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Etude de quelques modèles d'EDPS fractionnaires et leurs applications

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بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمْ

"وَقُلِ اعْمَلُواْ فَسَيَرَى اللّهُ عَمَلَكُمْ وَرَسُولُهُ وَالْمُؤْمِنُونَ وَسَتُرَدُّونَ إِلَى عَالِمِ الْغَيْبِ وَالشَّهَادَةِ فَيُنَبِّئُكُم بِمَا كُنتُمْ تَعْمَلُونَ" [التوبة: 105]

ويقول العماد الأصفهاني

(إني رأيت أنه لا يكتب أحد كتابا في يومه إلا قال فيه عنه : لو غير هذا لكان أحسن ، ولو زيد لكان يستحسن ، ولو قدم هذا لكان أفضل ، ولو ترك هذا لكان أجمل ، وهذا من أعظم العبر ، وهو دليل على استيلاء النقص على جملة البشر).

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Notation

$\mathbb{N}$	Natural numbers $\{0,1,2,3,\ldots\}$.
$\mathbb{N}^*$	Nonzero natural numbers $\{1,2,3,\ldots\}$.
$\mathbb{Z}$	Integers $\{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$.
$\mathbb{Z}_0^-$	Negative integers $\{\ldots,-3,-2,-1,0\}$.
$\mathbb{R}$	Real numbers $(-\infty,\infty)$.
$\mathbb{R}_+$	Positive real numbers $(0,\infty)$.
$\mathbb{R}^*$	Nonzero real numbers $(-\infty,0)\cup(0,\infty)$.
$\mathbb{C}$	Complex numbers, $z\in\mathbb{C}$, then $z=x+iy$, where $x,y\in\mathbb{R}$, and $i^2=-1$.
$L^1(\Omega)$	Space of LEBESGUE complex-valued measurable functions φ on Ω, for which $\\|\varphi\\|_{L^1}=\int_\Omega\|\varphi(\xi)\|\,d\xi<\infty$.
$C(\Omega)$	The BANACH space of all continuous functions φ on Ω, for which $\\|\varphi\\|_\infty=\sup\limits_{0\leq\eta\leq\ell}\|\varphi(\eta)\|$.
$\Gamma(\cdot)$	EULER gamma function.
$B(\cdot,\cdot)$	Beta function.
$E_\alpha(\cdot)$	Standard MITTAG-LEFFLER function.
$E_{\alpha,\beta}(\cdot)$	MITTAG-LEFFLER function in two arguments, α and β.
$\mathcal{I}_{0+}^{\alpha}\varphi$	RIEMANN-LIOUVILLE fractional integral of order α.
${}^{RL}\mathcal{D}_{0+}^{\alpha}\varphi$	RIEMANN-LIOUVILLE fractional derivative of order α.
${}^{C}\mathcal{D}_{0+}^{\alpha}\varphi$	CAPUTO fractional derivative of order α.
FDE	Fractional Differential Equation.
FPDE	Fractional-order's Partial Differential Equation.

Contents

Introduction

Historically, on the topic of fractional integrals and derivatives, we cite a particular date as that of the first appearance of the so called "Fractional Calculus". In a letter dated September 30^{th}, 1695, L'HÔSPITAL wrote to LEIBNIZ asking him about a particular notation he had used in his writings for the n^{th}-derivative of the linear function $u(t) = t$, $\frac{d^n t}{dt^n}$. L'HÔSPITAL wondered what the result would be if $n = 1/2$. LEIBNIZ response was: "An apparent paradox, from which one day useful consequences will be drawn." In these words fractional calculus was born.

In recent years, considerable interest in fractional calculus has been aroused by Fractional partial differential equations (FPDEs) Which is a valuable tool for modeling numerous tangible incidents that science attempts to explain and has approached more frequently. For further reading on their use, readers can refer to the following books (Diethelm 2010 [20], Kilbas et al. 2006 [28], Podlubny 1999 [36], Samko et al. 1993 [39]).

Exact solutions (or closed-forms) of fractional-order's PDEs are crucial for rendering many qualitative features of natural science processes and phenomena fathomable, where become obtainable using various methods including the residual power series, symmetry, spectral, Fourier transform, similarity, *etc*. (see [9, 10, 15, 25, 32, 33, 35, 37, 43]).

The existence and uniqueness of solutions for fractional differential equations or fractional-order's PDEs have been investigated in recent years. (see [7, 9, 10, 15, 28, 32, 33, 35, 43]) for further details. For this purpose, the technique used is to reduce the study of our problem to the research of a fixed point of an integral operator. The obtained results are based on some standard fixed point theorems such as Banach and Schauder [23].

Our directions in this thesis are based particularly on several works that study various FPDEs by converting them into fractional differential equations using several methods.

The lie group analysis has been discussed by Luchko et al. (see [15, 32]), who studied the space-time fractional diffusion/wave equation.

In 2020, B. Basti and N. Benhamidouche [9] studied the space-fractional heat equation by self-similar forms (see also [5]).

From 2014 to 2017, the authors (see [18, 24]) studied the nonlinear time-fractional Boussinesq equation by traveling wave transformation.

The main objective of this thesis is the study of the existence and uniqueness of solutions of CAPUTO-type partial differential equations of fractional order, by transforming it into differential equations of fractional order.

The originality of our work was by suggesting, studying and discussing new forms and methods transform FPDEs to FDEs, which were represented in: Firstly, traveling profile forms (see [11]), this method plays an important role in modeling all scientific fields: computer science, physics, biology, medicine..., because it contributes to the study of complex problems by transforming them into simple problems. Secondly, traveling wave forms (see [41]), which is a special case of traveling profile forms, and radially symmetric forms (for more details see [1, 2, 16, 19, 30, 42, 45]).

This thesis is divided into five chapters as follows

In the first chapter, we recall the basic notions related to the theory of fractional calculus that we will need in the rest of this work such as Gamma, Beta and Mittag-Leffler functions that play an important role in the theory of fractional differential equations, as well as the fixed point theorems such as Banach and Schauder. Two approaches (RIEMANN-LIOUVILLE and CAPUTO) to the generalization of notions of derivation will then be considered.

In the second chapter, we study the existence and uniqueness of solutions under the traveling wave forms

$$\omega\left(x,t\right)=\exp\left(-\kappa^{2}t\right)\varphi\left(x-\kappa t\right),\text{ with }\kappa\in\mathbb{R}^{*},$$

for a free boundary problem of higher-order space-fractional wave equations as follows

$$\begin{cases}\partial_{t}^{2}\omega=\kappa^{2}\partial_{x}^{\alpha}\omega, & (x,t)\in\Omega\subset\mathbb{R}^{2},\\ \omega\left(\kappa t,t\right)=c_{0}\exp\left(-\kappa^{2}t\right), & c_{0}\in\mathbb{R},\\ \partial_{x}^{k}\omega\left(\kappa t,t\right)=0, & k\in\left\{1,2,\ldots,m-1\right\}.\end{cases}$$

It does so by applying the properties of Schauder's and Banach's fixed point theorems.

In the third chapter, we study the existence and uniqueness of solutions under the traveling wave forms

$$\omega\left(x,t\right)=\exp\left(-\frac{\kappa^{2}}{\delta}t\right)\varphi\left(x-\kappa t\right),\text{ with }\kappa,\delta\in\mathbb{R}_{+}^{*},$$

for a free boundary Cauchy problem of space-fractional Jordan-Moore-Gibson-Thompson equations of nonlinear acoustics as follows

$$\begin{cases} \tau\omega_{ttt} + \mu\omega_{tt} - \kappa^2\partial_x^\alpha\omega - \delta\partial_x^\alpha\omega_t = F\left(x,t,\omega,\omega_t,\omega_{tt},\omega_{xx},(\omega_t)_{xx}\right), & (x,t)\in\Omega\subset\mathbb{R}^2, \\ \omega(x,0)=\omega_0(x),\ \omega_t(x,0)=\omega_1(x),\ \omega_{tt}(x,0)=\omega_2(x), & \omega_0,\omega_1,\omega_2\in\mathbb{C}, \\ \omega(\kappa t,t)=c_0\exp\left(-\frac{\kappa^2}{\delta}t\right),\ \omega_x(\kappa t,t)=(\omega_t)_x(\kappa t,t)=0, & \kappa>0,\ c_0\in\mathbb{C}. \end{cases}$$

It does so by applying the properties of Schauder's and Banach's fixed point theorems, while CAPUTO's fractional derivative is used as the differential operator. For application purposes, some examples of explicit solutions are provided to demonstrate the usefulness of our main results.

In the fourth chapter, we treat and discuss some analytical studies on the existence and uniqueness of global or blow-up solutions under the traveling profile forms

$$\omega(x,t)=c(t)\,\varphi\left(\frac{x-b(t)}{a(t)}\right),\ \text{with } a,c\in\mathbb{R}_+^*,\ b\in\mathbb{R},$$

for a free boundary problem of diffusion equations of moving fractional order as follows

$$\begin{cases} \partial_t\omega=\kappa\partial_x^\alpha\omega,\ (x,t)\in\Omega, & \kappa\in\mathbb{R}^*,\ \Omega\subset\mathbb{R}^2, \\ \omega(b(t),t)=c_0c(t), & c_0\in\mathbb{R},\ c\in\mathbb{R}_+^*, \\ \partial_x\omega(b(t),t)=c_1\frac{c(t)}{a(t)}, & c_1\in\mathbb{R},\ a,c\in\mathbb{R}_+^*, \\ \partial_x^k\omega(b(t),t)=0,\ k\in 2,3,\ldots,m-1, & \text{for } m\geq 3. \end{cases}$$

It does so by applying the properties of Schauder's and Banach's fixed point theorems. For application purposes, some examples of explicit solutions are provided to demonstrate the usefulness of our main results.

The fifth chapter, we treat and discuss some analytical studies on the existence of radially symmetric solutions

$$\omega(x,t)=|x|^\mu\,\varphi\left(|x|^\gamma t\right),\ (t,x)\in\Omega\subset\mathbb{R}_+\times\mathbb{R}^m,\ \mu,\gamma\in\mathbb{C},$$

for a multidimensional nonlinear time and space-fractional reaction-diffusion/wave equation as follows

$$\begin{cases} \partial_t^\alpha\omega-\kappa^2\Delta\omega=F\left(t,x,\omega,\partial_t^\beta\omega,(-\Delta)^s\omega\right), & (t,x)\in\Omega,\ \kappa\in\mathbb{R}^*, \\ \omega(0,x)=|x|^\delta c_0,\ \frac{\partial\omega}{\partial t}(0,x)=0, & \delta,c_0\in\mathbb{C}. \end{cases}$$

It does so by applying the properties of Schauder's and Banach's fixed point theorems. For application purposes, some examples of explicit solutions are provided to demonstrate the usefulness of our main results.

CHAPTER 1

PRELIMINARIES AND BACKGROUND MATERIALS

This chapter will be devoted to the primary definitions and basic concepts related to fractional calculus such as the EULER Gamma, Beta and MITTAG-LEFFLER functions. In addition to that, it will also present other elements of functional analysis, such as the fractional derivation, fractional integration, relative definitions of operators of fractional order, among others, which will all be at the core of this work.

1.1 Special Functions of the Fractional Calculus

In this section, we present the functions EULER gamma, Beta and MITTAG-LEFFLER. These functions play an important role in the theory of fractional calculus and its applications.

Euler Gamma Function

One of the basic functions of fractional calculus is EULER Gamma function $\Gamma(\alpha)$ which naturally extends the factorial to positive real numbers (and even to complex numbers with positive real parts).

Definition 1.1 ([28]). *For $\alpha > 0$ (or actually $Re(\alpha) > 0$), the* EULER *gamma function $\Gamma(\alpha)$ defined by*

$$\Gamma(\alpha) = \int_0^\infty e^{-\xi}\xi^{\alpha-1}d\xi, \tag{1.1}$$

this integral is convergent for all complex $Re(\alpha) > 0$, with $\Gamma(1) = 1$, $\Gamma(0^+) = +\infty$, $\Gamma(\alpha)$ is a monotonous and strictly decreasing function for $0 < \alpha \leq 1$.

An important property of the EULER *gamma function $\Gamma(\alpha)$ is the following recurrence relation*

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha), \ \text{Re}(\alpha) > 0, \tag{1.2}$$

Beta Function

It is one of the basic functions of fractional calculus. This function plays an important role when combined with the Gamma function.

Definition 1.2 ([28]). *The Beta function is a type of* EULER *integral defined by*

$$B\left(p,q\right)=\int_{0}^{1}\xi^{p-1}\left(1-\xi\right)^{q-1}d\xi,\ p,q\in\mathbb{C}\backslash\mathbb{Z}_{0}^{-}, \tag{1.3}$$

The Beta function is related to the Gamma function by the following relation

$$B\left(p,q\right)=\frac{\Gamma\left(p\right)\Gamma\left(q\right)}{\Gamma\left(p+q\right)}. \tag{1.4}$$

Mittag-Leffler Function

The MITTAG-LEFFLER function is an important function that is widely used in the field of fractional calculus. Just as the exponential naturally arises out of the solution to integer order differential equations, the MITTAG-LEFFLER function plays an analogous role in the solution of non-integer order differential equations. The generalization of the single-parameter exponential function has been introduced by G. M. MITTAG-LEFFLER and is designated by the following definition:

Definition 1.3 ([28]). *The standard definition of the* MITTAG-LEFFLER *function is given by*

$$E_{\alpha}\left(\eta\right)=\sum_{k=0}^{+\infty}\frac{\eta^{k}}{\Gamma\left(\alpha k+1\right)},\ \alpha>0. \tag{1.5}$$

It is also common to represent the MITTAG-LEFFLER *function in two arguments, α and β. Such that*

$$E_{\alpha,\beta}\left(\eta\right)=\sum_{k=0}^{+\infty}\frac{\eta^{k}}{\Gamma\left(\alpha k+\beta\right)},\ \alpha,\beta>0. \tag{1.6}$$

The last relation is the more generalized form of the function. For $\beta=1$, we find (1.5).

Example 1.1. *From the relation (1.6), we find that*

$$\begin{aligned}
E_{1,1}\left(\eta\right) &= \sum_{k=0}^{+\infty}\frac{\eta^{k}}{\Gamma\left(k+1\right)}=\sum_{k=0}^{+\infty}\frac{\eta^{k}}{k!}=e^{\eta},\\
E_{1,2}\left(\eta\right) &= \sum_{k=0}^{+\infty}\frac{\eta^{k}}{\Gamma\left(k+2\right)}=\sum_{k=0}^{+\infty}\frac{\eta^{k}}{\left(k+1\right)!}=\frac{1}{\eta}\sum_{k=0}^{+\infty}\frac{\eta^{k+1}}{\left(k+1\right)!}=\frac{1}{\eta}\left(e^{\eta}-1\right),
\end{aligned}$$

1.2 Elements From Fractional Calculus Theory

The purpose of this part is to introduce the two most important approaches to fractional calculus: in the sense of RIEMANN-LIOUVILLE and in the sense of CAPUTO, including some of their properties as well as the relationship between these two approaches. The majority of the definitions in this section are taken from [28] and [36], which we refer to for a thorough analysis of the subject.

Riemann-Liouville Fractional Integrals

Definition 1.4 (Left-sided Riemann-Liouville fractional integral [28]). *The left-sided* RIEMANN-LIOUVILLE *fractional integral of order $\alpha > 0$ of a continuous function $\varphi : [0, \ell] \to \mathbb{R}$ is given by*

$$\mathcal{I}_{0^+}^{\alpha}\varphi(\eta) = \frac{1}{\Gamma(\alpha)}\int_0^{\eta}(\eta-\xi)^{\alpha-1}\varphi(\xi)\,d\xi,\ \eta\in[0,\ell]. \tag{1.7}$$

$\Gamma(\alpha)$ *is the Euler gamma function (1.1).*

Example 1.2. *If $\alpha > 0$ and $\beta > -1$, then*

$$\mathcal{I}_{0^+}^{\alpha}\eta^{\beta} = \frac{1}{\Gamma(\alpha)}\int_0^{\eta}(\eta-\xi)^{\alpha-1}\xi^{\beta}d\xi, \tag{1.8}$$

By making the change of variable $\xi = \eta z$, then (1.8) becomes

$$\begin{aligned}\mathcal{I}_{0^+}^{\alpha}\eta^{\beta} &= \frac{1}{\Gamma(\alpha)}\int_0^{1}(\eta-\eta z)^{\alpha-1}(\eta z)^{\beta}\eta dz\\ &= \frac{\eta^{\alpha+\beta}}{\Gamma(\alpha)}\int_0^{1}(1-z)^{\alpha-1}z^{\beta}dz\\ &= \frac{\eta^{\alpha+\beta}}{\Gamma(\alpha)}\int_0^{1}(1-z)^{\alpha-1}z^{\beta+1-1}dz.\end{aligned}$$

Using the Beta function definition (1.3) then the relationship (1.4), we arrive at

$$\begin{aligned}\mathcal{I}^{\alpha}\eta^{\beta} &= \frac{\eta^{\alpha+\beta}}{\Gamma(\alpha)}B(\alpha,\beta+1)\\ &= \frac{\eta^{\alpha+\beta}}{\Gamma(\alpha)}\frac{\Gamma(\alpha)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\\ &= \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\eta^{\alpha+\beta}. \end{aligned}\tag{1.9}$$

In particular, the relationship (1.9) shows that the fractional integral in the sense of RIEMANN-LIOUVILLE *of ordre α of a constant is given by*

$$\mathcal{I}_{0^+}^{\alpha}C = \frac{C}{\Gamma(\alpha+1)}\eta^{\alpha},\ C = const.$$

Property 1.1 ([28]). *Let $\alpha, \beta \in \mathbb{C}$ ($Re(\alpha) > 0$ and $Re(\beta) > 0$), for any function $\varphi \in L^1([0,\ell])$ and $\ell > 0$ we have*

$$\mathcal{I}_{0^+}^{\alpha}\left(\mathcal{I}_{0^+}^{\beta}\varphi(\eta)\right) = \mathcal{I}_{0^+}^{\alpha+\beta}\varphi(\eta) = \mathcal{I}_{0^+}^{\beta}\left(\mathcal{I}_{0^+}^{\alpha}\varphi(\eta)\right),$$

for almost everything $\eta \in [0,\ell]$. If more $\varphi \in C([0,\ell])$, then this identity is true $\forall \eta \in [0,\ell]$.

Riemann-Liouville Fractional Derivatives

Definition 1.5 (Left-sided Riemann-Liouville fractional derivative [28]). *The left-sided* RIEMANN-LIOUVILLE *fractional derivative of order $\alpha > 0$ of a continuous function $\varphi : [0,\ell] \to \mathbb{R}$ is given by*

$$^{RL}\mathcal{D}_{0^+}^{\alpha}\varphi(\eta) = \begin{cases} \frac{d^m\varphi(\eta)}{d\eta^m}, \text{ for } \alpha = m \in \mathbb{N}, \\ \frac{d^m}{d\eta^m}\mathcal{I}_{0^+}^{m-\alpha}\varphi(\eta) = \frac{d^m}{d\xi^m}\int_0^\eta \frac{(\eta-\xi)^{m-\alpha-1}}{\Gamma(m-\alpha)}\varphi(\xi)\,d\xi, \text{ for } m-1 < \alpha < m \in \mathbb{N}^*, \end{cases}$$

Example 1.3 (Constant Function). *Let $m-1 < \alpha < m \in \mathbb{N}$, then*

$$^{RL}\mathcal{D}_{0^+}^{\alpha}C = \frac{C\eta^{-\alpha}}{\Gamma(1-\alpha)} \neq 0, \; C = const.$$

Example 1.4 (Power Fonction). *Let $m-1 < \alpha < m$, $m-1 < \beta \in \mathbb{R}$, then*

$$^{RL}\mathcal{D}_{0^+}^{\alpha}\eta^{\beta} = \frac{\Gamma(\beta+1)\,\eta^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)}.$$

Caputo Fractional Derivatives

Definition 1.6 (Left-sided Caputo fractional derivative [28]). *The left-sided* CAPUTO *fractional derivative of order $\alpha > 0$ of a function $\varphi : [0,\ell] \to \mathbb{R}$ is given by*

$$^{C}\mathcal{D}_{0^+}^{\alpha}\varphi(\eta) = \begin{cases} \frac{d^m\varphi(\eta)}{d\eta^m}, \text{ for } \alpha = m \in \mathbb{N}, \\ \mathcal{I}_{0^+}^{m-\alpha}\frac{d^m\varphi(\eta)}{d\eta^m} = \int_0^\eta \frac{(\eta-\xi)^{m-\alpha-1}}{\Gamma(m-\alpha)}\frac{d^m\varphi(\xi)}{d\xi^m}d\xi, \text{ for } m-1 < \alpha < m \in \mathbb{N}^*, \end{cases} \tag{1.10}$$

Property 1.2 ([28]). *Let $\alpha \in \mathbb{C}$ such as $m-1 < Re(\alpha) < m$, and be $m \in \mathbb{N}^*$ both functions φ and ψ such as $^{C}\mathcal{D}_{0^+}^{\alpha}\varphi$ and $^{C}\mathcal{D}_{0^+}^{\alpha}\psi$ exist.* CAPUTO *fractional derivation is a linear operator*

$$^{C}\mathcal{D}_{0^+}^{\alpha}(\lambda\varphi+\psi)(\eta) = \lambda\, {}^{C}\mathcal{D}_{0^+}^{\alpha}\varphi(\eta) + {}^{C}\mathcal{D}_{0^+}^{\alpha}\psi(\eta), \; \lambda \in \mathbb{R}.$$

Proof. We have according to (1.10)

$$\begin{aligned} ^{C}\mathcal{D}_{0^+}^{\alpha}(\lambda\varphi+\psi)(\eta) &= \mathcal{I}^{m-\alpha}\mathcal{D}^{m}(\lambda\varphi+\psi)(\eta) \\ &= \lambda\mathcal{I}^{m-\alpha}\mathcal{D}^{m}(\varphi+\psi)(\eta). \end{aligned}$$

As the m^{th} derivative and the integral are linear then

$$\begin{aligned} {}^{C}\mathcal{D}_{0+}^{\alpha}\left(\lambda\varphi+\psi\right)(\eta) &= \lambda\mathcal{I}^{m-\alpha}\mathcal{D}^{m}\varphi(\eta)+\mathcal{I}^{m-\alpha}\mathcal{D}^{m}\psi(\eta) \\ &= \lambda^{C}\mathcal{D}_{0+}^{\alpha}\varphi(\eta)+^{C}\mathcal{D}_{0+}^{\alpha}\psi(\eta). \end{aligned}$$

The proof is complete. □

Property 1.3 ([36]). *Assume that $m-1<Re(\alpha)<m\in\mathbb{N}^{*}$, and let the function φ such as ${}^{C}\mathcal{D}_{0+}^{\alpha}\varphi$ exist, then*

$${}^{C}\mathcal{D}_{0+}^{\alpha}\mathcal{D}^{m}\varphi(\eta) = {}^{C}\mathcal{D}_{0+}^{\alpha+m}\varphi(\eta) \neq \mathcal{D}^{m}\,{}^{C}\mathcal{D}_{0+}^{\alpha}\varphi(\eta).$$

Lemma 1.1 ([28, 36]). *Assume that ${}^{C}\mathcal{D}_{0+}^{\alpha}\varphi\in C([0,\ell],\mathbb{R})$, for all $\alpha>0$, then*

$$\mathcal{I}_{0+}^{\alpha}\,{}^{C}\mathcal{D}_{0+}^{\alpha}\varphi(\eta) = \varphi(\eta)-\sum_{k=0}^{m-1}\frac{\varphi^{(k)}(0)}{k!}\eta^{k},\ m-1<\alpha\leq m\in\mathbb{N}^{*}.$$

Example 1.5 (Constant Function). *the following example is one of the advantages of the* CAPUTO *derivative over the* RIEMANN-LIOUVILLE *derivative (see [36]).*

$${}^{C}\mathcal{D}_{0+}^{\alpha}C=0,\ C=const.$$

In fact, as usual $0<m-1<\alpha<m\in\mathbb{N}$, which means $m\geq 1$. Applying the definition of the CAPUTO derivative (1.10) and since the m^{th} derivative of a constant C equals 0 it follows

$${}^{C}\mathcal{D}_{0+}^{\alpha}C=\frac{1}{\Gamma(m-\alpha)}\int_{0}^{\eta}C^{(m)}(\eta-\xi)^{m-\alpha-1}d\xi=0.$$

Example 1.6 (Power Fonction).

$${}^{C}\mathcal{D}_{0+}^{\alpha}\eta^{\beta} = {}^{RL}\mathcal{D}_{0+}^{\alpha}\eta^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}\eta^{\beta-\alpha},\ m-1<\alpha<m,\ m-1<\beta\in\mathbb{R}.$$

In fact, let $m-1<\alpha<m$, $m-1<\beta\in\mathbb{R}$.
The direct way reads

$$\begin{aligned} {}^{C}\mathcal{D}_{0+}^{\alpha}\eta^{\beta} &= \frac{1}{\Gamma(m-\alpha)}\int_{0}^{\eta}\left(\xi^{\beta}\right)^{(m)}(\eta-\xi)^{m-\alpha-1}d\xi \\ &= \frac{1}{\Gamma(m-\alpha)}\int_{0}^{\eta}\frac{\Gamma(\beta+1)}{\Gamma(\beta-m+1)}\xi^{\beta-m}(\eta-\xi)^{m-\alpha-1}d\xi, \end{aligned}$$

and using the substitution $\xi = z\eta$, $0 \le z \le 1$

$$\begin{aligned}
{}^{C}\mathcal{D}_{0^+}^{\alpha}\eta^{\beta} &= \frac{\Gamma(\beta+1)}{\Gamma(m-\alpha)\Gamma(\beta-m+1)}\int_0^{\eta}(z\eta)^{\beta-m}\left((1-z)\eta\right)^{m-\alpha-1}\eta dz \\
&= \frac{\Gamma(\beta+1)\eta^{\beta-\alpha}}{\Gamma(m-\alpha)\Gamma(\beta-m+1)}\int_0^{\eta} z^{\beta-m}(1-z)^{m-\alpha-1}dz \\
&= \frac{\Gamma(\beta+1)\eta^{\beta-\alpha}}{\Gamma(m-\alpha)\Gamma(\beta-m+1)}B(\beta-m+1,m-\alpha) \\
&= \frac{\Gamma(\beta+1)\eta^{\beta-\alpha}}{\Gamma(m-\alpha)\Gamma(\beta-m+1)}\frac{\Gamma(\beta-m+1)\Gamma(m-\alpha)}{\Gamma(\beta-\alpha+1)} \\
&= \frac{\Gamma(\beta+1)\eta^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)}.
\end{aligned}$$

Example 1.7 (Exponential Fonction). *Let* $\alpha \in \mathbb{R}$, $m-1 < \alpha < m \in \mathbb{N}$, $\beta \in \mathbb{C}$. *Then the* CAPUTO *fractional derivative of the exponential function has the form*

$$ {}^{C}\mathcal{D}_{0^+}^{\alpha}e^{\beta\eta} = \beta^m\eta^{m-\alpha}E_{1,m-\alpha+1}(\beta\eta). $$

In fact;

$$\begin{aligned}
{}^{C}\mathcal{D}_{0^+}^{\alpha}e^{\beta\eta} &= {}^{C}\mathcal{D}_{0^+}^{\alpha}\sum_{k=0}^{\infty}\frac{(\beta\eta)^k}{k!} \\
&= \frac{1}{\Gamma(m-\alpha)}\int_0^{\eta}\left(\sum_{k=0}^{\infty}\frac{(\beta\xi)^k}{k!}\right)^{(m)}(\eta-\xi)^{m-\alpha-1}d\xi \\
&= \frac{1}{\Gamma(m-\alpha)}\int_0^{\eta}\sum_{k=0}^{\infty}\frac{\beta^{m+k}\xi^k}{k!}(\eta-\xi)^{m-\alpha-1}d\xi \\
&= \frac{\beta^m}{\Gamma(m-\alpha)}\sum_{k=0}^{\infty}\frac{\beta^k}{k!}\int_0^{\eta}\xi^k(\eta-\xi)^{m-\alpha-1}d\xi.
\end{aligned}$$

Let $\xi = z\eta$, $0 < z < 1$

$$\begin{aligned}
{}^{C}\mathcal{D}_{0^+}^{\alpha}e^{\beta\eta} &= \frac{\beta^m}{\Gamma(m-\alpha)}\sum_{k=0}^{\infty}\frac{\beta^k}{k!}\int_0^1(z\eta)^k(\eta-z\eta)^{m-\alpha-1}\eta dz \\
&= \frac{\beta^m}{\Gamma(m-\alpha)}\sum_{k=0}^{\infty}\frac{\beta^k}{k!}\eta^{m+k-\alpha}\int_0^1 z^k(\eta-z)^{m-\alpha-1}dz \\
&= \frac{\beta^m\eta^{m-\alpha}}{\Gamma(m-\alpha)}\sum_{k=0}^{\infty}\frac{\beta^k\eta^k}{k!}B(k+1,m-\alpha) \\
&= \frac{\beta^m\eta^{m-\alpha}}{\Gamma(m-\alpha)}\sum_{k=0}^{\infty}\frac{(\beta\eta)^k}{k!}\frac{\Gamma(k+1)\Gamma(m-\alpha)}{\Gamma(m+k-\alpha+1)} \\
&= \beta^m\eta^{m-\alpha}\sum_{k=0}^{\infty}\frac{(\beta\eta)^k}{\Gamma(m+k-\alpha+1)}.
\end{aligned}$$

Example 1.8 (Sine function). *Let* $z \in \mathbb{C}$, $\alpha \in \mathbb{R}$, $m-1<\alpha<m \in \mathbb{N}$. *Then*

$$^{C}\mathcal{D}_{0+}^{\alpha} \sin z\eta = -\frac{1}{2} i \left(iz\right)^{m} \eta^{m-\alpha} \left(E_{1,m-\alpha+1}\left(iz\eta\right) - \left(-1\right)^{m} E_{1,m-\alpha+1}\left(-iz\eta\right)\right).$$

In fact, the following representation of the sine function used

$$\sin \xi = \frac{e^{i\xi} - e^{-i\xi}}{2i}, \ \xi \in \mathbb{C}.$$

Now, using the linearity property of the CAPUTO fractional derivative and formula for the exponential function it can be shown that

$$\begin{aligned} ^{C}\mathcal{D}_{0+}^{\alpha} \sin z\eta &= {}^{C}\mathcal{D}_{0+}^{\alpha} \frac{e^{iz\eta} - e^{-iz\eta}}{2i} \\ &= \frac{1}{2i}\left({}^{C}\mathcal{D}_{0+}^{\alpha} e^{iz\eta} - {}^{C}\mathcal{D}_{0+}^{\alpha} e^{-iz\eta}\right) \\ &= \frac{1}{2i}\left(\left(iz\eta\right)^{m} \eta^{m-\alpha} E_{1,m-\alpha+1}\left(iz\eta\right) - \left(-iz\eta\right)^{m} \eta^{m-\alpha} E_{1,m-\alpha+1}\left(-iz\eta\right)\right) \\ &= -\frac{1}{2} i \left(iz\eta\right)^{m} \eta^{m-\alpha} \left(E_{1,m-\alpha+1}\left(iz\eta\right) - \left(-1\right)^{m} E_{1,m-\alpha+1}\left(-iz\eta\right)\right). \end{aligned}$$

Example 1.9 (Cosine function). *Let* $z \in \mathbb{C}$, $\alpha \in \mathbb{R}$, $m-1<\alpha<m \in \mathbb{N}$. *Then*

$$^{C}\mathcal{D}_{0+}^{\alpha} \cos z\eta = \frac{1}{2} \left(iz\right)^{m} \eta^{m-\alpha} \left(E_{1,m-\alpha+1}\left(iz\eta\right) + \left(-1\right)^{m} E_{1,m-\alpha+1}\left(-iz\eta\right)\right).$$

In fact, the following representation of the cosine function used

$$\cos \xi = \frac{e^{i\xi} + e^{-i\xi}}{2}, \ \xi \in \mathbb{C}.$$

Now, using the linearity property of the CAPUTO fractional derivative and formula for the exponential function it can be shown that

$$\begin{aligned} ^{C}\mathcal{D}_{0+}^{\alpha} \cos z\eta &= {}^{C}\mathcal{D}_{0+}^{\alpha} \frac{e^{iz\eta} - e^{-iz\eta}}{2} \\ &= \frac{1}{2}\left({}^{C}\mathcal{D}_{0+}^{\alpha} e^{iz\eta} + {}^{C}\mathcal{D}_{0+}^{\alpha} e^{-iz\eta}\right) \\ &= \frac{1}{2}\left(\left(iz\eta\right)^{m} \eta^{m-\alpha} E_{1,m-\alpha+1}\left(iz\eta\right) + \left(-iz\eta\right)^{m} \eta^{m-\alpha} E_{1,m-\alpha+1}\left(-iz\eta\right)\right) \\ &= \frac{1}{2} \left(iz\eta\right)^{m} \eta^{m-\alpha} \left(E_{1,m-\alpha+1}\left(iz\eta\right) + \left(-1\right)^{m} E_{1,m-\alpha+1}\left(-iz\eta\right)\right). \end{aligned}$$

Relation Between Riemann-Liouville and Caputo Derivatives

If $\alpha \notin \mathbb{N}$ and φ is a function for which the CAPUTO fractional derivatives $^{C}\mathcal{D}_{0+}^{\alpha}\varphi$ of order $\alpha>0$ exist together with the RIEMANN-LIOUVILLE fractional derivatives $^{RL}\mathcal{D}_{0+}^{\alpha}\varphi$, then, in accordance with (1.7), they are connected with each other through the following relations

$$^{C}\mathcal{D}_{0+}^{\alpha}\varphi\left(\eta\right) = {}^{RL}\mathcal{D}_{0+}^{\alpha}\varphi\left(\eta\right) - \sum_{k=0}^{m-1} \frac{\varphi^{(k)}\left(0\right)}{\Gamma\left(k-\alpha+1\right)} \eta^{k-\alpha}, \text{ where } m = [\alpha]+1,$$

then

$$^{C}\mathcal{D}_{0+}^{\alpha}\varphi\left(\eta\right)=\,^{RL}\mathcal{D}_{0+}^{\alpha}\varphi\left(\eta\right),\text{ if }\varphi\left(0\right)=\varphi'\left(0\right)=\cdots=\varphi^{(m-1)}\left(0\right)=0.$$

1.3 Caputo Fractional Order's PDEs

Partial differential equations (PDEs) with fractional order have recently become a valuable tool for modeling numerous tangible incidents that science attempts to explain and have approached more frequently in recent years. Their application spans studies of vibration and control, signal and image processing, and modeling earthquakes, among others (Diethelm 2010 [20], Kilbas et al. 2006 [28], Podlubny 1999 [36], Samko et al. 1993 [39]).

Exact solutions of fractional-order's PDEs are crucial for rendering many qualitative features of natural science processes and phenomena fathomable, where become obtainable using various methods including the residual power series, symmetry, spectral, Fourier transform, similarity, *etc*. (for more details see [1, 2, 16, 19, 30, 42, 45]).

What is a CAPUTO fractional-order's partial differential equation?

To answer this question, we present the following two definitions.

Definition 1.7 (Caputo FDEs). CAPUTO *fractional differential equations is a relationship of the type*

$$^{C}\mathcal{D}_{0+}^{\alpha}\varphi\left(\eta\right)=f\left(\eta,\varphi,\,^{C}\mathcal{D}_{0+}^{\alpha_1}\varphi\,,\,^{C}\mathcal{D}_{0+}^{\alpha_2}\varphi\,,\,^{C}\mathcal{D}_{0+}^{\alpha_3}\varphi\,,\ldots\right),\tag{1.11}$$

where the variable $\eta\in\mathbb{R}$, *and the fractional derivatives of order* $\alpha_1,\alpha_2,\alpha_3,\ldots$ *of the unknown function* φ *at the point* η. *Here* $^{C}\mathcal{D}_{0+}^{\alpha}\varphi$ *presents a* CAPUTO *fractional differential operator of order* $\alpha\geq\alpha_1\geq\alpha_2\geq\ldots>0$.

Definition 1.8 (Caputo FPDEs). CAPUTO *fractional order's partial differential equations are space or time fractional Defined by the following relation*

$$\partial_{*}^{\alpha}\omega=F\left(x_1,x_2,\ldots,t,\omega,\left(-\Delta\right)^{\alpha_1}\omega,\partial_t^{\alpha_2}\omega,\partial_{x_1}^{\alpha_3}\omega\ldots\right),\ \alpha\geq\alpha_1\geq\alpha_2\geq\ldots>0,\tag{1.12}$$

where $\left(-\Delta\right)^{\alpha_1}$ *defines the fractional Laplacian operator [29] and the symbol* $\partial_{*}^{\alpha}w$ *is the* CAPUTO *left-sided fractional derivative of order* α. *With*

$$\partial_{*}^{\alpha}\omega=\partial_{x_i}^{\alpha}\omega=\mathcal{I}_a^{m-\alpha}\frac{\partial^m\omega}{\partial x_i^m}\text{ or }\partial_{*}^{\alpha}\omega=\partial_t^{\alpha}\omega=\mathcal{I}_0^{m-\alpha}\frac{\partial^m\omega}{\partial t^m},$$

where $\omega=\omega\left(x,t\right)$ *is a scalar function of the time* $t\geq0$ *and space variables* $x\in\left(a,b\right)^m$, *and* a,b *may be finite constants or infinitie. The symbol* $\mathcal{I}_{*}^{\alpha}$ *presents the* RIEMANN-LIOUVILLE *fractional integral of order* α.

1.4 Different Solution Forms For Caputo FPDEs

Sometimes to prove that the FPDEs accept at least one or only one solution, the FPDEs are converted into an FDEs of form and then the solution of the form is said to be the solution of the given FPDEs.

What are the methods available to convert FPDEs into a form FDEs?

To answer this question, we present the following ways

Traveling wave Solutions

Traveling wave solution is important in application because it allows modeling the dynamics of many problems in physics, chemistry, engineering, medicine, economics, control theory, etc. We propose solutions in "Traveling wave" form for FPDEs as follows

$$\omega\left(x,t\right)=\exp\left(a\left(t\right)\right)\varphi\left(\eta\right),\ \text{with}\ \eta=x-\kappa t,\ \text{and}\ \kappa\in\mathbb{R}^{*}, \tag{1.13}$$

the function $a\left(t\right)$ depends on time t, and the basic profile φ are not known in advance and are to be identified.

Example 1.10. *Let $\kappa\in\mathbb{R}^{*}$ and $a\left(t\right)=-\kappa^{2}t$, we consider the space-fractional wave equations of higher order as follows*

$$\frac{\partial^{2}\omega}{\partial t^{2}}=\kappa^{2}\frac{\partial^{\alpha}\omega}{\partial x^{\alpha}},\ \textit{for}\ m-1<\alpha\leq m\in\mathbb{N}^{*}, \tag{1.14}$$

then the transformation (1.13) reduces the partial differential equation of space-fractional order (1.14) to the ordinary differential equation of fractional order of the form

$${}^{C}\mathcal{D}_{0+}^{\alpha}\varphi\left(\eta\right)=\kappa^{2}\varphi\left(\eta\right)+2\kappa\varphi'\left(\eta\right)+\varphi''\left(\eta\right), \tag{1.15}$$

For more details about using the method "traveling wave" to convert FPDEs (1.14) into a form FDEs (1.15), see page 17 in chapter two.

Example 1.11. *Let $\tau,\mu,\kappa,\ell,\delta\in\mathbb{R}_{+}^{*}$, $p,q,m\in\mathbb{R}$ and $a\left(t\right)=-\frac{\kappa^{2}}{\delta}t$, we consider the space-fractional equations of nonlinear acoustics as follows*

$$\tau\omega_{ttt}+\mu\omega_{tt}-\kappa^{2}\partial_{x}^{\alpha}\omega-\delta\partial_{x}^{\alpha}\omega_{t}=F\left(x,t,\omega,\omega_{t},\omega_{tt},\omega_{xx},\left(\omega_{t}\right)_{xx}\right),\ \textit{for}\ 1<\alpha\leq 2, \tag{1.16}$$

with F is a nonlinear continuous function that is invariant by the change of scale (1.13). It gives us

$$F\left(x,t,\omega,\omega_{t},\omega_{tt},\omega_{xx},\left(\omega_{t}\right)_{xx}\right)=\exp\left(-\frac{\kappa^{2}}{\delta}t\right)\left(\delta\kappa f\left(\eta,\varphi,\varphi',\varphi''\right)-\kappa^{3}\tau\varphi'''\right),$$

where $f : [0, \ell] \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a continuous function, then the transformation (1.13) reduces the partial differential equation of space-fractional order (1.16) to the ordinary differential equation of fractional order of the form

$$ {}^{C}\mathcal{D}_{0^+}^{\alpha+1}\varphi(\eta) = p\varphi(\eta) + q\varphi'(\eta) + m\varphi''(\eta) + f(\eta, \varphi(\eta), \varphi'(\eta), \varphi''(\eta)), \tag{1.17} $$

For more details about using the method "traveling wave" to convert FPDEs (1.16) into a form FDEs (1.17), see page 28 in chapter three.

Traveling Profile Solutions

Traveling profile solutions (see [11]), plays an important role in modeling all scientific fields: computer science, physics, biology, medicine... Because it contributes to the study of complex problems by transforming them into simple problems. We suggest finding the solution for FPDEs in the following "traveling profile" form

$$ \omega(x,t) = c(t)\varphi(\eta), \text{ with } \eta = \frac{x-b(t)}{a(t)}, \text{ for } a \neq 0. \tag{1.18} $$

The functions $a(t)$, $b(t)$ and $c(t)$ depends on time t and the basic profile φ are not known in advance and are to be identified.

Example 1.12. *Let $\kappa \in \mathbb{R}^*$, we consider a diffusion equation of moving fractional order as follows*

$$ \frac{\partial \omega}{\partial t} = \kappa \frac{\partial^\alpha \omega}{\partial x^\alpha}, \tag{1.19} $$

then the transformation (1.18) reduces the partial differential equation of space-fractional order (1.19) to the ordinary differential equation of fractional order of the form

$$ {}^{C}\mathcal{D}_{0^+}^{\alpha}\varphi(\eta) = \alpha\varphi(\eta) + \beta\eta\varphi'(\eta) + \gamma\varphi'(\eta). \tag{1.20} $$

For more details about using the method "traveling profile" to convert FPDEs (1.16) into a form FDEs (1.20), see page 43 in chapter four.

Radially Symmetric Solutions

Radially symmetric solutions (see [9, 10, 15, 25, 43]), are extremely useful solutions in different fields of physics and pure mathematics because they model phenomena that are independent of the scale of measurement. We propose solutions in "radially symmetric" form for FPDEs as follows

$$ \omega(x,t) = |x|^\mu \varphi(\eta), \text{with } \eta = |x|^\gamma t, \text{ and } \mu, \gamma \in \mathbb{C}. \tag{1.21} $$

Example 1.13. *Let $\kappa \in \mathbb{R}^*$ and $p, q \in \mathbb{R}$, we consider a multidimensional nonlinear time and space-fractional reaction-diffusion/wave equation as follows*

$$\partial_t^\alpha \omega - \kappa^2 \Delta \omega = F\left(t, x, \omega, \partial_t^\beta \omega, (-\Delta)^s \omega\right), \text{ for } 0 < s \leq 1 < \beta \leq \alpha \leq 2, \tag{1.22}$$

with F is a nonlinear continuous function that is invariant by the change of scale (1.21). It gives us:

$$F\left(t, x, \omega, \partial_t^\beta \omega, (-\Delta)^s \omega\right) = |x|^{\delta-2}\left(f\left(\eta, \varphi(\eta), \varphi'(\eta), {}^C\mathcal{D}_{0+}^\beta \varphi(\eta)\right) - \frac{4\kappa^2}{\alpha^2}\eta^2 \varphi''(\eta)\right),$$

where $\eta = |x|^{-\frac{2}{\alpha}} t$ and $f : [0, \ell] \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a continuous function, then the transformation (1.21) reduces the partial differential equation of space-fractional order (1.22) to the ordinary differential equation of fractional order of the form

$${}^C\mathcal{D}_{0+}^\alpha \varphi(\eta) = p\varphi(\eta) + q\eta\varphi'(\eta) + f\left(\eta, \varphi(\eta), \varphi'(\eta), {}^C\mathcal{D}_{0+}^\beta \varphi(\eta)\right), \tag{1.23}$$

For more details about using the method "radially symmetric" to convert FPDEs (1.22) into a form FDEs (1.23), see page 57 in chapter five.

1.5 Fixed Point Theorems

In the remainder of this section, we introduce the notations, definitions and theorems necessary for this study.

Definition 1.9 (Equicontinuous [4]). *Let E be a* BANACH *space. A part P in $C(E)$ is called equicontinuous if*

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall \eta_1, \eta_2 \in E,\ \forall \mathcal{A} \in P,\ \ \|\eta_1 - \eta_2\| < \delta \Rightarrow \|\mathcal{A}(\eta_1) - \mathcal{A}(\eta_2)\| < \varepsilon.$$

Theorem 1.1 (Ascoli-Arzelà [23]). *Let E be a compact space. If $\mathcal{A}$ is an equicontinuous, bounded subset of $C(E)$, then $\mathcal{A}$ is relatively compact.*

Definition 1.10 ([23]). *Let E be any space and $\mathcal{A}$ a map of E, or of a subset of E, into E.*
- The map $\mathcal{A}$ is called a contraction mapping if there exists $k \in (0, 1)$ such that

$$\forall \varphi, \varphi_1 \in E,\ \ \|\mathcal{A}\varphi - \mathcal{A}\varphi_1\| \leq k \|\varphi - \varphi_1\|.$$

- A point $\varphi \in E$ is called a fixed point for $\mathcal{A}$ if $\mathcal{A}\varphi = \varphi$.

Theorem 1.2 (Banach's fixed point [23]). *Let P be a non-empty closed subset of a* BANACH *space E, then any contraction mapping $\mathcal{A}$ of P into itself has a unique fixed point.*

Theorem 1.3 (Schauder's fixed point [23]). *Let E be a* BANACH *space, and P be a closed, convex and nonempty subset of E. Let $\mathcal{A} : P \to P$ be a continuous mapping such that $\mathcal{A}(P)$ is a relatively compact subset of E. Then $\mathcal{A}$ has at least one fixed point in P.*

Chapter 2

Existence of Traveling Wave Solutions for a Free Boundary Problem of Higher Order Space Fractional Wave Equations

This chapter has been publication in: Journal of Applied Mathematics E-Notes **22** (2022), (see [22]).

2.1 Introduction

This chapter investigates the problem of existence and uniqueness of solutions under the traveling wave forms

$$\omega(x,t) = \exp\left(-\kappa^2 t\right) \varphi(x - \kappa t), \text{ with } \kappa \in \mathbb{R}^*, \tag{2.1}$$

for a free boundary problem of higher-order space-fractional wave equations as follows

$$\partial_t^2 \omega = \kappa^2 \partial_x^\alpha \omega,\ \kappa \in \mathbb{R}^*,\ m - 1 \leq \alpha < m \in \mathbb{N} - \{0, 1, 2\}, \tag{2.2}$$

with

$$\partial_x^\alpha \omega = \begin{cases} \partial_x^m \omega, & \alpha = m \in \mathbb{N}, \\ \mathcal{I}_{\kappa t}^{m-\alpha} \partial_x^m \omega = \frac{1}{\Gamma(m-\alpha)} \int_{\kappa t}^{x} (x-\tau)^{m-\alpha-1} \frac{\partial^m}{\partial \tau^m} \omega(\tau, t)\, d\tau, & m - 1 < \alpha < m \in \mathbb{N}^*. \end{cases}$$

It does so by applying the properties of Schauder's and Banach's fixed point theorems.

Where the basic profile φ are not known in advance and are to be identified and $\omega = \omega(x,t)$ is a scalar function of a space and time variables $(x,t) \in \Omega$ with

$$\Omega = \{(x,t) \in \mathbb{R} \times [0,T];\ \kappa t \leq x \leq X\},\ \text{for } T > 0 \text{ and } X > |\kappa| T.$$

The higher-order space-fractional wave equation (2.2) becomes the wave equation for $\alpha = 2$ and the fourth-order wave equation for $\alpha = 4,$ (see [44]). This was, with a second member, the first model of surface waves in shallow water that takes into consideration the balance between the nonlinearity and dispersion, thus, keeping the wave's shape; it is properly termed currently the 'Boussinesq paradigm with a second member. This balance bears solitary waves that behave like quasi-particles, these waves behave as particles called Solitons. This concept can be crucial for the interpretation of the dualism wave-particle in physics.

This method permits us to reduce the fractional-order's PDE (2.2) to a fractional differential equation. This approach (2.1) is very promising and can also bring novel results for other applications in fractional-order's PDEs.

Throughout the rest of this chapter, we have $m \geq 3$ is a natural number and

$$\begin{aligned} &m-1 \leq \alpha < m,\ T > 0 \text{ and } X > |\kappa|\, T \text{ for some } \kappa \in \mathbb{R}^*. \\ &\text{and } J = [0, \ell] \text{ with } \ell = X + |\kappa|\, T. \end{aligned} \tag{2.3}$$

2.2 Main Results

Statement of the Free Boundary Problem

In this part, we first attempt to find the equivalent approximate to the following free boundary problem of the higher-order space-fractional wave equation

$$\begin{cases} \partial_t^2 \omega = \kappa^2 \partial_x^\alpha \omega, & (x,t) \in \Omega, \\ \omega(\kappa t, t) = c_0 \exp\left(-\kappa^2 t\right), & c_0 \in \mathbb{R}, \\ \partial_x^k \omega(\kappa t, t) = 0, & k \in \{1, 2, \ldots, m-1\}, \end{cases} \tag{2.4}$$

under the traveling wave form

$$\omega(x,t) = \exp\left(-\kappa^2 t\right) \varphi(\eta), \text{ with } \eta = x - \kappa t\ . \tag{2.5}$$

Main Theorems

Now, we give the principal theorems of this work.

Theorem 2.1. *Let $\alpha, \kappa, T, X \in \mathbb{R}$, be the real constants given by (2.3). If*

$$(X + |\kappa|\, T)^{\alpha-2} \left[\alpha\left(2\left(X + |\kappa|\, T\right)|\kappa| + \alpha - 1\right) + \kappa^2 \left(X + |\kappa|\, T\right)^2\right] < \Gamma(\alpha+1), \tag{2.6}$$

then the problem (2.4) has at least one solution in the traveling wave form (2.5).

Theorem 2.2. *Let $\alpha, \kappa, T, X \in \mathbb{R}$, be the real constants given by (2.3). If*

$$\Gamma(\alpha+1) > \alpha (X + |\kappa| T)^{\alpha-2} (2(X+|\kappa| T)|\kappa| + \alpha - 1)$$

and

$$\frac{\kappa^2 (X+|\kappa| T)^{\alpha}}{\Gamma(\alpha+1) - \alpha (X+|\kappa| T)^{\alpha-2} (2(X+|\kappa| T)|\kappa| + \alpha - 1)} < 1, \tag{2.7}$$

then the problem (2.4) admits a unique solution in the traveling wave form (2.5).

2.3 Compute of Traveling Wave Solutions

First, we should deduce the equation satisfied by the function φ in (2.5) and used for the definition of traveling wave solutions.

Theorem 2.3. *The transformation (2.5) reduces the partial differential equation problem of space-fractional order (2.4) to the ordinary differential equation of fractional order of the form*

$$^{C}\mathcal{D}_{0+}^{\alpha}\varphi(\eta) = g(\eta), \ \eta \in J, \tag{2.8}$$

where

$$g(\eta) = \kappa^2\varphi(\eta) + 2\kappa\varphi'(\eta) + \varphi''(\eta),$$

with the conditions

$$\varphi(0) = c_0 \text{ and } \varphi^{(k)}(0) = 0, \text{ for } k \in \{1, 2, \ldots, m-1\}. \tag{2.9}$$

Proof. The fractional equation resulting from the substitution of expression (2.5) in the original fractional-order's PDE (2.4), should be reduced to the standard bilinear functional equation (see [37]).
First, for $\eta = x - \kappa t$, we get $\eta \in J$ and

$$\partial_t^2\omega = \exp\left(-\kappa^2 t\right) g(\eta), \tag{2.10}$$

with

$$g(\eta) = \kappa^2\varphi(\eta) + 2\kappa\varphi'(\eta) + \varphi''(\eta).$$

In another way, for $\xi = \tau - \kappa t$, we get

$$\begin{aligned}\partial_x^{\alpha}\omega &= \frac{1}{\Gamma(m-\alpha)}\int_{\kappa t}^{x} (x-\tau)^{m-\alpha-1}\frac{\partial^m \omega(\tau,t)}{\partial \tau^m} d\tau \\ &= \frac{\exp(-\kappa^2 t)}{\Gamma(m-\alpha)}\int_{0}^{\eta} (\eta-\xi)^{m-\alpha-1}\frac{d^m}{d\xi^m}\varphi(\xi)\, d\xi \\ &= \exp\left(-\kappa^2 t\right) {}^{C}\mathcal{D}_{0+}^{\alpha}\varphi(\eta). \end{aligned} \tag{2.11}$$

If we replace (2.10) and (2.11) in the first equation of (2.4), we get

$$^{C}\mathcal{D}_{0^+}^{\alpha}\varphi(\eta) = g(\eta).$$

From the conditions in (2.4), we find for each $k \in \{1, \ldots, m-1\}$ that

$$\begin{aligned}\omega(\kappa t, t) &= \exp\left(-\kappa^2 t\right)\varphi(\kappa t - \kappa t) = \exp\left(-\kappa^2 t\right)\varphi(0),\\ \partial_x^k \omega(\kappa t, t) &= \exp\left(-\kappa^2 t\right)\varphi^{(k)}(\kappa t - \kappa t) = \exp\left(-\kappa^2 t\right)\varphi^{(k)}(0),\end{aligned}$$

which implies that

$$\varphi(0) = c_0 \text{ and } \varphi^{(k)}(0) = 0, \text{ for } k \in \{1, 2, \ldots, m-1\}.$$

The proof is complete. □

2.4 Existence and Uniqueness Results

In what follows, we present some significant lemmas to show the principal theorems.

Lemma 2.1. *The problem (2.8)–(2.9) is equivalent to the integral equation*

$$\varphi(\eta) = c_0 + \frac{1}{\Gamma(\alpha)} \int_0^{\eta} (\eta - \xi)^{\alpha - 1} g(\xi)\, d\xi, \ \forall \eta \in J,$$

where $g \in C(J, \mathbb{R})$ satisfies the functional equation

$$g(\eta) = \kappa^2 \left(c_0 + \mathcal{I}_{0^+}^{\alpha} g(\eta)\right) + \psi(g(\eta)),$$

with $\psi : \mathbb{R} \to \mathbb{R}$ is a function satisfying

$$\psi(g(\eta)) = 2\kappa \mathcal{I}_{0^+}^{\alpha-1} g(\eta) + \mathcal{I}_{0^+}^{\alpha-2} g(\eta).$$

Proof. Using Theorem 2.3, and applying $\mathcal{I}_{0^+}^{\alpha}$ to the equation (2.8), we obtain

$$\mathcal{I}_{0^+}^{\alpha}\, {}^{C}\mathcal{D}_{0^+}^{\alpha}\varphi(\eta) = \mathcal{I}_{0^+}^{\alpha} g(\eta).$$

From Lemma 1.1, we simply find

$$\mathcal{I}_{0^+}^{\alpha}\, {}^{C}\mathcal{D}_{0^+}^{\alpha}\varphi(\eta) = \varphi(\eta) - \sum_{k=0}^{m-1} \frac{\varphi^{(k)}(0)}{k!}\eta^k, \ m - 1 < \alpha \le m \in \mathbb{N}^*.$$

Substituting (2.9) gives us

$$\varphi(\eta) = c_0 + \mathcal{I}_{0^+}^{\alpha} g(\eta). \tag{2.12}$$

As

$$\varphi'(\eta) = \frac{d}{d\eta}\left(c_0 + \mathcal{I}_{0+}^{\alpha} g(\eta)\right) = \mathcal{I}_{0+}^{\alpha-1} g(\eta)$$

and

$$\varphi''(\eta) = \frac{d^2}{d\eta^2}\left(c_0 + \mathcal{I}_{0+}^{\alpha} g(\eta)\right) = \mathcal{I}_{0+}^{\alpha-2} g(\eta),$$

then

$$\begin{aligned} g(\eta) &= \kappa^2 \varphi(\eta) + 2\kappa\varphi'(\eta) + \varphi''(\eta) \\ &= \kappa^2\left(c_0 + \mathcal{I}_{0+}^{\alpha} g(\eta)\right) + 2\kappa \mathcal{I}_{0+}^{\alpha-1} g(\eta) + \mathcal{I}_{0+}^{\alpha-2} g(\eta) \\ &= \kappa^2\left(c_0 + \mathcal{I}_{0+}^{\alpha} g(\eta)\right) + \psi(g(\eta)). \end{aligned}$$

Otherwise, starting by applying ${}^{C}\mathcal{D}_{0+}^{\alpha}$ on both sides of the equation (2.12) and using the linearity of Caputo's derivative and the fact that ${}^{C}\mathcal{D}_{0+}^{\alpha} c_0 = 0$, we find easily (2.8). Furthermore;

$$\begin{aligned} \varphi(0) &= \left(c_0 + \mathcal{I}_{0+}^{\alpha} g\right)(0) = c_0 \\ \varphi^{(k)}(0) &= \mathcal{I}_{0+}^{\alpha-k} g(0) = 0, \text{ for any } k \in \{1, 2, \ldots, m-1\}. \end{aligned}$$

The proof is complete. □

Theorem 2.4. *If we put*

$$\ell^{\alpha-2}\left[\alpha\left(2\ell|\kappa| + \alpha - 1\right) + \kappa^2\ell^2\right] < \Gamma(\alpha+1), \tag{2.13}$$

then the problem (2.8)–(2.9) has at least one solution on J.

Proof. To begin the proof, we will transform the problem (2.8)–(2.9) into a fixed point problem. Let us define

$$\mathcal{A}u(\eta) = c_0 + \frac{1}{\Gamma(\alpha)}\int_0^{\eta} (\eta - \xi)^{\alpha-1} g(\xi)\, d\xi, \tag{2.14}$$

where

$$g(\eta) = \kappa^2 u(\eta) + \psi(g(\eta)),\ \eta \in J,$$

with

$$\psi(g(\eta)) = 2\kappa \mathcal{I}_{0+}^{\alpha-1} g(\eta) + \mathcal{I}_{0+}^{\alpha-2} g(\eta).$$

We first notice that if $g \in C(J, \mathbb{R})$, then $\mathcal{A}u$ is indeed continuous (see the step 1 in this proof); therefore, it is an element of $C(J, \mathbb{R})$, and is equipped with the standard norm

$$\|\mathcal{A}u\|_{\infty} = \sup_{\eta \in J} |\mathcal{A}u(\eta)|.$$

Clearly, the fixed points of $\mathcal{A}$ are solutions of the problem (2.8)–(2.9).

We demonstrate that $\mathcal{A}$ satisfies the assumption of Schauder's fixed point theorem (see [23]). This could be proved through three steps.

Step 1: $\mathcal{A}$ is a continuous operator.

Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence such that $\lim_{n\to\infty} u_n = u$ in $C(J,\mathbb{R})$. Then $\forall\eta \in J$,

$$|\mathcal{A}u_n(\eta) - \mathcal{A}u(\eta)| \leq \int_0^\eta \frac{(\eta-\xi)^{\alpha-1}}{\Gamma(\alpha)} |g_n(\xi) - g(\xi)| \, d\xi, \tag{2.15}$$

where

$$\begin{cases} g_n(\eta) = \kappa^2 u_n(\eta) + \psi(g_n(\eta)), \\ g(\eta) = \kappa^2 u(\eta) + \psi(g(\eta)). \end{cases}$$

We have

$$\begin{aligned} |g_n(\eta) - g(\eta)| &= \left|\kappa^2 (u_n(\eta) - u(\eta)) + \psi(g_n(\eta)) - \psi(g(\eta))\right| \\ &\leq \kappa^2 \|u_n - u\|_\infty + 2|\kappa| \left|\mathcal{I}_{0^+}^{\alpha-1}(g_n(\eta) - g(\eta))\right| + \left|\mathcal{I}_{0^+}^{\alpha-2}(g_n(\eta) - g(\eta))\right|. \end{aligned}$$

As

$$\begin{aligned} \left|\mathcal{I}_{0^+}^{\alpha-1}(g_n(\eta) - g(\eta))\right| &\leq \frac{1}{\Gamma(\alpha-1)} \int_0^\eta (\eta-\xi)^{\alpha-2} |g_n(\xi) - g(\xi)| \, d\xi \\ &\leq \frac{\ell^{\alpha-1}}{\Gamma(\alpha)} \|g_n - g\|_\infty \end{aligned}$$

and

$$\begin{aligned} \left|\mathcal{I}_{0^+}^{\alpha-2}(g_n(\eta) - g(\eta))\right| &\leq \frac{1}{\Gamma(\alpha-2)} \int_0^\eta (\eta-\xi)^{\alpha-3} |g_n(\xi) - g(\xi)| \, d\xi \\ &\leq \frac{\ell^{\alpha-2}}{\Gamma(\alpha-1)} \|g_n - g\|_\infty \\ &\leq \frac{\ell^{\alpha-2}(\alpha-1)}{\Gamma(\alpha)} \|g_n - g\|_\infty. \end{aligned}$$

Then we get

$$\|g_n - g\|_\infty \leq \kappa^2 \|u_n - u\|_\infty + \frac{\ell^{\alpha-2}(2\ell|\kappa| + \alpha - 1)}{\Gamma(\alpha)} \|g_n - g\|_\infty.$$

According to (2.13), we have $\Gamma(\alpha) - \ell^{\alpha-2}(2\ell|\kappa| + \alpha - 1) > \frac{\kappa^2\ell^\alpha}{\alpha} > 0$, thus

$$\|g_n - g\|_\infty \leq \frac{\kappa^2\Gamma(\alpha)}{\Gamma(\alpha) - \ell^{\alpha-2}(2\ell|\kappa| + \alpha - 1)} \|u_n - u\|_\infty.$$

Since $u_n \to u$, we get $g_n \to g$ when $n \to \infty$.

Now, let $\mu > 0$ be such that for each $\eta \in J$, we get

$$|g_n(\eta)| \leq \mu, \;\; |g(\eta)| \leq \mu.$$

Then, we have

$$\begin{aligned}\frac{(\eta-\xi)^{\alpha-1}}{\Gamma(\alpha)}|g_n(\eta)-g(\eta)| &\leq \frac{(\eta-\xi)^{\alpha-1}}{\Gamma(\alpha)}[|g_n(\eta)|+|g(\eta)|] \\ &\leq \frac{2\mu}{\Gamma(\alpha)}(\eta-\xi)^{\alpha-1}.\end{aligned}$$

For each $\eta \in J$, the function $\xi \to \frac{2\mu}{\Gamma(\alpha)}(\eta-\xi)^{\alpha-1}$ is integrable on $[0,\eta]$, then the Lebesgue dominated convergence theorem and (2.15) imply that

$$|\mathcal{A}u_n(\eta) - \mathcal{A}u(\eta)| \to 0 \text{ as } n \to \infty,$$

and hence

$$\lim_{n\to\infty} \|\mathcal{A}u_n - \mathcal{A}u\|_\infty = 0.$$

Consequently, $\mathcal{A}$ is continuous.

Step 2: According to (2.13), we put the positive real

$$r \geq \left(1 + \frac{\kappa^2 \ell^\alpha}{\Gamma(\alpha+1) - \ell^{\alpha-2}[\alpha(2\ell|\kappa| + \alpha - 1) + \kappa^2\ell^2]}\right)|c_0|$$

and define the subset H as follows

$$H = \{u \in C(J,\mathbb{R}) : \|u\|_\infty \leq r\}.$$

It is clear that H is a bounded, closed and convex subset of $C(J,\mathbb{R})$.

Let $\mathcal{A} : H \to C(J,\mathbb{R})$ be the integral operator defined by (2.14), then $\mathcal{A}(H) \subset H$.

Indeed, we have for each $\eta \in J$

$$\begin{aligned}|g(\eta)| &= \left|\kappa^2 u(\eta) + \psi(g(\eta))\right| \\ &\leq \kappa^2|u(\eta)| + \frac{\ell^{\alpha-2}(2\ell|\kappa| + \alpha - 1)}{\Gamma(\alpha)}\|g\|_\infty.\end{aligned}$$

According to (2.13), we get $\Gamma(\alpha) - \ell^{\alpha-2}(2\ell|\kappa| + \alpha - 1) > 0$ and

$$\|g\|_\infty \leq \frac{\kappa^2\Gamma(\alpha)}{\Gamma(\alpha) - \ell^{\alpha-2}(2\ell|\kappa| + \alpha - 1)} r.$$

Then

$$\begin{aligned}
|\mathcal{A}u(\eta)| &\leq |c_0| + \frac{1}{\Gamma(\alpha)}\int_0^\eta (\eta-\xi)^{\alpha-1}|g(\xi)|\,d\xi \\
&\leq |c_0| + \frac{\kappa^2 r}{\Gamma(\alpha) - \ell^{\alpha-2}(2\ell|\kappa| + \alpha - 1)}\int_0^\eta (\eta-\xi)^{\alpha-1}\,d\xi \\
&\leq |c_0| + \frac{\frac{\ell^\alpha}{\alpha}\kappa^2 r}{\Gamma(\alpha) - \ell^{\alpha-2}(2\ell|\kappa| + \alpha - 1)} \\
&\leq |c_0| + \frac{\kappa^2\ell^\alpha}{\Gamma(\alpha+1) - \alpha\ell^{\alpha-2}(2\ell|\kappa| + \alpha - 1)}r \\
&\leq \frac{|c_0|\left(1 + \frac{\kappa^2\ell^\alpha}{\Gamma(\alpha+1) - \ell^{\alpha-2}[\alpha(2\ell|\kappa|+\alpha-1)+\kappa^2\ell^2]}\right)}{1 + \frac{\kappa^2\ell^\alpha}{\Gamma(\alpha+1) - \ell^{\alpha-2}[\alpha(2\ell|\kappa|+\alpha-1)+\kappa^2\ell^2]}} \\
&+ \frac{\kappa^2\ell^\alpha}{\Gamma(\alpha+1) - \alpha\ell^{\alpha-2}(2\ell|\kappa| + \alpha - 1)}r \\
&\leq r.
\end{aligned}$$

Then $\mathcal{A}(H) \subset H$.

Step 3: $\mathcal{A}(H)$ is relatively compact.

Let $\eta_1, \eta_2 \in J$, $\eta_1 < \eta_2$, and $u \in H$. Then

$$\begin{aligned}
|\mathcal{A}u(\eta_2) - \mathcal{A}u(\eta_1)| &= \frac{1}{\Gamma(\alpha)}\left|\int_0^{\eta_2}(\eta_2-\xi)^{\alpha-1}g(\xi)\,d\xi - \int_0^{\eta_1}(\eta_1-\xi)^{\alpha-1}g(\xi)\,d\xi\right| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^{\eta_1}\left|\left((\eta_2-\xi)^{\alpha-1} - (\eta_1-\xi)^{\alpha-1}\right)g(\xi)\right|d\xi \\
&+ \frac{1}{\Gamma(\alpha)}\int_{\eta_1}^{\eta_2}(\eta_2-\xi)^{\alpha-1}|g(\xi)|\,d\xi \\
&\leq \frac{\kappa^2 r}{\Gamma(\alpha) - \ell^{\alpha-2}(2\ell|\kappa| + \alpha - 1)}\left[\int_0^{\eta_1}\left|(\eta_2-\xi)^{\alpha-1} - \right.\right. \\
&\left.\left.(\eta_1-\xi)^{\alpha-1}\right|d\xi + \int_{\eta_1}^{\eta_2}(\eta_2-\xi)^{\alpha-1}\,d\xi\right]. \qquad (2.16)
\end{aligned}$$

We have

$$(\eta_2-\xi)^{\alpha-1} - (\eta_1-\xi)^{\alpha-1} = -\frac{1}{\alpha}\frac{d}{d\xi}\left[(\eta_2-\xi)^\alpha - (\eta_1-\xi)^\alpha\right],$$

then

$$\int_0^{\eta_1}\left|(\eta_2-\xi)^{\alpha-1} - (\eta_1-\xi)^{\alpha-1}\right|d\xi \leq \frac{1}{\alpha}\left[(\eta_2-\eta_1)^\alpha + (\eta_2^\alpha - \eta_1^\alpha)\right],$$

we have also

$$\int_{\eta_1}^{\eta_2}(\eta_2-\xi)^{\alpha-1}\,d\xi = -\frac{1}{\alpha}\left[(\eta_2-\xi)^\alpha\right]_{\eta_1}^{\eta_2} \leq \frac{1}{\alpha}(\eta_2-\eta_1)^\alpha.$$

Then (2.16) gives us

$$|\mathcal{A}u(\eta_2) - \mathcal{A}u(\eta_1)| \leq \frac{\kappa^2 r \left(2(\eta_2-\eta_1)^\alpha + (\eta_2^\alpha - \eta_1^\alpha)\right)}{\Gamma(\alpha+1) - \alpha\ell^{\alpha-2}(2\ell|\kappa| + \alpha - 1)}.$$

As $\eta_1 \to \eta_2$, the right-hand side of the above inequality tends to zero.

As a consequence of steps 1 to 3, and by means of the Ascoli-Arzelà theorem, we deduce that $\mathcal{A}: H \to H$ is continuous, compact and satisfies the assumption of Schauder's fixed point theorem [23]. Then $\mathcal{A}$ has a fixed point which is a solution of the problem (2.8)–(2.9) on J. The proof is complete. □

Theorem 2.5. *If we put* $\Gamma(\alpha+1) > \alpha\ell^{\alpha-2}(2\ell|\kappa| + \alpha - 1)$ *and*

$$\frac{\kappa^2 \ell^\alpha}{\Gamma(\alpha+1) - \alpha\ell^{\alpha-2}(2\ell|\kappa| + \alpha - 1)} < 1, \tag{2.17}$$

then the problem (2.8)–(2.9) admits a unique solution on J.

Proof. In the previous Theorem 2.4, we transformed the problem (2.8)–(2.9) into a fixed point problem (2.14).

Let $u_1, u_2 \in C(J, \mathbb{R})$, then

$$\mathcal{A}u_1(\eta) - \mathcal{A}u_2(\eta) = \frac{1}{\Gamma(\alpha)} \int_0^\eta (\eta - \xi)^{\alpha-1} (g_1(\xi) - g_2(\xi))\, d\xi.$$

Where

$$g_i(\eta) = \kappa^2 u_i(\eta) + \psi(g_i(\eta)),$$
$$\psi(g_i(\eta)) = 2\kappa \mathcal{I}_{0^+}^{\alpha-1} g_i(\eta) + \mathcal{I}_{0^+}^{\alpha-2} g_i(\eta), \text{ for } i = 1, 2.$$

Also

$$|\mathcal{A}u_1(\eta) - \mathcal{A}u_2(\eta)| \leq \frac{1}{\Gamma(\alpha)} \int_0^\eta (\eta - \xi)^{\alpha-1} |g_1(\xi) - g_2(\xi)|\, d\xi. \tag{2.18}$$

We have

$$\|g_1 - g_2\|_\infty \leq \frac{\kappa^2 \Gamma(\alpha)}{\Gamma(\alpha) - \ell^{\alpha-2}(2\ell|\kappa| + \alpha - 1)} \|u_1 - u_2\|_\infty.$$

From (2.18) we find

$$\|\mathcal{A}u_1 - \mathcal{A}u_2\|_\infty \leq \frac{\kappa^2 \ell^\alpha}{\Gamma(\alpha+1) - \alpha\ell^{\alpha-2}(2\ell|\kappa| + \alpha - 1)} \|u_1 - u_2\|_\infty.$$

This implies that by (2.17), $\mathcal{A}$ is a contraction operator.

As a consequence Banach's contraction principle (see [23]), we deduce that $\mathcal{A}$ has a unique fixed point which is the unique solution of the problem (2.8)–(2.9) on J. The proof is complete. □

2.5 Proof of Main Theorems

In this part, we prove the existence and uniqueness of solutions of the following free boundary problem of the higher-order space-fractional wave equation

$$\begin{cases} \partial_t^2 \omega = \kappa^2 \partial_x^\alpha \omega, & (x,t) \in \Omega, \\ \omega(\kappa t, t) = c_0 \exp(-\kappa^2 t), & c_0 \in \mathbb{R}, \\ \partial_x^k \omega(\kappa t, t) = 0, & k \in \{1, 2, \ldots, m-1\}, \end{cases} \tag{2.19}$$

under the traveling wave form

$$\omega(x,t) = \exp\left(-\kappa^2 t\right) \varphi(\eta), \text{ with } \eta = x - \kappa t. \tag{2.20}$$

Proof of Theorem 2.1

The transformation (2.20) reduces the problem of the higher-order space-fractional wave equation (2.19) to the ordinary differential equation of fractional order of the form

$${}^{C}\mathcal{D}_{0+}^{\alpha} \varphi(\eta) = g(\eta), \tag{2.21}$$

where

$$g(\eta) = \kappa^2 \varphi(\eta) + 2\kappa \varphi'(\eta) + \varphi''(\eta),$$

with the conditions

$$\varphi(0) = c_0 \text{ and } \varphi^{(k)}(0) = 0, \text{ for } k \in \{1, 2, \ldots, m-1\}. \tag{2.22}$$

From (2.3) we get $\ell = X + |\kappa| T$, then the condition (2.6)

$$\kappa^2 (X + |\kappa| T)^\alpha + \alpha (X + |\kappa| T)^{\alpha-2} (2(X + |\kappa| T)|\kappa| + \alpha - 1) < \Gamma(\alpha+1),$$

becomes

$$\kappa^2 \ell^\alpha + \alpha \ell^{\alpha-2} (2\ell|\kappa| + \alpha - 1) < \Gamma(\alpha+1),$$

which is the condition (2.13).

We already proved the existence of a solution of the problem (2.21)–(2.22) in Theorem 2.4, provided that (2.13) holds true. Consequently, if (2.6) holds, then there exists at least one solution of the problem of the higher-order space-fractional wave equation (2.19) under the traveling wave form (2.20). The proof is complete.

Proof of Theorem 2.2

Based on Theorem 2.5, we use the same steps through which we proved Theorem 2.1 to prove the existence and uniqueness of a traveling wave solution to the problem (2.19), provided that the condition (2.7) holds true. The proof is complete.

CHAPTER 3

EXISTENCE OF TRAVELING WAVE SOLUTIONS FOR A CAUCHY PROBLEM OF JORDAN-MOORE-GIBSON-THOMPSON EQUATIONS

This chapter has been sent for publication.

3.1 Introduction and Statement of Results

In this chapter by applying the properties of Schauder's and Banach's fixed point theorems we examine the existence and uniqueness of solutions under the traveling wave forms for a free boundary problem of space-fractional Jordan-Moore-Gibson-Thompson (JMGT) equation as follows

$$\tau\omega_{ttt} + \mu\omega_{tt} - \kappa^2\partial_x^\alpha\omega - \delta\partial_x^\alpha\omega_t = F\left(x, t, \omega, \omega_t, \omega_{tt}, \omega_{xx}, (\omega_t)_{xx}\right), \text{ for } 1 < \alpha \leq 2, \tag{3.1}$$

this equation results from modeling high-frequency ultra sound waves and which describe sound propagation in thermo-viscous elastic terms, with

$$\partial_x^\alpha\omega = \begin{cases} \partial_x^2\omega, & \alpha = 2, \\ \mathcal{I}_{\kappa t}^{2-\alpha}\partial_x^2\omega = \frac{1}{\Gamma(2-\alpha)}\int_{\kappa t}^{x}(x-\tau)^{1-\alpha}\frac{\partial^2}{\partial\tau^2}\omega(\tau, t)\,d\tau, & 1 < \alpha < 2, \end{cases}$$

where the unknown scalar function $\omega = \omega(x, t)$ of a space and time variables $(x, t) \in \Omega$ with

$$\Omega = \{(x, t) \in \mathbb{R} \times [0, T]\,;\ \kappa t \leq x \leq \ell\}, \text{ for } T > 0 \text{ and } \ell > \kappa T,$$

denotes an acoustic velocity.

The fractional JMGT model (3.1) exhibits a variety of dynamical behaviors for solutions, which heavily depend on the positive physical parameters in the equation. To be specific concerning model (3.1), κ stands for the speed of sound, and τ denotes the thermal relaxation in the view of the physical context of acoustic waves. Moreover, the parameter δ concerns the diffusivity of the sound carrying. See the works of Moore, Gibson and Thompson [31], and Jordan [26], for a detailed insight into their derivation and physical background, and [27, 34, 13] for a selection of results that account for their mathematical analysis.

The space-fractional equation (3.1) appears as a generalization of the Kuznetsov equation (3.2) (see [17]), for $\mu = 1$, $\tau = 0$ and $\alpha = 2$,

$$\omega_{tt} - \kappa^2 \partial_x^2 \omega - \delta \partial_x^2 \omega_t = F\left(x, t, \omega, \omega_t, \omega_{tt}, \omega_{xx}, (\omega_t)_{xx}\right). \tag{3.2}$$

Both equations (3.1) and (3.2) are used as models in what is called nonlinear acoustics, and that deals with finite-amplitude wave propagation in fluids and solids and related phenomena. See the books of Beyer [12] or Rudenko and Soluyan [38].

Note that for $F \equiv 0$ and $\alpha = 2$, the PDE (3.1) represents the Moore-Gibson-Thompson equation:

$$\tau \omega_{ttt} + \mu \omega_{tt} - \kappa^2 \partial_x^2 \omega - \delta \partial_x^2 \omega_t = 0,$$

which have recently been approached from various points of view. The study of the controllability properties of Moore-Gibson-Thompson type equations can be found for instance in [14, 31].

We define the Cauchy problem for $1 < \alpha \leq 2$ as follows

$$\begin{cases} \tau \omega_{ttt} + \mu \omega_{tt} - \kappa^2 \partial_x^\alpha \omega - \delta \partial_x^\alpha \omega_t = F\left(x, t, \omega, \omega_t, \omega_{tt}, \omega_{xx}, (\omega_t)_{xx}\right), & (x,t) \in \Omega, \\ \omega(x,0) = \omega_0(x), \ \omega_t(x,0) = \omega_1(x), \ \omega_{tt}(x,0) = \omega_2(x), & \omega_0, \omega_1, \omega_2 \in \mathbb{C}, \\ \omega(\kappa t, t) = c_0 \exp\left(-\frac{\kappa^2}{\delta} t\right), \ \omega_x(\kappa t, t) = (\omega_t)_x(\kappa t, t) = 0, & \kappa > 0, \ c_0 \in \mathbb{C}, \end{cases} \tag{3.3}$$

where $\tau, \mu, \kappa, \delta \in \mathbb{R}_+^*$ and $F : \Omega \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a nonlinear function.

The major goal of this work is to determine the existence and uniqueness of the fractional-order's partial differential equation (3.1), under the traveling wave form

$$\omega(x,t) = \exp\left(-\frac{\kappa^2}{\delta} t\right) \varphi(x - \kappa t), \text{ with } \kappa, \delta \in \mathbb{R}_+^*. \tag{3.4}$$

The basic profile φ is not known in advance and is to be identified.

This method permits us to reduce the fractional-order's PDE (3.1) to a fractional differential equation; the idea is well illustrated with examples in our chapter. This approach (3.4) is promising and can also bring new results for other applications in FPDEs.

For the forthcoming analysis, we impose the following assumptions

$(A1)$ F is a continuous function that is invariant by the change of scale (3.4). It gives us

$$F\left(x,t,\omega,\omega_t,\omega_{tt},\omega_{xx},(\omega_t)_{xx}\right)=\exp\left(-\frac{\kappa^2}{\delta}t\right)\left(\delta\kappa f\left(\eta,\varphi,\varphi',\varphi''\right)-\kappa^3\tau\varphi'''\right), \tag{3.5}$$

where $\eta=x-\kappa t$ and $f:[0,\ell]\times\mathbb{C}\times\mathbb{C}\times\mathbb{C}\to\mathbb{C}$ is a continuous function.

$(A2)$ There exist three positive constants $\beta,\gamma,\lambda>0$ so that the function f given by (3.5) satisfies

$$\left|f\left(\eta,u,v,w\right)-f\left(\eta,\bar{u},\bar{v},\bar{w}\right)\right|\leq\beta\left|u-\bar{u}\right|+\gamma\left|v-\bar{v}\right|+\lambda\left|w-\bar{w}\right|,\ \forall\beta,\gamma,\lambda>0,$$

for any $u,v,w,\bar{u},\bar{v},\bar{w}\in\mathbb{C}$.

$(A3)$ There exist four nonnegative functions $a,b,c,d\in C\left([0,\ell],\mathbb{R}_+\right)$, such that

$$\left|f\left(\eta,u,v,w\right)\right|\leq a\left(\eta\right)+b\left(\eta\right)\left|u\right|+c\left(\eta\right)\left|v\right|+d\left(\eta\right)\left|w\right|,\ \forall\eta\in[0,\ell],$$

for any $u,v,w\in\mathbb{C}$ and $\eta\in[0,\ell]$.

We denote by ϖ the positive constant defined by

$$\varpi=\max\left\{\frac{\ell\left|q+\gamma\right|+\alpha\left|m+\lambda\right|}{\ell^{1-\alpha}\Gamma\left(\alpha+1\right)},\frac{\ell\left|q+c^*\right|+\alpha\left|m+d^*\right|}{\ell^{1-\alpha}\Gamma\left(\alpha+1\right)}\right\}.$$

Where $q=\frac{\kappa^2}{\delta^2}\left(\frac{3\tau\kappa^2}{\delta}-2\mu\right)$, $m=\frac{\kappa}{\delta}\left(\frac{3\tau\kappa^2}{\delta}-\mu\right)$, and

$$a^*=\sup_{\eta\in[0,\ell]}a\left(\eta\right),\ b^*=\sup_{\eta\in[0,\ell]}b\left(\eta\right),\ c^*=\sup_{\eta\in[0,\ell]}c\left(\eta\right),\ \text{and } d^*=\sup_{\eta\in[0,\ell]}d\left(\eta\right).$$

Throughout the rest of this chapter, we give $J=[0,\ell]$ and $p=\frac{\kappa^3}{\delta^3}\left(\frac{\tau\kappa^2}{\delta}-\mu\right)$.

Now, we give the principal theorems of this work.

Theorem 3.1. *Assume that the assumptions $(A1)-(A3)$ hold. If we put $\varpi\in(0,1)$ and*

$$\frac{\ell^{\alpha+1}\left|\frac{\kappa^3}{\delta^3}\left(\frac{\tau\kappa^2}{\delta}-\mu\right)+b^*\right|}{\Gamma\left(\alpha+2\right)\left(1-\varpi\right)}<1, \tag{3.6}$$

then, there is at least one solution of the Cauchy problem (3.3) on Ω in the traveling wave form (3.4).

Theorem 3.2. *Assume that the assumptions $(A1)$, $(A2)$ hold. If we put $\varpi\in(0,1)$ and*

$$\frac{\ell^{\alpha+1}\left|\frac{\kappa^3}{\delta^3}\left(\frac{\tau\kappa^2}{\delta}-\mu\right)+\beta\right|}{\Gamma\left(\alpha+2\right)\left(1-\varpi\right)}<1, \tag{3.7}$$

then the Cauchy problem (3.3) admits a unique solution in the traveling wave form (3.4) on Ω.

3.2 Compute of Traveling Wave Solutions

Our initial aim is to infer that the function φ in (3.4) satisfies an equation that is employed in the definition of traveling wave solutions.

Theorem 3.3. *If the assumption $(A1)$ holds, then the transformation (3.4) reduces the partial differential equation problem of space-fractional order (3.3) to the ordinary differential equation of fractional order of the form*

$$^{C}\mathcal{D}_{0^+}^{\alpha+1}\varphi\left(\eta\right)=g\left(\eta\right),\ \eta\in J, \tag{3.8}$$

where

$$g\left(\eta\right)=p\varphi\left(\eta\right)+q\varphi'\left(\eta\right)+m\varphi''\left(\eta\right)+f\left(\eta,\varphi\left(\eta\right),\varphi'\left(\eta\right),\varphi''\left(\eta\right)\right),$$

with the conditions

$$\varphi\left(0\right)=c_0 \text{ and } \varphi'\left(0\right)=\varphi''\left(0\right)=0. \tag{3.9}$$

Proof. The fractional equation resulting from the substitution of expression (3.4) in the original fractional-order's PDE (3.3), should be reduced to the standard bilinear functional equation (check [9, 10, 15, 25, 32, 37, 43]). First, for $\eta=x-\kappa t$, we get $\eta\in J$ and

$$\tau\omega_{ttt}+\mu\omega_{tt}=-\exp\left(-\frac{\kappa^2}{\delta}t\right)\left(\kappa\delta\left(p\varphi\left(\eta\right)+q\varphi'\left(\eta\right)+m\varphi''\left(\eta\right)\right)+\kappa^3\tau\varphi'''\left(\eta\right)\right). \tag{3.10}$$

On the other hand, for $\xi=\tau-\kappa t$, we get

$$\begin{aligned}\frac{\partial^\alpha\omega}{\partial x^\alpha}&=\int_{\kappa t}^{x}\frac{\left(x-\tau\right)^{1-\alpha}}{\Gamma\left(2-\alpha\right)}\frac{\partial^2\omega\left(\tau,t\right)}{\partial\tau^2}d\tau\\&=\exp\left(-\frac{\kappa^2}{\delta}t\right)\int_{\kappa t}^{x}\frac{\left(x-\tau\right)^{1-\alpha}}{\Gamma\left(2-\alpha\right)}\frac{d^2\varphi\left(\tau-\kappa t\right)}{d\tau^2}d\tau\\&=\exp\left(-\frac{\kappa^2}{\delta}t\right)\int_{0}^{\eta}\frac{\left(\eta-\xi\right)^{1-\alpha}}{\Gamma\left(2-\alpha\right)}\frac{d^2\varphi\left(\xi\right)}{d\xi^2}d\xi\\&=\exp\left(-\frac{\kappa^2}{\delta}t\right){}^{C}\mathcal{D}_{0^+}^{\alpha}\varphi\left(\eta\right).\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}\frac{\partial^\alpha\omega_t}{\partial x^\alpha}&=\int_{\kappa t}^{x}\frac{\left(x-\tau\right)^{1-\alpha}}{\Gamma\left(2-\alpha\right)}\frac{\partial^2\omega_t\left(\tau,t\right)}{\partial\tau^2}d\tau\\&=-\exp\left(-\frac{\kappa^2}{\delta}t\right)\int_{\kappa t}^{x}\frac{\left(x-\tau\right)^{1-\alpha}}{\Gamma\left(2-\alpha\right)}\frac{d^2}{d\tau^2}\left(\frac{\kappa^2}{\delta}\varphi\left(\tau-\kappa t\right)+\kappa\varphi'\left(\tau-\kappa t\right)\right)d\tau\\&=-\exp\left(-\frac{\kappa^2}{\delta}t\right)\left(\frac{\kappa^2}{\delta}\int_{0}^{\eta}\frac{\left(\eta-\xi\right)^{1-\alpha}}{\Gamma\left(2-\alpha\right)}\frac{d^2\varphi\left(\xi\right)}{d\xi^2}d\xi+\kappa\int_{0}^{\eta}\frac{\left(\eta-\xi\right)^{2-(\alpha+1)}}{\Gamma\left(3-\left(\alpha+1\right)\right)}\frac{d^3\varphi\left(\xi\right)}{d\xi^3}d\xi\right)\\&=-\exp\left(-\frac{\kappa^2}{\delta}t\right)\left(\frac{\kappa^2}{\delta}{}^{C}\mathcal{D}_{0^+}^{\alpha}\varphi\left(\eta\right)+\kappa{}^{C}\mathcal{D}_{0^+}^{\alpha+1}\varphi\left(\eta\right)\right).\end{aligned} \tag{3.12}$$

If we replace (3.5), (3.10), (3.11) and (3.12) in the first equation of (3.3), we get

$$^{C}\mathcal{D}_{0^+}^{\alpha+1}\varphi\left(\eta\right)=g\left(\eta\right).$$

From the conditions in (3.3), we find

$$\omega\left(\kappa t,t\right)=\exp\left(-\frac{\kappa^2}{\delta}t\right)\varphi\left(\kappa t-\kappa t\right)=\varphi\left(0\right)\exp\left(-\frac{\kappa^2}{\delta}t\right),$$

also

$$\omega_x\left(\kappa t,t\right)=\exp\left(-\frac{\kappa^2}{\delta}t\right)\varphi'\left(\kappa t-\kappa t\right)=\varphi'\left(0\right)\exp\left(-\frac{\kappa^2}{\delta}t\right),$$

and

$$\begin{aligned}\left(\omega_t\right)_x\left(\kappa t,t\right) &= -\left(\frac{\kappa^2}{\delta}\varphi'\left(\kappa t-\kappa t\right)+\kappa\varphi''\left(\kappa t-\kappa t\right)\right)\exp\left(-\frac{\kappa^2}{\delta}t\right)\\ &= -\left(\frac{\kappa^2}{\delta}\varphi'\left(0\right)+\kappa\varphi''\left(0\right)\right)\exp\left(-\frac{\kappa^2}{\delta}t\right),\end{aligned}$$

which implies that

$$\varphi\left(0\right)=c_0 \text{ and } \varphi'\left(0\right)=\varphi''\left(0\right)=0.$$

The proof is complete. □

3.3 Existence and Uniqueness Results

Lemma 3.1. *Assume that $f:J\times\mathbb{C}\times\mathbb{C}\times\mathbb{C}\to\mathbb{C}$ is a continuous function, then the problem (3.8)–(3.9) is equivalent to the integral equation*

$$\varphi\left(\eta\right)=c_0+\frac{1}{\Gamma\left(\alpha+1\right)}\int_0^{\eta}\left(\eta-\xi\right)^{\alpha}g\left(\xi\right)d\xi,\ \forall\eta\in J,$$

where $g\in C\left(J,\mathbb{C}\right)$ satisfies the functional equation

$$g\left(\eta\right)=p\left(c_0+\mathcal{I}_{0^+}^{\alpha+1}g\left(\eta\right)\right)+\psi\left(g\left(\eta\right)\right),$$

with $\psi:\mathbb{C}\to\mathbb{C}$ is a function satisfying

$$\psi\left(g\left(\eta\right)\right)=q\mathcal{I}_{0^+}^{\alpha}g\left(\eta\right)+m\mathcal{I}_{0^+}^{\alpha-1}g\left(\eta\right)+f\left(\eta,c_0+\mathcal{I}_{0^+}^{\alpha+1}g\left(\eta\right),\mathcal{I}_{0^+}^{\alpha}g\left(\eta\right),\mathcal{I}_{0^+}^{\alpha-1}g\left(\eta\right)\right).$$

Proof. Using Theorem 3.3, and applying $\mathcal{I}_{0^+}^{\alpha+1}$ to the equation (3.8), we obtain

$$\mathcal{I}_{0^+}^{\alpha+1}\,{}^{C}\mathcal{D}_{0^+}^{\alpha+1}\varphi\left(\eta\right)=\mathcal{I}_{0^+}^{\alpha+1}g\left(\eta\right).$$

From Lemma 1.1, we simply find

$$\mathcal{I}_{0^+}^{\alpha+1}\,{}^{C}\mathcal{D}_{0^+}^{\alpha+1}\varphi(\eta) = \varphi(\eta) - c_0 - \eta\varphi'(0) - \frac{1}{2}\eta^2\varphi''(0).$$

Substituting (3.9) gives us

$$\varphi(\eta) = c_0 + \mathcal{I}_{0^+}^{\alpha+1}g(\eta). \tag{3.13}$$

As

$$\varphi'(\eta) = \frac{d}{d\eta}\left(c_0 + \mathcal{I}_{0^+}^{\alpha+1}g(\eta)\right) = \mathcal{I}_{0^+}^{\alpha}g(\eta)$$

and

$$\varphi''(\eta) = \frac{d^2}{d\eta^2}\left(c_0 + \mathcal{I}_{0^+}^{\alpha+1}g(\eta)\right) = \mathcal{I}_{0^+}^{\alpha-1}g(\eta),$$

then

$$\begin{aligned} g(\eta) &= p\varphi(\eta) + q\varphi'(\eta) + m\varphi''(\eta) + f(\eta, \varphi(\eta), \varphi'(\eta), \varphi''(\eta)) \\ &= p\left(c_0 + \mathcal{I}_{0^+}^{\alpha+1}g(\eta)\right) + q\mathcal{I}_{0^+}^{\alpha}g(\eta) + m\mathcal{I}_{0^+}^{\alpha-1}g(\eta) \\ &\quad + f\left(\eta, c_0 + \mathcal{I}_{0^+}^{\alpha+1}g(\eta), \mathcal{I}_{0^+}^{\alpha}g(\eta), \mathcal{I}_{0^+}^{\alpha-1}g(\eta)\right) \\ &= p\left(c_0 + \mathcal{I}_{0^+}^{\alpha+1}g(\eta)\right) + \psi(g(\eta)). \end{aligned}$$

Otherwise, starting by applying ${}^{C}\mathcal{D}_{0^+}^{\alpha+1}$ on both sides of the equation (3.13) and using the linearity of Caputo's derivative and the fact that ${}^{C}\mathcal{D}_{0^+}^{\alpha+1}c_0 = 0$, we find easily (3.8). Furthermore;

$$\begin{aligned} \varphi(0) &= \left(c_0 + \mathcal{I}_{0^+}^{\alpha+1}g\right)(0) = c_0 \\ \varphi^{(k)}(0) &= \mathcal{I}_{0^+}^{\alpha-k-1}g(0) = 0, \text{ for each } k = 1, 2. \end{aligned}$$

The proof is complete. □

Theorem 3.4. *Assume the assumptions* $(A2)$, $(A3)$ *hold. If we put* $\varpi \in (0,1)$ *and*

$$\frac{\ell^{\alpha+1}|p + b^*|}{\Gamma(\alpha+2)(1-\varpi)} < 1, \tag{3.14}$$

then the problem (3.8)–(3.9) has at least one solution on J.

Proof. To begin the proof, we will transform the problem (3.8)–(3.9) into a fixed point problem. Let us define

$$\mathcal{A}u(\eta) = c_0 + \frac{1}{\Gamma(\alpha+1)}\int_0^\eta (\eta - \xi)^\alpha g(\xi)\,d\xi, \tag{3.15}$$

where

$$g(\eta) = pu(\eta) + \psi(g(\eta)), \ \eta \in J,$$

with

$$\psi(g(\eta)) = q\mathcal{I}_{0+}^{\alpha} g(\eta) + m\mathcal{I}_{0+}^{\alpha-1} g(\eta) + f\left(\eta, u(\eta), \mathcal{I}_{0+}^{\alpha} g(\eta), \mathcal{I}_{0+}^{\alpha-1} g(\eta)\right).$$

As the assumptions $(A2), (A3)$ hold, we notice that if $g \in C(J, \mathbb{C})$, then $\mathcal{A}u$ is indeed continuous (see the step 1 in this proof); therefore, it is an element of $C(J, \mathbb{C})$, and is equipped with the standard norm

$$\|\mathcal{A}u\|_{\infty} = \sup_{\eta \in J} |\mathcal{A}u(\eta)|.$$

Clearly, the fixed points of $\mathcal{A}$ are solutions of the problem (3.8)–(3.9).

We demonstrate that $\mathcal{A}$ satisfies the assumption of Schauder's fixed point theorem (see [23]). This could be proved through three steps.

Step 1: $\mathcal{A}$ is a continuous operator.

Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence such that $\lim_{n\to\infty} u_n = u$ in $C(J, \mathbb{C})$. Then $\forall \eta \in J$,

$$|\mathcal{A}u_n(\eta) - \mathcal{A}u(\eta)| \le \frac{1}{\Gamma(\alpha+1)} \int_0^{\eta} (\eta - \xi)^{\alpha} |g_n(\xi) - g(\xi)| \, d\xi, \tag{3.16}$$

where

$$\begin{cases} g_n(\eta) = pu_n(\eta) + \psi(g_n(\eta)), \\ g(\eta) = pu(\eta) + \psi(g(\eta)). \end{cases}$$

We have

$$\begin{aligned} |g_n(\eta) - g(\eta)| &= |p(u_n(\eta) - u(\eta)) + (\psi(g_n(\eta)) - \psi(g(\eta)))| \\ &\le |p+\beta| \|u_n - u\|_{\infty} + |q+\gamma| \left|\mathcal{I}_{0+}^{\alpha}(g_n(\eta) - g(\eta))\right| \\ &\quad + |m+\lambda| \left|\mathcal{I}_{0+}^{\alpha-1}(g_n(\eta) - g(\eta))\right|. \end{aligned}$$

As

$$\begin{aligned} \left|\mathcal{I}_{0+}^{\alpha}(g_n(\eta) - g(\eta))\right| &\le \frac{1}{\Gamma(\alpha)} \int_0^{\eta} (\eta - \xi)^{\alpha-1} |(g_n(\xi) - g(\xi))| \, d\xi \\ &\le \frac{\ell^{\alpha}}{\Gamma(\alpha+1)} \|g_n - g\|_{\infty} \end{aligned}$$

and

$$\left|\mathcal{I}_{0+}^{\alpha-1}(g_n(\eta) - g(\eta))\right| \le \frac{\alpha \ell^{\alpha-1}}{\Gamma(\alpha+1)} \|g_n - g\|_{\infty}.$$

Then we get

$$\begin{aligned}\|g_n - g\|_\infty &\leq |p+\beta| \|u_n - u\|_\infty + \frac{\ell |q+\gamma| + \alpha |m+\lambda|}{\ell^{1-\alpha}\Gamma(\alpha+1)} \|g_n - g\|_\infty \\ &\leq |p+\beta| \|u_n - u\|_\infty + \varpi \|g_n - g\|_\infty .\end{aligned}$$

As $\varpi \in (0,1)$, thus

$$\|g_n - g\|_\infty \leq \frac{|p+\beta|}{1-\varpi} \|u_n - u\|_\infty .$$

Since $u_n \to u$, we get $g_n \to g$ when $n \to \infty$.

Now, let $z > 0$ be such that for each $\eta \in J$, we get

$$|g_n(\eta)| \leq z, \ \ |g(\eta)| \leq z.$$

Then, we have

$$\begin{aligned}\frac{(\eta-\xi)^\alpha}{\Gamma(\alpha+1)} |g_n(\eta) - g(\eta)| &\leq \frac{(\eta-\xi)^\alpha}{\Gamma(\alpha+1)} [|g_n(\eta)| + |g(\eta)|] \\ &\leq \frac{2z}{\Gamma(\alpha+1)} (\eta-\xi)^\alpha .\end{aligned}$$

For each $\eta \in J$, the function $\xi \to \frac{2z}{\Gamma(\alpha+1)} (\eta-\xi)^\alpha$ is integrable on $[0,\eta]$, then the Lebesgue dominated convergence theorem and (3.16) imply that

$$|\mathcal{A}u_n(\eta) - \mathcal{A}u(\eta)| \to 0 \text{ as } n \to \infty,$$

and hence

$$\lim_{n\to\infty} \|\mathcal{A}u_n - \mathcal{A}u\|_\infty = 0.$$

Consequently, $\mathcal{A}$ is continuous.

Step 2: Using (3.14), we put the positive real

$$r \geq \left(|c_0| + \frac{a^* \ell^{\alpha+1}}{\Gamma(\alpha+2)(1-\varpi)}\right) \frac{\Gamma(\alpha+2)(1-\varpi)}{\Gamma(\alpha+2)(1-\varpi) - \ell^{\alpha+1}|p+b^*|},$$

and define the subset H as follows

$$H = \{u \in C(J,\mathbb{C}) : \|u\|_\infty \leq r\} .$$

It is clear that H is bounded, closed and convex subset of $C(J,\mathbb{C})$.

Let $\mathcal{A} : H \to C(J,\mathbb{C})$ be the integral operator defined by (3.15), then $\mathcal{A}(H) \subset H$. Indeed, we have for each $\eta \in J$

$$\begin{aligned}|g(\eta)| &= |pu(\eta) + \psi(g(\eta))| \\ &\leq a^* + |p+b^*| |u(\eta)| + \varpi \|g\|_\infty .\end{aligned}$$

Then, we get

$$\|g\|_{\infty} \leq \frac{a^* + |p + b^*| \, r}{1 - \varpi}.$$

Thus

$$\begin{aligned}
|\mathcal{A}u(\eta)| &\leq |c_0| + \frac{1}{\Gamma(\alpha+1)} \int_0^{\eta} (\eta - \xi)^{\alpha} |g(\xi)| \, d\xi \\
&\leq |c_0| + \frac{\ell^{\alpha+1}}{\Gamma(\alpha+2)} \frac{a^* + |p + b^*| \, r}{1 - \varpi} \\
&\leq |c_0| + \frac{a^* \ell^{\alpha+1}}{(1-\varpi)\Gamma(\alpha+2)} + \frac{\ell^{\alpha+1} |p + b^*| \, r}{(1-\varpi)\Gamma(\alpha+2)} \\
&\leq \frac{\left(|c_0| + \frac{a^* \ell^{\alpha+1}}{(1-\varpi)\Gamma(\alpha+2)}\right) \frac{\Gamma(\alpha+2)(1-\varpi)}{(1-\varpi)\Gamma(\alpha+2) - \ell^{\alpha+1}|p+b^*|}}{\frac{(1-\varpi)\Gamma(\alpha+2)}{(1-\varpi)\Gamma(\alpha+2) - \ell^{\alpha+1}|p+b^*|}} + \frac{\ell^{\alpha+1} |p + b^*| \, r}{(1-\varpi)\Gamma(\alpha+2)} \\
&\leq r.
\end{aligned}$$

Then $\mathcal{A}(H) \subset H$.

Step 3: $\mathcal{A}(H)$ is relatively compact.

Let $\eta_1, \eta_2 \in J$, $\eta_1 < \eta_2$, and $u \in H$. Then

$$\begin{aligned}
|\mathcal{A}u(\eta_2) - \mathcal{A}u(\eta_1)| &= \frac{1}{\Gamma(\alpha+1)} \left| \int_0^{\eta_2} (\eta_2 - \xi)^{\alpha} g(\xi) \, d\xi - \int_0^{\eta_1} (\eta_1 - \xi)^{\alpha} g(\xi) \, d\xi \right| \\
&\leq \frac{1}{\Gamma(\alpha+1)} \int_0^{\eta_1} |((\eta_2 - \xi)^{\alpha} - (\eta_1 - \xi)^{\alpha}) \, g(\xi)| \, d\xi \\
&\quad + \frac{1}{\Gamma(\alpha+1)} \int_{\eta_1}^{\eta_2} (\eta_2 - \xi)^{\alpha} |g(\xi)| \, d\xi \\
&\leq \frac{a^* + |p + b^*| \, r}{\Gamma(\alpha+1)(1-\varpi)} \left[\int_0^{\eta_1} |(\eta_2 - \xi)^{\alpha} - (\eta_1 - \xi)^{\alpha}| \, d\xi \right. \\
&\quad \left. + \int_{\eta_1}^{\eta_2} (\eta_2 - \xi)^{\alpha - 1} \, d\xi \right].
\end{aligned} \tag{3.17}$$

We have

$$(\eta_2 - \xi)^{\alpha} - (\eta_1 - \xi)^{\alpha} = -\frac{1}{\alpha+1} \frac{d}{d\xi} \left[(\eta_2 - \xi)^{\alpha+1} - (\eta_1 - \xi)^{\alpha+1} \right],$$

then

$$\int_0^{\eta_1} |(\eta_2 - \xi)^{\alpha} - (\eta_1 - \xi)^{\alpha}| \, d\xi \leq \frac{1}{\alpha+1} \left[(\eta_2 - \eta_1)^{\alpha+1} + \left(\eta_2^{\alpha+1} - \eta_1^{\alpha+1} \right) \right],$$

we also have

$$\int_{\eta_1}^{\eta_2} (\eta_2 - \xi)^{\alpha} \, d\xi = -\frac{1}{\alpha+1} \left[(\eta_2 - \xi)^{\alpha+1} \right]_{\eta_1}^{\eta_2} \leq \frac{1}{\alpha+1} (\eta_2 - \eta_1)^{\alpha+1}.$$

Thus (3.17) gives us

$$|\mathcal{A}u(\eta_2) - \mathcal{A}u(\eta_1)| \leq \frac{2(\eta_2-\eta_1)^{\alpha+1} + \left(\eta_2^{\alpha+1} - \eta_1^{\alpha+1}\right)}{\Gamma(\alpha+2)(1-\varpi)}\left(a^* + |p+b^*|\, r\right).$$

The right-hand side of the latter inequality tends to zero when $\eta_1 \to \eta_2$.

As a consequence of steps 1 to 3, and through Ascoli-Arzelà theorem, we infer the continuity of $\mathcal{A} : H \to H$, its compact nature and its satisfaction of the assumption of Schauder's fixed point theorem [23]. Therefore, $\mathcal{A}$ has a fixed point which solves the problem (3.8)–(3.9) on J. □

Example 3.1. *If we choose* $\tau = 1$, $\mu = 3$, $\alpha = \frac{3}{2}$, $\delta = 2$, $\kappa = 1$ *and* $\ell = 1$, *we get* $\Omega = \{(x,t) \in \mathbb{R} \times [0,1]\,;\ t \leq x \leq 1\}$. *Consequently, the considered problem will be stated as follows*

$$\begin{cases} \omega_{ttt} + 3\omega_{tt} - \partial_x^{\frac{3}{2}}\omega - 2\partial_x^{\frac{3}{2}}\omega_t = F\left(x,t,\omega,\omega_t,\omega_{tt},\omega_{xx},(\omega_t)_{xx}\right), & (x,t) \in \Omega, \\ \omega(x,0) = \omega_0(x),\ \omega_t(x,0) = \omega_1(x),\ \omega_{tt}(x,0) = \omega_2(x), & \omega_0,\omega_1,\omega_2 \in \mathbb{C}, \\ \omega(t,t) = c_0 \exp\left(-\frac{1}{2}t\right),\ \omega_x(t,t) = (\omega_t)_x(t,t) = 0, & c_0 \in \mathbb{C}, \end{cases} \tag{3.18}$$

where

$$F\left(x,t,\omega,\omega_t,\omega_{tt},\omega_{xx},(\omega_t)_{xx}\right) = \frac{\ln(x+e-t-1)\left[2\exp\left(-\frac{1}{2}t\right) + \frac{3}{2}|\omega| + |\omega_t| + |\omega_{xx}|\right]}{2\exp\left(x-\frac{1}{2}t\right)\left[\exp\left(-\frac{1}{2}t\right) + \frac{3}{2}|\omega| + |\omega_t| + |\omega_{xx}|\right]} + 2\omega_{tt} + 2(\omega_t)_{xx}.$$

The transformation

$$\omega(x,t) = \exp\left(-\frac{1}{2}t\right)\varphi(\eta),\ \textit{with}\ \eta = x-t,$$

reduces the partial differential equation problem of space-fractional order (3.18) to the ordinary differential equation of fractional order of the form

$$g(\eta) = -\frac{5}{16}\varphi(\eta) - \frac{9}{8}\varphi'(\eta) - \frac{3}{4}\varphi''(\eta) + f(\eta,\varphi,\varphi',\varphi''),\ \eta \in [0,1],$$

with the conditions

$$\varphi(0) = c_0\ \textit{and}\ \varphi'(0) = \varphi''(0) = 0,$$

where

$$p = -\frac{5}{16},\ q = -\frac{9}{8},\ \textit{and}\ m = -\frac{3}{4},$$

and

$$f(\eta,\varphi,\varphi',\varphi'') = \frac{\ln(\eta+e-1)\left[2 + |\varphi(\eta)| + |\varphi'(\eta)| + |\varphi''(\eta)|\right]}{4\exp(\eta)\left[1 + |\varphi(\eta)| + |\varphi'(\eta)| + |\varphi''(\eta)|\right]} + \frac{1}{4}\varphi(\eta) + \varphi'(\eta) + \frac{1}{2}\varphi''(\eta).$$

Because $\ln(\eta+e-1)$, $\exp(\eta)$ *are continuous positive functions* $\forall\eta\in[0,1]$, *the function* f *is jointly continuous. Then*

$$f(\eta,u,v,w)=\frac{\ln(\eta+e-1)\left[2+|u|+|v|+|w|\right]}{4\exp(\eta)\left[1+|u|+|v|+|w|\right]}+\frac{1}{4}u+v+\frac{1}{2}w,\ \eta\in[0,1],\ u,v,w\in\mathbb{C}.$$

Clearly, the function f *is jointly continuous. For any* $u,v,w,\bar{u},\bar{v},\bar{w}\in\mathbb{C}$ *and* $\eta\in[0,1]$, *we have*

$$|f(\eta,u,v,w)-f(\eta,\bar{u},\bar{v},\bar{w})|\leq\frac{1}{2}|u-\bar{u}|+\frac{5}{4}|v-\bar{v}|+\frac{3}{4}|w-\bar{w}|.$$

Therefore, the assumption $(A2)$ *is satisfied with*

$$\beta=\frac{1}{2},\ \gamma=\frac{5}{4}\ \textit{and}\ \lambda=\frac{3}{4}.$$

Also, we have

$$|f(\eta,u,v,w)|\leq\frac{\ln(\eta+e-1)}{4\exp(\eta)}\left(2+|u|+|v|+|w|\right)+\frac{1}{4}|u|+|v|+\frac{1}{2}|w|.$$

Thus, the assumption $(A3)$ *is satisfied with*

$$\begin{cases} a(\eta)=\frac{\ln(\eta+e-1)}{2\exp(\eta)}, \\ b(\eta)=\frac{\ln(\eta+e-1)}{4\exp(\eta)}+\frac{1}{4}, \\ c(\eta)=\frac{\ln(\eta+e-1)}{4\exp(\eta)}+1, \\ d(\eta)=\frac{\ln(\eta+e-1)}{4\exp(\eta)}+\frac{1}{2}. \end{cases}$$

We also have

$$a^*=\frac{1}{2},\ b^*=\frac{1}{2},\ c^*=\frac{5}{4},\ d^*=\frac{3}{4},$$

with

$$\varpi=\sup\left\{\frac{\ell|q+\gamma|+\alpha|m+\lambda|}{\ell^{1-\alpha}\Gamma(\alpha+1)},\frac{\ell|q+c^*|+\alpha|m+d^*|}{\ell^{1-\alpha}\Gamma(\alpha+1)}\right\}=\frac{\sqrt{\pi}}{6\pi}<1.$$

And the condition (3.6)

$$\frac{\ell^{\alpha+1}\left|\frac{\kappa^3}{\delta^3}\left(\frac{\tau\kappa^2}{\delta}-\mu\right)+b^*\right|}{\Gamma(\alpha+2)(1-\varpi)}=\frac{3\sqrt{\pi}}{30\pi-5\sqrt{\pi}}<1.$$

It follows from theorem 3.1, that the Cauchy problem (3.18) has at least one solution.

Theorem 3.5. *Assume the assumption* $(A2)$ *holds. If we put* $\varpi\in(0,1)$ *and*

$$\frac{\ell^{\alpha+1}|p+\beta|}{(1-\varpi)\Gamma(\alpha+2)}<1, \tag{3.19}$$

then the problem (3.8)–(3.9) admits a unique solution on J.

Proof. Theorem 3.4 states that (3.8)–(3.9) can be rendered a problem of a fixed point (3.15).

Let $u_1, u_2 \in C(J, \mathbb{C})$, then we get

$$\mathcal{A}u_1(\eta) - \mathcal{A}u_2(\eta) = \frac{1}{\Gamma(\alpha+1)} \int_0^{\eta} (\eta - \xi)^{\alpha} (g_1(\xi) - g_2(\xi))\, d\xi,$$

where

$$g_i(\eta) = pu_i(\eta) + \psi(g_i(\eta)), \text{ for } i = 1, 2,$$
$$\psi(g_i(\eta)) = q\mathcal{I}_{0^+}^{\alpha} g_i(\eta) + m\mathcal{I}_{0^+}^{\alpha-1} g_i(\eta) + f\left(\eta, u_i(\eta), \mathcal{I}_{0^+}^{\alpha} g_i(\eta), \mathcal{I}_{0^+}^{\alpha-1} g_i(\eta)\right).$$

Also

$$|\mathcal{A}u_1(\eta) - \mathcal{A}u_2(\eta)| \leq \frac{1}{\Gamma(\alpha+1)} \int_0^{\eta} (\eta - \xi)^{\alpha} |g_1(\xi) - g_2(\xi)|\, d\xi. \tag{3.20}$$

We have

$$\|g_1 - g_2\|_{\infty} \leq \frac{|p+\beta|}{1-\varpi} \|u_1 - u_2\|_{\infty}.$$

From (3.20) we find

$$\|\mathcal{A}u_1 - \mathcal{A}u_2\|_{\infty} \leq \frac{\ell^{\alpha+1} |p+\beta|}{(1-\varpi)\Gamma(\alpha+2)} \|u_1 - u_2\|_{\infty}.$$

Thus, according to (3.19), $\mathcal{A}$ is considered a contraction operator.

Banach's contraction principle (see [23]) helps us infer that $\mathcal{A}$ has only one fixed point which is the unique solution of the problem (3.8)–(3.9) on J. □

Example 3.2. *If we put* $\tau = 2$, $\mu = 8$, $\alpha = \frac{7}{4}$, $\delta = 3$, $\kappa = 1$ *and* $\ell = \frac{\pi}{3}$, *we get* $\Omega = \left\{(x,t) \in \mathbb{R} \times [0,1]\,;\ t \leq x \leq \frac{\pi}{3}\right\}$. *Thus, the studied problem will be written as follows*

$$\begin{cases} 2\omega_{ttt} + 8\omega_{tt} - \partial_x^{\frac{7}{4}}\omega - 3\partial_x^{\frac{7}{4}}\omega_t = F\left(x, t, \omega, \omega_t, \omega_{tt}, \omega_{xx}, (\omega_t)_{xx}\right), & (x,t) \in \Omega, \\ \omega(x,0) = \omega_0(x),\ \omega_t(x,0) = \omega_1(x),\ \omega_{tt}(x,0) = \omega_2(x), & \omega_0, \omega_1, \omega_2 \in \mathbb{C}, \\ \omega(t,t) = c_0 \exp\left(-\frac{1}{3}t\right),\ \omega_x(t,t) = (\omega_t)_x(t,t) = 0, & c_0 \in \mathbb{C}, \end{cases} \tag{3.21}$$

where $\Omega = \left\{(x,t) \in \mathbb{R} \times [0,1]\,;\ t \leq x \leq \frac{\pi}{3}\right\}$ *and*

$$F\left(x, t, \omega, \omega_t, \omega_{tt}, \omega_{xx}, (\omega_t)_{xx}\right) = \frac{3\exp\left(-\frac{2}{3}t\right)\cos(x-t)}{\exp\left(-\frac{1}{3}t\right) + \frac{4}{3}|\omega| + |\omega_t| + |\omega_{xx}|} + 3\omega_{tt} + 3(\omega_t)_{xx},$$

The transformation

$$\omega(x,t) = \exp\left(-\frac{1}{3}t\right)\varphi(\eta),\ \textit{with}\ \eta = x - t,$$

reduces the partial differential equation problem of space-fractional order (3.18) to the ordinary differential equation of fractional order of the form

$$g(\eta) = -\frac{23}{81}\varphi(\eta) - \frac{5}{3}\varphi'(\eta) - 2\varphi''(\eta) + f(\eta, \varphi, \varphi', \varphi''),\ \eta \in \left[0, \frac{\pi}{3}\right],$$

with the conditions

$$\varphi(0) = c_0 \textit{ and } \varphi'(0) = \varphi''(0) = 0,$$

where

$$p = -\frac{23}{81},\ q = -\frac{5}{3},\ \textit{and } m = -2,$$

and

$$f(\eta, \varphi, \varphi', \varphi'') = \frac{\cos(\eta)}{1 + |\varphi(\eta)| + |\varphi'(\eta)| + |\varphi''(\eta)|} + \frac{1}{9}\varphi(\eta) + \frac{2}{3}\varphi'(\eta) + \frac{2}{3}\varphi''(\eta).$$

Because $\cos(\eta)$ *is continuous positive function* $\forall \eta \in \left[0, \frac{\pi}{3}\right]$, *the function* f *is jointly continuous. Then*

$$f(\eta, u, v, w) = \frac{\cos(\eta)}{1 + |u| + |v| + |w|} + \frac{1}{9}u + \frac{2}{3}v + \frac{2}{3}w,\ \eta \in [0, 1],\ u, v, w \in \mathbb{C}.$$

Clearly, the function f *is jointly continuous. For any* $u, v, w, \bar{u}, \bar{v}, \bar{w} \in \mathbb{C}$ *and* $\eta \in \left[0, \frac{\pi}{3}\right]$, *we have*

$$|f(\eta, u, v, w) - f(\eta, \bar{u}, \bar{v}, \bar{w})| \leq \frac{10}{9}|u - \bar{u}| + \frac{5}{3}|v - \bar{v}| + \frac{5}{3}|w - \bar{w}|.$$

Therefore, the assumption $(A2)$ *is satisfied with*

$$\beta = \frac{10}{9},\ \gamma = \frac{5}{3} \textit{ and } \lambda = \frac{5}{3}.$$

Also, we have

$$\varpi = \frac{\ell|q + \gamma| + \alpha|m + \lambda|}{\ell^{1-\alpha}\Gamma(\alpha + 1)} = \frac{\frac{4}{9}\left(\frac{\pi}{3}\right)^{\frac{3}{4}}}{\Gamma\left(\frac{3}{4}\right)} < 1.$$

What remains is to show that the condition (3.7)

$$\frac{\ell^{\alpha+1}\left|\frac{\kappa^3}{\delta^3}\left(\frac{\tau\kappa^2}{\delta} - \mu\right) + \beta\right|}{\Gamma(\alpha + 2)(1 - \varpi)} \simeq 0.345 < 1,$$

is satisfied. It follows from theorem 3.2 that the Cauchy problem (3.21) has a unique solution.

3.4 Main Theorems' Proof

This section demonstrates the proof of the existence and uniqueness of solutions of the given Cauchy problem for a space-fractional JMGT equation of nonlinear acoustics, which is

$$\begin{cases} \tau\omega_{ttt} + \mu\omega_{tt} - \kappa^2\partial_x^\alpha\omega - \delta\partial_x^\alpha\omega_t = F\left(x, t, \omega, \omega_t, \omega_{tt}, \omega_{xx}, (\omega_t)_{xx}\right), & (x, t) \in \Omega, \\ \omega(x, 0) = \omega_0(x),\ \omega_t(x, 0) = \omega_1(x),\ \omega_{tt}(x, 0) = \omega_2(x), & \omega_0, \omega_1, \omega_2 \in \mathbb{C}, \\ \omega(\kappa t, t) = c_0 \exp\left(-\frac{\kappa^2}{\delta}t\right),\ \omega_x(\kappa t, t) = (\omega_t)_x(\kappa t, t) = 0, & \kappa > 0,\ c_0 \in \mathbb{C}, \end{cases} \tag{3.22}$$

under the traveling wave form

$$\omega(x,t) = \exp\left(-\frac{\kappa^2}{\delta}t\right)\varphi(x-\kappa t), \text{ with } \kappa,\delta \in \mathbb{R}_+^*. \tag{3.23}$$

Proof of Theorem 3.1

Assume the assumptions $(A1)-(A3)$ hold. Using transformation (3.23), the Cauchy problem (3.22) is reduced to fractional order's ordinary differential equation of the form

$$^{C}\mathcal{D}_{0^+}^{\alpha+1}\varphi(\eta) = g(\eta), \ \eta \in J, \tag{3.24}$$

where

$$g(\eta) = p\varphi(\eta) + q\varphi'(\eta) + m\varphi''(\eta) + f(\eta, \varphi(\eta), \varphi'(\eta), \varphi''(\eta)),$$

with

$$p = \frac{\kappa^3}{\delta^3}\left(\frac{\tau\kappa^2}{\delta} - \mu\right), \ q = \frac{\kappa^2}{\delta^2}\left(\frac{3\tau\kappa^2}{\delta} - 2\mu\right) \text{ and } m = \frac{\kappa}{\delta}\left(\frac{3\tau\kappa^2}{\delta} - \mu\right), \tag{3.25}$$

along with the conditions

$$\varphi(0) = c_0 \text{ and } \varphi'(0) = \varphi''(0) = 0. \tag{3.26}$$

By using (3.25), the condition (3.6) is equivalent to (3.14), which is

$$\frac{\ell^{\alpha+1}|p+b^*|}{\Gamma(\alpha+2)(1-\varpi)} < 1, \text{ with } \varpi \in (0,1).$$

Therefore, after proving that problem (3.24)–(3.26) has a solution in Theorem 3.4 when (3.14) holds, we can similarly prove the existence of at least a solution of the Cauchy problem for the space-fractional JMGT equation of nonlinear acoustics (3.22) under the traveling wave form (3.23). This can be achieved if (3.6) holds. The proof is complete.

Proof of Theorem 3.2

Similarly to the steps that we followed during the proof of Theorem 3.1, the existence and uniqueness of a traveling wave solution to problem (3.22) is demonstrated using Theorem 3.5, provided that the condition (3.7) holds true. The proof is complete.

3.5 Explicit Solutions

In this section, we present some explicit solutions on the traveling wave form of the Cauchy problem (3.22)

Solution 1: Let $p, q, m \in \mathbb{R}$, $y > 1$ and $\beta, \gamma, \lambda \in \mathbb{R}_+^*$, for $1 < \alpha \leq 2$, we get

$$\varphi(\eta) = \eta^y,$$

which represents a solution of the problem (3.24)–(3.26), where

$$f\left(\eta,\varphi\left(\eta\right),\varphi'\left(\eta\right),\varphi''\left(\eta\right)\right)=\frac{\Gamma\left(y+1\right)}{\Gamma\left(y-\alpha\right)}\eta^{y-\alpha-1}-p\varphi\left(\eta\right)-q\varphi'\left(\eta\right)-m\varphi''\left(\eta\right).$$

Then the solution of problem (3.22) is presented as follows

$$\omega\left(x,t\right)=\exp\left(-\frac{\kappa^{2}}{\delta}t\right)\left(x-\kappa t\right)^{y},$$

where

$$\begin{aligned}F\left(x,t,\omega,\omega_{t},\omega_{tt},\omega_{xx},\left(\omega_{t}\right)_{xx}\right)=&\frac{2\tau\kappa^{6}}{\delta^{3}}\omega+\frac{3\tau\kappa^{4}}{\delta^{2}}\omega_{t}+\mu\omega_{tt}-\frac{2\tau\kappa^{4}}{\delta}\omega_{xx}+\kappa^{2}\tau\left(\omega_{t}\right)_{xx}\\&+\exp\left(-\frac{\kappa^{2}}{\delta}t\right)\frac{\delta\kappa\Gamma\left(y+1\right)}{\Gamma\left(y-\alpha\right)}\left(x-\kappa t\right)^{y-\alpha-1}.\end{aligned}$$

Solution 2: Let $p,q,m\in\mathbb{R}$, $y\in\mathbb{C}$ and $\beta,\gamma,\lambda\in\mathbb{R}_{+}^{*}$, for $1<\alpha\leq 2$, we get

$$\varphi\left(\eta\right)=\sin\left(y\eta\right)-y\eta,$$

which represents a solution of the problem (3.24)–(3.26), where

$$f\left(\eta,\varphi\left(\eta\right),\varphi'\left(\eta\right),\varphi''\left(\eta\right)\right)=-\frac{1}{2}y^{3}\eta^{3-\alpha}\left(E_{1,4-\alpha}\left(iy\eta\right)+E_{1,4-\alpha}\left(-iy\eta\right)\right)-p\varphi\left(\eta\right)-q\varphi'\left(\eta\right)-m\varphi''\left(\eta\right).$$

Where $E_{1,4-\alpha}\left(\eta\right)$ is the function of Mittag-Leffler type. Then the solution of problem (3.22) is presented as follows

$$\omega\left(x,t\right)=\exp\left(-\frac{\kappa^{2}}{\delta}t\right)\left[\sin\left(yx-y\kappa t\right)-y\left(x-\kappa t\right)\right].$$

Where

$$\begin{aligned}F\left(x,t,\omega,\omega_{t},\omega_{tt},\omega_{xx},\left(\omega_{t}\right)_{xx}\right)=&-\frac{1}{2}y^{3}\left(x-\kappa t\right)^{3-\alpha}\exp\left(-\frac{\kappa^{2}}{\delta}t\right)E_{1,4-\alpha}\left(iy\left(x-\kappa t\right)\right)+\frac{\kappa^{2}\tau}{\delta}\omega_{tt}\\&-\frac{y^{3}\left(x-\kappa t\right)^{3-\alpha}E_{1,4-\alpha}\left(-iy\left(x-\kappa t\right)\right)}{2}\exp\left(-\frac{\kappa^{2}}{\delta}t\right)+\kappa^{2}\tau\left(\omega_{t}\right)_{xx}\\&+\left(\frac{3\tau\kappa^{6}}{\delta^{3}}-\frac{\kappa^{4}\mu}{\delta^{2}}\right)\omega+\left(\frac{5\kappa^{4}\tau}{\delta^{2}}-\frac{2\kappa^{2}\mu}{\delta}\right)\omega_{t}-\delta\kappa m\omega_{xx}.\end{aligned}$$

Solution 3: Let $p,q\in\mathbb{R}$, $\gamma\in\mathbb{C}$ and $\beta,\gamma,\lambda\in\mathbb{R}_{+}^{*}$, for $1<\alpha\leq 2$,we get

$$\varphi\left(\eta\right)=2\cos\left(y\eta\right)+y^{2}\eta^{2},$$

which represents a solution of the problem (3.24)–(3.26), where

$$f\left(\eta,\varphi\left(\eta\right),\varphi'\left(\eta\right),\varphi''\left(\eta\right)\right)=-\frac{i}{2}y^{3}\eta^{3-\alpha}\left(E_{1,4-\alpha}\left(iy\eta\right)-E_{1,4-\alpha}\left(-iy\eta\right)\right)-p\varphi\left(\eta\right)-q\varphi'\left(\eta\right)-m\varphi''\left(\eta\right).$$

Where $E_{1,4-\alpha}(\eta)$ is the function of Mittag-Leffler type. Then the solution of problem (3.22) is presented as follows

$$\omega(x,t) = 2\exp\left(-\frac{\kappa^2}{\delta}t\right)\cos(yx - y\kappa t) + y^2\exp\left(-\frac{\kappa^2}{\delta}t\right)(x-\kappa t)^2 .$$

Where

$$\begin{aligned} F\left(x,t,\omega,\omega_t,\omega_{tt},\omega_{xx},(\omega_t)_{xx}\right) = &-\frac{i}{2}y^3(x-\kappa t)^{3-\alpha}\exp\left(-\frac{\kappa^2}{\delta}t\right)E_{1,4-\alpha}(iy(x-\kappa t)) \\ &-\frac{i}{2}y^3(x-\kappa t)^{3-\alpha}\exp\left(-\frac{\kappa^2}{\delta}t\right)E_{1,4-\alpha}(-iy(x-\kappa t)) \\ &+\frac{2\tau\kappa^6}{\delta^3}\omega + \frac{3\tau\kappa^4}{\delta^2}\omega_t + \mu\omega_{tt} - \frac{2\tau\kappa^4}{\delta}\omega_{xx} + \kappa^2\tau(\omega_t)_{xx} . \end{aligned}$$

CHAPTER 4

ANALYTICAL STUDIES ON THE GLOBAL EXISTENCE AND BLOW-UP OF SOLUTIONS FOR A FREE BOUNDARY PROBLEM OF TWO DIMENSIONAL DIFFUSION EQUATIONS OF MOVING FRACTIONAL ORDER

This chapter has been sent for publication.

4.1 Introduction

This chapter particularly addresses and discusses some analytical studies on the existence and uniqueness of global or blow-up solutions under the traveling profile forms for a free boundary problem of diffusion equations of moving fractional order as follows

$$\partial_t \omega = \kappa \partial_x^\alpha \omega, \ \kappa \in \mathbb{R}^*, \ m-1 < \alpha \leq m \in \mathbb{N} - \{0,1\}, \tag{4.1}$$

where $\partial_x^\alpha \omega = \mathcal{I}_{b(t)}^{m-\alpha} \partial_x^m \omega$, with b is a real function of time t. Also $\omega = \omega(x,t)$ is a scalar function of a space and time variables $(x,t) \in \Omega$ with

$$\Omega = \{(x,t) \in \mathbb{R} \times [0,T]\, ; \ b(t) \leq x \leq a(t) + b(t)\},$$

$a(t) > 0$ for any $t \in [0,T]$ and T may be an infinite or a finite positive constant.

It does so by applying the properties of Schauder′s and Banach′s fixed point theorems.

The equation (4.1) becomes the transport equation for $\alpha = 1$ and the linear dispersive equations of Airy type for $\alpha = 3.$

Therefore, for $m = 2,$ the space-fractional diffusion equation (4.1) becomes a space-fractional heat equation, in which the existence problems of its self-similar solutions and its scale-invariant solutions have been discussed in [15, 32, 9].

Our main goal in this work is to determine the existence, uniqueness and main properties of the global or blow-up solution in time of the fractional-order's PDE (4.1), under the traveling profile form (see [11, 46]), which is

$$\omega(x,t) = c(t)\varphi\left(\frac{x - b(t)}{a(t)}\right), \text{ with } a, c \in \mathbb{R}_+^*,\ b \in \mathbb{R}, \tag{4.2}$$

the functions $a(t), b(t)$ and $c(t)$ depend on time t and the basic profile φ are not known in advance and are to be identified.

This method permits us to reduce the fractional-order's PDE (4.1) to a fractional differential equation; the idea is well illustrated with examples in our chapter. This approach (4.2) is very promising and can also bring new results for other applications in FPDEs.

4.2 Main Results

Throughout the rest of this paper, we have $J = [0, 1]$ and $m - 1 < \alpha \leq m$, with $m \geq 2$ is a natural number, $\kappa \in \mathbb{R}^*$ and $\lambda, \beta, \gamma, c_0, c_1 \in \mathbb{R}.$ Also the functions $a(t), b(t)$ and $c(t)$ depend on time t given by (4.2).

Statement of the Free Boundary Problem

In this part, we first attempt to find the equivalent approximate to the following free boundary problem of the diffusion equation of moving fractional order

$$\begin{cases} \partial_t \omega = \kappa \partial_x^\alpha \omega,\ (x,t) \in \Omega, & \kappa \in \mathbb{R}^*, \\ \omega(b(t), t) = c_0 c(t), & c_0 \in \mathbb{R}, \\ \partial_x \omega(b(t), t) = c_1 \frac{c(t)}{a(t)}, & c_1 \in \mathbb{R}, \\ \partial_x^k \omega(b(t), t) = 0,\ k = 2, 3, \ldots, m-1, & \text{for } m \geq 3, \end{cases} \tag{4.3}$$

under the traveling profile form

$$\omega(x,t) = c(t)\varphi(\eta), \text{ with } \eta = \frac{x - b(t)}{a(t)} \text{ and } a, c \in \mathbb{R}_+^*,\ b \in \mathbb{R}, \tag{4.4}$$

where

$$a(0) = c(0) = 1,\ b(0) = 0.$$

Main Theorems

Now, we give the principal theorems of this work.

Theorem 4.1. *Let $a(t)$, $b(t)$ and $c(t)$ be three real functions of time t, given by the traveling profile form (4.4). If*

$$\frac{a^{\alpha}(t)}{|\kappa|}\left[\alpha\left(\left|\frac{\dot{a}(t)}{a(t)}\right|+\left|\frac{\dot{b}(t)}{a(t)}\right|\right)+\left|\frac{\dot{c}(t)}{c(t)}\right|\right]<\Gamma(\alpha+1), \tag{4.5}$$

then the problem (4.3) has at least one solution in the traveling profile form (4.4), which is global in time when $\dot{a}(t)>0$, and it blows up in a finite time

$$0<t<T=-\frac{a^{1-\alpha}(t)}{\alpha\dot{a}(t)} \text{ when } \dot{a}(t)<0 \text{ and } \dot{c}(t)>0.$$

Theorem 4.2. *Let $a(t)$, $b(t)$ and $c(t)$ be three real functions of time t, given by the traveling profile form (4.4). If we put $\frac{\alpha a^{\alpha}(t)}{|\kappa|}\left(\left|\frac{\dot{a}(t)}{a(t)}\right|+\left|\frac{\dot{b}(t)}{a(t)}\right|\right)<\Gamma(\alpha+1)$ and*

$$\frac{\frac{a^{\alpha}(t)}{|\kappa|}\left|\frac{\dot{c}(t)}{c(t)}\right|}{\Gamma(\alpha+1)-\frac{\alpha a^{\alpha}(t)}{|\kappa|}\left(\left|\frac{\dot{a}(t)}{a(t)}\right|+\left|\frac{\dot{b}(t)}{a(t)}\right|\right)}<1, \tag{4.6}$$

then the problem (4.3) admits a unique solution in the traveling profile form (4.4), which is global in time when $\dot{a}(t)>0$, and it blows up in a finite time

$$0<t<T=-\frac{a^{1-\alpha}(t)}{\alpha\dot{a}(t)} \text{ when } \dot{a}(t)<0 \text{ and } \dot{c}(t)>0.$$

4.3 Compute of Traveling Profile Solutions

We should first deduce the equation satisfied by the function φ in (4.4) and used for the definition of traveling profile solutions.

Theorem 4.3. *The transformation (4.4) reduces the partial differential equation problem of space-fractional order (4.3) to the ordinary differential equation of fractional order of the form*

$$^{C}\mathcal{D}_{0+}^{\alpha}\varphi(\eta)=g(\eta), \eta\in J, \tag{4.7}$$

where

$$g(\eta)=\lambda\varphi(\eta)+(\beta\eta+\gamma)\varphi'(\eta),$$

with the conditions

$$\begin{cases}\varphi(0)=c_0,\ \varphi'(0)=c_1, \text{ for any } m\geq 2,\\ \varphi^{(k)}(0)=0,\ k=2,3,\ldots,m-1, \text{ for } m\geq 3,\end{cases} \tag{4.8}$$

where

$$(\lambda,\beta,\gamma)=\frac{a^{\alpha}(t)}{\kappa}\left(\frac{\dot{c}(t)}{c(t)},-\frac{\dot{a}(t)}{a(t)},-\frac{\dot{b}(t)}{a(t)}\right)\text{, for some }\lambda,\beta,\gamma\in\mathbb{R}. \tag{4.9}$$

Proof. The fractional equation resulting from the substitution of expression (4.4) in the original fractional-order's PDE (4.3), should be reduced to the standard bilinear functional equation (see [11]). First, for $\eta=\frac{x-b(t)}{a(t)}$, we get $\eta\in J$ and

$$\partial_t\omega(x,t)=\dot{c}(t)\varphi(\eta)-c(t)\frac{\dot{a}(t)}{a(t)}\eta\varphi'(\eta)-c(t)\frac{\dot{b}(t)}{a(t)}\varphi'(\eta). \tag{4.10}$$

In another way, we get for $a(t)\xi=\tau-b(t)$ that

$$\begin{aligned}\kappa\partial_x^{\alpha}\omega(x,t)&=\kappa\mathcal{I}_{b(t)}^{m-\alpha}\partial_x^{m}\omega(x,t)\\&=\frac{\kappa c(t)}{\Gamma(m-\alpha)}\int_{b(t)}^{x}(x-\tau)^{m-1-\alpha}\frac{d^m}{d\tau^m}\varphi\left(\frac{\tau-b(t)}{a(t)}\right)d\tau\\&=\frac{\kappa c(t)a^{-\alpha}(t)}{\Gamma(m-\alpha)}\int_0^{\eta}(\eta-\xi)^{m-1-\alpha}\frac{d^m}{d\xi^m}\varphi(\xi)\,d\xi\\&=\kappa c(t)a^{-\alpha}(t)\,{}^{C}\mathcal{D}_{0^+}^{\alpha}\varphi(\eta).\end{aligned} \tag{4.11}$$

If we replace (4.10) and (4.11) in the first equation of (4.3), we get

$$\begin{aligned}{}^{C}\mathcal{D}_{0^+}^{\alpha}\varphi(\eta)&=\frac{a^{\alpha}(t)}{\kappa}\left(\frac{\dot{c}(t)}{c(t)}\varphi(\eta)-\frac{\dot{a}(t)}{a(t)}\eta\varphi'(\eta)-\frac{\dot{b}(t)}{a(t)}\varphi'(\eta)\right)\\&=\lambda\varphi(\eta)+(\beta\eta+\gamma)\varphi'(\eta)\\&=g(\eta).\end{aligned}$$

From the conditions in (4.3), we find for each $k\in\{1,\ldots,m-1\}$ that

$$\begin{aligned}\omega(b(t),t)&=c(t)\varphi\left(\frac{b(t)-b(t)}{a(t)}\right)=\varphi(0)c(t),\\\partial_x^k\omega(b(t),t)&=\frac{c(t)}{a^k(t)}\varphi^{(k)}\left(\frac{b(t)-b(t)}{a(t)}\right)=\varphi^{(k)}(0)\frac{c(t)}{a^k(t)},\end{aligned}$$

which implies that

$$\varphi(0)=c_0,\ \varphi'(0)=c_1\text{ and }\varphi^{(k)}(0)=0,\ \text{for any }k\in\{2,\ldots,m-1\}.$$

The proof is complete. □

4.4 Existence and Uniqueness Results of the Basic Profile

In what follows, we present some significant lemmas to show the principal theorems.

Lemma 4.1. *The problem (4.7)–(4.8) is equivalent to the integral equation*

$$\varphi(\eta) = c_0 + c_1\eta + \mathcal{I}_{0+}^{\alpha} g(\eta), \ \forall \eta \in J. \tag{4.12}$$

where $g \in C(J, \mathbb{R})$ *satisfies the functional equation:*

$$g(\eta) = \lambda\left(c_0 + c_1\eta + \mathcal{I}_{0+}^{\alpha} g(\eta)\right) + (\beta\eta + \gamma)\left(c_1 + \mathcal{I}_{0+}^{\alpha-1} g(\eta)\right).$$

Proof. Using Theorem 4.3, and applying $\mathcal{I}_{0+}^{\alpha}$ to the equation (4.7), we obtain

$$\mathcal{I}_{0+}^{\alpha}\ {}^{C}\mathcal{D}_{0+}^{\alpha}\varphi(\eta) = \mathcal{I}_{0+}^{\alpha} g(\eta).$$

From Lemma 1.1, we simply find

$$\mathcal{I}_{0+}^{\alpha}\ {}^{C}\mathcal{D}_{0+}^{\alpha}\varphi(\eta) = \varphi(\eta) - \sum_{k=0}^{m-1} \frac{\varphi^{(k)}(0)}{k!}\eta^{k}, \ m-1 < \alpha \leq m \in \mathbb{N}^{*}.$$

Substituting (4.8) gives us

$$\varphi(\eta) = \left(c_0 + c_1\eta + \mathcal{I}_{0+}^{\alpha} g(\eta)\right). \tag{4.13}$$

And

$$\varphi'(\eta) = \frac{d}{d\eta}\left(c_0 + c_1\eta + \mathcal{I}_{0+}^{\alpha} g(\eta)\right) = c_1 + \mathcal{I}_{0+}^{\alpha-1} g(\eta),$$

then

$$\begin{aligned} g(\eta) &= \lambda\varphi(\eta) + (\beta\eta + \gamma)\varphi'(\eta) \\ &= \lambda\left(c_0 + c_1\eta + \mathcal{I}_{0+}^{\alpha} g(\eta)\right) + (\beta\eta + \gamma)\left(c_1 + \mathcal{I}_{0+}^{\alpha-1} g(\eta)\right). \end{aligned}$$

Otherwise, starting by applying ${}^{C}\mathcal{D}_{0+}^{\alpha}$ on both sides of the equation (4.13) and using the linearity of Caputo's derivative and the fact that ${}^{C}\mathcal{D}_{0+}^{\alpha}(c_0 + c_1\eta) = 0$, we find easily (4.7). Furthermore;

$$\begin{aligned} \varphi(0) &= \left(c_0 + c_1\eta + \mathcal{I}_{0+}^{\alpha} g\right)(0) = c_0 \\ \varphi'(0) &= \left(c_1 + \mathcal{I}_{0+}^{\alpha-1} g\right)(0) = c_1 \\ \varphi^{(k)}(0) &= \mathcal{I}_{0+}^{\alpha-k} g(0) = 0, \text{ for any } k \in \{2, \ldots, m-1\}. \end{aligned}$$

The proof is complete. □

Theorem 4.4. *If we put*

$$\alpha(|\beta| + |\gamma|) + |\lambda| < \Gamma(\alpha + 1), \tag{4.14}$$

then the problem (4.7)–(4.8) has at least one solution on J.

Proof. To begin the proof, we will transform the problem (4.7)–(4.8) into a fixed point problem. Let us define

$$\mathcal{A}u(\eta) = c_0 + c_1\eta + \frac{1}{\Gamma(\alpha)} \int_0^\eta (\eta-\xi)^{\alpha-1} g(\xi)\, d\xi, \tag{4.15}$$

where

$$g(\eta) = \lambda u(\eta) + (\beta\eta+\gamma)\left(c_1 + \mathcal{I}_{0^+}^{\alpha-1} g(\eta)\right).$$

We first notice that if $g \in C(J,\mathbb{R})$, then $\mathcal{A}u$ is indeed continuous (see the step 1 in this proof); therefore, it is an element of $C(J,\mathbb{R})$, and is equipped with the standard norm

$$\|\mathcal{A}u\|_\infty = \sup_{\eta\in J} |\mathcal{A}u(\eta)|.$$

Clearly, the fixed points of $\mathcal{A}$ are solutions of the problem (4.7)–(4.8).

We demonstrate that $\mathcal{A}$ satisfies the assumption of Schauder's fixed point theorem (see [23]). This could be proved through three steps.

Step 1: $\mathcal{A}$ is a continuous operator.

Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence such that $\lim_{n\to\infty} u_n = u$ in $C(J,\mathbb{R})$. Then $\forall\eta\in J$,

$$|\mathcal{A}u_n(\eta) - \mathcal{A}u(\eta)| \le \int_0^\eta \frac{(\eta-\xi)^{\alpha-1}}{\Gamma(\alpha)} |g_n(\xi) - g(\xi)|\, d\xi, \tag{4.16}$$

where

$$\begin{cases} g_n(\eta) = \lambda u_n(\eta) + (\beta\eta+\gamma)\mathcal{I}_{0^+}^{\alpha-1} g_n(\eta), \\ g(\eta) = \lambda u(\eta) + (\beta\eta+\gamma)\mathcal{I}_{0^+}^{\alpha-1} g(\eta). \end{cases}$$

We have

$$\begin{aligned} |g_n(\eta) - g(\eta)| &= \left|\lambda(u_n(\eta) - u(\eta)) + (\beta\eta+\gamma)\mathcal{I}_{0^+}^{\alpha-1}(g_n(\eta) - g(\eta))\right| \\ &\le \lambda\|u_n - u\|_\infty + (|\beta| + |\gamma|)\left|\mathcal{I}_{0^+}^{\alpha-1}(g_n(\eta) - g(\eta))\right|. \end{aligned}$$

As

$$\begin{aligned} \left|\mathcal{I}_{0^+}^{\alpha-1}(g_n(\eta) - g(\eta))\right| &\le \frac{1}{\Gamma(\alpha-1)} \int_0^\eta (\eta-\xi)^{\alpha-2} |g_n(\xi) - g(\xi)|\, d\xi \\ &\le \frac{1}{\Gamma(\alpha)} \|g_n - g\|_\infty, \end{aligned}$$

then

$$\|g_n - g\|_\infty \le |\lambda|\,\|u_n - u\|_\infty + \frac{|\beta| + |\gamma|}{\Gamma(\alpha)} \|g_n - g\|_\infty.$$

According to (4.14), we have $\Gamma(\alpha) - |\beta| - |\gamma| > \frac{1}{\alpha}|\lambda| \geq 0$, thus

$$\|g_n - g\|_\infty \leq \frac{|\lambda| \Gamma(\alpha)}{\Gamma(\alpha) - |\beta| - |\gamma|} \|u_n - u\|_\infty .$$

Since $u_n \to u$, then we get $g_n \to g$ as $n \to \infty$.

Now, let $\mu > 0$ be such that for each $\eta \in J$, we get

$$|g_n(\eta)| \leq \mu, \ \ |g(\eta)| \leq \mu.$$

Then, we have

$$\begin{aligned} \frac{(\eta - \xi)^{\alpha-1}}{\Gamma(\alpha)} |g_n(\eta) - g(\eta)| &\leq \frac{(\eta - \xi)^{\alpha-1}}{\Gamma(\alpha)} [|g_n(\eta)| + |g(\eta)|] \\ &\leq \frac{2\mu}{\Gamma(\alpha)} (\eta - \xi)^{\alpha-1} . \end{aligned}$$

For each $\eta \in J$, the function $\xi \to \frac{2\mu}{\Gamma(\alpha)} (\eta - \xi)^{\alpha-1}$ is integrable on $[0, \eta]$, then the Lebesgue dominated convergence theorem and (4.16) imply that

$$|\mathcal{A}u_n(\eta) - \mathcal{A}u(\eta)| \to 0 \text{ as } n \to \infty,$$

and hence

$$\lim_{n\to\infty} \|\mathcal{A}u_n - \mathcal{A}u\|_\infty = 0.$$

Consequently, $\mathcal{A}$ is continuous.

Step 2: According to (4.14), we put the positive real

$$r \geq \left(1 + \frac{|\lambda|}{\Gamma(\alpha+1) - (\alpha(|\beta| + |\gamma|) + |\lambda|)}\right) (|c_0| + |c_1|) ,$$

and define the subset H as follows

$$H = \{u \in C(J, \mathbb{R}) : \|u\|_\infty \leq r\} .$$

It is clear that H is a bounded, closed and convex subset of $C(J, \mathbb{R})$.

Let $\mathcal{A} : H \to C(J, \mathbb{R})$ be the integral operator defined by (4.15), then $\mathcal{A}(H) \subset H$.

In fact, we have

$$\|g\|_\infty \leq \frac{|\lambda| \Gamma(\alpha)}{\Gamma(\alpha) - |\beta| - |\gamma|} r. \tag{4.17}$$

Then

$$
\begin{aligned}
|\mathcal{A}u(\eta)| &\leq |c_0| + |c_1| + \frac{1}{\Gamma(\alpha)} \int_0^{\eta} (\eta-\xi)^{\alpha-1} |g(\xi)|\, d\xi \\
&\leq |c_0| + |c_1| + \frac{1}{\Gamma(\alpha+1)} \frac{|\lambda|\,\Gamma(\alpha)}{\Gamma(\alpha) - |\beta| - |\gamma|} r \\
&\leq \frac{(|c_0| + |c_1|)\left(1 + \frac{|\lambda|}{\Gamma(\alpha+1) - (\alpha(|\beta|+|\gamma|)+|\lambda|)}\right)}{1 + \frac{|\lambda|}{\Gamma(\alpha+1) - (\alpha(|\beta|+|\gamma|)+|\lambda|)}} + \frac{|\lambda|\, r}{\Gamma(\alpha+1) - \alpha(|\beta|+|\gamma|)} \\
&\leq \frac{\Gamma(\alpha+1) - (\alpha(|\beta|+|\gamma|)+|\lambda|)}{\Gamma(\alpha+1) - \alpha(|\beta|+|\gamma|)} r + \frac{|\lambda|\, r}{\Gamma(\alpha+1) - \alpha(|\beta|+|\gamma|)} \\
&\leq r.
\end{aligned}
$$

Then $\mathcal{A}(H) \subset H$.

Step 3: $\mathcal{A}(H)$ is relatively compact.

Let $\eta_1, \eta_2 \in J$, $\eta_1 < \eta_2$, and $u \in H$. Then

$$
\begin{aligned}
|\mathcal{A}u(\eta_2) - \mathcal{A}u(\eta_1)| &= \Bigg| c_1\eta_2 + \frac{1}{\Gamma(\alpha)} \int_0^{\eta_2} (\eta_2-\xi)^{\alpha-1} g(\xi)\, d\xi \\
&\quad - c_1\eta_1 - \frac{1}{\Gamma(\alpha)} \int_0^{\eta_1} (\eta_1-\xi)^{\alpha-1} g(\xi)\, d\xi \Bigg| \\
&\leq |c_1|(\eta_2-\eta_1) + \frac{1}{\Gamma(\alpha)} \int_0^{\eta_1} \left| \left((\eta_2-\xi)^{\alpha-1} - (\eta_1-\xi)^{\alpha-1}\right) g(\xi) \right| d\xi \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{\eta_1}^{\eta_2} (\eta_2-\xi)^{\alpha-1} |g(\xi)|\, d\xi \\
&\leq |c_1|(\eta_2-\eta_1) + \frac{|\lambda|\, r}{\Gamma(\alpha) - |\beta| - |\gamma|} \left[\int_0^{\eta_1} \left| (\eta_2-\xi)^{\alpha-1} - \right.\right. \\
&\quad \left.\left. (\eta_1-\xi)^{\alpha-1} \right| d\xi + \int_{\eta_1}^{\eta_2} (\eta_2-\xi)^{\alpha-1} d\xi \right]. \qquad (4.18)
\end{aligned}
$$

We have

$$
(\eta_2-\xi)^{\alpha-1} - (\eta_1-\xi)^{\alpha-1} = -\frac{1}{\alpha} \frac{d}{d\xi} \left[(\eta_2-\xi)^{\alpha} - (\eta_1-\xi)^{\alpha} \right],
$$

then

$$
\int_0^{\eta_1} \left| (\eta_2-\xi)^{\alpha-1} - (\eta_1-\xi)^{\alpha-1} \right| d\xi \leq \frac{1}{\alpha} \left[(\eta_2-\eta_1)^{\alpha} + (\eta_2^{\alpha} - \eta_1^{\alpha}) \right],
$$

we have also

$$
\int_{\eta_1}^{\eta_2} (\eta_2-\xi)^{\alpha-1} d\xi = -\frac{1}{\alpha} \left[(\eta_2-\xi)^{\alpha} \right]_{\eta_1}^{\eta_2} \leq \frac{1}{\alpha} (\eta_2-\eta_1)^{\alpha}.
$$

Then (4.18) gives us

$$
|\mathcal{A}u(\eta_2) - \mathcal{A}u(\eta_1)| \leq |c_1|(\eta_2-\eta_1) + \frac{|\lambda|\, r \left(2(\eta_2-\eta_1)^{\alpha} + (\eta_2^{\alpha} - \eta_1^{\alpha})\right)}{\Gamma(\alpha+1) - \alpha(|\beta|+|\gamma|)}.
$$

As $\eta_1 \to \eta_2$, the right-hand side of the above inequality tends to zero.

As a consequence of steps 1 to 3, and by means of the Ascoli-Arzelà theorem, we deduce that $\mathcal{A} : H \to H$ is continuous, compact and satisfies the assumption of Schauder's fixed point theorem 1.3. Then $\mathcal{A}$ has a fixed point which is a solution of the problem (4.7)–(4.8) on J. The proof is complete. □

Theorem 4.5. *If we put* $|\beta| + |\gamma| < \Gamma(\alpha)$ *and*

$$\frac{|\lambda|}{\Gamma(\alpha+1) - \alpha(|\beta| + |\gamma|)} < 1, \tag{4.19}$$

then the problem (4.7)–(4.8) admits a unique solution on J.

Proof. In the previous Theorem 4.4, we transformed the problem (4.7)–(4.8) into a fixed point problem (4.15).

Let $u_1, u_2 \in C(J, \mathbb{R})$, then

$$\mathcal{A}u_1(\eta) - \mathcal{A}u_2(\eta) = \frac{1}{\Gamma(\alpha)} \int_0^\eta (\eta - \xi)^{\alpha-1} (g_1(\xi) - g_2(\xi))\, d\xi,$$

where

$$g_i(\eta) = \lambda u_i(\eta) + (\beta\eta + \gamma)\, \mathcal{I}_{0^+}^{\alpha-1} g_i(\eta), \text{ for } i = 1, 2.$$

Also

$$|\mathcal{A}u_1(\eta) - \mathcal{A}u_2(\eta)| \leq \frac{1}{\Gamma(\alpha)} \int_0^\eta (\eta - \xi)^{\alpha-1} |g_1(\xi) - g_2(\xi)|\, d\xi. \tag{4.20}$$

We have

$$\|g_1 - g_2\|_\infty \leq |\lambda| \|u_1 - u_2\|_\infty + \frac{|\beta| + |\gamma|}{\Gamma(\alpha)} \|g_1 - g_2\|_\infty.$$

As $|\beta| + |\gamma| < \Gamma(\alpha)$, we obtain

$$\|g_1 - g_2\|_\infty \leq \frac{|\lambda|\,\Gamma(\alpha)}{\Gamma(\alpha) - |\beta| - |\gamma|} \|u_1 - u_2\|_\infty.$$

From (4.20) we find

$$\|\mathcal{A}u_1 - \mathcal{A}u_2\|_\infty \leq \frac{|\lambda|}{\Gamma(\alpha+1) - \alpha(|\beta| + |\gamma|)} \|u_1 - u_2\|_\infty.$$

This implies that by (4.19), $\mathcal{A}$ is a contraction operator.

As a consequence of Theorem 1.2, using Banach's contraction principle [23], we deduce that $\mathcal{A}$ has a unique fixed point which is the unique solution of the problem (4.7)–(4.8) on J. The proof is complete. □

4.5 Proof of Main Theorems

In this part, we prove the existence and uniqueness of solutions of the following free boundary problem of the diffusion equation of moving fractional order

$$\begin{cases} \partial_t \omega = \kappa \partial_x^\alpha \omega, \ \ (x,t) \in \Omega, & \kappa \in \mathbb{R}^*, \\ \omega\left(b\left(t\right),t\right) = c_0 c\left(t\right), & c_0 \in \mathbb{R}, \\ \partial_x \omega\left(b\left(t\right),t\right) = c_1 \frac{c(t)}{a(t)}, & c_1 \in \mathbb{R}, \\ \partial_x^k \omega\left(b\left(t\right),t\right) = 0, \ k = 2,3,\ldots,m-1, & \text{for } m \geq 3, \end{cases} \tag{4.21}$$

under the traveling profile form

$$\omega\left(x,t\right) = c\left(t\right)\varphi\left(\eta\right), \text{with } \eta = \frac{x-b\left(t\right)}{a\left(t\right)} \text{ and } a,c \in \mathbb{R}_+^*, \ b \in \mathbb{R}, \tag{4.22}$$

withwhere

$$a\left(0\right) = c\left(0\right) = 1, \ b\left(0\right) = 0.$$

We denote by $(z)_+$ the positive part of z, which is z if $z > 0$ and what remains is zero.

Proof of Theorem 4.1

The transformation (4.22) reduces the space-fractional diffusion equation in (4.21) to the ordinary differential equation of fractional order of the form

$${}^C\mathcal{D}_{0+}^\alpha \varphi\left(\eta\right) = g\left(\eta\right), \eta \in J, \tag{4.23}$$

where

$$g\left(\eta\right) = \lambda\varphi\left(\eta\right) + \left(\beta\eta + \gamma\right)\varphi'\left(\eta\right)$$

and $\lambda, \beta, \gamma \in \mathbb{R}$ are constant satisfying

$$\left(\lambda,\beta,\gamma\right) = \frac{a^\alpha\left(t\right)}{\kappa}\left(\frac{\dot{c}\left(t\right)}{c(t)}, -\frac{\dot{a}\left(t\right)}{a\left(t\right)}, -\frac{\dot{b}\left(t\right)}{a\left(t\right)}\right), \tag{4.24}$$

with the conditions

$$\begin{cases} \varphi\left(0\right) = c_0, \ \varphi'\left(0\right) = c_1, \ \text{for any } m \geq 2, \\ \varphi^{(k)}\left(0\right) = 0, \ k = 2,3,\ldots,m-1, \ \text{for } m \geq 3. \end{cases} \tag{4.25}$$

Now, to determine the functions $a\left(t\right)$, $b\left(t\right)$ and $c\left(t\right)$, we just solve the system (4.24). If $\beta = 0$, we have $a\left(t\right) = 1$ and

$$\begin{cases} b\left(t\right) = -\kappa\gamma t, \\ c\left(t\right) = \exp\left(\kappa\lambda t\right). \end{cases} \ \ t > 0.$$

If $\beta \neq 0$, after an integration from 0 to t we get

$$\begin{cases} a(t) = (1-\alpha\kappa\beta t)_+^{\frac{1}{\alpha}}, \\ b(t) = \frac{\gamma}{\alpha\kappa\beta}\left((1-\alpha\kappa\beta t)_+^{\frac{1}{\alpha}} - 1\right), \quad 0 < t < T, \\ c(t) = (1-\alpha\kappa\beta t)_+^{-\frac{\lambda}{\alpha\beta}}, \end{cases} \tag{4.26}$$

where $T > 0$ is the maximal existence value of time for the solution ω, which may be finite or infinite. Thereupon, we separate the following cases

1. If $\kappa\beta \leq 0$ (i.e., $\dot{a}(t) \geq 0$), the problem (4.21) admits a global solution in time under the traveling profile form (4.22); this solution is defined for all $t > 0$, (i.e., $T = \infty$).
   In addition, for $\lambda\beta > 0$ (i.e., $\dot{c}(t) < 0$), we have

   $$\lim_{t\to+\infty} \omega(x,t) = 0.$$

2. If $\kappa\beta > 0$ (i.e., $\dot{a}(t) < 0$), the functions $a(t)$, $b(t)$ and $c(t)$ are defined locally and are well-defined if and only if

   $$0 < t < T = \frac{1}{\alpha\kappa\beta} = -\frac{a^{1-\alpha}(t)}{\alpha\dot{a}(t)}.$$

   The moment $T = \frac{1}{\alpha\kappa\beta}$ represents the maximal existence value of the functions $a(t)$, $b(t)$ and $c(t)$. Moreover; if $\lambda\beta > 0$ (i.e., $\dot{c}(t) > 0$), the problem (4.21) admits a solution under the traveling profile form (4.22), which it blows up in a finite time. The solution is defined for all $t \in [0,T)$, the moment T represents the blow-up time of the solution such that:

   $$\text{for all } x \in \mathbb{R}, \ \lim_{t\to T^-} \omega(x,t) = \lim_{t\to T^-} c(t)\,\varphi\left(\frac{x-b(t)}{a(t)}\right) = +\infty.$$

   We recall that the solution blows up in a finite time if there exists a time $T < +\infty$, which we call the blow-up time, such that the solution is well defined for all $0 < t < T$, while

   $$\sup_{x\in\mathbb{R}} |\omega(x,t)| \to +\infty, \ \text{when } t \to T = \frac{1}{\alpha\kappa\beta}.$$

By using (4.24), the condition (4.5) is equivalent to (4.14), which is

$$\alpha(|\beta| + |\gamma|) + |\lambda| < \Gamma(\alpha+1).$$

We already proved the existence of a solution to the problem (4.23)–(4.25) in Theorem 4.4, provided that (4.14) holds true. Consequently, if (4.5) holds for any $t \in [0,T]$, then there exists at least one solution of the problem of the diffusion equation of moving fractional order (4.21) under the traveling profile form (4.22). The proof is complete.

Proof of Theorem 4.2

Based on Theorem 4.5, we use the same steps through which we proved Theorem 4.1 to prove the existence and uniqueness of global or blow-up traveling profile solution to the problem (4.21), provided that the condition (4.6) holds true. The proof is complete.

4.6 Explicit Solutions

Example 1: According to the proof of the Theorem 4.1, for $\beta = \gamma = 0$ and $\lambda, \kappa \in \mathbb{R}^*$, we get

$$a(t) = 1,\ b(t) = 0 \text{ and } c(t) = \exp(\kappa\lambda t).$$

In this case, for $m = 2$, (i.e., Space-fractional heat equation), we give new explicit solutions on the traveling profile form of the problem (4.21) as follows

$$\omega(x,t) = \exp(\kappa\lambda t)\left[c_0 E_\alpha(\lambda x^\alpha) + c_1 x E_{\alpha,2}\left(\lambda x^2\right)\right],\ c_0, c_1 \in \mathbb{R},$$

where $E_{\alpha,m}(\eta)$ is the function of Mittag-Leffler type. The solution defined for all $t > 0$.

Example 2: We present new explicit solutions on the traveling profile form of the problem (4.21):

For $m \geq 2$, if we put $\beta, \kappa, c_0, c_1 \in \mathbb{R}^*$, $\lambda = (1-m)\beta$ and $\gamma = \frac{(m-1)\beta c_0}{c_1}$, we get that

$$\varphi(\eta) = c_0 + \sum_{k=1}^{m-1} \frac{c_1^k (m-1)!}{c_0^{k-1} k!(m-k-1)!(m-1)^k}\eta^k,$$

is a solution of the problem (4.23)–(4.25). Then the traveling profile solution of the problem (4.21) is presented as follows

$$\omega(x,t) = c(t)\left[c_0 + \sum_{k=1}^{m-1} \frac{c_1^k (m-1)!}{c_0^{k-1} k!(m-k-1)!(m-1)^k}\left(\frac{x-b(t)}{a(t)}\right)^k\right], \tag{4.27}$$

where

$$\begin{cases} a(t) = (1-\alpha\kappa\beta t)_+^{\frac{1}{\alpha}}, \\ b(t) = \frac{\gamma}{\alpha\kappa\beta}\left((1-\alpha\kappa\beta t)_+^{\frac{1}{\alpha}} - 1\right),\ \ 0 < t < T, \\ c(t) = (1-\alpha\kappa\beta t)_+^{\frac{m-1}{\alpha}}. \end{cases}$$

According to the proof of the Theorem 4.1, we separate the following cases:

1. If $\kappa\beta < 0$, the problem (4.21) admits a global solution in time under the form (4.27), this solution is defined for all $t > 0$.

2. If $\kappa\beta > 0$, the functions $a\,(t)$, $b\,(t)$ and $c\,(t)$ are defined if and only if $0 < t < T = \frac{1}{\alpha\kappa\beta}$ and the solution does not blow up in the moment T, because $\lambda\beta = (1-m)\,\beta^2 < 0$. Moreover;
$$\sup_{x\in\mathbb{R}} |\omega\,(x,t)| \to 0, \text{ when } t \to T = \frac{1}{\alpha\kappa\beta}.$$

CHAPTER 5

THEORETICAL STUDIES ON THE EXISTENCE OF SOLUTIONS FOR A MULTIDIMENSIONAL NONLINEAR TIME AND SPACE FRACTIONAL REACTION-DIFFUSION/WAVE EQUATION

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5.1 Introduction and Statement of Results

This chapter discusses and theoretically studies the existence of radially symmetric solution

$$\omega\left(t,x\right)=\left|x\right|^{\delta}\varphi\left(\left|x\right|^{-\frac{2}{\alpha}}t\right),\text{ for }\left|x\right|=\sqrt{x_1^2+\cdots+x_m^2},\text{ and }\delta\in\mathbb{C},\tag{5.1}$$

for a multidimensional nonlinear time and space-fractional reaction-diffusion/wave equationas follows

$$\partial_t^{\alpha}\omega-\kappa^2\Delta\omega=F\left(t,x,\omega,\partial_t^{\beta}\omega,\left(-\Delta\right)^{s}\omega\right),\text{ for }0<s\leq 1<\beta\leq\alpha\leq 2,\tag{5.2}$$

that enables treating vibration and control, signal and image processing, and modeling earthquakes, among others other physical phenomena. Additionally, the application of Schauder's and Banach's fixed point theorems facilitates identifying the existence and uniqueness of solutions for the selected equation. The applicability of our main results is demonstrated through examples and explicit solutions.

Where $\omega = \omega(t, x)$ is a scalar function of the time $t \geq 0$ and space variables $x \in \mathbb{R}^m$, with $m \in \mathbb{N}^*$. Also $F : [0, \infty) \times \mathbb{R}^m \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a nonlinear function, $\kappa \in \mathbb{R}^*$ is a real constant and

$$\partial_t^\alpha \omega(t, x) = \begin{cases} \partial_t^n \omega, & \alpha = n \in \mathbb{N}^*, \\ \mathcal{I}_{0^+}^{n-\alpha} \partial_t^n \omega \ = \int_0^t \frac{(t-\tau)^{n-\alpha-1}}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial \tau^n} \omega(\tau, x)\, d\tau, & n-1 < \alpha < n. \end{cases}$$

The symbol $(-\Delta)^s$ defines the fractional Laplacian operator [29];

$$(-\Delta)^s \omega = C_{m,s} \text{P.V.} \int_{\mathbb{R}^m} \frac{\omega(t, x) - \omega(t, y)}{|x-y|^{m+2s}} dy, \text{ for } 0 < s < 1,$$

where P.V. stands for the Cauchy principal value, and the constant $C_{m,s}$ is given by

$$C_{m,s} = \frac{2^{2s} s \Gamma\left(\frac{m+2s}{2}\right)}{\pi^{m/2} \Gamma(1-s)}.$$

We take the fractional power of $(-\Delta)$ to obtain a positive operator. As a result, our definition of the fractional Laplacian $(-\Delta)^s$ is the negative generator of the standard isotropic s-stable Lévy process [29], which is reduced to $-\Delta = -\partial^2/\partial^2 x_1 - \partial^2/\partial^2 x_2 - \cdots - \partial^2/\partial^2 x_m$ when $s = 1$.

The Significance of the Equation

Equation (5.2) is a representation of a large class of linear and nonlinear equations fall under the name of the fractional reaction-diffusion/wave equation (see table 5.1).

Obviously, the development of accurate mathematical models for the description of complex anomalous systems depends on swapping the fractional Laplacian with integer-order Laplacian, e.g.

Fractional equation	Formula
Reaction-diffusion/wave [9, 10, 15, 25, 33, 35, 45]	$\partial_t^\alpha u + \kappa^2 (-\Delta)^s u + c(t, x) u = 0$
Quasi-geostrophic [16]	$\partial_t v + \theta . \bigtriangledown v + \kappa (-\Delta)^s v = f$
Cahn-Hilliard [2, 1]	$\partial_t w + (-\Delta)^s (-\varepsilon^2 \Delta w + f(w)) = 0$
Porous medium [2, 1, 19]	$\partial_t u + (-\Delta)^s \left(|u|^{m-1} \text{sign} u\right) = 0$
Schrödinger [30]	$i\hbar \partial_t \psi = \partial_t^\alpha (-\hbar^2 \Delta)^s \psi + c(t, x) \psi$
Ultrasound [42]	$\frac{1}{c_0^2} \partial_t^2 \theta = \nabla^2 \theta - \left\{\tau \partial_t (-\Delta)^s + \eta (-\Delta)^{s+\frac{1}{2}}\right\} \theta$

Table 5.1: Significant equations involving fractional Laplacian

Problem Statement and Main Results

Let $0 < s \leq 1$, $1 < \beta \leq \alpha \leq 2$, $\varepsilon, \ell > 0$, and $T_\varepsilon = \ell \varepsilon^{\frac{2}{\alpha}}$ be such that $\Omega = [0, T_\varepsilon] \times [\varepsilon/\sqrt{m}, +\infty)^m$. We consider

$$\begin{cases} \partial_t^\alpha \omega - \kappa^2 \Delta \omega = F\left(t, x, \omega, \partial_t^\beta \omega, (-\Delta)^s \omega\right), & (t,x) \in \Omega,\ \kappa \in \mathbb{R}^*, \\ \omega(0,x) = |x|^\delta c_0,\ \partial_t \omega(0,x) = 0, & \delta, c_0 \in \mathbb{C}, \end{cases} \tag{5.3}$$

where $F : \Omega \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a nonlinear function.

This chapter contribution regards determining the existence, uniqueness, and main properties of the general solution of stability problems obtained through replacing classical rules with fractional quadrature rules of the radially symmetric

$$\omega(t,x) = |x|^\delta \varphi\left(|x|^{-\frac{2}{\alpha}} t\right), \text{ for } |x| = \sqrt{x_1^2 + \cdots + x_m^2}, \text{ and } \delta \in \mathbb{C},$$

the basic profile φ is not known in advance and is to be identified.

Taking into consideration the regularization processes, our major aim is employing of the solutions' intermediate properties for the fractional order's PDE's problem (5.3). We consider the intermediacy of the multidimensional nonlinear reaction-diffusion equation and the wave equation.

We illustrate that using analytical techniques to obtain the existence and uniqueness of weak solutions via the use of form (5.1) is promising and can also bring new results for other applications in fractional-order's PDEs. It permits us to reduce the fractional-order's PDE (5.2) to a fractional differential equation; the idea is well illustrated in this paper through selected examples and explicit solutions.

For the forthcoming analysis, we impose the following hypotheses

$(hyp.1)$ $F : \Omega \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a continuous function that is invariant by the change of scale (5.1). It gives us

$$F\left(t, x, \omega, \partial_t^\beta \omega, (-\Delta)^s \omega\right) = |x|^{\delta - 2}\left(f\left(\eta, \varphi(\eta), \varphi'(\eta), {}^C\mathcal{D}_{0+}^\beta \varphi(\eta)\right) - \frac{4\kappa^2}{\alpha^2}\eta^2 \varphi''(\eta)\right), \tag{5.4}$$

where $\eta = |x|^{-\frac{2}{\alpha}} t$ and $f : J \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a continuous function.

$(hyp.2)$ There exist three positive constants $\gamma_1, \gamma_2, \gamma_3 > 0$ so that the continuous function f given by (5.4) satisfies

$$|f(\eta, u, v, w) - f(\eta, \bar{u}, \bar{v}, \bar{w})| \leq \gamma_1 |u - \bar{u}| + \gamma_2 |v - \bar{v}| + \gamma_3 |w - \bar{w}|,$$

for any $u, v, w, \bar{u}, \bar{v}, \bar{w} \in \mathbb{C}$.

$(hyp.3)$ There exist four positive functions $a, b, c, d \in C(J, \mathbb{R}_+)$ such that the continuous function f given by (5.4) satisfies

$$|f(\eta, u, v, w)| \leq a(\eta) + b(\eta)|u| + c(\eta)|v| + d(\eta)|w|,$$

for any $u, v, w \in \mathbb{C}$ and $\eta \in J$.

λ denotes the positive constant defined by

$$\lambda = \sup\left\{\frac{\alpha\ell^{\beta-1}|\ell q + c^*| + d^*}{\ell^{\beta-\alpha}\Gamma(\alpha-\beta+1)}, \frac{\alpha\ell^{\beta-1}|\ell q + \gamma_2| + \gamma_3}{\ell^{\beta-\alpha}\Gamma(\alpha-\beta+1)}\right\},$$

where $q = -\frac{4\kappa^2}{\alpha^2}\left(\alpha\delta + \alpha + 1 + \frac{m\alpha}{2}\right)$ and

$$a^* = \sup_{\eta\in J} a(\eta),\ b^* = \sup_{\eta\in J} b(\eta),\ c^* = \sup_{\eta\in J} c(\eta) \text{ and } d^* = \sup_{\eta\in J} d(\eta).$$

Throughout the rest of this chapter, we give $p = \delta\kappa^2(\delta + m - 2)$.

Now, we give the principal theorems of this work.

Theorem 5.1. *Assume the hypotheses $(hyp.1) - (hyp.3)$ hold. If we put $\lambda \in (0,1)$ and*

$$\frac{\ell^\alpha|\delta\kappa^2(\delta + m - 2) + b^*|}{(1-\lambda)\Gamma(\alpha+1)} < 1, \tag{5.5}$$

then, there is at least one solution to the problem (5.3) on Ω in the radially symmetric form (5.1).

Theorem 5.2. *Assume the hypotheses $(hyp.1), (hyp.2)$ hold. If we put $\lambda \in (0,1)$ and*

$$\frac{\ell^\alpha|\delta\kappa^2(\delta + m - 2) + \gamma_1|}{(1-\lambda)\Gamma(\alpha+1)} < 1. \tag{5.6}$$

then the problem (5.3) admits a unique solution in the radially symmetric form (5.1) on Ω.

5.2 Compute of Radially Symmetric Solutions

Our initial aim is to infer that the function φ in (5.1) satisfies an equation that is employed in the definition of radially symmetric solutions.

Theorem 5.3. *If the hypothesis $(hyp.1)$ holds, the problem of time and space-fractional order (5.3) is reduced by the transformation (5.1) to the fractional order's ordinary differential equation of the form*

$${}^C\mathcal{D}^\alpha_{0^+}\varphi(\eta) = g(\eta),\ \eta \in J, \tag{5.7}$$

where

$$g(\eta) = p\varphi(\eta) + q\eta\varphi'(\eta) + f\left(\eta, \varphi(\eta), \varphi'(\eta), {}^C\mathcal{D}^\beta_{0^+}\varphi(\eta)\right),$$

with the conditions

$$\varphi(0) = c_0 \text{ and } \varphi'(0) = 0. \tag{5.8}$$

Proof. The fractional equation resulting from the substitution of expression (5.1) in the original fractional-order's PDE (5.3), should be reduced to the standard bilinear functional equation (check [9, 10, 15, 25, 32, 37, 43]). First, for $\eta = |x|^{-\frac{2}{\alpha}} t$, we get $\eta \in J$ and

$$\Delta\omega(t,x) = |x|^{\delta-2}\left(\delta(\delta+m-2)\varphi - \frac{4}{\alpha^2}\left(\alpha\delta+\alpha+1+\frac{m\alpha}{2}\right)\eta\varphi' + \frac{4}{\alpha^2}\eta^2\varphi''\right). \tag{5.9}$$

On the other hand, for $\xi = |x|^{-\frac{2}{\alpha}} \tau$, we get

$$\begin{aligned}\frac{\partial^\alpha\omega}{\partial t^\alpha} &= \frac{1}{\Gamma(2-\alpha)}\int_0^t (t-\tau)^{1-\alpha}\frac{\partial^2\omega(\tau,x)}{\partial\tau^2}d\tau \\ &= \frac{|x|^\delta}{\Gamma(2-\alpha)}\int_0^t (t-\tau)^{1-\alpha}\frac{d^2}{d\tau^2}\varphi\left(|x|^{-\frac{2}{\alpha}}\tau\right)d\tau \\ &= \frac{|x|^{\delta-2}}{\Gamma(2-\alpha)}\int_0^\eta (\eta-\xi)^{1-\alpha}\frac{d^2}{d\xi^2}\varphi(\xi)\,d\xi \\ &= |x|^{\delta-2}\,{}^C\mathcal{D}_{0+}^\alpha\varphi(\eta).\end{aligned} \tag{5.10}$$

If we replace (5.4), (5.9) and (5.10) in the first equation of (5.3), we obtain

$${}^C\mathcal{D}_{0+}^\alpha\varphi(\eta) = g(\eta).$$

From the conditions in (5.3), we find

$$\omega(0,x) = |x|^\delta\varphi(0)$$

and

$$\partial_t\omega(0,x) = |x|^{\delta-\frac{2}{\alpha}}\varphi'(0),$$

which implies that

$$\varphi(0) = c_0 \text{ and } \varphi'(0) = 0.$$

The proof is complete. □

5.3 Basic-Profile's Existence and Uniqueness Results

Lemma 5.1. *Assume that $f : J \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a continuous function, then the problem (5.7)–(5.8) is equivalent to the integral equation*

$$\varphi(\eta) = c_0 + \frac{1}{\Gamma(\alpha)}\int_0^\eta (\eta-\xi)^{\alpha-1} g(\xi)\,d\xi,\ \forall\eta \in J,$$

where $g \in C(J,\mathbb{C})$ satisfies the functional equation

$$g(\eta) = p\left(c_0 + \mathcal{I}_{0+}^\alpha g(\eta)\right) + \psi(g(\eta)),$$

with $\psi : \mathbb{C} \to \mathbb{C}$ *is a function satisfying*

$$\psi\left(g\left(\eta\right)\right) = q\eta\mathcal{I}_{0^+}^{\alpha-1}g\left(\eta\right) + f\left(\eta, c_0 + \mathcal{I}_{0^+}^{\alpha}g\left(\eta\right), \mathcal{I}_{0^+}^{\alpha-1}g\left(\eta\right), \mathcal{I}_{0^+}^{\alpha-\beta}g\left(\eta\right)\right).$$

Proof. Using Theorem 5.3, and applying $\mathcal{I}_{0^+}^{\alpha}$ to the equation (5.3), we obtain

$$\mathcal{I}_{0^+}^{\alpha}\ {}^{C}\mathcal{D}_{0^+}^{\alpha}\varphi\left(\eta\right) = \mathcal{I}_{0^+}^{\alpha}g\left(\eta\right).$$

From Lemma 1.1, we simply find

$$\mathcal{I}_{0^+}^{\alpha}\ {}^{C}\mathcal{D}_{0^+}^{\alpha}\varphi\left(\eta\right) = \varphi\left(\eta\right) - c_0 - \eta\varphi'\left(0\right).$$

Substituting (5.8) gives us

$$\varphi\left(\eta\right) = c_0 + \mathcal{I}_{0^+}^{\alpha}g\left(\eta\right). \tag{5.11}$$

As

$$\varphi'\left(\eta\right) = \frac{d}{d\eta}\left[c_0 + \mathcal{I}_{0^+}^{\alpha}g\left(\eta\right)\right] = \mathcal{I}_{0^+}^{\alpha-1}g\left(\eta\right)$$

and

$${}^{C}\mathcal{D}_{0^+}^{\beta}\varphi\left(\eta\right) = {}^{C}\mathcal{D}_{0^+}^{\beta}\left[c_0 + \mathcal{I}_{0^+}^{\alpha}g\left(\eta\right)\right] = {}^{C}\mathcal{D}_{0^+}^{\beta}\mathcal{I}_{0^+}^{\alpha}g\left(\eta\right) = \mathcal{I}_{0^+}^{\alpha-\beta}g\left(\eta\right),$$

then

$$\begin{aligned} g\left(\eta\right) &= p\varphi\left(\eta\right) + q\eta\varphi'\left(\eta\right) + f\left(\eta, \varphi\left(\eta\right), \varphi'\left(\eta\right), {}^{C}\mathcal{D}_{0^+}^{\beta}\varphi\left(\eta\right)\right) \\ &= p\left(c_0 + \mathcal{I}_{0^+}^{\alpha}g\left(\eta\right)\right) + q\eta\mathcal{I}_{0^+}^{\alpha-1}g\left(\eta\right) \\ &\quad + f\left(\eta, c_0 + \mathcal{I}_{0^+}^{\alpha}g\left(\eta\right), \mathcal{I}_{0^+}^{\alpha-1}g\left(\eta\right), \mathcal{I}_{0^+}^{\alpha-\beta}g\left(\eta\right)\right) \\ &= p\left(c_0 + \mathcal{I}_{0^+}^{\alpha}g\left(\eta\right)\right) + \psi\left(g\left(\eta\right)\right). \end{aligned}$$

Otherwise, starting by applying ${}^{C}\mathcal{D}_{0^+}^{\alpha}$ on both sides of the equation (5.11) and using the linearity of Caputo's derivative and the fact that ${}^{C}\mathcal{D}_{0^+}^{\alpha}c_0 = 0$, we find easily (5.7). Furthermore;

$$\begin{aligned} \varphi\left(0\right) &= \left(c_0 + \mathcal{I}_{0^+}^{\alpha}g\right)\left(0\right) = c_0 \\ \varphi'\left(0\right) &= \mathcal{I}_{0^+}^{\alpha-1}g\left(0\right) = 0. \end{aligned}$$

The proof is complete. □

Theorem 5.4. *Assume the hypotheses* $(hyp.2)$, $(hyp.3)$ *hold. If we put* $\lambda \in (0,1)$ *and*

$$\frac{\ell^{\alpha}\left|p + b^*\right|}{\left(1-\lambda\right)\Gamma\left(\alpha+1\right)} < 1, \tag{5.12}$$

then the problem (5.7)–(5.8) has at least one solution on J.

Proof. To begin the proof, we will transform the problem (5.7)–(5.8) into a fixed point problem. Let us define

$$\mathcal{A}u(\eta) = c_0 + \frac{1}{\Gamma(\alpha)} \int_0^\eta (\eta - \xi)^{\alpha-1} g(\xi)\, d\xi, \tag{5.13}$$

where

$$g(\eta) = pu(\eta) + \psi(g(\eta)), \ \eta \in J,$$

with

$$\psi(g(\eta)) = q\eta \mathcal{I}_{0^+}^{\alpha-1} g(\eta) + f\left(\eta, c_0 + \mathcal{I}_{0^+}^{\alpha} g(\eta), \mathcal{I}_{0^+}^{\alpha-1} g(\eta), \mathcal{I}_{0^+}^{\alpha-\beta} g(\eta)\right).$$

As the hypotheses $(hyp.2)$, $(hyp.3)$ hold, we notice that if $g \in C(J, \mathbb{C})$, then $\mathcal{A}u$ is indeed continuous (see the step 1 in this proof); therefore, it is an element of $C(J, \mathbb{C})$, and is equipped with the standard norm

$$\|\mathcal{A}u\|_\infty = \sup_{\eta \in J} |\mathcal{A}u(\eta)|.$$

Clearly, the fixed points of $\mathcal{A}$ are solutions of the problem (5.7)–(5.8).

We demonstrate that $\mathcal{A}$ satisfies the assumption of Schauder's fixed point theorem (see [23]). This could be proved through three steps.

Step 1: $\mathcal{A}$ is a continuous operator.

Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence such that $\lim_{n\to\infty} u_n = u$ in $C(J, \mathbb{C})$. Then $\forall \eta \in J$,

$$|\mathcal{A}u_n(\eta) - \mathcal{A}u(\eta)| \leq \frac{1}{\Gamma(\alpha)} \int_0^\eta (\eta - \xi)^{\alpha-1} |g_n(\xi) - g(\xi)|\, d\xi,$$

where

$$\begin{cases} g_n(\eta) = pu_n(\eta) + \psi(g_n(\eta)), \\ g(\eta) = pu(\eta) + \psi(g(\eta)). \end{cases}$$

We have

$$\begin{aligned} |g_n(\eta) - g(\eta)| &= |p(u_n(\eta) - u(\eta)) + (\psi(g_n(\eta)) - \psi(g(\eta)))| \\ &\leq |p + \gamma_1| |u_n(\eta) - u(\eta)| + |q + \gamma_2| \left|\mathcal{I}_{0^+}^{\alpha-1}(g_n(\eta) - g(\eta))\right| \\ &\quad + \gamma_3 \left|\mathcal{I}_{0^+}^{\alpha-\beta}(g_n(\eta) - g(\eta))\right|. \end{aligned}$$

As

$$\left|\mathcal{I}_{0^+}^{\alpha-1}(g_n(\eta) - g(\eta))\right| \leq \frac{\ell^{\alpha-1}}{\Gamma(\alpha)} \|g_n - g\|_\infty.$$

We have $\Gamma(\alpha+1) > \Gamma(\alpha-\beta+1)$ for any $1 < \beta \leq \alpha \leq 2$, then

$$\left|\mathcal{I}_{0^+}^{\alpha-1}\left(g_n(\eta)-g(\eta)\right)\right| \leq \frac{\alpha\ell^{\alpha-1}}{\Gamma(\alpha-\beta+1)}\left\|g_n-g\right\|_\infty .$$

In another way, we have

$$\left|\mathcal{I}_{0^+}^{\alpha-\beta}\left(g_n(\eta)-g(\eta)\right)\right| \leq \frac{\ell^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\left\|g_n-g\right\|_\infty .$$

Then we get

$$\begin{aligned}\left\|g_n-g\right\|_\infty &\leq |p+\gamma_1|\left\|u_n-u\right\|_\infty + \frac{\alpha\ell^{\beta-1}|\ell q+\gamma_2|+\gamma_3}{\ell^{\beta-\alpha}\Gamma(\alpha-\beta+1)}\left\|g_n-g\right\|_\infty \\ &\leq |p+\gamma_1|\left\|u_n-u\right\|_\infty + \lambda\left\|g_n-g\right\|_\infty .\end{aligned}$$

As $\lambda \in (0,1)$, thus

$$\left\|g_n-g\right\|_\infty \leq \frac{|p+\gamma_1|}{1-\lambda}\left\|u_n-u\right\|_\infty .$$

Since $u_n \to u$, we get $g_n \to g$ when $n \to \infty$.

Now, let $\mu > 0$ be such that for each $\eta \in J$, we get

$$|g_n(\eta)| \leq \mu, \;\; |g(\eta)| \leq \mu.$$

Then, we have

$$\begin{aligned}\frac{(\eta-\xi)^{\alpha-1}}{\Gamma(\alpha)}|g_n(\eta)-g(\eta)| &\leq \frac{(\eta-\xi)^{\alpha-1}}{\Gamma(\alpha)}\left[|g_n(\eta)|+|g(\eta)|\right] \\ &\leq \frac{2\mu}{\Gamma(\alpha)}(\eta-\xi)^{\alpha-1}.\end{aligned}$$

The function $\xi \to \frac{2\mu}{\Gamma(\alpha)}(\eta-\xi)^{\alpha-1}$ is integrable on $[0,\eta]$, $\forall\eta \in J$; thus, what the dominated convergence theorem of Lebesgue implies is

$$|\mathcal{A}u_n(\eta)-\mathcal{A}u(\eta)| \to 0 \text{ as } n \to \infty,$$

and hence

$$\lim_{n\to\infty}\left\|\mathcal{A}u_n-\mathcal{A}u\right\|_\infty = 0.$$

This indicates the continuity of $\mathcal{A}$.

Step 2: Using (5.12), we put the positive real

$$r \geq \left(|c_0| + \frac{a^*\ell^\alpha}{(1-\lambda)\Gamma(\alpha+1)}\right)\frac{(1-\lambda)\Gamma(\alpha+1)}{(1-\lambda)\Gamma(\alpha+1)-\ell^\alpha|p+b^*|},$$

and define the subset H as follows

$$H = \{u \in C(J, \mathbb{C}) : \|u\|_\infty \leq r\}.$$

It is clear that H is bounded, closed and convex subset of $C(J, \mathbb{C})$.

Let $\mathcal{A} : H \to C(J, \mathbb{C})$ be the integral operator defined by (5.13), then $\mathcal{A}(H) \subset H$. Indeed, we have for each $\eta \in J$

$$\begin{aligned}|g(\eta)| &= |pu(\eta) + \psi(g(\eta))| \\ &\leq a^* + |p + b^*| |u(\eta)| + \lambda \|g\|_\infty.\end{aligned}$$

Then, we get

$$\|g\|_\infty \leq \frac{a^* + |p + b^*| r}{1 - \lambda}.$$

Thus

$$\begin{aligned}|\mathcal{A}u(\eta)| &\leq |c_0| + \frac{1}{\Gamma(\alpha)} \int_0^\eta (\eta - \xi)^{\alpha-1} |g(\xi)| d\xi \\ &\leq |c_0| + \frac{\ell^\alpha}{\Gamma(\alpha+1)} \frac{a^* + |p + b^*| r}{1 - \lambda} \\ &\leq |c_0| + \frac{a^* \ell^\alpha}{(1-\lambda)\Gamma(\alpha+1)} + \frac{\ell^\alpha |p + b^*| r}{(1-\lambda)\Gamma(\alpha+1)} \\ &\leq \frac{\left(|c_0| + \frac{a^*\ell^\alpha}{(1-\lambda)\Gamma(\alpha+1)}\right) \frac{(1-\lambda)\Gamma(\alpha+1)}{(1-\lambda)\Gamma(\alpha+1) - \ell^\alpha |p+b^*|}}{\frac{(1-\lambda)\Gamma(\alpha+1)}{(1-\lambda)\Gamma(\alpha+1) - \ell^\alpha |p+b^*|}} + \frac{\ell^\alpha |p + b^*| r}{(1-\lambda)\Gamma(\alpha+1)} \\ &\leq r.\end{aligned}$$

Then $\mathcal{A}(H) \subset H$.

Step 3: $\mathcal{A}(H)$ is relatively compact.

Let $\eta_1, \eta_2 \in J$, $\eta_1 < \eta_2$, and $u \in H$. Then

$$\begin{aligned}|\mathcal{A}u(\eta_2) - \mathcal{A}u(\eta_1)| &= \frac{1}{\Gamma(\alpha)} \left| \int_0^{\eta_2} (\eta_2 - \xi)^{\alpha-1} g(\xi) d\xi - \int_0^{\eta_1} (\eta_1 - \xi)^{\alpha-1} g(\xi) d\xi \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^{\eta_1} \left| \left((\eta_2 - \xi)^{\alpha-1} - (\eta_1 - \xi)^{\alpha-1} \right) g(\xi) \right| d\xi \\ &+ \frac{1}{\Gamma(\alpha)} \int_{\eta_1}^{\eta_2} (\eta_2 - \xi)^{\alpha-1} |g(\xi)| d\xi \\ &\leq \frac{a^* + |p + b^*| r}{\Gamma(\alpha)(1-\lambda)} \left[\int_0^{\eta_1} \left| (\eta_2 - \xi)^{\alpha-1} - (\eta_1 - \xi)^{\alpha-1} \right| d\xi \right. \\ &\left. + \int_{\eta_1}^{\eta_2} (\eta_2 - \xi)^{\alpha-1} d\xi \right]. \end{aligned} \tag{5.14}$$

We have

$$(\eta_2-\xi)^{\alpha-1}-(\eta_1-\xi)^{\alpha-1}=-\frac{1}{\alpha}\frac{d}{d\xi}\left[(\eta_2-\xi)^{\alpha}-(\eta_1-\xi)^{\alpha}\right],$$

then

$$\int_0^{\eta_1}\left|(\eta_2-\xi)^{\alpha-1}-(\eta_1-\xi)^{\alpha-1}\right|d\xi\leq\frac{1}{\alpha}\left[(\eta_2-\eta_1)^{\alpha}+(\eta_2^{\alpha}-\eta_1^{\alpha})\right],$$

we also have

$$\int_{\eta_1}^{\eta_2}(\eta_2-\xi)^{\alpha-1}d\xi=-\frac{1}{\alpha}\left[(\eta_2-\xi)^{\alpha}\right]_{\eta_1}^{\eta_2}\leq\frac{1}{\alpha}(\eta_2-\eta_1)^{\alpha}.$$

Thus (5.14) gives us

$$\left|\mathcal{A}u(\eta_2)-\mathcal{A}u(\eta_1)\right|\leq\frac{2(\eta_2-\eta_1)^{\alpha}+(\eta_2^{\alpha}-\eta_1^{\alpha})}{\Gamma(\alpha)(1-\lambda)}\left(a^*+|p+b^*|\,r\right).$$

The right-hand side of the latter inequality tends to zero when $\eta_1\to\eta_2$.

As a consequence of steps 1 to 3, and through Ascoli-Arzelà theorem, we infer the continuity of $\mathcal{A}:H\to H$, its compact nature and its satisfaction of the assumption of Schauder's fixed point theorem [23]. Therefore, $\mathcal{A}$ has a fixed point which solves the problem (5.7)–(5.8) on J. □

Example 5.1. *If we choose $s=1$, $\beta=\frac{3}{2}$, $\alpha=\frac{7}{4}$, $\delta=1$, $m=2$, $\varepsilon=1$, $\kappa=\sqrt{\frac{7}{96}}$ and $\ell=1$, we obtain $\Omega=[0,1]\times\left[\frac{1}{\sqrt{2}},+\infty\right)^2$. Consequently, the considered problem will be stated as follows*

$$\begin{cases}\partial_t^{\frac{7}{4}}\omega-\frac{7}{96}\Delta\omega=F\left(t,x,\omega,\partial_t^{\frac{3}{2}}\omega,\Delta\omega\right), & (x,y)\in\Omega,\\ \omega(0,x,y)=\sqrt{x^2+y^2}, & \partial_t\omega(0,x,y)=0,\end{cases}\tag{5.15}$$

where

$$\begin{aligned}F\left(t,x,\omega,\partial_t^{\frac{3}{2}}\omega,\Delta\omega\right)&=\frac{\exp\left(-|x|^{-\frac{8}{7}}t\right)\left[|\omega|+2|x|+|x|^2\left|\partial_t^{\frac{3}{2}}\omega\right|\right]}{\left(|x|^{-\frac{8}{7}}t+2\ln\left(|x|^{-\frac{8}{7}}t+e\right)\right)\left[|x||\omega|+|x|^2+|x|^3\left|\partial_t^{\frac{3}{2}}\omega\right|\right]}-\frac{7}{96}\Delta\omega\\&=|x|^{-1}\left[f\left(\eta,\varphi,\varphi',{}^{C}\mathcal{D}_{0^+}^{\frac{3}{2}}\varphi(\eta)\right)-\frac{2}{21}\eta^2\varphi''(\eta)\right],\end{aligned}$$

with $\eta\in[0,1]$ and

$$f(\eta,u,v,w)=\frac{\exp(-\eta)\left[2+|u|+|w|\right]}{(\eta+2\ln(\eta+e))\left[1+|u|+|w|\right]}-\frac{7}{96}u+\frac{1}{2}\eta v,\ \textit{for}\ \eta\in[0,1].$$

Clearly, the function f is jointly continuous. For any $u,v,w,\bar{u},\bar{v},\bar{w}\in\mathbb{C}$ and $\eta\in[0,1]$, we get

$$\left|f(\eta,u,v,w)-f(\eta,\bar{u},\bar{v},\bar{w})\right|\leq\frac{55}{96}|u-\bar{u}|+\frac{1}{2}|v-\bar{v}|+\frac{1}{2}|w-\bar{w}|.$$

Therefore, hypothesis $(hyp.2)$ *is satisfied with*

$$\gamma_1 = \frac{55}{96},\ \gamma_2 = \frac{1}{2} \text{ and } \gamma_3 = \frac{1}{2}.$$

Also, we have

$$|f(\eta, u, v, w)| \leq \frac{\exp(-\eta)}{\eta + 2\ln(\eta + e)}(2 + |u| + |w|) + \frac{7}{96}|u| + \frac{1}{2}|v|.$$

Thus, the hypothesis $(hyp.3)$ *is satisfied with*

$$a(\eta) = \frac{2\exp(-\eta)}{\eta + 2\ln(\eta + e)},\ b(\eta) = \frac{\exp(-\eta)}{\eta + 2\ln(\eta + e)} + \frac{7}{96},\ c(\eta) = \frac{1}{2} \text{ and } d(\eta) = \frac{\exp(-\eta)}{\eta + 2\ln(\eta + e)}.$$

Then

$$a^* = 1,\ b^* = \frac{55}{96},\ c^* = \frac{1}{2},\ d^* = \frac{1}{2},$$

and

$$\begin{aligned} \lambda &= \sup\left\{\frac{\alpha\ell^{\beta-1}|\ell q + c^*| + d^*}{\ell^{\beta-\alpha}\Gamma(\alpha - \beta + 1)}, \frac{\alpha\ell^{\beta-1}|\ell q + \gamma_2| + \gamma_3}{\ell^{\beta-\alpha}\Gamma(\alpha - \beta + 1)}\right\} \\ &\simeq 0.55163 \\ &< 1. \end{aligned}$$

Condition (5.5) gives

$$\frac{\ell^{\alpha}|\delta\kappa^2(\delta + m - 2) + b^*|}{(1-\lambda)\Gamma(\alpha + 1)} \simeq 0.89557 < 1.$$

It follows from theorem 5.1, that the problem (5.15) has at least one solution on Ω*.*

Theorem 5.5. *Assume the hypothesis* $(hyp.2)$ *holds. If we put* $\lambda \in (0,1)$ *and*

$$\frac{\ell^{\alpha}|p + \gamma_1|}{\Gamma(\alpha + 1)(1 - \lambda)} < 1, \tag{5.16}$$

then the problem (5.7)–(5.8) admits a unique solution on J*.*

Proof. Theorem 5.4 states that (5.7)–(5.8) can be rendered a problem of a fixed point (5.13).

Let $u_1, u_2 \in C(J, \mathbb{C})$, then we get

$$\mathcal{A}u_1(\eta) - \mathcal{A}u_2(\eta) = \frac{1}{\Gamma(\alpha)}\int_0^{\eta}(\eta - \xi)^{\alpha-1}(g_1(\xi) - g_2(\xi))\,d\xi.$$

Where $g_i \in C(J, \mathbb{C})$ be such that

$$\begin{aligned} g_i(\eta) &= p\left(c_0 + \mathcal{I}_{0+}^{\alpha}g_i(\eta)\right) + \psi(g_i(\eta)),\ \text{for } i = 1, 2. \\ \psi(g_i(\eta)) &= q\eta\mathcal{I}_{0+}^{\alpha-1}g_i(\eta) + f\left(\eta, c_0 + \mathcal{I}_{0+}^{\alpha}g_i(\eta), \mathcal{I}_{0+}^{\alpha-1}g_i(\eta), \mathcal{I}_{0+}^{\alpha-\beta}g_i(\eta)\right) \end{aligned}$$

Also

$$|\mathcal{A}u_1(\eta) - \mathcal{A}u_2(\eta)| \leq \frac{1}{\Gamma(\alpha)} \int_0^{\eta} (\eta - \xi)^{\alpha-1} |g_1(\xi) - g_2(\xi)| \, d\xi. \tag{5.17}$$

We have

$$\|g_1 - g_2\|_\infty \leq \frac{|p + \gamma_1|}{1 - \lambda} \|u_1 - u_2\|_\infty .$$

From (5.17) we find

$$\|\mathcal{A}u_1 - \mathcal{A}u_2\|_\infty \leq \frac{\ell^\alpha |p + \gamma_1|}{\Gamma(\alpha+1)(1-\lambda)} \|u_1 - u_2\|_\infty .$$

Thus, according to (5.16), $\mathcal{A}$ is considered a contraction operator.

Banach's contraction principle (see [23]) helps us infer that $\mathcal{A}$ has only one fixed point which is the unique solution of the problem (5.7)–(5.8) on J. □

Example 5.2. *If we put* $s = 1$, $\beta = \frac{5}{4}$, $\alpha = \frac{3}{2}$, $\delta = 2$, $m = 4$, $\varepsilon = \sqrt[\frac{4}{3}]{\frac{1}{\pi}}$, $\kappa = -\sqrt{\frac{9}{272}}$ *and* $\ell = \frac{\pi}{4}$, *we get* $\Omega = \left[0, \frac{1}{4}\right] \times \left[\frac{1}{2}\sqrt[\frac{4}{3}]{\frac{1}{\pi}}, \infty\right)^4$. *Thus, the studied problem will be written as follows*

$$\begin{cases} \partial_t^{\frac{3}{2}}\omega - \frac{9}{272}\Delta\omega = F\left(t, x, \omega, \partial_t^{\frac{5}{4}}\omega, \Delta\omega\right), (t, x_1, \ldots, x_4) \in \Omega \\ \omega(0, x_1, \ldots, x_4) = 2\left(x_1^2 + \cdots + x_4^2\right), \ \frac{\partial\omega}{\partial t}(0, x_1, \ldots, x_4) = 0, \end{cases} \tag{5.18}$$

where

$$\begin{aligned} F\left(t, x, \omega, \partial_t^{\frac{5}{4}}\omega, \Delta\omega\right) &= \frac{\pi |x|^2 \cos\left(|x|^{-\frac{4}{3}} t\right)}{\left(8 + \tan\left(|x|^{-\frac{4}{3}} t\right)\right)\left[|x|^2 + |\omega| + |x|^2 \left|\partial_t^{\frac{5}{4}}\omega\right|\right]} - \frac{9}{272}\Delta\omega \\ &= f\left(\eta, \varphi, \varphi', {}^C\mathcal{D}_{0^+}^{\frac{5}{4}}\varphi(\eta)\right) - \frac{1}{17}\eta^2\varphi''(\eta), \end{aligned}$$

with $\eta \in \left[0, \frac{\pi}{4}\right]$ *and*

$$f(\eta, u, v, w) = \frac{\pi \cos(\eta)}{(8 + \tan(\eta))[1 + |u| + |w|]} - \frac{9}{34}u + \frac{1}{2}\eta v, \text{ for } \eta \in \left[0, \frac{\pi}{4}\right].$$

As $\tan(\eta)$, $\cos(\eta)$ *are positive continuous functions for* $\eta \in \left[0, \frac{\pi}{4}\right]$, *the function* u *is jointly continuous. For any* $u, v, w, \bar{u}, \bar{v}, \bar{w} \in \mathbb{C}$ *and* $\eta \in \left[0, \frac{\pi}{4}\right]$, *we have* $\frac{\sqrt{2}}{2} \leq \cos(\eta) \leq 1$, *and* $0 \leq \tan(\eta) \leq 1$, *also*

$$|f(\eta, u, v, w) - f(\eta, \bar{u}, \bar{v}, \bar{w})| \leq \left(\frac{9}{34} + \frac{\pi}{8}\right)|u - \bar{u}| + \frac{\pi}{8}|v - \bar{v}| + \frac{\pi}{8}|w - \bar{w}|.$$

Hence, hypothesis $(hyp.2)$ is satisfied with

$$\gamma_1 = \frac{9}{34} + \frac{\pi}{8}, \ \gamma_2 = \frac{\pi}{8}, \ \gamma_3 = \frac{\pi}{8},$$

and

$$\begin{aligned}\lambda &= \frac{\alpha \ell^{\beta-1}\left|\ell q+\gamma_2\right|+\gamma_3}{\ell^{\beta-\alpha}\Gamma\left(\alpha-\beta+1\right)} \simeq 0.40786 \\ &< 1.\end{aligned}$$

What remains is to show that condition (5.6)

$$\frac{\ell^{\alpha}\left|\delta\kappa^2\left(\delta+m-2\right)+\gamma_1\right|}{\left(1-\lambda\right)\Gamma\left(\alpha+1\right)} \simeq 0.92005 < 1,$$

is satisfied. It follows from theorem 5.2 that the problem (5.18) has a unique solution on Ω.

5.4 Main Theorems' Proof

This section demonstrates the proof of the existence and uniqueness of solutions of the given problem for a multidimensional nonlinear time and space-fractional reaction-diffusion/wave equation, which is

$$\begin{cases} \partial_t^{\alpha}\omega-\kappa^2\Delta\omega=F\left(t,x,\omega,\partial_t^{\beta}\omega,\left(-\Delta\right)^s\omega\right), & (t,x)\in\Omega,\ \kappa\in\mathbb{R}^*, \\ \omega\left(0,x\right)=\left|x\right|^{\delta}c_0,\ \frac{\partial\omega}{\partial t}\left(0,x\right)=0, & \delta, c_0\in\mathbb{C}, \end{cases} \tag{5.19}$$

under the radially symmetric form

$$\omega\left(t,x\right)=\left|x\right|^{\delta}\varphi\left(\eta\right),\ \text{with } \eta=\left|x\right|^{-\frac{2}{\alpha}}t\,. \tag{5.20}$$

Proof of Theorem 5.1

Assume that hypotheses $(hyp.1)-(hyp.3)$ hold. Using transformation (5.20), problem (5.19) is reduced to fractional order's ordinary differential equation of the form

$${}^{C}\mathcal{D}_{0^+}^{\alpha}\varphi\left(\eta\right)=g\left(\eta\right),\ \eta\in J, \tag{5.21}$$

where

$$g\left(\eta\right)=p\varphi\left(\eta\right)+q\eta\varphi'\left(\eta\right)+f\left(\eta,\varphi\left(\eta\right),\varphi'\left(\eta\right),{}^{C}\mathcal{D}_{0^+}^{\beta}\varphi\left(\eta\right)\right)$$

with

$$p=\delta\kappa^2\left(\delta+m-2\right) \text{ and } q=-\frac{4}{\alpha^2}\left(\alpha\delta+\alpha+1+\frac{m\alpha}{2}\right)\kappa^2, \tag{5.22}$$

along with the conditions

$$\varphi\left(0\right)=c_0 \text{ and } \varphi'\left(0\right)=0. \tag{5.23}$$

By using (5.22), the condition (5.5) is equivalent to (5.12), which is

$$\frac{\ell^{\alpha}\left|p+b^*\right|}{\Gamma\left(\alpha+1\right)\left(1-\lambda\right)}<1,\ \text{with } \lambda\in\left(0,1\right).$$

Therefore, after proving that problem (5.21)–(5.23) has a solution in Theorem 5.4 when (5.12) holds, we can similarly prove the existence of at least a solution of the problem for the multidimensional nonlinear time and space-fractional reaction-diffusion/wave equation (5.19) under the radially symmetric form (5.20). This can be achieved if (5.5) holds. The proof is complete.

Proof of Theorem 5.2

Similarly to the steps that we followed during the proof of Theorem 5.1, the existence and uniqueness of a radically symmetric solution to problem (5.19) is demonstrated using Theorem 5.5, provided that the condition (5.6) holds true. The proof is complete.

5.5 Explicit Solutions

Now, we present some explicit solutions on the radially symmetric form of the problem (5.19).

Solution 1: Let $p, q, \rho \in \mathbb{C}$, for $1 < \beta \leq \alpha \leq 2$, we get that

$$\varphi(\eta) = \eta^{\rho}, \text{ with } \mathrm{Re}(\rho) > 1,$$

is a solution of (5.21)–(5.23), where

$$f\left(\eta, \varphi(\eta), \varphi'(\eta), {}^{C}\mathcal{D}_{0^+}^{\beta}\varphi(\eta)\right) = \frac{\eta^{\beta-\alpha}\Gamma(\rho-\beta+1)}{\Gamma(\rho-\alpha+1)} \, {}^{C}\mathcal{D}_{0^+}^{\beta}\varphi(\eta) - p\varphi(\eta) - q\eta\varphi'(\eta).$$

Then the radially symmetric solution of the problem (5.19) is presented as follows

$$\omega(t,x) = |x|^{\delta - \frac{2\rho}{\alpha}} t^{\rho},$$

where

$$F\left(t, x, \omega, \partial_t^{\beta}\omega, (-\Delta)^{s}\omega\right) = \frac{\Gamma(\rho-\beta+1)\,\omega(t,x)}{\Gamma(\rho-\alpha+1)\,t^{\alpha-\beta+\rho}} |x|^{\frac{2\rho}{\alpha}-\delta} \partial_t^{\beta}\omega(t,x) - \kappa^2\Delta\omega(t,x).$$

Solution 2: Let $p, q, \rho \in \mathbb{C}$, for $1 < \beta \leq \alpha \leq 2$, we have

$$\varphi(\eta) = \exp(\rho\eta) - \rho\eta,$$

which is a solution of (5.21)–(5.23), where

$$f\left(\eta, \varphi(\eta), \varphi'(\eta), {}^{C}\mathcal{D}_{0^+}^{\beta}\varphi(\eta)\right) = \frac{\eta^{\beta-\alpha}E_{1,3-\alpha}(\rho\eta)}{E_{1,3-\beta}(\rho\eta)} \, {}^{C}\mathcal{D}_{0^+}^{\beta}\varphi(\eta) - p\varphi(\eta) - q\eta\varphi'(\eta).$$

Here, $E_{\alpha,\beta}(\eta)$ is the Mittag-Leffler function. Then the solution of the problem (5.19) is presented as follows

$$\omega(t,x) = |x|^{\delta}\left(e^{\rho|x|^{-\frac{2}{\alpha}}t} - \rho|x|^{-\frac{2}{\alpha}}t\right),$$

where

$$F\left(t,x,\omega,\partial_t^{\beta}\omega,(-\Delta)^{s}\omega\right) = \frac{|x|^{-\delta}t^{\beta-\alpha}E_{1,3-\alpha}\left(\rho|x|^{-\frac{2}{\alpha}}t\right)\omega(t,x)}{\left(e^{\rho|x|^{-\frac{2}{\alpha}}t} - \rho|x|^{-\frac{2}{\alpha}}t\right)E_{1,3-\beta}\left(\rho|x|^{-\frac{2}{\alpha}}t\right)}\partial_t^{\beta}\omega(t,x) - \kappa^2\Delta\omega(t,x).$$

Solution 3: Let $p,q,\rho\in\mathbb{C}$, for $1<\beta\leq\alpha\leq 2$, we get that

$$\varphi(\eta) = \sin(\rho\eta) + \cos(\rho\eta) - \rho\eta,$$

is a solution of the problem (5.21)–(5.23), where

$$\begin{aligned} f\left(\eta,\varphi(\eta),\varphi'(\eta),{}^{C}\mathcal{D}_{0^+}^{\beta}\varphi(\eta)\right) &= \frac{\eta^{\beta-\alpha}\left[(i-1)E_{1,3-\alpha}(i\rho\eta) - (1+i)E_{1,3-\alpha}(-i\rho\eta)\right]}{\left[(i-1)E_{1,3-\beta}(i\rho\eta) - (1+i)E_{1,3-\beta}(-i\rho\eta)\right]}\,{}^{C}\mathcal{D}_{0^+}^{\beta}\varphi(\eta) \\ &\quad - p\varphi(\eta) - q\eta\varphi'(\eta). \end{aligned}$$

Then the solution of the problem (5.19) is presented as follows

$$\omega(t,x) = |x|^{\delta}\left(\sin\left(\rho|x|^{-\frac{2}{\alpha}}t\right) + \cos\left(\rho|x|^{-\frac{2}{\alpha}}t\right) - \rho|x|^{-\frac{2}{\alpha}}t\right),$$

where

$$\begin{aligned} F\left(t,x,\omega,\partial_t^{\beta}\omega,(-\Delta)^{s}\omega\right) &= -\kappa^2\Delta\omega(t,x) + \frac{|x|^{-\delta}t^{\beta-\alpha}\omega(t,x)\,\partial_t^{\beta}\omega(t,x)}{\sin\left(\rho|x|^{-\frac{2}{\alpha}}t\right) + \cos\left(\rho|x|^{-\frac{2}{\alpha}}t\right) - \rho|x|^{-\frac{2}{\alpha}}t}\times \\ &\quad \frac{\left[(i-1)E_{1,3-\alpha}\left(i\rho|x|^{-\frac{2}{\alpha}}t\right) - (1+i)E_{1,3-\alpha}\left(-i\rho|x|^{-\frac{2}{\alpha}}t\right)\right]}{\left[(i-1)E_{1,3-\beta}\left(i\rho|x|^{-\frac{2}{\alpha}}t\right) - (1+i)E_{1,3-\beta}\left(-i\rho|x|^{-\frac{2}{\alpha}}t\right)\right]}. \end{aligned}$$

Conclusion

In this thesis, we have studied the existence and uniqueness of solutions to four problems for fractional FPDEs with CAPUTO derivative operator:

Firstly, we have discussed the existence and uniqueness of solutions by traveling wave transformation for higher-order space-fractional wave equations

$$\partial_t^2 \omega = \kappa^2 \partial_x^\alpha \omega,\ \kappa \in \mathbb{R}^*,\ m-1 \leq \alpha < m \in \mathbb{N} - \{0,1,2\}.$$

Secondly, we have studied by traveling wave forms the space-fractional Jordan-Moore-Gibson-Thompson equations of nonlinear acoustics

$$\tau \omega_{ttt} + \mu \omega_{tt} - \kappa^2 \partial_x^\alpha \omega - \delta \partial_x^\alpha \omega_t = F\left(x, t, \omega, \omega_t, \omega_{tt}, \omega_{xx}, (\omega_t)_{xx}\right),\ \text{for } 1 < \alpha \leq 2.$$

Thirdly, we have discussed the existence of solutions by traveling profile forms for diffusion equations of moving fractional order

$$\partial_t \omega = \kappa \partial_x^\alpha \omega,\ \kappa \in \mathbb{R}^*.$$

Fourthly, we have studied by radially symmetric solution the multidimensional nonlinear time and space-fractional reaction-diffusion/wave equation

$$\partial_t^\alpha \omega - \kappa^2 \Delta \omega = F\left(t, x, \omega, \partial_t^\beta \omega, (-\Delta)^s \omega\right),\ \text{for } 0 < s \leq 1 < \beta \leq \alpha \leq 2.$$

In chosen Banach spaces. For this purpose, we have proposed new methods for transforming FPDEs to fractional order differential equations. We have used several fixed point theorems such as Banach and Schauder to prove the results. We have also provided an illustrative example of each of our considered problems to show the validity of the conditions and justify the efficiency of our established results.

The future prospects are:

1. Research of numerical and analytical methods to solve fractional order partial differential equations in time and space, more precise than those proposed in this thesis

2. The application of one of these proposed methods to solve the partial differential equation of fractional order, but with another fractional derivative operator (in the sense of RIEMANN-LIOUVILLE, of GRUNWALD LETNIKOV, and in the sense of HADAMARD).

3. Study the existence and uniqueness of solutions for some systems of PDEs of fractional order.

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تَــمَ بِحَمْدِ اللهِ

"الْحَمْدُ لِلَّهِ الَّذِي لَهُ مَا فِي السَّمَاوَاتِ وَمَا فِي الْأَرْضِ وَلَهُ الْحَمْدُ فِي الْآخِرَةِ وَهُوَ الْحَكِيمُ الْخَبِيرُ" [سبأ: 01]

الملخص:

نتطرق في هذه الأطروحة لدراسة عدة نتائج حول وجود ووحدانية الحلول لمعادلات ذات اشتقاقات جزئية غير خطية ذات رتبةكسرية بمفهوم كابتو، بشروط حدودية، بقيم ابتدائية في فضاء بناخي، وذلك باستعمال تقنية نظرية النقطة الثابتة لشودار ومبدأ بناخ للتقلص.

كلمات مفتاحية: معادلات ذات اشتقاقات جزئية ذات رتبةكسرية، معادلات تفاضلية ذات رتبةكسرية، مشتقات كسرية لكابتو، نقطة ثابتة، فضاء بناخي، مسألة بشروط حدودية، مسألة بقيم ابتدائية، وجود، وحدانية.

Résumé :

Dans cette thèse, nous allons discuter plusieurs résultats d'existence et d'unicité de solutions pour certaines équations aux dérivées partielles non linéaires d'ordre fractionnaire de type Caputo, avec des valeurs aux limites, valeurs initiales, dans un espace de Banach, en utilisant le principe de contraction de Banach et le théorème de point fixe de Schauder.

Mots clés: *Equations aux dérivés partielles fractionnaires, équation différentielle fractionnaire, dérivée fractionnaire de Caputo, point fixe, espace de Banach, problème aux limites, problème de valeurs initiales, existence, unicité.*

Abstract:

In this thesis, we discuss several existence and uniqueness results of solutions for some nonlinear partial differential equations of fractional order of Caputo type, with boundary value, initial values in Banach space, we use the Banach contraction principle and Schauder fixed point theorem.

Key words: *Fractional partial differential equation, fractional differential equations, fractional derivative of Caputo, fixed point, Banach space, boundary value problem, initial values problem, existence, uniqueness.*

A.M.S Classifications: *35R11, 35A01, 34A08, 35C06, 34K37.*