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Master memory

Field : Mathematics and Computer Sciences
Branch : Mathematics
Option : Partial Differential Equations and Applications

Theme

Some Existence and Regularity Results for Unilateral Problems with Degenerate Coercivity

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Work plan

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Introduction

Introduction

Unilateral Problems

$$\begin{cases} u \geq 0, & \text{a.e.in } \Omega, \\ \langle Au, u - v \rangle \leq \int_\Omega f(u - v), & \\ \forall v \in H_0^1(\Omega), v \geq 0, & \text{a.e.in}\Omega, \end{cases}$$

Assumptions

Here $\Omega \subset \mathbb{R}^N$, is a bounded, open ,$N > 2$, $Au = -div(a(x,u)Du)$ with $a(x,s) : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function
Satisfying the following conditions :

- $$\frac{\alpha}{(1+|s|)^\theta} \leq a(x,s) \leq \beta,$$
- $$0 \leq \theta < 1,$$

Where α and β are positive constants.

Introduction

The existence and regularity

$$\blacktriangleright\ f \in L^m(\Omega), \quad \forall m, \quad m > 1,$$

we will study the following cases :

- If $f \in L^m(\Omega), m > \frac{N}{2}$, then $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$.
- If $f \in L^m(\Omega), \frac{2N}{N+2-\theta(N-2)} \leq m < \frac{N}{2}$, then $u \in H_0^1(\Omega) \cap L^r(\Omega)$,with $r = \frac{Nm(1-\theta)}{N-2m}$.
- If $f \in L^m(\Omega), \frac{N(2-\theta)}{N+2-N\theta} \leq m < \frac{2N}{N+2-\theta(N-2)}$, then $u \in W_0^{1,q}(\Omega)$, with $q = \frac{Nm(1-\theta)}{N-m(1+\theta)}$.
- If $f \in L^m(\Omega), \frac{N}{N+1-\theta(N-1)} < m < \frac{N(2-\theta)}{N+2-N\theta}$, then $u \in W_0^{1,q}(\Omega)$, with $q = \frac{Nm(1-\theta)}{N-m(1+\theta)}$.

Sobolev Spaces

Sobolev Spaces

Spaces $L^p(\Omega)$:

let $1 \leq p < \infty$, such that
$L^p(\Omega) = \{f : \Omega \longrightarrow \mathbb{R}; f$ measurable in addition $\int_\Omega |f|^p < \infty\}$
$L^\infty(\Omega) = \{f : \Omega \longrightarrow \mathbb{R}; f$ measurable in addition $\sup|f| < \infty, \ a.e\}$

The norm of $L^p(\Omega)$

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p \, dx\right)^{\frac{1}{p}}.$$

$$\|f\|_{L^\infty(\Omega)} = \sup ess\, |f|.$$

Sobolev Spaces

Spaces $H^m(\Omega)$

$$H^m(\Omega) = \left\{ v \in L^2(\Omega) : D^\alpha v \in L^2(\Omega), \alpha \in \mathbb{N}^n, |\alpha| \leq m \right\}$$

$$H^1(\Omega) = \left\{ v \in L^2(\Omega) : \nabla v \in L^2(\Omega) \right\}$$

$$H_0^1(\Omega) = \left\{ v \in H^1(\Omega) \text{ with zero trace on } \partial\Omega \right\}$$

The norm of $H^m(\Omega)$

$$\|u\|_{H^m(\Omega)}^2 = \sum_{|\alpha| \leq m} \int_\Omega |D^\alpha u|^2 \, dx.$$

Sobolev Spaces

The dual of $H_0^1(\Omega)$

We denote by $H^{-1}(\Omega)$ the dual of $H_0^1(\Omega)$

The norm of dual $H_0^1(\Omega)$

$$\|F\|_{H^{-1}(\Omega)} = \sup \left\{ |Fv| : v \in H_0^1(\Omega), \|v\|_{H_0^1(\Omega)} \leq 1 \right\}.$$

Sobolev Spaces

The spaces $W^{j,p}(\Omega)$ and $W_0^{j,p}(\Omega)$

$W^{j,p}(\Omega) = \left\{ u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega)\ \forall \alpha \in \mathbb{N}^N \text{ such that } |\alpha| \leq j \right\}.$
$W_0^{j,p}(\Omega) = \left\{ u \in W^{j,p}(\Omega) \text{ with zero trace on } \partial\Omega \right\}$

The norm of $W^{j,p}(\Omega)$

$$\|u\|_{W^{j,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_{|\alpha| \leq j} \|D^\alpha u\|_{L^p(\Omega)}$$

Sobolev Spaces

The spaces $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$

$W^{1,p}(\Omega) = \{u \in L^p(\Omega) : Du \in L^p(\Omega)\ \}$.
$W_0^{1,p}(\Omega) = \left\{u \in W^{1,p}(\Omega) \text{ with zero trace on } \partial\Omega\right\}$

The norm of $W^{1,p}(\Omega)$

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|Du\|_{L^p(\Omega)}$$

The dual of $W_0^1(\Omega)$

We denote by $W^{-1,p'}(\Omega)$ the dual of $W_0^{1,p}(\Omega)$

Unilateral Problems with L^1 Data

Unilateral Problems with L^1 Data

Statement of the results

We consider the following problem

The problem

$$\begin{cases} Au = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

The operator A

$$Au = -div\ a(x, Du)$$

Let $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ be a Caratheodory function,

Unilateral Problems with L^1 Data

H1

$$a(x,\xi)\xi \geq \alpha|\xi|^p$$

H2

$$|a(x,\xi)| \leq \beta \left[h(x) + |\xi|^{p-1}\right]$$

H3

$$[a(x,\xi) - a(x,\eta)]\,[\xi - \eta] > 0$$

with $\alpha, \beta > 0$ and $h(x) \in L^{p'}(\Omega)$(where p' denotes the conjugate exponent of p)

Unilateral Problems with L^1 Data

Let's assume that

H4

$$f \in L^1(\Omega)$$

H5

$$\psi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$$

K

$$K = \left\{ v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) : v(x) \geq \psi(x) \text{ in } \Omega \right\}.$$

Unilateral Problems with L^1 Data

Theorem 1 For $2-\frac{1}{N}<p<N$. assuming conditions (H1),$\cdots$,(H5) there exists a unique solution u of the problem

Eq1

$$\begin{cases} u \in W_0^{1,q}(\Omega)\,,\ 1<q<\frac{N(p-1)}{N-1} \text{ with } 2-\frac{1}{N}<p<N \\ u(x)\geq \psi(x)\ \ in\ \Omega \\ T_k(u)\in W_0^{1,p}(\Omega)\ \ \forall K>0 \\ \langle Au, T_k(u-v)\rangle \leq \int_\Omega fT_k\,(u-v)\ \ \forall v\in K \end{cases}$$

Unilateral Problems with L^1 Data

Theorem 2 Let's assume that the hypotheses of Theorem 1 hold and that

H9

$$A\psi \in L^1(\Omega).$$

Assuming that u is the solution of problem , the following inequality holds

H10

$$f \leq Au \leq f + (f - A\psi)^- .$$

Unilateral Problems with L^1 Data

The compactness method
Step01 : Approximation
Satisfying the following conditions :

F1

$$\begin{cases} f_n \to f & \text{in } L^1(\Omega) \\ \|f_n\|_1 \leq \|f\|_1 , & \forall n \in \mathbb{N} \end{cases}$$

Consider u_n as the solution of the problem :

Eq3

$$\begin{cases} u_n \in W_0^{1,p}(\Omega) \; u_n(x) \geq \psi(x) & \text{in } \Omega \\ \langle Au_n, u_n - v\rangle \leq \int_\Omega f_n(u_n - v) & \\ \forall v \in W_0^{1,p}(\Omega), v(x) \geq \psi(x) & \text{in } \Omega. \end{cases}$$

Thanks to the hypotheses ,H1 ,H2 ,and H3 A is a nonlinear operator of Leray-Lions type, so the existence of u_n follows from the classical results of [1]

Unilateral Problems with L^1 Data

Step02 : Uniform estimates
Lemma 1 There exists a constant $c_0(q)$, independent on n, such that :

Estimat1

$$\|u_n\|_{W_0^{1,q}(\Omega)} \leq c_0(q) \ \forall n \in \mathbb{N}, 1 < q < \frac{N(p-1)}{N-1}.$$

Unilateral Problems with L^1 Data

Step 03 : Passage to the limit

$q < \frac{N(p-1)}{(N-1)}$:

Inj1

$$\begin{cases} u_n \rightharpoonup u & weakly - \; W_0^{1,q}(\Omega) \\ u_n \to u & strongly - \; L^q(\Omega) \\ u_n \to u & \text{almost everywhere in } \Omega \end{cases}$$

Since $u_n \geq \psi(x)$ in Ω ,$\forall n \in \mathbb{N}$

Inequality

$$u(x) \geq \psi(x) \; in \; \Omega.$$

Unilateral Problems with L^1 Data

H11

$$Du_n \to Du \text{ almost everywhere in } \Omega$$

H12

$$\langle Au_n - Aw, T_k(u_n - w)\rangle + \langle Aw, T_k(u_n - w)\rangle \leq \int_\Omega f_n T_k(u_n - w)$$

Thus, taking the limit as $n \to \infty$ in (H11),From this, it follows that u is a solution of (Eq2)

Unilateral Problems with L^1 Data

The penalization method

We assume that $\psi = 0$

F2

$$\begin{cases} f_\varepsilon \to f & \text{in} L^1(\Omega) \\ \|f_\varepsilon\|_1 \leq \|f\|_1 & \forall \varepsilon > 0. \end{cases}$$

J1

$$\beta(s) = |s|^{p-2}s$$

Eq4

$$\begin{cases} u_\varepsilon \in W_0^{1,p}(\Omega) \\ Au_\varepsilon - \beta\left(\frac{(u_\varepsilon)^-}{\varepsilon}\right) = f_\varepsilon \end{cases}$$

Unilateral Problems with L^1 Data

Lemma 4 There exists a constant $c(q) > 0$, independent on ε, such that :

Estimate 2

$$\|u_\varepsilon\|_{W_0^{1,q}(\Omega)} \leq c(q), \ \forall \varepsilon > 0 \quad (1)$$

with $1 < q < \frac{N(p-1)}{N-1}$.

Unilateral Problems with L^1 Data

Passage to the limit

Inj 2

$$\begin{cases} u_\varepsilon \rightharpoonup u & \text{Weakly- } W_0^{1,q}(\Omega) \\ u_\varepsilon \to u & \text{Strongly- } L^q(\Omega) \\ u_\varepsilon \to u & \text{almost everywhere in } \Omega \end{cases}$$

H15

$$Du_\varepsilon \to Du \text{ a.e. in } \Omega$$

H16

$$\langle Au_\varepsilon, T_k(u_\varepsilon - v)\rangle \leq \int_\Omega f_\varepsilon T_k(u_\varepsilon - v)$$

Taking the limit as $\varepsilon \to 0$ in (H16) we conclude the proof of Theorem 1

Unilateral Problems with L^1 Data

The omographic approximation

Let $\lambda > 0$ and $\{f_\lambda\}$ be a sequence of smooth function such that $\forall q < \frac{N(p-1)}{(N-1)}$:

F2

$$\begin{cases} f_\lambda \to f \quad in\ L^1(\Omega) \\ \|f_\lambda\|_1 \leq \|f\|_1 \quad \forall\lambda > 0 \end{cases}$$

Let us consider the following problem :

Eq5

$$\begin{cases} u_\lambda \in W_0^{1.p}(\Omega) \\ Au_\lambda + g\frac{u_\lambda - \psi}{\lambda + |u_\lambda - \psi|} = f_\lambda + g \quad in\ \Omega \end{cases}$$

We observe that g positive ,shat that

g

$$g = (f_\lambda - A\psi)^- .$$

$$\|g\|_1 \leq \|f\|_1 + \left\|(A\psi)^+\right\|_1 .$$

Unilateral Problems with L^1 Data

Lemma 3 Assume that hypotheses ,H1 ,H2 ,H3 ,and ,H16 are satisfied.

H17

$$u_\lambda \geq \psi \quad \forall \lambda > 0 \text{ a.e. in } \Omega$$

Also, $\exists c(q) > 0$, independent on λ, we have :

Estimate 3

$$\|u_\lambda\|_{W_0^{1,q}(\Omega)} \leq c(q) \;\; \forall \; 1 < q < \frac{N(p-1)}{N-1}.$$

Unilateral Problems with L^1 Data

Passage to the limit

Inj 3

$$\begin{cases} u_\lambda \rightharpoonup u & \text{weakly -}W_0^{1,q}(\Omega) \\ u_\lambda \to u & \text{strongly -}L^q(\Omega) \\ u_\lambda \to u & \text{almost everywhere in } \Omega. \end{cases}$$

H18

$$Du_\lambda \to Du \text{ a.e. in } \Omega$$

H19

$$\|T_k(u_\lambda)\|_{W_0^{1,p}(\Omega)} \leq ck \quad \forall k > 0.$$

H20

$$\|T_k(u_\lambda - v)\|_{W_0^{1,p}(\Omega)} \leq C, \tag{2}$$

Unilateral Problems with L^1 Data

$$\langle Au_\lambda, T_k(u_\lambda - v)\rangle = \int_\Omega f_\lambda T_k(u_\lambda - v + \int_\Omega \lambda \frac{g}{\lambda + (u_\lambda - \psi)} T_k(u_\lambda - v).$$

Letting $\lambda \to 0$ and taking into account (Inj3),(H18),and (H20) we conclude the proof of Theorem 1

H21

$$Au_\lambda \leq f_\lambda + (f_\lambda - A\psi)^-.$$

H22

$$Au_\lambda \geq f_\lambda.$$

Taking the limit as $\lambda \to 0$ in (H21) and (H22) we obtain (H10)

Unilateral Problems with L^m Data, $m > 1$

Unilateral Problems with L^m Data, $m > 1$

Main results

We consider the following problem

The problem

$$\begin{cases} Au = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

The operator A

$$Au = -div \ a(x,u)Du$$

Let $a : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function,

HH1

$$\frac{\alpha}{(1+|s|)^{\theta}} \leq a(x,s) \leq \beta$$

HH2

$$0 \leq \theta < 1$$

Unilateral Problems with L^m Data, $m > 1$

Theorem 3.1.1 Let $f \in L^m(\Omega), m > \frac{N}{2}$.
In that case, a function $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ exists as a solution to the following unilateral problem

Eq3.1

$$\begin{cases} u \geq 0 & \text{a.e.in } \Omega \\ \langle Au, u - v\rangle \leq \int_\Omega f(u-v) & \\ \forall v \in H_0^1(\Omega), v \geq 0 & \text{a.e. in } \Omega. \end{cases}$$

Fa1

(3)
$$f \leq Au \leq f^+$$

Unilateral Problems with L^m Data, $m > 1$

Theorem 3.1.2 Let $f \in L^m(\Omega)$, with m such that

HH3

$$\frac{2N}{N+2-\theta(N-2)} \leq m < \frac{N}{2}.$$

In that case, there exists a function $u \in H_0^1(\Omega) \cap L^r(\Omega)$,with

HH4

(4) $$r = \frac{Nm(1-\theta)}{N-2m}$$

Unilateral Problems with L^m Data, $m > 1$

Theorem 3.1.3 Let $f \in L^m(\Omega)$, with m such that

HH5

$$\frac{N(2-\theta)}{N+2-N\theta} \leq m < \frac{2N}{N+2-\theta(N-2)}.$$

In that case, there exists a function $u \in W_0^{1,q}(\Omega)$,with

HH6

$$q = \frac{Nm(1-\theta)}{N-m(1+\theta)} \tag{5}$$

HH7

$$a(x,u)|Du|^2 \in L^1(\Omega). \tag{6}$$

Unilateral Problems with L^m Data, $m > 1$

Theorem 3.1.4 Let $f \in L^m(\Omega)$, with m such that

HH8

$$\frac{N}{N+1-\theta(N-1)} < m < \frac{N(2-\theta)}{N+2-N\theta}.$$

In that case, there exists a function $u \in W_0^{1,q}(\Omega)$,with

HH9

$$q = \frac{Nm(1-\theta)}{N-m(1+\theta)}$$

Eq5

$$\begin{cases} u(x) \geq 0 & \text{a.e. } x \in \Omega \\ T_k(u) \in H_0^1 & \forall k > 0 \\ \langle Au, T_k(u-v)\rangle \leq \int_\Omega fT_k(u-v) & \\ \forall v \in H_0^1 \cap L^\infty(\Omega), v \geq 0 & \text{a.e. in } \Omega. \end{cases}$$

Unilateral Problems with L^m Data, $m > 1$

A priori estimates.
Let $f \in L^m(\Omega)$,and let f_n be a sequence of smooth functions such that

F_n

$$f_n \in L^{\frac{2N}{N+2}}(\Omega) f_n \to f \text{ strongly in } L^m(\Omega)$$

$\|F_n\|$

$$\|f_n\|_{L^m(\Omega)} \leq \|f\|_{L^m(\Omega)}, \forall n \in \mathbb{N}.$$

Eq6

$$\begin{cases} A_n u_n + f_n^- \frac{u_n}{\frac{1}{n} + |u_n|} = f_n^+ & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

The operator A_n

$$A_n u_n = -div(a(x, T_n(u_n)) Du_n).$$

Unilateral Problems with L^m Data, $m > 1$

HH10

$$a(x, T_n(s)) \geq \frac{\alpha}{(1+n)^\theta}, \text{ for a.e. } x \in \Omega, \forall s \in \mathbb{R},$$

and since $f_n \in H^{-1}$, by well-known results (look at [1]) there exists at least a solution u_n of problem Eq6 in the sense that

Eq7

$$\begin{cases} u_n \in H_0^1(\Omega) \\ \int_\Omega a(x, T_n(u_n)) Du_n Dv + \int_\Omega f_n^- \frac{u_n}{\frac{1}{n}+|u_n|} v = \int_\Omega f_n^+ v \\ \forall v \in H_0^1(\Omega). \end{cases}$$

HH11

$$u_n(x) \geq 0 \text{ for a.e. } x \in \Omega.$$

Unilateral Problems with L^m Data, $m > 1$

Lemma 3.1 Let $f \in L^m(\Omega)$, with $m > \frac{N}{2}$ and let u_n be a solution of Eq6.
Then, there exist two positive constants c_1, c_2, depending on $N, m, \alpha, \theta, |\Omega|, \|f\|_{L^m(\Omega)}$, such that, for any $n \in \mathbb{N}$,

HH12

$$\|u_n\|_{L^\infty(\Omega)} \leq c_1,$$

HH13

$$\|u_n\|_{H_0^1(\Omega)} \leq c_2,$$

Unilateral Problems with L^m Data, $m > 1$

Lemma 3.2 Let $f \in L^m(\Omega)$, with m satisfying hypothesis (HH1), and let u_n be a solution of problem (Eq6).
Subsequently, there exist two positive constants, namely c_3 and c_4, depending on $N, m, \alpha, \theta, |\Omega|, \|f\|_{L^m(\Omega)}$, such that, for any $n \in \mathbb{N}$,

HH14

$$\|u_n\|_{L^r(\Omega)} \leq c_3,$$

HH15

$$\|u_n\|_{H_0^1(\Omega)} \leq c_4,$$

where r is defined by (HH2).

Unilateral Problems with L^m Data, $m > 1$

Lemma 3.3 Assume $f \in L^m(\Omega)$ with

HH16

$$\frac{N}{N+1-\theta(N-1)} < m < \frac{2N}{N+2-\theta(N-2)}.$$

Let $\{f_n\}$ be a sequence of functions satisfying (HH12) and (HH13), and let u_n be a solution of (Eq6).
Then, for any $n \in \mathbb{N}$ and $K > 0$ we obtain

HH17

$$\int_\Omega |DT_k(u_n)|^2 dx \leq \frac{\|f\|_{L^1(\Omega)}}{\alpha} (1+K)^{\theta+1}.$$

HH18

$$\|u_n\|_{W_0^{1,q}(\Omega)} \leq c_5, \quad \forall n \in \mathbb{N},$$

Unilateral Problems with L^m Data, $m > 1$

Lemma 3.4 Let $\{v_n\}$ be a sequence of functions which is weakly convergent to v in $H_0^1(\Omega)$, and let u_n be a sequence of functions which is almost everywhere convergent to some function u in Ω. Then

HH19

$$\int_\Omega a(x,u)|Dv|^2 \leq \liminf_{n\to+\infty} \int_\Omega a(x,T_n(u_n))|Dv_n|^2 \leq c.$$

HH20

$$\|u_n\|_{H_0^1(\Omega)} \leq c_2,$$

Conclusion

Conclusion

We suggest to study the following problem

The problems

$$\begin{cases} Au = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is a bounded Lipschitz domain in $\mathbb{R}^N$ with $N > 2$, A is defined as a nonlinear operator

The operator A

$$Au = -diva(x, Du)$$

Conclusion

with $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Caratheodory function such that for *a.e.* $x \in \Omega$ and $\forall \xi, \eta \in \mathbb{R}^N, (\xi \neq \eta)$ the following assumptions hold :

H1

$$a(x,\xi)\xi \geq \alpha|\xi|^p$$

H2

$$|a(x,\xi)| \leq \beta \left(h(x) + |\xi|^{p-1}\right)$$

H3

$$\left(a(x,\xi) - a(x,\eta)\right)(\xi - \eta) > 0$$

with $\alpha, \beta > 0$, $h(x)$ is a non-negative function in $L^{p'}(\Omega)$(here p' denotes the conjugate exponent of p) ,and $f \in L_\rho(\Omega)$

Conclusion

Let $\rho : [0, +\infty) \longrightarrow [0, +\infty)$ be an N-function, i.e, a convex function such that

convex function $\rho(\eta)$

$$\lim_{\eta \longrightarrow 0^+} \frac{\rho(\eta)}{\eta} = 0 \text{ and } \lim_{\eta \longrightarrow +\infty} \frac{\rho(\eta)}{\eta} = +\infty.$$

Then it is possible to define the Orlicz space

The definition $L_\rho(\Omega)$

$$L_\rho(\Omega) = \left\{ f \text{ measurable on } \Omega \mid \exists M > 0 : \int_\Omega \rho(\frac{|f|}{M}) < +\infty \right\}.$$

where $L_\rho(\Omega)$ is a Banach space under the norm

The norm $L_\rho(\Omega)$

$$\|f\|_{L_\rho(\Omega)} = \inf \left\{ M > 0 : \int_\Omega \rho(\frac{|f|}{M}) \leq 1 \right\}$$

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Thank you for your attention.