Advances in the Theory of Nonlinear Analysis and its Applications 6 (2022) No. 3, 287–299.
https://doi.org/10.31197/atnaa.1031436
Available online at www.atnaa.org
Research Article

Advances in the Theory of Nonlinear Analysis and its Applications

ISSN: 2587-2648

Peer-Reviewed Scientific Journal

Analytical studies on the global existence and blow-up of solutions for a free boundary problem of two-dimensional diffusion equations of moving fractional order

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Abstract

This paper particularly addresses and discusses some analytical studies on the existence and uniqueness of global or blow-up solutions under the traveling profile forms for a free boundary problem of two-dimensional diffusion equations of moving fractional order. It does so by applying the properties of Schauder's and Banach's fixed point theorems. For application purposes, some examples of explicit solutions are provided to demonstrate the usefulness of our main results.

Keywords: Fractional diffusion, Moving fractional order, Free boundary, Blow-up, Global existence, Uniqueness.
2010 MSC: 35R11, 35A01, 34A08, 35C06, 34K37.

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1. Introduction

Partial and ordinary differential equations appear as a description of many observed evolution phenomena in different scientific areas. The adequacy of these equations motivates the researchers in the modeling of several real-world problems to investigate their qualitative and quantitative aspects.

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Received December 2, 2021, Accepted March 06, 2022, Online March 10, 2022

The non-local property for the derivative operators of fractional order is important in application because it allows to model the dynamics of many problems in physics, engineering, medicine, economics, control theory, etc. For further reading on the subject, readers can refer to the following books (Samko et al. 1993 [1], Podlubny 1999 [2], Kilbas et al. 2006 [3], Diethelm 2010 [4]).

Exact solutions (or closed-forms) of fractional PDEs have an important role in the appropriate understanding of many qualitative features of various phenomena and processes in different areas of natural sciences. Exact solutions of fractional equations demonstrate and allow for the unraveling of the mechanisms of various complex phenomena, such as the spatial localization of natural transfer processes, the existence of several peaking regimes among others (see [5]). Furthermore, simple solutions are often used in teaching many courses as specific examples illustrating basic tenets of a theory that admits mathematical formulation.

In this work, we shall give an example of a class of fractional-order's PDEs, which allow to describe the diffusion phenomena; it is a two-dimensional diffusion equation of moving fractional order, and is written as follows:

$$\frac{\partial u}{\partial t} = \kappa \left(\frac{\partial^s u}{\partial x^s} + \frac{\partial^s u}{\partial y^s} \right), \ \kappa \in \mathbb{R}^*, \tag{1}$$

for $m-1 < s \leq m \in \mathbb{N} - \{0,1\}$, with

$$\frac{\partial^s u}{\partial x^s} = \mathcal{I}^{m-s}_{f(y,t)} \frac{\partial^m u}{\partial x^m} \text{ and } \frac{\partial^s u}{\partial y^s} = \mathcal{I}^{m-s}_{f(x,t)} \frac{\partial^m u}{\partial y^m},$$

where $u = u(x,y,t)$ is a scalar function of free space variables

$$(x,y) \in \Omega = [f(y,t), g(y,t)] \times [f(x,t), g(x,t)]$$

and time $t \in [0,T)$, T may be a finite constant or infinite, f and g are two continuous functions to be identified. The symbol $\mathcal{I}^s_*$ presents the Riemann-Liouville's fractional integral of order s.

The fractional-order's PDE (1) generalizes and unifies several equations in a single form. For the one-dimensional case, the equation (1) becomes the transport equation for $s = 1$ and the linear dispersive equations of Airy type for $s = 3$.

The fractional operator given by (1) generalizes the traditional Laplace operator for $s = 2$. Therefore, for $m = 2$, the space-fractional diffusion equation (1) becomes a two-dimensional space-fractional heat equation, in which the existence problems of its self-similar solutions and its scale-invariant solutions have been discussed for the one-dimensional case in [6, 7, 8, 9, 10, 11].

The existence and uniqueness of solutions for fractional differential equations or fractional-order's PDEs have been investigated in recent years. For more on the subject, we refer the reader to the following works [6, 7, 8, 9, 10, 12, 13, 14, 15, 16].

Our main goal in this work is to determine the existence, uniqueness and main properties of the global or blow-up solution in time of the fractional-order's PDE (1), under the traveling profile form (see [17, 18]), which is:

$$u(x,y,t) = c(t)\,\varphi\left(\frac{x+y-b(t)}{a(t)} \right), \text{ with } a,c \in \mathbb{R}^*_+, \ b \in \mathbb{R}, \tag{2}$$

the functions $a(t)$, $b(t)$ and $c(t)$ depend on time t and the basic profile φ are not known in advance and are to be identified.

We represent the role of Free Boundary Problems in the real world as a significant source of new ideas in modern analysis. With the help of a model problem, we illustrate the use of analytical techniques to obtain the existence and uniqueness of weak solutions via the use of the traveling profile method. This method permits us to reduce the fractional-order's PDE (1) to a fractional differential equation; the idea is well illustrated with examples in our paper. This approach (2) is very promising and can also bring new results for other applications in fractional-order's PDEs.

2. Preliminary and necessary definitions

In this section, we present the necessary definitions from the fractional calculus theory. By $C(J,\mathbb{R})$, we denote the Banach space of continuous functions from $J=[0,1]$ into $\mathbb{R}$ with the norm:

$$\|\varphi\|_{\infty}=\sup_{\eta\in J}|\varphi(\eta)|.$$

We start with the definitions introduced in [3] with a slight modification in the notation.

Definition 2.1 ([3]). *The left-sided (arbitrary) fractional integral of order $s>0$ of a continuous function $\varphi:J\to\mathbb{R}$ is given by:*

$$\mathcal{I}_{0+}^{s}\varphi(\eta)=\frac{1}{\Gamma(s)}\int_0^{\eta}(\eta-\xi)^{s-1}\varphi(\xi)\,d\xi,\ \eta\in J.$$

$\Gamma(s)=\int_0^{\infty}\xi^{s-1}\exp(-\xi)\,d\xi$ *is the Euler gamma function.*

Definition 2.2 (Caputo fractional derivative [3]). *The left-sided Caputo fractional derivative of order $s>0$ of a function $\varphi:J\to\mathbb{R}$ is given by:*

$$^{C}\mathcal{D}_{0+}^{s}\varphi(\eta)=\begin{cases}\frac{d^m\varphi(\eta)}{d\eta^m}, & \text{for } s=m\in\mathbb{N},\\ \int_0^{\eta}\frac{(\eta-\xi)^{m-s-1}}{\Gamma(m-s)}\frac{d^m\varphi(\xi)}{d\xi^m}d\xi, & \text{for } m-1<s<m\in\mathbb{N}^*.\end{cases}\tag{3}$$

Lemma 2.3 ([3]). *Assume that $^{C}\mathcal{D}_{0+}^{s}\varphi\in C(J,\mathbb{R})$, for all $s>0$, then:*

$$\mathcal{I}_{0+}^{s}\,{}^{C}\mathcal{D}_{0+}^{s}\varphi(\eta)=\varphi(\eta)-\sum_{k=0}^{m-1}\frac{\varphi^{(k)}(0)}{k!}\eta^k,\ m-1<s\le m\in\mathbb{N}^*.$$

Remark 2.4. *Let $m\ge 2$ be a natural number and let $m-1<s\le m$, $\lambda\ge 1$, $^{C}\mathcal{D}_{0+}^{s}\varphi\in C(J,\mathbb{R})$ and $v\in\mathbb{R}$, be such that $\varphi'(0)=v$ and*

$$\Gamma(s)|v|\le(\lambda-1)\left\|^{C}\mathcal{D}_{0+}^{s}\varphi\right\|_{\infty}.\tag{4}$$

Then

$$\begin{aligned}\mathcal{I}_{0+}^{s-1}\,{}^{C}\mathcal{D}_{0+}^{s}\varphi(\eta)&=\frac{d}{d\eta}\mathcal{I}_{0+}^{s}\,{}^{C}\mathcal{D}_{0+}^{s}\varphi(\eta)\\&=\varphi'(\eta)-v-\varphi''(0)\eta-\cdots-\frac{1}{(m-2)!}\varphi^{(m-1)}(0)\eta^{m-2}.\end{aligned}$$

Moreover; if $m\ge 3$ and $\varphi^{(k)}(0)=0$ for each $k=2,3,\dots,m-1$, we get

$$\left|\varphi'(\eta)\right|=\left|\mathcal{I}_{0+}^{s-1}\,{}^{C}\mathcal{D}_{0+}^{s}\varphi(\eta)+v\right|\le\frac{\lambda}{\Gamma(s)}\left\|^{C}\mathcal{D}_{0+}^{s}\varphi\right\|_{\infty},\ \forall\eta\in J.\tag{5}$$

Theorem 2.5 (Schauder's fixed point [19]). *Let E be a Banach space, and P be a closed, convex and nonempty subset of E. Let $\mathcal{M}:P\to P$ be a continuous mapping such that $\mathcal{M}(P)$ is a relatively compact subset of E. Then $\mathcal{M}$ has at least one fixed point in P.*

Theorem 2.6 (Banach's fixed point [19]). *Let P be a non-empty closed subset of a Banach space E, then any contraction mapping $\mathcal{M}$ of P into itself has a unique fixed point.*

3. Main results

Throughout the rest of this paper, we have $m-1<s\le m$, with $m\ge 2$ is a natural number, $\kappa\in\mathbb{R}^*$, $\lambda\ge 1$ and $\alpha,\beta,\gamma,\omega,v\in\mathbb{R}$, where v satisfies the inequality (4) from Remark 2.4.

3.1. Statement of the free boundary problem and main theorems

In this part, we first attempt to find the equivalent approximate to the following free boundary problem of the two-dimensional diffusion equation of moving fractional order:

$$\begin{cases} \frac{\partial u}{\partial t} = \kappa\left(\frac{\partial^s u}{\partial x^s} + \frac{\partial^s u}{\partial y^s}\right), \ (x,y,t) \in \Omega \times [0,T), & \kappa \in \mathbb{R}^*, \\ u\left(f\left(y,t\right),y,t\right) = c\left(t\right)\omega, \ \frac{\partial u(x,f(x,t),t)}{\partial y} = v\frac{c(t)}{a(t)}, & a, c \in \mathbb{R}_+^*, \\ \frac{\partial^k u}{\partial y^k}\left(x, f\left(x,t\right),t\right) = 0, \ k = 2,3,\ldots,m-1, & \text{for } m \geq 3, \\ u\left(x,y,0\right) = \varphi\left(x+y\right), & \varphi \in C\left(J,\mathbb{R}\right), \end{cases} \tag{6}$$

under the traveling profile form

$$u\left(x,y,t\right) = c\left(t\right)\varphi\left(\eta\right), \text{ with } \eta = \frac{x+y-b\left(t\right)}{a\left(t\right)} \text{ and } a,c \in \mathbb{R}_+^*, \ b \in \mathbb{R}, \tag{7}$$

where

$$a\left(0\right) = c\left(0\right) = 1, \ b\left(0\right) = 0.$$

Now, we give the principal theorems of this work.

Theorem 3.1. *Let $a(t)$, $b(t)$ and $c(t)$ be three real functions of time t, given by the traveling profile form (7). If*

$$\frac{\left|\dot{c}\left(t\right)\right| + \lambda s a^{-1}\left(t\right) c\left(t\right)\left(\left|\dot{a}\left(t\right)\right| + \left|\dot{b}\left(t\right)\right|\right)}{\Gamma\left(s+1\right)} < 2\left|\kappa\right| \frac{c\left(t\right)}{a^s\left(t\right)}, \tag{8}$$

then the problem (6) has at least one solution in the traveling profile form (7), which is global in time when $\dot{a}(t) > 0$, and it blows up in a finite time:

$$0 < t < T = -\frac{a^{1-s}\left(t\right)}{s\dot{a}\left(t\right)} \text{ when } \dot{a}\left(t\right) < 0 \text{ and } \dot{c}\left(t\right) > 0.$$

Theorem 3.2. *Let $a(t)$, $b(t)$ and $c(t)$ be three real functions of time t, given by the traveling profile form (7) and satisfy the following inequality:*

$$\lambda a^{s-1}\left(t\right)\left(\left|\dot{a}\left(t\right)\right| + \left|\dot{b}\left(t\right)\right|\right) < \Gamma\left(s+1\right).$$

If

$$\frac{\left|\dot{c}\left(t\right)\right|}{\Gamma\left(s+1\right) - \lambda a^{s-1}\left(t\right)\left(\left|\dot{a}\left(t\right)\right| + \left|\dot{b}\left(t\right)\right|\right)} < 2\left|\kappa\right| \frac{c\left(t\right)}{a^s\left(t\right)}, \tag{9}$$

then the problem (6) admits a unique solution in the traveling profile form (7), which is global in time when $\dot{a}(t) > 0$, and it blows up in a finite time:

$$0 < t < T = -\frac{a^{1-s}\left(t\right)}{s\dot{a}\left(t\right)} \text{ when } \dot{a}\left(t\right) < 0 \text{ and } \dot{c}\left(t\right) > 0.$$

3.2. Existence and uniqueness results of the basic profile

We should first deduce the equation satisfied by the function φ in (7) and used for the definition of traveling profile solutions.

Theorem 3.3. *Let* $\kappa \in \mathbb{R}^*$, $(x, y, t) \in [f(y,t), g(y,t)] \times [f(x,t), g(x,t)] \times [0, T)$, *also let* $a(t)$, $b(t)$ *and* $c(t)$ *be three functions that satisfy the relation (7). By choosing:*

$$f(z,t) = b(t) - z,\ g(z,t) = f(z,t) + a(t),\ \text{where } z \text{ takes } x \text{ or } y,$$

then the transformation (7) reduces the partial differential equation problem of space-fractional order (6) to the ordinary differential equation of fractional order of the form:

$$^{C}\mathcal{D}_{0+}^{s}\varphi(\eta) = \alpha\varphi(\eta) + \beta\eta\varphi'(\eta) + \gamma\varphi'(\eta),\ \eta \in J, \tag{10}$$

with the conditions:

$$\begin{cases} \varphi(0) = \omega,\ \varphi'(0) = v,\ \text{for any } m \geq 2, \\ \varphi^{(k)}(0) = 0,\ k = 2, 3, \ldots, m-1,\ \text{for } m \geq 3, \end{cases} \tag{11}$$

where

$$(\alpha, \beta, \gamma) = \frac{a^s(t)}{2\kappa}\left(\frac{\dot{c}(t)}{c(t)}, -\frac{\dot{a}(t)}{a(t)}, -\frac{\dot{b}(t)}{a(t)}\right),\ \text{for some } \alpha, \beta, \gamma \in \mathbb{R}. \tag{12}$$

Proof. The fractional equation resulting from the substitution of expression (7) in the original fractional-order's PDE (1), should be reduced to the standard bilinear functional equation (see [17]). First, for $\eta = \frac{x+y-b(t)}{a(t)}$, we get $\eta \in J$ and

$$\frac{\partial u}{\partial t} = \dot{c}(t)\varphi(\eta) - c(t)\frac{\dot{a}(t)}{a(t)}\eta\varphi'(\eta) - c(t)\frac{\dot{b}(t)}{a(t)}\varphi'(\eta). \tag{13}$$

In another way, we get for $a(t)\xi_1 = \tau + y - b(t)$ and $a(t)\xi_2 = x + \tau - b(t)$ that:

$$\begin{aligned} \kappa\left(\frac{\partial^s u}{\partial x^s} + \frac{\partial^s u}{\partial y^s}\right) &= \kappa\left(\mathcal{I}_{f(y,t)}^{m-s}\frac{\partial^m u(x,y,t)}{\partial x^m} + \mathcal{I}_{f(x,t)}^{m-s}\frac{\partial^m u(x,y,t)}{\partial y^m}\right) \\ &= \frac{\kappa c(t)}{\Gamma(m-s)}\left(\int_{f(y,t)}^{x}(x-\tau)^{m-1-s}\frac{d^m\varphi\left(\frac{\tau+y-b(t)}{a(t)}\right)}{d\tau^m}d\tau\right. \\ &\quad \left.+ \int_{f(x,t)}^{y}(y-\tau)^{m-1-s}\frac{d^m}{d\tau^m}\varphi\left(\frac{x+\tau-b(t)}{a(t)}\right)d\tau\right) \\ &= \frac{\kappa c(t)a^{-s}(t)}{\Gamma(m-s)}\left(\int_0^{\eta}(\eta-\xi_1)^{m-1-s}\frac{d^m}{d\xi_1^m}\varphi(\xi_1)d\xi_1\right. \\ &\quad \left.+ \int_0^{\eta}(\eta-\xi_2)^{m-1-s}\frac{d^m}{d\xi_2^m}\varphi(\xi_2)d\xi_2\right) \\ &= 2\kappa c(t)a^{-s}(t)\, ^{C}\mathcal{D}_{0+}^{s}\varphi(\eta). \end{aligned} \tag{14}$$

If we replace (13) and (14) in (6), we get

$$^{C}\mathcal{D}_{0+}^{s}\varphi(\eta) = \frac{a^s(t)}{2\kappa}\left(\frac{\dot{c}(t)}{c(t)}\varphi(\eta) - \frac{\dot{a}(t)}{a(t)}\eta\varphi'(\eta) - \frac{\dot{b}(t)}{a(t)}\varphi'(\eta)\right),\ \kappa \in \mathbb{R}^*. \tag{15}$$

Now, let $\alpha, \beta, \gamma \in \mathbb{R}$ be such that the three functions $a(t)$, $b(t)$ and $c(t)$ of time variable $t \in [0, T)$, are solutions of the following system

$$\begin{cases} \dot{a}(t) = -2\kappa\beta a^{1-s}(t), \\ \dot{b}(t) = -2\kappa\gamma a^{1-s}(t), \\ \dot{c}(t) = 2\kappa\alpha c(t)a^{-s}(t), \end{cases}$$

which is equivalent to (12), and is resolved in the last part of this section (see the part 3.3).

If we replace (12) in (15), we obtain (10)–(11). The proof is complete. □

In what follows, we present some significant lemmas to show the principal theorems. We have:

Lemma 3.4. *Let $\varphi, \varphi', {}^{C}\mathcal{D}^{s}_{0+}\varphi \in C(J,\mathbb{R})$, then the problem (10)–(11) is equivalent to the integral equation:*

$$\varphi(\eta) = \omega + v\eta + \frac{1}{\Gamma(s)}\int_0^\eta (\eta-\xi)^{s-1}\left(\alpha\varphi(\xi) + \beta\xi\varphi'(\xi) + \gamma\varphi'(\xi)\right)d\xi. \tag{16}$$

Proof. Let $\varphi, \varphi', {}^{C}\mathcal{D}^{s}_{0+}\varphi \in C(J,\mathbb{R})$, then by using Lemma 2.3, we reduce the fractional equation (10) to an equivalent fractional integral equation. By applying $\mathcal{I}^{s}_{0+}$ to the equation (10) we obtain:

$$\mathcal{I}^{s}_{0+}\,{}^{C}\mathcal{D}^{s}_{0+}\varphi(\eta) = \mathcal{I}^{s}_{0+}\left(\alpha\varphi(\eta) + \beta\eta\varphi'(\eta) + \gamma\varphi'(\eta)\right). \tag{17}$$

From Lemma 2.3, we simply find:

$$\mathcal{I}^{s}_{0+}\,{}^{C}\mathcal{D}^{s}_{0+}\varphi(\eta) = \varphi(\eta) - \varphi(0) - \eta\varphi'(0) - \cdots - \frac{1}{(m-1)!}\eta^{m-1}\varphi^{(m-1)}(0).$$

By using (11), the fractional integral equation (17) gives us:

$$\varphi(\eta) = \mathcal{I}^{s}_{0+}\left(\alpha\varphi(\eta) + \beta\eta\varphi'(\eta) + \gamma\varphi'(\eta)\right) + \omega + v\eta. \tag{18}$$

The proof is complete. □

Theorem 3.5. *If we put*

$$\frac{|\alpha| + \lambda s\left(|\beta| + |\gamma|\right)}{\Gamma(s+1)} < 1, \tag{19}$$

then the problem (10)–(11) has at least one solution on J.

Proof. To begin the proof, we will transform the problem (10)–(11) into a fixed point problem $\mathcal{M}\varphi(\eta) = \varphi(\eta)$, with

$$\mathcal{M}\varphi(\eta) = \omega + v\eta + \int_0^\eta \frac{(\eta-\xi)^{s-1}}{\Gamma(s)}\left(\alpha\varphi(\xi) + \beta\xi\varphi'(\xi) + \gamma\varphi'(\xi)\right)d\xi. \tag{20}$$

We first notice that if $\varphi \in C(J,\mathbb{R})$, then $\mathcal{M}\varphi$ is actually continuous since it is an operator of a polynomial, a primitive of continuous functions and their derivatives. Hence, we can consider it an element of $C(J,\mathbb{R})$ and is equipped with the standard norm:

$$\|\mathcal{M}\varphi\|_\infty = \sup_{\eta\in J}|\mathcal{M}\varphi(\eta)|.$$

Because the problem (10)–(11) is equivalent to the fractional integral equation (20), the fixed points of $\mathcal{M}$ are solutions of the problem (10)–(11).

We demonstrate that $\mathcal{M}$ satisfies the assumption of Schauder's fixed point theorem 2.5. This could be proved through three steps:

Step 1 $\mathcal{M}$ is a continuous operator.

Let $(\varphi_n)_{n\in\mathbb{N}}$ be a real sequence such that $\lim_{n\to\infty}\varphi_n = \varphi$ in $C(J,\mathbb{R})$. Then for all $\eta \in J$,

$$\begin{aligned}|\mathcal{M}\varphi_n(\eta) - \mathcal{M}\varphi(\eta)| \le \int_0^\eta \frac{(\eta-\xi)^{s-1}}{\Gamma(s)}\big|\alpha\left(\varphi_n(\xi) - \varphi(\xi)\right) \\ + \beta\xi\left(\varphi_n'(\xi) - \varphi'(\xi)\right) \\ +\gamma\left(\varphi_n'(\xi) - \varphi'(\xi)\right)\big|\,d\xi,\end{aligned} \tag{21}$$

where φ_n and φ satisfy the problem (10)–(11). Then we have:

$$\begin{aligned}\left|{}^{C}\mathcal{D}_{0+}^{s}\varphi_n(\eta)-{}^{C}\mathcal{D}_{0+}^{s}\varphi(\eta)\right| &= |\alpha(\varphi_n(\eta)-\varphi(\eta)) \\ &\quad +(\beta\eta+\gamma)\left(\varphi_n'(\eta)-\varphi'(\eta)\right)| \\ &\le |\alpha|\,|\varphi_n(\eta)-\varphi(\eta)| \\ &\quad +(|\beta|+|\gamma|)\left|\varphi_n'(\eta)-\varphi'(\eta)\right|.\end{aligned}$$

By using (5) from Remark 2.4, we have:

$$\begin{aligned}\left\|{}^{C}\mathcal{D}_{0+}^{s}\varphi_n-{}^{C}\mathcal{D}_{0+}^{s}\varphi\right\|_\infty &\le |\alpha|\,\|\varphi_n-\varphi\|_\infty+\frac{\lambda(|\beta|+|\gamma|)}{\Gamma(s)} \\ &\quad \times\left\|{}^{C}\mathcal{D}_{0+}^{s}\varphi_n-{}^{C}\mathcal{D}_{0+}^{s}\varphi\right\|_\infty.\end{aligned}$$

According to (19), we have $\Gamma(s)-\lambda(|\beta|+|\gamma|)>\frac{1}{s}|\alpha|\ge 0$, thus:

$$\left\|{}^{C}\mathcal{D}_{0+}^{s}\varphi_n-{}^{C}\mathcal{D}_{0+}^{s}\varphi\right\|_\infty \le \frac{|\alpha|\,\Gamma(s)}{\Gamma(s)-\lambda(|\beta|+|\gamma|)}\|\varphi_n-\varphi\|_\infty.$$

Since $\varphi_n\to\varphi$, then we get ${}^{C}\mathcal{D}_{0+}^{s}\varphi_n\to{}^{C}\mathcal{D}_{0+}^{s}\varphi$ as $n\to\infty$ for each $\eta\in J$.

Now let $\mu>0$, be such that for each $\eta\in J$, we have:

$$\left|{}^{C}\mathcal{D}_{0+}^{s}\varphi_n(\eta)\right|\le\mu,\ \ \left|{}^{C}\mathcal{D}_{0+}^{s}\varphi(\eta)\right|\le\mu.$$

Then, we have:

$$\begin{aligned}|\mathcal{M}\varphi_n(\eta)-\mathcal{M}\varphi(\eta)| &\le \frac{1}{\Gamma(s)}\int_0^\eta(\eta-\xi)^{s-1}|\alpha(\varphi_n(\xi)-\varphi(\xi)) \\ &\quad +(\beta\xi+\gamma)\left(\varphi_n'(\xi)-\varphi'(\xi)\right)|\,d\xi, \\ &\le \frac{1}{\Gamma(s)}\int_0^\eta(\eta-\xi)^{s-1}\times \\ &\quad \left|{}^{C}\mathcal{D}_{0+}^{s}\varphi_n(\xi)-{}^{C}\mathcal{D}_{0+}^{s}\varphi(\xi)\right|d\xi \\ &\le \frac{2\mu}{\Gamma(s)}\int_0^\eta(\eta-\xi)^{s-1}d\xi.\end{aligned}$$

For each $\eta\in J$, the function $\xi\to\frac{2\mu}{\Gamma(s)}(\eta-\xi)^{s-1}$ is integrable on $[0,\eta]$, then the Lebesgue dominated convergence theorem and (21) imply that:

$$|\mathcal{M}\varphi_n(\eta)-\mathcal{M}\varphi(\eta)|\to 0 \text{ as } n\to\infty,$$

and hence:

$$\lim_{n\to\infty}\|\mathcal{M}\varphi_n-\mathcal{M}\varphi\|_\infty=0.$$

Consequently, $\mathcal{M}$ is continuous.

Step 2 According to (19), we put the positive real

$$r\ge\left(1+\frac{|\alpha|}{\Gamma(s+1)-(\lambda s(|\beta|+|\gamma|)+|\alpha|)}\right)(|\omega|+|v|)$$

and define the subset P as follows:

$$P=\{\varphi\in C(J,\mathbb{R}):\|\varphi\|_\infty\le r\}.$$

It is clear that P is a bounded, closed and convex subset of $C(J,\mathbb{R})$.

Let $\varphi \in P$ be a function which satisfies the problem (10)–(11) and $\mathcal{M} : P \to C(J,\mathbb{R})$ be the integral operator defined by (20), then $\mathcal{M}(P) \subset P$.

In fact, by using (5) from Remark 2.4, we have for each $\eta \in J$:

$$\begin{aligned}\left|{}^{C}\mathcal{D}_{0^+}^{s}\varphi(\eta)\right| &= \left|\alpha\varphi(\eta) + \beta\eta\varphi'(\eta) + \gamma\varphi'(\eta)\right| \\ &\le |\alpha|\,|\varphi(\eta)| + (|\beta| + |\gamma|)\left|\varphi'(\eta)\right|.\end{aligned}$$

According to (19), we get $\Gamma(s) > \lambda(|\beta| + |\gamma|)$ and

$$\left\|{}^{C}\mathcal{D}_{0^+}^{s}\varphi\right\|_{\infty} \le \frac{|\alpha|\,\Gamma(s)}{\Gamma(s) - \lambda(|\beta| + |\gamma|)}r. \tag{22}$$

Then

$$\begin{aligned}|\mathcal{M}\varphi(\eta)| &\le |\omega| + |v| + \frac{1}{\Gamma(s)}\int_0^{\eta}(\eta-\xi)^{s-1}\times \\ &\quad \left|\alpha\varphi(\xi) + \beta\xi\varphi'(\xi) + \gamma\varphi'(\xi)\right| d\xi \\ &\le \frac{(|\omega| + |v|)\left(1 + \frac{|\alpha|}{\Gamma(s+1) - (\lambda s(|\beta|+|\gamma|)+|\alpha|)}\right)}{1 + \frac{|\alpha|}{\Gamma(s+1) - (\lambda s(|\beta|+|\gamma|)+|\alpha|)}} \\ &\quad + \frac{|\alpha|\,r}{\Gamma(s+1) - \lambda s(|\beta| + |\gamma|)} \\ &\le r.\end{aligned}$$

Then $\mathcal{M}(P) \subset P$.

Step 3 $\mathcal{M}(P)$ is relatively compact.

Let $\eta_1, \eta_2 \in J$, $\eta_1 < \eta_2$, and $\varphi \in P$. Then

$$\begin{aligned}&|\mathcal{M}\varphi(\eta_2) - \mathcal{M}\varphi(\eta_1)| \le |v|(\eta_2 - \eta_1) \\ &+ \frac{1}{\Gamma(s)}\int_0^{\eta_1}\Big|\left((\eta_2-\xi)^{s-1} - (\eta_1-\xi)^{s-1}\right) \\ &\times \left(\alpha\varphi(\xi) + \beta\xi\varphi'(\xi) + \gamma\varphi'(\xi)\right)\Big| d\xi \\ &+ \frac{1}{\Gamma(s)}\int_{\eta_1}^{\eta_2}(\eta_2-\xi)^{s-1}\left|\left(\alpha\varphi(\xi) + \beta\xi\varphi'(\xi) + \gamma\varphi'(\xi)\right)\right| d\xi \\ &\le |v|(\eta_2 - \eta_1) + \frac{|\alpha|\,r}{\Gamma(s) - \lambda(|\beta| + |\gamma|)}\left[\int_0^{\eta_1}\Big|(\eta_2-\xi)^{s-1} - \right. \\ &\left.(\eta_1-\xi)^{s-1}\Big| d\xi + \int_{\eta_1}^{\eta_2}(\eta_2-\xi)^{s-1}d\xi\right].\end{aligned} \tag{23}$$

We have:

$$(\eta_2-\xi)^{s-1} - (\eta_1-\xi)^{s-1} = -\frac{1}{s}\frac{d}{d\xi}\left[(\eta_2-\xi)^{s} - (\eta_1-\xi)^{s}\right],$$

then

$$\int_0^{\eta_1}\left|(\eta_2-\xi)^{s-1} - (\eta_1-\xi)^{s-1}\right| d\xi \le \frac{1}{s}\left[(\eta_2-\eta_1)^{s} + (\eta_2^{s} - \eta_1^{s})\right],$$

we have also:

$$\int_{\eta_1}^{\eta_2}(\eta_2-\xi)^{s-1}d\xi = -\frac{1}{s}\left[(\eta_2-\xi)^{s}\right]_{\eta_1}^{\eta_2} \le \frac{1}{s}(\eta_2-\eta_1)^{s}.$$

Then (23) gives us:

$$\left|\mathcal{M}\varphi\left(\eta_2\right)-\mathcal{M}\varphi\left(\eta_1\right)\right| \leq |v|\left(\eta_2-\eta_1\right)+\frac{|\alpha|\, r\left(2\left(\eta_2-\eta_1\right)^s+\left(\eta_2^s-\eta_1^s\right)\right)}{\Gamma\left(s+1\right)-\lambda s\left(|\beta|+|\gamma|\right)}.$$

As $\eta_1 \to \eta_2$, the right-hand side of the above inequality tends to zero.

As a consequence of steps 1 to 3, and by means of the Ascoli-Arzelà theorem, we deduce that $\mathcal{M} : P \to P$ is continuous, compact and satisfies the assumption of Schauder's fixed point theorem 2.5. Then $\mathcal{M}$ has a fixed point which is a solution of the problem (10)–(11) on J. The proof is complete. □

Theorem 3.6. *If we put* $\lambda\left(|\beta|+|\gamma|\right)<\Gamma\left(s\right)$ *and:*

$$\frac{|\alpha|}{\Gamma\left(s+1\right)-\lambda s\left(|\beta|+|\gamma|\right)}<1, \tag{24}$$

then the problem (10)–(11) admits a unique solution on J.

Proof. In the previous Theorem 3.5, we already transform the problem (10)–(11) into a fixed point problem (20).

Let $\varphi, \psi \in C\left(J, \mathbb{R}\right)$ be two functions that satisfy the problem (10)–(11), then

$$\begin{aligned}\mathcal{M}\varphi\left(\eta\right)-\mathcal{M}\psi\left(\eta\right)=&\frac{1}{\Gamma\left(s\right)}\int_0^\eta\left(\eta-\xi\right)^{s-1}\left[\alpha\left(\varphi\left(\xi\right)-\psi\left(\xi\right)\right)\right.\\&\left.+\left(\beta\xi+\gamma\right)\left(\varphi'\left(\xi\right)-\psi'\left(\xi\right)\right)\right]d\xi.\end{aligned}$$

Also

$$\left|\mathcal{M}\varphi\left(\eta\right)-\mathcal{M}\psi\left(\eta\right)\right| \leq \int_0^\eta \frac{\left(\eta-\xi\right)^{s-1}}{\Gamma\left(s\right)}\left|{}^C\mathcal{D}_{0^+}^s\varphi\left(\xi\right)-{}^C\mathcal{D}_{0^+}^s\psi\left(\xi\right)\right|d\xi. \tag{25}$$

For all $\eta \in J$, we have:

$$\begin{aligned}\left|{}^C\mathcal{D}_{0^+}^s\varphi\left(\eta\right)-{}^C\mathcal{D}_{0^+}^s\psi\left(\eta\right)\right|&=\left|\alpha\left(\varphi\left(\eta\right)-\psi\left(\eta\right)\right)+\left(\beta\eta+\gamma\right)\left(\varphi'\left(\eta\right)-\psi'\left(\eta\right)\right)\right|\\&\leq|\alpha|\left|\varphi\left(\eta\right)-\psi\left(\eta\right)\right|+\left(|\beta|+|\gamma|\right)\left|\varphi'\left(\eta\right)-\psi'\left(\eta\right)\right|.\end{aligned}$$

By using (5) from Remark 2.4, we have:

$$\left\|{}^C\mathcal{D}_{0^+}^s\varphi-{}^C\mathcal{D}_{0^+}^s\psi\right\|_\infty \leq |\alpha|\left\|\varphi-\psi\right\|_\infty+\frac{\lambda\left(|\beta|+|\gamma|\right)}{\Gamma\left(s\right)}\left\|{}^C\mathcal{D}_{0^+}^s\varphi-{}^C\mathcal{D}_{0^+}^s\psi\right\|_\infty.$$

As $\Gamma\left(s\right)-\lambda\left(|\beta|+|\gamma|\right)>0$, we have:

$$\left\|{}^C\mathcal{D}_{0^+}^s\varphi-{}^C\mathcal{D}_{0^+}^s\psi\right\|_\infty \leq \frac{|\alpha|\,\Gamma\left(s\right)}{\Gamma\left(s\right)-\lambda\left(|\beta|+|\gamma|\right)}\left\|\varphi-\psi\right\|_\infty.$$

From (25) we find:

$$\left\|\mathcal{M}\varphi-\mathcal{M}\psi\right\|_\infty \leq \frac{|\alpha|}{\Gamma\left(s+1\right)-\lambda s\left(|\beta|+|\gamma|\right)}\left\|\varphi-\psi\right\|_\infty.$$

This implies that by (24), $\mathcal{M}$ is a contraction operator.

As a consequence of Theorem 2.6, using Banach's contraction principle [19], we deduce that $\mathcal{M}$ has a unique fixed point which is the unique solution of the problem (10)–(11) on J. The proof is complete. □

3.3. Proof of main theorems

In this part, we prove the existence and uniqueness of solutions of the following free boundary problem of the two-dimensional diffusion equation of moving fractional order:

$$\begin{cases} \frac{\partial u}{\partial t} = \kappa\left(\frac{\partial^s u}{\partial x^s} + \frac{\partial^s u}{\partial y^s}\right), \ (x,y,t) \in \Omega \times [0,T), & \kappa \in \mathbb{R}^*, \\ u(f(y,t), y, t) = c(t)\,\omega, \ \frac{\partial u(x,f(x,t),t)}{\partial y} = v\frac{c(t)}{a(t)}, & a, c \in \mathbb{R}^*_+, \\ \frac{\partial^k u}{\partial y^k}(x, f(x,t), t) = 0, \ k = 2, 3, \ldots, m-1, & \text{for } m \geq 3, \\ u(x,y,0) = \varphi(x+y), & \varphi \in C(J, \mathbb{R}), \end{cases} \tag{26}$$

under the traveling profile form:

$$u(x,y,t) = c(t)\,\varphi(\eta), \text{ with } \eta = \frac{x+y-b(t)}{a(t)} \text{ and } a, c \in \mathbb{R}^*_+, \ b \in \mathbb{R}, \tag{27}$$

with $\Omega = [f(y,t), g(y,t)] \times [f(x,t), g(x,t)]$, for

$$f(z,t) = b(t) - z, \ g(z,t) = f(z,t) + a(t), \text{ where } z \text{ takes } x \text{ or } y.$$

Proof of Theorem 3.1
The transformation (27) reduces the two-dimensional space-fractional diffusion equation in (26) to the ordinary differential equation of fractional order of the form:

$${}^C\mathcal{D}^s_{0+}\varphi(\eta) = \alpha\varphi(\eta) + \beta\eta\varphi'(\eta) + \gamma\varphi'(\eta), \ \alpha, \beta, \gamma \in \mathbb{R}, \tag{28}$$

where

$$\eta = \frac{x+y-b(t)}{a(t)}, \ \text{for } a \in \mathbb{R}^*_+, \ b \in \mathbb{R} \text{ and } (x,y) \in \Omega,$$

and

$$\begin{cases} \dot{a}(t) = -2\kappa\beta a^{1-s}(t), \\ \dot{b}(t) = -2\kappa\gamma a^{1-s}(t), \\ \dot{c}(t) = 2\kappa\alpha c(t)\, a^{-s}(t), \end{cases} \tag{29}$$

with the conditions:

$$\begin{cases} \varphi(0) = \omega, \ \varphi'(0) = v, \ \text{for any } m \geq 2, \\ \varphi^{(k)}(0) = 0, \ k = 2, 3, \ldots, m-1, \ \text{for } m \geq 3. \end{cases} \tag{30}$$

Now, to determine the functions $a(t)$, $b(t)$ and $c(t)$, we just solve the system (29).
We denote by $(z)_+$ the positive part of z, which is z if $z > 0$ and what remains is zero.
If $\beta = 0$, we have $a(t) = 1$ and

$$\begin{cases} b(t) = -2\kappa\gamma t, \\ c(t) = \exp(2\kappa\alpha t). \end{cases} \quad t > 0.$$

If $\beta \neq 0$, after an integration from 0 to t we get:

$$\begin{cases} a(t) = (1 - 2s\kappa\beta t)_+^{\frac{1}{s}}, \\ b(t) = \frac{\gamma}{2s\kappa\beta}\left((1 - 2s\kappa\beta t)_+^{\frac{1}{s}} - 1\right), \quad 0 < t < T, \\ c(t) = (1 - 2s\kappa\beta t)_+^{-\frac{\alpha}{s\beta}}, \end{cases} \tag{31}$$

where $T > 0$ is the maximal existence time for the solution u, which may be finite or infinite. Thereupon, we separate the following cases:

1. If $\kappa\beta \leq 0$ (i.e., $\dot{a}(t) \geq 0$), the problem (26) admits a global solution in time under the traveling profile form (27); this solution is defined for all $t > 0$, (i.e., $T = \infty$).
   In addition, for $\alpha\beta > 0$ (i.e., $\dot{c}(t) < 0$), we have:
$$\lim_{t\to+\infty} u(x,y,t) = 0, \text{ for all } (x,y) \in \Omega.$$
2. If $\kappa\beta > 0$ (i.e., $\dot{a}(t) < 0$), the functions $a(t)$, $b(t)$ and $c(t)$ are defined locally and are well-defined if and only if
$$0 < t < T = \frac{1}{2s\kappa\beta} = -\frac{a^{1-s}(t)}{s\dot{a}(t)}.$$
   The moment $T = \frac{1}{2s\kappa\beta}$ represents the maximal existence value of the functions $a(t)$, $b(t)$ and $c(t)$. Moreover; if $\alpha\beta > 0$ (i.e., $\dot{c}(t) > 0$), the problem (26) admits a solution under the traveling profile form (27), which blows up in a finite time. The solution is defined for all $t \in [0,T)$, the moment T represents the blow-up time of the solution such that:
$$\lim_{t\to T^-} u(x,y,t) = \lim_{t\to T^-} c(t)\,\varphi\left(\frac{x+y-b(t)}{a(t)}\right) = +\infty,$$
   for all $x,y \in \mathbb{R}$. We recall that the solution blows up in finite time if there exists a time $T < +\infty$, which we call the blow-up time, such that the solution is well defined for all $0 < t < T$, while:
$$\sup_{x,y\in\mathbb{R}} |u(x,y,t)| \to +\infty, \text{ when } t \to T^- = \frac{1}{2s\kappa\beta}.$$

Now, let $\varphi \in C(J,\mathbb{R})$ be a continuous function. By using (27), the condition (8) is equivalent to (19), which is:
$$\frac{|\alpha| + \lambda s(|\beta| + |\gamma|)}{\Gamma(s+1)} < 1.$$
We already proved the existence of a solution to the problem (28)–(30) in Theorem 3.5, provided that (19) holds true. Consequently, if (8) holds for any $(x,y,t) \in \Omega \times [0,T)$, then there exists at least one solution of the problem of the two-dimensional diffusion equation of moving fractional order (26) under the traveling profile form (27). The proof is complete.

Proof of Theorem 3.2

Based on Theorem 3.6, we use the same steps through which we proved Theorem 3.1 to prove the existence and uniqueness of global or blow-up traveling profile solution to the problem (26), provided that the condition (9) holds true. The proof is complete.

4. Explicit solutions

Example 1: According to the proof of the Theorem 3.1, for $\beta = \gamma = 0$ and $\alpha, \kappa \in \mathbb{R}^*$, we get
$$a(t) = 1, \; b(t) = 0 \text{ and } c(t) = \exp(2\kappa\alpha t).$$
In this case, for $m = 2$, (i.e., Space-fractional heat equation), we give new explicit solutions on the traveling profile form of the problem (26) as follows:
$$u(x,y,t) = \exp(2\kappa\alpha t)\left(\omega E_s(\alpha(x+y)^s) + v(x+y)\,E_{s,2}\left(\alpha(x+y)^2\right)\right),$$
for $\omega, v \in \mathbb{R}$, where $E_{s,m}(\eta)$ is the function of Mittag-Leffler type. The solution is defined for all $t > 0$.
Example 2: We present new explicit solutions on the traveling profile form of the problem (26):

For $m \geq 2$, if we put $\beta, \kappa, \omega, v \in \mathbb{R}^*$, $\alpha = (1-m)\beta$ and $\gamma = \frac{(m-1)\beta\omega}{v}$, we get that

$$\varphi(\eta) = \omega + \sum_{k=1}^{m-1} \frac{(m-1)! v^k}{k!\,(m-k-1)!\,(m-1)^k\,\omega^{k-1}} \eta^k,$$

is a solution of the problem (28)–(30). Then the traveling profile solution of the problem (26) is presented as follows:

$$u(x,y,t) = c(t)\left[\omega + \sum_{k=1}^{m-1} \frac{\frac{(m-1)!}{(m-k-1)!} v^k}{k!\omega^{k-1}(m-1)^k} \left(\frac{x+y-b(t)}{a(t)}\right)^k\right], \tag{32}$$

where

$$\begin{cases} a(t) = (1-2s\kappa\beta t)_+^{\frac{1}{s}}, \\ b(t) = \frac{\gamma}{2s\kappa\beta}\left((1-2s\kappa\beta t)_+^{\frac{1}{s}} - 1\right), \quad 0 < t < T, \\ c(t) = (1-2s\kappa\beta t)_+^{\frac{m-1}{s}}. \end{cases}$$

According to the proof of the Theorem 3.1, we separate the following cases:

1. If $\kappa\beta < 0$, the problem (26) admits a global solution in time under the form (32), this solution is defined for all $t > 0$.
2. If $\kappa\beta > 0$, the functions $a(t)$, $b(t)$ and $c(t)$ are defined if and only if $0 < t < T = \frac{1}{2s\kappa\beta}$ and the solution does not blow up in the moment T, because $\alpha\beta = (1-m)\beta^2 < 0$. Moreover;

$$\sup_{x,y\in\mathbb{R}} |u(x,y,t)| \to 0, \text{ when } t \to T = \frac{1}{2s\kappa\beta}.$$

5. Conclusion

In this paper, we have discussed the existence and uniqueness of solutions for a class of fractional-order's PDEs, which are known as two-dimensional diffusion equations of moving fractional order, with mixed free boundary conditions under the traveling profile form. The behavior of these solutions depends on some parameters that satisfy some conditions which make their existing global or local in a time T. For that, we used the Banach contraction principle and Schauder's fixed point theorem, while Caputo's fractional derivative is used as the differential operator.

Acknowledgment

The authors are deeply grateful to the reviewers and editors for their insightful comments that helped to improve the quality of this research, which was supported by the General Direction of Scientific Research and Technological Development (DGRSTD). They would also like to extend their appreciation to Ms. Ikhlas HADJI for proofreading the article.

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