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The P-adic Metric Space Over the Words

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Abstract

Let p be a prime number. In this paper, we investigate on the p-adic metric space over the words. More precisely, using this distance, we define a metric spaces over the free monoids and the free groups. After that, we study their quotient topologys and some properties.

Keywords: *Fre monoid, Free group, Metric space and relatives, The p-adic valuation.*

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1 Introduction

Combinatorics on words is a field that has grown separately within several branches of mathematics, such as number theory, group theory, and appears frequently in problems of theoretical computer science, as dealing with automata and formal languages [16]. Consider Σ be a finite set, called alphabet. We denote by Σ^* the free monoid over Σ, and by $F(\Sigma)$ the free group over Σ. The most basic fact about free groups in abstract group theory is every group is isomorphic to the quotient group of a free group.The Hall topology for the free group was introduced by M. Hall in [15]. This is coarset topology such that every group morphism from the free group onto a finite discrete group is continuous[10].
The remainder of this paper is organized as follows. In Section 2, we begin

with some elementary material concerning of toplological space, metric space and the p-adic valuation over the integers. In Section 3, we define a metric spaces over the free monoid and their quotient topology. In Section 4, we give a metric spaces over the free group and their quotient topologys. Finally, we draw our conclusion in Section 5.

2 Preliminaries

Given a set X, a topology on X is a collection $\tau \in \mathcal{P}(X)$ such that
(i) For all $A \subseteq \tau$, we have $\cup_{o \in A} o \in \tau$.
(ii) For all $o_1, o_2 \in \tau$, we have $o_1 \cap o_2 \in \tau$.
(iii) $\emptyset, X \in \tau$.
The elements of τ are called the open sets. The complement of an open set is called a closed set. A set is clopen if it is both open and closed. The closure of a subset Y of X, denoted by $\overline{Y}$, is the intersection of the closed sets containing Y. A subset of X is dense if its closure is equal to X. A topological space is a set X together with a topology on X. A topology τ_2 on a set is a refinement of a topology τ_1 on the same set, if each open set for τ_1 is also an open set for τ_2.
A map from a topological space to another one is continuous if the inverse image of each open set is an open set. It is an homeomorphism if it is a continuos bijection and the inverse bijection is also continuous. Two topological spaces are homeomorphic if there is an homeomorphism between them. A topological space (X, τ) is Hausdorff if for each $u, v \in X$ with $u \neq v$, there exist disjoint open sets U and V such that $u \in U$ and $v \in V$. If f is a continuous map from a topological space X to an Hausdorff space Y, then the graph of f is closed in $X \times Y$.
If τ is a topology on X and $\mathcal{B} \subseteq \tau$, then $\mathcal{B}$ is a basis for τ if and only if
(i) $X \subseteq \cup_{B \in \mathcal{B}} B$
(ii) For all $A \in \tau$ and for all $x \in A$, there exists $B \in \mathcal{B}$ with $x \in B \subseteq A$.
If $\mathcal{B}_1, \mathcal{B}_2$ are bases for the typologies on X_1, X_2 then $\mathcal{B}_{1\times 2} = \{B_1 \times B_1, B_i \in \mathcal{B}_i\}$ is a basis for $\tau_{X_1 \times X_2}$. The projection maps are defined as $\pi_i : X_1 \times X_2 \longrightarrow X_i$ where $(x_1, x_2) \longmapsto x_i$ for $i = 1, 2$. Note that π_1 and π_2 are continuous. Also, a function $f : Y \longrightarrow X_1 \times X_2$ is continuous if and only if $\pi_1 \circ f$ and $\pi_2 \circ f$ are continuous.
If $\sim$ is an equivalence relation on a topological space X, then $X/\sim = \{[x], x \in X\}$ and the quotient topology on $X/\sim$ is defined to be
$\tau_{X/\sim} = \{A \subseteq X/\sim, q^{-1}(A) \text{ is open in } X\}$ where $q : X \longrightarrow X/\sim$ defined by $x \longmapsto [x]$ is the quotient map. Moreover, q is continuous and a function $f : X/\sim \longrightarrow Y$ is continuous if and only if $f \circ q$ is continuous.
A metric space (M, d) is a set M associated with a metric function $d : M \times M \longrightarrow \mathbb{R}_+ = \{x \in \mathbb{R}, x \geq 0\}$ with conditions following for d hold for all

$x, y, z \in M$.
(i) (definiteness) $d(x, y) = 0$ if and only if $x = y$;
(ii) (symmetry) $d(x, y) = d(y, x)$;
(iii) (triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$.
The topology defined by d is obtained by taking as a basis the open ϵ-balls defined for $x \in M$ and $\epsilon > 0$ by $B(x, \epsilon) = \{y \in M, d(x, y) < \epsilon\}$.
A sequence $(u_n)_{n \geq 0}$ of elements of a metric space (M, d) is converging to a limit u if, for each $\epsilon > 0$, there exists an integer k such that for each $n \geq k, d(u_n, u) < \epsilon$.
A Cauchy sequence in a metric space (M, d) is a sequence $(u_n)_{n \geq 0}$ of elements of M such that for each $\epsilon > 0$, there exists an integer k such that for each $n \geq k$ and $m \geq k, d(u_n, u_m) < \epsilon$. Every convergent sequence is a Cauchy sequence, but the converse does not hold in general. A metric space in which each Cauchy sequence is convergent is said to be complete.
The completion of M can be constructed as follows. Let $C(M)$ be the set of Cauchy sequences in M. Define an equivalence relation $\sim C(M)$ as follows. Two Cauchy sequences $u = (u_n)_{n \geq 0}$ and $v = (v_n)_{n \geq 0}$ are equivalent if the interleaved sequence $u_0, v_0, u_1, v_1, ...$ is also a Cauchy sequence. The completion of M is defined to be the set $\widehat{M}$ of equivalence classes of $C(M)$. The metric d on M extends to a metric on $\widehat{M}$ defined by $d(u, v) = \lim_{n \longrightarrow \infty} d(u_n, v_n)$.
A map $f : (M, d) \longrightarrow (M', d')$ between two metric spaces is continuous if and only if, for every sequence $(u_n)_{n \geq 0}$ of elements of M converging to a limit u, the sequence $(f(u_n))_{n \geq 0}$ converges to $f(u)$. A function f from (M, d) to (M', d') is said to uniformuly continuous if for each $\epsilon > 0$, there exists $\delta > 0$ such that the relation $d(x, y) < \delta$ implies $d'(f(x), f(y)) < \epsilon$.
Let $(M_i, d_i)_{1 \leq i \leq n}$ be a finite family of metric spaces. Then $(M_1 \times ... \times M_n, d)$ is a metric space, where d, defined by
$d((x_1, ..., x_n), (y_1, ..., y_n)) = \max\{d_1(x_1, y_1), ..., d_n(x_n, y_n)\}$.
Let p be a prime number, the p-adic valuation over the integers is a function $v_p : \mathbb{Z} \longrightarrow \mathbb{N}$ such that

$$v_p(n) = \begin{cases} +\infty \text{ if } n = 0 \\ \max\left\{k \in \mathbb{N}, p^k \text{ divides } n\right\} \text{ if } n \neq 0 \end{cases}.$$

The p-adic valuation can be extended to the rationals as well, and the function $v_p : \mathbb{Q} \longrightarrow \mathbb{Z}$ is defined as $v_p\left(\frac{a}{b}\right) = v_p(a) - v_p(b)$.
The p-adic valuation is an auxillary function. It is used to clarify the definition of the p-adic norm, which is defined as $|\cdot|_p : \mathbb{Q} \longrightarrow \mathbb{N}$ such that

$$|n|_p = \begin{cases} 0 \text{ if } n = 0 \\ p^{-v_p(n)} \text{ if } n \neq 0 \end{cases}$$

The p-adic metric d_p on $\mathbb{Q}$ is defined by setting, for each pair (x, y) of rationals $d_p(x, y) = |x - y|_p$. It is well-known that d_p is an ultrametric.
For $x \in \mathbb{N}, x \neq 0$, let $\mathcal{E}$ represent the set of prime numbers and infinity $\mathcal{E} = \{\infty, 2, 3, 5, 7, 11, ...\}$ and define the ∞-adic norm such that $|x|_\infty = x$. Then

$_{p\in\mathcal{E}} |x|_p = 1$ [5, 13].

A monoid $(N, \cdot)$ consists of a set N together with a binary operation $\cdot$ on N such that

(i) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in N$. (associativity)

(ii) There exists an identity $1_N \in N$ such that $a \cdot 1_N = 1_N \cdot a = a$ for all $a \in N$. A monoid $(N, \cdot)$ is called commutative if the operation " $\cdot$" is commutative. Hence a semigroup $(S, \cdot)$ is just a set S together with an associative binary operation.

A congruence on a monoid N is an equivalence relation $\sim$ on N compatible with the operation of N, i.e, for all $n, n' \in N, u, v \in N$

$$n \sim n' \Longrightarrow unv \sim un'v$$

Let N_1 and N_2 be monoids. Then $f : N_1 \longrightarrow N_2$ is a homomorphism if and only if $f(n_1 n_2) = f(n_1) f(n_2)$ for all $n_1, n_2 \in N_1$ and $f(1_{N_1}) = f(1_{N_2})$.

We say that f is one-to-one if for every $n_1, n_2 \in N_1$ where $f(n_1) = f(n_2)$, we have $n_1 = n_2$.

We formally define an alphabet as a non-empty finite set. A word over an alphabet Σ is a finite sequence of symbols of Σ. Although one writes a sequence as $(\sigma_1, \sigma_2, ..., \sigma_n)$, in the present context, we prefer to write it as $\sigma_1\sigma_2...\sigma_n$. The set of all words on the alphabet Σ is denoted by $\Sigma^*$and is equipped with the associative operation defined by the concatenation of two sequences. The concatenation of two sequences $\alpha_1\alpha_2...\alpha_n$ and $\beta_1\beta_2...\beta_m$ is the sequence $\alpha_1\alpha_2...\alpha_n\beta_1\beta_2...\beta_m$.

The concatenation is an associative operation. The string consisting of zero letters is called the empty word, written λ. Thus, $\lambda, \alpha, \beta, \alpha\alpha\beta\alpha, \alpha\alpha\alpha\beta\alpha$ are words over the alphabet $\{\alpha, \beta\}$. Thus the set Σ^* of words is equipped with the structure of a monoid. The monoid Σ^* is called the free monoid on Σ. The length of a word w, denoted $|w|$, is the number of letters in w when each letter is counted as many times as it occurs. Again by definition, $|\lambda| = 0$. For example $|\alpha\alpha\beta\alpha| = 4$ and $|\alpha\alpha\alpha\beta\alpha| = 5$. Let w be a word over an alphabet Σ. For $\sigma \in \Sigma$, the number of occurrences of σ in w shall be denoted by $|w|_\sigma$. For example $|\alpha\alpha\beta\alpha|_\beta = 1$ and $|\alpha\alpha\alpha\beta\alpha|_\alpha = 4$.

A language is a subset of Σ^*, a language L of Σ^* is recognizable if and only if there exists a monoid morphism f from Σ^* onto some finite monoid N such that $L = f^{-1}(f(L))$. It is then said that N recognizes the language L [1, 19].

A binary reation on Σ^* is a subset $\mathcal{R} \subseteq \Sigma^* \times \Sigma^*$. The congruence generated by $\mathcal{R}$ is defined as follows:

- $urv \longleftrightarrow_{\mathcal{R}} ur'v$ whenever $u, v \in \Sigma^*$, and $r\mathcal{R}r'$ or $r'\mathcal{R}r$;

- $u \overset{*}{\underset{\mathcal{R}}{\leftrightarrow}} v$ whenever $u = u_0 \longleftrightarrow_{\mathcal{R}} u_1 \longleftrightarrow_{\mathcal{R}} ... \longleftrightarrow_{\mathcal{R}} u_n = v$.

A presentation of a monoid N is a pair $(\Sigma, \mathcal{R})$ such that N is isomorphic to the

quotient of Σ^* by the congruence noted $\underset{\mathcal{R}}{\overset{*}{\leftrightarrow}}$ generated by $\mathcal{R}$, i.e, $N \cong \Sigma^* / \underset{\mathcal{R}}{\overset{*}{\leftrightarrow}}$.
The elements of Σ are called generators, and those of $\mathcal{R}$ are called relations. If there are finitely many generators and relations, i.e. $\Sigma = \{\sigma_1, ..., \sigma_p\}$ and $\mathcal{R} = \left\{(r_1, r_1'), ..., (r_q, r_q')\right\}$, we say that the monoid N is finitely presentable, and we write $N \cong \langle \sigma_1, ..., \sigma_n / r_1 = r_1', ..., r_q = r_q' \rangle$.
Recall that Any monoid $(N, \cdot, 1_N)$ has a standard presentation $(\Sigma, \mathcal{R})$, where Σ consists of one symbol σ_n for each $n \in N$, and $\mathcal{R}$ is defined by $\mathcal{R} = \{(\sigma_{1_N}, \epsilon), (\sigma_m \sigma_n, \sigma_{m \bullet n})$ for all $m, n \in N\}$. In particular, any finite monoid is finitely presented [18].
We say that a morphism $f : \Sigma^* \longrightarrow N$ separates two words x and y of Σ^* if $f(x) \neq f(y)$,
Note that, any pair of distinct of Σ^* can be separated by a finite monoid.
Given two words $x, y \in \Sigma^*$, we set
$s(x, y) = \min\{|N|, N$ is a monoid that separates x and $y\}$ and $d_2(x, y) = 2^{-s(x,y)}$ with the usual convention $\min \emptyset = +\infty$ and $2^{-\infty} = 0$. The function $d_2 : \Sigma^* \times \Sigma^* \longrightarrow \mathbb{R}_+$ is an ultrametric, that is, satisfies the folowing properties, for all $x, y, z \in \Sigma^*$,
(i) $d_2(x, y) = 0$ if and only if $x = y$;
(ii) $d_2(x, y) = d_2(y, x)$;
(iii) $d_2(x, z) \leq \max\{d_2(x, y), d_2(y, z)\}$.
It also satisfies the property
(iv) $d_2(xz, yz) \leq d_2(x, y)$ and $d_2(zx, zy) \leq d_2(x, y)$.
A topological monoid is a monoid N equipped with a topology for which the monoid operation is continuous.
The topology defined on Σ^* by d_2 is discrete, i.e, every subset is clopen.

Let p be a positive prime number. A p-group is a group in which every element has order equal to a power of p. A finite group is a p-group if and only if its order is a power of p.
Let G be a group and let $S \subseteq G$. The subgroup generated by S (denoted $\langle S \rangle$) is the smallest subgroup (with respect to inclusion) of G that contains S. This subgroup always exists and can be expressed as follows:
$\langle S \rangle = \cap\{H, S \subseteq H, H \leq G\}$
$= \{s_1^{\alpha_1} ... s_n^{\alpha_n}, n \in \mathbb{N}, s_1, ..., s_n \in S, \alpha_1, ..., \alpha_n \in \{-1, 1\}\}$.
A group G is finitely generated if it contains a finite subset that generates G. The rank of a group G (denoted $rank(G)$) is the smallest cardinality of a generating set of G, that is
$rank(G) = \min\{|S|, S \subseteq G, \langle S \rangle = G\}$.
Let S be a set. Define $\Sigma = S \cup S^{-1}$, where $S^{-1} = \{s^{-1}, s \in S\}$. Define a nonempty word in S to mean a formal expression of the forme $s_1^{\alpha_1} ... s_n^{\alpha_n}$ where $s_i \in S, \alpha_i \in \{-1, 1\}$ and $n > 0$. The concatenation of two words $s_1^{\alpha_1} ... s_n^{\alpha_n}$ and $t_1^{\beta_1} ... t_l^{\beta_l}$ is the word $s_1^{\alpha_1} ... s_n^{\alpha_n} t_1^{\beta_1} ... t_l^{\beta_l}$.
The concatenation is an associative operation. The word consisting of zero

letters is called the empty word, written λ. Thus the set Σ^* of words is equipped with the structure of a monoid. Call $s_n^{-\alpha_n}...s_1^{-\alpha_1}$ to be the inverse of the nonempty word $s_1^{\alpha_1}...s_n^{\alpha_n}$, we also write $\lambda^{-1} = \lambda$. on the set Σ^*, we defin the following relation,
$w_1 \sim w_2$ if w_2 is obtained from w_1 by a finite sequence of the operations: inserting or deleting expressions like ss^{-1} or $s^{-1}s$ for $s \in S$. Call a word w reduced if it does not contain any s adjacent to s^{-1} with $s \in S$. In particular, the empty word is reduced. This is an equivalence relation and the set F of equivalence classes $[w]$ will be given the structure of a group now in an obvious manner. Define $[w_1]\,[w_2] = [w_1 w_2]$. Each equivalence class in F contains a unique reduced word. Recall that, a free group F generated by S satisfies the following universal property: for any group G and any function $f : S \longrightarrow G$ there is a unique group homomorphism $\widehat{f} : F \longrightarrow G$ extending f such that $f = i \circ \widehat{f}$ where the mapping i denotes the inclusion from S into F. Free groups are important in group theory, because any group is a quotient of a free group by a normal subgroup. This is called presentation of a group. Let $R \subset F$ be a subset, let $\langle R \rangle_F^{\triangleleft}$ the smallest normal subgroup of F contains R. It always exists and can be expressed as:
$\langle R \rangle_F^{\triangleleft} = \cap \{N, R \subseteq N, N \triangleleft F\}$
$= \left\{ {}_{i=1}^{n} f_i^{-1} r_i^{\alpha_i} f_i, f_i \in F, r_i \in R, \alpha_i \in \{-1, 1\} \right\}$.
We notice $\langle S/R \rangle = F/\langle R \rangle_F^{\triangleleft}$. A group G is finitely presented if there exists a finite generating set S and a finite set R suth that $G \approxeq \langle S/R \rangle$. Note that every group has a presentation.
The Hall (or profinite) topology for the free group was introduced by M. Hall in [6].
We say that a morphism $f : F \longrightarrow G$ separates two words u and v of F if $f(u) \neq f(v)$,
Note that, any pair of distinct of F can be separated by a finite group.
Given two words $u, v \in F$, we set
$v_p(u) = \min \{n,\ G \text{ is a } p\text{-group of order } p^n \text{ that separates } u \text{ and } \lambda\}$ and
$d_p(u, v) = p^{-v_p(uv^{-1})}$ with the usual convention $\min \emptyset = +\infty$ and $p^{-\infty} = 0$. The function $d_p : F \times F \longrightarrow \mathbb{R}_+$ is an ultrametric, that is, satisfies the folowing properties, for all $x, y, z \in \Sigma^*$,
(i) $d_p(x, y) = 0$ if and only if $x = y$;
(ii) $d_p(x, y) = d_p(y, x)$;
(iii) $d_p(x, z) \leq \max \{d_p(x, y), d_p(y, z)\}$.
It also satisfies the property
(iv) $d_p(xz, yz) = d_p(x, y)$ and $d_p(zx, zy) = d_p(x, y)$.
With the distance d_p, the free group operation is uniformly continuous, and for all $u \in F$, we have $\lim_{n \longrightarrow \infty} \left(u^{p^n - 1}\right) = u^{-1}$ [8, 9].

3 Main results

In the following propositions, using the metric d_2, we define a metrics and their open open ϵ-balls on the free monoid.

Consider the metric space (Σ^*, d_2). Then,
1. The function $\gamma : \Sigma^* \times \Sigma^* \longrightarrow \mathbb{R}_+$, defined by $\gamma(x,y) = \frac{d_2(x,y)}{1+d_2(x,y)}$ is a distance on Σ^*.
2. The open balls associates with γ is given by:
for all $x \in \Sigma^*$ and $\epsilon > 0, B_\gamma(x,\epsilon) = \begin{cases} B_{d_2}\left(x, \frac{\epsilon}{1-\epsilon}\right) \text{ if } \epsilon < 1 \\ \Sigma^* \text{ if } \epsilon \geq 1 \end{cases}$.
3. For all $L \subseteq \Sigma^*$, for all $x, y \in \Sigma^*$,
$|\gamma(x,L) - \gamma(y,L)| \leq \lambda(x,y)$.

1. Let $x, y, z \in \Sigma^*$, we
• $\gamma(x,y) = 0 \Longleftrightarrow \frac{d_2(x,y)}{1+d_2(x,y)} = 0 \Longleftrightarrow d_2(x,y) = 0 \Longleftrightarrow x = y$.
• $\gamma(x,y) = \frac{d_2(x,y)}{1+d_2(x,y)} = \frac{d_2(y,x)}{1+d_2(y,x)} = \gamma(y,x)$.
• $\gamma(x,y) = \frac{d_2(x,y)}{1+d_2(x,y)} = \frac{1+d_2(x,y)-1}{1+d_2(x,y)} = \frac{1+d_2(x,y)}{1+d_2(x,y)} - \frac{1}{1+d_2(x,y)}$
$= 1 - \frac{1}{1+d_2(x,y)} \leq 1 - \frac{1}{1+d_2(x,z)+d_2(z,y)} = \frac{d_2(x,z)+d_2(z,y)}{1+d_2(x,z)+d_2(z,y)}$
$= \frac{d_2(x,z)}{1+d_2(x,z)+d_2(z,y)} + \frac{d_2(z,y)}{1+d_2(x,z)+d_2(z,y)}$
$\leq \frac{d_2(x,z)}{1+d_2(x,z)} + \frac{d_2(z,y)}{1+d_2(z,y)} = \gamma(x,z) + \gamma(z,y)$.
Consequently (Σ^*, γ) is a metric space.
2. Let $x \in \Sigma^*, \epsilon > 0$, we have,
$B_\gamma(x,\epsilon) = \{u \in \Sigma^*, \gamma(x,u) < \epsilon\}$
$= \left\{u \in \Sigma^*, \frac{d_2(x,u)}{1+d_2(x,u)} < \epsilon\right\} = \{u \in \Sigma^*, d_2(x,u) < \epsilon + \epsilon d_2(x,u)\}$
$= \{u \in \Sigma^*, (1-\epsilon)\, d_2(x,u) < \epsilon\}$.
There is only two cases to be considered:
• if $\epsilon < 1$, then
$B_\gamma(x,\epsilon) = \left\{u \in \Sigma^*, d_2(x,u) < \frac{\epsilon}{(1-\epsilon)}\right\} = B_{d_2}\left(x, \frac{\epsilon}{(1-\epsilon)}\right)$.
• if $\epsilon \geq 1$, then $B_\gamma(x,\epsilon) = \Sigma^*$.
3. Let $x, y \in \Sigma^*$ and $L \subseteq \Sigma^*$, we know that $\gamma(x,L) = \inf_{z \in L} \gamma(x,z)$.
We have, for all $x, y \in \Sigma^*$, for all $z \in L$,
$\gamma(x,z) \leq \gamma(x,y) + \gamma(y,z)$
implies that $\inf_{z\in L} \gamma(x,z) \leq \gamma(x,y) + \inf_{z\in L} \gamma(y,z)$.
Then $\gamma(x,L) - \gamma(y,L) \leq \gamma(x,y) \ldots.. (\triangle)$.
By changing the roles between x and y, we get
$\gamma(y,L) - \gamma(x,L) \leq \gamma(x,y) \ldots.. (\triangle\triangle)$.
According the reults $(\triangle)$ and $(\triangle\triangle)$, we obtain $|\gamma(x,L) - \gamma(y,L)| \leq \gamma(x,y)$.

Consider the metric space (Σ^*, d_2). Let S be a set and $f : S \longrightarrow \Sigma^*$ be a one-to-one function. Then,
the function $\mu : S \times S \longrightarrow \mathbb{R}_+$, defined by $\mu(x,y) = d_2(f(x), f(y))$ is a distance on S.

Let $x, y, z \in S$, we have
• $\mu(x,y) = 0 \Longleftrightarrow d_2(f(x), f(y)) = 0 \Longleftrightarrow f(x) = f(y) \Longleftrightarrow x = y$.
• $\mu(x,y) = d_2(f(x), f(y)) = d_2(f(y), f(x)) = \mu(y,x)$.
• $\mu(x,y) = d_2(f(x), f(y)) \leq d_2(f(x), f(z)) + d_2(f(z), f(y)) = \mu(x,z) + \mu(z,y)$.

Consider the metric space (Σ^*, d_2). Let $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be an increasing function such that,
(i) $f(0) = 0$ and for all $x > 0, f(x) > 0$;
(ii) for all $x, y \in \mathbb{R}_+, f(x+y) \leq f(x) + f(y)$.
(iii) f is an increasing function.
Then, The function $\theta : \Sigma^* \times \Sigma^* \longrightarrow \mathbb{R}_+$, defined by $\theta(x,y) = f(d_2(x,y))$ is a distance on Σ^*.

Let $x, y, z \in \Sigma^*$, we have
• If $x = y$, then $d_2(x,y) = 0$, implies that $f(d_2(x,y)) = 0$, because $f(0) = 0$, consequently $\theta(x,y) = 0$. If $x \neq y$, then $d_2(x,y) > 0$, implies that $f(d_2(x,y)) > 0$, because for all $x > 0, f(x) > 0$, then $\theta(x,y) \neq 0$. Finally, we have $\theta(x,y) = 0 \Longleftrightarrow x = y$.
• $\theta(x,y) = f(d_2(x,y)) = f(d_2(y,x)) = \theta(y,x)$.
• $d_2(x,y) \leq d_2(x,z) + d_2(z,y)$, implies that $f(d_2(x,y)) \leq f(d_2(x,z) + d_2(z,y))$, then $f(d_2(x,y)) \leq f(d_2(x,z)) + f(d_2(z,y))$. Finally, we have $\theta(x,y) \leq \theta(x,z) + \theta(z,y)$.

1. For all $0 < \alpha \leq 1$, the function d_2^α is a distance on Σ^*: we have for all $x, y \in \Sigma^*, d_2^\alpha(x,y) = f(d_2(x,y))$, where $f(x) = x^\alpha$. It is clear that the function f satisfies the following conditions:
(i) $f(0) = 0$ and for all $x > 0, f(x) > 0$;
(ii) for all $x, y \in \mathbb{R}_+, f(x+y) \leq f(x) + f(y)$.
(iii) f is an increasing function.
2. the function $\ln(1 + d_2)$ is a distance on Σ^*: we have for all $x, y \in \Sigma^*, \ln(1 + d_2)(x,y) = f(d_2(x,y))$, where $f(x) = \ln(1+x)$. It is clear that the function f satisfies the following conditions:
(i) $f(0) = 0$ and for all $x > 0, f(x) > 0$;
(ii) for all $x, y \in \mathbb{R}_+, f(x+y) \leq f(x) + f(y)$.
(iii) f is an increasing function.
3. the function $min\{1, d_2\}$ is a distance on Σ^*: we have for all $x, y \in \Sigma^*, min(1, d_2)(x,y) = f(d_2(x,y))$, where $f(x) = \min\{1, x\}$. It is clear that the function f satisfies the following conditions:
(i) $f(0) = 0$ and for all $x > 0, f(x) > 0$;
(ii) for all $x, y \in \mathbb{R}_+, f(x+y) \leq f(x) + f(y)$.
(iii) f is an increasing function.

In the following proposition, we define the quotient topology on the free monoid.

Consider the metric space (Σ^*, d_2). Let $\mathcal{R} \subseteq \Sigma^* \times \Sigma^*$ be a binary reation on Σ^* and $\underset{\mathcal{R}}{\overset{*}{\leftrightarrow}}$ The congruence generated by $\mathcal{R}$. Let $\Sigma^*/\underset{\mathcal{R}}{\overset{*}{\leftrightarrow}} = \{[u], u \in \Sigma^*\}$ be the quotient of Σ^* by $\underset{\mathcal{R}}{\overset{*}{\leftrightarrow}}$. Then,

1. The topology defined on $\Sigma^*/\underset{\mathcal{R}}{\overset{*}{\leftrightarrow}}$ by d_2 is discrete.

2. The monoid operation $\rho : \Sigma^*/\underset{\mathcal{R}}{\overset{*}{\leftrightarrow}} \times \Sigma^*/\underset{\mathcal{R}}{\overset{*}{\leftrightarrow}} \longrightarrow \Sigma^*/\underset{\mathcal{R}}{\overset{*}{\leftrightarrow}}$ defined by $([u], [v]) \longmapsto [uv]$ is continuous.

1. Let $q : \Sigma^* \longrightarrow \Sigma^*/\underset{\mathcal{R}}{\overset{*}{\leftrightarrow}}$ defined by $u \longmapsto [u]$ be the quotient map, we have $\tau_{\Sigma^*/\underset{\mathcal{R}}{\overset{*}{\leftrightarrow}}} = \left\{A \subseteq \Sigma^*/\underset{\mathcal{R}}{\overset{*}{\leftrightarrow}}, q^{-1}(A) \text{ is open in } \Sigma^*\right\} = \mathcal{P}\left(_{\Sigma^*/\underset{\mathcal{R}}{\overset{*}{\leftrightarrow}}}\right)$.

2. Let O be an open set of $\tau_{\Sigma^*/\underset{\mathcal{R}}{\overset{*}{\leftrightarrow}}}$, then the inverse image of O by ρ is an open set on $\Sigma^*/\underset{\mathcal{R}}{\overset{*}{\leftrightarrow}} \times \Sigma^*/\underset{\mathcal{R}}{\overset{*}{\leftrightarrow}}$.

In the following propositions, using the metric d_p, we define a metrics and their open open ϵ-balls on the free group.

Proposition Consider the metric space (F, d_p). Then,

1. The function $\gamma : F \times F \longrightarrow \mathbb{R}_+$, defined by $\gamma(x, y) = \frac{d_2(x,y)}{1+d_2(x,y)}$ is a distance on F.

2. The open balls associates with γ is given by:
for all $x \in F$ and $\epsilon > 0, B_\gamma(x, \epsilon) = \begin{cases} B_{d_p}\left(x, \frac{\epsilon}{1-\epsilon}\right) \text{ if } \epsilon < 1 \\ \Sigma^* \text{ if } \epsilon \geq 1 \end{cases}$.

3. For all $L \subseteq F$, for all $x, y \in F$,
$|\gamma(x, L) - \gamma(y, L)| \leq \lambda(x, y)$.

4. Consider the metric space (F, d_p). Let $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be an increasing function such that,
(i) $f(0) = 0$ and for all $x > 0, f(x) > 0$;
(ii) for all $x, y \in \mathbb{R}_+, f(x + y) \leq f(x) + f(y)$.
(iii) f is an increasing function.

Proof A similar argument as in the case of free monoid.

In the following proposition, we define the quotient topology on the free group.

Consider the metric space (F, d_p). Let $F/\langle R\rangle_F^\triangleleft$ be the quotient of the free group F, $F/\langle R\rangle_F^\triangleleft = \{u\langle R\rangle_F^\triangleleft, u \in F\}$. Then,

1. The topology defined on $F/\langle R\rangle_F^\triangleleft$ by d_p is discrete.

2. The group operation $\rho : F/\langle R\rangle_F^\triangleleft \times F/\langle R\rangle_F^\triangleleft \longrightarrow F/\langle R\rangle_F^\triangleleft$ defined by $([u], [v]) \longmapsto [uv]$ is continuous.

A similar argument as in the case of free monoid.Then, The function $\theta : F \times F \longrightarrow \mathbb{R}_+$, defined by $\theta(x, y) = f(d_p(x, y))$ is a distance on F.

4 Open Problem

The open problem here is to examine under what conditions, we can introduce a distance over a free monoid and a free group?

5 Conclusion

In this work, we have defined a distances on a free monoids and free groups. After that, we give their the quotient topologys.

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