldsymbol{u}^0, \boldsymbol{u}^n)_N + \Delta t^\alpha(\boldsymbol{g}^n, \boldsymbol{u}^n)_N, \tag{3.22}$$

then

$$||\boldsymbol{u}^n||_N^2 \leq C_\alpha ||\boldsymbol{u}^0||_N^2 + C_1 \max_{0\leq k\leq n_T} ||\boldsymbol{g}^k||_N^2, \tag{3.23}$$

where C_α *is only dependent on* α *and* C_1 *is independent of* $n, \Delta t$*.*

Proof. Denote by $\mu = \Delta t^\alpha / b_0^{(\alpha)}$ and $c_k = b_k/b_0 = O(k^{-\alpha})$,so $c_0 = 1, |c_k| \leq 1$ and $\sum_{k=1}^{\infty} |c_k| \leq 1$. From (49) and the Cauchy inequality, we have

$$||\mathbf{u}^n||_N^2 \leq \frac{1}{2}\sum_{k=1}^{n-1} |c_{n-k}|(||\mathbf{u}^k||_N^2 + ||\mathbf{u}^n||_N^2) + \frac{|c_n|}{4}||\mathbf{u}^n||_N^2 + \frac{B_n^2}{b_0|c_n|}||\mathbf{u}^0||_N^2 + \frac{|c_n|}{4}||\mathbf{u}^n||_N^2 + \frac{\mu^2}{|c_n|}||\mathbf{g}^n||_N^2.$$

For $n = 1, 2, \ldots, m-1$.We get

$$\begin{aligned}
||\mathbf{u}^n||_N^2 &\leq \sum_{k=1}^{n-1} |c_{n-k}|||\mathbf{u}^k||_N^2 + \frac{2B_n^2}{b_0|c_n|}||\mathbf{u}^0||_N^2 + \frac{2\mu^2}{|c_n|}||\mathbf{g}^n||_N^2 \\
&\leq \sum_{k=1}^{n-1} |c_{n-k}|||\mathbf{u}^k||_N^2 + |c_n|(C_0||\mathbf{u}^0||_N^2 + C_1||\mathbf{g}^n||_N^2),
\end{aligned}$$

where we have used $\frac{2B_n^2}{b_0|c_n|} \leq C_\alpha |c_n|$ and $\frac{\mu^2}{c_n^2} \leq C$ here C_α is only dependent on α, and C_1 is independent of Δt and n.

Now, we use the mathematical induction method to prove that

$$||\mathbf{u}^1||_N^2 \leq C_\alpha ||\mathbf{u}^0||_N^2 + C_1 \max_{0\leq k\leq n_T} ||\mathbf{g}^k||_N^2 = E. \tag{3.24}$$

For $n = 1$, we have

$$||\mathbf{u}^1||_N^2 \leq |c_1|(C_0||\mathbf{u}^0||_N^2 + C_1||\mathbf{g}^0||_N^2) \leq E.$$

For $n = m$, we have

$$||\mathbf{u}^n||_N^2 \leq \sum_{k=1}^{n-1} |c_{n-k}|||\mathbf{u}^k||_N^2 + |c_n|E \leq E\sum_{k=1}^{n} |c_k| \leq c_0 E = E.$$

Therefore,$||\mathbf{u}^n||_N^2 \leq E$ holds for all n.

Theorem 4. *Let $U(x_i, t_n)$ and $u_i^n (i = 0, 1, \ldots, N; n = 1, 2, \ldots, n_T)$ are the solutions to the equations (3.19). Denote by $e_i^n = U(x_i, t_n) - u_i^n$ and $\boldsymbol{e}^n = (e_0^n, e_1^n, \ldots, e_N^n)^T$. Then there exists a positive constant C independent of $n, \Delta t$ and Δx, such that*

$$||\boldsymbol{e}^n||_N \leq C(\Delta t^{2-\alpha} + \Delta x^2).$$

Proof. We can get the error equation as follows

$$b_0^{(\alpha)} e_i^n - \sum_{k=1}^{n-1} (b_{n-k-1}^{(\alpha)} - b_{n-k}^{(\alpha)}) e_i^k - b_{n-1}^{(\alpha)} e_i^0 = K_\alpha \Delta t^\alpha \delta_x^2 e_i^n + \Delta t^\alpha R_i^n,$$

where $|R_i^n| \leq C(\Delta t^{2-\alpha} + \Delta x^2)$,and we have

$$||\mathbf{u}^n||_N^2 \leq 2||\mathbf{u}^0||_N^2 + C \max_{0 \leq k \leq n_T} ||\mathbf{g}^n||_N^2. \tag{3.25}$$

We get

$$||\mathbf{e}^n||_N^2 \leq C_\alpha ||\mathbf{e}^0||_N^2 + C \max_{0 \leq k \leq n_T} ||\mathbf{R}^n||_N^2 \leq C(\Delta t^{2-\alpha} + \Delta x^2).$$

3.1.1.3 Crank-Nicolson Type Methods

We know that the Crank-Nicolson (CN) method for the classical equation has second-order accuracy in time. The CN method for the classical diffusion equation can be constructed by the following direct methods:

Let $(x, t) = (x_i, t_n)$ in (3.1). Then we have

$${}^C D_{0,t}^\alpha u(x_i, t_n) = K_\alpha \partial_x^2 u(x_i, t_n) + g(x_i, t_n). \tag{3.26}$$

The space is discretized by the central difference scheme, i.e.,

$$\partial_x^2 u(x_i, t_n) = \partial_x^2 u_i^n + O(\Delta x^2) = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}. \tag{3.27}$$

We can get the following CN method for (3.1)

$$\begin{cases} {}^C D_{0,t}^\alpha u_i^n = K_\alpha \partial_x^2 u_i^n + g_i^n, & i = 1, 2, \ldots, N-1, \quad n = 0, 1, \ldots, n_t - 1, \\ u_i^0 = \phi_0(x_i), & i = 1, 2, \ldots, N, \\ u_0^n = U_a(t_n), u_N^n = U_b(t_n), \end{cases} \tag{3.28}$$

we have

$${}^C D_{0,t}^\alpha u_i^n = \sum_{k=0}^{n-1} b_{n-k}^\alpha \left(u_i^{k+1} - u_i^k \right), \tag{3.29}$$

where b_k^α is defined by (3.8).

We can obtain

$$\sum_{k=0}^{n-1} b_{n-k}^\alpha \left(u_i^{k+1} - u_i^k\right) = K_\alpha \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} + g_i^n. \tag{3.30}$$

Here we will show that the stability of the fractional numerical schemes can be analyzed very easily and efficiently with a method close to the well-known Von Neumann (or Fourier) method of non-fractional partial differential equations ([15], [20]).

Theorem 5. *The fractional Crank-Nicolson discretization, applied to the time-fractional diffusion equation (3.1) and defined by (3.30) is unconditionally stable for* $0 < \alpha < 1$.

Proof. Setting $\mu = \frac{1}{2\Delta x^2}$ and $b_1^\alpha = 1$, we finally get for $n = 1$:

$$-\mu u_{i-1}^1 + (K_\alpha b_n^\alpha + 2\mu)u_i^1 - \mu u_{i+1}^1 = (K_\alpha b_n^\alpha - 2\mu)u_i^0 + \mu\left(u_{i+1}^0 + u_{i-1}^0\right), i = 1, 2, \ldots, N-1, \tag{3.31}$$

for $n \geq 2, i = 1; 2; \ldots; N-1$,we have:

$$-\mu u_{i-1}^n + (K_\alpha b_n^\alpha + 2\mu)u_i^n - \mu u_{i+1}^n = (K_\alpha b_n^\alpha - 2\mu)u_i^{n-1} + \mu\left(u_{i+1}^{n-1} + u_{i-1}^{n-1}\right) - K_\alpha b_n^\alpha \sum_{k=0}^{n-2} b_{n-k}^\alpha \left(u_i^{k+1} - u_i^k\right). \tag{3.32}$$

To study the stability of the method, we look for a solution of the form $u_i^n = \rho_n e^{i\omega jh}; j = \sqrt{-1}; \omega$ real.

$$\begin{aligned} -\mu\rho_n e^{(i-1)\omega jh} + (K_\alpha b_k^\alpha + 2\mu)\rho_n e^{i\omega jh} - \mu\rho_n e^{(i+1)\omega jh} &= (K_\alpha b_n^\alpha - 2\mu)\rho_{n-1} e^{i\omega jh} \\ + \mu\left(\rho_{n-1} e^{(i+1)\omega jh} + \rho_{n-1} e^{(i-1)\omega jh}\right) - K_\alpha b_k^\alpha \sum_{k=0}^{n-2} b_{n-k}^\alpha \left(\rho_{k+1} e^{i\omega jh} - \rho_k e^{i\omega jh}\right). \end{aligned} \tag{3.33}$$

Simplifying and grouping like terms:

$$\left(1 + \frac{2\mu}{K_\alpha b_n^\alpha}(1 - \cos(\omega h))\right)\rho_n = (1 - \frac{2\mu}{K_\alpha b_n^\alpha})\rho_{n-1} + \frac{2\mu}{K_\alpha b_n^\alpha}\rho_{n-1}\cos(\omega h) - \sum_{k=0}^{n-2} b_{n-k}^\alpha \left(\rho_{k+1} - \rho_k\right), \tag{3.34}$$

this can be reduced to:

$$\rho_n = \frac{(1 - \frac{2\mu}{K_\alpha b_n^\alpha})\rho_{n-1} + \frac{2\mu}{K_\alpha b_n^\alpha}\rho_{n-1}\cos(\omega h) - \sum_{k=0}^{n-2} b_{n-k}^\alpha \left(\rho_{k+1} - \rho_k\right)}{\left(1 + \frac{2\mu}{K_\alpha b_n^\alpha}(1 - \cos(\omega h))\right)}. \tag{3.35}$$

We observe that from equation (3.35), since $\left(1+\frac{2\mu}{K_\alpha b_n^\alpha}(1-\cos(\omega h))\right) \geq 1$ for all $\alpha; n; \omega; h$ and k; it follows that:

$$\rho_1 \leq \rho_0 \left(1 - \frac{2\mu}{K_\alpha b_n^\alpha}(1-\cos(\omega h))\right), \tag{3.36}$$

and

$$\rho_n \leq \rho_{n-1} \left(1 - \frac{2\mu}{K_\alpha b_n^\alpha}(1-\cos(\omega h))\right) - \sum_{k=0}^{n-2} b_{n-k}^\alpha \left(\rho_{k+1} - \rho_k\right). \tag{3.37}$$

Repeating the process until $\rho_k \leq \rho_{k-1}; k = 1; 2; \ldots; n-1$; we finally have:

$$\rho_n \leq \rho_{n-1} \left(1 - \frac{2\mu}{K_\alpha b_n^\alpha}(1-\cos(\omega h))\right) - \sum_{k=0}^{n-2} b_{n-k}^\alpha \left(\rho_{k+1} - \rho_k\right) \leq \rho_{n-k}, \tag{3.38}$$

since each term in the summation is negative. This shows that the inequalities (3.37) and (3.38) imply $\rho_n \leq \rho_{n-1} \leq \rho_{k-2} \leq \ldots \rho_1 \leq \rho_0$.

Thus, $\rho_n = |u_i^n| \leq \rho_0 = |u_i^0| = |f_i|$; which entails $||u_i^n||_{L^2} \leq ||f_i||_{L^2}$ and we have stability.

Remark 1. *For $\alpha = 1$; the numerical scheme is reduced to the well-known convergent fully CN algorithm for the equation (3.30). Also, the proof of stability (and hence convergence) can be extended to other types of boundary conditions and more general time fractional diffusion equations in one and higher space dimensions.*

The truncation error $T(x,t)$ of the CN difference scheme is:

$$\begin{aligned}
T(x,t) &= \sum_{k=0}^{n-1} b_{n-k}^\alpha \left(u_i^{k+1} - u_i^k\right) - K_\alpha \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} - g_i^n \\
&= \sum_{k=0}^{n-1} b_{n-k}^\alpha \left[\left(u_i^n + (\Delta t - 1)u_t + \frac{(\Delta t-1)^2}{2}u_{tt} + \cdots\right) - \left(u_i^n + \Delta t u_t + \frac{\Delta t^2}{2}u_{tt} + \cdots\right)\right] + O(\Delta t) \\
&- \frac{K_\alpha}{\Delta x^2}\left[\left(u_i^n + \Delta x u_x + \frac{\Delta x^2}{2}u_{xx} + \cdots\right) - 2u_i^n + \left(u_i^n + \Delta x u_x + \frac{\Delta x^2}{2}u_{xx} + \cdots\right)\right] + O(\Delta x^2) - g_i^n \\
&= O(\Delta t) + O(\Delta x^2).
\end{aligned}$$

3.1.2 Riemann-Liouville Type Subdiffusion Equations

Consider the following type of time-fractional diffusion equation

$$\begin{cases} \partial_t u = {}^{RL}D_{0,t}^{1-\alpha}\left(K_\alpha \partial_x^2 u\right) + f(x,t), & (x,t) \in [a,b] \times [0,T], \\ u(x,0) = \phi_0(x), & a \leq x \leq b, \\ u(a,t) = u_a(t), u(b,t) = u_b(t), & 0 \leq t \leq T, \end{cases} \tag{3.39}$$

where $K_\alpha > 0$ and $0 < \alpha < 1$.

3.1.2.1 Explicit Euler Type Methods

Letting $(x,t)=(x_i,t_n)$ in (3.39) leads to

$$\partial_t u(x_i,t_n)=K_\alpha\left({}^{RL}D_{0,t}^{1-\alpha}\partial_x^2 u\right)(x_i,t_n)+f(x_i,t_n). \tag{3.40}$$

The integer-order time derivative and fractional derivative in (3.40) are discretized by the forward Euler method and the Grünwald-Letnikov formula, i.e.,

$$\partial_t u(x_i,t_n)=\frac{u(x_i,t_{n+1})-u(x_i,t_n)}{\Delta t}+O(\Delta t)=\delta_t u_i^{n+\frac{1}{2}}+O(\Delta t),$$

$$\left({}^{RL}D_{0,t}^{1-\alpha}\partial_x^2 u\right)(x_i,t_n)={}^{GL}\delta_t^{(1-\alpha)}(\partial_x^2 u^n(x_i))+O(\Delta t).$$

The space is discretized by the central difference scheme, i.e.,

$$\partial_x^2 u(x_i,t_n)=\partial_x^2 u^n(x_i)=\partial_x^2 u_i^n+O(\Delta x^2).$$

We can obtain

$$\partial_t u_i^{n+\frac{1}{2}}=K_\alpha^{GL}\delta_t^{(1-\alpha)}\partial_x^2 u_i^n+f_i^n+O(\Delta t+\Delta x^2). \tag{3.41}$$

We can get the following explicit Euler method for (3.39) as:

$$\begin{cases} \partial_t u_i^{n+\frac{1}{2}}=K_\alpha^{GL}\delta_t^{(1-\alpha)}\left(\partial_x^2 u_i^n\right)+f_i^n, & i=1,2,\ldots,N-1,\\ u_i^0=\phi_0(x_i), & i=1,2,\ldots,N,\\ u_0^n=U_a(t_n), u_N^n=U_b(t_n), & \end{cases} \tag{3.42}$$

where ${}^{GL}\delta_t^{(1-\gamma)}$ is defined by

$${}^{GL}\delta_t^{(\alpha)}u^n=\frac{1}{\Delta t^\alpha}\sum_{k=0}^{n}\omega_{n-k}^\alpha u^k, \omega_k^\alpha=\frac{\Gamma(k-\alpha)}{\Gamma(k+1)\Gamma(-\alpha)}. \tag{3.43}$$

If $\alpha\longrightarrow 1$,the method (3.42) is reduced to the classical forward Euler method.

Let $\mu=\frac{K_\alpha\Delta t^\alpha}{\Delta x^2}$ and we have

$$\begin{aligned} \delta_t u_i^{n+\frac{1}{2}} &= \frac{u_i^{n+1}-u_i^n}{\Delta t},\\ \delta_x^2 u_i^n &= \frac{u_{i+1}^n-2u_i^n+u_{i-1}^n}{\Delta x^2},\\ {}^{GL}\delta_t^{(1-\alpha)}\left(\partial_x^2 u_i^n\right) &= \frac{1}{\Delta t^{1-\alpha}\Delta x^2}\sum_{k=0}^{n}\omega_{n-k}^{1-\alpha}(u_{i+1}^k-2u_i^k+u_{i-1}^k), \end{aligned}$$

then

$$\frac{u_i^{n+1}-u_i^n}{\Delta t} = \frac{K_\alpha}{\Delta t^{1-\alpha}\Delta x^2}\sum_{k=0}^{n}\omega_{n-k}^{1-\alpha}(u_{i+1}^k - 2u_i^k + u_{i-1}^k) + f_i^n.$$

Then method (3.42) can be written as

$$u_i^{n+1} = u_i^n + \mu\sum_{k=0}^{n}\omega_{n-k}^{1-\alpha}(u_{i+1}^k - 2u_i^k + u_{i-1}^k) + \Delta t f_i^n. \tag{3.44}$$

Therefore, the unknowns u_i^{n+1} can be solved if $u_i^k(k=0,1,,n)$ and f_i^n are given.

The fractional **Von Neumann analysis** for the stability analysis of scheme (3.42) is illustrated below.

Let $f_i^n = 0$ and $u_i^k = \rho_k e^{ij\sigma\Delta x}(j^2=-1)$. Inserting u_i^k into (3.42) yields

$$\rho_{n+1} = \rho_n - 4\mu\sin^2\left(\frac{\sigma\Delta x}{2}\right)\sum_{k=0}^{n}\omega_{n-k}^{1-\alpha}\rho_k. \tag{3.45}$$

According to the Von Neumann method([5],[34]),we can first assume that $\rho_{n+1} = \xi(\sigma)\rho_n$ and $\xi(\sigma)$ is independent of time. Then (3.42) implies a closed equation for the amplification factor ξ as:

$$\xi = 1 - 4\mu\sin^2\left(\frac{\sigma\Delta x}{2}\right)\sum_{k=0}^{n}\omega_k^{1-\alpha}\xi^{-k}. \tag{3.46}$$

If $|\xi|>1$ for some σ, ρ_n grows to infinity and the method is unstable. Considering the extreme value $\xi=-1$, we obtain from (3.46) the following stability bound on μ:

$$\mu\sin^2\left(\frac{\sigma\Delta x}{2}\right) \le \frac{1}{2\sum_{k=0}^{n}\omega_k^{1-\alpha}(-1)^{-k}} \equiv S_{\alpha,n}. \tag{3.47}$$

The bound defined by () depends on the number n of iterations. Nevertheless, this dependence is weak: $S_{\alpha,n}$ approaches $S_\alpha = \lim_{n\longrightarrow\infty} S_{\alpha,n}$ in the form of oscillations with small decaying amplitudes([34]). Since $\sum_{k=0}^{\infty}\omega_k^{1-\alpha}z^{-k} = (1-z^{-1})^{1-\alpha} = w_1^{1-\alpha}(z^{-1})$. We find that the explicit method (3.42) is stable as long as

$$\mu\sin^2\left(\frac{\sigma\Delta x}{2}\right) \le S_\alpha = \frac{1}{2w_1^{1-\alpha}(-1)}. \tag{3.48}$$

Since $\sin^2\left(\frac{\sigma\Delta x}{2}\right)\le 1$, we can give a more conservative but simple bound: the explicit method (3.42) is stable when

$$\mu = \frac{K_\alpha\Delta t^\alpha}{\Delta x^2} \le S_\alpha = \frac{1}{2w_1^{1-\alpha}(-1)} = \frac{1}{2^{2-\alpha}}. \tag{3.49}$$

The stability bound in (3.49) is reduced to that of the forward Euler method if $\alpha \longrightarrow 1$.

For $p = 2$ with $w_2^{1-\alpha}(z) = (3/2 - 2z + z^2/2)^{1-\alpha}$, We can obtain that the explicit method (3.42) is stable when

$$\mu = \frac{K_\alpha \Delta t^\alpha}{\Delta x^2} \le S_\alpha = \frac{1}{2w_2^{1-\alpha}(-1)} = \frac{1}{4^{3/2-\alpha}}. \tag{3.50}$$

Next, we consider the convergence. Let $e_i^n = U(x_i, t_n) - u_i^n$. Then one can derive the error equation from (3.35) and (3.36) as

$$e_i^{n+1} = e^n + \mu \sum_{k=0}^{n-1} \omega_{n-k}^{1-\alpha} \left(e_{i+1}^k - 2e_i^k + e_{i-1}^k\right) + \Delta t R_i^n.$$

Let $e_i^n = \eta^n e^{j\sigma i \Delta x}$, $R_i^n = r^n e^{j\sigma i \Delta x}$, and $\mu^* = 4\mu \sin^2\left(\frac{\sigma \Delta x}{2}\right)$. Then one has

$$\eta_{n+1} = \eta_n - \mu^* \sum_{k=0}^{n} \omega_{n-k}^{1-\alpha} \eta_k + \Delta t r^n.$$

From (3.44), we find that the local truncation error of the method (3.42) is $O(\Delta t(\Delta t + \Delta x^2))$. It is a little difficult to prove the global truncation error. In the following sections, some techniques will be introduced to prove the convergence of the numerical schemes for the subdiffusion equation (3.42).

3.1.2.2 Implicit Euler Type Methods

In (3.39), if the integer time derivative, the Riemann-Liouville derivative, and the space derivative are approximated by the backward Euler formula, the Grünwald-Letnikov formula, and the central difference method, respectively, i.e.,

$$\partial_t u(x_i, t_n) = \frac{u(x_i, t_n) - u(x_i, t_{n-1})}{\Delta t} + O(\Delta t) = \delta_t u_i^{n-\frac{1}{2}} + O(\Delta t),$$

$$\left({}^{RL}D_{0,T}^{1-\alpha} \partial_x^2 u\right)(x_i, t_n) = {}^{GL}\, \delta_t^{(1-\alpha)}(\partial_x^2 u^n(x_i)) + O(\Delta t).$$

We can obtain the backward Euler method for (3.39) as:

$$\begin{cases} \partial_t u_i^{n-\frac{1}{2}} = K_\alpha^{GL} \delta_t^{(1-\alpha)}\left(\partial_x^2 u_i^n\right) + f_i^n, & i = 1, 2, \ldots, N-1, \\ u_i^0 = \phi_0(x_i), & i = 0, 1, 2, \ldots, N, \\ u_0^n = U_a(t_n), u_N^n = U_b(t_n), & \end{cases} \tag{3.51}$$

where ${}^{GL}\delta_t^{(1-\gamma)}$ is given by (3.43).

We give a simple implementation of the method (3.51). We have

$$\delta_t u_i^{n-\frac{1}{2}} = \frac{u_i^n - u_i^{n-1}}{\Delta t},$$
$$\delta_x^2 u_i^n = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2},$$
$${}^{GL}\delta_t^{(1-\alpha)}\left(\partial_x^2 u_i^n\right) = \frac{1}{\Delta t^{1-\alpha}\Delta x^2}\sum_{k=0}^{n}\omega_{n-k}^{1-\alpha}(u_{i+1}^k - 2u_i^k + u_{i-1}^k),$$

then

$$\frac{u_i^n - u_i^{n-1}}{\Delta t} = \frac{K_\alpha}{\Delta t^{1-\alpha}\Delta x^2}\sum_{k=0}^{n}\omega_{n-k}^{1-\alpha}(u_{i+1}^k - 2u_i^k + u_{i-1}^k) + f_i^n.$$

We first rewrite the scheme (3.39) as

$$u_i^n = u_i^{n-1} + \mu\sum_{k=0}^{n}\omega_{n-k}^{1-\alpha}(u_{i+1}^k - 2u_i^k + u_{i-1}^k) + \Delta t f_i^n, \tag{3.52}$$

where $\mu = K_\alpha \Delta t^\alpha / \Delta x^2$.

We consider the stability of the finite difference scheme (3.51). The Fourier method is powerful tool for the stability and convergence analysis of the numerical methods for fractional differential equations. We mainly focus on the stability analysis, and the convergence analysis is somewhat equivalent to the stability analysis for the linear problems.

. **Fourier method**

We first use the Fourier method ([34],[2],[35]) for the stability analysis of the method (3.50). Supposing that u_i^n has perturbation $\tilde{u}_i^n$, we can obtain the perturbation equation as follows

$$\tilde{u}_i^n = \tilde{u}_i^{n-1} + \mu\sum_{k=0}^{n}\omega_{n-k}^{1-\alpha}(\tilde{u}_{i+1}^k - 2\tilde{u}_i^k + \tilde{u}_{i-1}^k). \tag{3.53}$$

Let $\tilde{u}_i^n = \rho_n e^{j\sigma i\Delta x}(j^2 = -1)$ and inserting $\tilde{u}_i^n$ into (3.51),

$$\begin{aligned}
\rho_n e^{j\sigma i\Delta x} &= \rho_{n-1}e^{j\sigma i\Delta x} + \mu\sum_{k=0}^{n}\omega_{n-k}^{1-\alpha}(\rho_k e^{j\sigma(i+1)\Delta x} - 2\rho_k e^{j\sigma i\Delta x} + \rho_k e^{j\sigma(i-1)\Delta x})\\
\rho_n &= \rho_{n-1} + \mu\sum_{k=0}^{n}\omega_{n-k}^{1-\alpha}\rho_k(e^{j\sigma\Delta x} - 2 + e^{-j\sigma\Delta x})\\
&= \rho_{n-1} + 2\mu\sum_{k=0}^{n}\omega_{n-k}^{1-\alpha}\rho_k(\cos(\sigma\Delta x) - 1)\\
&= \rho_{n-1} - 4\mu\sin^2\left(\frac{\sigma\Delta x}{2}\right)\sum_{k=0}^{n}\omega_{n-k}^{1-\alpha}\rho_k.
\end{aligned}$$

We get

$$\left(1+4\mu\sin^2\left(\frac{\sigma\Delta x}{2}\right)\right)\rho_n=\rho_{n-1}-4\mu\sin^2\left(\frac{\sigma\Delta x}{2}\right)\sum_{k=1}^{n}\omega_k^{1-\alpha}\rho_{n-k}. \tag{3.54}$$

Next, we prove that $|\rho_n|\le|\rho_0|$ from (3.54).

Theorem 6. *The finite difference method (3.51) is unconditionally stable.*

Proof. We use the mathematical induction to complete the proof.

Let $\mu^*=4\mu\sin^2\left(\frac{\sigma\Delta x}{2}\right)$. Then we have from (3.54)

$$\rho_n=\frac{1}{1+\mu^*}\rho_{n-1}-\frac{\mu^*}{1+\mu^*}\sum_{k=0}^{n}\omega_k^{1-\alpha}\rho_{n-k}. \tag{3.55}$$

For $n=1$, it follows from (3.55) that

$$|\rho_1|=\frac{|1-\mu^*\omega_1^{1-\alpha}|}{1+\mu^*}|\rho_0|=\frac{|1-\mu^*(1-\alpha)|}{1+\mu^*}|\rho_0|\le|\rho_0|.$$

Suppose that $|\rho_k|\le|\rho_0|(0\le k\le n-1)$. For $k=n$, we get from (3.54)

$$\begin{aligned}\rho_n&\le\frac{1}{1+\mu^*}|\rho_{n-1}|+\frac{\mu^*}{1+\mu^*}\sum_{k=0}^{n}|\omega_{n-k}^{1-\alpha}||\rho_{n-k}|\\&\le\frac{1}{1+\mu^*}|\rho_0|+\frac{\mu^*}{1+\mu^*}\sum_{k=0}^{n}|\omega_{n-k}^{1-\alpha}||\rho_0|\\&=\frac{1}{1+\mu^*}|\rho_0|+\frac{\mu^*}{1+\mu^*}\left(-\sum_{k=0}^{n}\omega_{n-k}^{1-\alpha}\right)|\rho_0|\\&\le\frac{1}{1+\mu^*}|\rho_0|+\frac{\mu^*}{1+\mu^*}|\rho_0|=|\rho_0|.\end{aligned}$$

Therefore, $|\rho_n|\le|\rho_0|$.

Remark 2. *If $\omega_k^{(1-\gamma)}$ satisfies $\omega_0^{(1-\gamma)}>0$ and $\sum_{k=1}^{n}|\omega_k^{(1-\gamma)}|\le\omega_0^{(1-\gamma)}$, then the inequality (3.54) holds.*

Theorem 7. *Let be $U(x,t)$ and $u_i^n(i=0,1,2,\dots,N;n=1,2,\dots,n_T)$ solutions to equations (3.39) and (3.51). Denote by $e_i^n=u_i^n-U(x_i,t_n)$ and $\boldsymbol{e}^n=(e_0^n,e_1^n,\dots,e_N^n)^T$. Then there exists a positive constant C independent of $n,\Delta t$ and Δx, such that*

$$||\boldsymbol{e}^n||_N\le C(\Delta t+\Delta x^2).$$

Proof. We can get the error equation as follows

$$e_i^n = e_i^{n-1} + \Delta t^\gamma K_\gamma \sum_{k=0}^{n} \omega_{n-k}^{(1-\gamma)} \delta_x^2 e_i^k + \Delta t R_i^n,$$

in which R_i^n is the truncation error satisfying $R_i^n = O(\Delta t + \Delta x^2)$

we only need to estimate

$$||\mathbf{e}^0||_N^2 + \Delta t^\gamma K_\gamma |\mathbf{e}^0|_N^2 + C\Delta t \max_{0\le k\le n_T} ||\mathbf{R}^k||_N^2.$$

To get the error bound, where $\mathbf{R}^n = (R_0^n, R_1^n, \ldots, R_N^n)^T$ with $R_i^n = O(\Delta t + \Delta x^2)$. $||e^0||_N = |e^0|_{1,N} = 0$,and $||\mathbf{R}^k||_N \le C(\Delta t + \Delta x^2)$. Hence $||\mathbf{e}^n||_N \le C(\Delta t + \Delta x^2)$.

3.1.2.3 Crank-Nicolson Type Methods

The CN method for the classical diffusion equation can be constructed by the following direct methods:

1. **Method I:** Let $t = t_{n+\frac{1}{2}}$ in (3.39) with $\alpha = 1$ yields

$$\partial_t u(t_{n+\frac{1}{2}}) = \mu \partial_x^2 u(t_{n+\frac{1}{2}}) + f(t_{n+\frac{1}{2}}).$$

Note that $\partial_t u(t_{n+\frac{1}{2}}) = \delta_t u^{n+\frac{1}{2}} + O(\Delta t^2)$ and $u(t_{n+\frac{1}{2}}) = u^{n+\frac{1}{2}} + O(\Delta t^2)$. We have

$$\delta_t u^{n+\frac{1}{2}} = \mu \partial_x^2 u^{n+\frac{1}{2}} + f(t_{n+\frac{1}{2}}) + O(\Delta t^2).$$

Let $x = x_i$ and using $\partial_x^2 u^n = \delta_x^2 u_i^n + O(\Delta x^2)$,We have

$$\delta_t u^{n+\frac{1}{2}} = \mu \delta_x^2 u^{n+\frac{1}{2}} + f(x_i, t_{n+\frac{1}{2}}) + O(\Delta t^2 + \Delta x^2).$$

The classical CN method below:

$$\delta_t u^{n+\frac{1}{2}} = \mu \delta_x^2 u^{n+\frac{1}{2}} + f(x_i, t_{n+\frac{1}{2}}). \tag{3.56}$$

2. **Method II:** Let $x = x_i, t = t_k, k = n, n+1$, in (3.39) with $\alpha = 1$ gives

$$\partial_t u(x_i, t_n) = \mu \partial_x^2 u(x_i, t_n) + f(x_i, t_n),$$

$$\partial_t u(x_i, t_{n+1}) = \mu \partial_x^2 u(x_i, t_{n+1}) + f(x_i, t_{n+1}).$$

Adding the two equations leads to

$$\partial_t u(x_i,t_n)+\partial_t u(x_i,t_{n+1})=\mu\left(\partial_x^2 U(x_i,t_n)+\partial_x^2 U(x_i,t_{n+1})\right)+f(x_i,t_n)+f(x_i,t_{n+1}).$$

Noting that

$$\partial_t u(x_i,t_n)+\partial_t u(x_i,t_{n+1})=2\delta_t u^{n+\frac{1}{2}}+O(\Delta t^2),$$

and

$$\partial_x^2 u(x_i,t_n)=2\delta_x^2 u^{n+\frac{1}{2}}+O(\Delta x^2).$$

We have

$$\delta_t u_i^{n+\frac{1}{2}}=\mu\delta_x^2 u_i^{n+\frac{1}{2}}+f_i^{n+\frac{1}{2}}+O(\Delta t^2+\Delta x^2).$$

Dropping the truncation error, the following CN method:

$$\delta_t u_i^{n+\frac{1}{2}}=\mu\delta_x^2 u_i^{n+\frac{1}{2}}+f_i^{n+\frac{1}{2}}$$

Similar to (3.56), we first let $(x,t)=(x_i,t_{n-\frac{1}{2}})$ in (3.39), which gives

$$\partial_t u(x_i,t_{n-\frac{1}{2}})=K_\alpha\left({}^{RL}D_{0,t}^{1-\alpha}\partial_x^2 u\right)(x_i,t_{n-\frac{1}{2}})+f(x_i,t_{n-\frac{1}{2}}),\tag{3.57}$$

with $(1+\alpha)$th-order accuracy to approximate ${}^{RL}D_{0,t}^{1-\alpha}$ at $t=t_{n-\frac{1}{2}}$. We have

$$\delta_t u_i^{n-\frac{1}{2}}=K_\alpha\delta_t^{(1-\alpha)}u_i^{n-\frac{1}{2}}+f(x_i,t_{n-\frac{1}{2}})+O(\Delta t^{1+\alpha}+\Delta x^2),$$

where

$$\delta_t^{(1-\alpha)}u_i^{n-\frac{1}{2}}=\frac{1}{\Delta t^{1-\alpha}}\left[b_0 u_i^{n-\frac{1}{2}}-\sum_{k=1}^{n-1}(b_{n-k-1}-b_{n-k})u_i^{k-\frac{1}{2}}-(b_{n-1}-B_{n-1})u_i^{\frac{1}{2}}-A_{n-1}u_i^0\right],$$

in which $A_n=B_n-\frac{\alpha(n+1/2)^{\alpha-1}}{\Gamma(1+\alpha)}$,$b_n$ and B_n are defined by

$$b_n=\frac{(n+1)^\alpha-n^\alpha}{\Gamma(1+\alpha)},\ B_n=\frac{2[(n+1/2)^\alpha-n^\alpha]}{\Gamma(1+\alpha)}.$$

The first CN type method is given by:

$$\begin{cases}\partial_t u_i^{n-\frac{1}{2}}=K_\alpha\delta_t^{(1-\alpha)}\partial_x^2 u_i^n+f_i^{n-\frac{1}{2}}, & i=1,2,\ldots,N-1,\\ u_i^0=\phi_0(x_i), & i=0,1,2,\ldots,N,\\ u_0^n=U_a(t_n),u_N^n=U_b(t_n).\end{cases}\tag{3.58}$$

The CN type method (3.58) is reduced to the classical CN method (3.56) if $\alpha \longrightarrow 1$. Of course, we can also use

$$\frac{1}{2}\left[\left({}^{RL}D_{0,t}^{1-\alpha}\partial_x^2 u\right)(x_i,t_n) + \left({}^{RL}D_{0,t}^{1-\alpha}\partial_x^2 u\right)(x_i,t_{n-1})\right]$$

to replace $\left({}^{RL}D_{0,t}^{1-\alpha}\partial_x^2 u\right)(x_i,t_{n-\frac{1}{2}})$ in (3.57) as in the classical CN method (3.56). Then the appropriate discretization for the time fractional derivative operator ${}^{RL}D_{0,t}^{1\alpha}$ at $t=t_n, t_{n-1}$ is applied. So we can derive the following CN type method

$$\begin{cases} \partial_t u_i^{n-\frac{1}{2}} = \frac{K_\alpha}{2}\left[\delta_t^{(1-\alpha)}\partial_x^2 u_i^{n-1} + \delta_t^{(1-\alpha)}\partial_x^2 u_i^n\right] + f_i^{n+\frac{1}{2}}, & i=1,2,\ldots,N-1, \\ u_i^0 = \phi_0(x_i), & i=0,1,2,\ldots,N, \\ u_0^n = U_a(t_n), u_N^n = U_b(t_n), & \end{cases} \tag{3.59}$$

where $\delta_t^{(1-\alpha)}$ is the approximate operator of the time fractional derivative operator ${}^{RL}D_{0,t}^{1-\alpha}$. It is known that $\frac{1}{2}\left(\left[{}^{RL}D_{0,t}^{1-\alpha}\partial_x^2 u(t)\right]_{t=t_{n-1}} + \left[{}^{RL}D_{0,t}^{1-\alpha}\partial_x^2 u(t)\right]_{t=t_n}\right)$ is not a good approximation to $\left[{}^{RL}D_{0,t}^{1-\alpha}\partial_x^2 u(t)\right]_{t=t_{n-\frac{1}{2}}}$.

For example, $u(t)=t^\nu, \nu \geq 0$, so we can derive

$$\begin{aligned} &\frac{1}{2}\left(\left[{}^{RL}D_{0,t}^{1-\alpha}\partial_x^2 u(t)\right]_{t=t_{n-1}} + \left[{}^{RL}D_{0,t}^{1-\alpha}\partial_x^2 u(t)\right]_{t=t_n}\right) - \left[{}^{RL}D_{0,t}^{1-\alpha}\partial_x^2 u(t)\right]_{t=t_{n-\frac{1}{2}}} \\ &= \frac{\Gamma(\nu+1)}{2\Gamma(\nu+\alpha)}\left[t_{n-1}^{\nu+\alpha-1} + t_n^{\nu+\alpha-1} - 2t_{n-\frac{1}{2}}^{\nu+\alpha-1}\right) = O(\Delta t^2 t_n^{\nu+\alpha-3}). \end{aligned}$$

Let $(x,t)=(x_i,t_n)$ and $(x,t)=(x_i,t_{n-1})$ in (3.39)

$$\partial_t u(x_i,t_n) = K_\alpha\left({}^{RL}D_{0,t}^{1-\alpha}\partial_x^2 u\right)(x_i,t_n) + f(x_i,t_n), \tag{3.60}$$

$$\partial_t u(x_i,t_{n-1}) = K_\alpha\left({}^{RL}D_{0,t}^{1-\alpha}\partial_x^2 U\right)(x_i,t_{n-1}) + f(x_i,t_{n-1}). \tag{3.61}$$

Adding (3.60) and (3.61), we have

$$\begin{aligned} &\partial_t u(x_i,t_n) + \partial_t u(x_i,t_{n-1}) \\ &\quad = K_\alpha\left[\left({}^{RL}D_{0,t}^{1-\alpha}\partial_x^2 u\right)(x_i,t_n) + \left({}^{RL}D_{0,t}^{1-\alpha}\partial_x^2 u\right)(x_i,t_{n-1})\right] + f(x_i,t_n) + f(x_i,t_{n-1}). \end{aligned} \tag{3.62}$$

One choice is to use L1 method to discretize ${}^{RL}D_{0,t}^{1-\alpha}\partial_x^2 U(x,t)$ at $t=t_{n-1}$ and $t=t_n$, which gives

$$\partial_t u_i^{n-\frac{1}{2}} = \frac{K_\alpha}{2}\left[{}_{L1}^{RL}\delta_t^{(1-\alpha)}\partial_x^2 u_i^n + {}_{L1}^{RL}\delta_t^{(1-\alpha)}\partial_x^2 u_i^{n-1}\right] + \frac{1}{2}\left(f_i^n + f_i^{n-1}\right) + O(\Delta t^{1+\alpha} + \Delta x^2), n > 1, \tag{3.63}$$

Where ${}^{RL}_{L1}\delta_t^{(1-\alpha)}$ is defined by

$$ {}^{L1}_{RL}\delta_t^{(\alpha)}u^n = \frac{1}{\Delta t^\alpha}\left(\sum_{k=0}^{n-1} b_{n-k-1}^{(\alpha)}(u^{k+1}-u^K) + \frac{n^{-\alpha}}{\Gamma(1-\alpha)}u^0\right), b_k^{(\alpha)} = \frac{(K+1)^(1-\alpha) - k^(1-\alpha)}{\Gamma(2-\alpha)}. \tag{3.64} $$

For $n=1$,we can use the following relation

$$ \partial_t u_i^{\frac{1}{2}} = K_\alpha({}^{RL}_{L1}\delta_t^{(1-\alpha)}\partial_x^2 u_i^1) + f_i^1 + O(\Delta t + \Delta x^2). \tag{3.65} $$

We can get the following CN type method

$$ \begin{cases} \partial_t u_i^{\frac{1}{2}} = \frac{K_\alpha}{2}[\delta_t^{(1-\alpha)}\partial_x^2 u_i^1] + f_i^1, & i = 1,2,\ldots,N-1, \\ \partial_t u_i^{n-\frac{1}{2}} = \frac{K_\alpha}{2}\left[{}^{RL}_{L1}\delta_t^{(1-\alpha)}\partial_x^2 u_i^n + {}^{RL}_{L1}\delta_t^{(1-\alpha)}\partial_x^2 u_i^{n-1}\right] + \frac{1}{2}\left(f_i^n + f_i^{n-1}\right), & i = 1,2,\ldots,N-1 \\ u_i^0 = \phi_0(x_i), & i = 0,1,2,\ldots,N, \\ u_0^n = U_a(t_n), u_N^n = U_b(t_n). \end{cases} \tag{3.66} $$

Next, we consider the stability and convergence of the three CN methods

Theorem 8. *Let be $\boldsymbol{u}^n = (u_0^n, u_1^n, \ldots, u_N^n)^T$ the solutions to the finite difference scheme (3.59),$u_0^n = u_N^n = 0$,and $\boldsymbol{f}^{n-\frac{1}{2}} = (0, f_1^{n-\frac{1}{2}}, \ldots, f_{N-1}^{n-\frac{1}{2}}, 0)^T$. Then*

$$ ||\boldsymbol{u}^{n+1}||_N^2 \leq 2||\boldsymbol{u}^0||_N^2 + C_1\Delta t^\gamma|\boldsymbol{u}^0|_{1,N}^2 + C_2\Delta t\sum_{j=0}^{n}||\boldsymbol{f}^{k+\frac{1}{2}}||_N^2, $$

where C_1 is a positive constant independent of n, h, τ and T, and C_2 is a positive constant independent of n, h and τ.

Proof. Let $\delta_t\mathbf{u}^{n+\frac{1}{2}} = (\delta_t u_0^{n+\frac{1}{2}}, \delta_t u_1^{n+\frac{1}{2}}, \ldots, \delta_t u_N^{n+\frac{1}{2}})^T$ and $u^{n+\frac{1}{2}} = (u_0^{n+\frac{1}{2}}, u_1^{n+\frac{1}{2}}, \ldots, u_N^{n+\frac{1}{2}})^T$. Then from (3.59) we have

$$ (\delta_t\mathbf{u}^{n+\frac{1}{2}}, \mathbf{u}^{n+\frac{1}{2}})_N = \mu[-b_0(\delta_x\mathbf{u}^{n+\frac{1}{2}}, \delta_x\mathbf{u}^{n+\frac{1}{2}})_N + \sum_{j=2}^{n-1}(b_{n-j} - b_{n+1-j})(\delta_x\mathbf{u}^{j-\frac{1}{2}}, \delta_x\mathbf{u}^{n+\frac{1}{2}})_N \tag{3.67} $$

$$ +(b_{n-1} - B_n)(\delta_x\mathbf{u}^{\frac{1}{2}}, \delta_x\mathbf{u}^{n+\frac{1}{2}})_N + A_n(\delta_x\mathbf{u}^0, \delta_x\mathbf{u}^{n+\frac{1}{2}})_N] + (\mathbf{f}^{n+\frac{1}{2}}, \mathbf{u}^{n+\frac{1}{2}})_N. $$

Using Cauchy-Schwarz inequality

$$
\begin{aligned}
&(\delta_t \mathbf{u}^{n+\frac{1}{2}}, \mathbf{u}^{n+\frac{1}{2}})_N \\
&\leq \frac{\mu}{2}[-2b_0|\mathbf{u}^{n+\frac{1}{2}}|_{1,N}^2 + \sum_{j=2}^{n-1}(b_{n-j} - b_{n+1-j})(|\mathbf{u}^{j-\frac{1}{2}}|_{1,N}^2 + |\mathbf{u}^{n+\frac{1}{2}}|_{1,N}^2) \\
&+ (b_{n-1} - B_n)(|\mathbf{u}^{\frac{1}{2}}|_{1,N}^2 + |\mathbf{u}^{n+\frac{1}{2}}|_{1,N}^2) + A_n(|\mathbf{u}^0|_{1,N}^2 + |\mathbf{u}^{n+\frac{1}{2}}|_{1,N}^2)] + (\mathbf{f}^{n+\frac{1}{2}}, \mathbf{u}^{n+\frac{1}{2}})_N \\
&= \frac{\mu}{2}[(-b_0 - B_n + A_n)|\mathbf{u}^{n+\frac{1}{2}}|_{1,N}^2 + \sum_{j=2}^{n-1}(b_{n-j} - b_{n+1-j})|\mathbf{u}^{j-\frac{1}{2}}|_{1,N}^2 + (b_{n-1} - B_n)|\mathbf{u}^{n+\frac{1}{2}}|_{1,N}^2) \\
&+ A_n|\mathbf{u}^0|_{1,N}^2] + (\mathbf{f}^{n+\frac{1}{2}}, \mathbf{u}^{n+\frac{1}{2}})_N \\
&\leq \frac{\mu}{2}[-b_0|\mathbf{u}^{n+\frac{1}{2}}|_{1,N}^2 + \sum_{j=1}^{n-1}(b_{n-j} - b_{n+1-j})|\mathbf{u}^{j-\frac{1}{2}}|_{1,N}^2 + A_n|\mathbf{u}^0|_{1,N}^2] + (\mathbf{f}^{n+\frac{1}{2}}, \mathbf{u}^{n+\frac{1}{2}})_N.
\end{aligned}
$$

Writing

$$
\begin{aligned}
&||\mathbf{u}^{n+1}||_N^2 + \mu\Delta t \sum_{j=1}^{n+1} b_{n+1-j}|\mathbf{u}^{j-\frac{1}{2}}|_{1,N}^2 \\
&\leq |\mathbf{u}^n|_{1,N}^2 + \mu\Delta t \sum_{j=1}^{n+1} b_{n-j}|\mathbf{u}^{j-\frac{1}{2}}|_{1,N}^2 + \mu\Delta t A_n|\mathbf{u}^0|_{1,N}^2 + 2\Delta t(\mathbf{f}^{n+\frac{1}{2}}, \mathbf{u}^{n+\frac{1}{2}})_N.
\end{aligned}
$$

Denote by

$$
E^{n+1} = ||\mathbf{u}^{n+1}||_N^2 + \mu\Delta t \sum_{j=1}^{n+1} b_{n+1-j}|\mathbf{u}^{j-\frac{1}{2}}|_{1,N}^2.
$$

Then, we can obtain

$$
\begin{aligned}
E^{n+1} &\leq E^n + \mu\Delta t A_n|\mathbf{u}^0|_{1,N}^2 + 2\Delta t(\mathbf{f}^{n+\frac{1}{2}}, \mathbf{u}^{n+\frac{1}{2}})_N \\
&\leq E^{n-1} + \mu\Delta t(A_n + A_{n-1})|\mathbf{u}^0|_{1,N}^2 + 2\Delta t\left[(\mathbf{f}^{n+\frac{1}{2}}, \mathbf{u}^{n+\frac{1}{2}})_N + (\mathbf{f}^{n-\frac{1}{2}}, \mathbf{u}^{n-\frac{1}{2}})_N\right] \\
&\leq E^1 + \mu\Delta t|\mathbf{u}^0|_{1,N}^2 \sum_{j=1}^{n} A_n + 2\Delta t \sum_{j=1}^{n}(\mathbf{f}^{n+\frac{1}{2}}, \mathbf{u}^{k+\frac{1}{2}})_N \\
&\leq E^1 + \mu\Delta t|\mathbf{u}^0|_{1,N}^2 \sum_{j=1}^{n} A_n + \epsilon\mu\Delta t \sum_{j=1}^{n+1} b_{n+1-j}|\mathbf{u}^{j-\frac{1}{2}}|_{1,N}^2 + \sum_{j=1}^{n+1} \frac{\Delta t}{\epsilon\mu b_{n+1-j}}||\mathbf{f}^{j-\frac{1}{2}}||_N^2,
\end{aligned}
$$

we have used the Cauchy-Schwarz inequality and $\epsilon||\mathbf{u}^{j-\frac{1}{2}}||_N^2 \leq |\mathbf{u}^{j-\frac{1}{2}}|_{1,N}^2$. ϵ is a suitable positive constant independent of j, h, τ and u_h^j. we have

$$
||\mathbf{u}^{n+\frac{1}{2}}||_N^2 \leq E^1 + \mu\Delta t^{\alpha}||\mathbf{u}^0||_N^2 \sum_{j=1}^{n} A_n + C\Delta t||\mathbf{f}^{j-\frac{1}{2}}||_N^2,
$$

where $1/b_n \le C_\alpha n^{1-\alpha}\Delta t^{1-\alpha} \le C_\alpha T^{1-\alpha}, C_\alpha$ only depends on α. E^1 can be estimated in the following way. Let $n = 0$ in (3.67) and using the Cauchy-Schwarz inequality gives

$$||\mathbf{u}^1||_N^2 + \mu\Delta t B_0|\mathbf{u}^{\frac{1}{2}}|_{1,N}^2 = ||\mathbf{u}^0||_N^2 + \mu\Delta t A_0(\delta_x\mathbf{u}^{\frac{1}{2}}, \delta_x\mathbf{u}^0)_N + 2\Delta t(\mathbf{f}^{\frac{1}{2}}, \mathbf{u}^{\frac{1}{2}})_N$$
$$\le ||\mathbf{u}^0||_N^2 + \mu\Delta t A_0\left(\epsilon_1|\mathbf{u}^{\frac{1}{2}}|_{1,N}^2 + \frac{1}{4\epsilon_1}|\mathbf{u}^0|_{1,N}^2\right) + 2\Delta t\left(\frac{1}{4\epsilon_2}||\mathbf{f}^{\frac{1}{2}}||_N^2 + \epsilon_2||\mathbf{u}^{\frac{1}{2}}||_N^2\right),$$

where $\epsilon_1, \epsilon_2 > 0$ are suitable constants such that

$$\mu A_0\epsilon_1|\mathbf{u}^{\frac{1}{2}}|_{1,N}^2 + 2\epsilon_2||\mathbf{u}^{\frac{1}{2}}||_N^2 \le \frac{1}{2}\mu B_0|\mathbf{u}^{\frac{1}{2}}|_{1,N}^2.$$

We obtain

$$\begin{aligned} E^1 &= ||\mathbf{u}^1||_N^2 + \mu\Delta t B_0|\mathbf{u}^{\frac{1}{2}}|_{1,N}^2 \\ &\le 2||\mathbf{u}^1||_N^2 + \mu\Delta t B_0|\mathbf{u}^{\frac{1}{2}}|_{1,N}^2 \\ &\le 2||\mathbf{u}^0||_N^2 + \frac{\mu\Delta t A_0}{2\epsilon_1}|\mathbf{u}^0|_{1,N}^2 + \frac{\Delta t}{\epsilon_2}||\mathbf{f}^{\frac{1}{2}}||_N^2. \end{aligned}$$

Then

$$||\mathbf{u}^{n+1}||_N^2 \le 2||\mathbf{u}^0||_N^2 + C_1\Delta t^\alpha|\mathbf{u}^0|_{1,N}^2 + \Delta t C_2\sum_{j=0}^{n}||\mathbf{f}^{k+\frac{1}{2}}||_N^2,$$

in which C_1 is a positive constant independent of n, h, τ and T, and C_2 is a positive constant independent of n, h and τ.

Theorem 8 states that the CN type method is unconditionally stable.

Let $e_i^n = U(x_i, t_n) - u_i^n$. Then

$$\delta_t e_i^{n-\frac{1}{2}} = K_\alpha\delta_t^{(1-\alpha)}\delta_x^2 e_i^{n-\frac{1}{2}} + R_i^n, \tag{3.68}$$

where $R_i^n = O(\Delta t^{1+\alpha} + \Delta x^2)$ and $e_0^n = e_N^n = 0$ and $e_i^0 = 0, i = 0, 1, \ldots, N$. Denote $\mathbf{e}^n = (e_0^n, e_1^n, \ldots, e_N^n)^T$ and $\mathbf{R}^n = (R_0^n, R_1^n, \ldots, R_N^n)^T$. Then from (3.68) and Theorem 8, we can easily obtain

$$||\mathbf{e}^{n+1}||_N^2 \le 2||\mathbf{e}^0||_N^2 + C_1\Delta t^\alpha|\mathbf{e}^0|_{1,N}^2 + \Delta t C_2\sum_{k=0}^{n}||\mathbf{R}^k||_N^2 \le C(\Delta t^{1+\alpha} + \Delta x^2).$$

The CN type method can be seen as a special case of the following weighted average finite difference method

$$\begin{cases} \partial_t u_i^{n-\frac{1}{2}} = K_\alpha \left[(1-\theta)\delta_t^{(1-\alpha)}\partial_x^2 u_i^n + \theta\delta_t^{(1-\alpha)}\partial_x^2 u_i^{n-1}\right] + f(x_i, t_{n-\frac{1}{2}}), \\ i = 1, 2, \ldots, N-1, n = 1, 2, \ldots, n_T, \\ u_i^0 = \phi_0(x_i), i = 0, 1, 2, \ldots, N, \\ u_0^n = U_a(t_n), u_N^n = U_b(t_n), \end{cases} \tag{3.69}$$

where $0 \le \theta \le 1$, and the operator $\delta_t^{(1-\alpha)}$ can be defined by any approximation operator to the time fractional derivative operator ${}^{RL}D_{0,t}^{1-\alpha}$.

3.2 One-Dimensional Space-Fractional Differential Equations

3.2.1 One-Sided Space-Fractional Diffusion Equation

We consider the following space-fractional diffusion equation with Dirichlet boundary conditions([21])

$$\begin{cases} \partial_t u = d(x)^{RL}D_{a,x}^\alpha u + g(x,t), & (x,t) \in [a,b] \times [0,T], \\ u(x,0) = \phi_0(x), & x \in [a,b], \\ u(a,t) = u_a(t), u(b,t) = u_b(t), & t \in [0,T], \end{cases} \tag{3.70}$$

where $1 < \alpha \le 2$ and $d(x) > 0$.

The Grünwald-Letnikov derivative of a given function is convergent to the Riemann-Liouville derivative, a natural way to discretize the space-fractional Riemann-Liouville derivative is to use the definition of the Grünwald-Letnikov formula

$$({}^{RL}D_{a,x}^\alpha u)(x_i,t) = \frac{1}{\Delta x^\alpha}\sum_{j=0}^{i} \omega_j^\alpha u(t, x_{i-j}) + O(\Delta x). \tag{3.71}$$

The first-order time derivative in (3.70) can be discretized by the classical methods such as the explicit Euler method, the implicit Euler method and the Crank-Nicolson method, etc. Unfortunately, the explicit Euler method, the implicit Euler method, and the Crank-Nicolson method based on the standard Grünwald-Letnikov formula for (3.70) are often unstable([20]).

3.2.1.1 The weighted Euler Type Methods

The weighted difference methods can be derived in the following way.

Let $(x,t) = (x_i, t_{n+\frac{1}{2}})$ in (3.71)

$$\partial_t u(x_i, t_{n+\frac{1}{2}}) = d(x)({}^{RL}D_{a,x}^\alpha u)(x_i, t_{n+\frac{1}{2}}) + g(x_i, t_{n+\frac{1}{2}}). \tag{3.72}$$

The explicit Euler method, the implicit Euler method and the Crank-Nicolson method can be seen as the special cases of the following weighted difference methods

$$\begin{cases} \delta_t u_i^{n+\frac{1}{2}} = d_i \left[(1-\theta)_L\delta_x^{(\alpha)} u_{i+1}^{n+1} + \theta_L\delta_x^{(\alpha)} u_{i+1}^{n}\right] + (1-\theta)g(x_i, t_{n+1}) + \theta g(x_i, t_n), \quad i = 1, 2, \ldots, N-1, \\ u_i^0 = \phi_0(x_i), i = 1, 2, \ldots, N, \\ u_0^n = U_a(t_n), u_N^n = U_b(t_n), \end{cases} \tag{3.73}$$

where $0 \leq \theta \leq 1$ and ${}_L\delta_x^{(\alpha)} u_{i+1}^n$ is defined by

$${}_L\delta_x^{(\alpha)} u_{i+1}^n = \frac{1}{\Delta x^\alpha} \sum_{j=0}^{i+1} \omega_j^\alpha u_{i+1-j}^n ; \omega_j^\alpha = \frac{\Gamma(k-\alpha)}{\Gamma(k+1)\Gamma(-\alpha)}. \tag{3.74}$$

We have

$$\partial_t u_i^{n+\frac{1}{2}} = d_i \left[(1-\theta)_L\delta_x^{(\alpha)} u_{i+1}^{n+1} + \theta_L\delta_x^{(\alpha)} u_{i+1}^{n}\right] + (1-\theta)g(x_i, t_{n+1}) + \theta g(x_i, t_{n+1}). \tag{3.75}$$

The weighted finite difference method (3.75) is reduced to the explicit Euler method if $\theta = 1$, the implicit Euler method if $\theta = 0$, and the Crank-Nicolson method if $\theta = 1/2$.

1. **Explicit Euler Type Methods**

   If $\theta = 1$,solution to equation (3.75), based on the shifted (1shift) Grünwald-Letnikov approximation to the fractional derivative, is given by:

$$\begin{cases} \delta_t u_i^{n+\frac{1}{2}} = d_{iL}\delta_x^{(\alpha)} u_{i+1}^n + g_i^n, & i = 1, 2, \ldots, N-1, \\ u_i^0 = \phi_0(x_i), & i = 1, 2, \ldots, N, \\ u_0^n = U_a(t_n), u_N^n = U_b(t_n), \end{cases} \tag{3.76}$$

   where ${}_L\delta_x^{(\alpha)} u_{i+1}^n$ is defined by (3.74).

2. **Implicit Euler Type Methods**

   If $\theta = 0$,solution to equation (3.75), based on the shifted (1shift) Grünwald-Letnikov approximation to the fractional derivative, is given by:

$$\begin{cases} \delta_t u_i^{n+\frac{1}{2}} = d_{iL}\delta_x^{(\alpha)} u_{i+1}^{n+1} + g_i^{n+1}, & i = 1, 2, \ldots, N-1, \\ u_i^0 = \phi_0(x_i), & i = 1, 2, \ldots, N, \\ u_0^n = U_a(t_n), u_N^n = U_b(t_n), \end{cases} \tag{3.77}$$

   where ${}_L\delta_x^{(\alpha)} u_{i+1}^n$ is defined by (3.74).

3. **Crank-Nicolson Type Methods**

If $\theta = \frac{1}{2}$,solution to equation (3.75), based on the shifted (1shift) Grünwald-Letnikov approximation to the fractional derivative, is given by:

$$\begin{cases} \delta_t u_i^{n+\frac{1}{2}} = d_{iL}\delta_x^{(\alpha)} u_{i+1}^{n+\frac{1}{2}} + g(x_i, t_{n+\frac{1}{2}}), & i = 1, 2, \ldots, N-1, \\ u_i^0 = \phi_0(x_i), & i = 1, 2, \ldots, N, \\ u_0^n = U_a(t_n), u_N^n = U_b(t_n), \end{cases} \tag{3.78}$$

where ${}_L\delta_x^{(\alpha)} u_{i+1}^n$ is defined by (3.74).

We can similarly prove that the weighted finite difference methods (3.73) are unconditionally stable when $0 \leq \theta \leq 1/2$, conditionally stable when $1/2 \leq \theta \leq 1$ and $\frac{\Delta t}{\Delta x^\alpha} \leq \frac{1}{\alpha d_{\max}(2\theta - 1)}$.

Lemma 2. *([20]) Suppose that $f(x) \in L^1(R)$,${}^{RL}D_{-\infty,x}^{\alpha+2} f(x)$ and its Fourier transform belong to $L^1(R)$, and let*

$${}_L\delta_{\Delta x,p}^\alpha f(x) = \frac{1}{\Delta x^\alpha} \sum_{k=0}^{\infty} \omega_k^\alpha f(x - (k-p)\Delta x), \omega_k^\alpha = \frac{\Gamma(k-\alpha)}{\Gamma(k+1)\Gamma(-\alpha)},$$

where p is a non negative integer. Then

$${}_L\delta_{\Delta x,p}^\alpha f(x) = {}^{RL}D_{-\infty,x}^\alpha f(x) + C(p - \frac{\alpha}{2})\Delta x + O(\Delta x^2), \tag{3.79}$$

where C is a constant independent of p.

A more general second-order discretization of the left Riemann-Liouville operator was developed in ([31]), which can be given as

$$\frac{\alpha - 2q}{2(p-q)}{}_R\delta_{\Delta x,p}^\alpha f(x) + \frac{2p - \alpha}{2(p-q)}{}_R\delta_{\Delta x,q}^\alpha f(x) = {}^{RL}D_{x,\infty}^\alpha f(x) + O(\Delta x^2), \tag{3.80}$$

where p and q are integers.

Let $f(x)$ be well defined on the interval $[a, b]$. If $f(a) = 0$, then the left Riemann-Liouville operator ${}^{RL}D_{a,x}^\alpha$ at $x = x_i$ can be discretized by the following formula

$$\left[{}^{RL}D_{a,x}^\alpha f(x)\right]_{x=x_i} = \frac{\alpha - 2q}{2(p-q)}{}_L\delta_x^\alpha f_{i+p} + \frac{2p-\alpha}{2(p-q)}{}_L\delta_x^\alpha f_{i+q} + O(\Delta x^2), \tag{3.81}$$

where $f_j = f(x_j)$ and the operator ${}_L\delta_x^\alpha f_i$ is defined by

$${}_L\delta_x^\alpha f_i = \frac{1}{\Delta x^\alpha} \sum_{j=0}^{i} \omega_j^{(\alpha)} f_{i-j}, \omega_j^\alpha = \frac{\Gamma(k-\alpha)}{\Gamma(k+1)\Gamma(-\alpha)}.$$

If $f(b) = 0$, then the right Riemann-Liouville operator ${}^{RL}D^{\alpha}_{x,b}$ at $x = x_i$ can be similarly discretized as

$$\left[{}^{RL}D^{\alpha}_{x,b}f(x)\right]_{x=x_i} = \frac{\alpha-2q}{2(p-q)}\,{}_R\delta^{\alpha}_x f_{i-p} + \frac{2p-\alpha}{2(p-q)}\,{}_R\delta^{\alpha}_x f_{i-q} + O(\Delta x^2), \tag{3.82}$$

where $f_j = f(x_j)$ and the operator ${}_R\delta^{\alpha}_x f_i$ is defined by

$${}_R\delta^{\alpha}_x f_i = \frac{1}{\Delta x^{\alpha}}\sum_{j=0}^{N-i}\omega^{(\alpha)}_j f_{i+j}, \omega^{\alpha}_j = \frac{\Gamma(k-\alpha)}{\Gamma(k+1)\Gamma(-\alpha)}.$$

We are interested in the two cases of (p,q), in which (3.81) and (3.82) are reduced to the central difference when $\alpha = 2$.

1. **Case I:** $(p,q) = (1,0)$, the left and right Riemann-Liouville derivatives ${}^{RL}D^{\alpha}_{a,x}f(x)$ and ${}^{RL}D^{\alpha}_{x,b}f(x)$ at $x = x_i$ can be discretized by the following weighted shifted Grünwald formulas

$${}_L\delta^{(\alpha,1)}_x f_i = \frac{\alpha}{2}\,{}_L\delta^{(\alpha)}_x f_{i+1} + \frac{2-\alpha}{2}\,{}_L\delta^{(\alpha)}_x f_i = \sum_{j=0}^{i+1} g^{\alpha,1}_j f_{i+1-j}, \tag{3.83}$$

and

$${}_R\delta^{(\alpha,1)}_x f_i = \frac{\alpha}{2}\,{}_R\delta^{(\alpha)}_x f_{i-1} + \frac{2-\alpha}{2}\,{}_R\delta^{(\alpha)}_x f_i = \sum_{j=0}^{N-i+1} g^{\alpha,1}_j f_{i-1+j}, \tag{3.84}$$

where

$$g^{\alpha,1}_0 = \frac{\alpha}{2}, g^{\alpha,1}_k = \frac{\alpha}{2}\omega^{\alpha}_k + \frac{2-\alpha}{2}\omega^{\alpha}_{k-1}, k \geq 1. \tag{3.85}$$

2. **Case II:** $(p,q) = (1,-1)$, the left and right Riemann-Liouville derivatives ${}^{RL}D^{\alpha}_{a,x}f(x)$ and ${}^{RL}D^{\alpha}_{x,b}f(x)$ at $x = x_i$ can be discretized by the following weighted shifted Grünwald formulas

$${}_L\delta^{(\alpha,2)}_x f_i = \frac{2+\alpha}{4}\,{}_L\delta^{(\alpha)}_x f_{i+1} + \frac{2-\alpha}{4}\,{}_L\delta^{(\alpha)}_x f_{i-1} = \sum_{j=0}^{i+1} g^{\alpha,2}_j f_{i+1-j}, \tag{3.86}$$

and

$${}_R\delta^{(\alpha,2)}_x f_i = \frac{2+\alpha}{4}\,{}_R\delta^{(\alpha)}_x f_{i-1} + \frac{2-\alpha}{4}\,{}_R\delta^{(\alpha)}_x f_{i+1} = \sum_{j=0}^{N-i+1} g^{\alpha,1}_j f_{i-1+j}, \tag{3.87}$$

Where

$$g^{\alpha,2}_0 = \frac{2+\alpha}{4}, g^{\alpha,2}_1 = \frac{2+\alpha}{4}\omega^{\alpha}_1, g^{\alpha,2}_k = \frac{2+\alpha}{4}\omega^{\alpha}_k + \frac{2-\alpha}{4}\omega^{\alpha}_{k-2}, k \geq 1. \tag{3.88}$$

Lemma 3. *([31]) Let $g_k^{(\alpha,1)}$ and $g_k^{(\alpha,2)}$ be defined by (3.85) and (3.88),$1<\alpha\le 2$, and*

$$S_{N-1}^{(m,\alpha)}=\begin{pmatrix} g_1^{(\alpha,m)} & g_0^{(\alpha,m)} & 0 & \cdots & 0 \\ g_2^{(\alpha,m)} & g_1^{(\alpha,m)} & g_0^{(\alpha,m)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_{N-2}^{(\alpha,m)} & g_{N-3}^{(\alpha,m)} & g_{N-4}^{(\alpha,m)} & \cdots & g_0^{(\alpha,m)} \\ g_{N-1}^{(\alpha,m)} & g_{N-2}^{(\alpha,m)} & g_{N-3}^{(\alpha,m)} & \cdots & g_1^{(\alpha,m)} \end{pmatrix}_{(N-1)\times(N-1)}. \tag{3.89}$$

Then the real part of the eigenvalue λ of $S_{N-1}^{(m,\alpha)}$ is negative, and the eigenvalues of $S_{N-1}^{(m,\alpha)}+(S_{N-1}^{(m,\alpha)})^T$ are negative.

From (3.83),(3.84)and (3.86), we can obtain the following finite difference methods for (3.70)

$$\begin{cases} \delta_t u_i^{n+\frac{1}{2}}=d_i\left[(1-\theta)_L\delta_x^{(\alpha,m)}u_i^{n+1}+\theta_L\delta_x^{(\alpha,m)}u_i^n\right]+(1-\theta)g_i^{n+1}+\theta g_i^n, \quad i=1,2,\ldots,N-1, \\ u_i^0=\phi_0(x_i), i=0,1,\ldots,N, \\ u_0^n=U_a(t_n), u_N^n=U_b(t_n), \end{cases} \tag{3.90}$$

where $0\le\theta\le 1$, and $_L\delta_x^{(\alpha,m)}$ is defined by (3.83) for $m=1$ or by (3.86) for $m=2$.

If $U_a(t)=U_b(t)=0$, then the method (3.90) has second-order accuracy in space, the matrix representation of which is given by

$$(E-\mu(1-\theta)S^{(m)})\underline{\boldsymbol{u}}^{n+1}=(E+\theta\mu S^m)\underline{\boldsymbol{u}}^n+\Delta t((1-\theta)\underline{\boldsymbol{g}}^n+\theta\underline{\boldsymbol{g}}^{n+1}), \tag{3.91}$$

where $\mu=\Delta t/\Delta x^\alpha, \underline{\boldsymbol{u}}^n=(u_1^n,\ldots,u_{N-1}^n)^T, \underline{\boldsymbol{g}}^n=(g_1^n,\ldots,g_{N-1}^n)^T, E$ is an $(N-1)\times(N-1)$ identity matrix, $S^{(m)}$ is given by

$$S^{(m)}=diag(d_1,d_2,\ldots,d_{N-1})S_{N-1}^{(m,\alpha)}.$$

In which $S_{N-1}^{(m,\alpha)}$ is defined by (3.89).

One knows that $g_0^{(\alpha,1)}+\sum_{k=2}^n g_k^{(\alpha,1)}<-g_1^{(\alpha,1)}$ for $\frac{\sqrt{17}-1}{2}\le\alpha\le 2$. In such a case, the matrix $S^{(1)}$ has eigenvalues with negative parts. So we can easily prove that the method (3.70) with $m=1$ is unconditionally stable for $0\le\theta\le 1/2$, and conditionally stable for $1/2<\theta\le 1$.

Assume that $d_i=d$ is a constant. We can easily prove that weighted finite difference method (3.90) is unconditionally stable for $0\le\theta\le 1/2$,and conditionally stable for $1/2<\theta\le 1$ by the energy method.

For $\theta=1/2$, the method (3.90) has second order accuracy both in time and space([31]).

3.2.2 Two-Sided Space-Fractional Diffusion Equation

On consider the finite difference methods for two-sided space fractional partial differential equations. A class of two-sided space-fractional partial differential equations can be written as

$$\begin{cases} \partial_t u = c(x,t)^{RL}D^{\alpha}_{a,x}u(x,t) + d(x,t)^{RL}D^{\alpha}_{x,b}u(x,t) + g(x,t), & (x,t) \in [a,b] \times [0,T], \\ u(x,0) = \phi_0(x), & x \in [a,b], \\ u(a,t) = U_a(t), u(b,t) = U_b(t), & t \in [0,T], \end{cases} \tag{3.92}$$

Where $1 < \alpha < 2$ and $c(x,t), d(x,t) \geq 0$.

Letting $(x,t) = (x_i, t_n)$ in (3.92) leads to

$$\partial_t u(x_i,t_n) = c(x_i,t_n)^{RL}D^{\alpha}_{a,x}u(x_i,t_n) + d(x_i,t_n)^{RL}D^{\alpha}_{x,b}u(x_i,t_n) + g(x_i,t_n).$$

We have

$$\partial_t u(x_i,t_n) = \delta_t u_i^{n+\frac{1}{2}} + O(\Delta t).$$

If the left and right Riemann-Liouville fractional derivative operators are respectively discretized by the right and left shifted formulas with one shift, then the weighted average method for (3.92) is given by:

$$c_i^n({}^{RL}D^{\alpha}_{a,x}u_i^n) = (1-\theta)c_i^{n+1}({}_L\delta_x^{(\alpha)}u_{i+1}^{n+1}) + \theta c_i^n({}_L\delta_x^{(\alpha)}u_{i+1}^n),$$

$$d_i^n({}^{RL}D^{\alpha}_{x,b}u_i^n) = (1-\theta)d_i^{n+1}({}_R\delta_x^{(\alpha)}u_{i-1}^{n+1}) + \theta d_i^n({}_R\delta_x^{(\alpha)}u_{i-1}^n).$$

We can get the weighted average method for (3.92) as

$$\begin{cases} \delta_t u_i^{n+\frac{1}{2}} = \Big[(1-\theta)c_i^{n+1}({}_L\delta_x^{(\alpha)}u_{i+1}^{n+1}) + \theta c_i^n({}_L\delta_x^{(\alpha)}u_{i+1}^n)\Big] \\ + \Big[(1-\theta)d_i^{n+1}({}_R\delta_x^{(\alpha)}u_{i-1}^{n+1}) + \theta d_i^n({}_R\delta_x^{(\alpha)}u_{i-1}^n)\Big] + (1-\theta)g_i^{n+1} + \theta g_i^n, i = 1,2,\ldots,N-1, \\ u_i^0 = \phi_0(x_i), i = 0,1,\ldots,N, \\ u_0^n = U_a(t_n), u_N^n = U_b(t_n), \end{cases} \tag{3.93}$$

where ${}_L\delta_x^{(\alpha)}$ and ${}_R\delta_x^{(\alpha)}$ are defined by

$${}_L\delta_x^{(\alpha)}u_i = \frac{1}{\Delta x^{\alpha}}\sum_{j=0}^{i}\omega_j^{\alpha}u_{i-j}; \omega_j^{\alpha} = \frac{\Gamma(k-\alpha)}{\Gamma(k+1)\Gamma(-\alpha)}, \tag{3.94}$$

$${}_R\delta_x^{(\alpha)}u_i = \frac{1}{\Delta x^{\alpha}}\sum_{j=0}^{N-i}\omega_j^{\alpha}u_{i+j}; \omega_j^{\alpha} = \frac{\Gamma(k-\alpha)}{\Gamma(k+1)\Gamma(-\alpha)}. \tag{3.95}$$

1. **Explicit Euler Type Methods**

   If $\theta = 1$,solution to equation (3.93) is given by:

$$\begin{cases} \delta_t u_i^{n+\frac{1}{2}} = c_i^n({}_L\delta_x^{(\alpha)} u_{i+1}^n) + d_i^n({}_R\delta_x^{(\alpha)} u_{i-1}^n) + g_i^n, i = 1, 2, \ldots, N-1 \\ u_i^0 = \phi_0(x_i), i = 1, 2, \ldots, N, \\ u_0^n = U_a(t_n), u_N^n = U_b(t_n). \end{cases} \tag{3.96}$$

2. **Implicit Euler Type Methods**

   If $\theta = 0$,solution to equation(3.93) is given by:

$$\begin{cases} \delta_t u_i^{n+\frac{1}{2}} = c_i^{n+1}({}_L\delta_x^{(\alpha)} u_{i+1}^{n+1}) + d_i^{n+1}({}_R\delta_x^{(\alpha)} u_{i-1}^{n+1}) + g_i^{n+1}, i = 1, 2, \ldots, N-1 \\ u_i^0 = \phi_0(x_i), i = 1, 2, \ldots, N, \\ u_0^n = U_a(t_n), u_N^n = U_b(t_n). \end{cases} \tag{3.97}$$

3. **Crank-Nicolson Type Methods**

   If $\theta = \frac{1}{2}$,solution to equation (3.93) is given by:

$$\begin{cases} \delta_t u_i^{n+\frac{1}{2}} = c_i^{n+\frac{1}{2}}({}_L\delta_x^{(\alpha)} u_{i+1}^{n+\frac{1}{2}}) + d_i^{n+\frac{1}{2}}({}_R\delta_x^{(\alpha)} u_{i-1}^{n+\frac{1}{2}}) + g_i^{n+\frac{1}{2}}, i = 1, 2, \ldots, N-1 \\ u_i^0 = \phi_0(x_i), i = 1, 2, \ldots, N, \\ u_i^0 = \phi_0(x_i), i = 1, 2, \ldots, N, \\ u_0^n = U_a(t_n), u_N^n = U_b(t_n). \end{cases} \tag{3.98}$$

Let $\mu = \frac{\Delta t}{\Delta x^\alpha}$, $\mu_{c,i}^n = \mu c_i^n$,and $\mu_{d,i}^n = \mu d_i^n$. Then (3.93) can be written as

$$u_i^{n+1} - (1-\theta)\left[\mu_{c,i}^{n+1} L\delta_x^{(\alpha)} u_i^{n+1} + \mu_{d,i}^{n+1} R\delta_x^{(\alpha)} u_i^{n+1}\right]$$

$$= u_i^n - \theta\left[\mu_{c,i}^{n+1} L\delta_x^{(\alpha)} u_i^n + \mu_{d,i}^{n+1} R\delta_x^{(\alpha)} u_i^n\right] + \Delta t\left[(1-\theta) g_i^{n+1} + \theta g_i^n\right], i = 1, 2, \ldots, N-1. \tag{3.99}$$

The matrix can be given below

$$\left[E - (1-\theta)\mu S^{n+1}\right]\underline{\mathbf{u}}^{n+1} = \left[E + \theta\mu S^n\right]\underline{\mathbf{u}}^n + \Delta t\left[(1-\theta)\underline{\mathbf{g}}_i^{n+1} + \theta\underline{\mathbf{g}}_i^n\right], \tag{3.100}$$

where E is an $(N-1)\times(N-1)$ identity matrix and

$$S^n = diag(c_1^n, c_2^n, \ldots, c_{N-1}^n) S_{n-1}^{(\alpha)} + diag(c_1^n, c_2^n, \ldots, c_{N-1}^n)(S_{n-1}^{(\alpha)})^T. \tag{3.101}$$

Next, we consider the stability of the weighted finite difference methods (3.93). For simplicity, we suppose that $c(x,t)$ and $d(x,t)$ are time independent. And we denote that by $c_{\max} = \max_{0\le i\le N} c(x_i), d_{\max} = \max_{0\le i\le N} d(x_i)$. The matrix S^n is independent of n, so we denote it by

$S = S^n$.

We have

$$|\lambda - \omega_1^{(\alpha)}(c_i + d_i)| \le c_i \sum_{j=0, j\neq 1}^{i} |\omega_j^\alpha| + d_i \sum_{j=0, j\neq 1}^{N-i} |\omega_j^\alpha|,$$

Noticing that $\omega_j^\alpha > 0, j \neq 1$, and $\sum_{j=0}^{\infty} \omega_j^\alpha = 0$, one has $\sum_{j=0, j\neq 1}^{N} \omega_j^\alpha \le -\omega_1^\alpha$.

$$|\lambda - \omega_1^\alpha(c_i + d_i)| \le -\omega_1^\alpha(c_i + d_i).$$

The eigenvalues λ of the matrix S satisfy

$$-2\alpha(c_{\max} + d_{\max}) \le 2\omega_1^\alpha(c_i + d_i) \le \lambda \le 0.$$

Next, we are in a position to estimate the eigenvalues of the following matrix

$$[E - \mu(1-\theta)A]^{-1}(E + \theta\mu S).$$

Suppose that λ is the eigenvalue of the matrix S . Then the eigenvalue of $[E - \mu(1-\theta)S]^{-1}(E + \theta\mu S)$ is $\frac{1+\mu\theta\lambda}{1-\mu(1-\theta)\lambda}$.

If $0 \le \theta \le 1/2$, then we always have $|\frac{1+\mu\theta\lambda}{1-\mu(1-\theta)\lambda}| \le 1$, so the weighted finite difference method (3.93) is unconditionally stable.If $1/2 < \theta \le 1$, we deduce from $-1 \le \frac{1+\mu\theta\lambda}{1-\mu(1-\theta)\lambda} \le 1$ that $\mu = \frac{\Delta t}{\Delta x^\alpha} \le \frac{1}{\alpha(c_{\max}+d_{\max}(2\theta-1)}$.The weighted finite difference method (3.93) is conditionally stable for $1/2 < \theta \le 1$ and $\frac{\Delta t}{\Delta x^\alpha} \le \frac{1}{\alpha(c_{\max}+d_{\max}(2\theta-1)}$.The first-order method is used in the space discretization in (3.93).

Replacing the operators ${}_L\delta_x^{(\alpha)}$ and ${}_R\delta_x^{(\alpha)}$ in (3.93) by ${}_L\delta_x^{(\alpha,m)}$ and ${}_R\delta_x^{(\alpha,m)}, m = 1, 2,$

We can get the following difference method:

$$\begin{cases} \delta_t u_i^{n+\frac{1}{2}} = \left[(1-\theta)c_i^{n+1}({}_L\delta_x^{(\alpha,m)}u_{i+1}^{n+1}) + \theta c_i^n({}_L\delta_x^{(\alpha,m)}u_{i+1}^n)\right] \\ + \left[(1-\theta)d_i^{n+1}({}_R\delta_x^{(\alpha,m)}u_{i-1}^{n+1}) + \theta d_i^n({}_R\delta_x^{(\alpha,m)}u_{i-1}^n)\right] + (1-\theta)g_i^{n+1} + \theta g_i^n, i = 1, 2, \ldots, N-1, \\ u_i^0 = \phi_0(x_i), i = 0, 1, \ldots, N, \\ u_0^n = U_a(t_n), u_N^n = U_b(t_n), \end{cases} \tag{3.102}$$

where $0 \le \theta \le 1,_L \delta_x^{(\alpha,m)}$ is defined by (3.83) for $m = 1$ or by (3.86) for $m = 2$, and ${}_R\delta_x^{(\alpha,m)}$ is defined by (3.84) for $m = 1$ or by (3.87) for $m = 2$.

As in method (3.90), we can easily obtain that for $\frac{\sqrt{17}-1}{2} \le \alpha \le 2$, the method (3.102) with $m = 1$

is unconditionally stable for $0 \le \theta \le 1/2$, and conditionally stable for $1/2 < \theta \le 1$.

If $0 \le \theta \le 1/2$ and $c(x,t) = d(x,t) = K, K > 0$, then method (3.102) is unconditionally stable([31]).

3.3 One-Dimensional Time-Space Fractional Differential Equations

In this section, we numerically investigate the time-space fractional differential equations, where the time derivative and the spatial derivative are both fractional.

3.3.1 Time-Space Fractional Diffusion Equation with Caputo Derivative in Time

We consider the following time-space fractional diffusion equation

$$\begin{cases} {}^{C}D_{0,t}^{\gamma}u = (L^{\alpha}u)(x,t) + g(x,t), & (x,t) \in [a,b] \times [0,T], \\ u(x,0) = \phi_0(x), & x \in [a,b], \\ u(a,t) = U_a(t), u(b,t) = U_b(t), & t \in [0,T], \end{cases} \tag{3.103}$$

where $L^{\alpha} = c(x,t)^{RL}D_{a,x}^{\alpha} + d(x,t)^{RL}D_{x,b}^{\alpha}, 0 < \gamma \le 1, 1 < \alpha < 2$ and $c, d > 0$.

We introduce the notation $L_{\Delta x,q}^{(\alpha,n)}$ defined by

$$L_{\Delta x,q}^{(\alpha,n)}u_i^n = \begin{cases} d_i^n({}_L\delta_x^{(\alpha)}u_{i+1}^n) + c_i^n({}_R\delta_x^{(\alpha)}u_{i-1}^n), q = 1, \\ d_i^n({}_L\delta_x^{(\alpha,1)}u_i^n) + c_i^n({}_L\delta_x^{(\alpha,1)}u_i^n), q = 2, \\ d_i^n({}_L\delta_x^{(\alpha,2)}u_i^n) + c_i^n({}_R\delta_x^{(\alpha,2)}u_i^n), q = 3, \end{cases} \tag{3.104}$$

where ${}_L\delta_x^{(\alpha)}, {}_R\delta_x^{(\alpha)}, {}_L\delta_x^{(\alpha,1)}, {}_L\delta_x^{(\alpha,1)}, {}_L\delta_x^{(\alpha,2)}$ and ${}_R\delta_x^{(\alpha,2)}$ by (3.94),(3.95),(3.83),(3.86),(3.84) and (3.87).

It is known from the previous sections that

$$L_{\Delta x,q}^{(\alpha,n)}u_i^n = (L^{(\alpha)}u)(x_i,t_n) + O(\Delta t^p), \tag{3.105}$$

where

$$p = \begin{cases} 1, q = 1, \\ 2, q = 2, 3. \end{cases} \tag{3.106}$$

1. The time fractional derivative is discretized by the Grünwald-Letnikov formula and the space operator $L^{(\alpha)}$ in (3.103) is discretized as in (3.105); the fully discrete finite difference

method for (3.103) is given by:

$$\begin{cases} \delta_t^{(\gamma)}(u_i^n - u_i^0) = L_{\Delta x,q}^{(\alpha,n)} u_i^n + g_i^n, i = 1, 2, \ldots, N-1, \\ u_i^0 = \phi(x_i), i = 0, 1, \ldots, N, \\ u_0^n = u_n^n = 0, \end{cases} \tag{3.107}$$

where $\delta_t^{(\gamma)}$ is defined as in (3.7) and $L^{(\alpha,n)}$ is defined by (3.105).

2. The time fractional derivative is discretized and the space operator $L^{(\alpha)}$ in (3.103) is discretized as in (3.105); the fully discrete finite difference method for (3.103) is given by:

$$\begin{cases} {}_C^{L1}\delta_t^{(\gamma)} u_i^n = L_{\Delta x,q}^{(\alpha,n)} u_i^n + g_i^n, i = 1, 2, \ldots, N-1, \\ u_i^0 = \phi(x_i), i = 0, 1, \ldots, N, \\ u_0^n = u_n^n = 0, \end{cases} \tag{3.108}$$

where $L^{(\alpha,n)}$ is defined by (3.105) and

$${}_C^{L1}\delta_t^{(\gamma)} u^n = \frac{1}{\Delta t^\gamma}\left(\sum_{k=0}^{n-1} b_{n-k-1}^{(\gamma)}(u^{k+1} - u^k)\right), b_k^{(\gamma)} = \frac{(k+1)^{1-\gamma} - k^{1-\gamma}}{\Gamma(2-\gamma)}. \tag{3.109}$$

Next, we present the stability analysis for the methods (3.107)-(3.108). For simplicity, we suppose that $c(x,t) = d(x,t) = constant$. We first focus on the stability for (3.108). The matrix representation of (3.108) is given by:

$$(b_0^{(\gamma)}E - \mu S)\underline{\mathbf{u}}^n = \sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)})\underline{\mathbf{u}}^k + b_n^{(\gamma)}\underline{\mathbf{u}}^0 + \Delta t^\gamma \underline{\mathbf{g}}^n, \tag{3.110}$$

where $\mu = \frac{\Delta t^\gamma}{\Delta x^\alpha}$,E is an $(N-1)\times(N-1)$ identity matrix.u^n means

$$\mathbf{u}^n = (u_0^n, \ldots, u_N^n)^T, \underline{\mathbf{u}}^n = (u_1^n, \ldots, u_{N-1}^n)^T, n = 0, 1, \cdots .$$

$\mathbf{g}^n, \underline{\mathbf{g}}^n, \mathbf{e}^n, \underline{\mathbf{e}}^n, \mathbf{G}$ and $\mathbf{R}^n$ with $g_0^n = g_N^n = e_0^n = e_N^n = R_0^n = R_N^n = 0$ have the same meaning.It is known that all the eigenvalues of the matrix S have negative real parts. For any vector $u \in R^{N-1}$, we have $(S\mathbf{u}, \mathbf{u}) = \mathbf{u}^T S\mathbf{u} \le 0$. We have from $u_0^n = u_N^0 = 0$ that

$$\begin{aligned} b_0^{(\gamma)}||\mathbf{u}^n||_N^2 &\le b_0^{(\gamma)}||\mathbf{u}^n||_N^2 + \mu\Delta x(-S\underline{\mathbf{u}}^n, \underline{\mathbf{u}}^n) \\ &= \sum_{k=1}^{n-1}(b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)})(\mathbf{u}^k, \mathbf{u}^n)_N + b_n^{(\gamma)}(\mathbf{u}^0, \mathbf{u}^n)_N + \Delta t^\gamma(\mathbf{g}^n, \mathbf{u}^n)_N. \end{aligned}$$

We get

$$||\mathbf{u}^n||_N^2 \le 2||\mathbf{u}^0||_N^2 + C \max_{1\le n\le n_t} ||\mathbf{g}^n||_N^2. \tag{3.111}$$

For method (3.107), one can similarly obtain that the numerical solution of (3.107) satisfies (3.111).

Next, we consider the convergence analysis. Let $e_i^n = U(x_i, t_n) - u_i^n$. Then one gets the error equation for (3.108) as

$$ {}_C^{L1}\delta_t^{(\gamma)} e_i^n = L_{\Delta x,q}^{(\alpha,n)} e_i^n + R_i^n, \tag{3.112}$$

where R_i^n is the truncation error satisfying $|R_i^n| \le C(\Delta t^{2-\gamma} + \Delta x^p)$. We get

$$ ||e^n||_N^2 \le 2||e^0||_N^2 + C \max_{1\le n\le n_T} ||R^n||_N^2 \le C(\Delta t^{2-\gamma} + \Delta x^p). \tag{3.113}$$

The error bounds for the method (3.107) can be similarly obtained,which is given by

$$ ||e^n||_N^2 \le C(\Delta t + \Delta x^p). \tag{3.114}$$

3.3.2 Time-Space Fractional Diffusion Equation with Riemann-Liouville Derivative in Time

We consider the finite difference methods for the following time-space fractional diffusion equation

$$\begin{cases} \partial_t u = {}^{RL} D_{0,t}^{1-\gamma}(L^\alpha u) + g(x,t), & (x,t) \in [a,b]\times[0,T], \\ u(x,0) = \phi_0(x), & x \in [a,b], \\ u(a,t) = U_a(t), u(b,t) = U_b(t), & t \in [0,T], \end{cases} \tag{3.115}$$

where $L^\alpha = c(x,t)^{RL}D_{a,x}^\alpha + d(x,t)^{RL}D_{x,b}^\alpha, 0 < \gamma \le 1, 1 < \alpha < 2$ and $c, d > 0$.

We directly list several finite difference methods for (3.115).

1. **Explicit Euler type methods:**

   The time direction is discretized , the space operator $L^{(\alpha)}$ at $t = t_n$ is approximated by $L_{\Delta x,q}^{(\alpha,n)}$ which is defined as (3.105), and the fully discrete finite difference method for (3.115) is given by:

$$\begin{cases} \delta_t u_i^{n+\frac{1}{2}} = {}^{GL}\ \delta_t^{(1-\gamma)}(L_{\Delta x,q}^{(\alpha,n)} u_i^n) + f_i^n, i = 1,2,\ldots,N-1, \\ u_i^0 = \phi(x_i), i = 0,1,\ldots,N \\ u_0^n = u_n^n = 0. \end{cases} \tag{3.116}$$

   The time fractional derivative in (3.115) can be discretized by the $L1$ method or the fractional backward difference formula; we just need to replace ${}^{GL}\delta_t^{(1-\gamma)}$ in (3.116) by ${}_{L1}^{RL}\delta_t^{(1-\gamma)}$

   .

2. **Implicit Euler type methods:**

   The time direction is discretized , the space is discretized as in (3.116), the fully implicit Euler type method for (3.115) is given by:

$$\begin{cases} \delta_t u_i^{n-\frac{1}{2}} = ^{GL}\delta_t^{(1-\gamma)}(L_{\Delta x,q}^{(\alpha,n)} u_i^n) + f_i^n, i = 1, 2, \ldots, N-1, \\ u_i^0 = \phi(x_i), i = 0, 1, \ldots, N, \\ u_0^n = u_n^n = 0. \end{cases} \tag{3.117}$$

   The operator ${}^{GL}\delta_t^{(1-\gamma)}$ in (3.117) can be replaced by ${}^{RL}_{L1}\delta_t^{(1-\gamma)}$ or ${}^{B}_{p}\delta_t^{(1-\gamma)}$ when the $L1$ method is used in the discretization of the time fractional derivative, which yields various Euler type methods.

3. **Crank-Nicolson type methods:**

   The time direction is discretized as that in the CN, the space operator $L^{(\alpha)}$ at $t = t_n$ is approximated by $L_{\Delta x,q}^{(\alpha,n-\frac{1}{2})}$, the fully discrete Crank-Nicolson type method for (3.116) is given by:

$$\begin{cases} \delta_t u_i^{n-\frac{1}{2}} = ^{GL}\delta_t^{(1-\gamma)}(L_{\Delta x,q}^{(\alpha,n-\frac{1}{2})} u_i^{n-\frac{1}{2}}) + f_i^n, i = 1, 2, \ldots, N-1, \\ u_i^0 = \phi(x_i), i = 0, 1, \ldots, N, \\ u_0^n = u_n^n = 0, \end{cases} \tag{3.118}$$

   where ${}^{GL}\delta_t^{(1-\gamma)}(L_{\Delta x,q}^{(\alpha,n-\frac{1}{2})} u_i^{n-\frac{1}{2}})$ is defined by

$$\begin{aligned} {}^{GL}\delta_t^{(1-\gamma)}(L_{\Delta x,q}^{(\alpha,n-\frac{1}{2})} u_i^{n-\frac{1}{2}}) = \frac{1}{\Delta t^{1-\gamma}}[b_0 L_{\Delta x,q}^{(\alpha,n-\frac{1}{2})} u_i^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(b_{n-1-k} - b_{n-k}) L_{\Delta x,q}^{(\alpha,k-\frac{1}{2})} u^{k-\frac{1}{2}} \\ -(b_n - B_n) L_{\Delta x,q}^{(\alpha,\frac{1}{2})} u_i^{\frac{1}{2}} - A_n L_{\Delta x,q}^{(\alpha,0)} u_i^0]. \end{aligned}$$

   In which $A_n = B_n - \frac{\gamma(n+1/2)^{\gamma-1}}{\Gamma(1+\gamma)\Delta t^{1-\gamma}}, B_n = \frac{2\Delta t^{\gamma-1}}{\Gamma(1+\gamma)}[(n+1/2)^\gamma - n^\gamma], b_l = \frac{1}{\Gamma(1+\gamma)}[(l+1)^\gamma - l^\gamma]$, If $\gamma \longrightarrow 1$ and $\alpha \longrightarrow 2$, the explicit methods (3.116) is reduced to the classical forward Euler method, the implicit methods (3.117) is reduced to the classical backward Euler method, and the Crank-Nicolson type method (3.118) is reduced to the classical CN method.

   The stability and convergence analyses of the methods (3.116)-(3.118) are more complicated than their counterparts of the classical equations.

   If $c(x,t) = d(x,t) = K^\gamma > 0$, then the implicit method (3.117), the CN type method (3.118) are unconditionally stable and are convergent to order $(\Delta t + \Delta x^p), (\Delta t^{2-\gamma} + \Delta x^p)$.

Conclusion

In this thesis, we will be devoted to a some preliminary concepts will be introduced as the Euler Gamma function, Beta and Mittag-Leffler functions. we are also interested in elementary defenitions and basic notions relating to fractional calculus: the fractional integrals and fractional Derivatives, we also talked about some of their properties and the relationship between them. We also touched upon Partial Fractional Derivatives.

In addition, we studied the numerical ways of Approximations to Riemann-Liouville Derivatives using serval ways of them the Grünwald-Letnikov approximation, L1 approximation and similar methods to it.

In the last,we investigate the finite difference methods for the time-fractional equation in one spatial dimension, the space-fractional equations in one spatial dimension and time-space fractional equations in one space dimension .

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ملخص: في هذه المذكرة، درسنا طرق التحليل العددي للمشتقات الكسرية من بينها طرق ذات أنواع مختلفة للفروق المحدودة لحل معادلات تفاضلية جزئية كسرية أحادية البعد، وخاصة المعادلات الكسرية الزمنية، المعادلات الكسرية المكانية والمعادلات الكسرية الزمنية-المكانية.

كلمات مفتاحية: التكامل الكسري ، الاشتقاق الكسري ،الفروق المحدودة ، معادلات تفاضلية جزئية كسرية ، معادلات كسرية فرعية .

Abstract: In this thesis, we have studied several methods of numerical analysis for fractional derivatives; including the methods of different types of finite difference methods for solving one-dimensional fractional partial differential equations, especially time fractional equations, space fractional equations and time-space fractional equations.

Key-Words: Integrals Fractional, Fractional Derivatives, Finite Differences, Fractional Partial Differential Equations, Fractional Sub-diffusion equations.