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ORIGINAL RESEARCH

A fractional $p(x,\cdot)$-Laplacian problem involving a singular term

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Abstract This paper deals with a class of singular problems involving the fractional $p(x,\cdot)$-Laplace operator of the form

$$\begin{cases} (-\Delta)^s_{p(x,\cdot)}u(x) = \frac{\lambda}{u^{\gamma(x)}} + u^{q(x)-1} & \text{in } \Omega, \\ u > 0, \text{ in } \Omega \\ u = 0 \text{ on } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 3)$, $0 < s < 1$, λ is a positive parameter and $\gamma : \mathbb{R}^N \longrightarrow (0,1)$ is a continuous function, $p : \mathbb{R}^{2N} \longrightarrow (1,\infty)$ is a bounded, continuous and symmetric function, $q : \mathbb{R}^N \longrightarrow (1,\infty)$ is a continuous function. Using the direct method of minimization combined with the theory of fractional Sobolev spaces with variable exponents, we prove that the problem has one positive solution for $\lambda > 0$ small enough. To our best knowledge, this paper is one of the first attempts in the study of singular problems involving fractional $p(x,\cdot)$-Laplace operators.

Keywords Fractional $p(x,\cdot)$-Laplace operators · Singular equations · Minimization methods · Fractional Sobolev spaces

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ be an open bounded Lipschitz domain and let $s \in (0,1)$. We consider the following nonlocal singular problem

$$(P_\lambda) \qquad \begin{cases} (-\Delta)^s_{p(x,\cdot)} u(x) = \frac{\lambda}{u^{\gamma(x)}} + u^{q(x)-1}, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where λ is a positive parameter, $p : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow (1,\infty)$ and $q : \mathbb{R}^N \longrightarrow (1,\infty)$ are two continuous functions satisfying the following conditions

(H_1) For all $x, y \in \mathbb{R}^N$, $p(x,y) = p(y,x)$ (i.e. p is symmetric) and $sp(x,y) < N$ with

$$1 < p^- = \inf_{(x,y)\in\mathbb{R}^{2N}} p(x,y) \leq p(x,y) \leq \sup_{(x,y)\in\mathbb{R}^{2N}} p(x,y) = p^+ < \infty,$$

$$q(x) < p^*(x) = \frac{Np(x,x)}{N - sp(x,x)} \quad \text{and} \quad p^+ < q^- := \inf_{x\in\mathbb{R}^N} q(x),$$

and $\gamma \in C(\mathbb{R}^N)$ verifies

(H_2)

$$0 < \gamma^- := \inf_{x\in\mathbb{R}^N} \gamma(x) \leq \gamma(x) \leq \sup_{x\in\mathbb{R}^N} \gamma(x) =: \gamma^+ < 1.$$

The operator $(-\Delta)^s_{p(x,\cdot)}$ formally defined by

$$(-\Delta)^s_{p(x,\cdot)} u(x) = 2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))}{|x - y|^{N+sp(x,y)}} \, dy, \quad \forall x \in \mathbb{R}^N,$$

that generalizes the fractional p-Laplacian operator. In particular, if $s = 1$ then the fractional p-Laplace operator $(-\Delta)^s_p(.)$ becomes the p-Laplace operator $-\Delta_p(.) = -\mathrm{div}\left(|\nabla(.)|^{p-2}\nabla(.)\right)$ as usual. There is a large literature which deal with problem involving the fractional p-Laplacian with Dirichlet boundary and note that many papers deal with problem related to the constant case, see for example [7, 8, 20, 28, 34] and references cited therein. On the other hand, the operator $(-\Delta)^s_{p(x,\cdot)}$ is a fractional version of the well-know $p(\cdot)$-Laplacian defined by $\Delta_{p(\cdot)} u = \mathrm{div}\left(|\nabla u|^{p(\cdot)-2}\nabla u\right)$ which lead to Sobolev spaces with variable exponent; we refer the readers to [13, 15, 23]. In [21], the authors have extended the Sobolev spaces with variable exponents to include the fractional case which are called fractional Sobolev spaces with variable exponents noted by $W^{s,q(x),p(\cdot,\cdot)}$ and also have proved a Sobolev-type inequality (see Proposition 2.2); moreover, note that there are other that works have studied these type of problems; see [1, 4–6, 9, 10].

In the local setting case the problem (P_λ) becomes

$$\begin{cases} -\Delta_{p(x)} u = \frac{\lambda}{u^{\gamma(x)}} + u^{q(x)-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here $\Omega \subset \mathbb{R}^N$, $(N \geq 2)$ be a bounded domain with C^2 boundary, λ is a positive parameter. Unfortunately, results for $p(x)$-Laplace equations with singular non-linearity are rare. Zhang [35] obtain the existence and the boundary asymptotic behavior of solutions to the purely singular $p(x)$-Laplace equation. Saoudi [30, 33] has been extended the results of existence for more general problem. Fan [14] using the critical point theory, investigate the existence and multiplicity of solutions for $p(x)-$Laplacian Dirichlet problem with singular coefficients. Liu [25] generalized the results of [26] to the problem involving $p(x)$-Laplace operator by making the similar assumptions.

Problems (P_λ) has been also studied with different elliptic operators. We refer the reader to the monographs of Ghergu-Radulescu [17] for a more general presentation of these results and the survey article of Crandall-Rabinowitz-Tartar [12]. After this, many authors have been considered the problem above for Laplacian operators, p-Laplacian operators, fractional Laplacian or fractional$p$-Laplacian, using the technique used in [12] or a combination of this approach with the Nehari's and Perron's methods, among others, we would like to mention Coclite-Palmieri [11], Giacomoni-Saoudi [18], and references therein in the case of the laplacian equation. In Saoudi [32] the singular case with the p-Laplace operator equation is considered. The corresponding quasilinear and singular N-Laplacian equation is considered in Saoudi-Kratou [31].

The paper is organized as follows: In Section 2, we recall some results concerning the variable exponent Lebesgue spaces, $L^{p(x)}(\Omega)$, as well as the generalized Sobolev spaces, $W_0^{s,p(x,y)}(\Omega)$. Moreover, some properties of these spaces will be also exhibited to be used later. The proof of our results will be presented in Section 3.

2 Preliminary results

In order to deal with fractional $p(x,\cdot)$-Laplacian problems, we need to recall some definitions and basic properties of the generalized Lebesgue space $L^{r(x)}(\Omega)$ as well as the fractional Sobolev space with variable exponent $W^{s,r(x),p(x,\cdot)}(\Omega)$. For more detailed information on this topic, we refer to the papers [4, 6, 21] and the books [13, 29]. Let Ω be a Lipschitz bounded open set in $\mathbb{R}^N$. Firstly, we introduce the space $C_+(\overline{\Omega})$ as follows

$$C_+(\overline{\Omega}) := \left\{u \in C(\overline{\Omega}),\ \ u(x) > 1 \ \text{ for any } x \in \overline{\Omega}\right\}.$$

For all $r \in C_+(\overline{\Omega})$, we denote $r^+ := \inf_{x\in\overline{\Omega}} r(x)$ and $r^- := \sup_{x\in\overline{\Omega}} r(x)$. For any $r \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space

$$L^{r(x)}(\Omega) := \left\{u : \Omega \to \mathbb{R} \ \text{ is mesurable} : \int_\Omega |u(x)|^{r(x)}\, dx < +\infty\right\},$$

which is equipped with the following Luxemburg norm

$$|u|_{r(x)} := \inf\left\{\mu > 0 : \int_\Omega \left|\frac{u(x)}{\mu}\right|^{r(x)} dx \le 1\right\}.$$

The space $(L^{r(x)}(\Omega), |\cdot|_{r(x)})$ becomes a separable and reflexive Banach space. As in the classical case, its conjugate space is $L^{r'(x)}(\Omega)$, where $\frac{1}{r(x)} + \frac{1}{r'(x)} = 1$. Moreover, for any $u \in L^{q(x)}(\Omega)$ and $v \in L^{r'(x)}(\Omega)$, we recall the Hölder type inequality

$$\left|\int_\Omega uv\, dx\right| \le \left(\frac{1}{r^-} + \frac{1}{r'^-}\right)|u|_{r(x)}|v|_{r'(x)} \le 2|u|_{r(x)}|v|_{r'(x)}, \tag{2.1}$$

where $r^- = \inf_{x\in\overline{\Omega}} r(x)$ and $r'^- = \inf_{x\in\overline{\Omega}} r'(x)$.

Proposition 2.1 *Let $\rho_{r(x)}(u) = \int_\Omega |u(x)|^{r(x)}\, dx$. For any $u \in L^{r(x)}(\Omega)$, we have*

(1) For $u \neq 0$, $|u|_{r(x)} = \lambda \Leftrightarrow \rho\left(\frac{u}{\lambda}\right) = 1$.
(2) If $|u|_{r(x)} > 1$, then $|u|_{r(x)}^{r^-} \le \rho_{r(x)}(u) \le |u|_{r(x)}^{r^+}$.
(3) If $|u|_{r(x)} < 1$, then $|u|_{r(x)}^{r^+} \le \rho_{r(x)}(u) \le |u|_{r(x)}^{r^-}$.
(4) $\lim_{n\to\infty} \rho_{r(x)}(u_n - u) = 0 \Leftrightarrow \lim_{n\to\infty} |u_n - u|_{r(x)} = 0$.

Now, let Ω be a smooth open set in $\mathbb{R}^N$. Fix $s \in (0,1)$, let $p : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R}$ be a continuous function which satisfies

$$p \ \text{ is symmetric, that is, } \ p(x,y) = p(y,x), \quad \forall (x,y) \in \overline{\Omega} \times \overline{\Omega} \tag{2.2}$$

and

$$1 < \min_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x,y) \le p(x,y) \le \max_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x,y) < \infty. \tag{2.3}$$

For the above function $p : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R}$, we set $\overline{p}(x) := p(x,x)$ for all $x \in \overline{\Omega}$. Let us define the fractional Sobolev space $W^{s,p(x,y)}(\Omega)$ as follows

$$W^{s,p(x,y)}(\Omega) := \left\{ u \in L^{\overline{p}(x)}(\Omega) : \iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{\mu^{p(x,y)}|x-y|^{N+sp(x,y)}} < \infty \text{ for some } \mu > 0 \right\},$$

which is a reflexive Banach space under the norm

$$\|u\|_{W^{s,p(x,y)}(\Omega)} := |u|_{\overline{p}(x)} + [u]_{\Omega}^{s,p(x,y)},$$

where $[u]_{\Omega}^{s,p(x,y)}$ denotes the Gagliardo seminorm with variable exponent defined by

$$[u]_{\Omega}^{s,p(x,y)} := \inf \left\{ \mu > 0 : \iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{\mu^{p(x,y)}|x-y|^{N+sp(x,y)}} \le 1 \right\}.$$

We recall the following compact embedding theorem into variable exponent Lebesgue spaces which is established in [3], see also the papers [6, 21] for some related results.

Proposition 2.2 *Let $\Omega \subset \mathbb{R}^N$ be a Lipschitz bounded domain and $s \in (0,1)$, let $p : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R}$ be a continuous function such that $sp(x,y) < N$ for $(x,y) \in \overline{\Omega} \times \overline{\Omega}$ and the conditions (2.2) and (2.3) hold. Assume that $\alpha : \overline{\Omega} \to (1, +\infty)$ is a continuous function such that*

$$p_s^*(x) := \frac{Np(x,x)}{N - sp(x,x)} > \alpha(x) \ge \alpha_- > 1,$$

for $x \in \overline{\Omega}$, then there exists a constant $C = C(N, s, p, \alpha, \Omega)$ such that for every $u \in W^{s,p(x,y)}(\Omega)$ it holds that

$$|u|_{\alpha(x)} \le C\|u\|_{W^{s,p(x,y)}(\Omega)}.$$

That is, the space $W^{s,p(x,y)}(\Omega)$ is continuously embedded in $L^{\alpha(x)}(\Omega)$ for any $\alpha \in (1, p^*)$. Moreover, this embedding is compact.

In addition, when one considers function $u \in W^{s,p(x,y)}(\Omega)$ that is compactly supported inside Ω, it holds that

$$|u|_{\alpha(x)} \le C[u]_{\Omega}^{s,p(x,y)}.$$

Next, we consider the function space

$$W := \left\{ u : \mathbb{R}^N \to \mathbb{R} \text{ is measurable}, \ u|_{\Omega} \in L^{\overline{p}(x)}(\Omega) \text{ and } \exists \lambda > 0 : \int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{N+sp(x,y)}} \le 1 \right\},$$

where $Q = \mathbb{R}^{2N} \setminus (\Omega^c \times \Omega^c)$ and $\Omega^c = \mathbb{R}^N \setminus \Omega$. Then W is a separable reflexive Banach space under the norm

$$\|u\|_W = |u|_{\overline{p}(x)} + [u]_Q^{s,p(\cdot,\cdot)}.$$

Moreover, if $u \in W$ then $u \in W^{s,p(x,y)}(\Omega)$ and $\|u\|_{W^{s,p(x,y)}(\Omega)} \le \|u\|_W$. Now, we define the following space

$$W_0 = \left\{ u \in W \quad \text{and} \quad u = 0 \text{ a.e. on } \mathbb{R}^N \setminus \Omega \right\}.$$

which is also a separable reflexive Banach space.

As in Proposition 2.2, it is proved in [3] that, if $\alpha : \overline{\Omega} \to (1, +\infty)$ is a continuous function such that

$$p_s^*(x) := \frac{Np(x,x)}{N - sp(x,x)} > \alpha(x) \ge \alpha_- > 1,$$

for $x \in \overline{\Omega}$, then there exists a constant $C = C(N, s, p, \alpha, \Omega)$ such that for every $u \in W$ it holds that

$$|u|_{\alpha(x)} \le C\|u\|_W.$$

That is, the space W is continuously embedded in $L^{\alpha(x)}(\Omega)$ for any $\alpha \in (1, p^*)$. Moreover, this embedding is compact. We also note that the above results remain true if we replace W by W_0 and the norms $\|.\|_{W_0} =$

$[u]_Q^{s,p(x,y)}$ and $\|.\|_W$ are equivalent in W_0.

Proposition 2.3 *(see [3]) Let $u \in W_0$, we have*

(1) If $\|u\|_{W_0} > 1$, then $\|u\|_{W_0}^{p^-} \leq \int\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} \leq \|u\|_{W_0}^{p^+}$.
(2) If $\|u\|_{W_0} < 1$, then $\|u\|_{W_0}^{p^+} \leq \int\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} \leq \|u\|_{W_0}^{p^-}$.

3 Existence of positive solutions

In what follows, we denote by $f^+ := \max\{g, 0\}$ and $f^- := \max\{-g, 0\}$ respectively the positive and negative part of a function *f*. For brevity, we will use $C, C_i, i = 1, 2, \ldots$ to denote various positive constants throughout this paper.

Definition 3.1 We call $u \in W_0$ a weak solution of problem (P_λ) if it holds that

$$\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+sp(x,y)}}\, dxdy - \int_\Omega (u^+)^{q(x)-1}\phi\, dx \\ -\lambda \int_\Omega \frac{\phi}{(u^+)^{\gamma(x)}}\, dx = 0. \tag{3.1}$$

for all $\phi \in W_0$.

Theorem 3.1 *Assume that the assumptions (H_1) and (H_2) hold. Then there exists λ^* such that for any $\lambda \in (0, \lambda^*)$, problem (P_λ) admits a positive weak solution.*

To prove Theorem 3.1, let us define for each $\lambda > 0$ the associating functional $F_\lambda : W_0 \to \mathbb{R}$ with problem (P_λ) as

$$F_\lambda(u) = \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\, dxdy - \int_\Omega \frac{(u^+)^{q(x)}}{q(x)}\, dx - \lambda \int_\Omega \frac{(u^+)^{1-\gamma(x)}}{1-\gamma(x)}\, dx.$$

We first study the geometry conditions.

Lemma 3.1 *There exist $\rho > 0$, λ^* and $\alpha > 0$ such that for any $u \in W_0$ with $\|u\|_{W_0} = \rho$ and for any $\lambda \in (0, \lambda^*)$, we have*

$$F_\lambda(u) \geq \alpha.$$

Proof Let $u \in W_0$ be such that $\max\{\|u\|_{W_0}, |u|_{q(x)}\} \leq 1$. Using Proposition 2.1, Proposition 2.3 and the continuous embedding of W_0 into $L^{q(x)}$, we deduce that

$$F_\lambda(u) \geq \frac{1}{p^+}\|u\|_{W_0}^{p^+} - \frac{C}{q^-}\|u\|_{W_0}^{q^-} - \frac{\lambda}{1-\gamma^+}\int_\Omega (u^+)^{1-\gamma(x)}\, dx.$$

By the Hölder inequality (2.1) with exponents $\theta_\gamma(x) = \frac{q(x)}{1-\gamma(x)}$ and its conjugate $\theta'_\gamma(x) = \frac{q(x)}{q(x)+\gamma(x)-1}$; (note that both these exponents are greater than 1), there exists $C_1 > 0$ such that

$$\int_\Omega (u^+)^{1-\gamma(x)}\, dx \leq C_1 \left||u|^{1-\gamma(x)}\right|_{\frac{q(x)}{1-\gamma(x)}} |1|_{\theta'_\gamma(x)}.$$

On the other hand, by Proposition 2.1, we have

$$M = \int_\Omega \left| \frac{|u(x)|^{1-\gamma(x)}}{\left||u|^{1-\gamma(x)}\right|_{\frac{q(x)}{1-\gamma(x)}}} \right|^{\frac{q(x)}{1-\gamma(x)}} dx = 1. \tag{3.2}$$

Hence, by the mean value theorem, there exists $\tau \in \overline{\Omega}$ such that

$$M = \int_\Omega \frac{|u(x)|^{q(x)}}{|u|_{q(x)}^{q(x)}} \frac{|u|_{q(x)}^{q(x)}}{\left||u|^{1-\gamma(x)}\right|_{\frac{q(x)}{1-\gamma(x)}}^{\frac{q(x)}{1-\gamma(x)}}} \, dx = \frac{|u|_{q(x)}^{q(\tau)}}{\left||u|^{1-\gamma(x)}\right|_{\frac{q(x)}{1-\gamma(x)}}^{\frac{q(\tau)}{1-\gamma(\tau)}}} \int_\Omega \frac{|u(x)|^{q(x)}}{|u|_{q(x)}^{q(x)}} \, dx.$$

Since

$$\int_\Omega \frac{|u(x)|^{q(x)}}{|u|_{q(x)}^{q(x)}} \, dx = 1,$$

we can rewrite M as

$$M = \frac{|u|_{q(x)}^{q(\tau)}}{\left||u|^{1-\gamma(x)}\right|_{\frac{q(x)}{1-\gamma(x)}}^{\frac{q(\tau)}{1-\gamma(\tau)}}}. \tag{3.3}$$

From (3.2) and (3.3), it follows that

$$\left||u|^{1-\gamma(x)}\right|_{\frac{q(x)}{1-\gamma(x)}} = |u|_{q(x)}^{1-\gamma(\tau)}.$$

Therefore, by the continuous embeddings, we have

$$\int_\Omega (u^+)^{1-\gamma(x)} \, dx \le C_2 |u|_{q(x)}^{1-\gamma(\tau)} \le C_3 \|u\|_{W_0}^{1-\gamma(\tau)}, \quad \forall u \in W_0. \tag{3.4}$$

From (3.4), we conclude that

$$\begin{aligned} F_\lambda(u) &\ge \frac{1}{p^+}\|u\|_{W_0}^{p^+} - \frac{C}{q^-}\|u\|_{W_0}^{q^-} - \frac{\lambda}{1-\gamma^+} C_3 \|u\|_{W_0}^{1-\gamma(\tau)} \\ &\ge \frac{1}{p^+}\|u\|_{W_0}^{p^+} - \frac{C}{q^-}\|u\|_{W_0}^{q^-} - \frac{\lambda}{1-\gamma^+} C_3 \|u\|_{W_0}^{1-\gamma^+} \\ &= \|u\|_{W_0}^{1-\gamma^+} \left(\frac{1}{p^+}\|u\|_{W_0}^{p^+ + \gamma^+ - 1} - \frac{C}{q^-}\|u\|_{W_0}^{q^- + \gamma^+ - 1} - \frac{\lambda C_3}{1-\gamma^+} \right). \end{aligned}$$

Now, we consider the function

$$h(t) = \frac{1}{p^+} t^{p^+ + \gamma^+ - 1} - \frac{C}{q^-} t^{q^- + \gamma^+ - 1}, \quad t \in (0,1).$$

Since $p^+ < q^-$ then h admits a strictly positive maximum at some $\rho \in (0,1)$ which is sufficiently small. Hence, let us choose

$$\lambda^* = \frac{1-\gamma^+}{2C_2} \max_{t \in [0,1]} h(t) = \frac{1-\gamma^+}{2C_2} h(\rho) > 0,$$

then for any $u \in W_0$ with $\|u\|_{W_0} = \rho$ we have

$$F_\lambda(u) \ge \rho^{1-\gamma^+} \left(h(\rho) - \frac{\lambda C_3}{1-\gamma^+} \right) \ge \frac{\rho^{1-\gamma^+}}{2} h(\rho) = \alpha > 0$$

provided $\lambda \in (0, \lambda^*)$. This completes the proof of Lemma 3.1. □

Lemma 3.2 *For any $\lambda \in (0, \lambda^*)$, we have*

$$\inf_{\|u\|_{W_0} \le \rho} F_\lambda(u) < 0.$$

Proof Let $u_0 \in W_0$ be such that u_0^+ is not identically zero. We infer for any $t \in (0,1)$ that

$$\begin{aligned}F_\lambda(tu_0) =& \iint_{\mathbb{R}^{2N}} \frac{t^{p(x,y)}|u_0(x)-u_0(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\,dxdy - \int_\Omega \frac{t^{q(x)}|u_0|^{q(x)}}{q(x)}\,dx\\ &- \lambda\int_\Omega \frac{t^{1-\gamma(x)}(u_0^+)^{1-\gamma(x)}}{1-\gamma(x)}\,dx\\ \leq& \frac{t^{p^-}}{p^-}\max\left\{\|u_0\|_{W_0}^{p^-}, \|u_0\|_{W_0}^{p^+}\right\} - \frac{\lambda t^{1-\gamma^-}}{1-\gamma^-}\int_\Omega (u_0^+)^{1-\gamma(x)}\,dx.\end{aligned}$$

From the last inequality and the fact that $p^- > 1 > 1-\gamma^-$, there exists $t_0 > 0$ so small that $F_\lambda(t_0u_0) < 0$ and $t_0 < \frac{\rho}{\|u_0\|_{W_0}}$, which yields $\inf_{\|u\|_{W_0}\leq\rho} F_\lambda(u) \leq F_\lambda(t_0u_0) < 0$. □

Lemma 3.3 $\inf_{\|u\|_{W_0}\leq\rho} F_\lambda(u)$ *is achieved in some* $w \in W_0$.

Proof Fix $\lambda \in (0,\lambda^*)$ and let (u_n) be a minimizing sequence for F_λ with $\|u_n\|_{W_0} \leq \rho$. Since (u_n) is a bounded sequence in W_0, passing to a subsequence if necessary, there exists $w \in W_0$ such that $u_n \rightharpoonup w$ weakly in W_0 and $\|w\|_{W_0} \leq \rho$. We deduce from Proposition 2.2 that $u_n(x)$ converges to $w(x)$ a.e. $x\in\Omega$ and $u_n \to w$ in $L^{q(x)}(\Omega)$. This means that

$$\lim_{n\to\infty}\int_\Omega \frac{(u_n^+)^{q(x)}}{q(x)}\,dx = \int_\Omega \frac{(w^+)^{q(x)}}{q(x)}\,dx. \tag{3.5}$$

By Fatou's lemma, we get

$$\lim_{n\to\infty}\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\,dxdy \geq \iint_{\mathbb{R}^{2N}} \frac{|w(x)-w(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\,dxdy. \tag{3.6}$$

Recalling the elementary inequality

$$|a^\eta - b^\eta| \leq |a-b|^\eta, \quad \forall a,b>0, \quad \forall \eta\in(0,1),$$

we obtain

$$\left|\int_\Omega \frac{1}{1-\gamma(x)}(u_n^+)^{1-\gamma(x)}\,dx - \int_\Omega \frac{1}{1-\gamma(x)}(w^+)^{1-\gamma(x)}\,dx\right| \leq \int_\Omega \frac{1}{1-\gamma(x)}|u_n^+ - w^+|^{1-\gamma(x)}\,dx$$

From (3.4), there exist $C_3 > 0$ and $\tau' \in \overline{\Omega}$ such that

$$\int_\Omega \frac{1}{1-\gamma(x)}|u_n^+ - w^+|^{1-\gamma(x)}\,dx \leq \frac{C_3}{1-\gamma^+}|u_n - w|_{q(x)}^{1-\gamma(\tau')}.$$

The above information combined with Proposition 2.2 implies that

$$\lim_{n\to+\infty}\int_\Omega \frac{1}{1-\gamma(x)}(u_n^+)^{1-\gamma(x)}\,dx = \int_\Omega \frac{1}{1-\gamma(x)}(w^+)^{1-\gamma(x)}\,dx. \tag{3.7}$$

Now, using (3.5), (3.6) and (3.7) it follows that,

$$\begin{aligned}\inf_{\|u\|_{W_0}\leq\rho} F_\lambda(u) =& \lim_{n\to+\infty} F_\lambda(u_n)\\ \geq& \iint_{\mathbb{R}^{2N}} \frac{|w(x)-w(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\,dxdy - \int_\Omega \frac{(w^+)^{q(x)}}{q(x)}\,dx\\ &- \int_\Omega \frac{1}{1-\gamma(x)}(w^+)^{1-\gamma(x)}\,dx.\\ =& F_\lambda(w).\end{aligned}$$

Thus $\inf_{\|u\|_{W_0}\leq\rho} F_\lambda(u) = F_\lambda(w)$ and we completes the proof of Lemma 3.3. □

Now, we are in the position to prove our main result.

Proof of Theorem 3.1 As a consequence of Lemmas 3.1-3.3, there exists λ^* such that for any $\lambda \in (0, \lambda^*)$, we obtain a function $w \in W_0$ as a nontrivial local minimizer of the functional F_λ. We will prove that w is a positive weak solution of problem (P_λ).

Let $\phi \in W_0$ be a nonnegative function a.e. in $\mathbb{R}^N$, we can choose $t > 0$ sufficiently small such that $\|w + t\phi\|_{W_0} \leq \rho$. Observe that

$$F_\lambda(w + t\phi) - F_\lambda(w) \geq 0, \tag{3.8}$$

from which, using the mean-value theorem and Fatou's lemma, it holds that

$$\liminf_{n\to\infty} \int_\Omega \frac{\left[(w + t\phi)^+\right]^{1-\gamma(x)} - (w^+)^{1-\gamma(x)}}{(1-\gamma(x))t}\, dx \geq \int_\Omega (w^+)^{-\gamma(x)} \phi\, dx.$$

Hence,

$$\begin{aligned}\lim_{t\to 0^+} \frac{F_\lambda(w + t\phi) - F_\lambda(w)}{t} \leq \iint_{\mathbb{R}^{2N}} &\frac{|w(x) - w(y)|^{p(x,y)-2}(w(x) - w(y))(\phi(x) - \phi(y))}{|x - y|^{N+sp(x,y)}}\, dxdy \\ &- \int_\Omega (w^+)^{q(x)-1}\phi\, dx - \lambda \int_\Omega (w^+)^{-\gamma(x)}\phi\, dx.\end{aligned} \tag{3.9}$$

From (3.8) and (3.9), we have,

$$\begin{aligned}\iint_{\mathbb{R}^{2N}} &\frac{|w(x) - w(y)|^{p(x,y)-2}(w(x) - w(y))(\phi(x) - \phi(y))}{|x - y|^{N+sp(x,y)}}\, dxdy \\ &- \int_\Omega (w^+)^{q(x)-1}\phi\, dx - \lambda \int_\Omega (w^+)^{-\gamma(x)}\phi\, dx \geq 0,\end{aligned} \tag{3.10}$$

for any $\phi \in W_0$ and $\phi \geq 0$.

Clearly, if $\|w\|_{W_0} = \rho$, then, thanks to Lemma 3.1, we have $F_\lambda(w) \geq \alpha > 0$ which contradicts Lemma 3.2, and thus $\|w\|_{W_0} < \rho$. Now, by taking $\delta = \frac{\rho - \|w\|_{W_0}}{\|w\|_{W_0}} > 0$, we can see that

$$\|w + tw\|_{W_0} \leq \rho, \quad \forall t \in [-\delta, \delta].$$

Since the functional F_λ has a local minimum w, the application $t \mapsto F_\lambda(w + tw)$ has a minimum at $t_0 = 0$, that is,

$$\lim_{t\to 0} \frac{F_\lambda(w + tw) - F_\lambda(w)}{t} = 0,$$

or

$$\iint_{\mathbb{R}^{2N}} \frac{|w(x) - w(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}}\, dxdy - \int_\Omega (w^+)^{q(x)}\, dx - \lambda \int_\Omega (w^+)^{1-\gamma(x)}\, dx = 0. \tag{3.11}$$

Let $\varepsilon > 0$ and $\phi \in W_0$. By using $\phi_\varepsilon = w^+ + \varepsilon\phi$ as the test function in (3.10) we get

$$\begin{aligned}\iint_{\mathbb{R}^{2N}} &\frac{|w(x) - w(y)|^{p(x,y)-2}(w(x) - w(y))(\phi_\varepsilon^+(x) - \phi_\varepsilon^+(y))}{|x - y|^{N+sp(x,y)}}\, dxdy \\ &- \int_\Omega (w^+)^{q(x)-1}\phi_\varepsilon^+\, dx - \lambda \int_\Omega (w^+)^{-\gamma(x)}\phi_\varepsilon^+\, dx \geq 0.\end{aligned}$$

Note that $\phi_\varepsilon^+ = \phi_\varepsilon + \phi_\varepsilon^-$, the above inequality leads to

$$
\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \frac{|w(x)-w(y)|^{p(x,y)-2}(w(x)-w(y))(w^+(x)-w^+(y))}{|x-y|^{N+sp(x,y)}}\,dxdy \\
&+\varepsilon\iint_{\mathbb{R}^{2N}} \frac{|w(x)-w(y)|^{p(x,y)-2}(w(x)-w(y))(\phi(x)-\phi(y))}{|x-y|^{N+sp(x,y)}}\,dxdy \\
&+\iint_{\mathbb{R}^{2N}} \frac{|w(x)-w(y)|^{p(x,y)-2}(w(x)-w(y))(\phi_\varepsilon^-(x)-\phi_\varepsilon^-(y))}{|x-y|^{N+sp(x,y)}}\,dxdy \\
&-\int_\Omega (w^+)^{q(x)}\,dx-\varepsilon\int_\Omega (w^+)^{q(x)-1}\phi\,dx-\int_\Omega (w^+)^{q(x)-1}\phi_\varepsilon^-\,dx \\
&-\lambda\int_\Omega (w^+)^{1-\gamma(x)}\,dx-\varepsilon\lambda\int_\Omega (w^+)^{-\gamma(x)}\phi\,dx-\lambda\int_\Omega (w^+)^{-\gamma(x)}\phi_\varepsilon^-\,dx\geq 0.
\end{aligned}
$$

Observe that

$$
\lambda\int_\Omega (w^+)^{-\gamma(x)}\phi_\varepsilon^-\,dx\geq 0 \text{ and } \int_\Omega (w^+)^{q(x)-1}\phi_\varepsilon^-\,dx\geq 0 \tag{3.12}
$$

and

$$
(w(x)-w(y))(w^+(x)-w^+(y))\geq |w^+(x)-w^+(y)|^2 \tag{3.13}
$$

since $w^+(x)w^-(x)=0$ and $w^+(x)w^-(y)\geq 0$ a.e. $x,y\in\mathbb{R}^N$. From (3.11), (3.12) and (3.13), we have

$$
\begin{aligned}
&\varepsilon\left[\iint_{\mathbb{R}^{2N}} \frac{|w(x)-w(y)|^{p(x,y)-2}(w(x)-w(y))(\phi(x)-\phi(y))}{|x-y|^{N+sp(x,y)}}\,dxdy\right. \\
&\left.-\int_\Omega (w^+)^{q(x)-1}\phi\,dx-\lambda\int_\Omega (w^+)^{-\gamma(x)}\phi\,dx\right] \\
&+\iint_{\mathbb{R}^{2N}} \frac{|w(x)-w(y)|^{p(x,y)-2}(w(x)-w(y))(\phi_\varepsilon^-(x)-\phi_\varepsilon^-(y))}{|x-y|^{N+sp(x,y)}}\,dxdy\geq 0.
\end{aligned} \tag{3.14}
$$

Set

$$
I(w,\phi_\varepsilon^-)=\iint_{\mathbb{R}^{2N}} \frac{|w(x)-w(y)|^{p(x,y)-2}(w(x)-w(y))(\phi_\varepsilon^-(x)-\phi_\varepsilon^-(y))}{|x-y|^{N+sp(x,y)}}\,dxdy
$$

and

$$
\{\phi_\varepsilon\leq 0\}=\{x\in\mathbb{R}^N:\phi_\varepsilon(x)\leq 0\} \text{ and } \{\phi_\varepsilon\leq 0\}^c=\mathbb{R}^N\setminus\{\phi_\varepsilon\leq 0\},
$$

we then deduce by using some direct computations and (3.13) that

$$
\begin{aligned}
I(w,\phi_\varepsilon^-)=&-\int_{\{\phi_\varepsilon\leq 0\}}\int_{\{\phi_\varepsilon\leq 0\}} \frac{|w(x)-w(y)|^{p(x,y)-2}(w(x)-w(y))(\phi_\varepsilon(x)-\phi_\varepsilon(y))}{|x-y|^{N+sp(x,y)}}\,dxdy \\
&+2\int_{\{\phi_\varepsilon\leq 0\}}\int_{\{\phi_\varepsilon\leq 0\}^c} \frac{|w(x)-w(y)|^{p(x,y)-2}(w(x)-w(y))(\phi_\varepsilon^-(x)-\phi_\varepsilon^-(y))}{|x-y|^{N+sp(x,y)}}\,dxdy.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
I(w,\phi_\varepsilon^-)\le & -\varepsilon\int_{\{\phi_\varepsilon\le 0\}}\int_{\{\phi_\varepsilon\le 0\}}\frac{|w(x)-w(y)|^{p(x,y)-2}(w(x)-w(y))(\phi(x)-\phi(y))}{|x-y|^{N+sp(x,y)}}\,dxdy\\
&+2\int_{\{\phi_\varepsilon\le 0\}}\int_{\{\phi_\varepsilon\le 0\}^c}\frac{|w(x)-w(y)|^{p(x,y)-2}(w(x)-w(y))(\phi_\varepsilon^-(x)-\phi_\varepsilon^-(y))}{|x-y|^{N+sp(x,y)}}\,dxdy\\
\le & -\varepsilon\int_{\{\phi_\varepsilon\le 0\}}\int_{\{\phi_\varepsilon\le 0\}}\frac{|w(x)-w(y)|^{p(x,y)-2}(w(x)-w(y))(\phi(x)-\phi(y))}{|x-y|^{N+sp(x,y)}}\,dxdy\\
&-2\varepsilon\int_{\{\phi_\varepsilon\le 0\}}\int_{\{\phi_\varepsilon\le 0\}^c}\frac{|w(x)-w(y)|^{p(x,y)-2}(w(x)-w(y))(\phi(x)-\phi(y))}{|x-y|^{N+sp(x,y)}}\,dxdy,
\end{aligned}
$$

and thus,

$$
I(w,\phi_\varepsilon^-)\le 2\varepsilon\int_{\{\phi_\varepsilon\le 0\}}\int_{\mathbb{R}^N}\frac{|w(x)-w(y)|^{p(x,y)-1}|\phi(x)-\phi(y)|}{|x-y|^{N+sp(x,y)}}\,dxdy. \tag{3.15}
$$

Moreover, combining (3.14) and (3.15), we get

$$
\begin{aligned}
&\Bigg[\iint_{\mathbb{R}^{2N}}\frac{|w(x)-w(y)|^{p(x,y)-2}(w(x)-w(y))(\phi(x)-\phi(y))}{|x-y|^{N+sp(x,y)}}\,dxdy-\lambda\int_\Omega (w^+)^{-\gamma(x)}\phi\,dx\\
&-\int_\Omega (w^+)^{q(x)-1}\phi\,dx\Bigg]+2\int_{\{\phi_\varepsilon\le 0\}}\int_{\mathbb{R}^N}\frac{|w(x)-w(y)|^{p(x,y)-1}|\phi(x)-\phi(y)|}{|x-y|^{N+sp(x,y)}}\,dxdy\ge 0.
\end{aligned}
\tag{3.16}
$$

Next, we set

$$
J(w,\phi)=\frac{|w(x)-w(y)|^{p(x,y)-1}|\phi(x)-\phi(y)|}{|x-y|^{N+sp(x,y)}},
$$

for any $n\in\mathbb{N}$, it can be rewritten as

$$
\int_{\{\phi_\varepsilon\le 0\}}\int_{\mathbb{R}^N}J(w,\phi)\,dxdy=\int_{\{\phi_\varepsilon\le 0\}}\int_{B_n}J(w,\phi)\,dxdy+\int_{\{\phi_\varepsilon\le 0\}}\int_{B_n^c}J(w,\phi)\,dxdy, \tag{3.17}
$$

where $B_n=B(0,n)$ and $B_n^c=\mathbb{R}^N\setminus B_n$.

To estimate the terms in (3.17), we first remark that

$$
\iint_{\mathbb{R}^{2N}}J(w,\phi)\,dxdy<\infty \text{ and } meas\{\phi_\varepsilon\le 0\}\to 0,\quad \varepsilon\to 0,
$$

which yields

$$
\lim_{n\to\infty}\int_{\{\phi_\varepsilon\le 0\}}\int_{B_n^c}J(w,\phi)\,dxdy=0
$$

and

$$
\lim_{\varepsilon\to 0}\int_{\{\phi_\varepsilon\le 0\}}\int_{B_n}J(w,\phi)\,dxdy=0,
$$

for any $n\in\mathbb{N}$. In summary, we obtain from the above information that

$$
\lim_{\varepsilon\to 0}\int_{\{\phi_\varepsilon\le 0\}}\int_{\mathbb{R}^N}\frac{|w(x)-w(y)|^{p(x,y)-1}|\phi(x)-\phi(y)|}{|x-y|^{N+sp(x,y)}}\,dxdy=0. \tag{3.18}
$$

From (3.17) and (3.18), we infer that

$$\iint_{\mathbb{R}^{2N}} \frac{|w(x)-w(y)|^{p(x,y)-2}(w(x)-w(y))(\phi(x)-\phi(y))}{|x-y|^{N+sp(x,y)}}\, dxdy$$
$$-\int_\Omega (w^+)^{q(x)-1}\phi\, dx - \lambda \int_\Omega (w^+)^{-\gamma(x)}\phi\, dx \geq 0,$$

for any $\phi \in W_0$. The arbitrariness of ϕ gives that w is a nontrivial weak solution of (P_λ).

Now, we show that the solution w is positive. Indeed, testing (3.1) with $\phi = w^-$ and using the inequality

$$(w(x)-w(y))(w^-(x)-w^-(y)) \leq -|w^-(x)-w^-(y)|^2,$$

for any x, y a.e. in $\mathbb{R}^N$, we deduce that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|w(x)-w(y)|^{p(x,y)-2}(w^-(x)-w^-(y))^2}{|x-y|^{N+sp(x,y)}}\, dxdy + \int_\Omega (w^+)^{q(x)-1}w^-\, dx + \lambda\int_\Omega \frac{w^-}{(w^+)^{\gamma(x)}}\, dx \leq 0.$$

Therefore, $w^- = 0$ in Ω. In conclusion, problem (P_λ) admits a nontrivial positive weak solution $w \in W_0$. The proof of Theorem 3.1 is now completely proved. □

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