A New Class of Fractional Cumulative Residual Entropy – Some Theoretical Results

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Abstract — In this paper, by differentiating the entropy's generating function (i.e., $h(t) = \int_{S_X} \bar{F}_X^t(x)dx$) using a Caputo fractional-order derivative, we derive a generalized non-logarithmic fractional cumulative residual entropy (FCRE). When the order of differentiation $\alpha \to 1$, the ordinary Rao CRE is recovered, which corresponds to the results from first-order ordinary differentiation. Some properties and examples of the proposed FCRE are also presented.

Keywords — cumulative residual entropy (CRE), entropy's generating function, fractional calculus, information measure, Riemann-Liouville/Caputo fractional integral/derivative, Tsallis/Rényi entropy.

1. Introduction

In keeping with the concept of entropy proposed by Clausius [1] in thermodynamics and Boltzmann [2] in classical statistical mechanics, Shannon proposed, in [3], a similar concept, in the context of communication theory, as a measure of surprise or uncertainty associated with the probability distributions of a random variable (RV). For a discrete RV X taking values in $\mathcal{X} = \{x_1, x_2, \ldots, x_N\}$ and having a probability mass function $p_i = P(X = x_i)$ with $\sum_{i=1}^{N} p_i = 1$ and $p_i \geqslant 0$ for $i = 1, \ldots, N$, it is given by:

$$S^{(S)}(X) = -\sum_{i=1}^{N} p_i \log p_i \,. \tag{1}$$

Due to its properties, which agree with the intuitive notions of randomness, such a measure of uncertainty suggested in Eq. (1) has proved to be useful in solving statistical problems related to the communication theory.

In the literature concerned with information theory and mathematical statistics, one can frequently encounter many contributions to generalize the concept of entropy in an attempt to explore and exploit the outcomes of its applications in different fields of engineering and physics. Perhaps the most popular among these contributions are the two developed by Tsallis [4] and Rényi [5], given by:

$$S_q^{(T)} = \frac{1}{q-1}\left(1 - \sum_i p_i^q\right), \quad q \in \mathbb{R}\,, \tag{2}$$

and

$$S_q^{(R)} = \frac{1}{1-q} \ln\left(\sum_i p_i^q\right), \quad q > 0,\ q \neq 1\,, \tag{3}$$

respectively.

Both entropies ($S_q^{(T)}$ and $S_q^{(R)}$) reduce to Shannon's entropy defined in Eq. (1), within the limit $q \to 1$. The Tsallis entropy $S_q^{(T)}$ has applications in Fokker-Plank systems [6] and diffusion equations [7]. Since the Rényi entropy $S_q^{(R)}$ has an inverse power law equilibrium distribution [8], it satisfies the zeroth law of thermodynamics [9]. Based on entropy $S_q^{(T)}$ defined in Eq. (2), Tsallis *et al*. have developed the theory of non-extensive statistical mechanics [4], where parameter q classifies a particular statistics, such that for $q \to 1$ it reproduces the standard Boltzmann-Gibs statistics.

Abe in [10] has shown that while the Shannon entropy $S^{(S)}$ and the Tsallis entropy $S_q^{(T)}$ share the same generating function, $h(t) = \sum_i p_i^t$, the difference between their entropic forms is due to the differentiation operators considered. Namely, by applying the classical first-order differentiation operator, we can generate the following Shannon entropy $S^{(S)}$:

$$-\frac{d}{dt}\, h(t)|_{t=1} = -\sum_i p_i \log p_i \equiv S^{(S)}\,. \tag{4}$$

Otherwise, the Tsallis entropy $S_q^{(T)}$ can be generated by applying Jackson's q-differential operator given by:

$$\left({}^J D_q f\right)(x) = \frac{f(qx) - f(x)}{qx - x}\,, \tag{5}$$

instead of the first-order ordinary differentiation operator d/dt. This indicates that if one adopts other families of differentiation operators, new classes of entropies may be created.

Known as the differential entropy, the continuous analogue of the discrete entropy defined in Eq. (1), for a continuous RV, is given by [11]:

$$h(X) = -\int_{S_X} f_X(x) \log f_X(x), \tag{6}$$

where S_X and $f_X(x)$ are the support and probability density functions (PDF) of the RV X, respectively. However, this direct extension of the concept of entropy from discrete to continuous schemes raises some concerns, including the following:

- it may be negative,
- it may become infinitely large,
- it does not necessarily remain invariant under the transformation of the coordinate systems,
- it is so difficult to be estimated through empirical samples.

For further discussions on these and more concerns, please refer to [12].

Among the proposed solutions intended to overcome these limitations, is the one named cumulative residual entropy (CRE) [13], [14]. It is given by:

$$ {}^{\mathrm{CRE}}h(X) = -\int_0^{+\infty} \bar{F}_X(x) \log \bar{F}_X(x), \tag{7} $$

where $\bar{F}_X(x)$ is the complementary cumulative distribution function (CCDF) of the RV X. CRE is a non-negative quantity. It is well-defined for both discrete and continuous RVs and can be easily estimated using empirical entropy samples [13]. Some extensions and generalizations for the CRE have been proposed in [15]–[20]. Other CDF or CCDF-based information measures including the cumulative Shannon entropy [21]–[23], cumulative residual Tsallis/Rényi's entropy [24]–[26] and cumulative residual K-L divergence [27], [28], were proposed.

By benefiting from Abe's idea [10], Ubriaco in [29] seems to be the first who has proposed an FC-based entropy given by:

$$ H_q^U(X) = \sum_{i=1} p_i(-\log p_i)^q,\ 0 \leqslant q \leqslant 1. \tag{8} $$

In the limit $q \longrightarrow 1$, we get Shannon's entropy $S^{(S)}(X)$ defined in Eq. (1).

Following Ubriaco's approach, Mitsohiro in [30] proposed a new class of fractional entropy.

In this paper, inspired by Mitsohiro's work in [30], we propose a new class of FCRE by relying on the CCDF-based entropy's generating function $h(t) := \int_{S_X} \bar{F}_X^t(x)\, dx$ and using the theory of fractial calculus (FC). The use of FC allows us to measure the information in a generalized metric space. The main area in which FCRE may be used directly is information theory. It can also be applied in computer vision solutions, where information-related measures, and in particular those related to CE, are of great importance. Among the recent works on FCRE are [31] and [32].

The remainder of this paper is organized as follows. In Section 2, from the theory of FC, we recall some definitions concerning the RL fractional integral and the Caputo fractional derivative. The generalized FCRE derived from a Caputo fractional-order differentiation of the aforementioned entropy's generating function with order α along with some of its properties and examples are then presented in Section 3. Finally, Section 4 concludes the paper.

2. Review of Fractional Integrals and Derivatives

Fractional calculus (FC) is a mathematical analysis branch which studies different possible approaches to define fractional-order integrals and derivatives. Based on FC, the theory of classical integer-order differential equations has been then generalized to the broader theory of fractional-order differential equations. FC can be traced back to a letter written to l'Hopital by Leibniz in 1695 [33]. In 1832, Liouville carried out a heavy-handed investigation on FC [34]. After that, the Riemann-Liouville (RL) fractional integrodifferential operator was introduced by Riemann in [35], along with a comprehensive theory of FC. FC has led to many breakthroughs in different fields of physics and engineering, where various processes can be modeled in a more accurate and authentic way [36].

The left-sided Rieman-Liouville (RL) fractional integral ${}^{\mathrm{RL}}I_{a+}^{\alpha} f$ of order $\alpha \in \mathbb{R}(\alpha > 0)$ of an integrable function $f : [a, b] \rightarrow \mathbb{R}$, $(0 \leqslant a < b \leqslant \infty)$ is defined as [37]:

$$ \begin{aligned} \left({}^{\mathrm{RL}}I_{a+}^{\alpha} f\right)(t) &= {}^{\mathrm{RL}}I_{a+}^{\alpha}\left[f(x)\right](t) \\ &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-x)^{\alpha-1} f(x) dx\,, \end{aligned} \tag{9} $$

with $a \in \mathbb{R}$, $t > a$, $\alpha > 0$. $\Gamma(.)$ is Euler's gamma function defined as $\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1}e^{-x}dx, \alpha \in \mathbb{R}(\alpha > 0)$.

The left-sided Caputo fractional derivative ${}^{\mathrm{C}}D_{a+}^{\alpha} f$ of order $\alpha \in \mathbb{R}(\alpha > 0)$ of an integrable and differentiable function $f : [a, b] \rightarrow \mathbb{R}$, $(0 \leqslant a < b \leqslant \infty)$ is defined as [37]:

$$ \begin{aligned} \left({}^{\mathrm{C}}D_{a+}^{\alpha} f\right)(t) &= {}^{\mathrm{C}}D_{a+}^{\alpha}\big[f(x)\big](t) \\ &= \left({}^{\mathrm{RL}}I_{a+}^{n-\alpha}\left(\frac{d}{dt}\right)^n f\right)(t)\,, \end{aligned} \tag{10} $$

with $a \in \mathbb{R}$, $t > a$, $\alpha > 0$, $n = [\alpha] + 1$.

When $0 < \alpha < 1$, we get:

$$ \left({}^{\mathrm{C}}D_{a+}^{\alpha} f\right)(t) = \left({}^{\mathrm{RL}}I_{a+}^{1-\alpha}\frac{d}{dt} f\right)(t), \quad (a \in \mathbb{R}, t > a)\ . \tag{11} $$

3. Caputo Fractional Derivative-based FCRE

Definition 1 – CRE [13]. The FCRE of a non-negative continuous RV X with a PDF $f_X(x)$, a CCDF $\bar{F}_X(x)$ and support $S_X = [0, +\infty]$ is defined as:

$$ \begin{aligned} {}^{\mathrm{CRE}}h(X) :&= -\int_{S_X} \bar{F}_X(x) \log \bar{F}_X(x) dx \\ &= E_X\left[-\frac{\bar{F}_X(X)\log\bar{F}_X(X)}{f_X(X)}\right], \end{aligned} \tag{12} $$

when the integral exists.

$E_X[X]$ is the expected value of the RV X.

Remark. Throughout the entire paper, the base of the logarithm will be set to Euler's number $e = \sum\limits_{n=0}^{\infty} \frac{1}{n!}$.

Our basic idea consists of rewriting Eq. (12) as follows:

$$ {}^{CRE}h(X) := -\lim_{t\to 1}\frac{d}{dt}\int_{S_X}\bar{F}_X^t(x)\,dx\,. \tag{13}$$

Then, we deform the ordinary differential operator $\frac{d}{dt}$ in Eq. (13) to the Caputo fractional differential operator ${}^{\mathrm{C}}D_{a+}^{\alpha}$ defined in Eq. (11) (which reduces to $\frac{d}{dt}$ in the limit $\alpha\to 1$).

Definition 2 – FCRE. The FCRE of a non-negative continuous RV X with a PDF $f_X(x)$, a CCDF $\bar{F}_X(x)$ and support $S_X = [0,+\infty]$ is defined as:

$$ {}^{\mathrm{FCRE}}h^{\alpha}(X) := -\lim_{t\to 1}{}^{\mathrm{C}}D_{a+}^{\alpha}\int_{S_X}\bar{F}_X^t(x)\,dx\,. \tag{14}$$

Based on these ideas, we derive in the following theorem a new class of FCRE.

3.1. Fractional CRE

Theorem 1 – FCRE. The finite-value FCRE of order α of a non-negative continuous RV X with a CCDF $\bar{F}_X(x)$ and support $S_X\subseteq[0,+\infty]$ is defined as:

$$ {}^{\mathrm{FCRE}}h^{\alpha}(X) :=\int_{S_X}\tfrac{(\alpha-1)}{\Gamma(2-\alpha)}{}_1F_1\left(1;1-\alpha;\log\bar{F}_X(x)\right)dx\,,\quad 0<\alpha\leqslant 1\,, \tag{15}$$

where ${}_1F_1\,(p;q;x)$ is the confluent hypergeometric function of the first kind – see Eq. (13.2.2) in [38].

Proof. Using the operator defined in Eq. (11), where the lower limit of the RL-integral is taken to zero, i.e. $a=0$, without loss of generality, Eq. (13) can be re-written as follows:

$$ {}^{\mathrm{FCRE}}h^{\alpha}(X) := -\lim_{t\to 1}\tfrac{d}{dt}\left({}^{\mathrm{RL}}I_0^{1-\alpha}\left[\int_{S_X}e^{t\log\bar{F}_X(x)}dx\right](t)\right),(0<\alpha\leqslant 1).$$

Therefore, we need to solve the following integral:

$$ {}^{\mathrm{FCRE}}h^{\alpha}(X) := \tfrac{-1}{\Gamma(1-\alpha)}\lim_{t\to 1}\tfrac{d}{dt}\int_{S_X}\left(\int_0^t(t-y)^{-\alpha}e^{y\log\bar{F}_X(x)}dy\right)dx\,,$$

By letting $t-y=\frac{z}{\log\bar{F}_X(x)}$ and using the definition of the incomplete gamma function of the first kind, i.e. $\gamma(\mu,x)=\int_0^x e^{-t}t^{\mu-1}dt$, we get:

$$ {}^{\mathrm{FCRE}}h^{\alpha}(X) := \tfrac{-1}{\Gamma(1-\alpha)} \times\lim_{t\to 1}\tfrac{d}{dt}\int_{S_X}\left[(\log\bar{F}_X(x))^{\alpha-1}e^{t\log\bar{F}_X(x)}\gamma(1-\alpha,t\log\bar{F}_X(x))\right]dx.$$

Now, we use the following relation:

$$\gamma(\mu,x)=\mu^{-1}x^{\mu}e^{-x}{}_1F_1(1;1+\mu;x)\,.$$

Then, we refer to Eq. (13.3.18) in [38] to take the ordinary first derivative and we set $t=1$ to get the results in Eq. (15). ■

The standard CRE defined in Eq. (12) can be recovered from the FCRE in Eq. (15), for $\alpha\to 1$, with the help of the limit:

$$\lim_{b\to 0} b\;{}_1F_1(1;b;x)=xe^{x}\,.$$

The expression in Eq. (15) allows us to calculate the FCRE ($h^{\alpha}(X)$) for different values of α and compare them to each other and to the conventional CRE ($\alpha=1$), simultaneously get further insights on its behavior.

3.2. Properties of FCRE

The FCRE defined in Eq. (15) is a generalized entropy. In the following section, we consider some of its mathematical properties.

3.2.1. Positivity of the FCRE

Let us consider the integrand of Eq. (15):

$$h^{\alpha}=\frac{(\alpha-1)}{\Gamma(2-\alpha)}{}_1F_1\left(1;1-\alpha;\log\bar{F}_X(x)\right),\ 0<\alpha\leqslant 1\,. \tag{16}$$

With the help of the asymptotic expansion of ${}_1F_1\,(.;.;.)$ for a large argument (see Sect. 13.7 in [38]) for the case where $\bar{F}_X(x)=0$, we can easily find that $h^{\alpha}=0$. For $\bar{F}_X(x)^{\alpha}=1$, however, we get:

$$h^{\alpha}=\frac{(\alpha-1)}{\Gamma(2-\alpha)}\,.$$

In this case, h^{α} is negative for $\alpha\in(0,1)$ and it is equal to zero for $\alpha\to 1$. These results imply that, as a unique property, the FCRE introduced in Eq. (15) is not positive unless $\alpha\to 1$.

3.2.2. Concavity of FCRE

The second ordinary derivative with respect to $\bar{F}_X(x)$ of the integrand given in Eq. (16) is:

$$\frac{\partial^2 h^{\alpha}}{\partial\bar{F}_X(x)^2}=\frac{-\alpha}{\Gamma(3-\alpha)\bar{F}_X(x)}{}_1F_1\left(1-\alpha;3-\alpha;-\log\bar{F}_X(x)\right), \tag{17}$$

which is less than zero for $\alpha\in(0,1)$, $0<\bar{F}_X(x)\leqslant 1$. This means that the FCRE defined in Eq. (15) is concave.

3.2.3. Homogeneity and shift-independence of FCRE

Let X be an RV. If $Y=aX+b$, with $a>0$ and $b\geqslant 0$, then:

$$ {}^{\mathrm{FCRE}}h^{\alpha}(Y)=a\,{}^{\mathrm{FCRE}}h^{\alpha}(X)\,. \tag{18}$$

Proof. Recall that $\bar{F}_Y(y)=1-F_X\left(\frac{y-b}{a}\right)$, $x\in[0,+\infty]$. From Eq. (15), the equality in Eq. (18) is obtained. ■

The result of Eq. (18) shows the effect of linear transformation of the RV on the FCRE. It demonstrates that the FCRE is homogeneous and shift-independent.

3.3. Examples on the FCRE

In the following section, we give a few examples of the FCRE for some common continuous probability distributions.

Example 1. Let X be a uniformally distributed RV in $[0,a]$, $a>0$, i.e. $X\sim\text{Uniform}\,(0,a)$. Then, its FCRE is given by:

$$ {}^{\mathrm{FCRE}}h^{\alpha}(X)=\frac{a\alpha(1-\alpha)}{2^{\alpha+1}\Gamma(2-\alpha)}\mathrm{B}_{\frac{1}{2}}\,(\alpha;1)\,, \tag{19}$$

where $\mathrm{B}_{.}\,(.;.)$ is the incomplete beta function (see Eq. (8.17.1) in [38]). To get further insights on the behavior of ${}^{\mathrm{FCRE}}h^{\alpha}(X)$

as a function of α, one may refer to Fig. 1 where Eq. (19) is plotted.

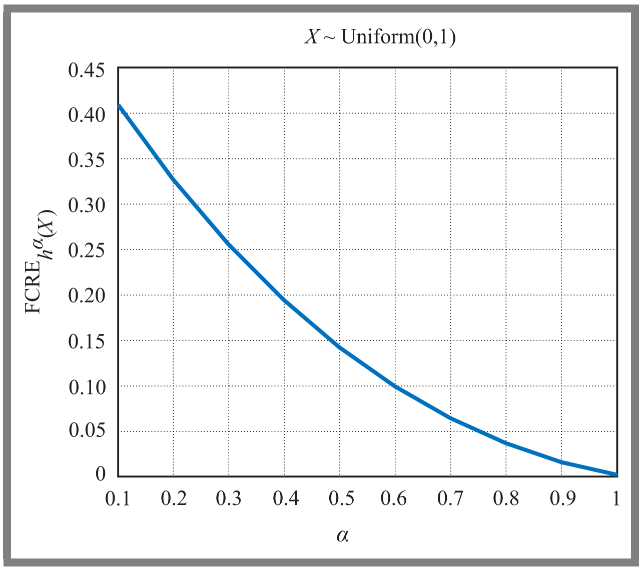

Fig. 1. The ${}^{\mathrm{FCRE}}h^{\alpha}(X)$ as a function of α of a uniformally distributed RV in $[0, a]$, with $a = 1$.

Example 2. Let X be an exponentially distributed RV with a rate parameter λ, i.e. $X \sim \mathrm{e}^{\lambda}$. Then, its FCRE is given by:

$$ {}^{\mathrm{FCRE}}h^{\alpha}(X) = \frac{(\alpha-1)}{\lambda\Gamma(2-\alpha)} \int_{0}^{+\infty} {}_1F_1\left(1; 1-\alpha; -x\right) dx \ , \tag{20} $$

where $\Gamma(.)$ is the Euler's gamma function (see Eq. (5.2.1) in [38]).

Example 3. Let X be a normally distributed RV with a mean μ and a standard deviation σ, i.e. $X \sim \mathcal{N}(\mu, \sigma)$. Then, its FCRE is given by:

$$ {}^{\mathrm{FCRE}}h^{\alpha}(X) = $$
$$ \frac{(\alpha-1)}{\Gamma(2-\alpha)} \int_{0}^{+\infty} {}_1F_1\left[1; 1-\alpha; \log\left(\mathrm{erfc}\left(\frac{x-\mu}{\sigma}\right)\right)\right] dx \ , \tag{21} $$

where $\mathrm{erfc}(.)$ is the complementary error function (see Eq. (7.2.2) in [38]).

4. Conclusion

In this paper, we have introduced a new entropy functional ${}^{\mathrm{FCRE}}h^{\alpha}(X)$ which generalizes the conventional CRE, ${}^{\mathrm{CRE}}h^{\alpha}(X)$, by applying the Caputo fractional-order derivative. The FCRE has inherited the merits of the conventional CRE. We have verified that this entropy has useful features. Other properties, e.g. stability, are worth exploring separately in the future. The possibility of estimating it from empirical random samples merits attention. Testing its reliability through simulations on the logistic map and its applications in computer vision-related fields are also activities that are worthy of consideration.

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