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Linear fractional differential equations with generalized operators

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Abstract – The aim of this work is the study of linear fractional differential equations with generalized operators, represented in the initial values problems whose elements belong to the Banach space, as well as linear differential equations of the type.

$$y^{(n\alpha)} + \sum_{k=0}^{n-1} a_k(x) y^{(k\alpha)} = f(x),$$

where $0 < \alpha \leq 1$ and a_k, f are continuous functions. This is through the use of ordinary linear differential equations techniques.

Keywords – fractional derivative, fractional differential equation, generalized operators.

I. Introduction

Ordinary differential equations are considered one of the oldest branches of mathematics, as it has its roots in the works of Newton and Leibniz, and the most important works revolving around this branch, we find the existence and oneness of elementary value issues, which are classic works around which many works revolve (see for example [3, 11]).

There is another branch that appeared old and was worked on recently, which is the branch of fractional differential equations, where it appeared with the famous question that Leibnitz asked, which is the definition of the derivative of the order 1/2, to develop the work with Riemann, Liouville, Caputo, Hadamard ...

Fractional-order differential equations have been used in the study of models of many phenomena in various fields of science and engineering, such as viscoelasticity, fluid mechanics, electrochemistry, control, porous media, mathematical biology and electromagnetic bio-engineering. More details are available, for instance, in the books Samko et al. 1993 [23], Podlubny 1999 [18], Kilbas et al. 2006 [16], Sabatier et al. 2007 [22], Das 2008 [8], Diethelm 2010 [9], Mathai and Haubold 2018 [17].

In the year 2011 Katugampola introduced a new definition of the fractional integral [13], which generalizes the Riemann-Liouville fractional integral and the Hadamard fractional integral, and then introduces the fractional derivative corresponding to this fractional integral and some properties (see [14]). Also in a recent paper [2], Almeida et al. define the Caputo version of the generalized fractional derivative. More details on this fractional derivatives mentioned above can be found in [12, 19, 20].

Many authors have conducted studies on the problems of elementary values of fractional differential equations with nonlinear data, but few of them dealt with linear data, and among those works we find those related to linear problems of the Riemann-Liouville pattern (see [7, 21]).

In our work, we will deal with the study of the existence and the uniqueness of solutions of the fractional differential equations with Caputo-Katigampula type according to the following methodology: In section one, we give some preliminaries according the generalized operators, and some important spaces. In section two, we study linear fractional differential equation with initial value condition, whose solutions belong to a Benach space, which generalizes the study presented in [2]. In the section three, we will deal with high-order fractional differential equation, attached to elementary values that enable us to return them to a differential equation of fractional order with one initial value condition, to be studied as in the second section. Finally, we present a practical method for finding solutions of a linear fractional differential equation with constant coefficients, with some examples.

II. Preliminaries

In [13], the author introduces a fractional integral operator. Next, in [14] he gave a fractional operator derivative. After that, in [2] the authors introduced a new derivative operator in the Caputo type. In this section, we give some notions properties for those operators and special functional spaces.

Let $[a,b]$ $(0<a<b<+\infty)$ be an interval of the real axis $\mathbb{R}, c\in\mathbb{R}$ and p such that $1\leq p\leq +\infty$. All functions in this paper considered real-valued functions. Let α,ρ be positive real numbers, we set $\delta = x\frac{d}{dx}$.

Definition 2.1 [16]

i) *$AC[a,b]$ is the space of primitive Lebesgue integrable functions, i.e there exists $\varphi\in L^1(a,b)$ such that*

$$f(x)=f(a)+\int_a^b \varphi(t)dt.$$

ii) $AC^n[a,b]$, $(n\in\mathbb{N})$ is the space of functions f which have continuous derivatives up to order n on $[a,b]$ such that $f^{(n-1)}\in AC[a,b]$. In particular $AC^1[a,b]=AC[a,b]$.

iii) $AC^n_{\delta,\rho}[a,b]$, $(n\in\mathbb{N})$ is the space of measurable functions f on (a,b) such that $x^\rho f(x)$ has $\delta -$ derivative up to order $n-1$ on $[a,b]$ and $\delta^{n-1}(x^\rho g(x))\in AC[a,b]$.

In particular $AC^1_{\delta,\rho}[a,b]=AC_{\delta,\rho}[a,b]$.

iv) $X^p_c(a,b)$ is the space of Lebesgue measurable functions f on (a,b) which $\| f \|_{X^p_c}<\infty$, with

$$\| f \|_{X^p_c}=\left(\int_a^b |t^c f(t)|^p\right)^{\frac{1}{p}}\ (1\leq p<\infty),$$

and

$$\| f \|_{X^\infty_c}= ess\sup_{a\leq x\leq b} x^c|f(x)|.$$

Now, let $f\in X^p_c(a,b)$.

Definition 2.2 [13] *We define a generalized fractional integral operator by*

$$\mathcal{J}^{\alpha,\rho}_{a^+}f(x)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^x \frac{t^{\rho-1}}{(x^\rho-t^\rho)^{1-\alpha}}f(t)dt.$$

where $\Gamma(\alpha)=\int_0^{+\infty} e^{-t}t^{\alpha-1}dt$, is the Euler gamma function.

Definition 2.3 [14] *We define a generalized fractional derivative by*

$$\mathcal{D}^{\alpha,\rho}_{a^+}f(x)=\left(x^{1-\rho}\frac{d}{dx}\right)^n(\mathcal{J}^{n-\alpha,\rho}_a f)(x)$$

$$=\frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\left(x^{1-\rho}\frac{d}{dx}\right)^n\int_a^x \frac{t^{\rho-1}f(t)}{(x^\rho-t^\rho)^{\alpha-n+1}}dt,$$

where $n=[\alpha]+1$.

Definition 2.4 [2] *We define a generalized fractional derivative with Caputo-type by*

$$\begin{aligned}&{}^c\mathcal{D}^{\alpha,\rho}_{a^+}f(x)\\ &=\mathcal{D}^{\alpha,\rho}_{a^+}[f(x)\\ &-\sum_{k=0}^{n-1}\frac{\left(x^{1-\rho}\frac{d}{dx}\right)^n f(a)}{k!}\left(\frac{x^\rho-a^\rho}{\rho}\right)^k].\end{aligned}$$

In case $0<\alpha<1$, we have

$${}^c\mathcal{D}^{\alpha,\rho}_{a^+}f(x)=\mathcal{D}^{\alpha,\rho}_{a^+}[f(x)-f(a)].$$

Theorem 2.1 [12, 19] *Let $\alpha,\rho>0$ and $n=[\alpha]+1$. If $f\in AC^n_{\delta,\rho}[a,b]$ then*

$${}^c\mathcal{D}^{\alpha,\rho}_{a^+}f(x)={}^\rho\mathcal{J}^{n-\alpha}_{a^+}\left(x^{1-\rho}\frac{d}{dx}\right)^n f(x)$$

$$= \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \int_a^x \frac{t^{\rho-1}\left(t^{1-\rho}\frac{d}{dt}\right)^n f(t)}{(x^\rho - t^\rho)^{\alpha-n+1}} dt.$$

Theorem 2.2 [12, 19] *Let* $\alpha, \beta, \rho > 0$ *and* $n = [\alpha] + 1, m = [\beta] + 1$. *If* $f \in AC_{\delta,\rho}^{n+m}[a, b]$ *then*

$$({}^C\mathcal{D}_{a^+}^{\alpha,\rho}\, {}^C\mathcal{D}_{a^+}^{\beta,\rho} f)(x) = ({}^C\mathcal{D}_{a^+}^{\alpha+\beta,\rho} f)(x).$$

Definition 2.5 [2] *The Mittag-Leffler function with dependence with two parameters* $\alpha > 0, \beta > 0$ *is defined by the series*

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{+\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}, \quad x \in \mathbb{C},$$

in particular

$$E_{\alpha,1}(x) = E_\alpha(x), \quad E_1(x) = e^x.$$

Proposition 2.3 [2] *For* $\lambda, x \in \mathbb{C}$ *we have*

$$\begin{aligned}({}^C\mathcal{D}_{a^+}^{\alpha,\rho} E_\alpha)(\lambda(x^\rho - a^\rho)^\alpha) \\ = \lambda \rho^\alpha E_\alpha(\lambda(x^\rho - a^\rho)^\alpha).\end{aligned}$$

III. Cauchy problem for a linear vectorial fractional differential equation

In this section, we give a study of existence and uniqueness of solution of a differential equation of fractional order with initial condition. This study informs us in the following paragraph that is concerned with the existence and unity of the solution of a linear differential equation of higher order with initial conditions on the solution and its successive derivatives.

Let E be a reflexive Banach space with a norm $\| . \|_E$, and $C([a, b], E)$ the space of continuous functions from $[a, b]$ to E, with the norm

$$\| F \|_{C([a,b],E)} = \sup_{a \le x \le b} \| F(x) \|_E.$$

We use the notations $X_{c,E}^p(a, b)$, $AC_E[a, b], AC_E^n[a, b], AC_{\delta,\rho,E}^n[a, b]$ for the spaces of the vector functions from $[a, b]$ in E. We can then generalize the notions of fractional derivatives and integrals to the preceding sense.

The following results are immediate generalizations of the results obtained in [2], [19].

Theorem 3.1 *The* $\mathcal{I}_{a^+}^{\alpha,\rho}$ *is linear and bounded from* $C([a, b], E)$ *to* $C([a, b], E)$, *i.e*

$$\| \mathcal{I}_{a^+}^{\alpha,\rho} F \|_{C([a,b],E)} \le K_{\alpha,\rho} \| F \|_{C([a,b],E)}.$$

where $K_{\alpha,\rho} = \frac{\rho^{-\alpha}(b^\rho - a^\rho)}{\Gamma(\alpha+1)}$.

Theorem 3.2 *Let* $\alpha, \beta, \rho > 0$ *and* $n = [\alpha] + 1,\ m = [\beta] + 1$. *Assume that there exists* k *such that* $k - 1 < \alpha + \beta < k$. *If* $F \in AC_{\delta,\rho,E}^k[a, b]$ *then*

$$({}^C\mathcal{D}_{a^+}^{\alpha,\rho}\ {}^C\mathcal{D}_{a^+}^{\alpha,\rho} F)(x) = ({}^C\mathcal{D}_{a^+}^{\alpha+\beta,\rho} F)(x).$$

Let $A \in \mathcal{L}(C([a, b], E))$ and $B \in C([a, b], E)$, we set $\| A \| = \| A \|_{\mathcal{L}(C([a,b],E))}$ and $\| B \| = \| B \|_{C([a,b],E)}$.

Consider the vectorial fractional differential equation with initial condition

$$\begin{cases}({}^C\mathcal{D}_{a^+}^{\alpha,\rho} Y)(x) = A(x).Y(x) + B(x) \\ Y(a) = Y_a\end{cases}, \qquad (3.1)$$

where $0 < \alpha \le 1$.

As in [2], the problem (3.1) equivalent to the following Volterra equation

$$Y(x) = Y_a + \mathcal{I}_{a^+}^{\alpha,\rho}(A.Y + B)(x). \qquad (3.2)$$

It should be noted that the initial condition $Y(a) = Y_a$ can be replaced by any other initial condition $Y(x_0) = Y_0$ where $x_0 \in [a, b[$.

The following theorem assure the existence and uniqueness of solution at the equation (3.2), consequently the problem (3.1).

Theorem 3.3 *The problem* (*3.2*) *have a unique solution in the space*

$$\begin{aligned}\mathcal{E}_\alpha([a, b], E) = \{F \in C([a, b], E);\ \mathcal{D}_{a^+}^{\alpha,\rho} F \\ \in C([a, b], E)\}.\end{aligned}$$

Proof. We shall proof the existence of solution in $C([a, z])$. For this, we consider (Y_n) the sequence defined by

$$Y_0(x) = Y_a,$$

$$\forall n \ge 0; \quad Y_{n+1}(x) = Y_a + I_{a^+}^{\alpha,\rho}(A.Y_n + B)(x).$$

We will show by induction that

$$\forall n \ge 1,\ \forall x \in a, b];\ \| Y_n(x) - Y_{n-1}(x) \|_E \le \frac{\rho^{-n\alpha}\|A\|^{n-1}(\|A\|\|Y_a\|_E + \|B\|)}{\Gamma(n\alpha+1)}(x^\rho - a^\rho)^{n\alpha}.$$

First, we have

$$
\begin{aligned}
\| Y_1(x) - Y_0(x) \|_E &= \| I^{\alpha,\rho}_{a^+}(A.Y_a + B)(x) \|_E \\
&\leq (\| A \|.\| Y_a \| + \| B \|) I^{\alpha,\rho}_{a^+} 1 \\
&= \frac{\rho^{-\alpha}(\|A\|\|Y_a\|_E + \|B\|)}{\Gamma(\alpha+1)} (x^\rho - a^\rho)^\alpha .
\end{aligned}
$$

Assuma that for $n \geq 1$

$$\| Y_n(x) - Y_{n-1}(x) \|_E \leq \frac{\rho^{-n\alpha}\|A\|^{n-1}(\|A\|\|Y_a\|_E + \|B\|)}{\Gamma(n\alpha+1)} (x^\rho - a^\rho)^{n\alpha}.$$

Then

$$
\begin{aligned}
\| Y_{n+1}(x) - Y_n(x) \|_E &= \| I^{\alpha,\rho}_{a^+}[A(Y_n - Y_{n-1})](x) \|_E \\
&\leq \| A \| \| I^{\alpha,\rho}_{a^+}(Y_n - Y_{n-1})(x) \|_E \\
&\leq \frac{\rho^{-n} \ \|A\|^n(\|A\|\|Y_a\|_E + \|B\|)}{\Gamma(\alpha+1)} I^{\alpha,\rho}_{a^+}(x^\rho - a^\rho)^{n\alpha} \\
&= \frac{\rho^{-(n+1)\alpha}\|A\|^n(\|A\|\|Y_a\|_E + \|B\|)}{\Gamma((n+1)\alpha+1)} (x^\rho - a^\rho)^{(n+1)\alpha}
\end{aligned}
$$

Now, we look that

$$\forall n \geq 1,\ \forall x \in a,b];\ \| Y_n(x) \|_E \leq \| Y_a \|_E + \sum_{k=1}^{n} \| Y_k(x) - Y_{k-1}(x) \|_E,$$

Taking into account the convergence of the numerical series

$$\frac{\rho^{-n\alpha}\|A\|^{n-1}(\|A\|\|Y_a\|_E + \|B\|)}{\Gamma(n\alpha+1)} (x^\rho - a^\rho)^{(n+1)\alpha},$$

thus involves the normal convergence of the function series

$$\sum_{n=0}^{+\infty} (Y_{n+1}(x) - Y_n(x)),$$

and as the space E is a Banach space, we deduce that this series is uniformly convergence on $[a,b]$. The sequence of functions $(Y_n(x))$ is therefore uniformly convergent and since the functions Y_n are continuous their limit Y is continues, as well as $A(x).Y(x)$. since the continuity of the operator $I^{\alpha,\rho}_{a^+}$, we get

$$
\begin{aligned}
Y(x) &= \lim_{n\to+\infty} Y_{n+1}(x) \\
&= \lim_{n\to+\infty} [Y_a + I^{\alpha,\rho}_{a^+}(A.Y_n + B)(x)] \\
&= Y_a + I^{\alpha,\rho}_{a^+}\left[\lim_{n\to+\infty}(A.Y_n + B)(x)\right] \\
&= Y_a + I^{\alpha,\rho}_{a^+}(A.Y + B)(x).
\end{aligned}
$$

Hence, Y is a solution of (3.2).

Now, let Y^1, Y^2 two solutions of (3.2), hence $T = Y^1 - Y^2$ is a solution of the problem

$$\begin{cases} ({}^C\mathcal{D}^{\alpha,\rho}_{a^+}T)(x) = A(x).T(x) \\ T(a) = 0 \end{cases}, \tag{3.3}$$

Using the same arguments as before, we obtain

$$\forall x \in [a,b];\ T(x) = \mathcal{I}^{\alpha,\rho}_{a^+}(A.T)(x).$$

Hence;

$$\forall x \in [a,b];\ \| T(x) \|_E \leq \mathcal{I}^{\alpha,\rho}_{a^+} \| A(x) \|_E \| T(x) \|_E.$$

Using the integral form of Grönwall's inequality, we deduce that $T = 0$,

so $Y_1 = Y_2$.

Now, we consider the homogeneous equation

$$({}^C\mathcal{D}^{\alpha,\rho}_{a^+}Y)(x) = A(x).Y(x). \tag{3.4}$$

The following theorem shows that the space of solutions of the homogeneous equation (3.4) is isomorphic to space E.

Theorem 3.4 *Let $\mathcal{S}$ be the space of solutions of the equation* (3.4). *Then, for all $x_0 \in a,b[$ the map φ from $\mathcal{S}$ to E defined by $\varphi(Y) = Y(x_0)$ is an isomorphism.*

Proof. It is obvious that the map φ is linear, it is bijective because of Theorem 2.3, as well as the inverse of φ.

The following corollary is important for studying a special linear differential equation of fractional order in the next section.

Corollary 3.5 *If space E is of finite dimension n, then $\mathcal{S}$ is of the same dimension.*

Definition 3.1 *Suppose that E is of finite dimension n, and let $Y_1, Y_2, \cdots, Y_n$ be solutions of the homogeneous equation* (3.4). *We say that $(Y_1, Y_2, \ldots, Y_n)$ is a fundamental system of the equation (3.4) if and only if $(Y_1, Y_2, \cdots, Y_n)$ is linearly independent.*

Proposition 3.6 *Let $(Y_1, Y_2, \cdots, Y_p)$ be solutions of* (2.4). *$(Y_1, Y_2, \cdots, Y_p)$ are linearly independent in*

$\mathcal{S}$ *if and only if their values at a point* $x_0 \in [a, b]$ *are linearly independent vectors in* E. *Consequently, their values at each point* $x \in [a, b]$ *are linearly independent vectors in* E.

Proposition 3.7 *Let* Y_p *be a particular solution of the inhomogeneous equation*

$$({}^C\mathcal{D}_{a^+}^{\alpha,\rho} Y)(x) = A(x).Y(x) + B(x), \tag{3.5}$$

and let $(Y_1, Y_2, \cdots, Y_n)$ be a fundamental system of the associated homogeneous equation. Any solution Y of the equation (3.5) is written as follows

$$Y = Y_p + \sum_{k=0}^{n} \lambda_k Y_k,$$

where $\lambda_1, \lambda_2, \cdots, \lambda_n \in \mathbb{R}$.

Remark 3.8 *Definition 3.1, propositions 3.6 and 3.7 are generalizations of the concepts and results obtained in the ordinary case* (*see for example [10] Ch1*).

Let us consider the following system

$$\begin{cases} ({}^C\mathcal{D}_{a^+}^{\alpha,\rho} Y_i)(x) = \sum_{k=1}^{n} a_{ik}.Y_i(x) \\ Y_i(a) = Y_a \end{cases} (1 \le i \le n), \tag{3.6}$$

where $a_{ij} \in \mathbb{R}, 1 \le i, j \le n$. We can write this system in the following form

$$\begin{cases} ({}^C\mathcal{D}_{a^+}^{\alpha,\rho} Y)(x) = A.Y(x) \\ Y(a) = Y_a \end{cases}, \tag{3.7}$$

where $A = (a_{ij})$ is the $n \times n$ matrix with real constant coefficients.

In the following, we give an explicit form of the solution of Cauchy problem (3.7), for that we will extend the definition of the exponential of a matrix to a definition adapted to the fractional case.

We then have the following definition

Definition 3.2 *The Mittag-Leffler operator associated with the operator A with two parameters* $\alpha > 0, \beta > 0$ *is defined by the series*

$$E_{\alpha,\beta}(A) = \sum_{k=0}^{+\infty} \frac{A^k}{\Gamma(\alpha k + \beta)},$$

where $A^{=}I_n$ and $A^{k+} = AA^k$

in particular

$$E_{\alpha,1}(A) = E_\alpha(A), \quad E_1(A) = e^A.$$

Remark 3.9 $E_{\alpha,\beta}(A)$ *is well defined, indeed let us set* $E_{\alpha,\beta}^m(A) = \sum_{k=0}^{m} \frac{A^k}{\Gamma(\alpha k + \ )}$.

We have $\| E_{\alpha,\beta}^m(A) \| \le \sum_{k=0}^{m} \frac{\|A\|^k}{\Gamma(\alpha k + \beta)}$, then

$\| E_{\alpha,\beta}(A) \| = \lim_{n \to +\infty} \| E_{\alpha,\beta}^m(A) \| \le \lim_{n \to +\infty} \sum_{k=0}^{m} \frac{\|A\|^k}{\Gamma(\alpha k + \beta)} = E_{\alpha,\beta}(\| A \|)$.

as in the usual Mittag-Leffler function, we have

Proposition 3.10 *For* $\lambda \in \mathbb{C}$ *we have*

$$\begin{aligned} &({}^C\mathcal{D}_{a^+}^{\alpha,\rho} E_\alpha)(\lambda(x^\rho - a^\rho)^\alpha A) \\ &\qquad = \lambda \rho^\alpha A.E_\alpha(\lambda(x^\rho - a^\rho)^\alpha A). \end{aligned}$$

The following theorem gives an explicit form of the solution of the problem (3.7)

Theorem 3.11 *The unique solution of the problem* (2.7) *is given by*

$$\Phi(x) = E_\alpha\left(\left(\frac{x^\rho - a^\rho}{\rho}\right)^\alpha .A\right).Y_a.$$

Proof. It is easy to verify that $\Phi(a) = Y_a$. We have also

$$\begin{aligned} {}^C\mathcal{D}_{a^+}^{\alpha,\rho} \Phi(x) &= {}^C\mathcal{D}_{a^+}^{\alpha,\rho} (E_\alpha (x^\rho - a^\rho)^\alpha.\rho^{-\alpha}.A).Y_a \\ &= A.E_\alpha\left(\left(\frac{x^\rho - a^\rho}{\rho}\right)^\alpha .A\right).Y_a \\ &= A.\Phi(x) \end{aligned}$$

IV. SEQUENTIAL LINEAR FRACTIONAL DIFFERENTIAL EQUATION

In this section, we will study a particular case of the differential equations of fractional order, it is the linear equation with continuous coefficients, depends only on variable.

Let $n \ge 2$, be an integer number and $0 < \alpha \le 1$, we set

$$\begin{aligned} &y^0 = y, \\ &y^{(k\alpha)} = {}^C\mathcal{D}_{a^+}^{\alpha,\rho} y^{((k-1)\alpha)}, \ 1 \le k \le n. \end{aligned} \tag{4.1}$$

We consider the differential equation of fractional order

$$y^{(n\alpha)} + \sum_{k=0}^{n-1} a_k(x) y^{(k\alpha)} = f(x), \qquad (4.2)$$

where $f(x), a_k(x)$ $(0 \leq a_k \leq n-1)$ are continuous functions defined in $[a,b]$.

We associate to this equation the following homogeneous equation

$$y^{(n\alpha)} + \sum_{k=0}^{n-1} a_k(x) y^{(k\alpha)} = 0. \qquad (4.3)$$

The following theorem allowed us to study the existence and the uniqueness of a solution of equation (4.2), by giving initial conditions on the successive derivatives $y^{(k\alpha)}$ at the point a.

Theorem 4.1 *The equation* (4.1) *have a unique solution in*

$$\mathcal{E}_{n\alpha}([a,b],\mathbb{R}) = \{F \in C^n([a,b],\mathbb{R}); {}^C\mathcal{D}_{a^+}^{n\alpha,\rho} F \in C([a,b],\mathbb{R})\},$$

satisfies the conditions:

$$y(a) = y_a \quad y^{(\alpha)}(a) = y_a^{\alpha} \quad \cdots \quad y^{(k\alpha)}(a) = y_a^{k\alpha} \quad \cdots \quad y^{(n-1)\alpha}(a) = y_a^{((n-1)\alpha)}. \qquad (4.4)$$

Proof. The idea of proof is to transform the equation (4.2) to a differential equation of fractional order of type (3.1), for that we set

$$y_1 = y \quad y_2 = y^{(2\alpha)} \quad \cdots \quad y_k = y^{(k\alpha)} \quad \cdots \quad y_n = y^{((n-1)\alpha)} \qquad (4.5)$$

According to theorem 2.2, we can write

$$y_1^{(\alpha)} = y_2 \quad y_2^{(\alpha)} = y_3 \quad \cdots \quad y_k^{(\alpha)} = y_{k+1} \quad \cdots \quad y_{n-1}^{(\alpha)} = y_n. \qquad (4.6)$$

Set $Y = (y_1, y_2, \cdots, y_n)$, and

${}^C\mathcal{D}_{a^+}^{\alpha,\rho} Y = ({}^C\mathcal{D}_{a^+}^{\alpha,\rho} y_1, {}^C\mathcal{D}_{a^+}^{\alpha,\rho} y_2, \cdots, {}^C\mathcal{D}_{a^+}^{\alpha,\rho} y_n)$, we get

$$\begin{aligned} {}^C\mathcal{D}_{a^+,\rho}^{\alpha} Y &= (y_2, y_3, \cdots, y^{(n\alpha)}) \\ &= \left(y_2, y_3, \cdots, f(x) - \sum_{k=1}^{n-1} a_k(x) y_k\right), \end{aligned}$$

hence

$$\begin{cases} ({}^C\mathcal{D}_{a^+}^{\alpha,\rho} Y)(x) = A(x).Y(x) + B(x) \\ Y(a) = Y_a \end{cases}, \qquad (4.7)$$

where $A(x)$ is $n \times n$ continuous matrix defined by

$$A(x) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{pmatrix},$$

and $B(x)$ is a continuous vectorial function defined by $B(x) = (0,0,\cdots,f(x))$.

If we set $E = (C[a,b],\mathbb{R})^n$, the existence and uniqueness of the solution is assured according to theorem 2.3. Note that $Y \in \mathcal{E}_{\alpha}([a,b],E)$, which give that ${}^C\mathcal{D}_{a^+}^{\alpha,\rho} Y \in C([a,b],E)$, hence ${}^C\mathcal{D}_{a^+}^{\alpha,\rho} y_n = {}^C\mathcal{D}_{a^+}^{n\alpha,\rho} y \in C([a,b],\mathbb{R})$, then $y \in \mathcal{E}_{n\alpha}([a,b],\mathbb{R})$.

The following proposition immediately becomes from corollary 3.5

Proposition 4.2 *Let* $u_1, u_2, \cdots, u_n$ *be* n *solutions of of the homogeneous equation* (4.3). *Then,* $(u_1, u_2, \cdots, u_n)$ *is a fundamental system of* (4.3) *if and only if*

$$\begin{pmatrix} u_1 & u_2 & \cdots & u_n \\ u_1^{(\alpha)} & u_2^{(\alpha)} & \cdots & u_n^{(\alpha)} \\ \vdots & \vdots & \ddots & \vdots \\ u_1^{((n-1)\alpha)} & u_2^{((n-1)\alpha)} & \cdots & u_n^{((n-1)\alpha)} \end{pmatrix},$$

is linearly independent.

Remark 4.3 *The system* $(u_1, u_2, \cdots, u_n)$ *is a fundamental system of* (4.3) *if and only if* $W_\alpha(u_1, u_2, \cdots, u_n)(x) \neq 0$,

where α −Woronskian is given by

$$W_\alpha(u_1, u_2, \cdots, u_n)(x) = \begin{vmatrix} u_1(x) & u_2(x) & \cdots & u_n(x) \\ u_1^{\alpha}(x) & u_2^{\alpha} & \cdots & u_n^{\alpha}(x) \\ \vdots & \vdots & \ddots & \vdots \\ u_1^{((n-1)\alpha)}(x) & u_2^{((n-1)\alpha)}(x) & \cdots & u_n^{((n-1)\alpha)}(x) \end{vmatrix}. \qquad (4.8)$$

Remark 4.4 *Consequently proposition 3.6,* $W_\alpha(u_1, u_2, \cdots, u_n)(x) \neq 0$ *if and only if there exists* $x_0 \in [a,b]$ *such that* $W_\alpha(u_1, u_2, \cdots, u_n)(x_0) \neq 0$.

V. Sequential Linear Fractional Differential Equation with Constant Coefficients

In this section we present a direct method for giving an explicit solution to an homogeneous linear fractional differential equation with constant coefficients:

$$y^{(n\alpha)} + \sum_{k=0}^{n-1} a_k y^{(k\alpha)} = 0, \qquad (5.1)$$

where a_k are real constants, and $y^{(k\alpha)}$ is the fractional derivative which is presented by (4.1).

First, we will give the following definition which generalized the ordinary case.

Definition 5.1 *We call the characteristic polynomial associated with equation* (4.1) *the polynomial defined by*

$$P_n(\lambda) = \lambda^n + \sum_{k=0}^{n-1} a_k \lambda^k.$$

The following theorem gives a relation between a root of the polynomial P_n and a solution of the equation (5.1).

Theorem 5.1 *Let $\lambda \in \mathbb{C}$ be a root of P_n, then*

$$\varphi_\lambda(x) = E_\alpha\left(\lambda\left(\frac{x^\rho - a^\rho}{\rho}\right)^\alpha\right),$$

is a solution of the equation (5.1).

Proof. Let $k = 1 \dots n$. According to proposition 2.3, we have

$$\left({}^{C}\mathcal{D}_{a^+}^{\alpha,\rho} \varphi_\lambda \right)(x) = {}^{C}\mathcal{D}_{a^+}^{\alpha,\rho} E_\alpha\left(\lambda\left(\frac{x^\rho - a^\rho}{\rho}\right)^\alpha\right)$$

$$= \lambda E_\alpha\left(\lambda\left(\frac{x^\rho - a^\rho}{\rho}\right)^\alpha\right)$$

$$= \lambda\varphi_\lambda(x)$$

hence, by recurrence we get

$$(\varphi_\lambda)^{(k\alpha)}(x) = \lambda^k \varphi_\lambda(x)$$

Applying this result to the equation (5.1), we obtain

$$\varphi_\lambda^{(n\alpha)} + \sum_{k=0}^{n-1} a_k \varphi_\lambda^{(k\alpha)} = \left(\lambda^n + \sum_{k=0}^{n-1} a_k \lambda^k\right)\varphi_\lambda = 0.$$

The following proposition immediately becomes of the definition of the Mittag-Leffler function

Proposition 5.2 *Let $\lambda_1, \lambda_2 \in \mathbb{C}$ be two distinct roots of P_n, then $\varphi_{\lambda_1}, \varphi_{\lambda_2}$ are two linearly independent solutions of the equation* (5.1).

Remark 5.3 *If $\lambda = |\lambda|e^{i\theta}$ $(\theta \in]0,2\pi])$ is a root of P_n, then $\frac{\varphi_\lambda + \varphi_{\bar\lambda}}{2}, \frac{\varphi_\lambda - \varphi_{\bar\lambda}}{2i}$ are two linearly independent real solutions of the equation* (5.1).

Indeed, since the coefficients a_k are real coefficients, $\bar\lambda$ is a root of P_n, then φ_λ and $\varphi_{\bar\lambda}$ are two solutions of (5.1). Taking into account the definition of the Mittag-Leffler function, we obtain the result.

VI. EXAMPLES

Now, we give some examples in different cases.

Example 6.1 *Let* $0 < \alpha \leq 1$. *Consider the following equation:*

$$y^{(2\alpha)} + 2y^{(\alpha)} - 3y = 0. \tag{6.1}$$

The characteristic polynomial associated with equation (6.1) is

$$P(\lambda) = \lambda^2 + 2\lambda - 3.$$

$P(\lambda)$ has two distinct roots -3 and 1. Hence the equation (6.1) has two linearly independent solutions,

$$\varphi_1(x) = E_\alpha\left(-3\left(\frac{x^\rho - a^\rho}{\rho}\right)^\alpha\right),$$

$$\text{and } \varphi_1(x) = E_\alpha\left(\left(\frac{x^\rho - a^\rho}{\rho}\right)^\alpha\right).$$

Indeed,

$$W_\alpha(\varphi_1, \varphi_2)(x) = \begin{vmatrix} \varphi_1(x) & \varphi_2(x) \\ -3\varphi_1(x) & \varphi_2(x) \end{vmatrix} = 4\varphi_1(x).\varphi_2(x) \neq 0.$$

If φ is a solution of the equation (6.1), then

$$\varphi(x) = C_1 E_\alpha\left(-3\left(\frac{x^\rho - a^\rho}{\rho}\right)^\alpha\right) + C_2 E_\alpha\left(\left(\frac{x^\rho - a^\rho}{\rho}\right)^\alpha\right), C_1, C_2 \in \mathbb{R}.$$

Example 6.2 *Let* $0 < \alpha \leq 1$. *Consider the following equation:*

$$y^{(2\alpha)} + y = 0. \tag{6.2}$$

The characteristic polynomial associated with equation (6.2) is

$$P(\lambda) = \lambda^2 + 1.$$

$P(\lambda)$ has two distinct roots $-i$ and i. Hence the equation (5.2) has two solutions

$$\varphi_1(x) = E_\alpha\left(-i\left(\frac{x^\rho - a^\rho}{\rho}\right)^\alpha\right),$$

$$\text{and}\varphi_2(x) = E_\alpha\left(i\left(\frac{x^\rho - a^\rho}{\rho}\right)^\alpha\right).$$

Set

$$\psi_1(x) = \frac{\varphi_1(x)+\varphi_2(x)}{2} = \sum_{k=0}^{+\infty} \frac{\cos(k\theta).(x^\rho - a^\rho)^{k\alpha}}{\rho^{k\alpha}(k\alpha+1)},$$

$$\psi_2(x) = \frac{\varphi_2(x)-\varphi_1(x)}{2i} = \sum_{k=0}^{+\infty} \frac{\sin(k\theta).(x^\rho - a^\rho)^{k\alpha}}{\rho^{k\alpha}(k\alpha+1)}.$$

We have

$$W_\alpha(\psi_1,\psi_2)(x) = \begin{vmatrix} \psi_1(x) & \psi_2(x) \\ \psi_1^{(\alpha)}(x) & \psi_2^{(\alpha)}(x) \end{vmatrix} = \begin{vmatrix} \frac{\varphi_1+\varphi_2}{2}(x) & \frac{\varphi_1-\varphi_2}{2i}(x) \\ \frac{i(\varphi_1-\varphi_2)}{2}(x) & \frac{\varphi_1+\varphi_2}{2}(x) \end{vmatrix} = \frac{\varphi_1 \cdot \varphi_2}{4}(x) \neq 0.$$

If φ is a solution of the equation (6.2), then

$$\varphi(x) = C_1\psi_1(x) + C_2\psi_2(x), \qquad C_1, C_2 \in \mathbb{R}.$$

Example 6.3 *Let* $0 < \alpha \leq 1,\ r \in \mathbb{R}^*$. *Consider the following equation:*

$$y^{(2\alpha)} - 2ry^{(\alpha)} + r^2 y = 0. \tag{6.3}$$

The characteristic polynomial associated with equation (6.3) is

$$P(\lambda) = \lambda^2 - 2r\lambda + r^2.$$

$P(\lambda)$ has a doubled root r. Hence the function

$$\varphi_1(x) = E_\alpha\left(r\left(\frac{x^\rho - a^\rho}{\rho}\right)^\alpha\right),$$

is a solution to the equation (6.3). We can proof that the function

$$\varphi_2(x) = \left(\frac{x^\rho - a^\rho}{\rho}\right)^\alpha E_{\alpha,\alpha}\left(r\left(\frac{x^\rho - a^\rho}{\rho}\right)^\alpha\right),$$

is also a solution of (6.3), linearly independent to φ_1 . Indeed

$$\begin{aligned}
\varphi_2^{(\alpha)} &= \sum_{k=0}^{+\infty} \frac{r^k}{\Gamma(\alpha k+\alpha)} D_{a^+}^{\alpha,\rho}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{(k+1)\alpha} \\
&= \sum_{k=0}^{+\infty} \frac{r^k \Gamma(\alpha k+\alpha+1)}{\Gamma(\alpha k+\alpha).\Gamma(\alpha k+1)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{k\alpha} \\
&= \alpha\sum_{k=0}^{+\infty} \frac{(k+1)r^k}{\Gamma(\alpha k+1)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{k\alpha} \\
&= \alpha + \alpha.r\sum_{k=1}^{+\infty} \frac{(k+1)r^{k-1}}{\Gamma(\alpha(k-1)+\alpha+1)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{k\alpha} \\
&= \alpha + \alpha r\sum_{k=0}^{+\infty} \frac{(k+2)r^k}{\Gamma(\alpha k+\alpha+1)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{(k+1)\alpha} \\
&= \alpha + r\sum_{k=0}^{+\infty} \frac{(k+2)r^k}{(k+1)\Gamma(\alpha k+\alpha)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{(k+1)\alpha},
\end{aligned}$$

and

$$\begin{aligned}
\varphi_2^{(2\alpha)} &= r\sum_{k=0}^{+\infty} \frac{(k+2)r^k}{(k+1)\Gamma(\alpha k+\alpha)} D_{a^+}^{\alpha,\rho}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{(k+1)\alpha} \\
&= r\sum_{k=0}^{+\infty} \frac{(k+2)r^k\Gamma(\alpha k+\alpha+1)}{(k+1)\Gamma(\alpha k+\alpha).\Gamma(\alpha k+1)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{k\alpha} \\
&= \alpha r\sum_{k=0}^{+\infty} \frac{(k+2)r^k}{\Gamma(\alpha k+1)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{k\alpha} \\
&= 2\alpha r + \alpha.r^2\sum_{k=1}^{+\infty} \frac{(k+2)r^{k-1}}{\Gamma(\alpha(k-1)+\alpha+1)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{k\alpha} \\
&= 2\alpha r + \alpha r^2\sum_{k=0}^{+\infty} \frac{(k+3)r^k}{\Gamma(\alpha k+\alpha+1)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{(k+1)\alpha} \\
&= 2\alpha r + r^2\sum_{k=0}^{+\infty} \frac{(k+3)r^k}{(k+1)\Gamma(\alpha k+1)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{(k+1)\alpha}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\varphi_2^{(2\alpha)} - 2r\varphi_2^{(\alpha)} + r^2\varphi_2 \\
&= r^2\sum_{k=0}^{+\infty}\left[\frac{k+3}{k+1} - \frac{2(k+2)}{k+1} + 1\right]\frac{r^k}{\Gamma(\alpha k+\alpha)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{(k+1)\alpha} \\
&= 0.
\end{aligned}$$

We have

$$W_\alpha(\varphi_1,\varphi_2)(x) = \begin{vmatrix} \varphi_1(x) & \varphi_2(x) \\ r\varphi_1(x) & \varphi_2^{(\alpha)}(x) \end{vmatrix} = \varphi_1(x)[\varphi_2^{(\alpha)}(x) - r\varphi_2(x)],$$

and

$$\begin{aligned}
\varphi_2^{(\alpha)}(x) - r\varphi_2(x) &= \alpha + \sum_{k=0}^{+\infty}\left(\frac{k+2}{k+1} - 1\right).\frac{r^{k+1}}{\Gamma(\alpha k+\alpha)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{(k+1)\alpha} \\
&= \alpha + \sum_{k=0}^{+\infty} .\frac{r^{k+1}}{(k+1)\Gamma(\alpha k+\alpha)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{(k+1)\alpha} \neq 0.
\end{aligned}$$

Hence, if φ is a solution of the equation (6.3), then

$$\varphi(x) = C_1\varphi_1(x) + C_2\varphi_2(x), C_1, C_2 \in \mathbb{R}.$$

Example 6.4 *Let* $0 < \alpha \leq 1$. *Consider the following equation*

$$y^{(3\alpha)} - 2y^{(2\alpha)} - y^{(\alpha)} + 2y = 0. \quad (6.4)$$

The characteristic polynomial associated with equation (6.4) is

$$P(\lambda) = \lambda^3 - 2\lambda^2 - \lambda + 2.$$

$P(\lambda)$ has three distinct roots $-1, 1, 2$. Hence the equation (6.4) has three linearly independent solutions

$$\varphi_1(x) = E_\alpha\left(-\left(\frac{x^\rho - a^\rho}{\rho}\right)^\alpha\right), \varphi_2(x) = E_\alpha\left(\left(\frac{x^\rho - a^\rho}{\rho}\right)^\alpha\right),$$

$$\varphi_3(x) = E_\alpha\left(2\left(\frac{x^\rho - a^\rho}{\rho}\right)^\alpha\right).$$

$$W_\alpha(\varphi_1, \varphi_2, \varphi_3)(x) = \begin{vmatrix} \varphi_1(x) & \varphi_2(x) & \varphi_3(x) \\ \varphi_1^{(\alpha)}(x) & \varphi_2^{(\alpha)}(x) & \varphi_3^{(\alpha)}(x) \\ \varphi_1^{(2\alpha)}(x) & \varphi_2^{(2\alpha)} & \varphi_3^{(2\alpha)}(x) \end{vmatrix}$$

$$= \begin{vmatrix} \varphi_1(x) & \varphi_2(x) & \varphi_3(x) \\ -\varphi_1^{(\alpha)}(x) & \varphi_2^{(\alpha)}(x) & 2\varphi_3^{(\alpha)}(x) \\ \varphi_1^{(2\alpha)}(x) & \varphi_2^{(2\alpha)} & 4\varphi_3^{(2\alpha)}(x) \end{vmatrix}$$

$$= -12\varphi_1(x).\varphi_2(x).\varphi_3(x) \neq 0.$$

If φ is a solution of the equation (6.4), then

$$\varphi(x) = C_1\varphi_1(x) + C_2\varphi_2(x) + C_3\varphi_3(x), C_1, C_2, C_3 \in \mathbb{R}.$$

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