The Australian Journal of Mathematical Analysis and Applications

AJMAA

Volume 16, Issue 2, Article 10, pp. 1-12, 2019

EXTENSION OF FACTORIZATION THEOREMS OF MAUREY TO $s-$POSITIVELY HOMOGENEOUS OPERATORS

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Received 13 January, 2019; accepted 23 August, 2019; published 4 November, 2019.

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ABSTRACT. In the present work, we prove that the class of $s-$positively homogeneous operators is a Banach space. As application, we give the generalization of some Maurey factorization theorems to T which is a $s-$positively homogeneous operator from X a Banach space into L_p. Where we establish necessary and sufficient conditions to proof that T factors through L_q. After this we give extend result of dual factorization theorem to same class of operators above.

Key words and phrases: Banach lattice; Factorization; Non-linear; Homogeneous operators; Sublinear operator.

2000 *Mathematics Subject Classification.* 46B42, 46B40, 47B460, 47B65.

ISSN (electronic): 1449-5910

1. INTRODUCTION

In this paper, we start with the classic definition of the usual class of the operators which are the non linear mappings $s-$positively homogeneous. Some results of this type of operators that was shown but under the need to the development wanted in this paper. We study the structure of the above classical class and prove that such a class is Banach space.

In the same circle ideas of Nikishin in [7]. B Maurey in his thesis [5], led to the idea of factorization and he guarantees with some necessary and sufficient conditions that every linear operator $u : X \longrightarrow L_p(\Omega,\mu)$ factors through $L_q(\Omega,\mu)$ like this $X \stackrel{\widetilde{u}}{\longrightarrow} L_q(\Omega,\mu) \stackrel{M_{g_u}}{\longrightarrow} L_p(\Omega,\mu)$. Where $\widetilde{u}$ is a bounded linear operator, M_{g_u} is the bounded linear operator of multiplication by a function g_u in $L_r(\Omega,\mu)$ and p,q,r real numbers such that $0 < p \leq q \leq \infty$ with $\frac{1}{p} = \frac{1}{r} + \frac{1}{q}$. In [2] A. Defant presented these results generalizing the type of Maurey theorems to a positive homogeneous operators.

As second part in this work we also make use of this occasion to prove of a famous theorems of B. Maurey, by a showing the generalization of last theorems of factorization in [2] to $s-$positively homogeneous operators space. Remark that in this part our generalization is only if we take the departs spaces $L_q(\Omega,\mu)$ or the arrived spaces $L_q(\Omega,\mu)$ in the Maurey theorem [2]. We can considered the generalization to sublinear operators given in [8] as particular case of this in [2]. In [8]. The authors study the generalization of the last theorem factorization where are establish necessary and sufficient conditions to give the result: If that for $0 < p \leq q \leq +\infty$, and T be a sublinear operator from a Banach space X into $L_p(\Omega,\mu)$ then T factors through $L_q(\Omega,\mu)$. But let cited that the proof in [8] is an other method of this in Defant see [2]. In [1]. The authors had proved the extension of the work in [10], died the dual problem factorization of sublinear operators from $L_s(S,\lambda)$ into a Banach space Y by $L_{q1}(S,\nu)$, for $1 \leq q < s < \infty$). Also in the present paper, we will study the dual problem of our factorization [5]. Anther words. Let s; p,q and r real numbers such that, $s = 1$, $0 < p \leq q \leq +\infty$, with $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. We shall proof that with some necessary and sufficient conditions over the $1-$positively homogeneous operator T from S_q closed subspace of $L_q(\Omega,\mu)$ into an X a complete Banach lattice, that the operator T factors through S_p closed subspace of $L_p(\Omega,\mu)$. ((Ω,μ) any measure space).

In Section 2, we give some basic definitions, some preliminaries on the Banach lattices, sublinear operators, positively homogeneous operators. We also give results of relations between linear and sublinear operators. We shall use this preliminaries in the sequel of the present paper.

In Section 3, We define the classical operators class of $s-$positively homogeneous operators and prove that the above class is a Banach space. At second main results. We show with a necessary and sufficient conditions and some conditions over real numbers p,q which T is a $s-$positively homogeneous operator from X a Banach space into L_p that the operator T factors through L_q. After this, we give the extend result of dual problem factorization to same operators space above or we can say that if T from $S_q \subset L_q$ into X a Banach space then T factors through $S_p \subset L_p$.

2. FUNDAMENTAL PROPERTIES AND PRELIMINARIES

We collect below some properties and definitions which shall use in the sequel of this work. For more information about positively homogeneous operator, sublinear operators, Banach spaces and lattice spaces. We refer the reader to [2], [9] , [4],[3] and [6].

Definition 2.1. Let X A real vector space is partially ordered by a partial order and denoted by $\leq$ is called an order vector space if

$$x \leq y \text{ implies } x+z \leq y+z \text{ for every } z \in X;$$
$$x \geq 0 \text{ implies } \alpha x \geq 0 \text{ for every } \alpha \geq 0 \text{ in } \mathbb{R}^+.$$

Remark 2.1. A subset A of X is called simply bounded if there exists an element y in X such that $x \leq y$ for all $x \in A$ and y is then called an upper bound for A or the supremum of A. If A is bounded then z is called the least upper bound of A if z is an upper bounded for A and $z \leq y$ for every upper bound of A. An order vector space in which each pair of elements has least upper bound is called a vector lattice. In this case denoted by $\sup\{x, y\}$ or $x \vee y$. For which every non-empty order bounded subset has a least upper bound, we mean an order space or complete vector lattice. Let recall that an Banach lattice space X is called a Banach K-space if each bounded from above subset of X has upper bound and $\|.\|$ is monotone or if $|x| \leq |y| \Rightarrow \|x\| \leq \|y\|$.

Example 2.1. *The spaces $L_1([0.1], \mu)$, $C(K)$ is a Banach lattices. The spaces $L_p\ (1 \leq p \leq \infty)$ is a complete Banach lattices and if $(1 < p < \infty)$ is Banach K-spaces.*

Definition 2.2. Let T be an operator from a Banach space Xinto Y a Banach lattice.

T *is said to be sublinear see* [8] *if for all x, y in X and λ in $\mathbb{R}_+$, we have,*

(1) $T(\lambda x) = \lambda T(x)$ (i.e. positively homogeneous),
(2) $T(x+y) \leq T(x) + T(y)$ (i.e. subadditive).

Note that the multiplication by a positive number is also a sublinear operator. The sum of two sublinear operators is sublinear operator.

Definition 2.3. Let T be an operator from a Banach space X into Y a Banach space T is said positively homogeneous operator see [2] if for all x in X and λ in $\mathbb{R}_+$, we have,

$$T(\lambda x) = \lambda T(x) \text{ (i.e. positively homogeneous).}$$

Let us denote by
$SL(X, Y) = \{$sublinear operators $T : X \longrightarrow Y\}$,
and $L(X, Y) =\{$ linear operators $u : X \longrightarrow Y\}$.
We equip it with the natural order induced by Y we take that T_1 and T_2 is in $SL(X, Y)$ then

$$T_1 \leq T_2 \Longleftrightarrow T_1(x) \leq T_2(x), \quad \forall x \in X. \tag{2.1}$$

Now, we will give the following properties.
Let $T \in SL(X, Y)$ or $T \in L(X, Y)$ be a bounded operator from a Banach space X into a Banach lattice Y. if and only if $\exists C > 0 : \forall x \in X$, $\|T(x)\| \leq \|T\| \|x\|$ in this case we put,

$$\|T\| = \sup\{\|T(x)\| : \|x\|_{B_X} = 1\}. \tag{2.2}$$

Remark 2.2. [8]. Let X be an arbitrary Banach space. Let Y, Z be Banach lattices. We have,

(1) Consider $T \in SL(X, Y)$ and $u\ inL(Y, Z)$. Assume that u is positive. Then, $u \circ T \in SL(X, Z)$.
(2) Consider $u \in L(X, Y)$ and T in $SL(Y, Z)$. Then, $T \circ u \in SL(X, Z)$.

3. MAIN RESULTS

3.1. **The Banach space of $s-$ positively homogeneous operators.** In this part we introduce the concept of non linear $s-$positively homogeneous operators. Next, we prove that the class of this operators is a Banach space, and Banach lattice if Y be Banach lattice space and give some properties of characterization of this space.

Definition 3.1. Let $s \in [1, +\infty[$. *Let* T be an operator from a Banach space X into a Banach space Y. T is called $s-$ positively homogeneous operator. If for all $x \in X$ and $(\lambda \in \mathbb{R}_+)$ we have,

$$T(\lambda x) = \lambda^s T(x) \text{ (i.e . s-positively homogeneous).} \tag{3.1}$$

If we take $s = 1$, see [2] we are in the type of positively homogenous operators.

Remark 3.1. Note that the multiplication by a real number is also a $s-$positively homogeneous operator. The sum of two $s-$positively homogeneous operators is $s-$positively homogeneous operator. The multiplication of two $s-$positively homogeneous operators is $2s-$positively homogeneous operator.

Let us denote by
$H_s(X, Y) = \{ \, s - \text{positively homogeneous operators } T : X \longrightarrow Y \, \}$,
and we equip it with the natural order induced by Y If Y be Banach lattice, we take that T_1 and T_2 is in $H_s(X, Y)$ then,

$$T_1 \leq T_2 \Longleftrightarrow T_1(x) \leq T_2(x), \quad \forall x \in X.$$

Now, we will give the following properties.

Remark 3.2. Let X be an arbitrary Banach space. Let Y, Z be Banach spaces. We have
(1) If $T \in H_s(X, Y)$ and $u \in L(Y, Z)$. Then, $u \circ T \in H_s(X, Z)$.
(2) If $u \in L(X, Y)$ and $T \in H_s(Y, Z)$. Then, $T \circ u \in H_s(X, Z)$.

Proposition 3.1. *Let* $s \in [1, +\infty[$ *and* X Y *be a Banach spaces and let* T *be* $s-$*positively homogeneous operator from* X *into an* Y. *If* T *is discontinuous at* x_0, *then* T *is discontinuous at all* $x_a = |a| \, x_{0,} \, \forall a \in \mathbb{R}^*$.

Proof. We have T is discontinuous at x_0 $\left(\text{i.e} \lim\limits_{x \to x_0} T\left(x\right) \neq T\left(x_0\right) \right)$ we have that the operator $T_1 = |a|^s \, T$ is discontinuous at x_0, then also the operator T_2 is discontinuous at x_0 such that

$$T_2\left(x\right) = T\left(|a| \, x\right) = T_1\left(x\right) = |a|^s \, T\left(x\right),$$

then we have $\lim\limits_{x \to x_0} T_2\left(x\right) \neq T_2\left(x_0\right)$, but

$$\begin{aligned} \lim_{x \to x_0} T_2\left(x\right) = \lim_{x \to x_0} T\left(|a| \, x\right) &\overset{y = |a| x}{=} \lim_{y \to |a| x_0} T\left(y\right) \neq T_2\left(x_0\right) \\ &\neq |a|^s \, T\left(x_0\right) = T\left(|a| \, x_0\right) \\ \text{then, } \lim_{y \to |a| x_0} T\left(y\right) \neq T\left(|a| \, x_0\right). \end{aligned}$$

Consequently T is discontinuous at all $x_a = |a| \, x_{0,} \, \forall a \in \mathbb{R}^*$. □

Proposition 3.2. *Let* $s \in [1, +\infty[$ *and* X Y *be a Banach spaces and let* T *be* $s-$*positively homogeneous operator from* X *into an* Y. T *is discontinuous over* X, *then,* T *is discontinuous at* 0.

Proof. Assume that T discontinuous over X. This implies that $\exists x_0 \in X$ such that T discontinuous at x_0 consequently we have the statement:

$$\forall \eta > 0, \exists \epsilon > 0, .x \neq x_0, \|x - x_0\|_X < \eta \Rightarrow \|T(x) - T(x_0)\|_Y \geq \epsilon. \tag{3.2}$$

Noted that if T is discontinuous at x_0, then also by Proposition (3.1) T is discontinuous at all $x_a = ax_{0,}\ \forall a \in \mathbb{R}_+^*$. We have

$$\forall \eta > 0, \exists \epsilon_a > 0, .x \neq ax_0, \text{ if } \\ \|x - ax_0\|_X < \eta \Rightarrow \|T(x) - T(ax_0)\|_Y \geq \epsilon_a. \tag{3.3}$$

Let $x_0 \neq 0$, assume that T discontinuous at x_0 and continuous at 0 then

$$T \text{ continuous at } 0 \Leftrightarrow \\ \forall \epsilon > 0,\ \ \exists \eta_0 > 0, .x \neq 0, \|x\|_X < \eta_0 \Rightarrow \|T(x)\|_Y < \varepsilon. \tag{3.4}$$

First if $\|x_0\|_X < \eta_0$,by (3.4)

$$\|x_0\|_X < \eta_0 \Rightarrow \|T(x_0)\|_Y < \varepsilon,\ \forall \epsilon > 0, \tag{3.5}$$

and in (3.2) if we put $\eta = \eta_1 = \dfrac{\|x_0\|\,\eta_0}{2}$, this implies that,

$$0 \in V\left(x_0\right) \subset V\left(0\right) \Rightarrow \exists \epsilon_1 > 0 \Rightarrow \|T(x_0)\|_Y \geq \epsilon_{1,} \tag{3.6}$$

by (3.5) and (3.6) contradiction.

Secondly if $\|x_0\|_X \geq \eta_0$. We get $a_0 = \dfrac{\|x_0\|_x\,\eta_0}{2}$, $\eta_2 = \dfrac{\eta_0}{2}$, and $x_1 = \dfrac{\|x_0\|_x\,\eta_0}{2}x_0$ since $\|x_1\| < \frac{\eta_0}{2} = \eta_2 < \eta_0$. We have

$$T \text{ continuous at } 0 \Leftrightarrow \\ \forall \epsilon > 0, \exists \eta_0 > 0, .\ \|x_1\|_X < \eta_0 \Rightarrow \|T(x_1)\|_Y < \varepsilon. \tag{3.7}$$

This implies

$$\|x_1\|_X < \eta_0 \\ \Rightarrow \|T(x_0)\|_Y < \frac{2^s}{\eta_0^s\,\|x_0\|^s}\varepsilon = B, \forall \epsilon > 0 \text{ or } \forall B > 0. \tag{3.8}$$

T a discontinuous at x_1 in this case we choose $(\forall \eta)$ that $\eta = \eta_2$ we have. that for $x_2 = \frac{x_1}{2}, \exists \epsilon_{a_0} > 0, , \|x_2\|_X < \eta_2$ then by (3.3) we have,

$$\left\| T\left(\frac{\|x_0\|_x\,\eta_0}{4}x_0\right) - T\left(\frac{\|x_0\|_x\,\eta_0}{2}x_0\right) \right\|_Y \geq \epsilon_{a_0},$$

then

$$\|T(x_0)\|_Y \geq \frac{1}{\left(\left|\frac{\|x_0\|_x\,\eta_0}{4} - \frac{\|x_0\|_x\,\eta_0}{2}\right|\right)^s}\epsilon_a = A > 0 \\ \text{or } \exists A > 0. \tag{3.9}$$

By (3.8) and (3.9) contradiction. Now if $x_0 = 0$. Evident. Consequently the result is proved. □

Theorem 3.3. *Let $s \in [1, +\infty[$, X and Y be a Banach spaces and let T be $s-$positively homogeneous operator from X into Y. Then, the following conditions are equivalent.*

1- *T is continuous over X.*
2- *T is continuous at 0.*
3- *There is a finite positive constant C such that,*

$$\|T\left(x\right)\|_Y \leq C\left\|x\right\|_X^s. \tag{3.10}$$

Proof. **(1)**$\Longrightarrow$**(2)**.Evident.

(2)$\Longrightarrow$**(3)**. Let T be a continuous $s-$positively homogeneous operator then, it is at 0, in this case we have

$$\exists\eta>0:\forall x(\neq 0)\in X,\|x\|_X<\eta\Rightarrow\|T(x)\|_Y\leq 1. \tag{3.11}$$

Take $y=\dfrac{x}{\|x\|_X}\eta$. Then $\|T(y)\|_Y\leq 1$. Hence $\|T(x)\|_Y\leq\dfrac{1}{\eta^s}\|x\|_X^s$. Put $C=\dfrac{1}{\eta^s}$.

(3)$\Longrightarrow$**(1)** Assume that T discontinuous over X. by the Proposition (3.2) we study only that T discontinuous at 0 consequently we have the statement:

$$\forall\eta,1>\eta>0,\exists\epsilon_0>0,.x\neq 0,\|x\|_X<\eta\Rightarrow\|T(x)\|_Y\geq\epsilon_0. \tag{3.12}$$

We define $V(0)=\{x\in X/\|x\|_X<\eta\}$. Take $z\in V(0)$ we define $x_1=\dfrac{z\eta}{C^{\frac{1}{s}}\|z\|_X\frac{1}{\epsilon_0}}$, $\eta_1=\dfrac{\eta}{C^{\frac{1}{s}}\|z\|_X\frac{1}{\epsilon_0}+1}$ and $V_1(0)=\{x\in X/\ \ \|x\|_X<\eta_1\}$ we have that $V_1(0)\subset V(0)$. We can see that $x_1\in V_1(0)\Rightarrow x_1\in V(0)$. We deduce by (3.12) that

$$\text{For }\eta,1>\eta>0,\exists\epsilon_0>0,.x\neq 0,\|x_1)\|_X<\eta\Rightarrow\|T(x_1)\|_Y\geq\epsilon_0.$$

We have $(V_1(0)\subset V(0))$ then

$$\text{For},\eta_1>0,\exists\epsilon_1>0,.\ \|x_1\|<\eta_1\Rightarrow\|T(x_1)\|_Y\geq\epsilon_1\geq\epsilon_0. \tag{3.13}$$

By (3.13)

$$\|T(x_1)\|_Y\geq\epsilon_0\Rightarrow\left\|T(\frac{z\eta}{C^{\frac{1}{s}}\|z\|_X\frac{1}{\epsilon_0}})\right\|_Y\geq\epsilon_0,$$
$$\Rightarrow\|T(z)\|_Y\geq\frac{1}{\eta}C\|z\|_X^s\geq C\|z\|_X^s\ /\ \left(\frac{1}{\eta}\geq 1\right),$$

we deduce that, there is $z\in X$ such that,$\|T(z)\|_Y>C\|z\|_X^s$. Finally, immediate T is continuous over X and $(\mathbf{3})\Longrightarrow(\mathbf{1})$ is shown. Consequently the result is proved. □

Proposition 3.4. *Let T be a continuous $s-$ positively homogeneous operator from a Banach space X into a Banach space Y. T is bounded if and only if $\exists C>0:\forall x\in X$, $\|T(x)\|\leq C\|x\|^s$ in this case we put*

$$C=\|T\|=\sup\{\|T(x)\|:\|x\|_{B_X}=1\}. \tag{3.14}$$

$\|T\|$ is a norm over $H(X,Y)$.

Proof. Let the set $\{\dfrac{\|T(x)\|}{\|x\|^s}/\forall x\neq 0\}$, we have,

$$\begin{aligned}\{\frac{\|T(x)\|}{\|x\|^s}/\forall x\neq 0\} &= \{\left\|T\left(\tfrac{x}{\|x\|}\right)\right\|/x\neq 0\}\\ &\subset \{\|T(x)\|/\|x\|_{B_X}=1\}\\ &\subset \{\|T(x)\|/\|x\|_{B_X}\leq 1\}.\end{aligned}$$

Put

$$\begin{aligned}C_1 &= \sup\{\frac{\|T(x)\|}{\|x\|^s}/\forall x\neq 0\},\\ C_2 &= \sup\{\|T(x)\|/\|x\|_{B_X}=1\},\\ C_3 &= \sup\{\|T(x)\|/\|x\|_{B_X}\leq 1\}.\end{aligned}$$

The above set inclusions show that $C_1 \leq C_2 \leq C_3$.

$$\text{If } \forall x \neq 0 \frac{\|T(x)\|}{\|x\|^s} \leq C_1 \Rightarrow \|T(x)\| \leq C_1 \|x\|^s .$$

$$\text{If } \|x\|_{B_X} = 1 \Rightarrow \|T(x)\| \leq C_1 \leq C_2 \|x\|_{B_X} .$$

$$\text{If } \|x\|_{B_X} \leq 1 \Rightarrow \|T(x)\| \leq C_1 \leq C_2 \leq C_3 \|x\|_{B_X} \leq C_3.$$

Then, $C_1 \leq C_2 \leq C_3 \Longrightarrow C_1 = C_2 = C_3 = C$.
For $\|T\| = \sup\{\|T(x)\| : \|x\|_{B_X} = 1\}$. $\|T\|$ is a norm over $HB_s(X,Y)$ evident. □

Notation . We denoted by
$HB_s(X,Y) =$ bounded $s-$ positively homogeneous operator. $T : X \longrightarrow Y$,
$HB_1(X,Y) =$ bounded $1-$ positively homogeneous operator. $T : X \longrightarrow Y$,
$SB(X,Y) =$ bounded sublinear operators $T : X \longrightarrow Y$,
and by
$B(X,Y) =$ bounded linear operators $u : X \longrightarrow Y$.

We have, $B(X,Y) \subset SB(X,Y) \subset HB_p(X,Y) \subset HB(X,Y)$.

Example 3.1. *Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x^n, \quad /n \in \mathbb{N}^*$, is it continuous, in our case if to answer we want to give that: we have that $f(x)$ is $n-$ positively homogeneous function, and we have also $|x^n| \leq 1 |x|^n$ then $\exists C = 1$ by Theorem (3.3) f is continuous.*

Example 3.2. *Same of Example (3.1) . Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $X = (x,y) \mapsto f(x,y) = (x^n, y^n), \quad /n \in \mathbb{N}^*$, We have that $f(X)$ is $n-$positively homogeneous function, and we have also*

$$\|f(x,y)\| = \sqrt{x^{2n} + y^{2n}} \leq \qquad \sqrt{(x^2+y^2)^n = x^{2n} + y^{2n} + \sum_{i=1}^{n-1} C_n^k (x^2)^k (y^2)^{n-k}}$$

$$\textit{such that } \sum_{i=1}^{n-1} C_n^k (x^2)^k (y^2)^{n-k} \geq 0 \quad \textit{then } \|f(x,y)\| \leq \sqrt{(x^2+y^2)^n} = \|(x,y)\|^n$$

then $\exists C = 1$ by Theorem (3.3) f is continuous.

Remark 3.3. Unfortunately in this time we can not give discontinuous linear or discontinuous $s-$ positively homogeneous for an application of Theorem (3.3).

Corollary 3.5. *Let $U_1, : X \to Y, \quad U_2, : Y \to Z$ be a bounded $s-$positively homogeneous operators, then $U_2 \circ U_1$ is bounded $2s-$positively homogeneous operator, $\|U_2 \circ U_1(x)\| \leq C \|x\|^{2s}$.*

Proposition 3.6. *Let $s \in [1, +\infty[$ and X be Y a Banach spaces. Then $HB_s(X,Y)$ is complete space.*

Proof. Let $(T_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $H_S(X,Y)$. Take $K \geq 1$ then there is N_K such that $\|T_n - T_m\|_Y \leq \frac{1}{K}$, for all $n;m \geq N_K$. Take the sequence $\{T_n(x)\}_{n\in\mathbb{N}}$ such that $x \in B_X$ is Cauchy sequence in Y this implies that $\|T_n(x) - T_m(x)\|_Y \leq \frac{1}{K}$ for all $n;m \geq N_K$. We deduce that is function $T(x)$ such that $T_n(x) \to T(x)$. Therefore $\|T - T_n\|_Y \leq \frac{1}{K}, \forall n \geq N_K$. As consequence $\|T - T_n\|_Y \to 0$. Proved now that T is in $HB_S(X,Y)$. Indeed, let $\lambda \in \mathbb{R}_+$, then, $T(\lambda x) = \lim_n T_n(\lambda x) = \lambda^S \lim_n T_n(x) = \lambda^S T(x)$, hence $T(\lambda x) = \lambda^S T(x)$. Remark that $\lim_{x\to x_0} \lim_n T(x) = \lim_n \lim_{x\to x_0} T_n(x) = \lim_n T_n(x_0) = T(x_0) \Rightarrow T$ is continuous at any $x_0 \in X$. □

If we according Remark (3.2), Theorem (3.3), Proposition (3.4) and Proposition (3.6), we give the below Corollary.

Corollary 3.7. *$HB_s(X,Y)$ is a Banach space.*

Remark 3.4. We can see that $B(X,Y)$, $HB_1(X,Y)$ are a subspaces of $HB_s(X,Y)$ but $SB(X,Y)$ is only a positive cone in $HB_s(X,Y)$.

3.2. **Generalized factorization theorems to elements of $H_s(X,Y)$.** In this subsection we start by giving extend of the same result of Theorem 4.2 in [8]

Theorem 3.8. *Let $s \in [1,+\infty[\,, p, q, r in]0,+\infty]$ such that $0 < p \leq q \leq +\infty$ and $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. Let X be a Banach space and let T be a $s-$positively homogeneous operator from X into an $L_p(\Omega,\mu)$. Let C a finite constant. The following assertions are equivalent.*

(i) *There is a finite positive constant C such that for all finite sequence $(x_i)_{1\leq i\leq n} \in X$, we have,*

$$(\int_\Omega \left[\sum_{i=1}^{n} |T(x_i)|^q\right]^{\frac{p}{q}} d\mu)^{\frac{1}{p}} \quad \leq \quad C \left[\sum_{i=1}^{n} ((\|x_i\|_X^s))^q\right]^{\frac{1}{q}}.$$

(ii) *There is a function $g \in B^+_{L_r(\Omega,\mu)}$ such that for all $x \in X$, we have*

$$(\int_\Omega \left|\frac{T(x)}{g}\right|^q d\mu)^{\frac{1}{q}} \leq C \|x\|_X^s.$$

(iii) *There is a function $g \in B^+_{L_r(\Omega,\mu)}$ and a continuous $s-$ positively homogeneous operator S from X into $L_q\Omega,\mu)$, and $T = T_g o S$ such that $T_g : L_q(\Omega,\mu) \to L_p(\Omega,\mu)$ is the linear operator of multiplication by g*

$$\begin{array}{ccc} X & \xrightarrow{T} & L_p(\Omega,\mu) \\ {\scriptstyle S}\searrow & & \nearrow{\scriptstyle T_g} \\ & L_q(\Omega,\mu) & \end{array}.$$

Proof. **(i)$\Longrightarrow$(ii).** With Similar idea of Theorem 4.2 in [8]. It suffices to take in [5, Theorem 2] the real values $\alpha_i = \|x_i\|_X^s$ where $1 \leq i \leq n$ and $f_i = T\left(\frac{x_i}{\|x_i\|_X}\right)$. We have

$$(\int_\Omega \left[\sum_{i=1}^{n} |T(x_i)|^q\right]^{\frac{p}{q}} d\mu(\omega))^{\frac{1}{p}} = (\int_\Omega \left[\sum_{i=1}^{n} \left|\frac{\|x_i\|_X^s}{\|x_i\|_X^s} T\left(x_i\right)\right|^q\right]^{\frac{p}{q}} d\mu(\omega))^{\frac{1}{p}}$$

$$(\int_\Omega \left[\sum_{i=1}^{n} \left|\|x_i\|_X^s\, T\left(\frac{x_i}{\|x_i\|}\right)\right|^q\right]^{\frac{p}{q}} d\mu(\omega))^{\frac{1}{p}}$$

$$\leq C\left[\sum_{i=1}^{n} ((\|x_i\|_X^s))^q\right]^{\frac{1}{q}}.$$

By [5, Theorem 2] $\exists g$ in $B^+_{L_r(\Omega,\mu)}$ such that,

$$\forall i \in I \quad (\int_\Omega \left|\frac{T\left(\frac{x_i}{\|x_i\|_X}\right)}{g}\right|^q d\mu(\omega))^{\frac{1}{q}} \leq C. \tag{3.15}$$

Consequently for all x_i in X, $\forall i \in I \quad (\int\limits_{\Omega} \left|\frac{T(x_i)}{g}\right|^q d\mu(\omega))^{\frac{1}{q}} \leq C \|x_i\|_X^s$. Then for all x in X

$$(\int\limits_{\Omega} \left|\frac{T(x)}{g}\right|^q d\mu(\omega))^{\frac{1}{q}} \leq C \|x\|_X^s . \tag{3.16}$$

(ii)⇒(iii). Used (3.15) we show that: $\dfrac{T}{g}(x_i) \in L_q(\Omega,\mu)$ for all x_i in X. Then we define S such that $S : X \to L_q(\Omega,\mu)$ by $S(x) = \dfrac{T(x)}{g}$. T by Theorem (3.3) is continuous $s-$positively homogeneous operator satisfied the statement that $\|S(x)\| \leq \delta \|x\|^s$. Finally, clearly we have $T = T_g o S$ by Remark (3.2) the diagram is commutative.

(iii)⇒(i). By Hölder inequality implies for $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$

$$\begin{aligned}
\int\limits_{\Omega} (\sum_{i=1}^{n} |T(x_i)|^q)^{\frac{p}{q}} d\mu &= \int\limits_{\Omega} \left((\sum_{i=1}^{n} |g|^q |S(x_i)|^q)^{\frac{1}{q}} \right)^{\frac{p}{q}} d\mu(\omega) \\
&\leq (\int\limits_{\Omega} \sum_{i=1}^{n} |S(x_i(\omega))|^q d\mu(\omega))^{\frac{p}{q}} (\int\limits_{\Omega} (|g|^q)^{\frac{r}{q}} d\mu(\omega))^{\frac{p}{r}} \\
&\leq \left(\int\limits_{\Omega} \sum_{i=1}^{n} (|S(x_i)|^q d\mu(\omega)) \right)^{\frac{p}{q}} (\int\limits_{\Omega} |g|^r d\mu(\omega))^{\frac{1}{r}} \\
\text{Using Theorem(3.3)} \quad &\leq C^p (\sum_{i=1}^{n} (\|x_i\|_X^s)^q)^{\frac{p}{q}}.
\end{aligned}$$

□

Remark 3.5. In our Theorem (3.8) if $s = 1$ same result in [2]. If $s = 1$ and T subadditive same result in [8]. If $s = 1$ and T additive same result in [5].

In this part of our work we study the transposed problem of the last factorization see [5]. We give an analogous factorization to $1-$positively homogeneous operators.

Lemma 3.9. [5] *Let p, q and r three real numbers such that $0 < p \leq q < +\infty$ and $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. Let (Ω,μ) a measure space, I a set of indices and $\{f_i\}_{i\in I} \subset L_q(\Omega,\mu)$. Then, the following assertions are equivalent.*

a)- *There is a measurable function $g \in B_{Lr(\Omega,\mu)}$ such that*

$$\forall i \in I \quad \int_{\Omega} |gf_i|^p (w) d\mu(w) \geq 1. \tag{3.17}$$

b)- *For all $\{\alpha_i\}_{i\in I}$ in $\mathbb{R}^{(I)}$ there is a finite constant C such that*

$$(\sum_{i\in I} |\alpha_i|^p)^{\frac{1}{p}} \leq C(\int_{\Omega} (\sum_{i\in I} |\alpha_i f_i(w)|^p)^{\frac{q}{p}} d\mu(w))^{\frac{1}{q}}. \tag{3.18}$$

Theorem 3.10. *Let $s = 1$, Let p, q and $r \in \mathbb{R}$, such that $p \leq q$ and , $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. Let (Ω,μ) a measure space, S_q closed subspace of $L_q(\Omega,\mu)$, T be a $1-$positively homogeneous operator from S_q into X a complete Banach lattice and a positive finite constant C. Then, the following assertions are equivalent.*

a) *There exists S_p closed subspace of $L_p(\Omega,\mu)$ such that T admits the following factorization*

$$\begin{array}{ccc} S_q & \xrightarrow{T} & X \\ {}_{T_g} \searrow & S_p & \nearrow v \end{array}$$

v is a continuous $1-$ homogenous operator such that $\|v\,(T_g f)\| \leq C\,\|T_g f\|^1$.
g a function in $B_{L_r(\Omega,\mu)}$.
T_g is the induced operator on S_q by the multiplication operator T_g.
b) *For all finite sequences $\{f_i\}_{1\leq i\leq n}$ in S_q, we have*

$$(\sum_{i=1}^{n}\|T(f_i)\|_X^p)^{\frac{1}{p}} \leq C(\int_\Omega(\sum_{i=1}^{n}|f_i|^p)^{\frac{q}{p}}d\mu(w))^{\frac{1}{q}}. \tag{3.19}$$

Proof. **a)**$\Rightarrow$**b)**. By Hölder inequality,we have

$$\begin{aligned}
\sum_{i=1}^{n}\|T(f_i)\|_X^p &= (\sum_{i=1}^{n}\|vJ_g(f_i)\|_X^p\,, \\
&\leq C\sum_{i=1}^{n}\|J_g(f_i)\|_{S_p}^p\,, \\
&\leq C\sum_{i=1}^{n}\|g(w)f_i(w)\|_{S_p}^p\,, \\
&\leq C\sum_{i=1}^{n}(\int_\Omega |g^p(w)f_i^p(w)|\,d\mu(w) \\
&\leq C\int_\Omega |g(w)^p|\left(\sum_{i=1}^{n}|f_i(w)|^p\right)d\mu(w), \\
&\leq C\left(\|g\|_{L_r}\right)\left(\int_\Omega\left(\sum_{i=1}^{n}|f_i(w)|^p\right)^{\frac{q}{p}}d\mu(w)\right)^{\frac{1}{q}}, \\
&\leq C\left(\int_\Omega\left(\sum_{i=1}^{n}|f_i(w)|^p\right)^{\frac{q}{p}}d\mu(w)\right)^{\frac{1}{q}}.
\end{aligned}$$

b)$\Rightarrow$**a)**. Let $\{\alpha_i\}_{i\in I}\in\mathbb{R}^{(I)}$.Writing

$$\begin{aligned}
|\alpha_i| &= \|T(f_i)\|_X\frac{|\alpha_i|}{\|T(f_i)\|_X} \\
&= \||\alpha_i|\,T(f_i)\|_X\frac{1}{\|T(f_i)\|_X} \\
&= \left\|T(\frac{|\alpha_i|\,f_i}{\|T(f_i)\|_X})\right\|_X.
\end{aligned}$$

Then we

$$
\begin{array}{rcl}
(\sum\limits_{i=1}^{n}|\alpha_i|^p)^{\frac{1}{p}} & = & (\sum\limits_{i=1}^{n}\left\|T(\frac{|\alpha_i|f_i}{\|T(f_i)\|_X})\right\|_X^p)^{\frac{1}{p}} \\
\text{by (3.19)} & \leq & C(\int_\Omega(\sum\limits_{i=1}^{n}(|\alpha_i|\,\dfrac{|f_i|}{\|T(f_i)\|_X})^p)^{\frac{q}{p}}d\mu(w))^{\frac{1}{q}},
\end{array}
$$

take $F_i = \dfrac{|f_i|}{\|T(f_i)\|_X}$ in $L_q(\Omega,\mu)$ therefore according to Lemma (3.9). We have

$$\exists g \in B_{L_r(\Omega,\mu)} \text{ such that } \int_\Omega |gF_i|^p\, d\mu(w) \geq 1. \tag{3.20}$$

By (3.20) implies that

$$\int_\Omega \left| g\frac{|f_i|}{\|T(f_i)\|}\right|^p d\mu(w) \geq 1,$$

hence

$$\forall f_i \in S_{q,} \qquad \|T(f_i)\|_X^p \leq \int_\Omega |gf_i|^p\, d\mu(w);$$

then

$$\forall f \in S_q, \quad \left(\|T(f)\|_{L_p}^p\right)^{\frac{1}{p}} \leq (\int_\Omega |gf|^p\, d\mu(w))^{\frac{1}{p}},$$

and

$$\forall f_i \in S_{q,} \qquad \|T(f_i)\|_X^p \leq \int_\Omega |gf_i|^p\, d\mu(w)^{\frac{1}{p}};$$

Then,

$$\forall f \in S_q, \quad \|T(f)\|_{L_p} \leq (\int_\Omega |gf|^p\, d\mu(w))^{\frac{1}{p}}.$$

By Hölder inequality we have

$$\forall f \in S_q, \quad \|T(f)\|_{L_p} \leq \|g\|_{L_r}\, \|f\|_{L_q}^1\,. \tag{3.21}$$

We can get

$$
\begin{array}{rrcl}
T_g: & L_q(\Omega,\mu) & \longrightarrow & L_p(\Omega,\mu) \\
 & f & \longmapsto & fg.
\end{array}
$$

Defining v on $T_g(S_q)$ by $v(fg) = T(f)$, T continuous then v also, by inequality (3.21) and Theorem (3.3) implies that $\|v\,(x)\| \leq C\,\|x\|^1$. By extending v to $S_p = \overline{T_g(S_q)}^{L_p}$. We have the result. $\square$

Remark 3.6. With similar supposition of above Theorem (3.10), only in this one T be a sublinear operator and with similar prove if $s = 1$ and subadditive we have the same factorization.

Proposition 3.11. *Let $1 < p \leq q \leq \infty$ and $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$. Let X Banach K-space and T_1, T_2 be sublinear operators from S_q closed subspace of $L_q(\Omega,\mu)$ into X such that $T_1 \leq T_2$. If T_2 factors through $S_p(\Omega,\mu)$, then T_1 factors through $S_p(\Omega,\mu)$.*

Proof. If T_2 factors through $S_p(\Omega,\mu)$. According Remark (3.6) there is a finite positive constant $C\,(T_2\,)$ such that for all finite sequences $(f_i)_{1\leq i\leq n}$ in S_q, we have

$$(\sum_{i=1}^{n}\|T_2(f_i)\|_X^p)^{\frac{1}{p}} \leq C\,(T_2\,)\,(\int_\Omega(\sum_{i=1}^{n}|f_i|^p)^{\frac{p}{q}}d\mu(w))^{\frac{1}{q}}. \tag{3.22}$$

By (2.1) all f_i in S_q (T_2 is sublinear), we have,

$$|T_1(f_i)| \leq |T_2(f_i)| + |T_2(-f_i)|, \quad \forall f \in S_q.$$

We have that X Banach K-space. Hence

$$\|T_1(f_i)\|_X \ \leq \ \|T_2(f_i)\|_X + \|T_2(-f_i)\|_X).$$

Therefore by the Minkowski inequality ,

$$\left\|\|T_1(f_i)\|_X\right\|_{l_p(X)} \leq \left\|\|T_2(f_i)\|_X\right\|_{l_p(X)} + \left\|\|T_2(-f_i)\|_X\right\|_{l_p(X)}.$$

By similar of (3.19) in Remark (3.6), for all finite sequences $(f_i)_{1\leq i\leq n}$ in S_q, we have

$$(\sum_{i=1}^{n} \|T_1(f_i)\|_X^p) d\mu)^{\frac{1}{p}} \leq C\,(T_2\,)\,(\int_\Omega (\sum_{i=1}^{n} |f_i|^p)^{\frac{p}{q}} d\mu(w))^{\frac{1}{q}},$$

where $C` = 2C\,(T_2\,)$ by Remark (3.6) we deduce that T_1 factors by $S_p(\Omega, \mu)$ and this concludes the proof.

□

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