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Existence Of Traveling Wave Solutions For A Free Boundary Problem Of Higher-Order Space-Fractional Wave Equations*

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Abstract

The fractional wave equation of higher order is presented as a generalization of the higher-order wave equation when arbitrary fractional-order derivatives are involved. This paper investigates the problem of existence and uniqueness of solutions under the traveling wave forms for a free boundary problem of higher-order space-fractional wave equations. It does so by applying the properties of Schauder's and Banach's fixed point theorems.

1 Introduction

The partial differential equations (PDEs) of fractional order appear as a natural description of observed evolution phenomena in various scientific areas. The fractional derivative operators are non-local and this property is important in application because it allows modeling the dynamics of many problems in physics, chemistry, engineering, medicine, economics, control theory, etc. For further reading on the subject, readers can refer to the following books (Samko et al. 1993 [1], Podlubny 1999 [2], Kilbas et al. 2006 [3], Diethelm 2010 [4]).

In this work, we shall give an example of a class of well-known fractional-order's PDEs; such equations are space-fractional wave equations of higher order and are written as follows:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^\alpha u}{\partial x^\alpha},\ c \in \mathbb{R}^*,\ m-1 \le \alpha < m \in \mathbb{N} - \{0,1,2\}, \tag{1}$$

with

$$\frac{\partial^\alpha u}{\partial x^\alpha} = \begin{cases} \frac{\partial^m u}{\partial x^m}, & \alpha = m \in \mathbb{N}, \\ \frac{\partial^\alpha u}{\partial x^\alpha} = \mathcal{I}_{ct}^{m-\alpha} \frac{\partial^m u}{\partial x^m} = \int_{ct}^{x} \frac{(x-\tau)^{m-\alpha-1}}{\Gamma(m-\alpha)} \frac{\partial^m}{\partial \tau^m} u(\tau,t)\, d\tau, & m-1 < \alpha < m \in \mathbb{N}^*, \end{cases}$$

where $u = u(x,t)$ is a scalar function of a space variable $x \in [ct, X]$ and time $t \in [0,T]$, with $T > 0$ and $X > |c|\,T$. The symbol $\mathcal{I}_*^\alpha$ presents the Riemann-Liouville's fractional integral of order α.

The higher-order space-fractional wave equation (1) becomes the wave equation for $\alpha = 2$, (see [5]) and the fourth-order wave equation for $\alpha = 4$, (see [6]). This was, with a second member, the first model of surface waves in shallow water that takes into consideration the balance between the nonlinearity and dispersion, thus, keeping the wave's shape; it is properly termed currently the 'Boussinesq paradigm with a second member. This balance bears solitary waves that behave like quasi-particles, these waves behave as particles called Solitons. This concept can be crucial for the interpretation of the dualism wave-particle in physics.

The existence and uniqueness of solutions for fractional differential equations or fractional-order's PDEs have been investigated in recent years. For more on the subject, we refer the reader to the following works [2]–[21].

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Our main goal in this work is to determine the existence, uniqueness and main properties of solutions of the space-fractional PDE (1), under the traveling wave form:

$$u(x,t) = a(t) f(x-ct), \text{ with } c \in \mathbb{R}^*, \tag{2}$$

the function $a(t) \neq 0$, which depends on time t, and the basic profile f are not known in advance and are to be identified.

We exemplify the role of Free Boundary Problems as an important source of ideas in modern analysis. With the help of a model problem, we illustrate the use of analytical techniques to obtain the existence and uniqueness of weak solutions via the use of the traveling wave method. This method permits us to reduce the fractional-order's PDE (1) to a fractional differential equation. This approach (2) is very promising and can also bring novel results for other applications in fractional-order's PDEs.

2 Definitions and Preliminary Results

In this section, we present the necessary definitions from the fractional calculus theory. By $C(J,\mathbb{R})$, we denote the Banach space of continuous functions from $J=[0,\lambda]$ into $\mathbb{R}$ with the norm:

$$\|f\|_\infty = \sup_{\eta\in J} |f(\eta)|.$$

We start with the definitions introduced in [3] with a slight modification in the notation.

Definition 1 ([3]) *The left-sided (arbitrary) fractional integral of order $\alpha > 0$ of a continuous function $f : J \to \mathbb{R}$ is given by:*

$$\mathcal{I}_{0^+}^{\alpha} f(\eta) = \frac{1}{\Gamma(\alpha)} \int_0^\eta (\eta-\xi)^{\alpha-1} f(\xi)\, d\xi, \ \eta \in J.$$

$\Gamma(\alpha) = \int_0^\infty \xi^{\alpha-1} \exp(-\xi)\, d\xi$ *is the Euler gamma function.*

Definition 2 (Caputo fractional derivative [3]) *The Caputo's left-sided fractional derivative of order $\alpha > 0$ of a function $f : J \to \mathbb{R}$ is given by:*

$${}^{C}\mathcal{D}_{0^+}^{\alpha} f(\eta) = \begin{cases} \frac{d^m f(\eta)}{d\eta^m}, \text{ for } \alpha = m \in \mathbb{N}, \\ \mathcal{I}_{0^+}^{m-\alpha} \frac{d^m f(\eta)}{d\eta^m} = \int_0^\eta \frac{(\eta-\xi)^{m-\alpha-1}}{\Gamma(m-\alpha)} \frac{d^m f(\xi)}{d\xi^m} d\xi, \text{ for } m-1 < \alpha < m \in \mathbb{N}^*. \end{cases}$$

Lemma 1 ([3]) *Assume that ${}^{C}\mathcal{D}_{0^+}^{\alpha} f \in C(J,\mathbb{R})$, for all $\alpha > 0$, then:*

$$\mathcal{I}_{0^+}^{\alpha}\ {}^{C}\mathcal{D}_{0^+}^{\alpha} f(\eta) = f(\eta) - \sum_{k=0}^{m-1} \frac{f^{(k)}(0)}{k!} \eta^k, \ m-1 < \alpha \leq m \in \mathbb{N}^*.$$

3 Main Results

Throughout the rest of this paper, we have $m \geq 3$ is a natural number and

$$m-1 \leq \alpha < m, \ T > 0 \text{ and } X > |c|\, T \text{ for some } c \in \mathbb{R}^*. \tag{3}$$

3.1 Statement of the Free Boundary Problem and Main Theorems

In this part, we first attempt to find the equivalent approximate to the following free boundary problem of the higher-order space-fractional wave equation:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^\alpha u}{\partial x^\alpha}, \ (x,t) \in [ct, X] \times [0,T], & m-1 \leq \alpha < m, \ c \in \mathbb{R}^*, \\ \frac{\partial^k u}{\partial x^k}(ct,t) = 0, & k = \{1,2,\ldots,m-1\}, \\ u(x,0) = f(x), \ \frac{\partial u}{\partial t}(x,0) = -c\,(cf(x) + f'(x)) & f, f' \in C(J,\mathbb{R}), \end{cases} \tag{4}$$

under the traveling wave form

$$u(x,t) = a(t) f(\eta), \text{ with } \eta = x - ct \text{ and } a \in \mathbb{R}^*, \text{ where } a(0) = 1. \tag{5}$$

Now, we give the principal theorems of this work.

Theorem 1 *Let $\alpha, c, T, X \in \mathbb{R}$, be the real constants given by (3). If*

$$\frac{c^2 (X + |c| T)^{\alpha} + \alpha (X + |c| T)^{\alpha - 2} (2 (X + |c| T) |c| + \alpha - 1)}{\Gamma(\alpha + 1)} < 1, \tag{6}$$

then the problem (4) has at least one solution in the traveling wave form (5).

Theorem 2 *Let $\alpha, c, T, X \in \mathbb{R}$, be the real constants given by (3) which satisfy the following inequality:*

$$0 < X + |c| T < \left(c^{-2} \Gamma(\alpha + 1)\right)^{\frac{1}{\alpha}}.$$

If

$$\frac{\alpha (X + |c| T)^{\alpha} [2 (X + |c| T) |c| + \alpha - 1]}{\Gamma(\alpha + 1) - c^2 (X + |c| T)^{\alpha}} < (X + |c| T)^2, \tag{7}$$

then the problem (4) admits a unique solution in the traveling wave form (5).

3.2 Existence and Uniqueness Results of the Basic Profile

First, we should deduce the equation satisfied by the function f in (5) and used for the definition of traveling wave solutions.

Theorem 3 *Let $(x,t) \in [ct, X] \times [0, T]$, then the transformation (5) reduces the partial differential equation problem of space-fractional order (4) to the ordinary differential equation of fractional order of the form:*

$${}^{C}\mathcal{D}^{\alpha}_{0+} f(\eta) = c^2 f(\eta) + 2c f'(\eta) + f''(\eta), \; \eta \in J = [0, \lambda], \; \text{for } \lambda = X + |c| T, \tag{8}$$

with the conditions:

$$f^{(k)}(0) = 0, \; \text{for } k = \{1, 2, \ldots, m-1\} \tag{9}$$

and we get $a(t) = \exp\left(-c^2 t\right)$.

Proof. The fractional equation resulting from the substitution of expression (5) in the original fractional-order's PDE (1), should be reduced to the standard bilinear functional equation (see [22]). First, for $\eta = x - ct$, we get $\eta \in J$ and

$$\frac{\partial^2 u}{\partial t^2} = \ddot{a}(t) f(\eta) - 2c\dot{a}(t) f'(\eta) + c^2 a(t) f''(\eta). \tag{10}$$

In another way, for $\xi = \tau - ct$, we get:

$$\begin{aligned} \frac{\partial^{\alpha} u}{\partial x^{\alpha}} &= \frac{1}{\Gamma(m-\alpha)} \int_{ct}^{x} (\eta - \tau)^{m-\alpha-1} \frac{\partial^m u(\tau, t)}{\partial \tau^m} d\tau \\ &= \frac{a(t)}{\Gamma(m-\alpha)} \int_{ct}^{x} (x - \tau)^{m-\alpha-1} \frac{d^m}{d\tau^m} f(\tau - ct) \, d\tau \\ &= \frac{a(t)}{\Gamma(m-\alpha)} \int_{0}^{\eta} (\eta - \xi)^{m-\alpha-1} \frac{d^m}{d\xi^m} f(\xi) \, d\xi \\ &= a(t) \, {}^{C}\mathcal{D}^{\alpha}_{0+} f(\eta). \end{aligned} \tag{11}$$

If we replace (10) and (11) in (1), we get:

$$^{C}\mathcal{D}_{0+}^{\alpha}f(\eta)=\frac{\ddot{a}(t)}{c^{2}a(t)}f(\eta)-\frac{2\dot{a}(t)}{ca(t)}f'(\eta)+f''(\eta).$$

From the conditions of the problem (4), we obtain $a(t)=\exp\left(-c^{2}t\right)$ and therefore the problem (8)–(9). The proof is complete. ■

In what follows, we present some significant lemmas to show the principal theorems.

We have:

Lemma 2 *Let $f,\ f',\ f'',{}^{C}\mathcal{D}_{0+}^{\alpha}f\in C(J,\mathbb{R})$, then the problem (8)–(9) is equivalent to the integral equation:*

$$f(\eta)=f(0)+\frac{1}{\Gamma(\alpha)}\int_{0}^{\eta}(\eta-\xi)^{\alpha-1}\left(c^{2}f(\xi)+2cf'(\xi)+f''(\xi)\right)d\xi,\ \forall\eta\in J.$$

Proof. Let $f,f',f'',{}^{C}\mathcal{D}_{0+}^{\alpha}f\in C(J,\mathbb{R})$, then by using Lemma 1, we reduce the fractional equation (8) to an equivalent fractional integral equation. By applying $\mathcal{I}_{0+}^{\alpha}$ to the equation (8), we obtain:

$$\mathcal{I}_{0+}^{\alpha}\ {}^{C}\mathcal{D}_{0+}^{\alpha}f(\eta)=\mathcal{I}_{0+}^{\alpha}\left(c^{2}f(\eta)+2cf'(\eta)+f''(\eta)\right).\tag{12}$$

From Lemma 1, we simply find:

$$\mathcal{I}_{0+}^{\alpha}\ {}^{C}\mathcal{D}_{0+}^{\alpha}f(\eta)=f(\eta)-\sum_{k=0}^{m-1}\frac{f^{(k)}(0)}{k!}\eta^{k},\ m-1<\alpha\leq m\in\mathbb{N}^{*}.$$

by using (9), the fractional integral equation (12) gives us:

$$f(\eta)=\mathcal{I}_{0+}^{\alpha}\left(c^{2}f(\eta)+2cf'(\eta)+f''(\eta)\right)+f(0).$$

The proof is complete. ■

Lemma 3 *Let $f,{}^{C}\mathcal{D}_{0+}^{\alpha}f\in C(J,\mathbb{R})$ be such that $f^{(k)}(0)=0$ exist $\forall k=\{1,2,\ldots,m-1\}$. Then:*

$$|2cf'(\eta)|+|f''(\eta)|\leq\frac{\lambda^{\alpha-2}(2\lambda|c|+\alpha-1)}{\Gamma(\alpha)}\left\|{}^{C}\mathcal{D}_{0+}^{\alpha}f\right\|_{\infty},\ \forall\eta\in[0,\lambda].\tag{13}$$

Proof. By using Lemma 1, for all ${}^{C}\mathcal{D}_{0+}^{\alpha}f\in C(J,\mathbb{R})$, we get:

$$\begin{aligned}\mathcal{I}_{0+}^{\alpha-1}\ {}^{C}\mathcal{D}_{0+}^{\alpha}f(\eta)&=\frac{d}{d\eta}\mathcal{I}_{0+}^{\alpha}\ {}^{C}\mathcal{D}_{0+}^{\alpha}f(\eta)\\&=f'(\eta)-f'(0)-\eta f''(0)+\cdots+\frac{\eta^{m-2}}{(m-2)!}f^{(m-1)}(0).\end{aligned}$$

and

$$\begin{aligned}\mathcal{I}_{0+}^{\alpha-2}\ {}^{C}\mathcal{D}_{0+}^{\alpha}f(\eta)&=\frac{d^{2}}{d\eta^{2}}\mathcal{I}_{0+}^{\alpha}\ {}^{C}\mathcal{D}_{0+}^{\alpha}f(\eta)\\&=f''(\eta)-f''(0)+\cdots+\frac{\eta^{m-3}}{(m-3)!}f^{(m-1)}(0).\end{aligned}$$

Moreover; if $f^{(k)}(0)=0$ exist $\forall k=\{1,2,\ldots,m-1\}$, then:

$$\mathcal{I}_{0+}^{\alpha-1}\ {}^{C}\mathcal{D}_{0+}^{\alpha}f(\eta)=f'(\eta)\ \text{and}\ \mathcal{I}_{0+}^{\alpha-2}\ {}^{C}\mathcal{D}_{0+}^{\alpha}f(\eta)=f''(\eta),$$

also, we have for any $\eta \in J$,

$$\begin{aligned} \left|2cf'(\eta)\right| + \left|f''(\eta)\right| &= 2\left|c\right|\left|\mathcal{I}_{0^+}^{\alpha-1}\, {}^{C}\mathcal{D}_{0^+}^{\alpha} f(\eta)\right| + \left|\mathcal{I}_{0^+}^{\alpha-2}\, {}^{C}\mathcal{D}_{0^+}^{\alpha} f(\eta)\right| \\ &\leq \int_0^{\eta} \frac{2\left|c\right|(\eta-\xi)^{\alpha-2}\left|{}^{C}\mathcal{D}_{0^+}^{\alpha} f(\xi)\right|}{\Gamma(\alpha-1)} d\xi + \int_0^{\eta} \frac{(\eta-\xi)^{\alpha-3}\left|{}^{C}\mathcal{D}_{0^+}^{\alpha} f(\xi)\right|}{\Gamma(\alpha-2)} d\xi \\ &\leq \frac{\lambda^{\alpha-2}\left(2\lambda\left|c\right| + \alpha - 1\right)}{\Gamma(\alpha)} \left\|{}^{C}\mathcal{D}_{0^+}^{\alpha} f\right\|_{\infty}. \end{aligned}$$

The proof is complete. ■

Theorem 4 *If we put*

$$\frac{c^2\lambda^{\alpha} + \alpha\lambda^{\alpha-2}\left(2\lambda\left|c\right| + \alpha - 1\right)}{\Gamma(\alpha+1)} < 1, \tag{14}$$

then the problem (8)–(9) has at least one solution on J.

Proof. To begin the proof, we will transform the problem (8)–(9) into a fixed point problem $\mathcal{A}f(\eta) = f(\eta)$, with

$$\mathcal{A}f(\eta) = f(0) + \frac{1}{\Gamma(\alpha)} \int_0^{\eta} (\eta-\xi)^{\alpha-1}\left(c^2 f(\xi) + 2cf'(\xi) + f''(\xi)\right) d\xi. \tag{15}$$

We first notice that if $f, {}^{C}\mathcal{D}_{0^+}^{\alpha} f \in C(J,\mathbb{R})$, then $\mathcal{A}f$ is being an operator of a polynomial and a primitive of continuous functions and its derivatives is indeed continuous (see (13) and the step 1 in this proof); therefore, it is an element of $C(J,\mathbb{R})$, and is equipped with the standard norm:

$$\left\|\mathcal{A}f\right\|_{\infty} = \sup_{\eta\in J}\left|\mathcal{A}f(\eta)\right|.$$

Because the problem (8)–(9) is equivalent to the fractional integral equation (15), the fixed points of $\mathcal{A}$ are solutions of the problem (8)–(9).

We demonstrate that $\mathcal{A}$ satisfies the assumption of Schauder's fixed point theorem (see [23]). This could be proved through three steps:

Step 1: $\mathcal{A}$ is a continuous operator.

Let $(f_n)_{n\in\mathbb{N}}$ be a real sequence such that $\lim_{n\to\infty} f_n = f$ in $C(J,\mathbb{R})$. Then $\forall \eta \in J$,

$$\begin{aligned} \left|\mathcal{A}f_n(\eta) - \mathcal{A}f(\eta)\right| \leq \int_0^{\eta} \frac{(\eta-\xi)^{\alpha-1}}{\Gamma(\alpha)} &\left|c^2\left(f_n(\xi) - f(\xi)\right)\right. \\ &\left. + 2c\left(f_n'(\xi) - f'(\xi)\right) + f_n''(\xi) - f''(\xi)\right| d\xi, \end{aligned} \tag{16}$$

where f_n and f satisfy the problem (8)–(9). Then we have:

$$\begin{aligned} \left|{}^{C}\mathcal{D}_{0^+}^{\alpha} f_n(\eta) - {}^{C}\mathcal{D}_{0^+}^{\alpha} f(\eta)\right| &= \left|c^2\left(f_n(\eta) - f(\eta)\right) + 2c\left(f_n'(\eta) - f'(\eta)\right)\right. \\ &\quad \left. + f_n''(\eta) - f''(\eta)\right| \\ &\leq c^2\left|f_n(\eta) - f(\eta)\right| + 2\left|c\right|\left|f_n'(\eta) - f'(\eta)\right| \\ &\quad + \left|f_n''(\eta) - f''(\eta)\right|. \end{aligned}$$

By using (13) from Lemma 3, we have:

$$\begin{aligned} \left\|{}^{C}\mathcal{D}_{0^+}^{\alpha} f_n - {}^{C}\mathcal{D}_{0^+}^{\alpha} f\right\|_{\infty} &\leq c^2\left\|f_n - f\right\|_{\infty} \\ &\quad + \frac{\lambda^{\alpha-2}\left(2\lambda\left|c\right| + \alpha - 1\right)}{\Gamma(\alpha)} \left\|{}^{C}\mathcal{D}_{0^+}^{\alpha} f_n - {}^{C}\mathcal{D}_{0^+}^{\alpha} f\right\|_{\infty}. \end{aligned}$$

According to (14), we have $\Gamma(\alpha) - \lambda^{\alpha-2}(2\lambda|c| + \alpha - 1) > \frac{c^2\lambda^\alpha}{\alpha} > 0$, thus:

$$\left\|{}^C\mathcal{D}^\alpha_{0+}f_n - {}^C\mathcal{D}^\alpha_{0+}f\right\|_\infty \leq \frac{c^2\Gamma(\alpha)}{\Gamma(\alpha) - \lambda^{\alpha-2}(2\lambda|c| + \alpha - 1)}\|f_n - f\|_\infty .$$

Since $f_n \to f$, we get ${}^C\mathcal{D}^\alpha_{0+}f_n \to {}^C\mathcal{D}^\alpha_{0+}f$ when $n \to \infty$ for each $\eta \in J$.

Now let $\mu > 0$, be such that for each $\eta \in J$, we have:

$$\left|{}^C\mathcal{D}^\alpha_{0+}f_n(\eta)\right| \leq \mu, \quad \left|{}^C\mathcal{D}^\alpha_{0+}f(\eta)\right| \leq \mu.$$

Then, we have:

$$\begin{aligned}
|\mathcal{A}f_n(\eta) - \mathcal{A}f(\eta)| &\leq \frac{1}{\Gamma(\alpha)}\int_0^\eta (\eta-\xi)^{\alpha-1} \times \\
&\quad \left|c^2(f_n(\xi) - f(\xi)) + 2c(f_n'(\xi) - f'(\xi)) + (f_n''(\xi) - f''(\xi))\right| d\xi, \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^\eta (\eta-\xi)^{\alpha-1}\left|{}^C\mathcal{D}^\alpha_{0+}f_n(\xi) - {}^C\mathcal{D}^\alpha_{0+}f(\xi)\right| d\xi \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^\eta (\eta-\xi)^{\alpha-1}\left[\left|{}^C\mathcal{D}^\alpha_{0+}f_n(\xi)\right| + \left|{}^C\mathcal{D}^\alpha_{0+}f(\xi)\right|\right] d\xi \\
&\leq \frac{2\mu}{\Gamma(\alpha)}\int_0^\eta (\eta-\xi)^{\alpha-1} d\xi.
\end{aligned}$$

For each $\eta \in J$, the function $\xi \to \frac{2\mu}{\Gamma(\alpha)}(\eta-\xi)^{\alpha-1}$ is integrable on $[0,\eta]$, then the Lebesgue dominated convergence theorem and (16) imply that:

$$|\mathcal{A}f_n(\eta) - \mathcal{A}f(\eta)| \to 0 \text{ as } n \to \infty,$$

and hence:

$$\lim_{n\to\infty}\|\mathcal{A}f_n - \mathcal{A}f\|_\infty = 0.$$

Consequently, $\mathcal{A}$ is continuous.

Step 2: According to (14), we put the positive real

$$r \geq \left(1 + \frac{c^2\lambda^\alpha}{\Gamma(\alpha+1) - \lambda^{\alpha-2}\left[\alpha(2\lambda|c| + \alpha - 1) + c^2\lambda^2\right]}\right)|f(0)|$$

and define the subset P as follows:

$$P = \left\{f \in C(J,\mathbb{R}) : \|f\|_\infty \leq r\right\}.$$

It is clear that P is a bounded, closed and convex subset of $C(J,\mathbb{R})$.

Let $f \in P$ be a function that satisfies the problem (8)–(9) and $\mathcal{A} : P \to C(J,\mathbb{R})$ be the integral operator defined by (15), then $\mathcal{A}(P) \subset P$.

In fact, by using (13) from Lemma 3, we have for each $\eta \in J$:

$$\begin{aligned}
\left|{}^C\mathcal{D}^\alpha_{0+}f(\eta)\right| &= \left|c^2f(\eta) + 2cf'(\eta) + f''(\eta)\right| \\
&\leq c^2|f(\eta)| + \frac{\lambda^{\alpha-2}(2\lambda|c| + \alpha - 1)}{\Gamma(\alpha)}\left\|{}^C\mathcal{D}^\alpha_{0+}f\right\|_\infty .
\end{aligned}$$

According to (14), we get $\Gamma(\alpha) - \lambda^{\alpha-2}(2\lambda|c| + \alpha - 1) > 0$ and

$$\left\|{}^C\mathcal{D}^\alpha_{0+}f\right\|_\infty \leq \frac{c^2\Gamma(\alpha)}{\Gamma(\alpha) - \lambda^{\alpha-2}(2\lambda|c| + \alpha - 1)}r.$$

Then

$$
\begin{aligned}
|\mathcal{A}f(\eta)| &\le |f(0)| + \frac{1}{\Gamma(\alpha)} \int_0^{\eta} (\eta-\xi)^{\alpha-1} \left|c^2 f(\xi) + 2cf'(\xi) + f''(\xi)\right| d\xi \\
&\le \frac{|f(0)| \left(1 + \frac{c^2\lambda^{\alpha}}{\Gamma(\alpha+1) - \lambda^{\alpha-2}[\alpha(2\lambda|c|+\alpha-1)+c^2\lambda^2]}\right)}{1 + \frac{c^2\lambda^{\alpha}}{\Gamma(\alpha+1) - \lambda^{\alpha-2}[\alpha(2\lambda|c|+\alpha-1)+c^2\lambda^2]}} \\
&+ \frac{c^2\lambda^{\alpha}}{\Gamma(\alpha+1) - \alpha\lambda^{\alpha-2}(2\lambda|c|+\alpha-1)} r \\
&\le r.
\end{aligned}
$$

Then $\mathcal{A}(P) \subset P$.

Step 3: $\mathcal{A}(P)$ is relatively compact.
Let $\eta_1, \eta_2 \in J$, $\eta_1 < \eta_2$, and $f \in P$. Then

$$
\begin{aligned}
|\mathcal{A}f(\eta_2) - \mathcal{A}f(\eta_1)| &= \left| \frac{1}{\Gamma(\alpha)} \int_0^{\eta_2} (\eta_2-\xi)^{\alpha-1} \left(c^2 f(\xi) + 2cf'(\xi) + f''(\xi)\right) d\xi \right. \\
&\quad \left. - \frac{1}{\Gamma(\alpha)} \int_0^{\eta_1} (\eta_1-\xi)^{\alpha-1} \left(c^2 f(\xi) + 2cf'(\xi) + f''(\xi)\right) d\xi \right| \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^{\eta_1} \left| \left((\eta_2-\xi)^{\alpha-1} - (\eta_1-\xi)^{\alpha-1}\right) \right. \\
&\quad \times \left. \left(c^2 f(\xi) + 2cf'(\xi) + f''(\xi)\right) \right| d\xi \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{\eta_1}^{\eta_2} (\eta_2-\xi)^{\alpha-1} \left|\left(c^2 f(\xi) + 2cf'(\xi) + f''(\xi)\right)\right| d\xi \\
&\le \frac{c^2 r}{\Gamma(\alpha) - \lambda^{\alpha-2}(2\lambda|c|+\alpha-1)} \left[\int_0^{\eta_1} \left|(\eta_2-\xi)^{\alpha-1} - \right. \right. \\
&\quad \left. \left. (\eta_1-\xi)^{\alpha-1}\right| d\xi + \int_{\eta_1}^{\eta_2} (\eta_2-\xi)^{\alpha-1} d\xi \right].
\end{aligned} \tag{17}
$$

We have:

$$
(\eta_2-\xi)^{\alpha-1} - (\eta_1-\xi)^{\alpha-1} = -\frac{1}{\alpha}\frac{d}{d\xi}\left[(\eta_2-\xi)^{\alpha} - (\eta_1-\xi)^{\alpha}\right],
$$

then

$$
\int_0^{\eta_1} \left|(\eta_2-\xi)^{\alpha-1} - (\eta_1-\xi)^{\alpha-1}\right| d\xi \le \frac{1}{\alpha}\left[(\eta_2-\eta_1)^{\alpha} + (\eta_2^{\alpha} - \eta_1^{\alpha})\right],
$$

we have also:

$$
\int_{\eta_1}^{\eta_2} (\eta_2-\xi)^{\alpha-1} d\xi = -\frac{1}{\alpha}\left[(\eta_2-\xi)^{\alpha}\right]_{\eta_1}^{\eta_2} \le \frac{1}{\alpha}(\eta_2-\eta_1)^{\alpha}.
$$

Then (17) gives us:

$$
|\mathcal{A}f(\eta_2) - \mathcal{A}f(\eta_1)| \le \frac{c^2 r \left(2(\eta_2-\eta_1)^{\alpha} + (\eta_2^{\alpha} - \eta_1^{\alpha})\right)}{\Gamma(\alpha+1) - \alpha\lambda^{\alpha-2}(2\lambda|c|+\alpha-1)}.
$$

As $\eta_1 \to \eta_2$, the right-hand side of the above inequality tends to zero.

As a consequence of steps 1 to 3, and by means of the Ascoli-Arzelà theorem, we deduce that $\mathcal{A} : P \to P$ is continuous, compact and satisfies the assumption of Schauder's fixed point theorem [23]. Then $\mathcal{A}$ has a fixed point which is a solution of the problem (8)–(9) on J. The proof is complete. ■

Theorem 5 *If we put* $0 < \lambda < \left(c^{-2}\Gamma(\alpha+1)\right)^{\frac{1}{\alpha}}$ *and*

$$\frac{\alpha\lambda^{\alpha}\left(2\lambda|c|+\alpha-1\right)}{\Gamma(\alpha+1)-c^{2}\lambda^{\alpha}} < \lambda^{2}, \tag{18}$$

then the problem (8)–(9) admits a unique solution on J.

Proof. In the previous Theorem 4, we transformed the problem (8)–(9) into a fixed point problem (15).
Let $f, g \in C(J, \mathbb{R})$ be two functions that satisfy the problem (8)–(9), then

$$\mathcal{A}f(\eta) - \mathcal{A}g(\eta) = \frac{1}{\Gamma(\alpha)}\int_0^{\eta}(\eta-\xi)^{\alpha-1} \times \left[c^{2}\left(f(\xi)-g(\xi)\right) + 2c\left(f'(\xi)-g'(\xi)\right) + \left(f''(\xi)-g''(\xi)\right)\right] d\xi.$$

Also

$$\left|\mathcal{A}f(\eta) - \mathcal{A}g(\eta)\right| \leq \frac{1}{\Gamma(\alpha)}\int_0^{\eta}(\eta-\xi)^{\alpha-1}\left|{}^{C}\mathcal{D}_{0^+}^{\alpha}f(\xi) - {}^{C}\mathcal{D}_{0^+}^{\alpha}g(\xi)\right| d\xi. \tag{19}$$

By using (13) from Lemma 3, we have:

$$\left\|{}^{C}\mathcal{D}_{0^+}^{\alpha}f - {}^{C}\mathcal{D}_{0^+}^{\alpha}g\right\|_{\infty} \leq c^{2}\|f-g\|_{\infty} + \frac{\lambda^{\alpha-2}\left(2\lambda|c|+\alpha-1\right)}{\Gamma(\alpha)}\left\|{}^{C}\mathcal{D}_{0^+}^{\alpha}f - {}^{C}\mathcal{D}_{0^+}^{\alpha}g\right\|_{\infty}.$$

As $\Gamma(\alpha) - \lambda^{\alpha-2}\left(2\lambda|c|+\alpha-1\right) > 0$, we have:

$$\left\|{}^{C}\mathcal{D}_{0^+}^{\alpha}f - {}^{C}\mathcal{D}_{0^+}^{\alpha}g\right\|_{\infty} \leq \frac{c^{2}\Gamma(\alpha)}{\Gamma(\alpha)-\lambda^{\alpha-2}\left(2\lambda|c|+\alpha-1\right)}\|f-g\|_{\infty}.$$

From (19) we find:

$$\|\mathcal{A}f - \mathcal{A}g\|_{\infty} \leq \frac{c^{2}\lambda^{\alpha}}{\Gamma(\alpha+1)-\alpha\lambda^{\alpha-2}\left(2\lambda|c|+\alpha-1\right)}\|f-g\|_{\infty}.$$

This implies that by (18), $\mathcal{A}$ is a contraction operator.

As a consequence Banach's contraction principle (see [23]), we deduce that $\mathcal{A}$ has a unique fixed point which is the unique solution of the problem (8)–(9) on J. The proof is complete. ■

3.3 Proof of Main Theorems

In this part, we prove the existence and uniqueness of solutions of the following free boundary problem of the higher-order space-fractional wave equation:

$$\begin{cases} \frac{\partial^{2}u}{\partial t^{2}} = c^{2}\frac{\partial^{\alpha}u}{\partial x^{\alpha}}, \ (x,t) \in [ct, X]\times[0,T], & m-1 \leq \alpha < m, \ c \in \mathbb{R}^{*}, \\ \frac{\partial^{k}u}{\partial x^{k}}(ct,t) = 0, & k = \{1,2,\ldots,m-1\}, \\ u(x,0) = f(x), \ \frac{\partial u}{\partial t}(x,0) = -c\left(cf(x)+f'(x)\right) & f, f' \in C(J,\mathbb{R}). \end{cases} \tag{20}$$

Under the traveling wave form:

$$u(x,t) = \exp\left(-c^{2}t\right)f(\eta), \text{ with } \eta = x - ct. \tag{21}$$

Proof of Theorem 1. The transformation (21) reduces the problem of the higher-order space-fractional wave equation (20) to the ordinary differential equation of fractional order of the form:

$${}^{C}\mathcal{D}_{0^+}^{\alpha}f(\eta) = c^{2}f(\eta) + 2cf'(\eta) + f''(\eta), \tag{22}$$

with the conditions:

$$f^{(k)}(0) = 0, \text{ for } k = \{1, 2, \ldots, m-1\}. \tag{23}$$

Let $f \in C(J, \mathbb{R})$ be a continuous function. By using Theorem 3, the condition (6) is equivalent to (14), which is:

$$\frac{c^2\lambda^\alpha + \alpha\lambda^{\alpha-2}(2\lambda|c| + \alpha - 1)}{\Gamma(\alpha+1)} < 1.$$

We already proved the existence of a solution of the problem (22)–(23) in Theorem 4, provided that (14) holds true. Consequently, if (6) holds for any $(x, t) \in [ct, X] \times [0, T]$, then there exists at least one solution of the problem of the higher-order space-fractional wave equation (20) under the traveling wave form (21). The proof is complete. ■

Proof of Theorem 2. Based on Theorem 5, we use the same steps through which we proved Theorem 1 to prove the existence and uniqueness of a traveling wave solution to the problem (20), provided that the condition (7) holds true. The proof is complete. ■

4 Conclusion

In this paper, we have discussed the existence and uniqueness of solutions for a class of space-fractional PDEs, which is known as a space-fractional wave equation of higher order with free boundary conditions, under the traveling wave form. For that, we used the Banach contraction principle and Schauder's fixed point theorem, while Caputo's fractional derivative is used as the differential operator.

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