Acta Math. Univ. Comenianae
Vol. LXXXIX, 2 (2020), pp. 243–260

EXISTENCE RESULTS FOR NONLINEAR KATUGAMPOLA FRACTIONAL DIFFERENTIAL EQUATIONS WITH AN INTEGRAL CONDITION

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Abstract. This work studies the existence and uniqueness of solutions for a class of nonlinear fractional differential equations via the Katugampola fractional derivatives with an integral condition. The arguments for the study are based upon the Banach contraction principle, Schauder's fixed point theorem, and the nonlinear alternative of Leray-Schauder type.

1. Introduction

Fractional calculus is a mathematical branch which investigates the properties of derivatives and integrals of non-integer orders (also known as fractional derivatives and integrals, briefly differ-integrals). We refer the interested readers in the subject to the books (Samko et al. 1993 [**19**], Podlubny 1999 [**18**], Kilbas et al. 2006 [**15**], Diethelm 2010 [**7**]).

The differential equations of fractional order are generalizations of classical differential equations of integer order they are increasingly used in such fields as fluid flow, control theory of dynamical systems, diffusive transport akin to diffusion, probability and statistics etc. The boundary value problem of fractional differential equations was recently approached by various researchers ([**6**], [**17**], [**13**], [**4**]).

In [**6**], Benchohra and Lazreg applied the Banach contraction principle with Schauder fixed-point theorem and Leray-Schauder type to show the existence and uniqueness of solutions for an initial value problem of the nonlinear implicit fractional differential equation

$$\begin{cases} {}^{c}\mathcal{D}_{0^+}^{\alpha} u(t) = f\left(t, u(t), {}^{c}\mathcal{D}_{0^+}^{\alpha} u(t)\right), \quad t \in [0,T]\,,\ T>0,\ 0<\alpha\leq 1, \\ u\left(0\right) = u_0, \end{cases}$$

where ${}^{c}\mathcal{D}_{0^+}^{\alpha} u$ is the Caputo fractional derivative, $f\colon\ [0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a given function, and $u_0\in\mathbb{R}$.

Received June 23, 2019; revised October 1, 2019.
2010 *Mathematics Subject Classification.* Primary 34A08, 34A37.
Key words and phrases. Fractional equation; integral condition; fixed point theorems; existence; uniqueness.

In [**17**], by means of Schauder fixed-point theorem and the Banach contraction principle, Murad and Hadid, considered the boundary value problem of the fractional differential equation

$$\begin{cases} \mathcal{D}_{0^+}^{\alpha} u(t) = f\big(t, u(t), \mathcal{D}_{0^+}^{\beta} u(t)\big), & t \in (0,1),\ 1 < \alpha \leq 2,\ 0 < \beta < 1, \\ u(0) = 0,\ u(1) = \mathcal{I}_{0^+}^{\gamma} u(s), & 0 < \gamma \leq 1, \end{cases}$$

where $\mathcal{D}_{0^+}^{\alpha} u$ (resp., $\mathcal{I}_{0^+}^{\alpha} u$) is the Riemann-Liouville fractional derivative (resp., fractional integral), and $f\colon [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

In this work, our objective is to study in a general manner the existence and uniqueness of solutions of nonlinear fractional differential equations

$$ {}^{\rho}\mathcal{D}_{0^+}^{\alpha} u(t) = f\big(t, u(t), {}^{\rho}\mathcal{D}_{0^+}^{\beta} u(t)\big), \qquad t \in [0,T], \tag{1}$$

with the integral condition

$$\left({}^{\rho}\mathcal{I}_{0^+}^{1-\alpha} u\right)\left(0^+\right) = 0, \tag{2}$$

where $0 < \beta < \alpha \leq 1$, $\rho > 0$, and for any $1 \leq p \leq \infty$, $c > 0$, $T \leq (pc)^{\frac{1}{pc}}$ is a finite positive constant. The symbol ${}^{\rho}\mathcal{D}_{0^+}^{\alpha}$ (resp., ${}^{\rho}\mathcal{I}_{0^+}^{\alpha}$) presents the Katugampola fractional derivative (resp., integral) operator and $f\colon [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

We obtain several existence and uniqueness results for the problem (1)–(2).

2. Preliminaries

In this section, we present the necessary definitions from fractional calculus theory. As in [**15**], consider the space $X_c^p[0,T]$, $(c \in \mathbb{R},\ 1 \leq p \leq \infty)$ of those complex-valued Lebesgue measurable functions u on $[0,T]$, for which $\|u\|_{X_c^p} < \infty$, where the norm is defined by

$$\|u\|_{X_c^p} = \Big(\int_0^T |s^c u(s)|^p \,\frac{\mathrm{d}s}{s}\Big)^{\frac{1}{p}} < \infty$$

for $1 \leq p < \infty$, $c \in \mathbb{R}$. For the case $p = \infty$,

$$\|u\|_{X_c^\infty} = \operatorname{ess}\sup_{0 \leq t \leq T} \left[t^c \left|u(t)\right|\right] \qquad (c \in \mathbb{R}).$$

By $C[0,T]$, we denote the Banach space of all continuous functions from $[0,T]$ into $\mathbb{R}$ with the norm

$$\|u\|_\infty = \sup_{0 \leq t \leq T} |u(t)|.$$

Remark. Let $p, c, T \in \mathbb{R}_+^*$ be such that $p \geq 1$, $c > 0$, and $T \leq (pc)^{\frac{1}{pc}}$,
It's clear that for all $u \in C[0,T]$

$$\|u\|_{X_c^p} = \Big(\int_0^T |s^c u(s)|^p \,\frac{\mathrm{d}s}{s}\Big)^{\frac{1}{p}} \leq \Big(\|u\|_\infty^p \int_0^T s^{pc-1} \mathrm{d}s\Big)^{\frac{1}{p}} = \frac{T^c}{(pc)^{\frac{1}{p}}} \|u\|_\infty$$

and

$$\|u\|_{X_c^\infty} = \operatorname{ess}\sup_{0 \leq t \leq T} \left[t^c \left|u(t)\right|\right] \leq T^c \|u\|_\infty,$$

which implies that $C[0,T] \hookrightarrow X_c^p[0,T]$ and

$$\|u\|_{X_c^p} \leq \|u\|_\infty \qquad \text{for all } \; T \leq (pc)^{\frac{1}{pc}} \,.$$

We start with the definitions introduced in [**15**], with a slight modification in the notation.

Definition 2.1 (Riemann-Liouville fractional integral [**15**])**.** The left-sided Riemann-Liouville fractional integral of order $\alpha \in \mathbb{R}_+$ of a continuous function $u\colon [0,T] \to \mathbb{R}$ is given by

$${}^{RL}\mathcal{I}_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) \mathrm{d}s, \qquad t \in [0,T].$$

Definition 2.2 (Riemann-Liouville fractional derivative [**15**])**.** The left-sided Riemann Liouville fractional derivative of order $\alpha \in \mathbb{R}_+$ of a continuous function $u\colon [0,T] \to \mathbb{R}$ is given by

$${}^{RL}\mathcal{D}_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_0^t (t-s)^{n-\alpha-1} u(s) \mathrm{d}s, \quad t \in [0,T], \; n = [\alpha]+1,$$

$[\alpha]$ denotes the integer part of α.

Definition 2.3 (Hadamard fractional integral [**15**])**.** The left-sided Hadamard fractional integral of order $\alpha \in \mathbb{R}_+$ of a continuous function $u\colon [0,T] \to \mathbb{R}$ is given by

$${}^{H}\mathcal{I}_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{u(s)}{s} \mathrm{d}s, \qquad t \in [0,T].$$

Definition 2.4 (Hadamard fractional derivative [**15**])**.** Left-sided Hadamard fractional derivative of order $\alpha \in \mathbb{R}_+$ of a continuous function $u\colon [0,T] \to \mathbb{R}$ is given by

$${}^{H}\mathcal{D}_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_0^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \frac{u(s)}{s} \mathrm{d}s, \quad t \in [0,T], \; n = [\alpha]+1,$$

$[\alpha]$ denotes the integer part of α.

A recent generalization, introduced by Udita Katugampola (2011) [**14**], generalizes the Riemann-Liouville fractional integral and the Hadamard fractional integral (see [**15**]).

The integral is now known as the Katugampola fractional integral, it is given in the following definition

Definition 2.5 (Katugampola fractional integral [**14**])**.** The *Katugampola* fractional integrals of order $\alpha \in \mathbb{R}_+$ of a function $u \in X_c^p[0,T]$ is defined by

$$(3) \qquad {}^{\rho}\mathcal{I}_{0^+}^{\alpha} u(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{s^{\rho-1} u(s)}{(t^\rho - s^\rho)^{1-\alpha}} \mathrm{d}s, \qquad t \in [0,T],$$

for $\rho > 0$. This integral is called left-sided integral.

Similarly, we can define right-sided integrals [**15**], [**12**]–[**14**]. In a similar way, we have the following definition.

Definition 2.6 (Katugampola fractional derivatives [**12**])**.** If the integral exists, the generalized fractional derivatives of order $\alpha \in \mathbb{R}_+$, corresponding to the *Katugampola* fractional integrals (3), defined for any $t \in [0, T]$, by

$$\begin{aligned} {}^{\rho}\mathcal{D}^{\alpha}_{0+} y(t) &= \left(t^{1-\rho}\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n} \left({}^{\rho}\mathcal{I}^{n-\alpha}_{0+} y\right)(t) \\ &= \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\left(t^{1-\rho}\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n} \int_0^t \frac{s^{\rho-1}y(s)}{(t^{\rho}-s^{\rho})^{\alpha-n+1}}\mathrm{d}s, \end{aligned} \tag{4}$$

$[\alpha]$ denotes the integer part of α, $n = [\alpha]+1$, and $\rho > 0$.

Remark ([**12**]–[**14**])**.** As a basic example, for $\alpha, \rho > 0$, and $\mu > -\rho$, we quote

$${}^{\rho}\mathcal{D}^{\alpha}_{0+} t^{\mu} = \frac{\rho^{\alpha-1}\Gamma\left(1+\frac{\mu}{\rho}\right)}{\Gamma\left(1-\alpha+\frac{\mu}{\rho}\right)} t^{\mu-\alpha\rho}.$$

Let us give in particular

$${}^{\rho}\mathcal{D}^{\alpha}_{0+} t^{\rho(\alpha-m)} = 0 \qquad \text{for each } m = 1, 2, \ldots, n.$$

In fact, for $\alpha, \rho > 0$ and $\mu > -\rho$, we have

$${}^{\rho}\mathcal{D}^{\alpha}_{0+} t^{\mu} = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\left(t^{1-\rho}\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n} \int_0^t s^{\rho+\mu-1}\left(t^{\rho}-s^{\rho}\right)^{n-\alpha-1}\mathrm{d}s$$

$$= \frac{\rho^{\alpha-1}\Gamma\left(1+\frac{\mu}{\rho}\right)}{\Gamma\left(1+n-\alpha+\frac{\mu}{\rho}\right)}\left[n-\alpha+\frac{\mu}{\rho}\right]\cdots\left[1-\alpha+\frac{\mu}{\rho}\right]t^{\mu-\alpha\rho} \tag{5}$$

$$= \frac{\rho^{\alpha-1}\Gamma\left(1+\frac{\mu}{\rho}\right)}{\Gamma\left(1-\alpha+\frac{\mu}{\rho}\right)} t^{\mu-\alpha\rho}. \tag{6}$$

If we put $m = \alpha - \frac{\mu}{\rho}$ from (5), we obtain

$${}^{\rho}\mathcal{D}^{\alpha}_{0+} t^{\rho(\alpha-m)} = \rho^{\alpha-1}\frac{\Gamma(\alpha-m+1)}{\Gamma(n-m+1)}(n-m)(n-m-1)\ldots(1-m)t^{-\rho m}.$$

So, for $m = 1, 2, \ldots, n$, we have ${}^{\rho}\mathcal{D}^{\alpha}_{0+} t^{\rho(\alpha-m)} = 0$ for all $\alpha, \rho > 0$.
Similarly, for all $\alpha, \rho > 0$, we have

$${}^{\rho}\mathcal{I}^{\alpha}_{0+} t^{\mu} = \frac{\rho^{-\alpha}\Gamma\left(1+\frac{\mu}{\rho}\right)}{\Gamma\left(1+\alpha+\frac{\mu}{\rho}\right)} t^{\mu+\alpha\rho} \qquad \text{for all } \mu > -\rho. \tag{7}$$

In the theorem below, we present some properties of Katugampola fractional integrals and derivatives.

Theorem 2.7 ([**12**]–[**14**])**.** *Let $\alpha, \beta, \rho, c \in \mathbb{R}$ be such that $\alpha, \beta, \rho > 0$. Then for any $u, v \in X^p_c[0, T]$, where $1 \le p \le \infty$, we have:*
– Index property:

$${}^{\rho}\mathcal{I}^{\alpha}_{0+}\,{}^{\rho}\mathcal{I}^{\beta}_{0+} u(t) = {}^{\rho}\mathcal{I}^{\alpha+\beta}_{0+} u(t) \qquad \textit{for all } \alpha, \beta > 0,$$

$${}^{\rho}\mathcal{D}^{\alpha}_{0+}\,{}^{\rho}\mathcal{D}^{\beta}_{0+} u(t) = {}^{\rho}\mathcal{D}^{\alpha+\beta}_{0+} u(t) \qquad \textit{for all } 0 < \alpha, \beta < 1,$$

– *Inverse property:*

$$ {}^{\rho}\mathcal{D}_{0^+}^{\alpha}\, {}^{\rho}\mathcal{I}_{0^+}^{\alpha} u(t) = u(t) \qquad \text{for all } \alpha \in (0,1) \tag{8} $$

– *Linearity property: for all* $\alpha \in (0,1)$, *we have*

$$ \begin{cases} {}^{\rho}\mathcal{D}_{0^+}^{\alpha}\,(u+v)(t) = {}^{\rho}\mathcal{D}_{0^+}^{\alpha} u(t) + {}^{\rho}\mathcal{D}_{0^+}^{\alpha} v(t)\,, \\ {}^{\rho}\mathcal{I}_{0^+}^{\alpha}\,(u+v)(t) = {}^{\rho}\mathcal{I}_{0^+}^{\alpha} u(t) + {}^{\rho}\mathcal{I}_{0^+}^{\alpha} v(t)\,. \end{cases} \tag{9} $$

From Definitions 2.5 and 2.6, and Theorem 2.7, we deduce that

$$ \begin{aligned} {}^{\rho}\mathcal{I}_{0^+}^{1}\Big(t^{1-\rho}\frac{\mathrm{d}}{\mathrm{d}t}\Big)\, {}^{\rho}\mathcal{I}_{0^+}^{\alpha+1} u(t) &= \int_0^t s^{\rho-1}\Big(s^{1-\rho}\frac{\mathrm{d}}{\mathrm{d}s}\Big)\, {}^{\rho}\mathcal{I}_{0^+}^{\alpha+1} u(s)\,\mathrm{d}s = \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\, {}^{\rho}\mathcal{I}_{0^+}^{\alpha+1} u(s)\,\mathrm{d}s \\ &= \Big[\frac{1}{\rho^{\alpha}\Gamma(\alpha+1)}\int_0^s \tau^{\rho-1}\left(t^{\rho}-\tau^{\rho}\right)^{\alpha} u(\tau)\,\mathrm{d}\tau\Big]_0^t = {}^{\rho}\mathcal{I}_{0^+}^{\alpha+1} u(t). \end{aligned} $$

Consequently,

$$ \Big(t^{1-\rho}\frac{\mathrm{d}}{\mathrm{d}t}\Big)\, {}^{\rho}\mathcal{I}_{0^+}^{\alpha+1} u(t) = {}^{\rho}\mathcal{I}_{0^+}^{\alpha} u(t) \qquad \text{for all } \alpha > 0. \tag{10} $$

Theorem 2.8 (Ascoli-Arzelà [**1**]). *Let E be a compact space. If $\mathcal{A}$ is an equicontinuous, bounded subset of $C(E)$, then $\mathcal{A}$ is relatively compact.*

Definition 2.9 (Completely continuous [**9**]). We say $\mathcal{A}\colon E \to E$ is completely continuous if for any bounded subset $P \subset E$, the set $\mathcal{A}(P)$ is relatively compact.

Lemma 2.10 (Gronwall [**11**]). *Let $u(t)$ and $v(t)$ be nonnegative, continuous functions on $0 \le t \le T$ for which the inequality*

$$ u(t) \le \mu + \int_0^t v(s)u(s)\mathrm{d}s, \qquad 0 \le t \le T, $$

where μ is a nonnegative constant, holds. Then

$$ u(t) \le \mu \exp\Big(\int_0^t v(s)\mathrm{d}s\Big), \qquad 0 \le t \le T. $$

Theorem 2.11 (Banach's fixed point [**10**]). *Let P be a non-empty closed subset of a Banach space E, then any contraction mapping $\mathcal{A}$ of P into itself has a unique fixed point.*

Theorem 2.12 (Schauder's fixed point [**10**]). *Let E be a Banach space and P be a closed, convex, and nonempty subset of E. Let $\mathcal{A}\colon P \to P$ be a continuous mapping such that $\mathcal{A}(P)$ is a relatively compact subset of E. Then $\mathcal{A}$ has at least one fixed point in P.*

Theorem 2.13 (Nonlinear Alternative of Leray-Schauder type [**10**]). *Let E be a Banach space with $P \subset E$ being closed and convex. Assume U is a relatively open subset of P with $0 \in U$ and $\mathcal{A}\colon \bar{U} \to P$ is a compact map. Then either*

(i) *$\mathcal{A}$ has a fixed point in $\bar{U}$ or*
(ii) *there is a point $u \in \partial U$ and $\mu \in (0,1)$ with $u = \mu\mathcal{A}(u)$.*

3. Main results

Throughout the remaining of this paper T, p and c are real constants such that

$$p \geq 1, \ c > 0, \quad \text{and} \quad T \leq (pc)^{\frac{1}{pc}}.$$

In what follows, we present some significant lemmas to show the principal theorems.

Lemma 3.1. *Let $\alpha, \rho \in \mathbb{R}$ be such that $0 < \alpha \leq 1$ and $\rho > 0$. We define*

$$P = \left\{ u \in C[0,T] \mid \left({}^{\rho}\mathcal{I}_{0^+}^{1-\alpha} u \right) \left(0^+ \right) = 0 \right\}. \tag{11}$$

Then $(P, \|\cdot\|_{\infty})$ is a Banach space.

Proof. Let $0 < \alpha \leq 1$ and $\rho > 0$.
It is clear that the space P with the norm $\|\cdot\|_{\infty}$ is a subspace of $C[0,T]$ which is a Banach space.

It remains to prove that P is a closed subspace in $C[0,T]$. Let $(u_n)_{n \in \mathbb{N}} \in P$ be a real sequence such that $\lim\limits_{n \to \infty} u_n = u$ in $C[0,T]$. Then for each $t \in [0,T]$, we have

$$|u_n(t)| \leq K_0, \ |u(t)| \leq K_0 \text{ for some } K_0 > 0.$$

Since $u_n \to u$, then we get ${}^{\rho}\mathcal{I}_{0^+}^{1-\alpha} u_n(t) \to {}^{\rho}\mathcal{I}_{0^+}^{1-\alpha} u(t)$ as $n \to \infty$ for each $t \in [0,T]$. In fact,

$$\begin{aligned}
\left| {}^{\rho}\mathcal{I}_{0^+}^{1-\alpha} u_n(t) - {}^{\rho}\mathcal{I}_{0^+}^{1-\alpha} u(t) \right| &= \left| \frac{\rho^{\alpha}}{\Gamma(1-\alpha)} \int_0^t \frac{s^{\rho-1} u_n(s)}{(t^{\rho} - s^{\rho})^{\alpha}} \mathrm{d}s \right. \\
&\quad \left. - \frac{\rho^{\alpha}}{\Gamma(1-\alpha)} \int_0^t \frac{s^{\rho-1} u(s)}{(t^{\rho} - s^{\rho})^{\alpha}} \mathrm{d}s \right| \\
&\leq \frac{\rho^{\alpha}}{\Gamma(1-\alpha)} \int_0^t \frac{s^{\rho-1}}{(t^{\rho} - s^{\rho})^{\alpha}} \left| u_n(s) - u(s) \right| \mathrm{d}s.
\end{aligned} \tag{12}$$

Then

$$\begin{aligned}
\left| {}^{\rho}\mathcal{I}_{0^+}^{1-\alpha} u_n(t) - {}^{\rho}\mathcal{I}_{0^+}^{1-\alpha} u(t) \right| &\leq \frac{\rho^{\alpha}}{\Gamma(1-\alpha)} \int_0^t s^{\rho-1} (t^{\rho} - s^{\rho})^{-\alpha} \left(|u_n(s)| + |u(s)| \right) \mathrm{d}s \\
&\leq \frac{2\rho^{\alpha} K_0}{\Gamma(1-\alpha)} \Big[- \frac{(t^{\rho} - s^{\rho})^{1-\alpha}}{1-\alpha} \Big]_0^t \leq \frac{2\rho^{\alpha} T^{\rho(1-\alpha)} K_0}{\Gamma(2-\alpha)}.
\end{aligned}$$

Thus, for each $t \in [0,T]$, the Lebesgue dominated convergence theorem and (12) imply that

$$\left| {}^{\rho}\mathcal{I}_{0^+}^{1-\alpha} u_n(t) - {}^{\rho}\mathcal{I}_{0^+}^{1-\alpha} u(t) \right| \to 0 \quad \text{as } n \to \infty.$$

Hence

$$\lim_{n \to \infty} \left\| {}^{\rho}\mathcal{I}_{0^+}^{1-\alpha} u_n(t) - {}^{\rho}\mathcal{I}_{0^+}^{1-\alpha} u(t) \right\|_{\infty} = 0$$

and for $t \to 0^+$, we have also

$$\lim_{n \to \infty} \left({}^{\rho}\mathcal{I}_{0^+}^{1-\alpha} u_n \right) \left(0^+ \right) = \left({}^{\rho}\mathcal{I}_{0^+}^{1-\alpha} u \right) \left(0^+ \right) = 0, \quad \text{then } u \in P.$$

Consequently, P is closed in $C[0,T]$, and hence $(P, \|\cdot\|_{\infty})$ is a Banach space. The proof is complete. ∎

Lemma 3.2. *Let $\alpha, \rho > 0$. If $u \in C[0,T]$, then:*

(i) *The fractional deferential equation ${}^{\rho}\mathcal{D}^{\alpha}_{0^+} u(t) = 0$ has a unique solution*

$$u(t) = C_1 t^{\rho(\alpha-1)} + C_2 t^{\rho(\alpha-2)} + \cdots + C_n t^{\rho(\alpha-n)},$$

where $C_m \in \mathbb{R}$ with $m = 1, \ldots, n$.

(ii) *If ${}^{\rho}\mathcal{D}^{\alpha}_{0^+} u \in C[0,T]$ and $0 < \alpha \leq 1$, then*

$$ {}^{\rho}\mathcal{I}^{\alpha}_{0^+} {}^{\rho}\mathcal{D}^{\alpha}_{0^+} u(t) = u(t) + C t^{\rho(\alpha-1)} \tag{13}$$

for some constant $C \in \mathbb{R}$.

(iii) *Let $0 < \beta < \alpha \leq 1$ be such that ${}^{\rho}\mathcal{D}^{\alpha}_{0^+} u \in C[0,T]$, then*

$$ {}^{\rho}\mathcal{I}^{\alpha-\beta}_{0^+} {}^{\rho}\mathcal{D}^{\alpha}_{0^+} u(t) = {}^{\rho}\mathcal{D}^{\beta}_{0^+} u(t) - \frac{\rho^{1-\alpha+\beta} \left({}^{\rho}\mathcal{I}^{1-\alpha}_{0^+} u\right)(0^+)}{\Gamma(\alpha-\beta)} t^{\rho(\alpha-\beta-1)}. \tag{14}$$

Moreover, for each $u \in P$, we have for every $t \in [0,T]$ that

$$ \left| {}^{\rho}\mathcal{D}^{\beta}_{0^+} u(t) \right| \leq \frac{T^{\rho(\alpha-\beta)}}{\rho^{\alpha-\beta}\Gamma(1+\alpha-\beta)} \left\| {}^{\rho}\mathcal{D}^{\alpha}_{0^+} u \right\|_{\infty}. \tag{15}$$

Proof. (i) Let $\alpha, \rho > 0$, from remark 2, we have

$$ {}^{\rho}\mathcal{D}^{\alpha}_{0^+} t^{\rho(\alpha-m)} = 0 \qquad \text{for each } m = 1, 2, \ldots, n.$$

Then the fractional differential equation ${}^{\rho}\mathcal{D}^{\alpha}_{0^+} u(t) = 0$ has a particular solution as follows

$$ u(t) = C_m t^{\rho(\alpha-m)}, \qquad C_m \in \mathbb{R} \qquad \text{for each } m = 1, 2, \ldots, n. \tag{16}$$

Thus, the given general solution of ${}^{\rho}\mathcal{D}^{\alpha}_{0^+} u(t) = 0$ is a sum of particular solutions (16), i.e.,

$$ u(t) = C_1 t^{\rho(\alpha-1)} + C_2 t^{\rho(\alpha-2)} + \cdots + C_n t^{\rho(\alpha-n)}, \qquad C_m \in \mathbb{R} \ (m = 1, 2, \ldots, n).$$

(ii) Let ${}^{\rho}\mathcal{D}^{\alpha}_{0^+} u \in C\,[0,T]$ be the fractional derivatives (4) of order $0 < \alpha \leq 1$.

If we apply the operator ${}^{\rho}\mathcal{D}^{\alpha}_{0^+}$ to ${}^{\rho}\mathcal{I}^{\alpha}_{0^+} {}^{\rho}\mathcal{D}^{\alpha}_{0^+} u(t) - u(t)$ and use the properties (8), (9), we have

$$\begin{aligned} {}^{\rho}\mathcal{D}^{\alpha}_{0^+} \left[{}^{\rho}\mathcal{I}^{\alpha}_{0^+} {}^{\rho}\mathcal{D}^{\alpha}_{0^+} u(t) - u(t) \right] &= {}^{\rho}\mathcal{D}^{\alpha}_{0^+} {}^{\rho}\mathcal{I}^{\alpha}_{0^+} {}^{\rho}\mathcal{D}^{\alpha}_{0^+} u(t) - {}^{\rho}\mathcal{D}^{\alpha}_{0^+} u(t) \\ &= {}^{\rho}\mathcal{D}^{\alpha}_{0^+} u(t) - {}^{\rho}\mathcal{D}^{\alpha}_{0^+} u(t) = 0. \end{aligned}$$

After the step (i), we deduce there exists $C \in \mathbb{R}$ such that

$$ {}^{\rho}\mathcal{I}^{\alpha}_{0^+} {}^{\rho}\mathcal{D}^{\alpha}_{0^+} u(t) - u(t) = C t^{\rho(\alpha-1)},$$

which implies the law of composition (13).

(iii) Let ${}^{\rho}\mathcal{D}^{\alpha}_{0^+} u \in C\,[0,T]$ be the fractional derivative (4) of order $0 < \alpha \leq 1$. If we apply the operator ${}^{\rho}\mathcal{I}^{\alpha-\beta}_{0^+}$ to ${}^{\rho}\mathcal{D}^{\alpha}_{0^+} u(t)$, where $0 < \beta < \alpha$, and use Definitions 2.5, 2.6, Theorem 2.7, and equation (10), we get

$$\begin{aligned} {}^{\rho}\mathcal{I}^{\alpha-\beta}_{0^+} {}^{\rho}\mathcal{D}^{\alpha}_{0^+} u(t) &= \left(t^{1-\rho} \frac{\mathrm{d}}{\mathrm{d}t} \right) {}^{\rho}\mathcal{I}^{\alpha-\beta+1}_{0^+} {}^{\rho}\mathcal{D}^{\alpha}_{0^+} u(t) \\ &= \left(t^{1-\rho} \frac{\mathrm{d}}{\mathrm{d}t} \right) \left[\frac{\rho^{\beta-\alpha}}{\Gamma(1+\alpha-\beta)} \int_0^t (t^{\rho} - s^{\rho})^{\alpha-\beta} \frac{\mathrm{d}}{\mathrm{d}s} {}^{\rho}\mathcal{I}^{1-\alpha}_{0^+} u(s)\, \mathrm{d}s \right] \end{aligned}$$

$$= \left(t^{1-\rho}\frac{\mathrm{d}}{\mathrm{d}t}\right)\frac{\rho^{\beta-\alpha}}{\Gamma\left(1+\alpha-\beta\right)}\Big[\left[\,(t^{\rho}-s^{\rho})^{\alpha-\beta}\ {}^{\rho}\mathcal{I}_{0+}^{1-\alpha}u(s)\,\right]_0^t$$
$$+\rho\left(\alpha-\beta\right)\int_0^t s^{\rho-1}\left(t^{\rho}-s^{\rho}\right)^{\alpha-\beta-1}\ {}^{\rho}\mathcal{I}_{0+}^{1-\alpha}u(s)\,\mathrm{d}s\Big].$$

Then

$$\begin{aligned}
{}^{\rho}\mathcal{I}_{0+}^{\alpha-\beta}\ {}^{\rho}\mathcal{D}_{0+}^{\alpha}u(t) &= \left(t^{1-\rho}\frac{\mathrm{d}}{\mathrm{d}t}\right)\frac{\rho^{1-\alpha+\beta}}{\Gamma\left(\alpha-\beta\right)}\int_0^t s^{\rho-1}\left(t^{\rho}-s^{\rho}\right)^{\alpha-\beta-1}\ {}^{\rho}\mathcal{I}_{0+}^{1-\alpha}u(s)\,\mathrm{d}s \\
&\quad -\frac{\rho^{\beta-\alpha}\left({}^{\rho}\mathcal{I}_{0+}^{1-\alpha}u\right)(0^{+})}{\Gamma\left(1+\alpha-\beta\right)}\left(t^{1-\rho}\frac{\mathrm{d}}{\mathrm{d}t}\right)t^{\rho(\alpha-\beta)} \\
&= \left(t^{1-\rho}\frac{\mathrm{d}}{\mathrm{d}t}\right)\ {}^{\rho}\mathcal{I}_{0+}^{\alpha-\beta}\ {}^{\rho}\mathcal{I}_{0+}^{1-\alpha}u(s) \\
&\quad -\frac{\rho^{\beta-\alpha}\left({}^{\rho}\mathcal{I}_{0+}^{1-\alpha}u\right)(0^{+})}{\Gamma\left(1+\alpha-\beta\right)}\rho\left(\alpha-\beta\right)t^{1-\rho}t^{\rho(\alpha-\beta)-1} \\
&= {}^{\rho}\mathcal{D}_{0+}^{\beta}u(t)-\frac{\rho^{1-\alpha+\beta}\left({}^{\rho}\mathcal{I}_{0+}^{1-\alpha}u\right)(0^{+})}{\Gamma\left(\alpha-\beta\right)}t^{\rho(\alpha-\beta-1)}.
\end{aligned}$$

Moreover, for each $u\in P$, we have $\left({}^{\rho}\mathcal{I}_{0+}^{1-\alpha}u\right)(0^{+})=0$. Then for every $t\in[0,T]$,

$${}^{\rho}\mathcal{I}_{0+}^{\alpha-\beta}\ {}^{\rho}\mathcal{D}_{0+}^{\alpha}u(t) = {}^{\rho}\mathcal{D}_{0+}^{\beta}u(t)$$

and

$$\begin{aligned}
\left|{}^{\rho}\mathcal{D}_{0+}^{\beta}u(t)\right| &= \left|{}^{\rho}\mathcal{I}_{0+}^{\alpha-\beta}\ {}^{\rho}\mathcal{D}_{0+}^{\alpha}u(t)\right| \\
&\leq \frac{\rho^{1-\alpha+\beta}}{\Gamma\left(\alpha-\beta\right)}\int_0^t s^{\rho-1}\left(t^{\rho}-s^{\rho}\right)^{\alpha-\beta-1}\left|{}^{\rho}\mathcal{D}_{0+}^{\alpha}u(s)\right|\mathrm{d}s \\
&\leq \Big[-\frac{\rho^{\beta-\alpha}}{\left(\alpha-\beta\right)\Gamma\left(\alpha-\beta\right)}\left(t^{\rho}-s^{\rho}\right)^{\alpha-\beta}\Big]_0^t\Big\{\sup_{0\leq t\leq T}\left|{}^{\rho}\mathcal{D}_{0+}^{\alpha}u(t)\right|\Big\} \\
&\leq \frac{T^{\rho(\alpha-\beta)}}{\rho^{\alpha-\beta}\Gamma\left(1+\alpha-\beta\right)}\left\|{}^{\rho}\mathcal{D}_{0+}^{\alpha}u\right\|_{\infty}.
\end{aligned}$$

The proof is complete. □

Based on the previous lemma, we define the integral solution of the problem (1)–(2).

Lemma 3.3. *Let $\alpha,\beta,\rho\in\mathbb{R}$ be such that $0<\beta<\alpha\leq 1$, and $\rho>0$. We give u, ${}^{\rho}\mathcal{D}_{0+}^{\alpha}u\in C[0,T]$, and a continuous function $f\left(t,u,v\right)$. Then the problem (1)–(2) is equivalent to the integral equation*

$$u(t)=\int_0^t G\left(t,s\right)f\left(s,u(s),\,{}^{\rho}\mathcal{D}_{0+}^{\beta}u(s)\right)\mathrm{d}s, \tag{17}$$

where

$$G\left(t,s\right)=\frac{\rho^{1-\alpha}}{\Gamma\left(\alpha\right)}s^{\rho-1}\left(t^{\rho}-s^{\rho}\right)^{\alpha-1}. \tag{18}$$

Proof. Let $0 < \beta < \alpha \leq 1$ and $\rho > 0$, we may apply Lemma 3.2 to reduce the fractional equation (1) to an equivalent fractional integral equation.
By applying ${}^{\rho}\mathcal{I}^{\alpha}_{0+}$ to equation (1), we obtain

$$ {}^{\rho}\mathcal{I}^{\alpha}_{0+}\ {}^{\rho}\mathcal{D}^{\alpha}_{0+}u(t) = {}^{\rho}\mathcal{I}^{\alpha}_{0+} f\big(t, u(t), {}^{\rho}\mathcal{D}^{\beta}_{0+}u(t)\ \big)\,. \tag{19}$$

From Lemma 3.2, we find easily

$$ {}^{\rho}\mathcal{I}^{\alpha}_{0+}\ {}^{\rho}\mathcal{D}^{\alpha}_{0+}u(t) = u(t) + Ct^{\rho(\alpha-1)} $$

for some $C \in \mathbb{R}$. Then, the fractional integral equation (19) gives

$$ u(t) = {}^{\rho}\mathcal{I}^{\alpha}_{0+} f\big(t, u(t), {}^{\rho}\mathcal{D}^{\beta}_{0+}u(t)\ \big) - Ct^{\rho(\alpha-1)}. \tag{20}$$

From (7), we have

$$ {}^{\rho}\mathcal{I}^{1-\alpha}_{0+} t^{\rho(\alpha-1)} = \rho^{\alpha-1}\Gamma(\alpha)\,. $$

If we use the condition (2) in equation (20), we find

$$ \left({}^{\rho}\mathcal{I}^{1-\alpha}_{0+}u\right)\left(0^{+}\right) = 0 = -C\rho^{\alpha-1}\Gamma(\alpha) \quad \Longrightarrow \quad C = 0. $$

Therefore, the problem (1)–(2) is equivalent to

$$ u(t) = \int_0^t G(t,s)\, f\big(s, u(s), {}^{\rho}\mathcal{D}^{\beta}_{0+}u(s)\ \big)\mathrm{d}s, \tag{21}$$

where $G(t,s)$ is given by the equality (18). The proof is complete. □

Lemma 3.4. *Let $\mathcal{A}\colon P \to C[0,T]$ be an integral operator, which is defined by*

$$ \mathcal{A}u(t) = \int_0^t G(t,s)\, f\big(s, u(s), {}^{\rho}\mathcal{D}^{\beta}_{0+}u(s)\ \big)\mathrm{d}s, \tag{22}$$

equipped with the standard norm

$$ \|\mathcal{A}u\|_{\infty} = \sup_{0\leq t\leq T} |\mathcal{A}u(t)|\,. $$

Then $\mathcal{A}(P) \subset P$.

Proof. Let $u \in P$ be such that ${}^{\rho}\mathcal{D}^{\alpha}_{0+}u(t) = f\big(t, u(t), {}^{\rho}\mathcal{D}^{\beta}_{0+}u(t)\ \big)$. From (22), we have

$$ \begin{aligned} \left({}^{\rho}\mathcal{I}^{1-\alpha}_{0+}\mathcal{A}u\right)(t) &= {}^{\rho}\mathcal{I}^{1-\alpha}_{0+}\ {}^{\rho}\mathcal{I}^{\alpha}_{0+} f\big(t, u(t), {}^{\rho}\mathcal{D}^{\beta}_{0+}u(t)\ \big) \\ &= {}^{\rho}\mathcal{I}^{1}_{0+} f\big(t, u(t), {}^{\rho}\mathcal{D}^{\beta}_{0+}u(t)\ \big) = {}^{\rho}\mathcal{I}^{1}_{0+}\ {}^{\rho}\mathcal{D}^{\alpha}_{0+}u(t)\ . \end{aligned} $$

If we use (10) and (4), we have

$$ \left({}^{\rho}\mathcal{I}^{1-\alpha}_{0+}\mathcal{A}u\right)(t) = {}^{\rho}\mathcal{I}^{1}_{0+}\ {}^{\rho}\mathcal{D}^{\alpha}_{0+}u(t) = {}^{\rho}\mathcal{I}^{1}_{0+}\left(t^{1-\rho}\frac{\mathrm{d}}{\mathrm{d}t}\right){}^{\rho}\mathcal{I}^{1-\alpha}_{0+}u(t) = {}^{\rho}\mathcal{I}^{1-\alpha}_{0+}u(t)\ . $$

Thus $\left({}^{\rho}\mathcal{I}^{1-\alpha}_{0+}\mathcal{A}u\right)(0^{+}) = 0$. Consequently, $\mathcal{A}(P) \subset P$. The proof is complete. □

Now, we prove our first existence result for the problem (1)–(2) which is based on Banach's fixed point theorem.

We impose the following hypotheses:

(H1) $f\colon [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

(H2) For all $0 < \beta < \alpha \leq 1$, there exist two constants $\lambda, \gamma > 0$, where $\gamma < \frac{\rho^{\alpha-\beta}\Gamma(1+\alpha-\beta)}{T^{\rho(\alpha-\beta)}}$ such that

$$|f(t,u,v) - f(t,\tilde{u},\tilde{v})| \leq \lambda |u - \tilde{u}| + \gamma |v - \tilde{v}|$$

for any $u, v, \tilde{u}, \tilde{v} \in \mathbb{R}$ and $t \in [0,T]$.

(H3) There exist three positive functions $a, b, c \in C[0,T]$ such that

$$|f(t,u,v)| \leq a(t) + b(t)|u| + c(t)|v| \qquad \text{for all } t \in [0,T] \text{ and } u, v \in \mathbb{R}.$$

We denote

$$M_0 = \frac{\rho^{\alpha-\beta}\Gamma(1+\alpha-\beta)\, a^*}{\rho^{\alpha-\beta}\Gamma(1+\alpha-\beta) - c^* T^{\rho(\alpha-\beta)}}$$

and

$$M_1 = \frac{\rho^{\alpha-\beta}\Gamma(1+\alpha-\beta)\, b^*}{\rho^{\alpha-\beta}\Gamma(1+\alpha-\beta) - c^* T^{\rho(\alpha-\beta)}},$$

where $0 < \beta < \alpha \leq 1$ and

$$a^* = \sup_{t\in[0,T]} a(t), \qquad b^* = \sup_{t\in[0,T]} b(t), \qquad c^* = \sup_{t\in[0,T]} c(t)$$

with $c^* < \frac{\rho^{\alpha-\beta}\Gamma(1+\alpha-\beta)}{T^{\rho(\alpha-\beta)}}$.

In what follows, we present the principal theorems.

Theorem 3.5. *Assume the hypotheses* (H1)–(H2) *hold. We give* $0 < \beta < \alpha \leq 1$ *and* $\rho > 0$. *If*

$$\frac{\lambda T^{\rho\alpha}\Gamma(1+\alpha-\beta)}{\Gamma(\alpha+1)\left[\rho^{\alpha}\Gamma(1+\alpha-\beta) - \gamma\rho^{\beta}T^{\rho(\alpha-\beta)}\right]} < 1. \tag{23}$$

Then the problem (1)–(2) *admits a unique solution on* $[0,T]$.

Proof. To begin the proof, we transform the problem (1)–(2) into a fixed point problem. Define the operator $\mathcal{A}\colon P \to P$ by

$$\mathcal{A}u(t) = \int_0^t G(t,s)\, f\big(s, u(s), {}^{\rho}\mathcal{D}^{\beta}_{0+}u(s)\big)\mathrm{d}s. \tag{24}$$

Because the problem (1)–(2) is equivalent to the fractional integral equation (24), the fixed points of $\mathcal{A}$ are solutions of the problem (1)–(2).

Let $u, v \in P$ be such that

$${}^{\rho}\mathcal{D}^{\alpha}_{0+}u(t) = f\big(t, u(t), {}^{\rho}\mathcal{D}^{\beta}_{0+}u(t)\big), \qquad {}^{\rho}\mathcal{D}^{\alpha}_{0+}v(t) = f\big(t, v(t), {}^{\rho}\mathcal{D}^{\beta}_{0+}v(t)\big).$$

Which implies that

$$\mathcal{A}u(t) - \mathcal{A}v(t) = \int_0^t G(t,s)\left[f\big(s, u(s), {}^{\rho}\mathcal{D}^{\beta}_{0+}u(s)\big) - f\big(s, v(s), {}^{\rho}\mathcal{D}^{\beta}_{0+}v(s)\big)\right]\mathrm{d}s.$$

Then, for all $t \in [0,T]$,

$$|\mathcal{A}u(t) - \mathcal{A}v(t)| \leq \int_0^t G(t,s)\left|{}^{\rho}\mathcal{D}^{\alpha}_{0+}u(s) - {}^{\rho}\mathcal{D}^{\alpha}_{0+}v(s)\right|\mathrm{d}s. \tag{25}$$

By (H2), we have

$$\begin{aligned}|{}^{\rho}\mathcal{D}^{\alpha}_{0+}u(t) - {}^{\rho}\mathcal{D}^{\alpha}_{0+}v(t)| &= \left|f\left(t, u(t), {}^{\rho}\mathcal{D}^{\beta}_{0+}u(t)\right) - f\left(t, v(t), {}^{\rho}\mathcal{D}^{\beta}_{0+}v(t)\right)\right| \\ &\le \lambda\,|u(t)-v(t)| + \gamma\left|{}^{\rho}\mathcal{D}^{\beta}_{0+}u(t) - {}^{\rho}\mathcal{D}^{\beta}_{0+}v(t)\right|.\end{aligned}$$

By using (15) from Lemma 3.2, we have

$$\|{}^{\rho}\mathcal{D}^{\alpha}_{0+}u - {}^{\rho}\mathcal{D}^{\alpha}_{0+}v\|_{\infty} \le \lambda\,\|u-v\|_{\infty} + \frac{\gamma T^{\rho(\alpha-\beta)}}{\rho^{\alpha-\beta}\Gamma(1+\alpha-\beta)}\,\|{}^{\rho}\mathcal{D}^{\alpha}_{0+}u - {}^{\rho}\mathcal{D}^{\alpha}_{0+}v\|_{\infty},$$

thus

$$\|{}^{\rho}\mathcal{D}^{\alpha}_{0+}u - {}^{\rho}\mathcal{D}^{\alpha}_{0+}v\|_{\infty} \le \frac{\lambda\rho^{\alpha-\beta}\Gamma(1+\alpha-\beta)}{\rho^{\alpha-\beta}\Gamma(1+\alpha-\beta) - \gamma T^{\rho(\alpha-\beta)}}\,\|u-v\|_{\infty}.$$

From (25) we have

$$\|\mathcal{A}u - \mathcal{A}v\|_{\infty} \le \frac{\lambda T^{\rho\alpha}\Gamma(1+\alpha-\beta)}{\Gamma(\alpha+1)\left[\rho^{\alpha}\Gamma(1+\alpha-\beta) - \gamma\rho^{\beta}T^{\rho(\alpha-\beta)}\right]}\,\|u-v\|_{\infty}.$$

This implies that by (23), $\mathcal{A}$ is a contraction operator.

As a consequence of Theorem 2.11, using Banach's contraction principle [**10**], we deduce that $\mathcal{A}$ has a unique fixed point which is the unique solution of the problem (1)–(2) on $[0, T]$. □

Theorem 3.6. *Assume that hypotheses* (H1)–(H3) *hold. We give* $0 < \beta < \alpha \le 1$, *and* $\rho > 0$. *If we put*

$$\frac{M_1 T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} < 1,$$

then the problem (1)–(2) *has at least one solution on* $[0, T]$.

Proof. In the previous theorem, we already transform the problem (1)–(2) into a fixed point problem

$$\mathcal{A}u(t) = \int_0^t G(t,s)\,f\big(s, u(s), {}^{\rho}\mathcal{D}^{\beta}_{0+}u(s)\big)\,\mathrm{d}s.$$

We demonstrate that $\mathcal{A}$ satisfies the assumption of Schauder's fixed point Theorem 2.12. This could be proved through three steps:

<u>*Step 1.*</u> $\mathcal{A}$ is a continuous operator.

Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence such that $\lim_{n\to\infty} u_n = u$ in P. Then for each $t \in [0, T]$,

$$\begin{aligned}|\mathcal{A}u_n(t) - \mathcal{A}u(t)| \le &\int_0^t G(t,s) \\ &\times \left|f\big(s, u_n(s), {}^{\rho}\mathcal{D}^{\beta}_{0+}u_n(s)\big) - f\big(s, u(s), {}^{\rho}\mathcal{D}^{\beta}_{0+}u(s)\big)\right|\mathrm{d}s,\end{aligned} \tag{26}$$

where

$${}^{\rho}\mathcal{D}^{\alpha}_{0+}u_n(t) = f\big(t, u_n(t), {}^{\rho}\mathcal{D}^{\beta}_{0+}u_n(t)\big) \quad\text{and}\quad {}^{\rho}\mathcal{D}^{\alpha}_{0+}u(t) = f\big(t, u(t), {}^{\rho}\mathcal{D}^{\beta}_{0+}u(t)\big).$$

As a consequence of (H2), we find easily ${}^{\rho}\mathcal{D}_{0^+}^{\alpha} u_n \to^{\rho} \mathcal{D}_{0^+}^{\alpha} u$ in P. In fact, we have

$$\begin{aligned}|{}^{\rho}\mathcal{D}_{0^+}^{\alpha} u_n(t) - {}^{\rho}\mathcal{D}_{0^+}^{\alpha} u(t)\,| &= \left|f\big(t, u_n(t), {}^{\rho}\mathcal{D}_{0^+}^{\beta} u_n(t)\ \big) - f\big(t, u(t), {}^{\rho}\mathcal{D}_{0^+}^{\beta} u(t)\ \big)\right| \\ &\leq \lambda\, |u_n(t) - u(t)| + \gamma \left|{}^{\rho}\mathcal{D}_{0^+}^{\beta} u_n(t) - {}^{\rho}\mathcal{D}_{0^+}^{\beta} u(t)\ \right|.\end{aligned}$$

By using (15) from Lemma 3.2, we have

$$\|{}^{\rho}\mathcal{D}_{0^+}^{\alpha} u_n - {}^{\rho}\mathcal{D}_{0^+}^{\alpha} u\,\|_\infty \leq \lambda\, \|u_n - u\|_\infty + \frac{\gamma T^{\rho(\alpha-\beta)}}{\rho^{\alpha-\beta}\Gamma\,(1+\alpha-\beta)}\, \|{}^{\rho}\mathcal{D}_{0^+}^{\alpha} u_n - {}^{\rho}\mathcal{D}_{0^+}^{\alpha} u\,\|_\infty\,,$$

thus

$$\|{}^{\rho}\mathcal{D}_{0^+}^{\alpha} u_n - {}^{\rho}\mathcal{D}_{0^+}^{\alpha} u\,\|_\infty \leq \frac{\lambda \rho^{\alpha-\beta}\Gamma\,(1+\alpha-\beta)}{\rho^{\alpha-\beta}\Gamma\,(1+\alpha-\beta) - \gamma T^{\rho(\alpha-\beta)}}\, \|u_n - u\|_\infty\,.$$

Since $u_n \to u$, then we get ${}^{\rho}\mathcal{D}_{0^+}^{\alpha} u_n(t) \to {}^{\rho}\mathcal{D}_{0^+}^{\alpha} u(t)$ as $n \to \infty$ for each $t \in [0, T]$.

Now let $K_1 > 0$ be such that for each $t \in [0, T]$, we have

$$|{}^{\rho}\mathcal{D}_{0^+}^{\alpha} u_n(t)| \leq K_1, |{}^{\rho}\mathcal{D}_{0^+}^{\alpha} u(t)| \leq K_1.$$

Then, we have

$$\begin{aligned}|\mathcal{A}u_n(t) - \mathcal{A}u(t)| &\leq \int_0^t G\,(t,s) \\ &\quad \times \left|f\big(s, u_n(s), {}^{\rho}\mathcal{D}_{0^+}^{\beta} u_n(s)\ \big) - f\big(s, u(s), {}^{\rho}\mathcal{D}_{0^+}^{\beta} u(s)\ \big)\right| \mathrm{d}s \\ &\leq \int_0^t G\,(t,s)\, |{}^{\rho}\mathcal{D}_{0^+}^{\alpha} u_n(s) - {}^{\rho}\mathcal{D}_{0^+}^{\alpha} u(s)\,|\, \mathrm{d}s \\ &\leq \int_0^t G\,(t,s)\, [|{}^{\rho}\mathcal{D}_{0^+}^{\alpha} u_n(s)| + |{}^{\rho}\mathcal{D}_{0^+}^{\alpha} u(s)|]\, \mathrm{d}s \\ &\leq \int_0^t 2K_1 G\,(t,s)\, \mathrm{d}s.\end{aligned}$$

For each $t \in [0, T]$, the function $s \to 2K_1 G\,(t,s)$ is integrable on $[0, t]$, then the Lebesgue dominated convergence theorem and (26) imply that

$$|\mathcal{A}u_n(t) - \mathcal{A}u(t)| \to 0 \qquad \text{as } n \to \infty,$$

and hence

$$\lim_{n\to\infty} \|\mathcal{A}u_n - \mathcal{A}u\|_\infty = 0.$$

Consequently, $\mathcal{A}$ is continuous.

<u>*Step 2.*</u> Let $r \geq \frac{M_0 T^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1) - M_1 T^{\rho\alpha}}$, and define

$$P_r = \{u \in P : \|u\|_\infty \leq r\}\,.$$

It is clear that P_r is a bounded, closed, and convex subset of P.

Let $u \in P_r$, and $\mathcal{A}$: $P_r \to P$ be the integral operator defined in (24), then $\mathcal{A}\,(P_r) \subset P_r$.

In fact, by using (15) from Lemma 3.2, and (H3), we have for each $t \in [0, T]$,

$$|{}^{\rho}\mathcal{D}_{0^+}^{\alpha} u(t)| = \left|f\big(t, u(t), {}^{\rho}\mathcal{D}_{0^+}^{\beta} u(t)\ \big)\right| \leq a(t) + b(t)\, |u(t)| + c(t) \left|{}^{\rho}\mathcal{D}_{0^+}^{\beta} u(t)\right|$$

$$\leq a^* + b^* |u(t)| + \frac{c^* T^{\rho(\alpha-\beta)}}{\rho^{\alpha-\beta}\Gamma(1+\alpha-\beta)} \|{}^{\rho}\mathcal{D}_{0+}^{\alpha} u\|_{\infty}.$$

Then

$$\begin{aligned} \|{}^{\rho}\mathcal{D}_{0+}^{\alpha} u\|_{\infty} \leq & \frac{\rho^{\alpha-\beta}\Gamma(1+\alpha-\beta) a^*}{\rho^{\alpha-\beta}\Gamma(1+\alpha-\beta) - c^* T^{\rho(\alpha-\beta)}} \\ & + \frac{\rho^{\alpha-\beta}\Gamma(1+\alpha-\beta) b^*}{\rho^{\alpha-\beta}\Gamma(1+\alpha-\beta) - c^* T^{\rho(\alpha-\beta)}} r \leq M_0 + M_1 r. \end{aligned} \tag{27}$$

Thus

$$\begin{aligned} |\mathcal{A}u(t)| &\leq \int_0^t G(t,s) \left| f\left(s, u(s), {}^{\rho}\mathcal{D}_{0+}^{\beta} u(s)\right) \right| \mathrm{d}s \\ &\leq \frac{M_0 T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} + \frac{M_1 T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} r \\ &\leq \frac{[\rho^{\alpha}\Gamma(\alpha+1) - M_1 T^{\rho\alpha}] \frac{M_0 T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1) - M_1 T^{\rho\alpha}} + M_1 T^{\rho\alpha} r}{\rho^{\alpha}\Gamma(\alpha+1)} \\ &\leq \frac{[\rho^{\alpha}\Gamma(\alpha+1) - M_1 T^{\rho\alpha}] r + M_1 T^{\rho\alpha} r}{\rho^{\alpha}\Gamma(\alpha+1)} \leq r. \end{aligned}$$

Then $\mathcal{A}(P_r) \subset P_r$.

Step 3. $\mathcal{A}(P_r)$ is relatively compact.

Let $t_1, t_2 \in [0, T]$, $t_1 < t_2$, and $u \in P_r$. Then

$$\begin{aligned} |\mathcal{A}u(t_2) - \mathcal{A}u(t_1)| = & \Big| \int_0^{t_2} G(t_2, s) f\left(s, u(s), {}^{\rho}\mathcal{D}_{0+}^{\beta} u(s)\right) \mathrm{d}s \\ & - \int_0^{t_1} G(t_1, s) f\left(s, u(s), {}^{\rho}\mathcal{D}_{0+}^{\beta} u(s)\right) \mathrm{d}s \Big| \\ \leq & \int_0^{t_1} \left| [G(t_2, s) - G(t_1, s)] f\left(s, u(s), {}^{\rho}\mathcal{D}_{0+}^{\beta} u(s)\right) \right| \mathrm{d}s \\ & + \int_{t_1}^{t_2} G(t_2, s) \left| f\left(s, u(s), {}^{\rho}\mathcal{D}_{0+}^{\beta} u(s)\right) \right| \mathrm{d}s \\ \leq & (M_0 + M_1 r) \\ & \times \left[\int_0^{t_1} |(G(t_2, s) - G(t_1, s))| \mathrm{d}s + \int_{t_1}^{t_2} G(t_2, s) \mathrm{d}s \right]. \end{aligned} \tag{28}$$

We have

$$\begin{aligned} G(t_2, s) - G(t_1, s) &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} s^{\rho-1} \left[(t_2^{\rho} - s^{\rho})^{\alpha-1} - (t_1^{\rho} - s^{\rho})^{\alpha-1} \right] \\ &= \frac{-1}{\alpha \rho^{\alpha}\Gamma(\alpha)} \frac{\mathrm{d}}{\mathrm{d}s} \left[(t_2^{\rho} - s^{\rho})^{\alpha} - (t_1^{\rho} - s^{\rho})^{\alpha} \right], \end{aligned}$$

then

$$\int_0^{t_1} |(G(t_2, s) - G(t_1, s))| \mathrm{d}s \leq \frac{1}{\rho^{\alpha}\Gamma(\alpha+1)} \left[(t_2^{\rho} - t_1^{\rho})^{\alpha} + (t_2^{\rho\alpha} - t_1^{\rho\alpha}) \right].$$

We also have

$$\begin{aligned}\int_{t_1}^{t_2} G(t_2,s)\,\mathrm{d}s &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{t_1}^{t_2} s^{\rho-1}\left(t_2^{\rho}-s^{\rho}\right)^{\alpha-1}\mathrm{d}s\\ &= \frac{-1}{\alpha\rho^{\alpha}\Gamma(\alpha)}\left[\left(t_2^{\rho}-s^{\rho}\right)^{\alpha}\right]_{t_1}^{t_2}\\ &\leq \frac{1}{\rho^{\alpha}\Gamma(\alpha+1)}\left(t_2^{\rho}-t_1^{\rho}\right)^{\alpha}.\end{aligned}$$

Then (28) gives

$$|\mathcal{A}u(t_2)-\mathcal{A}u(t_1)| \leq \frac{M_0+M_1 r}{\rho^{\alpha}\Gamma(\alpha+1)}\left[2\left(t_2^{\rho}-t_1^{\rho}\right)^{\alpha}+\left(t_2^{\rho\alpha}-t_1^{\rho\alpha}\right)\right].$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero.

As a consequence of steps 1 to 3 together, and by means of the Ascoli-Arzelà Theorem 2.8, we deduce that $\mathcal{A}\colon P_r \to P_r$ is continuous, compact, and satisfies the assumption of Schauder's fixed point Theorem 2.12. Then $\mathcal{A}$ has a fixed point which is a solution of the problem (1)–(2) on $[0,T]$. □

Our next existence result is based on the nonlinear alternative of Leray-Schauder type.

Theorem 3.7. *Assume* (H1)–(H3) *holds. Then the problem* (1)–(2) *has at least one solution on* $[0,T]$.

Proof. Let $\alpha,\beta,\rho>0$, be such that $\beta<\alpha\leq 1$.

We show that the operator $\mathcal{A}$ defined in (24), satisfies the assumption of Leray-Schauder fixed point Theorem 2.13. The proof is given in several steps.

Step 1. Clearly $\mathcal{A}$ is continuous.

Step 2. $\mathcal{A}$ maps bounded sets into bounded sets in P.

Indeed, it is enough to show that for any $\omega>0$, there exists a positive constant ℓ such that for each $u\in B_{\omega}=\{u\in P:\|u\|_{\infty}\leq\omega\}$, we have $\|\mathcal{A}u\|_{\infty}\leq\ell$.

For $u\in B_{\omega}$, we have for each $t\in[0,T]$,

$$|\mathcal{A}u(t)| \leq \int_0^t G(t,s)\left|f\left(s,u(s),{}^{\rho}\mathcal{D}_{0^+}^{\beta}u(s)\right)\right|\mathrm{d}s. \tag{29}$$

By (H3), similarly to (27), for each $t\in[0,T]$, we have

$$\left|f\left(t,u(t),{}^{\rho}\mathcal{D}_{0^+}^{\beta}u(t)\right)\right| \leq M_0+M_1\omega.$$

Thus (29) implies that

$$\|\mathcal{A}u\|_{\infty} \leq \frac{M_0 T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}+\frac{M_1 T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\omega=\ell.$$

Step 3. Clearly, $\mathcal{A}$ maps bounded sets into equicontinuous sets of P.

We conclude that $\mathcal{A}\colon P\to P$ is continuous and completely continuous.

Step 4. A priori bounds.

We now show there exists an open set $U\subset P$ with $u\neq\mu\mathcal{A}(u)$ for $\mu\in(0,1)$ and $u\in\partial U$.

Let $u \in P$ and $u = \mu\mathcal{A}(u)$ for some $0 < \mu < 1$. Thus for each $t \in [0, T]$, we have

$$u(t) \leq \mu \int_0^t G(t,s) \left| f\big(s, u(s), {}^{\rho}\mathcal{D}^{\beta}_{0^+} u(s)\ \big) \right| \mathrm{d}s.$$

By (H3), for all solution $u \in P$ of the problem (1)–(2), we have

$$|u(t)| = \Big| \int_0^t G(t,s)\, f\big(s, u(s), {}^{\rho}\mathcal{D}^{\beta}_{0^+} u(s)\ \big) \mathrm{d}s \Big| \leq \int_0^t G(t,s) \left| {}^{\rho}\mathcal{D}^{\alpha}_{0^+} u(s) \right| \mathrm{d}s.$$

Then for each $t \in [0, T]$, we have

$$\begin{aligned} \left| {}^{\rho}\mathcal{D}^{\alpha}_{0^+} u(t) \right| &= \left| f\big(t, u(t), {}^{\rho}\mathcal{D}^{\beta}_{0^+} u(t)\ \big) \right| \leq a(t) + b(t)\,|u(t)| + c(t) \left| {}^{\rho}\mathcal{D}^{\beta}_{0^+} u(t) \right| \\ &\leq a^* + b^*\,|u(t)| + \frac{c^* T^{\rho(\alpha-\beta)}}{\rho^{\alpha-\beta}\Gamma(1+\alpha-\beta)} \sup_{0\leq t\leq T} \left| {}^{\rho}\mathcal{D}^{\alpha}_{0^+} u(t) \right|. \end{aligned}$$

Then

$$\begin{aligned} \sup_{0\leq t\leq T} \left| {}^{\rho}\mathcal{D}^{\alpha}_{0^+} u(t) \right| &\leq \frac{\rho^{\alpha-\beta}\Gamma(1+\alpha-\beta)}{\rho^{\alpha-\beta}\Gamma(1+\alpha-\beta) - c^* T^{\rho(\alpha-\beta)}} \Big(a^* + b^* \sup_{0\leq t\leq T} |u(t)| \Big) \\ &\leq M_0 + M_1 \sup_{0\leq t\leq T} |u(t)|. \end{aligned}$$

Hence

$$\sup_{0\leq t\leq T} |u(t)| \leq \frac{M_0 T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} + \int_0^t M_1 G(t,s) \Big\{ \sup_{0\leq s\leq T} |u(s)| \Big\} \mathrm{d}s.$$

After the Gronwall Lemma [**11**], we have

$$\sup_{0\leq t\leq T} |u(t)| \leq \frac{M_0 T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} \exp\Big(\frac{M_1 T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} \Big).$$

Thus

$$\|u\|_{\infty} \leq \frac{M_0 T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} \exp\Big(\frac{M_1 T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} \Big) = M_2.$$

Let

$$U = \{ u \in P : \|u\|_{\infty} < M_2 + 1 \}.$$

By choosing U, there is no $u \in \partial U$ such that $u = \mu\mathcal{A}(u)$ for $\mu \in (0, 1)$. As a consequence of Leray-Schauder's Theorem 2.13, $\mathcal{A}$ has a fixed point u in U, which is a solution to (1)–(2). □

4. Examples

Example 1. Consider the following problem

(30)

$$\begin{cases} {}^{1}\mathcal{D}^{\frac{1}{2}}_{0^+} u(t) = \dfrac{\cos(t)}{\pi\left(\sqrt{2}\cos(t) + \sin(t)\right)\left[1 + |u(t)| + \left| {}^{1}\mathcal{D}^{\frac{1}{4}}_{0^+} u(t) \right| \right]}, \quad t \in \left[0, \dfrac{\pi}{4}\right], \\ \left({}^{1}\mathcal{I}^{\frac{1}{2}}_{0^+} u \right)(0^+) = 0. \end{cases}$$

Set

$$f(t,u,v)=\frac{\cos(t)}{\pi\left(\sqrt{2}\cos(t)+\sin(t)\right)\left[1+|u|+|v|\right]},\qquad t\in\left[0,\frac{\pi}{4}\right],\ u,v\in\mathbb{R}.$$

Because $\sin(t)$, $\cos(t)$ are continuous positive functions for all $t\in\left[0,\frac{\pi}{4}\right]$, the function f is jointly continuous. For any $u,v,\tilde{u},\tilde{v}\in\mathbb{R}$ and $t\in\left[0,\frac{\pi}{4}\right]$, we have $\frac{\sqrt{2}}{2}\leq\cos(t)\leq 1$ and $0\leq\sin(t)\leq\frac{\sqrt{2}}{2}$, then

$$|f(t,u,v)-f(t,\tilde{u},\tilde{v})|\leq\frac{1}{\pi}\left(|u-\tilde{u}|+|v-\tilde{v}|\right).$$

Hence, the condition (H2) is satisfied with

$$\lambda=\gamma=\frac{1}{\pi}\simeq 0.3183<\frac{\rho^{\alpha-\beta}\Gamma(1+\alpha-\beta)}{T^{\rho(\alpha-\beta)}}=\left(\frac{\pi}{4}\right)^{-\frac{1}{4}}\Gamma\left(\frac{5}{4}\right)\simeq 0.9628.$$

It remains to show that the condition (23)

$$\begin{aligned}\frac{\lambda T^{\rho\alpha}\Gamma(1+\alpha-\beta)}{\Gamma(\alpha+1)\left[\rho^{\alpha}\Gamma(1+\alpha-\beta)-\gamma\rho^{\beta}T^{\rho(\alpha-\beta)}\right]}&=\frac{\left(\frac{1}{\pi}\right)\left(\frac{\pi}{4}\right)^{\frac{1}{2}}\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{2}\right)\left[\Gamma\left(\frac{5}{4}\right)-\frac{1}{\pi}\left(\frac{\pi}{4}\right)^{\frac{1}{4}}\right]}\\&=\frac{\sqrt{\pi}\Gamma\left(\frac{5}{4}\right)}{2\Gamma\left(\frac{3}{2}\right)\left[\pi\Gamma\left(\frac{5}{4}\right)-\left(\frac{\pi}{4}\right)^{\frac{1}{4}}\right]}\\&\simeq 0.4755<1,\end{aligned}$$

is satisfied. It follows from Theorem 3.5 that the problem (30) has a unique solution.

Example 2. Consider the following problem

$$\left\{\begin{array}{l}{}^{1}\mathcal{D}_{0^+}^{\frac{1}{2}}u(t)=\dfrac{\cos(t)\left[2+|u(t)|+\left|{}^{1}\mathcal{D}_{0^+}^{\frac{1}{4}}u(t)\right|\right]}{\pi\left(\sqrt{2}\cos(t)+\sin(t)\right)\left[1+|u(t)|+\left|{}^{1}\mathcal{D}_{0^+}^{\frac{1}{4}}u(t)\right|\right]},\qquad t\in\left[0,\frac{\pi}{4}\right],\\ \left({}^{1}\mathcal{I}_{0^+}^{\frac{1}{2}}u\right)(0^+)=0.\end{array}\right.\tag{31}$$

Set

$$f(t,u,v)=\frac{\cos(t)\left[2+|u|+|v|\right]}{\pi\left(\sqrt{2}\cos(t)+\sin(t)\right)\left[1+|u|+|v|\right]},\qquad t\in\left[0,\frac{\pi}{4}\right],\ u,v\in\mathbb{R}.$$

Clearly, the function f is jointly continuous. For any $u,v,\tilde{u},\tilde{v}\in\mathbb{R}$ and $t\in\left[0,\frac{\pi}{4}\right]$, we have

$$|f(t,u,v)-f(t,\tilde{u},\tilde{v})|\leq\frac{1}{\pi}\left(|u-\tilde{u}|+|v-\tilde{v}|\right).$$

Therefore, the condition (H2) is satisfied with

$$\lambda=\gamma=\frac{1}{\pi}\simeq 0.3183<\frac{\rho^{\alpha-\beta}\Gamma(1+\alpha-\beta)}{T^{\rho(\alpha-\beta)}}=\left(\frac{\pi}{4}\right)^{-\frac{1}{4}}\Gamma\left(\frac{5}{4}\right)\simeq 0.9628.$$

Also, we have

$$|f(t,u,v)|\leq\frac{\cos(t)}{\pi\left(\sqrt{2}\cos(t)+\sin(t)\right)}\left(2+|u|+|v|\right).$$

Thus, the condition (H3) is satisfied with

$$a(t) = \frac{2\cos(t)}{\pi\left(\sqrt{2}\cos(t) + \sin(t)\right)} \quad \text{and} \quad b(t) = c(t) = \frac{\cos(t)}{\pi\left(\sqrt{2}\cos(t) + \sin(t)\right)}.$$

We also have

$$a^* = \frac{2}{\pi},$$

$$b^* = c^* = \frac{1}{\pi} \simeq 0.3183 < \frac{\rho^{\alpha-\beta}\Gamma\left(1+\alpha-\beta\right)}{T^{\rho(\alpha-\beta)}} = \left(\frac{\pi}{4}\right)^{-\frac{1}{4}}\Gamma\!\left(\frac{5}{4}\right) \simeq 0.9628,$$

$$M_0 = \frac{\rho^{\alpha-\beta}\Gamma\left(1+\alpha-\beta\right)a^*}{\rho^{\alpha-\beta}\Gamma\left(1+\alpha-\beta\right) - c^* T^{\rho(\alpha-\beta)}} = \frac{2\Gamma\left(\frac{5}{4}\right)}{\pi\Gamma\left(\frac{5}{4}\right) - \left(\frac{\pi}{4}\right)^{\frac{1}{4}}},$$

$$M_1 = \frac{\rho^{\alpha-\beta}\Gamma\left(1+\alpha-\beta\right)b^*}{\rho^{\alpha-\beta}\Gamma\left(1+\alpha-\beta\right) - c^* T^{\rho(\alpha-\beta)}} = \frac{\Gamma\left(\frac{5}{4}\right)}{\pi\Gamma\left(\frac{5}{4}\right) - \left(\frac{\pi}{4}\right)^{\frac{1}{4}}}$$

and the condition

$$\frac{M_1 T^{\rho\alpha}}{\rho^{\alpha}\Gamma\left(\alpha+1\right)} = \frac{\left(\frac{\Gamma\left(\frac{5}{4}\right)}{\pi\Gamma\left(\frac{5}{4}\right) - \left(\frac{\pi}{4}\right)^{\frac{1}{4}}}\right)\left(\frac{\pi}{4}\right)^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}+1\right)} = \frac{\sqrt{\pi}\Gamma\left(\frac{5}{4}\right)}{2\Gamma\left(\frac{3}{2}\right)\left[\pi\Gamma\left(\frac{5}{4}\right) - \left(\frac{\pi}{4}\right)^{\frac{1}{4}}\right]} \simeq 0.4755 < 1.$$

It follows from Theorem 3.6 and Theorem 3.7, that the problem (31) has at least one solution.

5. Conclusion

In this paper, we have discussed the existence and uniqueness of solutions for a class of nonlinear fractional differential equations with an integral condition, we made use of the Banach contraction principle, Schauder's fixed point theorem, and the nonlinear alternative of Leray-Schauder type. The differential operator used is extended by Katugampola, which generalizes the Riemann-Liouville and the Hadamard fractional derivatives into a single form.

Acknowledgment. The authors are deeply grateful to the reviewers and editors for their insightful comments that have helped to improve the quality of this research work that is supported by the General Direction of Scientific Research and Technological Development (DGRSTD) – Algeria.

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