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Analytical method for solving a time-conformable fractional telegraph equation

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Abstract. In this paper, we present an analytical method to solve a time-conformable fractional telegraph equation with three boundary conditions namely, Dirichlet, Neumann and Robin. This method based on Fourier method and conformable fractional calculus properties. We give three examples to validate this method.

1. Introduction

Fractional calculus generalized the classical calculus to an arbitrary (non-integer) order. The history of this theory was begun from a letter written by L'Hôspital to Leibniz in 1695 asking him if $n = \frac{1}{2}$, what does it mean $\frac{d^n f}{dx^n}$. Leibniz then responded saying "*An apparent paradox*", different explanations of the fractional derivative are presented. There are almost 25 definitions of the fractional derivative. The most famous of them are Caputo and R-L derivatives. Both these definitions include integral in their definitions. Few properties of these fractional-order derivatives are similar to the classical order derivatives. However, there are a few complications, see [20].

☞ These definitions are non-local, which makes them unsuitable for investigating properties related to local scaling or fractional differentiability.

☞ Riemann Liouville's derivative does not fulfil $\mathcal{D}^{\alpha}(1) = 0$.

☞ For Caputo's derivative, we have to assume that the function is differentiable. Otherwise, we cannot apply this definition.

All fractional derivatives are deficient in some mathematical properties like product rule, chain rule, and quotient rule. Therefore, the solution of differential equations is not easy to obtain using these definitions.

A new type of fractional derivative was introduced by Khalil et al., [12] and developed by Abdeljawad, [2] called "conformable fractional derivative". This definition is different from other fractional derivatives and similar to the classical definition of the derivative. It depends on the limit definition of the derivative

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of a function. So this definition seems to be a natural extension of the ordinary derivative. Other fractional derivatives do not have geometrical interpretation but conformable derivative has [11]. This theory has attracted many researchers to work within and so many new concepts are introduced in conformable fractional calculus, see [1, 3, 7, 18, 19, 24]. Recently, the authors in [4] introduced a fuzzy conformable derivative of order Ψ and extended by Younus et al. in [21–23]. In [16] the authors introduced a new class of mixed fractional differential equations involving the conformable and Caputo derivatives with integral boundary conditions. The exact solutions of conformable time-fractional modified non-linear Schrödinger equation is obtained by Direct algebraic method and Sine-Gordon expansion method [5].

Modelling of real-life problems with conformable derivatives was done in [8] and [13] where the authors modelled the dynamic cobweb and the Gray system, respectively.

We consider the following non-homogeneous time-fractional telegraph equation:

$$\mathcal{D}_t^{(2\alpha)} u(x,t) + 2a\mathcal{D}_t^{(\alpha)} u(x,t) + b^2 u(x,t) = k^2 \frac{\partial^2 u}{\partial x^2}(x,t) + f(x,t), \ (x,t) \in \Omega_T, \ 0 < \alpha \le 1 \tag{1}$$

and initial conditions

$$u(x,0) = \phi(x), \ \mathcal{D}_t^{(\alpha)} u(x,0) = \psi(x), \ 0 \le x \le \ell, \tag{2}$$

where $\Omega_T := \{(x,t) : 0 < x < \ell, \ 0 < t \le T\}$ with $\ell, T > 0$ are given, $\mathcal{D}_t^{(\alpha)}$ represents the left-conformable fractional derivative of order $0 < \alpha \le 1$ with respect to t and $\mathcal{D}_t^{(2\alpha)} = \mathcal{D}_t^{(\alpha)}\left(\mathcal{D}_t^{(\alpha)}\right)$. $f(x,t)$ is the source term and a, b, k are constants such that $a \ge b > 0$, $k > 0$ and $\phi, \psi \in C(0,\ell)$.

For $\alpha = 1$, equation (1) is the classical telegraph equation introduced by Oliver Heaviside [9]. This equation is a second-order linear hyperbolic equation and it models several phenomena in many different fields such as signal analysis [10], wave propagation [17], random walk theory [6].

In this paper, we derive the analytical solution of equation (1) under three types of non-homogeneous boundary conditions using the Fourier method and the properties of conformable fractional calculus.

We organized this paper as follows: in Section 2, we give some concepts on the conformable fractional calculus. In Section 3, we derive the analytical solution of equation (1) with Dirichlet boundary conditions. In Section 4 and Section 5, we discuss the analytical solution equation (1) with Neumann and Robin boundary conditions, respectively. Some conclusions are drawn in Section 6.

2. Preliminaries on conformable fractional calculus

In this section, we start by recalling some concepts about conformable fractional calculus.

Definition 2.1 ([2]). *Let $\varphi : [a, +\infty[\to \mathbb{R}$ is a given function and $\alpha \in]0, 1]$. Then, the left-conformable fractional derivative of order α is defined by:*

$$\mathcal{D}_t^{(\alpha)}(\varphi)(t) := \lim_{\varepsilon \to 0} \frac{\varphi\left(t + \varepsilon (t-a)^{1-\alpha}\right) - \varphi(t)}{\varepsilon}. \tag{3}$$

If $\mathcal{D}_t^{(\alpha)}(\varphi)(t)$ exists on $]a, +\infty[$, then $\mathcal{D}_t^{(\alpha)}(\varphi)(a) = \lim_{t \to a^+} \mathcal{D}_t^{(\alpha)}(\varphi)(t)$. If $a = 0$, the definition (3) is introduced by Khalil et al. [12]. In this case, we say that φ is α-differentiable.

Definition 2.2 ([12]). *Let $\alpha \in]0, 1]$ and $\varphi : [0, +\infty[\to \mathbb{R}$ be real valued function. The left-conformable fractional integral of φ of order α from zero to t is defined by:*

$$\mathcal{I}_\alpha \varphi(t) := \int_0^t s^{\alpha-1} \varphi(s)\, ds, \ \ t \ge 0, \tag{4}$$

Lemma 2.1 ([12]). *Let $\varphi : [0, +\infty[\to \mathbb{R}$ is a given function and $0 < \alpha \le 1$. Then, for all $t > 0$, we have:*

1. *If φ is continuous, then $\mathcal{D}_t^{(\alpha)}\left[\mathcal{I}_\alpha \varphi(t)\right] = \varphi(t)$.*
2. *If φ is α-differentiable, then $\mathcal{I}_\alpha\left[\mathcal{D}_t^{(\alpha)}(\varphi)(t)\right] = \varphi(t) - \varphi(0)$.*

Definition 2.3 ([2]). *Let $0 < \alpha \leq 1$ and $\varphi : [0, +\infty[\to \mathbb{R}$ be real valued function. Then, the fractional Laplace transform of order α starting from zero of φ is defined by:*

$$\mathcal{L}_\alpha\left[\varphi(t)\right](s) = \int_0^{+\infty} t^{\alpha-1}\varphi(t)\, e^{-s\frac{t^\alpha}{\alpha}}\, dt. \tag{5}$$

Proposition 2.1 ([2]). 1. *The conformable fractional Laplace transform is a linear operator:*

$$\mathcal{L}_\alpha\left\{\mu f(t) + \lambda g(t)\right\}(s) = \mu\mathcal{L}_\alpha\left\{f(t)\right\} + \lambda\mathcal{L}_\alpha\left\{g(t)\right\}, \tag{6}$$

where μ and λ are real constants.

2. *We have,*

$$\mathcal{L}_\alpha\left\{f(t)\, e^{-k\frac{t^\alpha}{\alpha}}\right\}(s) = \mathcal{L}\left\{f\left((\alpha t)^{1/\alpha}\right)\right\}\Big|_{s=s+k}, \quad s > -k. \tag{7}$$

3. *Let, $\alpha \in\,]0, 1\,]$ and $f(t)$, $g(t)$ are functions.*
The conformable fractional Laplace transform of the convolution product of f and g is given by,

$$\mathcal{L}_\alpha\left\{(f * g)(t)\right\} = \mathcal{F}_\alpha(s)\, \mathcal{G}_\alpha(s), \tag{8}$$

where, $(f * g)(t) = \int_0^t f\left((t^\alpha - \tau^\alpha)^{1/\alpha}\right) g(\tau)\, \frac{d\tau}{\tau^{1-\alpha}}$.

Theorem 2.1 ([2]). *Let $0 < \alpha \leq 1$ and $\varphi : [0, +\infty[\to \mathbb{R}$ be differentiable real valued function. Then, we have:*

$$\mathcal{L}_\alpha\left[\mathcal{D}_t^{(\alpha)}\varphi(t)\right](s) = s\mathcal{L}_\alpha\left[\varphi(t)\right](s) - \varphi(0). \tag{9}$$

We introduce the following theorem, which is used further in this paper.

Theorem 2.2 ([15]). *Let $g : [0, +\infty[\to \mathbb{R}$ is a continuous function and $\eta, \gamma \in \mathbb{R}_+$ such that $\eta < \gamma$. For all $0 < \alpha \leq 1$, the initial value problem:*

$$\begin{cases} \mathcal{D}_t^{(2\alpha)} y(t) + 2\eta\mathcal{D}_t^{(\alpha)} y(t) + \gamma^2 y(t) = g(t), \\ y(0) = y_0, \;\; \mathcal{D}_t^{(\alpha)} y(0) = y_\alpha. \end{cases} \tag{10}$$

admits a unique solution given by

$$\begin{aligned} y(t) &= y_0 e^{-\eta\frac{t^\alpha}{\alpha}} \cos\left(\sqrt{\gamma^2 - \eta^2}\frac{t^\alpha}{\alpha}\right) + \frac{y_0\eta + y_\alpha}{\sqrt{\gamma^2 - \eta^2}} e^{-\eta\frac{t^\alpha}{\alpha}} \sin\left(\sqrt{\gamma^2 - \eta^2}\frac{t^\alpha}{\alpha}\right) \\ &\quad + \frac{1}{\sqrt{\gamma^2 - \eta^2}} \int_0^t g\left((t^\alpha - \tau^\alpha)^{1/\alpha}\right) e^{-\eta\frac{\tau^\alpha}{\alpha}} \sin\left(\sqrt{\gamma^2 - \eta^2}\frac{\tau^\alpha}{\alpha}\right) \frac{d\tau}{\tau^{1-\alpha}}. \end{aligned} \tag{11}$$

3. Conformable fractional telegraph equation with Dirichlet boundary condition

In this section, we determine the analytical solution of the time-fractional telegraph equation (1) with the initial conditions (2) and the non-homogeneous Dirichlet boundary conditions

$$u(0, t) = \mu_1(t), \;\; u(\ell, t) = \mu_2(t), \;\; t > 0, \tag{12}$$

where $\mu_1, \mu_2 \in C^1(0, T)$ satisfying

$$\phi(0) = \mu_1(0) \;\; \text{and} \;\; \phi(\ell) = \mu_2(0).$$

We Assume

$$u(x,t) = W_1(x,t) + V_1(x,t),$$

where $V_1(x,t)$ is given by:

$$V_1(x,t) = \mu_1(t) + \frac{[\mu_2(t) - \mu_1(t)]\,x}{\ell}$$

in which satisfies the Dirichlet boundary conditions:

$$V_1(0,t) = \mu_1(t), \quad V_1(\ell,t) = \mu_2(t).$$

and the function $W_1(x,t)$ is the solution of the following problem:

$$\begin{cases} \mathcal{D}_t^{(2\alpha)} W_1(x,t) + 2a\mathcal{D}_t^{(\alpha)} W_1(x,t) + b^2 W_1(x,t) = k^2 \frac{\partial^2 W_1(x,t)}{\partial x^2} + \tilde{f}(x,t), \\ W_1(x,0) = \phi_1(x), \quad \mathcal{D}_t^{(\alpha)} W_1(x,0) = \psi_1(x), \\ W_1(0,t) = W_1(\ell,t) = 0, \end{cases} \tag{13}$$

where

$$\tilde{f}(x,t) = -\mathcal{D}_t^{(2\alpha)} V_1(x,t) - 2a\mathcal{D}_t^{(\alpha)} V_1(x,t) - b^2 V_1(x,t) + f(x,t),$$

$$\phi_1(x) = \phi(x) - \mu_1(0) - \frac{[\mu_2(0) - \mu_1(0)]\,x}{\ell},$$

$$\psi_1(x) = \psi(x) - \mathcal{D}_t^{(\alpha)}\mu_1(0) - \frac{\left[\mathcal{D}_t^{(\alpha)}\mu_2(0) - \mathcal{D}_t^{(\alpha)}\mu_1(0)\right]x}{\ell}.$$

We firstly assume the solution of the homogeneous equation in (13) (putting $\tilde{f}(x,t) = 0$) has the form:

$$W_1(x,t) = X(x)\,Y(t).$$

Substituting in (13) we obtain the Sturm-Liouville problem:

$$\begin{cases} X''(x) + \lambda X(x) = 0, \\ X(0) = X(\ell) = 0, \end{cases} \tag{14}$$

This problem has eigenvalues

$$\lambda_n = \frac{n^2\pi^2}{\ell^2}, \quad n \in \mathbb{N}^*$$

and corresponding eigenfunctions

$$X_n(x) = \sin\left(\frac{n\pi x}{\ell}\right), \quad n \in \mathbb{N}^*.$$

Now we seek a solution of the non-homogeneous problem (13) of the form

$$W_1(x,t) = \sum_{n=1}^{+\infty} B_n(t) \sin\left(\frac{n\pi x}{\ell}\right). \tag{15}$$

In order to determine $B_n(t)$, we expand $\tilde{f}(x,t)$ as a Fourier series by the eigenfunctions $X_n(x)$:

$$\tilde{f}(x,t) = \sum_{n=1}^{+\infty} \tilde{f}_n(t) \sin\left(\frac{n\pi x}{\ell}\right), \text{ where } \tilde{f}_n(t) = \frac{2}{\ell}\int_0^\ell \tilde{f}(x,t) \sin\left(\frac{n\pi x}{\ell}\right) dx. \tag{16}$$

Substituting (15), (16) into (13) yields

$$\mathcal{D}_t^{(2\alpha)} B_n(t) + 2a\mathcal{D}_t^{(\alpha)} B_n(t) + \left(b^2 + \lambda_n k^2\right) B_n(t) = \tilde{f}_n(t). \tag{17}$$

Since $W_1(x,t)$ satisfies the initial conditions in (13), we must have

$$\begin{cases} \sum\limits_{n=0}^{+\infty} B_n(0) \sin\left(\frac{n\pi x}{\ell}\right) = \phi_1(x), \quad 0 < x < \ell, \\ \sum\limits_{n=0}^{+\infty} \mathcal{D}_t^{(\alpha)} B_n(0) \sin\left(\frac{n\pi x}{\ell}\right) = \psi_1(x), \quad 0 < x < \ell, \end{cases}$$

which yields

$$\begin{cases} B_n(0) = \frac{2}{\ell}\int_0^\ell \phi_1(x) \sin\left(\frac{n\pi x}{\ell}\right) dx, \quad n \in \mathbb{N}^*, \\ \mathcal{D}_t^{(\alpha)} B_n(0) = \frac{2}{\ell}\int_0^\ell \psi_1(x) \sin\left(\frac{n\pi x}{\ell}\right) dx, \quad n \in \mathbb{N}^*. \end{cases} \tag{18}$$

We assume that

$$0 \le \frac{a^2 - b^2}{k^2} < \lambda_1, \tag{19}$$

where $\lambda_1 = \pi^2/\ell^2$ is the smallest eigenvalue of the Strum-Liouville problem (14).

Using condition (19), Theorem 2.2, (17), (18) and (15) we obtain the solution of problem (13) as

$$\begin{aligned} W_1(x,t) = \sum_{n=1}^{+\infty} \Bigg[& B_n(0) e^{-a\frac{t^\alpha}{\alpha}} \cos\left(\sqrt{b^2 - a^2 + \lambda_n k^2}\frac{t^\alpha}{\alpha}\right) \\ & + \frac{B_n(0)a + \mathcal{D}_t^{(\alpha)} B_n(0)}{\sqrt{b^2 - a^2 + \lambda_n k^2}} e^{-a\frac{t^\alpha}{\alpha}} \sin\left(\sqrt{b^2 - a^2 + \lambda_n k^2}\frac{t^\alpha}{\alpha}\right) \\ & + \frac{1}{\sqrt{b^2 - a^2 + \lambda_n k^2}} \int_0^t \tilde{f}_n\left((t^\alpha - \tau^\alpha)^{1/\alpha}\right) e^{-a\frac{\tau^\alpha}{\alpha}} \sin\left(\sqrt{b^2 - a^2 + \lambda_n k^2}\frac{\tau^\alpha}{\alpha}\right) \frac{d\tau}{\tau^{1-\alpha}} \Bigg] \sin\left(\frac{n\pi x}{\ell}\right). \end{aligned} \tag{20}$$

Example 3.1. *([14, Chapter 2]) In this example, we consider the following data:*

$$\begin{cases} a = b = k = \ell = 1, \ \alpha = \frac{1}{2} \text{ and } f(x,t) = 0, \\ \phi(x) = 1, \ \psi(x) = 0, \\ \mu_1(t) = \mu_2(t) = 0. \end{cases}$$

The problem (1), (2) and (12) becomes:

$$\begin{cases} \mathcal{D}_t^{(1/2)}\left(\mathcal{D}_t^{(1/2)} u(x,t)\right) + 2\mathcal{D}_t^{(1/2)} u(x,t) + u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}, \\ u(x,0) = 1, \ \mathcal{D}_t^{(1/2)} u(x,0) = 0, \quad 0 \le x \le 1, \\ u(0,t) = u(1,t) = 0. \end{cases} \tag{21}$$

The eigenvalues and the eigenfunctions are given by

$$\lambda_n = \pi^2 n^2, \ \text{ and } \ X_n(x) = \sin(\pi n x), \ n \in \mathbb{N}^*.$$

From (18), we get

$$B_n(0) = \frac{2\left(1 - (-1)^n\right)}{\pi n} \ \text{ and } \quad \mathcal{D}_t^{(1/2)} B_n(0) = 0.$$

According to (20), the analytical solution of problem (21) is given by

$$u(x,t) = \sum_{n=0}^{+\infty} \frac{4e^{-2\sqrt{t}}}{\pi(2n+1)}\left(\cos(4n+2)\pi\sqrt{t} + \frac{1}{\pi(2n+1)}\sin(4n+2)\pi\sqrt{t}\right)\sin(2n+1)\pi x.$$

4. Conformable fractional telegraph equation with Neumann boundary condition

In this section, we determine analytical solution of the time-fractional telegraph equation (1) with initial conditions (2) and non-homogeneous Neumann boundary conditions

$$u_x(0,t) = \mu_1(t), \quad u_x(\ell,t) = \mu_2(t), \quad t > 0, \tag{22}$$

where $\mu_1, \mu_2 \in C(0,T)$ satisfying

$$\mu_1(0) = \phi'(0) \text{ and } \mu_2(0) = \phi'(\ell).$$

Let

$$u(x,t) = W_2(x,t) + V_2(x,t),$$

where $W_2(x,t)$ is given by

$$V_2(x,t) = \mu_1(t) + \mu_1(t)x + \frac{[\mu_2(t) - \mu_1(t)]x^2}{2\ell},$$

in which satisfies the Neumann boundary conditions

$$\frac{\partial V_2(0,t)}{\partial x} = \mu_1(t), \quad \frac{\partial V_2(\ell,t)}{\partial x} = \mu_2(t)$$

and the function $W_2(x,t)$ is the solution of the following problem:

$$\begin{cases} \mathcal{D}_t^{(2\alpha)} W_2(x,t) + 2a\mathcal{D}_t^{(\alpha)} W_2(x,t) + b^2 W_2(x,t) = k^2 \frac{\partial^2 W_2(x,t)}{\partial x^2} + \tilde{f}(x,t), \\ W_2(x,0) = \phi_2(x), \quad \mathcal{D}_t^{(\alpha)} W_2(x,0) = \psi_2(x), \\ \frac{\partial W_2(0,t)}{\partial x} = \frac{\partial W_2(\ell,t)}{\partial x} = 0, \end{cases} \tag{23}$$

in which

$$\tilde{f}(x,t) = -\mathcal{D}_t^{(2\alpha)} V_2(x,t) - 2a\mathcal{D}_t^{(\alpha)} V_2(x,t) - b^2 V_2(x,t) + f(x,t),$$

$$\phi_2(x) = \phi(x) - \mu_1(0)x - \frac{[\mu_2(0) - \mu_1(0)]x^2}{2\ell},$$

$$\psi_2(x) = \psi(x) - \mathcal{D}_t^{(\alpha)}\mu_1(0)x - \frac{\left[\mathcal{D}_t^{(\alpha)}\mu_2(0) - \mathcal{D}_t^{(\alpha)}\mu_1(0)\right]x^2}{2\ell}.$$

We assume the solution of the homogeneous equation in (23), by taking $\tilde{f}(x,t) = 0$, has the form:

$$W_2(x,t) = X(x)Y(t)$$

and substitute in (23), we obtain the Sturm-Liouville problem:

$$\begin{cases} X''(x) + \lambda X(x) = 0, \\ X'(0) = X'(\ell) = 0. \end{cases} \tag{24}$$

A simple calculation shows that the eigenvalues of the Sturm-Liouville problem (24) are

$$\lambda_n = \frac{\pi^2 n^2}{\ell^2}, \quad n \in \mathbb{N}$$

and corresponding eigenfunctions

$$X_n(x) = \cos\left(\frac{\pi n x}{\ell}\right), \quad n \in \mathbb{N}.$$

Consider the solution of the problem (23) is the form

$$W_2(x,t) = \sum_{n=1}^{+\infty} B_n(t)\cos\left(\frac{n\pi x}{\ell}\right). \tag{25}$$

In order to determine $B_n(t)$, we expand $\tilde{f}(x,t)$ as a Fourier series by the eigenfunctions $\left\{\cos\left(\frac{n\pi x}{\ell}\right)\right\}$:

$$\tilde{f}(x,t) = \sum_{n=1}^{+\infty} \tilde{f}_n(t)\cos\left(\frac{n\pi x}{\ell}\right) \text{ where } \tilde{f}_n(t) = \frac{2}{\ell}\int_0^{\ell} \tilde{f}(x,t)\cos\left(\frac{n\pi x}{\ell}\right)dx. \tag{26}$$

Substituting (25), (26) into (23) yields

$$\mathcal{D}_t^{(2\alpha)}B_n(t) + 2a\mathcal{D}_t^{(\alpha)}B_n(t) + \left(b^2 + \lambda_n k^2\right)B_n(t) = \tilde{f}_n(t). \tag{27}$$

Since $W_2(x,t)$ satisfies the initial conditions in (23), we must have

$$\begin{cases} \sum_{n=0}^{+\infty} B_n(0)\cos\left(\frac{n\pi x}{\ell}\right) = \phi_2(x), \\ \sum_{n=0}^{+\infty} \mathcal{D}_t^{(\alpha)}B_n(0)\cos\left(\frac{n\pi x}{\ell}\right) = \psi_2(x), \end{cases}$$

which yields

$$\begin{cases} B_n(0) = \frac{2}{\ell}\int_0^{\ell}\phi_2(x)\cos\left(\frac{n\pi x}{\ell}\right)dx, \\ \mathcal{D}_t^{(\alpha)}B_n(0) = \frac{2}{\ell}\int_0^{\ell}\psi_2(x)\cos\left(\frac{n\pi x}{\ell}\right)dx. \end{cases} \tag{28}$$

From the condition (19), Theorem 2.2, (27), (28) and (25) we obtain the solution of problem (23) as

$$\begin{aligned} W_2(x,t) = \sum_{n=1}^{+\infty} \Bigg[& B_n(0)e^{-a\frac{t^\alpha}{\alpha}}\cos\left(\sqrt{b^2 - a^2 + \lambda_n k^2}\frac{t^\alpha}{\alpha}\right) \\ & + \frac{B_n(0)a + \mathcal{D}_t^{(\alpha)}B_n(0)}{\sqrt{b^2 - a^2 + \lambda_n k^2}}e^{-a\frac{t^\alpha}{\alpha}}\sin\left(\sqrt{b^2 - a^2 + \lambda_n k^2}\frac{t^\alpha}{\alpha}\right) \\ & + \frac{1}{\sqrt{b^2 - a^2 + \lambda_n k^2}}\int_0^t \tilde{f}_n\left((t^\alpha - \tau^\alpha)^{1/\alpha}\right)e^{-a\frac{\tau^\alpha}{\alpha}}\sin\left(\sqrt{b^2 - a^2 + \lambda_n k^2}\frac{\tau^\alpha}{\alpha}\right)\frac{d\tau}{\tau^{1-\alpha}}\Bigg]\cos\left(\frac{n\pi x}{\ell}\right). \end{aligned} \tag{29}$$

Example 4.1. *([14, Chapter 2]) In this example, we consider the following data:*

$$\begin{cases} a = b = k = \ell = 1,\ \alpha = \frac{1}{2} \text{ and } f(x,t) = 0, \\ \phi(x) = x,\ \psi(x) = 0, \\ \mu_1(t) = \mu_2(t) = 0. \end{cases}$$

The problem (1), (2) and (22) becomes:

$$\begin{cases} \mathcal{D}_t^{(1/2)}\left(\mathcal{D}_t^{(1/2)}u(x,t)\right) + 2\mathcal{D}_t^{(1/2)}u(x,t) + u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}, \\ u(x,0) = x,\ \mathcal{D}_t^{(1/2)}u(x,0) = 0, \\ u_x(0,t) = u_x(1,t) = 0. \end{cases} \tag{30}$$

The eigenvalues and the eigenfunctions are given by

$$\lambda_n = \pi^2 n^2, \text{ and } X_n(x) = \cos(\pi n x),\ n \in \mathbb{N}.$$

From (28), *we get*

$$B_n(0) = \frac{2}{\pi^2 n^2}\left((-1)^n - 1\right) \text{ and } \quad \mathcal{D}_t^{(1/2)} B_n(0) = 0.$$

From (29), *the analytical solution of problem* (30) *is given by*

$$u(x,t) = \sum_{n=0}^{+\infty} \frac{-4e^{-2\sqrt{t}}}{\pi^2 (2n+1)^2}\left[\cos\pi(4n+2)\sqrt{t} + \frac{\sin\pi(4n+2)\sqrt{t}}{\pi(2n+1)}\right]\cos\pi(2n+1)x.$$

5. Conformable fractional telegraph equation with Robin boundary condition

In this section, we find the analytical solution for the conformable fractional telegraph equation (1) with initial conditions (2) and the non-homogeneous Robin boundary conditions

$$u(0,t) + \rho u_x(0,t) = \mu_1(t), \quad u(\ell,t) + \sigma u_x(\ell,t) = \mu_2(t), \quad t > 0, \tag{31}$$

where $\rho < 0$, $\sigma > 0$ and $\mu_1, \mu_2 \in C(0,T)$ satisfying

$$\begin{cases} \phi(0) + \rho\phi'(0) = \mu_1(0), \\ \phi(\ell) + \rho\phi'(\ell) = \mu_2(0). \end{cases}$$

We assume that

$$u(x,t) = W_3(x,t) + V_3(x,t),$$

where $V_3(x,t)$ is given by

$$V_3(x,t) = \frac{\mu_1(t) - \mu_2(t)}{\rho - \sigma - \ell}x - \frac{(\ell+\sigma)\mu_1(t) - \rho\mu_2(t)}{\rho - \sigma - \ell},$$

in which satisfies the Robin boundary conditions

$$\begin{cases} V_3(0,t) + \rho\frac{\partial V_3(0,t)}{\partial x} = \mu_1(t), \\ V_3(\ell,t) + \sigma\frac{\partial V_3(\ell,t)}{\partial x} = \mu_2(t) \end{cases}$$

and the function $W_3(x,t)$ is the solution of the following problem

$$\begin{cases} \mathcal{D}_t^{(2\alpha)} W_3(x,t) + 2a\mathcal{D}_t^{(\alpha)} W_3(x,t) + b^2 W_3(x,t) = k^2\frac{\partial^2 W_3(x,t)}{\partial x^2} + \tilde{f}(x,t), \\ W_3(x,0) = \phi_3(x), \quad \mathcal{D}_t^{(\alpha)} W_3(x,0) = \psi_3(x), \quad 0 \le x \le \ell \\ W_3(0,t) + \rho\frac{\partial V_3(0,t)}{\partial x} = 0, \qquad t \ge 0, \\ W_3(\ell,t) + \sigma\frac{\partial V_3(\ell,t)}{\partial x} = 0, \qquad t \ge 0, \end{cases} \tag{32}$$

i which

$$\begin{aligned} \tilde{f}(x,t) &= -\mathcal{D}_t^{(2\alpha)} V_3(x,t) - 2a\mathcal{D}_t^{(\alpha)} V_3(x,t) - b^2 V_3(x,t) + f(x,t), \\ \phi_3(x) &= \phi(x) - V_3(x,0) \\ \psi_3(x) &= \psi(x) - \mathcal{D}_t^{(\alpha)} V_3(x,0). \end{aligned}$$

We assume the solution of the homogeneous equation in (32), by taking $\tilde{f}(x,t) = 0$, has the form:

$$W_3(x,t) = X(x)\,Y(t)$$

and substitute in (32), we obtain the Sturm-Liouville problem

$$\begin{cases} X''(x) + \lambda X(x) = 0, \\ X(0) + \rho X'(0) = 0, \\ X(\ell) + \sigma X'(\ell) = 0, \end{cases} \tag{33}$$

To study the Sturm-Liouville problem (33), we have the following lemma:

Lemma 5.1. *Let (λ_n) be the eigenvalues and $(X_n(x))$ be the corresponding eigenfunctions of the Sturm-Liouville problem (33). Then, the sequence of functions $(X_n(x))$ is orthogonal on $[0,\ell]$, i.e.*

$$\langle X_n, X_m \rangle = \int_0^\ell X_n(x) X_m(x)\, dx = \begin{cases} 0 & \text{if } n = m, \\ R_n & \text{if } n \neq m. \end{cases} \tag{34}$$

Proof. Let λ_i and λ_j , $i \neq j$ be eigenvalues, and $X_i(x)$ and $X_j(x)$ the corresponding eigenfunctions of problem (33). we have

$$X_i''(x) + \lambda_i X_i(x) = 0, \tag{35}$$
$$X_j''(x) + \lambda_j X_j(x) = 0. \tag{36}$$

By multiplying (35) by $X_j(x)$, (36) by $X_i(x)$ and with the difference, we obtain

$$\left(X_j(x) X_i'(x) - X_i(x) X_j'(x)\right)' + \left(\lambda_i - \lambda_j\right) X_i(x) X_j(x) = 0.$$

By integration on $[0,\ell]$, we obtain

$$\left(\lambda_i - \lambda_j\right) \int_0^\ell X_i(x) X_j(x)\, dx = \left[X_j(x) X_i'(x) - X_i(x) X_j'(x)\right]_0^\ell . \tag{37}$$

$X_i(x)$ and $X_j(x)$ satisfy the Robin boundary conditions in (33), we have

$$\begin{cases} X_i(0) + \rho X_i'(0) = 0, \\ X_j(0) + \sigma X_j'(0) = 0, \end{cases} \quad \text{and} \quad \begin{cases} X_i(\ell) + \rho X_i'(\ell) = 0, \\ X_j(\ell) + \sigma X_j'(\ell) = 0, \end{cases}$$

it is necessary that

$$X_i(0) X_j'(0) - X_j(0) X_i'(0) = X_i(\ell) X_j'(\ell) - X_j(\ell) X_i'(x) = 0.$$

Thus, equation (37) reduces to

$$\left(\lambda_i - \lambda_j\right) \int_0^\ell X_i(x) X_j(x)\, dx = 0,$$

but $\lambda_i \neq \lambda_j$, so $\int_0^\ell X_i(x) X_j(x)\, dx = 0$. □

Lemma 5.2. *We have:*

1. *All eigenvalues of the Sturm-Liouville problem (33) are strictly positive.*
2. *The eigenvalues of the Sturm-Liouville problem (33) satisfy the following algebraic equation:*

$$\tan\left(\ell\sqrt{\lambda}\right) = \frac{(\rho - \sigma)\sqrt{\lambda}}{1 + \rho\sigma\lambda}. \tag{38}$$

3. *The Sturm-Liouville problem (33) admits a strictly increasing sequence of eigenvalues (λ_n) such that*

$$\lambda_n \xrightarrow[n\to+\infty]{} +\infty.$$

Proof. 1. Let λ be an eigenvalue and $X(x) \neq 0$ be an eigenfunction associated of problem (33). We have:

$$X''(x) + \lambda X(x) = 0. \tag{39}$$

By multiplying (39) by $X(x)$ and using integration by parts, we obtain:

$$\lambda \int_0^\ell X^2(x)\,dx = -\int_0^\ell X''(x)\,X(x)\,dx$$
$$= X'(0)\,X(0) - X'(\ell)\,X(\ell) + \int_0^\ell (X')^2(x)\,dx.$$

$X(x)$ satisfies the Robin boundary conditions in (33), so we have:

$$\lambda = \frac{-\frac{X^2(0)}{\rho} + \frac{X^2(\ell)}{\sigma} + \int_0^\ell (X'(x))^2\,dx}{\int_0^\ell X^2(x)\,dx} > 0$$

2. From (39), we obtain:

$$X(x) = c_1\cos\left(\sqrt{\lambda}x\right) + c_2\sin\left(\sqrt{\lambda}x\right). \tag{40}$$

Using (40) and (33), we get the following linear system

$$\begin{cases} c_1 + \rho\sqrt{\lambda}c_2 = 0, \\ \left(\cos\left(\sqrt{\lambda}\ell\right) - \sigma\sqrt{\lambda}\sin\left(\sqrt{\lambda}\ell\right)\right)c_1 + \left(\sin\left(\sqrt{\lambda}\ell\right) + \sigma\sqrt{\lambda}\cos\left(\sqrt{\lambda}\ell\right)\right)c_2 = 0. \end{cases} \tag{41}$$

The eigenfunction $X(x) \neq 0$ is equivalent the determinant of the linear system (41) is zero, i.e.

$$\begin{vmatrix} 1 & \rho\sqrt{\lambda} \\ \cos\left(\sqrt{\lambda}\ell\right) - \sigma\sqrt{\lambda}\sin\left(\sqrt{\lambda}\ell\right) & \sin\left(\sqrt{\lambda}\ell\right) + \sigma\sqrt{\lambda}\cos\left(\sqrt{\lambda}\ell\right) \end{vmatrix} = 0, \tag{42}$$

with simplification of (42), we obtain (38).

3. Consider the function κ defined by:

$$\kappa(\lambda) = \tan\left(\ell\sqrt{\lambda}\right) - \frac{(\rho-\sigma)\sqrt{\lambda}}{1+\rho\sigma\lambda}.$$

✓ The function κ is differentiable on $]0,+\infty[$ and its derivative given by:

$$\kappa'(\lambda) = \frac{\ell\left(1+\tan^2\left(\ell\sqrt{\lambda}\right)\right)}{2\sqrt{\lambda}} + \frac{(\sigma-\rho)(1-\rho\sigma\lambda)}{2\sqrt{\lambda}(1+\rho\sigma\lambda)^2} > 0,$$

where, κ is strictly increasing on $]0,+\infty[$.

✓ We show the following proposition:

$$\forall n\in\mathbb{N},\ \exists!\lambda_n \in \left]\frac{(2n+1)^2\pi^2}{4\ell^2}, \frac{(2n+3)^2\pi^2}{4\ell^2}\right[\text{ such that } \kappa(\lambda_n) = 0.$$

We have:

$$\kappa(\lambda) \xrightarrow[\lambda \overset{>}{\to} \frac{(2n+1)^2\pi^2}{4\ell^2}]{} -\infty \quad\text{and}\quad \kappa(\lambda) \xrightarrow[\lambda \overset{<}{\to} \frac{(2n+3)^2\pi^2}{4\ell^2}]{} +\infty.$$

According to the intermediate value theorem and the increasing of the function κ there exists unique λ_n.

The sequence (λ_n) is positive and strictly increasing, then $\lambda_n \xrightarrow[n\to+\infty]{} +\infty$. $\square$

Now, we seek a solution $W_3(x,t)$ of the non-homogeneous problem (32) in the following form:

$$W_3(x,t) = \sum_{n=0}^{+\infty} B_n(t) X_n(x). \tag{43}$$

To determine $B_n(t)$, we expand $\tilde{f}(x,t)$ as a Fourier series by the eigenfunctions $X_n(x)$,

$$\tilde{f}(x,t) = \sum_{n=0}^{+\infty} f_n(t) X_n(x) \quad \text{with} \quad f_n(t) = \frac{1}{R_n}\int_0^{\ell} \tilde{f}(x,t) X_n(x)\, dx. \tag{44}$$

By substituting (43) and (44) in (32), we obtain:

$$\mathcal{D}_t^{(2\alpha)} B_n(t) + 2a\mathcal{D}_t^{(\alpha)} B_n(t) + \left(b^2 + \lambda_n k^2\right) B_n(t) = f_n(t). \tag{45}$$

Since $W_3(x,t)$ satisfies the initial conditions of problem (32), we have:

$$\begin{cases} \phi_3(x) = \sum_{n=0}^{+\infty} B_n(0) X_n(x), \\ \psi_3(x) = \sum_{n=0}^{+\infty} \mathcal{D}_t^{(\alpha)} B_n(0) X_n(x), \end{cases}$$

where

$$\begin{cases} B_n(0) = \frac{1}{R_n}\int_0^{\ell} \phi_3(x) X_n(x)\, dx, \\ \mathcal{D}_t^{(\alpha)} B_n(0) = \frac{1}{R_n}\int_0^{\ell} \psi_3(x) X_n(x)\, dx. \end{cases} \tag{46}$$

On the other hand, we assume that

$$0 \le \frac{a^2 - b^2}{k^2} < \lambda_0, \tag{47}$$

where λ_0 is the smallest eigenvalue of problem (33).

From the condition (47), Theorem 2.2, (45), (46) and (43), we obtain the analytical solution of problem (32) id given by:

$$\begin{aligned} W_3(x,t) = \sum_{n=0}^{+\infty} \Bigg[& B_n(0) e^{-a\frac{t^\alpha}{\alpha}} \cos\left(\sqrt{b^2 - a^2 + \lambda_n k^2}\, \frac{t^\alpha}{\alpha}\right) \\ & + \frac{B_n(0)\, a + \mathcal{D}_t^{(\alpha)} B_n(0)}{\sqrt{b^2 - a^2 + \lambda_n k^2}} e^{-a\frac{t^\alpha}{\alpha}} \sin\left(\sqrt{b^2 - a^2 + \lambda_n k^2}\, \frac{t^\alpha}{\alpha}\right) \\ & + \frac{1}{\sqrt{b^2 - a^2 + \lambda_n k^2}} \int_0^t \tilde{f}_n\left(\left(t^\alpha - \tau^\alpha\right)^{1/\alpha}\right) e^{-a\frac{\tau^\alpha}{\alpha}} \sin\left(\sqrt{b^2 - a^2 + \lambda_n k^2}\, \frac{\tau^\alpha}{\alpha}\right) \frac{d\tau}{\tau^{1-\alpha}} \Bigg] X_n(x). \end{aligned} \tag{48}$$

Example 5.1. *([14, Chapter 2]) We consider the following data:*

$$\begin{cases} a = b = k = \ell = 1, \ \alpha = \frac{1}{2}, \ \text{and} \ \ f(x,t) = 0. \\ \phi(x) = 1, \ \ \psi(x) = 0, \\ \rho = -1, \ \sigma = 1 \ \text{and} \ \ \mu_1(t) = \mu_2(t) = 0. \end{cases}$$

The problem (1), (2), (31) becomes:

$$\begin{cases} \mathcal{D}_t^{(1/2)}\left(\mathcal{D}_t^{(1/2)} u(x,t)\right) + 2\mathcal{D}_t^{(1/2)} u(x,t) + u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}, \\ u(x,0) = 1, \ \ \mathcal{D}_t^{(1/2)} u(x,0) = 0, \ \ 0 \le x \le 1, \\ u(0,t) - u_x(0,t) = 0, \ \ u(1,t) + u_x(1,t) = 0. \end{cases} \tag{49}$$

Using (38), we obtain the eigenvalues satisfying the following algebraic equation:

$$\tan\left(\sqrt{\lambda}\right) = -\frac{2\sqrt{\lambda}}{1-\lambda}.$$

Figure 1 represents the graphs of the functions $x \mapsto \tan\left(\sqrt{x}\right)$ *and* $x \mapsto -\frac{2\sqrt{x}}{1-x}$ *with the points of the intersection being the eigenvalues* λ_n.

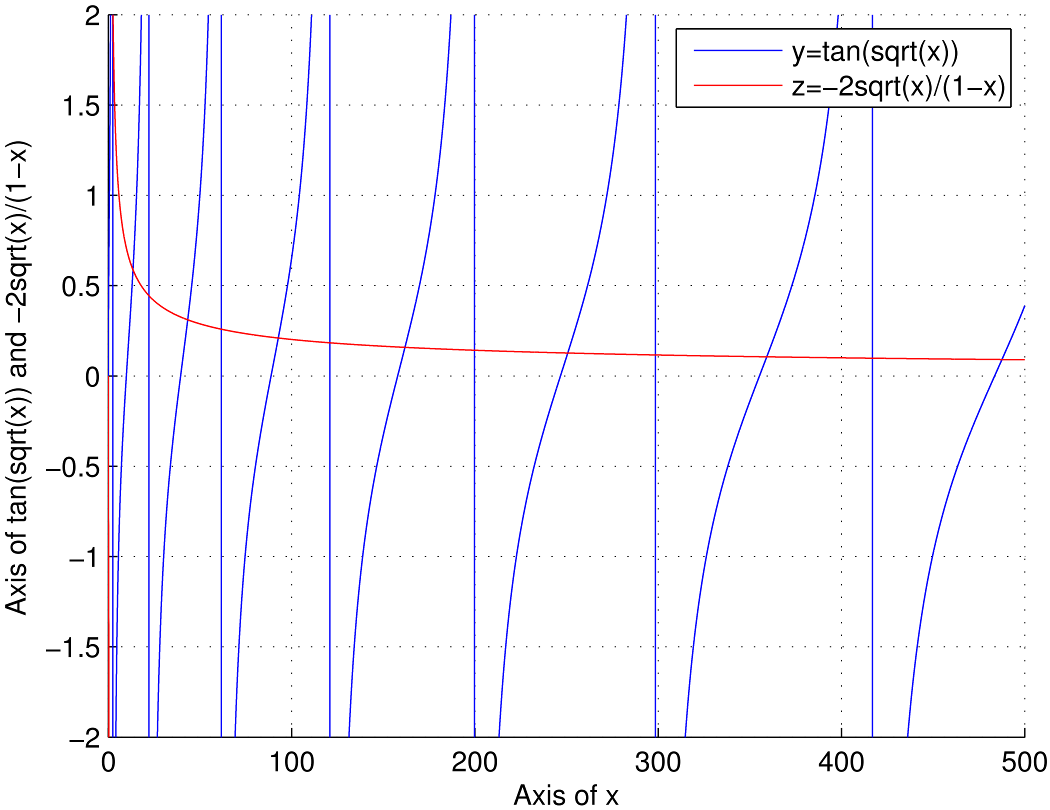

Figure 1: Graphs of two functions $x \mapsto \tan\left(\sqrt{x}\right)$ et $x \mapsto -\frac{2\sqrt{x}}{1-x}$.

From (40) and (41), the eigenfunctions are given by:

$$X_n(x) = \sin\left(\sqrt{\lambda_n}x\right) - \sqrt{\lambda_n}\cos\left(\sqrt{\lambda_n}x\right).$$

From (34) and (46), we have:

$$R_n = \frac{\lambda_n}{2} + \left(\frac{\lambda_n}{2} - \frac{1}{2}\right)\frac{\sin\left(2\sqrt{\lambda_n}\right)}{2\sqrt{\lambda_n}} + \frac{\cos\left(2\sqrt{\lambda_n}\right)}{2},$$

$$B_n(0) = \frac{-1}{R_n}\left(\frac{\cos\left(\sqrt{\lambda_n}\right)}{\sqrt{\lambda_n}} + \sin\left(\sqrt{\lambda_n}\right) - \frac{1}{\sqrt{\lambda_n}}\right),$$

$$\mathcal{D}_t^{(1/2)}\left(B_n(0)\right) = 0.$$

Thanks to (48), *the analytical solution of problem* (49) *is given by:*

$$u(x,t) = \sum_{n=0}^{+\infty} B_n(0)\, e^{-2\sqrt{t}} \left[\cos\left(2\sqrt{\lambda_n}\sqrt{t}\right) + \frac{\sin\left(2\sqrt{\lambda_n}\sqrt{t}\right)}{\sqrt{\lambda_n}} \right] X_n(x).$$

6. Conclusion

We have derived the analytical solution of a time-fractional telegraph equation with three boundary conditions using the Fourier method. The time-fractional derivative are considered in the conformable sense. Three examples are presented.

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