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Theme

Some Existence and Regularity Results for Unilateral Problems with Degenerate Coercivity

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Dedication

To my remarkable family,

On this momentous occasion of my graduation, I am overwhelmed with profound gratitude for the unwavering support, boundless love, and endless encouragement you have bestowed upon me throughout my university journey. Today is not just a celebration of my individual achievements but a testament to the remarkable strength and unity that defines our family, I dedicate my graduation to all of you.

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Abstract

In this work we study the existence and regularity of the solutions of unilateral problems

$$\begin{cases} u \geq 0, & \text{a.e.in } \Omega, \\ \langle Au, u-v\rangle \leq \int_\Omega f(u-v), & \\ \forall v \in H_0^1(\Omega), v \geq 0, & \text{a.e.in } \Omega, \end{cases} \quad (1)$$

Here Ω is a bounded, open subset of $\mathbb{R}^N$, with $N > 2$, and $Au = -div(a(x,u)Du)$ with $a(x,s)$: $\Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, satisfying the following conditions:

$$\frac{\alpha}{(1+|s|)^\theta} \leq a(x,s) \leq \beta. \quad (2)$$

for some real number θ such that.

$$0 \leq \theta < 1 \quad (3)$$

for almost every $x \in \Omega$, for every $s \in \mathbb{R}$, where α and β are positive constants. The objective of this contribution is to study the existence and regularity of the solutions of unilateral problems (1) associated to A under assumptions (2)-(3) and with data f belonging to various Lebesgue space $L^m(\Omega)$, for some $m > 1$.

Key words: Unilateral Problems, Degenerate Coercivity, Existence, Regularity.

ملخص

في هذا العمل ندرس وجود و إنتظام حلول المسائل الأحادية الجانب

$$\begin{cases} u \geq 0, & a.e.in \ \Omega, \\ \langle Au, u-v\rangle \leq \int_\Omega f(u-v), & \\ \forall v \in H_0^1(\Omega), \ \ v \geq 0, & a.e.in \ \Omega, \end{cases} \quad (4)$$

هنا Ω مجال جزئي مفتوح من $\mathbb{R}^N$ مع $N>2$ ، و $Au=-div(a(x,u)Du)$ و $a(x,s):\Omega\times\mathbb{R}\to\mathbb{R}$ هي دالة Carathéodory ، مع تحقيق الشرط التالي:

$$\frac{\alpha}{(1+|s|)^\theta} \leq a(x,s) \leq \beta.$$

بالنسبة لعدد الحقيقي θ

$$0 \leq \theta < 1.$$

لكل $x\in\Omega$ ، ولكل $s\in\mathbb{R}$ ، حيث يمثل α و β ثابتين موجبين. الهدف من هذه العمل هو دراسة الوجود و الإنتظام لحلول المسائل الأحادية الجانب (1)
المرتبطة بالفرضيات (2) ـ (3) ومع f تنتمي الى فضاء لوبيغ L^m ، حيث $m>1$.

الكلمات الرئيسية: المسائل الأحادية الجانب ، وجود الحل و إنتظامه ، القهرية الإنحلالية

Résumé

Dans ce travail, nous étudions l'existence et la régularité des solutions du problèmes unilatéraux

$$\begin{cases} u \geq 0, & \text{a.e.in } \Omega, \\ \langle Au, u-v\rangle \leq \int_\Omega f(u-v), & \\ \forall v \in H_0^1(\Omega), v \geq 0, & \text{a.e.in } \Omega, \end{cases}$$

Avac Ω un sous-ensemble borné et ouvert de $\mathbb{R}^N$, avec $N > 2$, et $Au = -div(a(x,u)Du)$ avec $a(x,s)$: $\Omega \times \mathbb{R} \to \mathbb{R}$ est une fonction de Carathéodory, satisfaisant les conditions suivantes:

$$\frac{\alpha}{(1+|s|)^\theta} \leq a(x,s) \leq \beta.$$

pour un nombre réel θ tel que.

$$0 \leq \theta < 1.$$

pour presque tout $x \in \Omega$,pour tout $s \in \mathbb{R}$, oú α et β sont des constantes positives. L'objectif de cette contribution est d'étudier l'existence et la régularité des solutions du problème unilatéraux (1) associès á A sous les hypothèses (3)-(3) et avec des données f appartenant á des espaces de Lebesgue L^m, pour un certain $m > 1$.

mots-clés: problèmes unilatéraux, coercivité dégénérée, la régularité, l'existence

List of Symbols

In what follows, we will use the following notations.

$\mathbb{R}^n$	Euclidean, n-dimensional space.
x	Vecteur de $\mathbb{R}^n$, $x=(x_1,x_2,...,x_n)$, $x_i \in \mathbb{R}$, $1 \leq i \leq n$.
$d\mu$	or dx Lebesgue measure N-dimensional.
$\lvert\Omega\rvert$	Measure of the set Ω.
Ω	Open set in $\mathbb{R}^n$.
$\overline{\Omega}$	The closure set of in $\mathbb{R}^n$.
$\partial\Omega$	The border of Ω.
B	Open ball.
$B(x,r)$	Open ball with center x and radius $r>0$.
B_E	The closed unit ball of E .
B_E	$=\{x \in E$ in which $\lVert x\rVert = 1\}$.
$W^{k,p}(\Omega)$	$=\left\{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega)\ \forall \alpha \in \mathbb{N}^n \text{ such that } \lvert\alpha\rvert \leq j\right\}$.
$W_0^{k,p}(\Omega)$	Sobolev space with 0 on $\partial\Omega$.
$W^{-k,p'}(\Omega)$	Dual space of $W_0^{k,p}(\Omega)$.
$D_i u$	$=\frac{\partial u}{\partial x_i}$ The partial derivative of u with respect to x_i.
$\mathscr{D}(\Omega)$	Space of indefinitely differentiable functions on Ω.
p'	The conjugate exponent of p.
p^*	$=\frac{Np}{N-p}$ Sobolev conjugate.
$C^\infty(\Omega)$	Is the set of functions in $C^k(\Omega)$ for all k.
$C_0^\infty(\Omega)$ or $D(\Omega)$	The space of smooth functions with compact support in Ω.
$D'(\Omega)$	The dual space of $D(\Omega)$; space of real distributions on Ω.
$suppf$	$=\left\{\overline{x \in \Omega : f(x) \neq 0}\right\}$ The support of f.
∇u	The gradient of u.
Δu	The Laplacian of u.
$C(\Omega)$	Is the set of functions continuous in Ω.
$C(\bar{\Omega})$	Is the set of functions continuous in $\bar{\Omega}$.
$C^k(\Omega)$	Is the set of functions which have derivatives of $order \leq k$ that are continuous in Ω.
$C^k(\bar{\Omega})$	Is the set of functions in $C(\bar{\Omega})$ which have derivatives in Ω of order is lass than or equals k.

Introduction

This memory master is devoted to the study of the existence and regularity of the solutions of unilateral problems like

$$\begin{cases} u \geq 0, & \text{a.e.in } \Omega, \\ \langle Au, u-v \rangle \leq \int_\Omega f(u-v), \\ \forall v \in H_0^1(\Omega), v \geq 0, & \text{a.e.in } \Omega, \end{cases}$$

Here Ω is a bounded, open subset of $\mathbb{R}^N$, with $N > 2$, and $Au = -div(a(x,u)Du)$ with $a(x,s)$: $\Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function (i.e. is measurable with respect to x for every $s \in \mathbb{R}$, and continuous with respect to s for almost every $x \in \Omega$), satisfying the following conditions:

$$\frac{\alpha}{(1+|s|)^\theta} \leq a(x,s) \leq \beta$$

for some real number θ such that.

$$0 \leq \theta < 1$$

for almost every $x \in \Omega$, for every $s \in \mathbb{R}$, where α, β are positive constants, and f belonging to various Lebesgue space $L^m(\Omega)$, for some $m > 1$. The regularity of solutions of the unilateral problems in terms of the summability of the datum f can be summarized as follows (look at [11])

- If $f \in L^m(\Omega), m > \frac{N}{2}$, then $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

- If $f \in L^m(\Omega)$, $\frac{2N}{N+2-\theta(N-2)} \leq m < \frac{N}{2}$, then $u \in H_0^1(\Omega) \cap L^r(\Omega)$,with $r = \frac{Nm(1-\theta)}{N-2m}$.

- If $f \in L^m(\Omega)$, $\frac{N(2-\theta)}{N+2-N\theta} \leq m < \frac{2N}{N+2-\theta(N-2)}$, then $u \in W_0^{1,q}(\Omega)$, with $q = \frac{Nm(1-\theta)}{N-m(1+\theta)}$.

- If $f \in L^m(\Omega)$, $\frac{N}{N+1-\theta(N-1)} < m < \frac{N(2-\theta)}{N+2-N\theta}$, then $u \in W_0^{1,q}(\Omega)$, with $q = \frac{Nm(1-\theta)}{N-m(1+\theta)}$.

The first chapter deals with Sobolev spaces constitute one of the most relevant functional settings for the treatment of boundary value problems. These spaces also allows us to study an existence result for the problem in Chapter 2. We refer to [33] and [32] for the theory of these spaces.

In the second chapter of the memory we study an existence and uniqueness theorem for the solution of unilateral problems with L^1 data associated to differential operators $Au = -div\ a(x, Du)$ of monotone satisfies some conditions. Moreover, we give more details about the regularly of the solutions of problem. The principal and more interesting subject of the second chapter of this memory concerns the regularity of solutions. We recall that problems have been studied by [10].

In Chapter 3, we study the question of existence and regularity of the solutions of unilateral problems associated to (2.1) and with data f belonging to various Lebesgue space $L^m(\Omega)$, for some $m > 1$, we will discuss the cases as mentioned before.

We highlight that all the results in Chapter 3 can be found in [11]. As already mentioned in [11], the classical method used in order to prove the existence of solutions to unilateral problems cannot be applied, although the datum f is regular. So we can get the best of this difficulty by considering a sequence of nondegenerate Dirichlet problems, having nonnegative solutions. However, there is another difficulty appears when $L^m(\Omega)$, for some $m > 1$. To overcome this difficulty we use another formulation that introduced in [11]. We highlight that all studies of the results in Chapter 2 and 3, are different if we substitute the operator Au with another one (see [4] or [28]).

Chapter 1

Some Results about Sobolev Spaces

In this chapter we recall some facts on Sobolev spaces and we give some of their properties. For further details on the Lebesgue and Sobolev spaces, we refer to [33],[20],[23],[3] ,and [32]. We remind that Sobolev spaces constitute one of the most relevant functional settings for the treatment of boundary value problems.

1.1 The spaces $W^{j,p}(\Omega)$ and $W_0^{j,p}(\Omega)$

Definition 1.1.1.

let $1 \leq p < \infty$, such that
$L^p(\Omega) = \{f : \Omega \longrightarrow \mathbb{R}; f$ measurable in addition $\int_\Omega |f|^p < \infty\}$
Note

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p \, dx\right)^{\frac{1}{p}}.$$

$L^\infty(\Omega) = \left\{f : \Omega \longrightarrow \mathbb{R}; f \text{ measurable in addition } \exists C \text{ such that } |f| < C a.e\right\}$
Note

$$\|f\|_{L^\infty(\Omega)} = \sup ess|f|.$$

Definition 1.1.2.

Suppose $1 \leq p < \infty$. Then

- $L^p_{loc}(\Omega) = \left\{u : u \in L^p(K) \text{ for every compact subset } K \text{ of } \Omega\right\}$,
- u is locally integrable in Ω if $u \in L^1_{loc}(\Omega)$.
- Let u and v be locally integrable functions defined in Ω. We define $v$as the weak derivative of u with respect to α if, for every $\phi \in C_0^\infty(\Omega)$

$$\int_\Omega u D^\alpha \phi dx = (-1)^{|\alpha|} \int_\Omega v \phi dx.$$

  and we say that $D^\alpha u = v$ in the weak sense.
- Let u and v be in $L^p_{loc}(\Omega)$. We define v as the strong derivative of u with respect to α if, for every compact subset K of Ω, there exists a sequence $\{\phi_j\}$ in $C^{|\alpha|}(K)$ such that $\phi_j \to u$ in $L^p(K)$ and $D^\alpha \phi_j \to v$ in $L^p(K)$.

Theorem 1.1.1.

If $D^{\alpha}u = v$ and $D^{\beta}v = w$ in the weak sense then $D^{\alpha+\beta}u = w$
in the weak sense.

Proof. let $\psi \in C_0^{\infty}(\Omega)$ and $\phi = D^{\beta}\psi$.Then

$$\int_{\Omega} uD^{\alpha+\beta}\psi dx = (-1)^{|\alpha|}\int_{\Omega}\phi v dx = (-1)^{|\alpha|}\int_{\Omega} vD^{\beta}\psi dx = (-1)^{|\alpha|+|\beta|}\int_{\Omega}\psi w dx.$$

□

Definition 1.1.3.

Let $\mu \in C_0^{\infty}(\mathbb{R}^n)$ be such that

1. $supp\mu \subset B_1(0)$,(recall that "supp" denotes the support of a function, and $B_r(c)$ denotes an open ball of radius r and center c).
2. $\int \mu(x)dx = 1$.
3. $\mu(x) \geq 0$.

   if $\varepsilon > 0$ then we set (provided that the integral exists)

$$J_{\varepsilon}u(x) = \frac{1}{\varepsilon^n}\int_{\Omega}\mu(\frac{x-y}{\varepsilon})u(y)dy.$$

$J_{\varepsilon}u$ is called a mollifier of u. Note that if u is locally integrable in Ω and if K is a compact subset of Ω then $J_{\varepsilon}u \in C^{\infty}(K)$ provided that $\varepsilon < dist(K, \partial\Omega)$.Suppose now that $u \in L^p_{loc}(\Omega)$.

$$J_{\varepsilon}u(x) = \int_{B_1(0)}\mu(y)u(x-\varepsilon y)dy,$$

so for $p > 1$ we have (if $\frac{1}{p} + \frac{1}{q} = 1$)

$$\begin{aligned} |J_{\varepsilon}u(x)| &\leq \int_{B_1(0)}\{\mu(y)\}^{\frac{1}{q}}\{\mu(y)\}^{\frac{1}{p}}|u(x-\varepsilon y)|dy \\ &\leq (\int_{B_1(0)}(\{\mu(y)\}^{\frac{1}{q}})^q dx)^{\frac{1}{q}}(\int_{B_1(0)}(\{\mu(y)\}^{\frac{1}{p}}|u(x-\varepsilon y)|)^p dy)^{\frac{1}{p}}. \end{aligned}$$

Hence $|J_{\varepsilon}u(x)|^p \leq \int_{B_1(0)}\mu(y)|u(x-\varepsilon y)|^p dy$, and this trivially holds if $p = 1$ too. Integrating this, we see that

$$\begin{aligned} \int_K |J_{\varepsilon}u(x)|^p dx &\leq \int_{B_1(0)}\mu(y)\int_K |u(x-\varepsilon y)|^p dxdy \\ &\leq \int_{B_1(0)}\mu(y)\int_{K_0}|u(x)|^p dxdy \\ &= \int_{K_0}|u(x)|^p dx, \end{aligned}$$

where K_0 is a compact subset of Ω, $K \subset Interior(K_0)$ and $\varepsilon < dist(K, \partial K_0)$ i.e. we have

$$\|J_{\varepsilon}u\|_{L^p(K)} \leq \|u\|_{L^p(K_0)}. \tag{1.1}$$

Lemma 1.1.1.

If $u \in L^P_{loc}(\Omega)$ and K is a compct subset of Ω then $\|J_\varepsilon u - u\|_{L^p(K)} \to 0$ as $\varepsilon \to 0$

Proof. Let K_0 be a compact subset of Ω where $K \subset Interior(K_0)$ and let $\varepsilon < dist(K, \partial K_0)$. Let $\delta > 0$ and let $w \in C^\infty(K_0)$ be such that $\|u - w\|_{L^p(K_0)} < \delta$. Then applying (1.1) to $u - w$, we get

$$\|J_\varepsilon u - J_\varepsilon w\|_{L^p(K)} < \delta. \tag{1.2}$$

However $J_\varepsilon w(x) - w(x) = \int_{B_1(0)} \mu(y)\left\{w(x - \varepsilon y) - w(x)\right\} dy$, and this goes to zero uniformly on K as $\varepsilon \to 0$. Hence, if ε is sufficiently small, we have

$$\|J_\varepsilon w - w\|_{L^p(K)} < \delta. \tag{1.3}$$

Hence, by (1.2) and (1.3)

$$\|J_\varepsilon u - u\|_{L^p(K)} \leq \|w - u\|_{L^p(K)} + \|J_\varepsilon u - J_\varepsilon w\|_{L^p(K)} + \|J_\varepsilon w - w\|_{L^p(K)} < 3\delta. \tag{1.4}$$

Since δ is arbitrary, $\|J_\varepsilon u - u\|_{L^p(K)} \to 0$ as $\varepsilon \to 0$. □

Theorem 1.1.2.

Suppose that u and v are in $L^p_{loc}(\Omega)$. Then $D^\alpha u = v$ in the weak sense if and only if $D^\alpha u = v$ in the strong $L^P(\Omega)$ sense.

Proof. Suppose that $D^\alpha u = v$. Let $\phi \in C_0^\infty(\Omega)$ and let $K = supp\phi$. Let $\varepsilon > 0$ and take $\psi \in C^{|\alpha|}(K)$ so that $\left\|\psi - u\right\|_{L^p(K)} < \varepsilon$ and $\left\|D^\alpha \psi - v\right\|_{L^p(K)} < \varepsilon$. Then

$$\begin{aligned}
\left|\int_K u D^\alpha \phi dx - (-1)^{|\alpha|}\int_K v\phi dx\right| &\leq \left|\int_K \psi D^\alpha \phi dx - (-1)^{|\alpha|}\int_K \phi D^\alpha \psi dx\right| \\
&\quad + \left|\int_K (u - \psi) D^\alpha \phi dx\right| + \left|\int_K (v - D^\alpha \psi)\phi dx\right| \\
&\leq \left\|u - \psi\right\|_{L^p(K)} \left\|D^\alpha \phi\right\|_{L^q(K)} + \left\|v - D^\alpha \psi\right\|_{L^p(K)} \left\|\phi\right\|_{L^q(K)} \\
&\leq \varepsilon(\left\|D^\alpha \phi\right\|_{L^q(K)} + \left\|\phi\right\|_{L^q(K)}),
\end{aligned}$$

where q is the conjugate exponent of p (if $p = 1$ then $q = \infty$ and if $p > 1$ then $\frac{1}{p} + \frac{1}{q} = 1$). But ε is arbitrary, so the LHS must be zero. So $D^\alpha u = v$ in the weak sense.

Conversely, suppose that $D^\alpha u = v$ in the weak sense and let K be a compact subset of Ω. Then $J_\varepsilon u \in C^\infty(K)$ if $\varepsilon < dist(K, \partial\Omega)$ and we have for all x in K

$$\begin{aligned}
D^\alpha J_\varepsilon u(x) &= \varepsilon^{-n}\int_\Omega D^\alpha_x \mu(\frac{x - y}{\varepsilon}) u(y) dy \\
&= \varepsilon^{-n}(-1)^{|\alpha|}\int_\Omega D^\alpha_y \mu(\frac{x - y}{\varepsilon}) u(y) dy \\
&= \varepsilon^{-n}\int_\Omega \mu(\frac{x - y}{\varepsilon}) v(y) dy \\
&= J_\varepsilon v(x).
\end{aligned}$$

But by Lemma 1.1.1, $\|J_\varepsilon u - u\|_{L^p(K)} \to 0$ and $\|D^\alpha J_\varepsilon u - v\|_{L^p(K)} = \|J_\varepsilon v - v\|_{L^p(K)} \to 0$ as $\varepsilon \to 0$.
Thus $D^\alpha u = v$ in the strong sense. □

Definition 1.1.4.

1. $\|u\|_{H^{j,p}(\Omega)}=\left(\sum_{|\alpha|\le j}\int_\Omega |D^\alpha u(x)|^p dx\right)^{1/p}$.

2. $\hat{C}^{j,p}(\Omega)=\{u\in C^j(\Omega): |u|_{H^{j,p}(\Omega)}<\infty\}$.

3. $H^{j,p}(\Omega)$ =completion of $\hat{C}^{j,p}(\Omega)$ with respect to the norm $|\ |_{H^{j,p}(\Omega)}$.

$H^{j,p}(\Omega)$ is called a Sobolev space. We will encounter other such spaces as well. Recall that for $1\le p<\infty$, $L^p(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the usual "p norm". This knowledge allows us to see what members of $H^{j,p}(\Omega)$. Suppose that u_m is a Cauchy sequence in $\hat{C}^{j,p}(\Omega)$ .Then for $|\alpha|\le j$, $D^\alpha u_m$ is a Cauchy sequence in $L^p(\Omega)$. Hence, there are members u^α of $L^p(\Omega)$ such that $D^\alpha u_m \to u^\alpha$ in $L^p(\Omega)$. Hence, according to our definition of strong derivatives, u^0 is in $L^p(\Omega)$ and u^α is the α strong derivative of u^0. Hence we see that,

$H^{j,p}(\Omega)=\{u\in L^p(\Omega):$ u has strong $L^p(\Omega)$ derivatives of order is less than or equals j in $L^p(\Omega)$ and there exists a sequence u_m in $\hat{C}^{j,p}(\Omega)$ such that $D^\alpha u_m\to D^\alpha u$ in $L^p(\Omega)\}$.

Definition 1.1.5.

$W^{j,p}(\Omega)=\{u\in L^p(\Omega): D^\alpha u\in L^p(\Omega)\ \forall\alpha\in\mathbb{N}^n \text{ such that } |\alpha|\le j\}$

Note

$$\|u\|_{W^{j,p}(\Omega)}=\|u\|_{L^p(\Omega)}+\sum_{|\alpha|\le j}\left\|D^\alpha u\right\|_{L^p(\Omega)} \tag{1.5}$$

Lemma 1.1.2.

Let $E\subset\mathbb{R}^n$ and let G be a collection of open sets U such that $E\subset\{\bigcup U: U\in G\}$. Then there exists a family F of non-negative functions $f\in C_0^\infty(\mathbb{R}^n)$ such that $0\le f(x)\le 1$ and

(i) for each $f\in F$, there exists $U\in G$ such that $supp f\subset U$

(ii) if $K\subset E$ is compact then $supp f\cap K$ is non-empty for only finitely many $f\in F$,

(iii) $\sum_{f\in F} f(x)=1$ for each $x\in E$,

(IV) if $G=\{\Omega_1,\Omega_2,\ldots\}$ where each Ω_i is bounded and $\bar{\Omega}_i\subset E$ then the family F of such functions can be constructed so that $F=\{f_1,f_2,\ldots\}$ and $supp f_j\subset\Omega_i$.

The family of functions F is called a partition of unity subordinate to the cover G.

Theorem 1.1.3. *(Meyers and Serrin, 1964)* $H^{j,p}(\Omega) = W^{j,p}(\Omega)$.

Proof. We already know that $H^{j,p}(\Omega) \subset W^{j,p}(\Omega)$.The opposite inclusion follows if we can show that for every $u \in W^{j,p}$ and for every $\varepsilon > 0$ we can find $w \in \hat{C}^{j,p}$ such that for $|\alpha| \le j$,$\|D^\alpha w - D^\alpha u\|_{L^p(\Omega)} < \varepsilon$.

For $m \ge 1$ let

$$\Omega_m = \left\{ x \in \Omega : \|x\| < m \qquad , dist(x, \partial\Omega) > \frac{1}{m} \right\}$$

and let $\Omega_0 = \Omega_{-1} = \emptyset$. let $\{\psi_m\}$ be the partition of unity of part (iv), Theorem (1.1.3) ,subordinate to the cover $\left\{\Omega_{m+2} - \overline{\Omega}_m\right\}$ .Each $u\psi_m$ is j times weakly differentiable and has support in $\Omega_{m+2} - \overline{\Omega}_m$ .As in the "conversely" part of the proof of Theorem (1.1.2), we can pick $\varepsilon_m > 0$ so small that $w_m = J_{\varepsilon m}(u\psi_m)$ has support in $\Omega_{m+3} - \overline{\Omega}_{m-1}$ and $\|w_m - u\psi_m\|_{W^{j,p}(\Omega)} < \frac{\varepsilon}{2^m}$. Let $w = \sum_{m=1}^{\infty} w_m$ .This is a C^∞ function because on each set $\Omega_{m+2} - \overline{\Omega}_m$ we have $w = w_{m-2} + w_{m-1} + w_m + w_{m+1} + w_{m+2}$.

Further .

$$\begin{aligned} \|D^\alpha w - D^\alpha u\|_{L^p(\Omega)} &= \left\| \sum_{m=1}^{\infty} D^\alpha (w_m - u\psi_m) \right\|_{L^p(\Omega)} \\ &\le \sum_{m=1}^{\infty} \|D^\alpha (w_m - u\psi_m)\|_{L^p(\Omega)} \\ &\le \sum_{m=1}^{\infty} \frac{\varepsilon}{2^m} = \varepsilon. \end{aligned}$$

□

Remarks 1.1.1.

(i) The proof shows that in fact $C^\infty(\Omega) \cap \hat{C}^{j,p}(\Omega)$ is dense in $W^{j,p}(\Omega)$.

(ii) Clearly members of $C^\infty(\Omega) \cap \hat{C}^{j,p}(\Omega)$ are not necessarily continuous on $\partial\Omega$ or even bounded near $\partial\Omega$. It would be very useful to have the knowledge that $C^\infty(\bar{\Omega}) \cup \hat{C}^{j,p}(\Omega)$ or $C^j(\bar{\Omega}) \cup \hat{C}^{j,p}(\Omega)$ is also dense in $W^{j,p}(\Omega)$.

Theorem 1.1.4.

If Ω has the segment property then the set of restrictions to Ω of functions in $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{m,p}(\Omega)$.

Theorem 1.1.5. ***Change of Variables and the Chain Rule.***

Let V, Ω be domains in $\mathbb{R}^n$ and let $T : V \to \Omega$ be invertible. Suppose that T and T^{-1} have continuous, bounded derivatives of order is j . Then if $u \in W^{j,p}(\Omega)$we have $v = u \circ T \in W^{j,p}(V)$ and the derivatives of v are given by the chain rule.

Proof. Let y denote coordinates in Ω and let x denote coordinates in V ($y = T(x)$). If $f \in L^p(\Omega)$ then $f \circ T \in L^p(V)$ because

$$\int_V |f \circ T|^p dx = \int_\Omega |f|^p J dy \leq const. \int_\Omega |f|^p dy \tag{1.6}$$

(Here J is the Jacobian of T^{-1}).

If $u \in W^{j,p}(\Omega)$, let $\{u_m\}$ be a sequence in $\hat{C}^{j,p}(\Omega)$ converging to u in $W^{j,p}(\Omega)$ and set $v_m = u_m \circ T$. By the chain rule, if $|\alpha| \leq j$

$$D_x^\alpha v_m = \sum_{\beta \leq \alpha} (D_y^\beta u_m) \circ T R_{\alpha,\beta}$$

Where the $R_{\alpha,\beta}$ are bounded terms involving T and its derivatives. But for $|\beta| \leq j$ $D_y^\beta u \in L^p(\Omega) \Rightarrow (D_y^\beta u) \circ T \in L^p(V) \Rightarrow (D_y^\beta u) \circ T R_{\alpha,\beta} \in L^p(V)$ since the $R_{\alpha,\beta}$ are bounded.
Further,

$$\begin{aligned}
\left\| D_x^\alpha v_m - \sum_{\beta \leq \alpha} (D_y^\beta u) \circ T R_{\alpha,\beta} \right\|_{L^p(V)} &= \left\| \sum_{\beta \leq \alpha} (D_y^\beta u_m - D_y^\beta u) \circ T R_{\alpha,\beta} \right\|_{L^p(V)} \\
&\leq \sum_{\beta \leq \alpha} \left\| (D_y^\beta u_m - D_y^\beta u) \circ T R_{\alpha,\beta} \right\|_{L^p(V)} \\
&\leq const. \sum_{\beta \leq \alpha} \left\| (D_y^\beta u_m - D_y^\beta u) \circ T \right\|_{L^p(V)} \\
&\leq const. \sum_{\beta \leq \alpha} \left\| (D_y^\beta u_m - D_y^\beta u) \right\|_{L^p(V)}
\end{aligned}$$

by (1.6). So ($\alpha = 0$ case), $v_m \to v = u \circ T$ in $L^p(V)$ and $D_x^\alpha v_m \to \sum_{\beta \leq \alpha} (D_y^\beta u) \circ T R_{\alpha,\beta}$ in $L^p(V)$. This shows that $v \in W^{j,p}(V)$ and $D_x^\alpha v = \sum_{\beta \leq \alpha} (D_y^\beta u) \circ T R_{\alpha,\beta}$. □

Definition 1.1.6.

$$W_0^{j,p}(\Omega) = \left\{ completion\ of\ C_0^\infty(\Omega)\ with\ respect\ to\ the\ norm\ |\ |_{W^{j,p}(\Omega)} \right\}$$

Proposition 1.1.1.

Let $\Omega \subset \mathbb{R}^N$ be an open set. Then, the following statemente hold :

(i) For each $1 \leq p \leq \infty$, $W^{1,p}(\Omega)$ is a Banach space.

(ii) For each $1 < p < \infty$, $W^{1,p}(\Omega)$ is refelexive.

(iii) For each $1 \leq p < \infty$, $W^{1,p}(\Omega)$ is a separable.

Proof. [1]

(i) Let $\{u_n\}_{n\in N}$ be a Cauchy sequence in $W^{1,p}(\Omega)$, with $1 \le p \le \infty$, Then, from (1.5) it follows that $\{u_n\}_{n\in N}$ and $\left\{(u_n)_{x_i}\right\}_{n\in N}$, with $1 \le i \le N$, are Cauchy sequences in $L^p(\Omega)$. Thus, since $L^p(\Omega)$ is a Banach space , it follows that $u_n \to n$ and $(u_n)_{x_i} \to g_i$ in $L^p(\Omega)$ with $u, g_i \in L^p(\Omega)$ Therefore, since

$$\int u_n \varphi_{x_i} = -\int_\Omega (u_n)_{x_i} \varphi \qquad \forall \varphi \in C_0^\infty(\Omega).$$

Letting $n \to +\infty$

$$\int u \varphi_{x_i} = -\int_\Omega g_i \varphi \qquad \forall \varphi \in C_0^\infty(\Omega)$$

Therefore, we obtain that $u \in W^{1,p}(\Omega)$, $u_{xi} = g_i$ and thus

$$\|u_n - u\|_{W^{1,p}(\Omega)} = \|u_n - u\|_{L^p(\Omega)} + \sum_{i=1}^{N} \left\|U_n - g_i\right\|_{L^p(\Omega)} \to 0$$

as desired.

(ii) Consider the space $E = L^p(\Omega) \times L^p(\Omega)$ which is reflexive since it is the product of reflexive spaces. Set the operator $T : W^{1,p}(\Omega) \to E$ defined by $Tu = (u, \nabla u)$ Then, T is an isometry, and since $W^{1,p}(\Omega)$ is a Banach space, $M = T\left(W^{1,p}(\Omega)\right)$ is a closed subspace of E since E is reflexive, B_E is compact in the weak topology $\sigma\left(E, E^\star\right)$, and M is closed in the topology $\sigma\left(E, E^\star\right)$ Therefore, B_M is compact in $\sigma\left(E, E^\star\right)$, and Therefore $T\left(W^{1,p}(\Omega)\right)$ is reflexive .As a consequence, $W^{1,p}(\Omega)$ is also reflexive .

(iii) Under the notation of (ii), and taking into account that E is separable , it follows that $T\left(W^{1,p}(\Omega)\right)$ is separable and therefore $W^{1,p}(\Omega)$ is also separable . □

Remarks 1.1.2.

(i) Saying that $f \in W_0^{j,p}(\Omega)$ is a generalized way of saying that f and its derivatives of order is less than or equals $j-1$ vanish on $\partial\Omega$. e.g. $W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$ is a useful space for studying solutions of the Dirichlet problem for second order elliptic PDE's.

(ii) $C_0^j(\Omega) \subset W_0^{j,p}(\Omega)$ because if $f \in C_0^j(\Omega)$, we know that if ε is sufficiently small then $J_\varepsilon f \in C_0^\infty(\Omega)$ and $J_\varepsilon f \to f$ in $|\ |_{W^{j,p}(\Omega)}$ norm.

[1]As mentioned in the Preface, whenever a black circle precedes some content, this content is original.

1.2 Extension Theorems

Most of the important Sobolev inequalities and imbedding theorems that we will derive in the next section are most easily derived for the space $W_0^{j,p}(\Omega)$ which can be viewed as being a subspace of $W^{j,p}(\mathbb{R}^n)$.

Lemma 1.2.1.

Let $u \in \mathbb{R}^n$ and $f \in L^p(\mathbb{R}^n)$. Set $f_\delta(x) = f(x+\delta u)$. Then $\lim_{\delta\to 0} f_\delta = f$ in $L^p(\mathbb{R}^n)$.

Proof. Given $\varepsilon > 0$, let $\phi \in C_0^\infty(\mathbb{R}^n)$ be such that $\|f-\phi\|_{L^p(\mathbb{R}^n)} < \varepsilon$. Since $\phi_\delta \to \phi$ uniformly on a sufficiently large ball containing the supports of all ϕ_δ (say, for $\delta \le 1$), we can pick δ so small that $\|\phi-\phi_\delta\|_{L^p(\mathbb{R}^n)} < \varepsilon$. Then

$$\|f-f_\delta\|_{L^p(\mathbb{R}^n)} \le \|f-\phi\|_{L^p(\mathbb{R}^n)} + \|\phi-\phi_\delta\|_{L^p(\mathbb{R}^n)} + \|\phi_\delta - f_\delta\|_{L^p(\mathbb{R}^n)} < 3\varepsilon.$$

□

Lemma 1.2.2.

Let $\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x_i > 0\}$.$C^\infty(\bar{\mathbb{R}}_+^n) \cap \hat{C}^{j,p}(\mathbb{R}_+^n)$ is dense in $W^{j,p}(\mathbb{R}_+^n)$.

Proof. Suppose f is in $W^{j,p}(\mathbb{R}_+^n)$ let $\varepsilon > 0$ and pick $\phi \in C^\infty(\mathbb{R}_+^n) \cap \hat{C}^{j,p}(\mathbb{R}_+^n)$ so that $\|D^\alpha\phi - D^\alpha f\|_{L^p(\mathbb{R}_+^n)} < \varepsilon$ for all $|\alpha| \le j$.We take the vector of Lemma (1.2.1) to be $u = (0,0,0,\dots,1)$ and define functions $\psi^\alpha \in L^p(\mathbb{R}^n)$ as

$$\psi^\alpha(x) = \begin{cases} D^\alpha\phi(x) & ,x_i > 0 \\ 0 & ,x_i \le 0 \end{cases}$$

Observe that for each $\delta > 0$,$\phi_\delta \in C^\infty(\bar{\mathbb{R}}_+^n) \cap \hat{C}^{j,p}(\mathbb{R}_+^n)$. By Lemma (1.2.1), we can pick $\delta > 0$ so that, for all $|\alpha| \le j$,$\|\psi_\delta^\alpha - \psi^\alpha\|_{L^p(\mathbb{R}^n)} < \varepsilon$. But this implies that $\|D^\alpha\phi_\delta - D^\alpha\phi\|_{L^p(\mathbb{R}_+^n)} < \varepsilon$.
Hence

$$\|D^\alpha\phi_\delta - D^\alpha f\|_{L^p(\mathbb{R}_+^n)} \le \|D^\alpha\phi_\delta - D^\alpha\phi\|_{L^p(\mathbb{R}_+^n)} + \|D^\alpha\phi - D^\alpha f\|_{L^p(\mathbb{R}_+^n)} < 2\varepsilon.$$

□

Lemma 1.2.3.

There exists a linear mapping $E_0 : W^{j,p}(\mathbb{R}_+^n) \to W^{j,p}(\mathbb{R}^n)$ such that $E_0 f = f$ in $\mathbb{R}_+^n$ and $\|E_0 f\|_{W^{j,p}(\mathbb{R}^n)} \le C\|f\|_{W^{j,p}(\mathbb{R}_+^n)}$, where C depends on only n and p.

Proof. If $f \in C^\infty(\bar{\mathbb{R}}^n{}_+)$, define

$$E_0 f(x) = \begin{cases} f(x) & ,x_n \ge 0 \\ \sum_{k=1}^{j+1} c_k f(x_1, x_2, \dots, x_{n-1}, -kx_n) & ,x_n < 0 \end{cases}$$

Where the constants c_k are chosen so that $E_0 f(x) \in C^j(\mathbb{R}^n)$, i.e.

$$\sum_{k=1}^{j+1} (-k)^m c_k = 1, \ m = 0, 1, 2, \ldots, j.$$

It is easy to check that there is a constant C depending on only n and p such that

$$\left\| D^\alpha E_0 f \right\|_{L^p(\mathbb{R}^n)} \leq C \left\| D^\alpha f \right\|_{L^p(R^n_+)}. \tag{1.7}$$

If now $f \in W^{j,p}(\mathbb{R}^n_+)$, take a sequence $f_m \in C^\infty(\bar{\mathbb{R}}^n_+) \cap \hat{C}^{j,p}(\mathbb{R}^n_+)$ converging to f in $W^{j,p}(\mathbb{R}^n_+)$ (we can do this by Lemma 1.2.2). Then f_m is a Cauchy sequence and (1.7) implies that $E_0 f_m$ is a Cauchy sequence in $W^{j,p}(\mathbb{R}^n)$. We denote the limit by $E_0 f$. Since $\left\| D^\alpha E_0 f_m \right\|_{L^p(\mathbb{R}^n)} \leq C \left\| D^\alpha f_m \right\|_{L^p(\mathbb{R}^n_+)}$ taking limits shows that f satisfies (1.7). □

Definition 1.2.1.

A domain Ω is of class C^m if $\partial\Omega$ can be covered by bounded open sets Ω_j such that there are mappings $\psi_j : \bar{\Omega}_j \to \bar{B}$,where B is the unit ball centered at the origin and

(i) $\psi_j(\Omega_j \cap \Omega) = B \cap R^n_+$

(ii) $\psi_j(\Omega_j \cap \partial\Omega) = B \cap \partial R^n_+$

(iii) $\psi_j \in C^m(\bar{\Omega}_j)$ and $\psi_j^{-1} \in C^m(\bar{B})$.

(Because of (iii), all derivatives of order is less than or equals m of ψ_j and its inverse are bounded).

Theorem 1.2.1.

If Ω is a bounded domain of class C^m then there exists a bounded linear extension operator $E : W^{m,p}(\Omega) \to W^{m,p}(\mathbb{R}^n)$.

Definition 1.2.2.

A domain Ω is said to satisfy the cone property if there exist positive constants α , h such that for each $x \in \Omega$ there exists a right spherical cone $V_x \subset \Omega$ with height h and opening α.

1.3 Sobolev Inequalities and Imbedding Theorems

Theorem 1.3.1.

If $\Omega \subset \mathbb{R}^n$ satisfies the cone condition (with height h and opening α) and if $P > 1$, $mp > n$ then $W^{m,p}(\Omega) \subset C_B(\Omega)$ and there is a constant C depending on only α , h, n and p such that for all $u \in W^{m,p}(\Omega)$, $\sup|u| \leq C \|u\|_{W^{m,p}(\Omega)}$.

Proof. Initially, suppose that u is in $\hat{C}^{m,p}(\Omega)$ .Let $g \in C^{\infty}(\mathbb{R})$ be such that $g(t) = 1$ if $t \le \frac{1}{2}$ and $g(t) = 0$ if $t \ge 1$.Let $x \in \Omega$ and let (r,θ) denote polar coordinates centered at x. Here, $\theta = (\theta_1, \theta_2, \dots, \theta_{n-1})$ denotes the angular coordinates and we can describe the cone with vertex x in polar coordinates as $V_x = \{(r,\theta) : 0 \le r \le h, \theta \in A\}$ .Clearly, we have

$$u(x) = -\int_0^h \frac{\partial}{\partial r}\left\{g(\frac{r}{h})u(r,\theta)\right\} dr,$$
$$= \frac{(-1)^m}{(m-1)!}\int_0^h r^{m-1}\frac{\partial^m}{\partial r^m}\left\{g(\frac{r}{h})u(r,\theta)\right\} dr,$$

After $m-1$ integrations by parts. Next, we integrate with respect to the angular measure dS_θ ,noting that the left-hand-side becomes a constant times $u(x)$.

$$u(x) = c\int_A\int_0^h r^{m-1}\frac{\partial^m}{\partial r^m}\left\{g(\frac{r}{h})u(r,\theta)\right\} dr dS_\theta$$
$$= c\int_A\int_0^h r^{m-n}\frac{\partial^m}{\partial r^m}\left\{g(\frac{r}{h})u(r,\theta)\right\} r^{n-1} dr dS_\theta$$
$$= \int_{V_x} r^{m-n}\frac{\partial^m}{\partial r^m}\left\{g(\frac{r}{h})u(r,\theta)\right\} dV.$$

Applying Hölder's inequality to this, we obtain

$$|u(x)| \le c\left\|r^{m-n}\right\|_{L^q(V_x)}\left\|\frac{\partial^m}{\partial r^m}\left\{g(\frac{r}{h})u(r,\theta)\right\}\right\|_{L^q(V_x)}$$
$$\le c\left\|r^{m-n}\right\|_{L^q(V_x)}\|u\|_{W^{m,p}(\Omega)}.$$

But r^{m-n} is in $L^q(V_x)$ if $n-1+(m-n)q > -1$, which is the case because $q = \frac{p}{p-1}$ and $mp > n$ Thus, we obtain $\sup|u| \le C\|u\|_{W^{m,p}(\Omega)}$. To extend this result to arbitrary $u \in W^{m,p}(\Omega)$,take a sequence $\{u_k\}$ of functions in $\hat{C}^{m,p}(\Omega)$ converging to u in the $|\ |_{W^{m,p}(\Omega)}$ norm.
Then $\sup|u_j - u_k| \le C\left\|u_j - u_k\right\|_{W^{m,p}(\Omega)}$, showing that the sequence is a Cauchy sequence in $C_B(\Omega)$. Thus u is in $C_B(\Omega)$ and taking the limit of $\sup|u_j| \le C\left\|u_j\right\|_{W^{m,p}(\Omega)}$shows that u satisfies the same inequality. □

Corollary 1.3.1.

If $\Omega \subset \mathbb{R}^n$ satisfies the cone condition (with height h and opening α) and if $p > 1, (m-k)p > n$ then $W^{m,p}(\Omega) \subset C_B^k(\Omega)$and there is a constant C depending on only α, h, n, k and p such that for all $u \in W^{m,p}(\Omega)$ $\quad \sup_{|\alpha|\le k} |D^\alpha u| \le C\|u\|_{W^{m,p}(\Omega)}$.

Theorem 1.3.2.

If $\Omega \subset \mathbb{R}^n$ is any domain and $p > n$ then $W_0^{1,p}(\Omega) \subset C^{0,\alpha}(\bar{\Omega})$, where $\alpha = 1 - \frac{n}{p}$ and there exists α constant C depending on only p and n such that for all $u \in W_0^{1,p}(\Omega)$

$$\frac{|u(x) - u(y)|}{\left\|x - y\right\|^\alpha} \le C\sum_{i=1}^n \|D_i u\|_{L^p(\Omega)}.$$

Theorem 1.3.3.

If $\Omega \subset \mathbb{R}^n$ is any domain and $p < n$ then $W_0^{1,p}(\Omega) \subset L^r(\Omega)$ where $r = \frac{np}{n-p}$ and there exists a constant C depending on only p and n such that for all $u \in W_0^{1,p}(\Omega)$

$$\|u\|_{L^r(\Omega)} \leq C \sum_{i=1}^{n} \|D_i u\|_{L^p(\Omega)}.$$

Remark 1.3.1.

Suppose that $a, b \geq 0$ and $1 < p, q < \infty$ in addition $\frac{1}{p} + \frac{1}{q} = 1$, the Young inequality is expressed by

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

which is more general than the previous one

$$ab \leq \frac{(a\varepsilon)^p}{p} + \frac{(\frac{b}{\varepsilon})^q}{q} = \delta a^p + C(\delta) b^q.$$

for all $\delta = \frac{\varepsilon^p}{p}$

Remark 1.3.2.

Suppose that $u_i \in L^{p_i}(\Omega), i = (1, 2, 3, \ldots, m)$ and $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \ldots + \frac{1}{p_m} = 1$.
The Hölder's inequality is expressed by

$$\int_\Omega |u_1 u_2 u_3 \ldots u_m| \, dx \leq \|u_1\|_{L^{p_1}(\Omega)} \|u_2\|_{L^{p_2}(\Omega)} \ldots \|u_m\|_{L^{p_m}(\Omega)} \tag{1.8}$$

*Proof. of Theorem*1.3.3 It suffices to prove the result for $u \in C_0^1(\mathbb{R}^n)$. First we prove the result for the case $p = 1$. For each i we have

$$|u(x)| \leq \int_{-\infty}^{x_i} |D_i u| dx_i \leq \int_{-\infty}^{\infty} |D_i u| dx_i.$$

Multiplying these n inequalities together and taking the $n-1$ the root gives

$$|u(x)|^{\frac{n}{n-1}} \leq \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |D_i u| dx_i \right)^{\frac{1}{n-1}} \tag{1.9}$$

Observe that $\int_{-\infty}^{\infty} |D_i x| dx_i$ does not depend on x_i , but it does depend on all $n-1$ of the remaining variables. We integrate each side of (1.9) with respect to x_1 and use the generalized Hölder inequality with $p_i = m = n-1$ to obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}} dx_1 &\leq \left(\int_{-\infty}^{\infty} |D_1 u| dx_1 \right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} |D_i u| dx_i \right)^{\frac{1}{n-1}} dx_1 \\
&\leq \left(\int_{-\infty}^{\infty} |D_1 u| dx_1 \right)^{\frac{1}{n-1}} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |D_i u| dx_i dx_1 \right)^{\frac{1}{n-1}}.
\end{aligned}$$

The RHS is still a product of $n-1$ functions of x_2 , so we integrate each side with respect to x_2, again applying (1.8) with $p_i = m = n-1$.Continuing in this manner, we finally obtain

$$\int_{\mathbb{R}^n} |u(x)|^{\frac{n}{n-1}} dx \leq \left(\prod_{i=1}^{n} \int_{\mathbb{R}^n} |D_i u| dx \right)^{\frac{1}{n-1}}$$

i.e.

$$\begin{aligned} \|u\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} &\leq \left(\prod_{i=1}^{n} \int_{\mathbb{R}^n} |D_i u| dx \right)^{\frac{1}{n}} \\ &\leq \frac{1}{n} \sum_{i=1}^{n} \int_{\mathbb{R}^n} |D_i u| dx \end{aligned}$$

Here we have used the fact that an arithmetic mean is no less than a geometric mean of the same numbers. This proves the result for the case $p = 1$.
For $p > 1$, let $\gamma = \frac{(n-1)p}{n-p} = 1 + \frac{n(p-1)}{n-p}$,Since $\gamma > 1$ and $u \in C_0^1(\mathbb{R}^n)$, it follows that $|u|^\gamma \in C_0^1(\mathbb{R}^n)$.

$$D_i |u|^\gamma = \frac{(n-1)p}{n-p} |u|^{\frac{n(p-1)}{n-p}} (\pm D_i u).$$

We apply the $p = 1$ case to $|u|^\gamma$ and obtain

$$\begin{aligned} \left(\int_{\mathbb{R}^n} |u|^{\frac{np}{n-p}} dx \right)^{\frac{n-1}{n}} &\leq \sum_{i=1}^{n} \frac{1}{n} \int_{\mathbb{R}^n} \frac{(n-1)p}{n-p} |u|^{\frac{n(p-1)}{n-p}} |D_i u| \, dx \\ &\leq \frac{(n-1)p}{n(n-p)} \sum_{i=1}^{n} \left(\int_{\mathbb{R}^n} (|u|^{\frac{n(p-1)}{n-p}})^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \|D_i u\|_{L^p(\mathbb{R}^n)} \\ &= \frac{(n-1)p}{n(n-p)} \sum_{i=1}^{n} \left(\int_{\mathbb{R}^n} |u|^{\frac{np}{n-p}} dx \right)^{\frac{p-1}{p}} \|D_i u\|_{L^p(\mathbb{R}^n)} \end{aligned}$$

Hence

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{np}{n-p}} dx \right)^{\frac{n-p}{np}} \leq \frac{(n-1)p}{n(n-p)} \sum_{i=1}^{n} \|D_i u\|_{L^p(\mathbb{R}^n)}$$

which is the desired result. As usual, to obtain the same result for a function $u \in W_0^{1,p}(\Omega)$,we just take a sequence of functions in $C_0^1(\mathbb{R}^n)$ converging to u. □

Remark 1.3.3.

$W_0^{1,p}(\Omega) \subset L^r(\Omega)$,where r is given above. But obviously $W_0^{1,p}(\Omega) \subset L^p(\Omega)$,so by the following interpolation lemma, $W_0^{1,p}(\Omega) \subset L^q(\Omega)$ for all q satisfying $p \leq q \leq r$. If Ω is bounded then clearly this holds for all q satisfying $1 \leq q \leq r$.

Lemma 1.3.1. *If $s \leq q \leq r$ and $\phi \in L^s(\Omega) \cap L^r(\Omega)$, then $\phi \in L^q(\Omega)$ and*

$$\|\phi\|_{L^q(\Omega)} \leq \|\phi\|_{L^s(\Omega)}^{\lambda} \|\phi\|_{L^r(\Omega)}^{1-\lambda},$$

where $\lambda = \frac{s(r-q)}{q(r-s)}$.

Corollary 1.3.2.

For every domain Ω in R^n there exists a constant C depending on only n and p such that

(i) if $kp < n$ then $W_0^{k,p}(\Omega) \subset L^{\frac{np}{n-kp}}(\Omega)$ and for each $u \in W_0^{k,p}(\Omega)$

$$\|u\|_{L^{\frac{np}{n-kp}}(\Omega)} \leq C\|u\|_{W^{k,p}(\Omega)}$$

(ii) if $kp > n$ then $W_0^{k,p}(\Omega) \subset C^{m,\alpha}(\bar{\Omega})$, where m is the integer satisfying $0 < k - m - \frac{n}{p} < 1$ and $\alpha = k - m - \frac{n}{p}$. Further, if $u \in W_0^{k,p}(\Omega)$ then

$$\|u\|_{C^{m,\alpha}(\bar{\Omega})} \leq C\|u\|_{W^{k,p}(\Omega)}.$$

Remarks 1.3.1.

(i) If $kp = n$ and $p > 1$ then $W_0^{k,p}(\Omega) \subset L^q(\Omega)$ for all q satisfying $p \leq q < \infty$

(ii) If $kp > n, p > 1$ and $\frac{n}{p}$ is an integer then $W_0^{k,p}(\Omega) \subset W_0^{k-\frac{n}{p},q}(\Omega)$ for all q satisfying $p \leq q < \infty$.

(iii) If $kp > n$ and $p = 1$ (so $\frac{n}{p}$ is obviously an integer) then $W_0^{k,p}(\Omega) \subset C_\beta^{k-n}(\bar{\Omega})$.

Corollary 1.3.3.

If Ω is a bounded C^1 domain in $\mathbb{R}^n$ (or any other domain such that there exists a bounded extension operator $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$) then the statements concerning the spaces $W_0^{k,p}(\Omega)$ in Corollary (1.3.2) and in the remark following the corollary also apply to the spaces $W^{k,p}(\Omega)$. However, the constant C may also depend on Ω.

Proof. The cases for $k = 1$ dealt with in Theorems (1.3.2) and (1.3.3) are easily seen to have their counterparts here because of the extension operator. Inspection of the proof of Corollary (1.3.2) shows how the results for $k > 1$ may be derived from the results for $k = 1$ without any additional assumptions on the domain. □

Definition 1.3.1.

Let A and B be Banach spaces. If $A \subset B$, we say that A is continuously imbedded in B ((in symbols, this is written $A \hookrightarrow B$) if there is a constant C such that $\|x\|_B \leq C\|x\|_A$.

The theorems in this section provide examples of imbeddings and are called Sobolev Imbedding Theorems.e.g. $W_0^{1,p}(\Omega) \hookrightarrow L^{\frac{np}{n-p}}(\Omega)$ for $p > n$.

It is easy to see that $A \hookrightarrow B$ is equivalent to the identity mapping from A into B being continuous.

1.4 Compactness Theorems

Lemma 1.4.1.

Suppose that Ω is a bounded domain. If

1. $0 < \lambda \leq 1$ then $C^{m,\lambda}(\bar{\Omega})$ is compactly imbedded in $C^{m}(\bar{\Omega})$.

2. $0 < \nu < \lambda \leq 1$ then $C^{m,\lambda}(\bar{\Omega})$ is compactly imbedded in $C^{m,\nu}(\bar{\Omega})$.

Proof. It suffices to prove the results for $m = 0$ because, once this is done, we can apply this case to the derivatives of the functions and deduce the result for general m. Let $\{f_j\}$ be a sequence in $C^{0,\lambda}(\bar{\Omega})$ such that $\|f_j\|_{C^{0,\lambda}(\bar{\Omega})} \leq M$. But this implies $|f_j(x) - f_j(y)| \leq M\|x-y\|^{\lambda}$, showing that the sequence is a bounded, equicontinuous set of functions. By the Arzela-Ascoli Theorem, there exists a subsequence $\{f_{jk}\}$ that converges in $C(\bar{\Omega})$. Thus $C^{0,\lambda}(\bar{\Omega})$ is compactly imbedded in $C(\bar{\Omega})$.

We show below that the same subsequence also converges in $C^{0,\nu}(\bar{\Omega})$. Suppose that $\psi \in C^{0,\lambda}(\bar{\Omega})$. Then

$$\begin{aligned}
[\psi]_{0,\nu} &= \sup \frac{|\psi(x)-\psi(y)|}{\|x-y\|^{\nu}} \\
&= \sup \left(\frac{|\psi(x)-\psi(y)|}{\|x-y\|^{\lambda}} \right)^{\frac{\nu}{\lambda}} |\psi(x)-\psi(y)|^{1-\frac{\nu}{\lambda}} \\
&\leq 2^{1-\frac{\nu}{\lambda}} \left([\psi]_{0,\lambda} \right)^{\frac{\nu}{\lambda}} (\max|\psi|)^{1-\frac{\nu}{\lambda}}
\end{aligned}$$

We apply this to $f_{jk} - f_{jr}$, noting that $[f_{jk} - f_{jr}]_{0,\lambda} \leq [f_{jk}]_{0,\lambda} + [f_{jr}]_{0,\lambda} \leq 2M$, and obtain

$$[f_{jk} - f_{jr}]_{0,\nu} \leq 2M^{\frac{\nu}{\lambda}} (\max|f_{jk} - f_{jr}|)^{1-\frac{\nu}{\lambda}},$$

showing that the subsequence is a Cauchy sequence in $C^{0,\nu}(\bar{\Omega})$ (because it converges in $C(\bar{\Omega})$). Thus the subsequence converges in $C^{0,\nu}(\bar{\Omega})$. □

Corollary 1.4.1.

If Ω is bounded, $kp > n$ and $0 < k - m - \frac{n}{p} < 1$ then $W_0^{k,p}(\Omega)$ is compactly imbedded in $C^{m,\beta}(\bar{\Omega})$ if $\beta < k - m - \frac{n}{p}$.

Proof. Let $\alpha = k - m - \frac{n}{p}$ Then $W_0^{k,p}(\Omega) \hookrightarrow C^{m,\alpha}(\bar{\Omega}) \hookrightarrow C^{m,\beta}(\bar{\Omega})$,and the second, imbedding is compact. □

Corollary 1.4.2.

If Ω is a bounded C^1 domain (or any other domain for which there is a bounded extension operator $E : W^{1.p}(\Omega) \to W^{1.p}(R^n)$),$Kp > n$ and $0 < k - m - \frac{n}{p} < 1$ then $W^{k,p}(\Omega)$is compactly imbedded in $C^{m,\beta}\left(\overline{\Omega}\right)$ $if\, \beta < k - m - \frac{n}{p}$.

Proof. Let $\phi \in C_0^\infty(\mathbb{R}^n)$ be such that supp ϕ is contained in some ball B containing Ω and $\phi = 1$ on Ω.Then we can define $\widetilde{E} : W^{1,p}(\Omega) \to W_0^{1,p}(B)$ by $\widetilde{E}(f) = \phi E(f)$.
By Corollary (1.4.1),$W_0^{1,p}(B)$ is compactly imbedded in $C^{0,\beta}(\overline{B})$.Hence $W^{1,p}(\Omega)$is compactly imbedded in $C^{0,\beta}(\overline{\Omega})$.The result for general k can be deduced from the $k = 1$ case by considering derivatives of the functions (as in the proof of Corollary (1.3.2) (b), deduce that if $u \in W^{k,p}(\Omega)$ and $|\beta| \le m$ then $D^\beta u \in W^{1,\frac{n}{(1-\alpha)}}(\Omega)$,which is contained in $C^{0,\alpha}(\overline{\Omega})$). □

Definition 1.4.1.

A subset E of a metric space is said to be totally bounded if $\forall \varepsilon > 0$, E can be covered by finitely many balls of radius ε .

Theorem 1.4.1.

Let E be a subset of a complete metric space X. Then the following statements are equivalent.

(i) $\overline{E}$ is compact.

(ii) Every sequence in E has a convergent subsequence.

(iii) E is totally bounded.

Theorem 1.4.2.

if Ω is bounded and $p < n$, then $W_0^{1,p}(\Omega)$ is compactly imbedded in $L^q(\Omega)$ for all $q = \frac{np}{n-p}$.

Proof. Consider first the case $q = 1$. Let A be a bounded set in $W_0^{1,p}(\Omega)$.We may consider the members of A as members of $W^{1,p}(\mathbb{R}^n)$with supports contained in $\bar{\Omega}$. let $A_h = \{J_h u : u \in A\}$.Note that we have

$$|J_h u(x)| \le h^{-n} \int_\Omega \rho\left(\frac{x-z}{h}\right) |u(z)| dz \le h^{-n} (\max \rho) \|u\|_{L^1(\Omega)}$$

and

$$|D_i J_h u(x)| \le h^{-n-1} \int_\Omega \left| D_i \rho\left(\frac{x-z}{h}\right)\right| |u(z)| dz \le h^{-n-1} (\max |D_i \rho|) \|u\|_{L^1(\Omega)}.$$

Since Ω is bounded, $\|u\|_{L^1(\Omega)} \le const. \|u\|_{L^p(\Omega)}$. The inequalities above show that A_h is a bounded equicontinuous set of functions in $C(\bar{\Omega})$. By the Arzela-Ascoli Theorem, every sequence in A_h has a subsequence that converges in $C(\bar{\Omega})$. Obviously, such subsequences also converge in $L^1(\Omega)$, so we see that A_h is totally bounded in $L^1(\Omega)$.

If $u \in A$ then

$$\begin{aligned} u(x) - J_h u(x) &= \int_{|z|\le 1} \rho(z)(u(x) - u(x - hz)) dz \\ &= \int_{|z|\le 1} \rho(z) \int_0^{h\|z\|} -\frac{\partial}{\partial r} u\left(x - r\frac{z}{\|z\|}\right) dr dz. \end{aligned}$$

Thus

$$|u(x) - J_h u(x)| \le \int_{|z|\le 1} \rho(z) \int_0^{h\|z\|} \sum_{i=1}^n \left| D_i u\left(x - r\frac{z}{\|z\|}\right)\right| drdz.$$

Integrating this with respect to x, we find

$$\begin{aligned}
\int_\Omega |u(x) - J_h u(x)|\, dx &\le \int_{|z|\le 1} \rho(z) \int_0^{h\|z\|} \sum_{i=1}^n \int_{\mathbb{R}^n} \left| D_i u\left(x - r\frac{z}{\|z\|}\right)\right| dxdrdz \\
&= \int_{|z|\le 1} \rho(z) \int_0^{h\|z\|} \sum_{i=1}^n \int_\Omega |D_i u(x)|\, dxdrdz \\
&= \int_{|z|\le 1} \rho(z) h\|z\| \sum_{i=1}^n \int_\Omega |D_i u(x)|\, dxdz \\
&\le h \sum_{i=1}^n \int_\Omega |D_i u(x)|\, dx \\
&\le hB,
\end{aligned} \tag{1.10}$$

Where B is a constant depending on our bound of members of A in $W_0^{1,p}(\Omega)$.

Let $\varepsilon > 0$. Since A_h is totally bounded in $L^1(\Omega)$, we can cover A_h by a finite number of balls B_i of radius $\frac{\varepsilon}{2}$. Let $h = \frac{\varepsilon}{2B}$ .By (1.10), if $J_h u \in B_i$,then u is contained in a ball of radius ε centered at the center of B_i . Thus, A is covered by a finite number of balls of radius ε. i.e. A is totally bounded in $L^1(\Omega)$. Thus $W_0^{1,p}(\Omega)$ is compactly imbedded in $L^1(\Omega)$.

Suppose $\phi \in W_0^{1,p}(\Omega)$. Then $\phi \in L^{\frac{np}{n-p}}(\Omega)$ by Theorem 1.3.3 and we get from Lemma 1.3.1 $\left(\text{with } s = 1 \text{ and } r = \frac{np}{n-p}\right)$ that

$$\|\phi\|_{L^q(\Omega)} \le \|\phi\|_{L^1(\Omega)}^{\lambda} \|\phi\|_{L^{\frac{np}{n-p}}(\Omega)}^{1-\lambda} \le C\|\phi\|_{L^1(\Omega)}^{\lambda} \left(\sum_{i=1}^n \|D_i\phi\|_{L^p(\Omega)}\right)^{1-\lambda}$$

Now let $\{u_m\}$ be a bounded sequence in $W_0^{1,p}(\Omega)$ and assume $\|u_m\|_{W_0^{1,p}(\Omega)} \le M$.Since $W_0^{1,p}(\Omega)$ is compactly imbedded in $L^1(\Omega)$, we can extract a subsequence $\{u_m\}$ that converges in $L^1(\Omega)$. Applying the inequality above to $u_{m_f} - u_{m_k}$,noting that $\left\|u_{m_f} - u_{m_k}\right\|_{W_0^{1,p}(\Omega)} \le 2M$,
We obtain

$$\left\|u_{m_f} - u_{m_k}\right\|_{L^q(\Omega)} \le const. \left\|u_{m_f} - u_{m_k}\right\|_{L^1(\Omega)}^{\lambda},$$

showing that the subsequence is a Cauchy sequence in $L^q(\Omega)$. Hence the subsequence converges in $L^q(\Omega)$ and $W_0^{1,p}(\Omega)$ is compactly imbedded in $L^q(\Omega)$. □

Corollary 1.4.3.

If $kp < n$ and Ω is bounded then $W_0^{k,p}(\Omega)$ is compactly imbedded in $L^q(\Omega)$ for all $q < \frac{np}{n-kp}$.

Proof. $W_0^{k,p}(\Omega)$ is continuously imbedded in $W_0^{1,\frac{np}{n-(k-1)p}}(\Omega)$, which is compactly imbedded in $L^q(\Omega)$ if $q < \frac{np}{n-kp}$,by Theorem (1.4.2). □

Corollary 1.4.4.

The same compactness results hold for $W^{k,p}(\Omega)$ if Ω is a bounded, C^1 domain (or any other type of bounded domain for which there is an extension operator $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$).

1.5 Interpolation Results

The following results are very useful in PDE theory. We make use of Theorem 1.5.1 in our proof of Gårding's Inequality in our study of elliptic problems.

Theorem 1.5.1.

Let $u \in W_0^{k,p}(\Omega)$.Then for any $\varepsilon > 0$ and any $0 < |\beta| < k$

$$\left\| D^\beta u \right\|_{L^p(\Omega)} < \varepsilon \|u\|_{W^{k,p}(\Omega)} + C\varepsilon^{\frac{-|\beta|}{k-|\beta|}} \|u\|_{L^p(\Omega)}$$

Where C is a constant depending only on k.

Proof. We prove the result for $|\beta| = 1$, $k = 2$. The general result is easily obtained from this case by induction. In fact, we show that for each i

$$\left\| \frac{\partial u}{\partial x_i} \right\|_{L^p} \le \varepsilon \left\| \frac{\partial^2 u}{\partial x_i^2} \right\|_{L^p} + \frac{72}{\varepsilon} \|u\|_{L^p} \tag{1.11}$$

First suppose that $u \in C_0^2(R)$ and consider an interval (a, b) of length $b - a = \varepsilon$. If $y \in (a, \frac{a+\varepsilon}{3})$ and $z \in (\frac{b-\varepsilon}{3}, b)$, then by the Mean Value Theorem there is a $p \in (a, b)$ such that

$$|u'(p)| = \left| \frac{u(z) - u(y)}{z - y} \right| \le \frac{3}{\varepsilon} \left(|u(z)| + |u(y)|\right)$$

Consequently, for every $x \in (a, b)$, we obtain

$$|u'(x)| = \left| u'(p) + \int_p^x u''(t)\,dt \right| \le \frac{3}{\varepsilon}\left(|u(z)| + |u(y)|\right) + \int_a^b |u''(t)|\,dt.$$

Integrating with respect to y and z over the intervals $\left(a, \frac{a+\varepsilon}{3}\right)$ and $\left(\frac{b-\varepsilon}{3}, b\right)$ respectively, we obtain

$$\left|u'(x)\right| \le \int_a^b \left|u''(t)\right| dt + \frac{18}{\varepsilon^2} \int_a^b |u(t)|\,dt,$$

so by Hölder's inequality and the inequality $(A + B)^p \le 2^{p-1}(A^p + B^p)$,

$$\begin{aligned}
\left|u'(x)\right|^p &\le 2^{p-1} \left(\left(\int_a^b \left|u''(t)\right| dt \right)^p + \frac{(18)^p}{\varepsilon^{2p}} \left(\int_a^b |u(t)|\,dt \right)^p \right) \\
&\le 2^{p-1} \left(\left(\int_a^b \left|u''(t)\right|^p dt \right) \left(\int_a^b 1\,dt \right)^{p-1} + \frac{(18)^p}{\varepsilon^{2p}} \left(\int_a^b |u(t)|^p\,dt \right) \left(\int_a^b 1\,dt \right)^{p-1} \right) \\
&= 2^{p-1} \left(\varepsilon^{p-1} \int_a^b \left|u''(t)\right|^p dt + \frac{(18)^p}{\varepsilon^{p+1}} \int_a^b |u(t)|^p\,dt \right).
\end{aligned}$$

Integrating this with respect to x over the interval (a,b) gives

$$\int_a^b \left|u'(x)\right|^p dx = 2^{p-1}\left(\varepsilon^p \int_a^b \left|u''(t)\right|^p dt + \frac{(18)^p}{\varepsilon^p}\int_a^b |u(t)|^p dt\right).$$

We now subdivide $\mathbb{R}$ into intervals of length ϵ and obtain by adding all of these inequalities that

$$\int_{-\infty}^{\infty} \left|u'(x)\right|^p dx \leq 2^{p-1}\left(\varepsilon^p \int_{-\infty}^{\infty} \left|u''(t)\right|^p dt + \frac{(18)^p}{\varepsilon^p}\int_{-\infty}^{\infty} |u(t)|^p dt\right). \tag{1.12}$$

Suppose now that $u \in C_0^{\infty}(\mathbb{R}^n)$. Then we can apply (1.12) to u regarded as a function of x_i and integrate with respect to the remaining variables to obtain

$$\int_{\mathbb{R}^n} \left|\frac{\partial u}{\partial x_i}\right|^p dx \leq 2^{p-1}\left(\varepsilon^p \int_{\mathbb{R}^n} \left|\frac{\partial^2 u}{\partial x_i^2}\right|^p dx + \frac{(18)^p}{\varepsilon^p}\int_{\mathbb{R}^n} |u|^p dx\right)$$

Taking the pth root of this and using $(A^p+B^p)^{1/p} \leq A+B$, we obtain (1.11). (Actually, we don't quite obtain (1.11). We actually obtain the inequality (1.11) for 2ε instead of ε . But since ε is an arbitrary positive constant, (1.11) holds). Finally, to obtain the result for $u \in W_0^{\infty}(\Omega)$, we take a sequence of functions in C_0^{∞} converging to u. □

Corollary 1.5.1.

The interpolation inequality stated in Theorem 1.5.1 also applies to members of $W^{k,p}(\Omega)$, provided that Ω is a bounded C^2 domain (or any other domain for which there is a bounded extension operator $E: W^{2,p}(\Omega) \longrightarrow W^{2,p}(\mathbb{R}^n)$. Here the constant C may also depend on p and Ω.

Proof. Because of the extension operator, an inequality of the form (1.11) holds for functions in $W^{2,p}(\Omega)$. □

1.6 The Spaces $H^m(\Omega)$ and $H_0^m(\Omega)$

The following abstract theorem is a flexible tool for generating Sobolev Spaces. The ingredients of the construction are:

(i) The space $D'(\Omega;\mathbb{R}^n)$, in particular, for $n=1, D'(\Omega)$.

(ii) Two Hilbert spaces H and Z with $Z \hookrightarrow D'(\Omega;\mathbb{R}^n)$ for some $n \geq 1$. In particular

$$v_k \longrightarrow v \text{ in } Z \qquad \text{implies} \qquad v_k \longrightarrow v \text{ in } D'(\Omega;\mathbb{R}^n). \tag{1.13}$$

(iii) A linear continuous operator $L: H \longrightarrow D'(\Omega;\mathbb{R}^n)$ (such as a gradient or a divergence).

Theorem 1.6.1.

Define

$$W = \{v \in H : Lv \in Z\}$$

and

$$(u, v)_W = (u, v)_H + (Lu, Lv)_Z. \tag{1.14}$$

Then W is a Hilbert space with inner product given by (1.14). The embedding of W in H is continuous and the restriction of L to W is continuous from W into Z.

Proof. Thus W is an inner-product space. It remains to check its completeness. Let $\{v_k\}$ be a Cauchy sequence in W. We must show that there exists $v \in H$ such that

$$v_k \longrightarrow v \text{ in } H$$

and

$$Lv_k \longrightarrow Lv \text{ in } Z.$$

Observe that $\{v_k\}$ and $\{Lv_k\}$ are Cauchy sequences in H and Z, respectively. Thus, there exist $v \in H$ and $z \in Z$such that

$$v_k \longrightarrow v \text{ in } H \text{ and } Lv_k \longrightarrow z \text{ in } Z.$$

The continuity of L and (1.13) yield

$$Lv_k \longrightarrow Lv \text{ in } D'(\Omega;\mathbb{R}^n)\text{and}Lv_k \longrightarrow z \text{ in } D'(\Omega;\mathbb{R}^n).$$

Since the limit of a sequence in $D'(\Omega;\mathbb{R}^n)$ is unique, we infer that $Lv = z$. Therefore

$$Lv_k \longrightarrow Lv \text{ in } Z$$

and W is a Hilbert space.

The continuity of the embedding $W \subset H$ follows from

$$\|u\|_H \le \|u\|_W$$

while the continuity of $L_{|W} : W \longrightarrow Z$ follows from

$$\|Lu\|_Z \le \|u\|_W.$$

□

The space $H^1(\Omega)$

Let $\Omega \subseteq \mathbb{R}^n$ be a domain. Choose in Theorem 1.6.1:

$$H = L^2(\Omega), Z = L^2\left(\Omega;\mathbb{R}^n\right) \hookrightarrow D'\left(\Omega;\mathbb{R}^n\right)$$

and $L : H \to D'(\Omega;\mathbb{R}^n)$given by

$$L = \nabla$$

where the gradient is considered in the sense of distributions. Then, W is the Sobolev space of the functions in $L^2(\Omega)$, whose first derivatives in the sense of distributions are functions in $L^2(\Omega)$. For this space we use the symbol $H^1(\Omega)$. Thus:

$$H^1(\Omega) = \left\{v \in L^2(\Omega) : \nabla v \in L^2(\Omega;\mathbb{R}^n)\right\}.$$

In other words, if $v \in H^1(\Omega)$, every partial derivative $\partial_{x_i} v$ is a function $v_i \in L^2(\Omega)$. This means that

$$\langle \theta_{x_i} v, \varphi\rangle = -\left(v, \theta_{x_i}\varphi\right)_{L^2}(\Omega) = \left(v_i, \varphi\right)_{L^2}(\Omega), \quad \forall \varphi \in D(\Omega)$$

Or, more explicitly,

$$\int_\Omega v(x)\theta_{x_i}\varphi(x)dx = -\int_\Omega v_i(x)\varphi(x)dx, \quad \forall \varphi \in D(\Omega).$$

In many applied situations, the Dirichlet integral

$$\int_\Omega |\nabla v|^2$$

represents an energy. The functions in $H^1(\Omega)$ are therefore associated with configurations having finite energy. From Theorem 1.6.1 and the separability of $L^2(\Omega)$, we have:

Proposition 1.6.1.

$H^1(\Omega)$ is a separable Hilbert space, continuously embedded in $L^2(\Omega)$. The gradient operator is continuous from $H^1(\Omega)$ into $L^2(\Omega;\mathbb{R}^n)$.

The inner product and the norm in $H^1(\Omega)$ are given, respectively, by

$$(u, v)_{H^1(\Omega)} = \int_\Omega uvdx + \int_\Omega \nabla u.\nabla vdxand \|u\|^2_{H^1(\Omega)} = \int_\Omega u^2dx + \int_\Omega |\nabla u|^2dx.$$

Exemple 1.6.1.

Let $\Omega = B_{1/2}(O) = \left\{x \in \mathbb{R}^2 : |x| < 1/2\right\}$ and $u(x) = (-log|x|)^a, x \neq O$. We have, using polar coordinates,

$$\int_{B_{1/2}(0)} u^2dx = 2\pi\int_0^{1/2}(-\log r)^{2a}rdr < \infty, \text{ for every a } \in \mathbb{R},$$

so that $u \in L^2(B_{1/2}(0))$ for every $a \in \mathbb{R}$. Also:

$$u_{x_i} = -a x_i |x|^{-2}(-\log|x|)^{a-1}, i = 1,2$$

and therefore

$$|\nabla u| = \left|a(-\log|x|)^{a-1}\right| |x|^{-1}.$$

Using polar coordinates, we get

$$\int_{B_{1/2}(0)} |\nabla u|^2 dx = 2\pi a^2 \int_0^{1/2} |\log r|^{2a-2} r^{-1} dr.$$

This integral is finite only if $2-2a > 1$ or $a < 1/2$. In particular, ∇u represents the gradient of u in the sense of distribution as well. We conclude that $u \in H^1(B_1(0))$ only if $a < 1/2$. We point out that when $a > 0$, u is unbounded near 0.

Proposition 1.6.2.

Let $u \in L^2(a,b)$. Then $u \in H^1(a,b)$ if and only if u is continuous in $[a,b]$ and there exists $w \in L^2(a,b)$ such that

$$u(y) = u(x) + \int_x^y w(s) ds, \ \forall x, y \in [a,b]. \tag{1.15}$$

Also $u' = w$

Proof. Assume that u is continuous in $[a,b]$ and that (1.15) holds with $w \in L^2(a,b)$. Choose $x = a$. Replacing, if necessary, u by $u - u(a)$, we may assume $u(a) = 0$, so that

$$u(y) = \int_a^y w(s) ds, \ \forall x, y \in [a,b].$$

Let $\varphi \in D(a,b)$. We have:

$$\begin{aligned}\langle u', \varphi \rangle = -\langle u, \varphi' \rangle &= -\int_a^b u(s)\varphi'(s) ds \\ &= -\int_a^b \left[\int_a^b w(t) dt\right] \varphi'(s) ds\end{aligned}$$

(exchanging the order of integration)

$$\begin{aligned}&= -\int_a^b \left[\int_a^b \varphi'(s) ds\right] w(t) dt \\ &= \int_a^b \varphi(t) w(t) dt = \langle w, \varphi \rangle.\end{aligned}$$

Thus $u' = w$ in $D'(a,b)$ and therefore $u \in H^1(a,b)$. From the Lebesgue Differentiation Theorem we deduce that $u' = w$ a.e. as well. Viceversa, let $u \in H^1(a,b)$. Define

$$v(x) = \int_a^x u'(s) ds, \ x \in [a,b]. \tag{1.16}$$

The function v is continuous in $[a,b]$ and the above proof shows that $v' = u'$ in $D'(a,b)$.

$$u = v + C, \ C \in \mathbb{R},$$

and therefore u is continuous in $[a,b]$ as well. Moreover, (1.16) yields

$$u(y)-u(x)=v(y)-v(x)=\int_x^y u'(s)ds$$

which is (1.15). □

Since a function $u \in H^1(a,b)$ is continuous in $[a,b]$, the value $u(x_0)$ at every point $x_0 \in [a,b]$ makes perfect sense. In particular the trace of u at the end points of the interval is given by the values $u(a)$ and $u(b)$.

The space $H_0^1(\Omega)$

Let $\Omega \subseteq \mathbb{R}^n$ be a domain. We study an important subspace of $H^1(\Omega)$.

Definition 1.6.1.

We denote by $H_0^1(\Omega)$ the closure of $D(\Omega)$ in $H^1(\Omega)$.

Thus $u \in H_0^1(\Omega)$ if and only if there exists a sequence $\{\varphi_k\} \subset D(\Omega)$ such that $\varphi_k \longrightarrow u$ in $H^1(\Omega)$, that is, such that both $\|\varphi_k - u\|_{L^2(\Omega)} \longrightarrow 0$ and $\|\nabla\varphi_k - \nabla u\|_{L^2(\Omega;\mathbb{R}^n)} \longrightarrow 0$ as $k \longrightarrow \infty$.

Since the test functions in $D(\Omega)$ have zero trace on $\partial\Omega$, every $u \in H_0^1(\Omega)$ inherits this property and it is reasonable to consider the elements $H_0^1(\Omega)$ as the functions in $H^1(\Omega)$ with zero trace on $\partial\Omega$. Clearly, $H_0^1(\Omega)$ is a Hilbert subspace of $H^1(\Omega)$.

An important property that holds in $H_0^1(\Omega)$, particularly useful in the solution of boundary value problems, is expressed by the following inequality of Poincaré. Recall that the diameter of a set Ω is given by

$$diam(\Omega) = \sup_{x,y\in\Omega} |x-y|.$$

Theorem 1.6.2.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. There exists a positive constant C_P (a Poincaré's constant) depending only on n and $diam(\Omega)$, such that, for every $u \in H_0^1(\Omega)$,

$$\|u\|_{L^2(\Omega)} \le C_p \|\nabla u\|_{L^2(\Omega;\mathbb{R}^n)}. \tag{1.17}$$

Proof. We use a strategy which is rather common for proving formulas in $H_0^1(\Omega)$. First, we prove the formula for $v \in D(\Omega)$; then, if $u \in H_0^1(\Omega)$, we select a sequence $vk \subset D(\Omega)$ converging to u in the $H^1(\Omega)$ norm as $k \to \infty$, that is

$$\|vk-u\|_{L^2(\Omega)} \to 0, \quad \|\nabla vk - \nabla u\|_{L^2(\Omega;\mathbb{R}^n)} \to 0$$

In particular

$$\|v_k\|_{L^2(\Omega)} \to \|u\|_{L^2(\Omega)}, \quad \|\nabla v_k\|_{L^2(\Omega;\mathbb{R}^n)} \to \|\nabla u\|_{L^2(\Omega;\mathbb{R}^n)}$$

Since (1.17) holds for every v_k, we have

$$\|v_k\|_{L^2(\Omega)} \leq Cp\,\|\nabla v_k\|_{L^2(\Omega;\mathbb{R}^n)}$$

Letting $k \to \infty$ we obtain (1.17) for u. Thus, it is enough to prove (1.17) for $v \in D(\Omega)$. Assume without loss of generality that $0 \in \Omega$, and set $\max_{x\in\Omega}|x| \leq M = diam(\Omega) < \infty$. Applying the Gauss Divergence Theorem, we can write

$$\int_\Omega div(v^2 x)dx = 0, \tag{1.18}$$

Since $v = 0$ on $\theta\Omega$. Now,

$$div(v^2 x) = 2v\nabla v \cdot x + nv^2$$

So that (1.18) yields

$$\int_\Omega^2 dx = -\frac{2}{n}\int_\Omega v\nabla v \cdot x dx$$

Since Ω is bounded, using Schwarz's inequality, we get

$$\int_\Omega v^2 dx = \frac{2}{n}|\int_\Omega v\nabla v \cdot x dx| \leq \frac{2M}{n}\left(\int_\Omega v^2 dx\right)^{1/2}\left(\int_\Omega |\nabla v|^2 dx\right)^{1/2}$$

Simplifying, it follows that

$$\|v\|_{L^2(\Omega)} \leq Cp\,\|\nabla v\|_{L^2(\Omega;\mathbb{R}^n)}$$

with $Cp = \frac{2M}{n}$. Inequality (1.17) implies that in $H_0^1(\Omega)$ the norm $\|u\|_{H^1(\Omega)}$ is equivalent to $\|\nabla u\|_{L^2(\Omega;\mathbb{R}^n)}$.Indeed

$$\|u\|_{H^1(\Omega)} = \sqrt{\|u\|^2_{L^2}(\Omega) + \|\nabla u\|^2_{L^2}(\Omega;\mathbb{R}^n)}$$

and from (1.17),

$$\|\nabla u\|_{L^2(\Omega;\mathbb{R}^n)} \leq \|u\|_{H^1(\Omega)} \leq \sqrt{C^2 p + 1}\,\|\nabla u\|_{L^2(\Omega;\mathbb{R}^n)}$$

Unless explicitly stated, we will choose in $H_0^1(\Omega)$

$$(u, v)_{H_0^1(\Omega)} = (\nabla u, \nabla v)_{L^2(\Omega;\mathbb{R}^n)} \quad and \quad \|u\|_{H_0^1(\Omega)} = \|\nabla u\|_{L^2(\Omega;\mathbb{R}^n)}$$

as inner product and norm, respectively.

□

The dual of $H_0^1(\Omega)$

In the applications of the Lax-Milgram theorem to boundary value problems, the dual of $H_0^1(\Omega)$ plays an important role. In fact it deserves a special symbol.

Definition 1.6.2.

We denote by $H^{-1}(\Omega)$ the dual of $H_0^1(\Omega)$ with the norm

$$\|F\|_{H^{-1}(\Omega)} = \sup\left\{|Fv| : v \in H_0^1(\Omega), \|v\|_{H_0^1(\Omega)} \leq 1\right\}.$$

The first thing to observe is that, since $D(\Omega)$ is dense (by definition) and continuously embedded in $H_0^1(\Omega)$, $H^{-1}(\Omega)$ is a space of distributions. This means two things:

(a) If $F \in H^{-1}(\Omega)$, its restriction to $D(\Omega)$ is a distribution.

(b) If $F, G \in H^{-1}(\Omega)$ and $F\varphi = G\varphi$ for every $\varphi \in D(\Omega)$, then $F = G$.

To prove (a) it is enough to note that if $\varphi_k \longrightarrow \varphi$ in $D(\Omega)$, then $\varphi k \longrightarrow \varphi$in $H_0^1(\Omega)$ as well, and therefore $F\varphi k \longrightarrow F\varphi$ Thus $F \in D(\Omega)$. To prove (a) let $u \in H_0^1(\Omega)$ and $\varphi_k \longrightarrow u$ in $H_0^1(\Omega)$, with $\varphi k \in D(\Omega)$. Then, since $F\varphi_k = G\varphi_k$, we may write

$$Fu = \lim_{h \longrightarrow +\infty} F\varphi_k = \lim_{h \longrightarrow +\infty} G\varphi_k = G$$

whence $F = G$.
Thus, $H^{-1}(\Omega)$ is in one-to-one correspondence with a subspace of $D'(\Omega)$ and in this sense we will write $H^{-1}(\Omega) \subset D'(\Omega)$. Which distributions belong to $H^{-1}(\Omega)$? The following theorem gives a satisfactory answer.

Theorem 1.6.3.

$H^1(\Omega)$ is the set of distributions of the form

$$F = f_0 + div f \tag{1.19}$$

where $f_0 \in L^2(\Omega)$ and $f = (f_1, \cdots, f_n) \in L^2(\Omega; \mathbb{R}^n)$. Moreover:

$$\|F\|_{H^{-1}(\Omega)} \leq \left\{C_p \left\|f_0\right\|_{L^2(\Omega)} + \left\|f\right\|_{L^2(\Omega;\mathbb{R}^n)}\right\}. \tag{1.20}$$

Proof. Let $F \in H^{-1}(\Omega)$. From Riesz's Representation Theorem, there exists a unique $u \in H_0^1(\Omega)$ such that

$$(u, v)_{H_0^1(\Omega)} = Fv, \ \ \forall v \in H_0^1(\Omega)$$

Since

$$(u, v)_{H_0^1(\Omega)} = (\nabla u, \nabla v)_{L^2(\Omega;\mathbb{R}^n)} = -\langle div \nabla u, v\rangle$$

in $D'(\Omega)$, it follows that (1.19) holds with $f_0 = 0$ and $f = -\nabla u$. Moreover,

$$\|F\|_{H^{-1}(\Omega)} = \|u\|_{H_0^1(\Omega)} = \|f\|_{L^2(\Omega;\mathbb{R}^n)}$$

Viceversa, let $F = f_0 + div f$, with $f_0 \in L^2(\Omega)$ and $f = L^2(\Omega;\mathbb{R}^n)$. Then $F \in D'(\Omega)$ and, letting $\langle F, v\rangle = Fv$, we have,

$$Fv = \int_\Omega f_0 v dx + \int_\Omega f \cdot \nabla v dx, \quad \forall v \in D(\Omega)$$

From the Schwarz and Poincaré inequalities, we have

$$|Fv| \leq \left\{Cp\,\|f_0\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega;\mathbb{R}^n)}\right\}\|v\|_{H_0^1(\Omega)}. \tag{1.21}$$

Thus, F is continuous in the H_0^1-norm. It remains to show that F has a unique continuous extension to all $H_0^1(\Omega)$. Take $u \in H_0^1(\Omega)$ and $\{v_k\} \subset D(\Omega)$ such that $\|v_k - u\|_{H_0^1(\Omega)}$. Then, (1.21) yields

$$|Fv_k - Fv_h| \leq \left\{Cp\,\|f_0\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega;\mathbb{R}^n)}\right\}\|v_k - v_h\|_{H_0^1(\Omega)}$$

Therefore $\{Fv_k\}$ is a Cauchy sequence in $\mathbb{R}$ and converges to a limit we may denote by Fu, which is independent of the sequence approximating u, as it is not difficult to check. Finally, since

$$|Fu| = \lim_{k\to\infty}|Fv_k| \qquad and \qquad \|u\|_{H_0^1(\Omega)} = \lim_{k\to\infty}\|v_k\|_{H_0^1(\Omega)},$$

from (1.21) we get:

$$|Fu| \leq \left\{Cp\,\|f_0\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega;\mathbb{R}^n)}\right\}\|u\|_{H_0^1(\Omega)}$$

showing that $F \in H^{-1}(\Omega)$. □

Theorem 1.6.4.

says that the elements of $H^{-1}(\Omega)$ are represented by a linear combination of functions in $L^2(\Omega)$ and their first derivatives (in the sense of distributions). In particular, $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$.

The spaces $H^m(\Omega), m > 1$

By involving higher order derivatives, we may construct new Sobolev spaces. Let $\mathbb{N}$ be the number of multi-indexes $\alpha = (\alpha_1, \cdots, \alpha_n)$ such that $|\alpha| = \sum_{i=1}^{n} \alpha_i \leq m$. Choose in Theorem (1.6.1)

$$H = L^2(\Omega), \;\; Z = L^2(\Omega;\mathbb{R}^N) \subset D'(\Omega;\mathbb{R}^N),$$

and $L : L^2(\Omega) \longrightarrow D'(\Omega;\mathbb{R}^N)$ given by

$$Lv = \{D^\alpha v\}_{|\alpha|\leq m}.$$

Then W is the Sobolev space of the functions in $L^2(\Omega)$, whose derivatives (in the sense of distributions) up to order m included, are functions in $L^2(\Omega)$. For this space we use the symbol $H^m(\Omega)$. Thus:

$$H^m(\Omega) = \left\{v \in L^2 : D^\alpha v \in L^2(\Omega), \; \forall \alpha \leq m\right\}.$$

From Theorem (1.6.1) and the separability of $L^2(\Omega)$, we deduce:

Proposition 1.6.3.

$H^m(\Omega)$ is a separable Hilbert space, continuously embedded in $L^2(\Omega)$. The operators $D^\alpha, |\alpha| \le m$, are continuous from $H^m(\Omega)$ into $L^2(\Omega)$.
The inner product and the norm in $H^m(\Omega)$ are given, respectively, by

$$(u,v)_{H^m(\Omega)} = \sum_{|\alpha|\le m} \int_\Omega D^\alpha u D^\alpha v dx.$$

and

$$\|u\|^2_{H^m(\Omega)} = \sum_{|\alpha|\le m} \int_\Omega |D^\alpha u|^2 dx.$$

If $u \in H^m(\Omega)$, any derivative of u of order $k \le m$
belongs to $H^{m-k}(\Omega)$ and $H^m(\Omega) \hookrightarrow H^{m-k}(\Omega), k \ge 1$.

1.7 Trace Theorems

In the following results, a vector x in $\mathbb{R}^n$ is denoted by $x = (x', x_n)$, where x' belongs to $\mathbb{R}^{n-1}$.

Lemma 1.7.1.

If $u \in W^{1,1}(\mathbb{R}^n)$, then for every $\zeta \in \mathbb{R}$, the function $v(x') = u(x',\zeta)$ is in $L^1(\mathbb{R}^{n-1})$, and

$$\|v\|_{L^1(\mathbb{R}^{n-1})} \le \|u\|_{L^1(\mathbb{R}^n)} + \|D_n u\|_{L^1(\mathbb{R}^n)}$$

Proof. It suffices to prove the result for the case $\zeta = 0$ and $u \in C_0^\infty(\mathbb{R}^n)$. By the Mean Value Theorem for integrals

$$\int_0^1 \int_{\mathbb{R}^{n-1}} \left|u(x', x_n)\right| dx' dx_n = \int_{\mathbb{R}^{n-1}} \left|u(x',\sigma)\right| dx'$$

for some $\sigma \in [0,1]$. But

$$\begin{aligned} |u(x',0)| &= |u(x',\sigma) - \int_0^\sigma D_n u(x',t)dt| \\ &\le |u(x',\sigma)| + \int_0^1 |D_n u(x',t)|dt. \end{aligned}$$

Integrating this over $\mathbb{R}^{n-1}$ gives

$$\begin{aligned} \|v\|_{L^1(\mathbb{R}^{n-1})} &\le \int_{\mathbb{R}^{n-1}} |u(x',\sigma)|dx' + \int_{\mathbb{R}^{n-1}} \int_0^1 |D_n u(x',t)|dtdx' \\ &= \int_0^1 \int_{\mathbb{R}^{n-1}} |u(x',\sigma)|dx'dt + \int_{\mathbb{R}^{n-1}} \int_0^1 |D_n u(x',t)|dtdx'. \end{aligned}$$

□

Lemma 1.7.2.

If $u \in W^{1,p}(\mathbb{R}^n)$ where $p < n$, then for every $\zeta \in \mathbb{R}$, the function $v(x') = u(x', \zeta)$ is in $L^r(\mathbb{R}^{n-1})$, where

$$r = \frac{(n-1)p}{n-p} = 1 + \frac{n(p-1)}{n-p}$$

and there is a constant C depending on only n and p such that

$$\|v\|_{L^r(\mathbb{R}^{n-1})} \leq C \|u\|_{W^{1,p}(\mathbb{R}^n)}.$$

Proof. We can assume that $p > 1$ because the $p = 1$ case is dealt with in the previous lemma. We first show that if $u \in W^{1,p}(\mathbb{R}^n)$ then $w = |u|^r \in W^{1,1}(\mathbb{R}^n)$ and

$$\|w\|_{L^1(\mathbb{R}^n)} \leq C \|Du\|^{r-1}_{L^p(\mathbb{R}^n)} \|u\|_{L^p(\mathbb{R}^n)}, \|D_i w\|_{L^1(\mathbb{R}^n)} \leq C \|Du\|^r_{L^p(\mathbb{R}^n)}. \tag{1.22}$$

It suffices to prove this result for the case $u \in C_0^\infty(\mathbb{R}^n)$. Let $q = \frac{p}{(p-1)}$.Then $(r-1)\,q = \frac{np}{(n-p)}$, so by the Sobolev Imbedding Theorem 1.3.3,

$$\left\| |u|^{r-1} \right\|^q_{L^p(\mathbb{R}^n)} \leq const. \|Du\|^{\frac{np}{(n-p)}}_{L^p(\mathbb{R}^n)}$$

and combining this with Hölder's Inequality, we get the first of (1.22):

$$\|w\|_{L^1(\mathbb{R}^n)} = \int |u|^r dx = \int |u|^{r-1} |u| dx \leq \|u\|_{L^p(\mathbb{R}^n)} \left\| |u|^{r-1} \right\|_{L^q(\mathbb{R}^n)} \leq const. \|Du\|^{r-1}_{L^p(\mathbb{R}^n)} \|u\|_{L^p(\mathbb{R}^n)}.$$

Since $D_i w = \pm r |u|^{r-1} D_i u$, we obtain the second of (1.22):

$$\|D_i w\|_{L^1(\mathbb{R}^n)} = r \left\| |u|^{r-1} \right\|_{L^p(\mathbb{R}^n)} \|D_i u\|_{L^p(\mathbb{R}^n)} \leq const. \|Du\|^r_{L^p(\mathbb{R}^n)}.$$

We now apply Lemma 1.7.1 to w and immediately obtain the inequality

$$\begin{aligned} \|v\|_{L^r(\mathbb{R}^{n-1})} &\leq const. \left(\|Du\|^{r-1}_{L^p(\mathbb{R}^{n-1})} \|u\|_{L^p(\mathbb{R}^{n-1})} + \|Du\|^r_{L^p(\mathbb{R}^{n-1})} \right)^{\frac{1}{r}} \\ &\leq const. \left(\|Du\|^{1-\frac{1}{r}}_{L^p(\mathbb{R}^{n-1})} \|u\|^{\frac{1}{r}}_{L^p(\mathbb{R}^{n-1})} + \|Du\|_{L^p(\mathbb{R}^{n-1})} \right) \\ &\leq const. \left(\|u\|_{L^p(\mathbb{R}^{n-1})} + \|Du\|_{L^p(\mathbb{R}^{n-1})} \right). \end{aligned}$$

□

Lemma 1.7.3.

If $u \in W^{k,p}(\mathbb{R}^n)$ where $kp < n$, then for every $\xi \in \mathbb{R}$,the function $v(x') = u(x', \xi)$ is in $L^r(\mathbb{R}^{n-1})$, where

$$r = \frac{(n-1)\,p}{n - kp}$$

and there is a constant C depending on only n, k and p such that

$$\|v\|_{L^r(\mathbb{R}^{n-1})} \leq C \|u\|_{W^{k,p}(\mathbb{R}^n)}.$$

1.8 Penalization operators

Let E be a reflexive Banach space, we will always assume that the norm of E and that of its dual $E^{'}$ are strictly convex.

Definition 1.8.1.

We call penalization operator (attached to K) any operator β of $E \longrightarrow E^{'}$ having the following properties:

$$\beta \text{ is monotone bounded and semicontinuous of } E \longrightarrow E^{'}$$

$$S = \left\{\beta(v) = 0, v \in E\right\}.$$

Theorem 1.8.1.

We assume E defined as above and let F a duality operator of $E \longrightarrow E^{'}$ related to Φ we therefore have $(F(u), u) = \|F(u)\|_* \|u\|$, $\|F(u)\|_* = \Phi(\|u\|)$, where $\|\cdot\|_*$ is the norm in $E^{'}$,dual of $\|\cdot\|$ .So if P_S denotes the projection operator of $E \longrightarrow S$ such that $u \in E$,$P_S u$ is the only element of S such as

$$\|u - P_S u\| \leq \|u - s\| \ \ \forall s \in S.$$

the operator β given by

$$\beta(u) = F(u - P_S u),$$

is a penalty operator.

Penalty application

Theorem 1.8.2.

We assume E defined as before. Let A be an operator of $E \longrightarrow E^{'}$,pseudomonotonic and coercive in the sense :

$$\begin{cases} \exists v_0 \in S \text{ such as} \\ \frac{\langle A(v), v - v_0\rangle}{\|v\|} \longrightarrow +\infty, \text{ if } \|v\| \longrightarrow \infty. \end{cases}$$

So for everything $f \in E^{'}$,$\exists u \in S$ such as

$$\langle A(u), v - u\rangle \geq \langle f, v - u\rangle \ \ \forall v \in S.$$

Remark 1.8.1.

Let W be a Banach space of strictly convex norm as well as that of its dual and suppose that

1. $V \subset W$ with continuous injection , $V \subset W$ (so $W' \subset V'$)

2. K is a closed convex set in V and in W. We can then consider a penalization operator attached to K in the space W, such that:

$$\beta \text{ is monotone bounded and semicontinuous of } W \longrightarrow W'$$

$$k = \{\beta(w) = 0, w \in W\}.$$

Exemple 1.8.1.

We take $V = H_0^1(\Omega)$, A given by

$$\begin{cases} A(\varphi) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial \varphi}{\partial x_j}) + a_0\varphi, & a_0, a_{ij} \in L^{\infty}(\Omega), \\ \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geq \alpha(\xi_1^2 + \cdots + \xi_n^2), \alpha > 0, & \forall \xi_i \in \mathbb{R}, \text{ a.e. of } \Omega, \\ a_0(x) \geq \alpha_0, & \text{a.e. of } \Omega, \end{cases}$$

and either

$$k = \{\beta(w) = 0, w \in H_0^1\}.$$

We can still apply the remark 1.8.1.
we project in $L^2(\Omega)$, with $W = L^2(\Omega)$ and $k = \{\beta(w) = 0, w \in L^2(\Omega)\}$.
we choose

$$\beta(w) = F(w - P_k w)$$

with

$$F = identity$$

and $P_k w = w^+$ such as

$$\begin{cases} w(x) & \text{if } w(x) \geq 0 \\ 0 & \text{if } w(x) < 0 \end{cases}$$

The corresponding equation is therefore:

$$\begin{cases} Au_\varepsilon - \frac{1}{\varepsilon}u_\varepsilon^- = f \\ u_\varepsilon \in H_0^1(\Omega). \end{cases}$$

Unilateral Problems with L^1 Data

In this chapter, we study an existence and uniqueness theorem for the solution of unilateral problems with L^1 data. Also, we discuss on details the existence and regularity of distributional solutions in appropriate Sobolev spaces. The results of this chapter, that have been published by [10].

2.1 Statement of the results

The problem we are considering is as follows.

$$\begin{cases} Au = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

Where Ω be an open bounded subset of $\mathbb{R}^N$ where $N \geq 2$, The operator A is defined as a nonlinear operator

$$Au = -div\ a(x, Du) \tag{2.2}$$

Let $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ be a Caratheodory function satisfying the following assumptions for almost every $x \in \Omega$ and for all $\xi, \eta \in \mathbb{R}^N$ where $\xi \neq \eta$ and $1 < p < \infty$:

$$a(x,\xi)\xi \geq \alpha|\xi|^p \tag{2.3}$$

$$|a(x,\xi)| \leq \beta\left[h(x) + |\xi|^{p-1}\right] \tag{2.4}$$

$$\left[a(x,\xi) - a(x,\eta)\right]\left[\xi - \eta\right] > 0 \tag{2.5}$$

With $\alpha, \beta > 0$ and $h(x) \in L^{p'}(\Omega)$(where p' denotes the conjugate exponent of p).
Let's assume that

$$f \in L^1(\Omega) \tag{2.6}$$

and

$$\psi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \tag{2.7}$$

We will define the set K as follows:

$$K = \left\{v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) : v(x) \geq \psi(x) \text{ in } \Omega\right\}.$$

In the following, we will denote by $T_k(s)$ the truncation defined by

$$T_k(s) = \begin{cases} k & \text{if } s > k \\ s & \text{if } |s| \leq k \\ -k & \text{if } s < -k \end{cases}$$

Our main purpose in this session is to study the proof the following theorems.

Theorem 2.1.1.

For $2 - \frac{1}{N} < p < N$. assuming conditions (2.3),$\cdots$,(2.7) there exists a unique solution u of the problem

$$\begin{cases} u \in W_0^{1,q}(\Omega),\ 1 < q < \frac{N(p-1)}{N-1} \text{ with } 2 - \frac{1}{N} < p < N \\ u(x) \geq \psi(x)\ in\, \Omega \\ T_k(u) \in W_0^{1,p}(\Omega)\ \forall K > 0 \\ \langle Au, T_k(u-v)\rangle \leq \int_\Omega f T_k(u-v)\ \forall v \in K \end{cases} \tag{2.8}$$

The problem we are considering is $Au = -div(a(x, Du)), f \in L^1(\Omega)$ where A satisfies the assumptions mentioned in 2.1

Theorem 2.1.2.

Let's assume that the hypotheses of Theorem 2.1.1 hold and that

$$A\psi \in L^1(\Omega). \tag{2.9}$$

Assuming that u is the solution of problem (2.8), the following inequality holds

$$f \leq Au \leq f + \left(f - A\psi\right)^-. \tag{2.10}$$

2.2 The compactness method

Step01: Approximation

We define the following.
Consider a sequence $\{f_n\}$ of smooth functions satisfying the following conditions:

$$\begin{cases} f_n \to f & \text{in } L^1(\Omega) \\ \|f_n\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}, & \forall n \in \mathbb{N} \end{cases} \tag{2.11}$$

Consider u_n as the solution of the problem:

$$\begin{cases} u_n \in W_0^{1,p}(\Omega)\ u_n(x) \geq \psi(x) & \text{in } \Omega \\ \langle Au_n, u_n - v\rangle \leq \int_\Omega f_n(u_n - v) \\ \forall v \in W_0^{1,p}(\Omega), v(x) \geq \psi(x) & \text{in } \Omega. \end{cases} \tag{2.12}$$

Thanks to the hypotheses (2.3),(2.4),(2.5) A is a nonlinear operator of Leray-Lions type, so the existence of u_n follows from the classical results of [25].

Step02: Uniform estimates

We consider the following proof

Lemma 2.2.1.

There exists a constant $c_0(q)$, independent on n, such that:

$$\|u_n\|_{W_0^{1,q}(\Omega)} \leq c_0(q)\ \forall n \in \mathbb{N}, 1 < q < \frac{N(p-1)}{N-1}. \tag{2.13}$$

Proof. Let $k \in \mathbb{N}^*$ and $\varphi_k : \mathbb{R} \to \mathbb{R}$ be the function defined by:

$$\varphi_k(s) = \begin{cases} 1 & \text{if } s \geq k+1 \\ s-k & \text{if } k \leq s < k+1 \\ 0 & \text{if } 0 \leq s < k \\ -\varphi_k(-s) & \text{if } s < 0 \end{cases} \tag{2.14}$$

Let $k \geq \|\psi\|_{L^\infty(\Omega)}$:taking as test function in (2.12) $v = u_n - \varphi_k(u_n)$, We find

$$\langle Au_n, (u_n - v)\rangle \leq \int_\Omega f_n(u_n - v)$$
$$\left\langle Au_n, (u_n - u_n + \varphi_k(u_n))\right\rangle \leq \int_\Omega f_n(u_n - u_n + \varphi_k(u_n))$$
$$\left\langle Au_n, \varphi_k(u_n)\right\rangle \leq \int_\Omega f_n\varphi_k(u_n)\ \forall k \geq \|\psi\|_{L^\infty(\Omega)}.$$

Thanks to assumption (2.3) we obtain

$$\left\langle Au_n, \varphi_k(u_n)\right\rangle \leq \int_\Omega f_n\varphi_k(u_n)$$
$$\int_\Omega a(x, Du_n)D\varphi_k(u_n) \leq \int_\Omega f_n\varphi_k(u_n)$$

We have $\varphi_k(s) = s - k / \varphi_k(s) \in B_k^n$

$$\int_{B_k^n} a(x, Du_n)D(u_n - k) \leq \int_{B_k^n} f_n(u_n - k)$$
$$\int_{B_k^n} a(x, Du_n)Du_n \leq \int_{B_k^n} f_n(u_n - k)$$

Applying (2.3) we obtain

$$\alpha \int_{B_k^n} |Du_n|^p \leq \int_{B_k^n} f_n(u_n - k)$$

We have $k \leq |u_n| < k+1, |u_n| < k+1$

$$\alpha \int_{B_k^n} |Du_n|^p \leq \int_{B_k^n} f_n(u_n - k) \leq \int_{B_k^n} f_n$$
$$\alpha \int_{B_k^n} |Du_n|^p \leq \int_{B_k^n} f_n \leq \int_{B_k^n} |f_n|$$
$$\alpha \int_{B_k^n} |Du_n|^p \leq \int_{B_k^n} |f_n| \leq \|f_n\|_{L^1(\Omega)}$$

Applying (2.11) we obtain

$$\int_{B_k^n} |Du_n|^p \leq \frac{\|f\|_{L^1(\Omega)}}{\alpha} \ \forall k \geq \|\psi\|_{L^\infty(\Omega)}, \tag{2.15}$$

Where

$$B_k^n = \{x \in \Omega : k \leq |u_n(x)| < k+1\}. \tag{2.16}$$

Let $v \in W_0^{1,p}(\Omega)$ and $v(x) \geq \psi(x)$ in Ω, taking as test function in (2.12), $v = u_n - T_k(u_n - \psi)$, $k > 0$

$$\langle Au_n, (u_n - v)\rangle \leq \int_\Omega f_n(u_n - v)$$

$$\left\langle Au_n, (u_n - u_n - T_k(u_n-\psi))\right\rangle \leq \int_\Omega f_n(u_n - u_n - T_k(u_n-\psi))$$

$$\left\langle Au_n, (T_k(u_n-\psi))\right\rangle \leq \int_\Omega f_n(T_k(u_n-\psi))$$

$$\int_\Omega a(x, Du_n) D(T_k(u_n-\psi)) \leq \int_\Omega f_n(T_k(u_n-\psi))$$

Applying $u_n < \psi + k$ and (2.11) we obtain

$$\int_{\{u_n-\psi<k\}} a(x, Du_n) D(u_n - \psi) \leq k\|f\|_{L^1(\Omega)}.$$

By virtue of hypotheses (2.3),(2.4) and using Young's inequality (with exponents p, p') we get

$$\alpha \int_{\{u_n-\psi<k\}} |Du_n|^p \leq$$

$$k\|f\|_{L^1(\Omega)} + \frac{\alpha}{2}\int_{\{u_n-\psi<k\}} (|h(x)|^{p'} + |Du_n|^p) + c_1\|\psi\|^p_{W_0^{1,p}(\Omega)}$$

and finally

$$\begin{aligned} \int_{\{|u_n|<k\}} |Du_n|^p &\leq \int_{\{u_n-\psi<k+\|\psi\|_{L^\infty(\Omega)}\}} |Du_n|^p \\ &\leq c_2(k + \|\psi\|_{L^\infty(\Omega)} + \|\psi\|^p_{W_0^{1,p}(\Omega)} + \|h\|^p_{L^{p'}(\Omega)}), \forall k > 0 \end{aligned} \tag{2.17}$$

Let $1 < q < \frac{N(p-1)}{N-1}$ and $\bar{k} \geq \|\psi\|_{L^\infty(\Omega)}$.

It results:

$$\int_\Omega |Du_n|^q \leq$$

$$\left(\int_{\{u_n<\bar{k}\}} |Du_n|^p\right)^{\frac{q}{p}} |\Omega|^{1-\frac{q}{p}} + \sum_{j=\bar{k}}^{\infty} \left(\frac{1}{(1+j)^\lambda}\int_{B_j^n} |Du_n|^p\right)^{\frac{q}{p}} \left(\int_{B_j^n} (1+|u_n|)^{\frac{\lambda q}{p-q}}\right)^{1-\frac{q}{p}}$$

Where $\lambda = \frac{N(p-q)}{N-q}$, observe that $\lambda > 1$, since $1 < q < \frac{N(p-1)}{N-1}$.

From this inequality, using (2.15) in addition (2.17) we get

$$\int_\Omega |Du_n|^q \leq c_3 + c_4\left(\int_\Omega |u_n|^{q*}\right)^{1-\frac{q}{p}}. \tag{2.18}$$

Where $q^* = \frac{Nq}{N-q}$, From this estimate, by Sobolev's inequality, we obtain

$$\int_\Omega |u_n|^{q*} \leq c_5.$$

Finally, from (2.18) we get (2.13). □

Step03: Passage to the limit

First Proof of Theorem 2.1.1 The estimate (2.13) ensures the existence of a subsequence, denoted as $\{u_n\}$, for which the following condition holds for all $q < \frac{N(p-1)}{(N-1)}$:

$$\begin{cases} u_n \rightharpoonup u \;\; weakly - \; W_0^{1,q}(\Omega) \\ u_n \to u \;\; strongly - \; L^q(\Omega) \\ u_n \to u \;\; \text{almost everywhere in } \Omega \end{cases} \tag{2.19}$$

Since $u_n \geq \psi(x)$ in Ω ,$\forall n \in \mathbb{N}$

$$u(x) \geq \psi(x) \, in \, \Omega.$$

Additionally, we have looked at or investigated the idea that

$$Du_n \to Du \text{ almost everywhere in } \Omega \tag{2.20}$$

We take $u_n - T_k(u_n - u_m)$ and then $u_m + T_k(u_n - u_m)$, as test function in (2.12) we get

$$\langle Au_n, u_n - u_n + T_k(u_n - u_m)\rangle \leq \int_\Omega f_n(u_n - u_n + T_k(u_n - u_m))$$

$$\langle Au_n, T_k(u_n - u_m)\rangle \leq \int_\Omega f_n T_k(u_n - u_m)$$

in addition

$$\langle Au_m, u_m - u_m - T_k(u_n - u_m)\rangle \leq \int_\Omega f_n(u_m - u_m - T_k(u_n - u_m))$$

$$-\langle Au_m, T_k(u_n - u_m)\rangle \leq -\int_\Omega f_m T_k(u_n - u_m).$$

The following expression can be derived or obtained by adding together these inequalities.

$$\langle Au_n - Au_m, T_k(u_n - u_m)\rangle \leq \int_\Omega \left(f_n - f_m\right) T_k(u_n - u_m) \tag{2.21}$$

The right-hand side of equation (2.21) tends to zero as $(n, m) \longrightarrow \infty$ .As a result, we can infer that $\langle Au_n - Au_m, T_k(u_n - u_m)\rangle \longrightarrow 0$ due to the monotonicity of the operator A.
Therefore, according to Lemma 1 in [16] ,it follows that

$$Du_n \to Du \qquad \text{a.e. in } \Omega. \tag{2.22}$$

Consider $w \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. We note that for any $k > 0$, the sequence $\|Tk(u_n - w)\|_{W_0^{1,p}(\Omega)} \leq C$. This can be shown as follows:

$$\int_\Omega |DT_k(u_n - w)|^p \leq c_6 \int_{\{|u_n| < k + \|w\|_{L^\infty(\Omega)}\}} |Du_n|^p + c_7 \int_\Omega |Dw|^p$$

Equation (2.15) imposes a restriction on the upper limit of the right-hand side of the preceding inequality, ensuring its bounded nature.
Consider $w \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that $w(x) \geq \psi(x)$ in Ω. The function

$$u_n - T_k(u_n - w)$$

Is an admissible test function in (2.12).

This choice yields

$$\begin{aligned}\langle Au_n, u_n - u_n + T_k(u_n - w)\rangle &\le \int_\Omega f_n(u_n - u_n + T_k(u_n - w))\\ \langle Au_n - Aw + Aw, T_k(u_n - w)\rangle &\le \int_\Omega f_n T_k(u_n - w)\\ \langle Au_n - Aw, T_k(u_n - w)\rangle + \langle Aw, T_k(u_n - w)\rangle &\le \int_\Omega f_n T_k(u_n - w)\end{aligned} \tag{2.23}$$

Since $\|T_k(u_n - w)\|_{W_0^{1,p}(\Omega)} \le Const$ and in $L^\infty(\Omega)$, we deduce:

$$\liminf \langle Au_n - Aw, T_k(u_n - w)\rangle \ge \langle Au - Aw, T_k(u - w)\rangle$$

$$\lim \langle Aw, T_k(u_n - w)\rangle = \langle Aw, T_k(u - w)\rangle$$

$$\lim \int_\Omega f_n T_k(u_n - w) = \int_\Omega f T_k(u - w)$$

Thus, taking the limit as $n \to \infty$ in (2.23),From this, it follows that u is a solution of (2.8).

2.3 The penalization method

We assume that

$$\psi = 0. \tag{2.24}$$

Let $\{f_n\}$ be a sequence of smooth functions such that

$$\begin{cases} f_\varepsilon \to f & \text{in } L^1(\Omega)\\ \|f_\varepsilon\|_{L^1(\Omega)} \le \|f\|_{L^1(\Omega)} & \forall \varepsilon > 0.\end{cases} \tag{2.25}$$

In addition

$$\beta(s) = |s|^{p-2}s. \tag{2.26}$$

We consider the following problem:

$$\begin{cases} u_\varepsilon \in W_0^{1,p}(\Omega)\\ Au_\varepsilon - \beta\left(\frac{(u_\varepsilon)^-}{\varepsilon}\right) = f_\varepsilon.\end{cases} \tag{2.27}$$

Then the result in [25], provide us with the existence of u_ε

For continue, we shall study the prove of following lemma:

Lemma 2.3.1.

There exists a constant $c(q) > 0$, independent on ε, such that:

$$\|u_\varepsilon\|_{W_0^{1,q}}(\Omega) \le c(q), \ \forall \varepsilon > 0 \tag{2.28}$$

With $1 < q < \frac{N(p-1)}{N-1}$.

Proof. Let $k > 0$. We select the test function in the weak formulation of (2.27)

$$v = \varphi_k(u_\varepsilon),$$

Where $\varphi_k(s)$ is the function defined in the proof of Lemma 2.2.1.

This selection leads to

$$\begin{aligned}
\langle Au_\varepsilon, v\rangle - \int_\Omega \beta\left(\frac{(u_\varepsilon)^-}{\varepsilon}\right) v &= \int_\Omega f_\varepsilon v \\
\big\langle Au_\varepsilon, \varphi_k(u_\varepsilon)\big\rangle - \int_\Omega \beta\left(\frac{(u_\varepsilon)^-}{\varepsilon}\right)\varphi_k(u_\varepsilon) &= \int_\Omega f_\varepsilon, \varphi_k(u_\varepsilon) \\
\big\langle Au_\varepsilon, \varphi_k(u_\varepsilon)\big\rangle - \int_\Omega \left(\frac{(u_\varepsilon)^-}{\varepsilon}\right)^{p-2}(u_\varepsilon)^-\varphi_k(u_\varepsilon) &= \int_\Omega f_\varepsilon \varphi_k(u_\varepsilon)
\end{aligned}$$

So that $\varphi_k(u_\varepsilon) \in B_k^\varepsilon$

$$\begin{aligned}
&\big\langle Au_\varepsilon, \varphi_k(u_\varepsilon)\big\rangle - \int_\Omega \left(\frac{(u_\varepsilon)^-}{\varepsilon}\right)^{p-1}\varphi_k(u_\varepsilon) \le \int_{B_k^\varepsilon} f_\varepsilon(u_\varepsilon - k) \\
&\big\langle Au_\varepsilon, \varphi_k(u_\varepsilon)\big\rangle - \int_\Omega \left(\frac{(u_\varepsilon)^-}{\varepsilon}\right)^{p-1}\varphi_k(u_\varepsilon) \le \int_{B_k^\varepsilon} f_\varepsilon(u_\varepsilon - k) \le \int_{B_k^\varepsilon} f_\varepsilon(|u_\varepsilon| - k) \\
&\big\langle Au_\varepsilon, \varphi_k(u_\varepsilon)\big\rangle - \int_\Omega \left(\frac{(u_\varepsilon)^-}{\varepsilon}\right)^{p-1}\varphi_k(u_\varepsilon) \le \int_{B_k^\varepsilon} f_\varepsilon(k+1-k) \le \int_{B_k^\varepsilon} |f_\varepsilon| \le \int_\Omega |f_\varepsilon| \le \|f_\varepsilon\|_{L^1(\Omega)}
\end{aligned}$$

Applying (2.11) we get

$$\big\langle Au_\varepsilon, \varphi_k(u_\varepsilon)\big\rangle - \int_\Omega \left(\frac{(u_\varepsilon)^-}{\varepsilon}\right)^{p-1}\varphi_k(u_\varepsilon) \le \|f\|_{L^1(\Omega)}. \tag{2.29}$$

Since $k > 0$, we have:

$$\int_\Omega \left(\frac{(u_\varepsilon)^-}{\varepsilon}\right)^{p-1}\varphi_k(u_\varepsilon) \le 0.$$

and from (2.29) we deduce

$$\int_{B_k^\varepsilon} |Du_\varepsilon|^p \le c_1 \ \forall k > 0, \forall \varepsilon > 0, \tag{2.30}$$

Where B_k^ε is the set defined in (2.16) and $c_1 > 0$ independent on ε.

We take $v = T_k(u_\varepsilon)$ as test function in (2.27) , we find

$$\begin{aligned}
\langle Au_\varepsilon, v\rangle - \int_\Omega \beta\left(\frac{(u_\varepsilon)^-}{\varepsilon}\right) v &= \int_\Omega f_\varepsilon v \\
\langle Au_\varepsilon, T_k(u_\varepsilon)\rangle - \int_\Omega \beta\left(\frac{(u_\varepsilon)^-}{\varepsilon}\right) T_k(u_\varepsilon) &= \int_\Omega f_\varepsilon, T_k(u_\varepsilon) \\
\langle Au_\varepsilon, T_k(u_\varepsilon)\rangle - \int_\Omega \left(\frac{(u_\varepsilon)^-}{\varepsilon}\right)^{p-2}(u_\varepsilon)^- T_k(u_\varepsilon) &= \int_\Omega f_\varepsilon T_k(u_\varepsilon) \\
\int_\Omega a(x, Du_\varepsilon) DT_k(u_\varepsilon) - \int_\Omega \left(\frac{(u_\varepsilon)^-}{\varepsilon}\right)^{p-1} T_k(u_\varepsilon) &= \int_\Omega f_\varepsilon T_k(u_\varepsilon)
\end{aligned}$$

Since $T_k(u_\varepsilon) \in |u_\varepsilon| < k \Rightarrow T_k(u_\varepsilon) = u_\varepsilon$ we find

$$\begin{aligned}
\int_{|u_\varepsilon|<k} a(x, Du_\varepsilon) DT_k(u_\varepsilon) - \int_{|u_\varepsilon|<k} \left(\frac{(u_\varepsilon)^-}{\varepsilon}\right)^{p-1} T_k(u_\varepsilon) &= \int_{|u_\varepsilon|<k} f_\varepsilon T_k(u_\varepsilon) \\
\int_{|u_\varepsilon|<k} a(x, Du_\varepsilon) Du_\varepsilon - \int_{|u_\varepsilon|<k} \left(\frac{(u_\varepsilon)^-}{\varepsilon}\right)^{p-1} u_\varepsilon &\le \int_{|u_\varepsilon|<k} \big|f_\varepsilon u_\varepsilon\big|
\end{aligned}$$

Applying (2.11) we get

$$\int_{|u_\varepsilon|<k} a(x, Du_\varepsilon) Du_\varepsilon \leq \int_{|u_\varepsilon|<k} |f_\varepsilon| |u_\varepsilon|$$
$$\int_{|u_\varepsilon|<k} a(x, Du_\varepsilon) Du_\varepsilon \leq \int_{|u_\varepsilon|<k} k |f_\varepsilon| < k \|f_\varepsilon\|_{L^1(\Omega)}$$
$$\int_{\{|u_\varepsilon|<k\}} a(x, Du_\varepsilon) Du_\varepsilon \leq k \|f\|_{L^1(\Omega)}.$$

From this estimate, by the ellipticity condition (2.3), we obtain:

$$\int_\Omega |DT_k(u_\varepsilon)|^p \leq c_4 K \quad \forall K > 0. \tag{2.31}$$

Let $1 < q < \frac{N(p-1)}{N-1}$ and $\bar{k} \geq \|\psi\|_{L^\infty(\Omega)}$.
It results:

$$\int_\Omega |Du_n|^q \leq$$
$$\left(\int_{\{u_n<\bar{k}\}} |Du_n|^p\right)^{\frac{q}{p}} |\Omega|^{1-\frac{q}{p}} + \sum_{j=\bar{k}}^{\infty} \left(\frac{1}{(1+j)^\lambda} \int_{B_j^n} |Du_n|^p\right)^{\frac{q}{p}} \left(\int_{B_j^n} (1+|u_n|)^{\frac{\lambda q}{p-q}}\right)^{1-\frac{q}{p}}$$

Where $\lambda = \frac{N(p-q)}{N-q}$, observe that $\lambda > 1$, since $1 < q < \frac{N(p-1)}{N-1}$.
From this inequality, using (2.30) in addition (2.31) we get

$$\int_\Omega |Du_n|^q \leq c_3 + c_4 \left(\int_\Omega |u_n|^{q*}\right)^{1-\frac{q}{p}}. \tag{2.32}$$

Where $q^* = \frac{Nq}{N-q}$, From this estimate, by Sobolev's inequality, we obtain

$$\int_\Omega |u_n|^{q*} \leq c_5.$$

Finally, from (2.32) we obtain the estimate (2.28). □

Second proof of Theorem 2.1.1. Since $\|u_\varepsilon\|_{W_0^{1,q}(\Omega)} \leq c$, there exists a subsequence, still denoted by $\{u_\varepsilon\}$, such that $\forall q < \frac{N(p-1)}{(N-1)}$:

$$\begin{cases} u_\varepsilon \rightharpoonup u & \text{Weakly- } W_0^{1,q}(\Omega) \\ u_\varepsilon \to u & \text{Strongly- } L^q(\Omega) \\ u_\varepsilon \to u & \text{almost everywhere in } \Omega \end{cases} \tag{2.33}$$

Now, we will move forward with the proof in order to show that

$$u \geq \psi \text{ a.e. in } \Omega \tag{2.34}$$

Consider a sequence of increasing functions $\{\theta_n(s)\}$ that converges to

$$\theta(s) = \begin{cases} 1 & \text{if } s > 0 \\ 0 & \text{if } s = 0 \\ -1 & \text{if } s < 0 \end{cases}$$

We choose $v=\theta_n(u_\varepsilon-\psi)$ as test function in the weak formulation of problem (2.27); then, there exists $\nu\in N$ such that :

$$\langle Au_\varepsilon,v\rangle-\int_\Omega\beta\left(\frac{(u_\varepsilon)^-}{\varepsilon}\right)v=\int_\Omega f_\varepsilon v$$

$$\langle A(u_\varepsilon-\phi),\theta_n(u_\varepsilon-\psi)\rangle-\int_\Omega\beta\left(\frac{(u_\varepsilon-\phi)^-}{\varepsilon}\right)\theta_n(u_\varepsilon-\psi)=\int_\Omega f_\varepsilon\theta_n(u_\varepsilon-\psi)$$

So that

$$-\int_\Omega\beta\left(\frac{(u_\varepsilon-\phi)^-}{\varepsilon}\right)\theta_n(u_\varepsilon-\psi)\le\int_\Omega f_\varepsilon\theta_n(u_\varepsilon-\psi)$$

$$-\left(\frac{1}{\varepsilon}\right)^{p-1}\int_\Omega\left((u_\varepsilon-\psi)^-\right)^{p-1}\theta_n(u_\varepsilon-\psi)\le\|f\|_{L^1(\Omega)}\quad\forall n>\nu$$

Passing to the limit as $n\to\infty$ we get:

$$\int_\Omega\left((u_\varepsilon-\psi)^-\right)^{p-1}\le\varepsilon^{p-1}\|f\|_{L^1(\Omega)}.\tag{2.35}$$

From this inequality we deduce

$$\left((u_\varepsilon-\psi)^-\right)^{p-1}\to 0\text{ strongly in }L^1(\Omega).$$

Thanks to (2.33) we obtain:

$$(u-\psi)^-=0\text{ a.e. in }\Omega$$

Which proves (2.34).

Moreover, by (2.35), $\left\|\beta\left(\frac{(u_\varepsilon-\psi)^-}{\varepsilon}\right)+f_\varepsilon\right\|_{L^1(\Omega)}\le C$, and from the results of [16] it thus follows

$$Du_\varepsilon\to Du\text{ a.e. in }\Omega\tag{2.36}$$

Let $v\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, $v\ge\psi$ a.e.in Ω, and $k>0$.

Using the estimate (2.28) In our study, we have examined $\|T_k(u_\varepsilon-v)\|_{W_0^{1,p}(\Omega)}\le c$, moreover, we take $T_k(u_\varepsilon-v)$ as test function in the weak formulation of problem (2.27), We obtain:

$$\langle Au_\varepsilon,T_k(u_\varepsilon-v)\rangle-\int_\Omega\beta\left(\frac{(u_\varepsilon)^-}{\varepsilon}\right)T_k(u_\varepsilon-v)=\int_\Omega f_\varepsilon T_k(u_\varepsilon-v)$$

$$\langle Au_\varepsilon-Av,T_k(u_\varepsilon-v)\rangle+\langle Av,T_k(u_\varepsilon-v)\rangle\le\int_\Omega f_\varepsilon T_k(u_\varepsilon-v)$$

$$\langle Au_\varepsilon+Av-Av,T_k(u_\varepsilon-v)\rangle\le\int_\Omega f_\varepsilon T_k(u_\varepsilon-v)$$

$$\langle Au_\varepsilon,T_k(u_\varepsilon-v)\rangle\le\int_\Omega f_\varepsilon T_k(u_\varepsilon-v),$$

Taking the limit as $\varepsilon\to0$ in the last inequality we conclude the proof of Theorem 2.1.1

2.4 The omographic approximation

Let $\lambda>0$ and $\{f_\lambda\}$ be a sequence of smooth function such that $\forall q<\frac{N(p-1)}{(N-1)}$:

$$\begin{cases}f_\lambda\to f\quad in\ L^1(\Omega)\\ \|f_\lambda\|_{L^1(\Omega)}\le\|f\|_{L^1(\Omega)}\quad\forall\lambda>0\end{cases}$$

Let us consider the following problem:

$$\begin{cases} u_\lambda \in W_0^{1,p}(\Omega) \\ Au_\lambda + g\frac{u_\lambda-\psi}{\lambda+|u_\lambda-\psi|} = f_\lambda + g \;\; in\,\Omega \end{cases} \tag{2.37}$$

Shat that

$$g = \left(f_\lambda - A\psi\right)^- .$$

We observe that g positive and $\|g\|_{L^1(\Omega)} \le \|f\|_{L^1(\Omega)} + \left\|(A\psi)^+\right\|_{L^1(\Omega)}$.

The existence of u_λ follows from the results of [24].

In order to prove Theorem (2.1.1) we need the following:

Lemma 2.4.1.

Assume that hypotheses (2.3),(2.4),(2.5) and (2.9) are satisfied.

then:

$$u_\lambda \ge \psi \;\; \forall \lambda > 0 \text{ a.e. in } \Omega \tag{2.38}$$

Also, $\exists c\left(q\right) > 0$, independent on λ, we have:

$$\|u_\lambda\|_{W_0^{1,q}(\Omega)} \le c\left(q\right) \;\forall 1 < q < \frac{N\left(p-1\right)}{N-1}. \tag{2.39}$$

Proof. We take $\left(u_\lambda - \psi\right)^-$ as test function in the weak formulation of (2.37) , such that $\forall v \in W_0^{1,p}$

$$\langle Au_\lambda, v\rangle + \int_\Omega g(v)\frac{(u_\lambda-\psi)^-}{\lambda+|u_\lambda-\psi|} = \langle f_\lambda + g, v\rangle$$

$$\left\langle Au_\lambda, (u_\lambda-\psi)^-\right\rangle + \int_\Omega g(u_\lambda-\psi)\frac{(u_\lambda-\psi)^-}{\lambda+|u_\lambda-\psi|} = \left\langle f_\lambda + g, (u_\lambda-\psi)^-\right\rangle$$

Since

$$\int_\Omega g(u_\lambda-\psi)\frac{(u_\lambda-\psi)^-}{\lambda+|u_\lambda-\psi|} \le 0,$$

Furthermore

$$\left\langle f_\lambda + g - A\psi, (u_\lambda-\psi)^-\right\rangle \ge 0,$$

We get

$$\int_{\{u_\lambda-\psi<0\}} (a(x,Du_\lambda) - a(x,D\psi))D(u_\lambda-\psi) \le 0.$$

From this estimate, using also assumption (2.5)

We have

$$D(u_\lambda(x) - \psi(x)) = 0 \text{ a.e. in } \left\{x \in \Omega : u_\lambda(x) < \psi(x)\right\}.$$

Then we obtain:

$$\left\|(u_\lambda-\psi)^-\right\|_{W_0^{1,p}(\Omega)} = 0$$

From which easily follows(2.38).

Thanks to (2.38), u_λ is solution of the following equation

$$Au_\lambda - g\frac{\lambda}{\lambda+(u_\lambda-\psi)} = f_\lambda \;\text{ in } \Omega \tag{2.40}$$

Let $k>0$, choosing $\varphi_k(u_\lambda)(\varphi_k(s)$ is the function defined (2.14)) as test function in the weak formulation of the last equation, we have

$$\begin{aligned}
\langle Au_\lambda, v\rangle - \int_\Omega g\frac{\lambda}{\lambda+(u_\lambda-\psi)}(v) &= \langle f_\lambda, v\rangle\\
\left\langle Au_\lambda, \varphi_k(u_\lambda)\right\rangle - \int_\Omega g\frac{\lambda}{\lambda+(u_\lambda-\psi)}(\varphi_k(u_\lambda)) &= \left\langle f_\lambda, \varphi_k(u_\lambda)\right\rangle\\
\int_\Omega a(x,Du_\lambda)D(\varphi_k(u_\lambda)) - \int_\Omega g\frac{\lambda}{\lambda+(u_\lambda-\psi)}(\varphi_k(u_\lambda)) &= \int_\Omega f_\lambda\varphi_k(u_\lambda)\\
\int_{B_k^\lambda} a(x,Du_\lambda)D(u_\lambda) - \int_{B_k^\lambda} g\frac{\lambda}{\lambda+(u_\lambda-\psi)}(u_\lambda) &= \int_{B_k^\lambda} f_\lambda(u_\lambda)\\
\int_{B_k^\lambda} a(x,Du_\lambda)Du_\lambda &\le \int_{B_k^\lambda}\left(\left|f_\lambda + g\frac{\lambda}{\lambda+(u_\lambda-\psi)}\right|\right)(k+1)\\
\int_{B_k^\lambda} a(x,Du_\lambda)Du_\lambda &\le Const\left\|f_\lambda + g\frac{\lambda}{\lambda+(u_\lambda-\psi)}\right\|_{L^1(\Omega)}
\end{aligned}$$

We get $\left(\left\|f_\lambda + g\frac{\lambda}{\lambda+(u_\lambda-\psi)}\right\|_{L^1(\Omega)} \le C\right)$ and using also assumption (2.3)

$$\int_{B_k^\lambda} |Du_\lambda|^p \le c_1,$$

Where B_k^λ is the set defined (2.16) .

From this estimate, we get the proof of (2.39).

□

Proof of Theorem 2.1.1 We point out that we shall use $A\psi \in L^1(\Omega)$. Using Lemma (2.4.1) there exists a subsequence, still denoted by $\{u_\lambda\}$. such that:

$$\begin{cases} u_\lambda \rightharpoonup u & \text{weakly -} W_0^{1,q}(\Omega)\\ u_\lambda \to u & \text{strongly -} L^q(\Omega)\\ u_\lambda \to u & \text{almost everywhere in } \Omega. \end{cases} \tag{2.41}$$

Also, since u_λ is solution of (2.40) and $\left\|f_\lambda + \lambda\frac{g}{\lambda+(u_\lambda-\psi)}\right\|_{L^1(\Omega)} \le C$, reasoning as before we obtain

$$Du_\lambda \to Du \text{ a.e. in } \Omega \tag{2.42}$$

Let $k>0$. Choosing $T_k(u_\lambda)$ as test function in the weak formulation of (2.40),and using also assumption (2.3)

$$\langle Au_\lambda, v\rangle - \int_\Omega g\frac{(\lambda}{\lambda+(u_\lambda-\psi)}(v) = \langle f_\lambda + g, v\rangle$$

$$\langle Au_\lambda, T_k u_\lambda\rangle - \int_\Omega g\frac{(\lambda}{\lambda+(u_\lambda-\psi)}(T_k(u_\lambda)) = \langle f_\lambda + g, T_k(u_\lambda)\rangle$$

$$\int_\Omega a(x, Du_\lambda)DT_k(u_\lambda) - \int_\Omega g\frac{(\lambda}{\lambda+(u_\lambda-\psi)}(T_k(u_\lambda)) = \int_\Omega f_\lambda T_k(u_\lambda)$$

$$\int_\Omega a(x, Du_\lambda)DT_k(u_\lambda) \le \int_{B_k^\lambda}\left(\left|f_\lambda + g\frac{\lambda}{\lambda+(u_\lambda-\psi)}\right|\right)(k)$$

$$\int_\Omega a(x, Du_\lambda)DT_k(u_\lambda) \le k\left\|f_\lambda + g\frac{\lambda}{\lambda+(u_\lambda-\psi)}\right\|_{L^1(\Omega)}$$

$$\int_\Omega |DT_k(u_\lambda)|^p \le \frac{constk}{\alpha}$$

$$\|DT_k(u_\lambda)\|_{L^p(\Omega)} \le c$$

We get

$$\|T_k(u_\lambda)\|_{W_0^{1,p}(\Omega)} \le ck \quad \forall k>0. \tag{2.43}$$

From this estimate we can prove that

$$\|T_k(u_\lambda - v)\|_{W_0^{1,p}(\Omega)} \le C, \tag{2.44}$$

For any $v \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$.
Let $v \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, $v \ge \psi$ a.e. in Ω, then

$$\langle Au_\lambda - Av, T_k(u_\lambda - v)\rangle + \langle Av, T_k(u_\lambda - v)\rangle = \int_\Omega f_\lambda T_k(u_\lambda - v) + \int_\Omega \lambda\frac{g}{\lambda+(u_\lambda-\psi)}T_k(u_\lambda - v).$$

Letting $\lambda \to 0$, and taking into account (2.41),(2.42) and (2.44) we can conclude the proof of Theorem (2.1.1)

Proof of Theorem 2.1.2

Since u_λ satisfies (2.37) shat that

$$Au_\lambda \le f_\lambda + (f_\lambda - A\psi)^-. \tag{2.45}$$

Also, thanks to (2.40) we get

$$Au_\lambda \ge f_\lambda. \tag{2.46}$$

Taking the limit as $\lambda \to 0$ in (2.45) and (2.46) we obtain (2.10).

Unilateral Problems with L^m Data, $m > 1$

This chapter deals with whose main results are contained in [11], where we study the proof of some existence and regularity results for unilateral problems with degenerate coercivity.

3.1 Main results

The problem we are considering is as follows.

$$\begin{cases} Au = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Here Ω is a bounded, open subset of $\mathbb{R}^N$, with $N > 2$, and $Au = -div(a(x,u)Du)$ with $a(x,s) : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function , satisfying the following conditions:

$$\frac{\alpha}{(1+|s|)^{\theta}} \leq a(x,s) \leq \beta \tag{3.1}$$

For some real number θ such that.

$$0 \leq \theta < 1 \tag{3.2}$$

For almost every $x \in \Omega$, for every $s \in \mathbb{R}$, where α and β are positive constants. we define $Au = -div(a(x,u)Du)$. Here, we assume that hypotheses (3.1) and (3.2) holds.

Theorem 3.1.1.

Let $f \in L^m(\Omega), m > \frac{N}{2}$.
In that case, a function $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ exists as a solution to the following unilateral problem

$$\begin{cases} u \geq 0 & \text{a.e.in } \Omega \\ \langle Au, u-v \rangle \leq \int_\Omega f(u-v) & \\ \forall v \in H_0^1(\Omega), v \geq 0 & \text{a.e. in } \Omega. \end{cases} \tag{3.3}$$

Additionally, u fulfills the inequality

$$f \leq Au \leq f^+ \tag{3.4}$$

The following result pertains to data f that yield unbounded solutions in $H_0^1(\Omega)$.

Theorem 3.1.2.

Let $f \in L^m(\Omega)$, with m such that

$$\frac{2N}{N+2-\theta(N-2)} \leq m < \frac{N}{2}. \tag{3.5}$$

In that case, there exists a function $u \in H_0^1(\Omega) \cap L^r(\Omega)$,with

$$r = \frac{Nm(1-\theta)}{N-2m} \tag{3.6}$$

That serves as a solution to problem (3.3).

Additionally u satisfies the inequality (3.4).

Theorem 3.1.3.

Let $f \in L^m(\Omega)$, with

$$\frac{N(2-\theta)}{N+2-N\theta} \leq m < \frac{2N}{N+2-\theta(N-2)}. \tag{3.7}$$

In that case, there exists a function $u \in W_0^{1,q}(\Omega)$, with

$$q = \frac{Nm(1-\theta)}{N-m(1+\theta)}. \tag{3.8}$$

Such that

$$a(x,u)|Du|^2 \in L^1(\Omega). \tag{3.9}$$

Furthermore, u is a solution to the unilateral problem (3.3) and simultaneously satisfies the inequality (3.4).

Remark 3.1.1.

In order to introduce the new formulation of unilateral problem let us recall the definition of the truncature function.

Given a constant $k > 0$ let $T_k : \mathbb{R} \to \mathbb{R}$ the function defined by

$$T_k(s) = \max\{-k, \min\{k, s\}\}.$$

Our focus of study will be on the following result.

Theorem 3.1.4. *[11]*

Let $f \in L^m(\Omega)$, with $m > 1$ such that

$$\frac{N}{N+1-\theta(N-1)} < m < \frac{N(2-\theta)}{N+2-N\theta}. \tag{3.10}$$

In that case, there exists a function $u \in W_0^{1,q}(\Omega)$, with q as in (3.8) we have

$$\begin{cases} u(x) \geq 0 & \text{a.e. } x \in \Omega \\ T_k(u) \in H_0^1 & \forall k > 0 \\ \langle Au, T_k(u-v)\rangle \leq \int_\Omega f T_k(u-v) & \\ \forall v \in H_0^1 \cap L^\infty(\Omega), v \geq 0 & \text{a.e. in } \Omega. \end{cases} \tag{3.11}$$

Additionally u satisfies the inequality (3.4).

3.2 A priori estimates.

Let $f \in L^m(\Omega)$, where m is defined in the statement of the theorems, and let f_n be a sequence of smooth functions such that

$$f_n \in L^{\frac{2N}{N+2}}(\Omega) \qquad f_n \to f \qquad \text{strongly in } L^m(\Omega) \tag{3.12}$$

We have $2^* = \frac{2N}{N-2}, L^{2^{*'}}(\Omega) = L^{\frac{2N}{N+2}}(\Omega)$, and $H_0^1(\Omega) \subset L^{2^*}(\Omega) \subset L^{2^{*'}}(\Omega) \subset H^{-1}(\Omega)$

$$\|f_n\|_{L^m(\Omega)} \le \|f\|_{L^m(\Omega)}, \forall n \in \mathbb{N}. \tag{3.13}$$

We define the following sequence of Dirichlet problems

$$\begin{cases} A_n u_n + f_n^- \frac{u_n}{\frac{1}{n}+|u_n|} = f_n^+ & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.14}$$

Where

$$A_n u_n = -div(a(x, T_n(u_n))Du_n).$$

For every $n \in \mathbb{N}$, the function $a(x, T_n(s))$ satisfies the condition (3.1). Also, since

$$a(x, T_n(s)) \ge \frac{\alpha}{(1+n)^\theta}, \text{ for a.e. } x \in \Omega, \forall s \in \mathbb{R}, \tag{3.15}$$

and since $f_n \in H^{-1}$, by well-known results (look at [25]) there exists at least a solution u_n of problem (3.14) in the sense that

$$\begin{cases} u_n \in H_0^1(\Omega) \\ \int_\Omega a(x, T_n(u_n))Du_n Dv + \int_\Omega f_n^- \frac{u_n}{\frac{1}{n}+|u_n|} v = \int_\Omega f_n^+ v \\ \forall v \in H_0^1(\Omega). \end{cases} \tag{3.16}$$

Note that, $\forall n \in \mathbb{N}$

$$u_n(x) \ge 0 \text{ for a.e. } x \in \Omega. \tag{3.17}$$

We choose as test function in (3.16) $v = u_n^-$, we obtain

$$\int_\Omega a(x, T_n(u_n))Du_n Dv + \int_\Omega f_n^- \frac{u_n}{\frac{1}{n}+|u_n|} v = \int_\Omega f_n^+ v$$

$$\int_\Omega a(x, T_n(u_n))Du_n Du_n^- + \int_\Omega f_n^- \frac{u_n}{\frac{1}{n}+|u_n|} u_n^- = \int_\Omega f_n^+ u_n^-$$

$$-\int_\Omega a(x, T_n(u_n))|Du_n^-|^2 = \int_\Omega f_n^+ u_n^- + \int_\Omega f_n^- \frac{(u_n^-)^2}{\frac{1}{n}+|u_n|}.$$

Since the right hand side is non negative and using condition (3.15), we have

$$\frac{\alpha}{(1+n)^\theta} \int_\Omega |Du_n^-|^2 dx \le 0,$$

This leads to the implication of (3.17), thereby establishing that u_n is a solution to the problem.

$$\begin{cases} u_n \in H_0^1(\Omega) \\ \int_\Omega a(x,T_n(u_n))Du_nDv + \int_\Omega f_n^- \frac{u_n}{\frac{1}{n}+u_n} v = \int_\Omega f_n^+ v \\ \forall v \in H_0^1(\Omega). \end{cases} \tag{3.18}$$

In order to prove Theorem (3.1.1) we need the following L^∞ a priori estimate

Lemma 3.2.1. *[11]*

Let $f \in L^m(\Omega)$, with $m > \frac{N}{2}$ and let u_n be a solution of (3.14). Then, there exist two positive constants c_1, c_2, depending on $N, m, \alpha, \theta, |\Omega|, \|f\|_{L^m(\Omega)}$, such that, for any $n \in \mathbb{N}$,

$$\|u_n\|_{L^\infty(\Omega)} \le c_1, \tag{3.19}$$

$$\|u_n\|_{H_0^1(\Omega)} \le c_2, \tag{3.20}$$

Proof. Let us define, for s in $\mathbb{R}$ and $k > 0$,

$$G_k(s) = s - T_k(s),$$

and set, for n in $\mathbb{N}$

$$A_k = \{x \in \Omega : u_n(x) > k\}. \tag{3.21}$$

We take $G_k(u_n)$ as test function in (3.18),and using Hölder's inequality (with exponents m, m')

$$\int_\Omega a(x,T_n(u_n))Du_nDv + \int_\Omega f_n^- \frac{u_n}{\frac{1}{n}+u_n} v = \int_\Omega f_n^+ v$$
$$\int_\Omega a(x,T_n(u_n))Du_nDG_k(u_n) + \int_\Omega f_n^- \frac{u_n}{\frac{1}{n}+u_n} G_k(u_n) = \int_\Omega f_n^+ G_k(u_n)$$
$$\int_{A_k} a(x,T_n(u_n))|Du_n|^2 \le \int_\Omega |f| G_k(u_n)$$
$$\int_{A_k} a(x,T_n(u_n))|Du_n|^2 \le \|f\|_{L^m(\Omega)} \left(\int_\Omega (G_k(u_n))^{m'}\right)^{\frac{1}{m'}},$$

and use assumption (3.1) and condition (3.17), we obtain

$$\begin{aligned} \alpha \int_{A_k} \frac{|Du_n|^2}{(1+u_n)^\theta} &\le \int_\Omega |f| G_k(u_n) \\ &\le \|f\|_{L^m(\Omega)} \left(\int_\Omega (G_k(u_n))^{m'}\right)^{\frac{1}{m'}}, \end{aligned} \tag{3.22}$$

Where $m' = \frac{m}{m-1}$. Thanks to estimate (3.22) we get the L^∞-estimate as in the proof of Lemma 2.2 of [12].

We study proof the estimate (3.20), we take u_n as test function in (3.18), we obtain

$$\int_\Omega a(x,T_n(u_n))Du_nDv+\int_\Omega f_n^-\frac{u_n}{\frac{1}{n}+u_n}v=\int_\Omega f_n^+v$$

$$\int_\Omega a(x,T_n(u_n))Du_nDu_n+\int_\Omega f_n^-\frac{u_n}{\frac{1}{n}+u_n}u_n=\int_\Omega f_n^+u_n$$

$$\int_\Omega a(x,T_n(u_n))|Du_n|^2=-\int_\Omega f_n^-\frac{u_n}{\frac{1}{n}+u_n}u_n+\int_\Omega f_n^+u_n$$

$$\int_\Omega a(x,T_n(u_n))|Du_n|^2\le-\int_\Omega f_n^-\frac{u_n}{\frac{1}{n}+u_n}u_n+\int_\Omega f_n^+u_n$$

Using hypothesis (3.1) we get

$$\alpha\int_\Omega\frac{|Du_n|^2}{(1+u_n)^\theta}\le\int_\Omega f^+u_n-\int_\Omega f^-\frac{u_n}{\frac{1}{n}+u_n}u_n\le\int_\Omega|f|u_n.$$

From this estimate, using (3.19) we obtain

$$\int_\Omega|Du_n|^2\le\frac{(1+c_1)^{\theta+1}}{\alpha}\|f\|_{L^m(\Omega)}.$$

□

The subsequent result will be utilized in the proof of Theorem 3.1.2.

Lemma 3.2.2.

Let $f\in L^m(\Omega)$, with m satisfying hypothesis (3.5), and let u_n be a solution of problem (3.14). Subsequently, there exist two positive constants, namely c_3 and c_4, depending on $N,m,\alpha,\theta,|\Omega|,\|f\|_{L^m(\Omega)}$, such that, for any $n\in\mathbb{N}$,

$$\|u_n\|_{L^r(\Omega)}\le c_3, \tag{3.23}$$

$$\|u_n\|_{H_0^1(\Omega)}\le c_4, \tag{3.24}$$

Where r is defined by 3.6.

Proof. Let $k>0$. Following the outline of the proof of Lemma 2.3 of [12] we have to prove the following estimate

$$\alpha\int_{B_k}|Du_n|^2\le(2+k)^\theta\int_{A_k}|f|, \tag{3.25}$$

Where A_k is the set defined in (3.21) and

$$B_k=\{x\in\Omega: k\le u_n<k+1\}. \tag{3.26}$$

If we take in (3.18) $v=T_1(G_k(u_n))$,

$$\int_\Omega a(x,T_n(u_n))Du_nDv+\int_\Omega f_n^-\frac{u_n}{\frac{1}{n}+u_n}v=\int_\Omega f_n^+v$$

$$\int_\Omega a(x,T_n(u_n))Du_nDT_1(G_k(u_n))+\int_\Omega f_n^-\frac{u_n}{\frac{1}{n}+u_n}T_1(G_k(u_n))=\int_\Omega f_n^+T_1(G_k(u_n))$$

$$\int_{B_k} a(x,T_n(u_n))Du_nD(G_k(u_n)) \le \int_\Omega f_n^+ T_1(G_k(u_n)) - \int_\Omega f_n^- \frac{u_n}{\frac{1}{n}+u_n} T_1(G_k(u_n))$$

$$\int_{B_k} a(x,T_n(u_n))Du_nDu_n =\le \int_{A_k} f_n^+ T_1(G_k(u_n)) - \int_{A_k} f_n^- \frac{u_n}{\frac{1}{n}+u_n} T_1(G_k(u_n))$$

$$\int_{B_k} a(x,T_n(u_n))|Du_n|^2 \le \int_{A_k} f_n^+ T_1(G_k(u_n)) - \int_{A_k} f_n^- T_1(G_k(u_n))$$

$$\int_{B_k} a(x,T_n(u_n))|Du_n|^2 \le \int_{A_k} |f_n| T_1(G_k(u_n))$$

Thanks to hypothesis (3.1), (3.17) and $u_n \le k+1$ we get

$$\alpha \int_{B_k} \frac{|Du_n|^2}{(1+u_n)^\theta} \le \int_{A_k} |f| T_1(G_k(u_n))$$

$$\alpha \int_{B_k} \frac{|Du_n|^2}{(1+k+1)^\theta} \le \int_{A_k} |f| T_1(G_k(u_n)),$$

Which implies (3.25). □

The next lemma deals with the case in which the sequence $\{u_n\}$ is not bounded in H_0^1 and will be used in the proof of Theorems (3.1.3), (3.1.4).

Lemma 3.2.3.

Assume $f \in L^m(\Omega)$ with

$$\frac{N}{N+1-\theta(N-1)} < m < \frac{2N}{N+2-\theta(N-2)}. \tag{3.27}$$

Let $\{f_n\}$ be a sequence of functions satisfying (3.12) and (3.13), and let u_n be a solution of (3.14). Then, for any $n \in \mathbb{N}$ and $K > 0$ we obtain

$$\int_\Omega |DT_k(u_n)|^2 dx \le \frac{\|f\|_{L^1(\Omega)}}{\alpha} (1+K)^{\theta+1}. \tag{3.28}$$

Also

$$\|u_n\|_{W_0^{1,q}(\Omega)} \le c_5, \quad \forall n \in \mathbb{N}, \tag{3.29}$$

Where c_5 depends on $N, m, \theta, \alpha, |\Omega|, \|f\|_{L^m(\Omega)}$ and q is defined by (3.8).

Proof. Let us take $T_k(u_u)$ as test function in (3.18)

$$\int_\Omega a(x,T_n(u_n))Du_nDv + \int_\Omega f_n^- \frac{u_n}{\frac{1}{n}+u_n} v = \int_\Omega f_n^+ v$$

$$\int_\Omega a(x,T_n(u_n))Du_nDT_k(u_u) + \int_\Omega f_n^- \frac{u_n}{\frac{1}{n}+u_n} T_k(u_u) = \int_\Omega f_n^+ T_k(u_u)$$

$$\int_{B_k} a(x,T_n(u_n))Du_nDu_n \le \int_\Omega f_n^+ T_k(u_u) - \int_\Omega f_n^- \frac{u_n}{\frac{1}{n}+u_n} T_k(u_u)$$

$$\int_{B_k} a(x,T_n(u_n))Du_nDu_n \le \int_\Omega f_n^+ T_k(u_u) - \int_{A_k} f_n^- \frac{u_n}{\frac{1}{n}+u_n} T_k(u_u)$$

$$\int_{B_k} a(x,T_n(u_n))|Du_n|^2 \le \int_\Omega f_n^+ T_k(u_u) - \int_{A_k} f_n^- T_k(u_u)$$

$$\int_{B_k} a(x,T_n(u_n))|Du_n|^2 \le \int_\Omega |f_n| T_k(u_u)$$

Using (3.1) and condition (3.17) we get

$$\alpha\int_{B_k}\frac{|Du_n|^2}{(1+u_n)^\theta}\leq\int_\Omega|f|T_k(u_n)$$
$$\frac{\alpha}{(1+k)^\theta}\int_\Omega|DT_k(u_n)|^2\leq\int_\Omega|f_n|(k+1)$$
$$\int_\Omega|DT_k(u_n)|^2\leq\frac{\|f\|_{L^1(\Omega)}}{\alpha}(1+k)^{\theta+1}$$

Which implies (3.28).

The estimate (3.29) follows working as in the proof of Lemma 2.5 of [12]. □

Before proving the theorems we state the following result (for the proof see Lemma 2.8, [12]).

Lemma 3.2.4.

Let $\{v_n\}$ be a sequence of functions which is weakly convergent to v in $H_0^1(\Omega)$, and let u_n be a sequence of functions which is almost everywhere convergent to some function u in Ω. Then

$$\int_\Omega a(x,u)|Dv|^2\leq\liminf_{n\to+\infty}\int_\Omega a(x,T_n(u_n))|Dv_n|^2\leq c.$$

We are now in position to prove the Theorems.

3.3 Proof of the Theorems

Let $f\in L^m(\Omega)$, with m as in the statements of the theorems and let $\{u_n\}$ be a sequence of solutions of (3.14). Using the results of Lemmas (3.2.1) in addition (3.2.2) we get that the sequence $\{u_n\}$ is bounded in $H_0^1(\Omega)$ and in the Lebesgue spaces as in the statements of the theorems.

Then, there exists a subsequence, still denoted by $\{u_n\}$, which is weakly convergent to some function u in $H_0^1(\Omega)$. Moreover, $\{u_n\}$ converges to u almost everywhere in Ω as a consequence of the Rellich theorem.

Let us prove that u is a solution of the unilateral problem 3.3.

Since $u_n(x)\geq 0$ a.e. $x\in\Omega$ for any $n\in\mathbb{N}$ we have

$$u(x)\geq 0 \text{ a.e. } x\in\Omega.$$

Let $w\in H_0^1(\Omega)$, $w\geq 0$, and take u_n-w as test function in (3.18). We obtain

$$\int_\Omega a(x,T_n(u_n))Du_nDv+\int_\Omega f_n^-\frac{u_n}{\frac{1}{n}+u_n}v=\int_\Omega f_n^+v$$
$$\int_\Omega a(x,T_n(u_n))Du_nD(u_n-w)+\int_\Omega f_n^-\frac{u_n}{\frac{1}{n}+u_n}(u_n-w)=\int_\Omega f_n^+(u_n-w)$$
$$\int_\Omega a(x,T_n(u_n))Du_nD(u_n-w)=\int_\Omega f_n(u_n-w)+\frac{1}{n}\int_\Omega f_n^-(u_n-w). \tag{3.30}$$

Applying Lemma (3.2.4), with $v_n = u_n$, we have

$$\int_\Omega a(x,u)|Du|^2 \le \liminf_{n\to+\infty} \int_\Omega a(x,T_n(u_n))|Du_n|^2.$$

Thanks to the boundeness and the continuity of $a(x,s)$, and since u_n converges to u weakly in $H_0^1(\Omega)$ and almost everywhere in Ω we have

$$\lim_{n\to+\infty} \int_\Omega a(x,T_n(u_n))Du_n D_w = \int_\Omega a(x,u)DuDw.$$

Hence, taking the limit as $n\to+\infty$ in (3.30), since the right hand side converges to

$$\int_\Omega f(u-w)dx,$$

u is a solution of (3.3).

In order to prove the inequality (3.4) we note that, since u_n is non negative, from (3.14) we derive

$$f \le A_n u_n \le f^+.$$

Thanks to the linearity of $A_n u_n$ with respect to Du_n , letting $n\to+\infty$ in the previous inequality, we obtain inequality (3.4).

Proof. of Theorem 3.1.3

Let $\{f_n\}$ be a sequence of functions satisfying (3.12) and (3.13), with m as in the statement of Theorem (3.1.3),and let $\{u_n\}$ be a sequence of solutions of problem (3.14). By Lemma (3.2.3) the sequence $\{T_k(u_n)\}$ is bounded in $H_0^1(\Omega)$. Also the sequence $\{u_n\}$ is bounded in $W_0^{1,q}(\Omega)$ and in $L^r(\Omega)$, with q and r defined by (3.8), (3.6), respectively. Thus, there exists a subsequence, denoted by $\{u_n\}$ such that

$$\begin{cases} u_n \rightharpoonup u \ weakly - W_0^{1,q}(\Omega) \\ u_n \to u \ strongly - L^q, and\ a.e. x\in\Omega, \\ T_k u_n \rightharpoonup T_k u \ weakly - H_0^1(\Omega). \end{cases} \tag{3.31}$$

Let's study proof that u satisfies (3.9).

Taking $T_k(u_n)$ as test function in (3.18),

$$\int_\Omega a(x,T_n(u_n))Du_nDv + \int_\Omega f_n^- \frac{u_n}{\frac{1}{n}+u_n} v = \int_\Omega f_n^+ v$$

$$\int_\Omega a(x,T_n(u_n))Du_nDT_k(u_n) + \int_\Omega f_n^- \frac{u_n}{\frac{1}{n}+u_n} T_k(u_n) = \int_\Omega f_n^+ T_k(u_n)$$

$$\int_\Omega a(x,T_n(u_n))|DT_k(u_n)|^2 \le \int_\Omega f_n^+ u_n - \int_\Omega f_n^- \frac{u_n}{\frac{1}{n}+u_n} u_n$$

$$\int_\Omega a(x,T_n(u_n))|DT_k(u_n)|^2 \le \int_\Omega f_n(T_n(u_n)) + \frac{1}{n}\int_\Omega f_n^-(T_n(u_n))$$

We have

$$\int_\Omega a(x,T_n(u_n))|DT_k(u_n)|^2 \le \int_\Omega f_n T_k(u_n) + \frac{1}{n}\|f_n^-\|_{L^1(\Omega)}. \tag{3.32}$$

Applying Lemma (3.2.4) with $v_n = T_k(u_n)$, we thus have

$$\int_\Omega a(x,u)\,|DT_k(u)|^2 \le \liminf_{n\to+\infty} \int_\Omega a(x,T_n u_n)\,|DT_k(u_n)|^2. \tag{3.33}$$

Passing to the limit as $n \to +\infty$ in 3.32 we obtain

$$\int_\Omega a(x,u)\,|DT_k(u)|^2 \le \int_\Omega f\,T_k(u).$$

Letting $k \longrightarrow \infty$, we obtain

$$\int_\Omega a(x,u)\,|Du|^2 \le \int_\Omega f\,u \le c. \tag{3.34}$$

Now we can prove that u is a solution of the unilateral problem (3.3). First of all we note that $u(x) \ge 0$ almost everywhere $x \in \Omega$.

Let φ be a function in $C_0^\infty(\Omega)$, $\varphi(x) \ge 0$ a.e. $x \in \Omega$ and $k > 0$. Taking $T_k(u_n) - \varphi$ as test function in (3.18)

$$\int_\Omega a(x,T_n(u_n))Du_n Dv + \int_\Omega f_n^- \frac{u_n}{\frac{1}{n}+u_n} v = \int_\Omega f_n^+ v$$

$$\int_\Omega a(x,T_n(u_n))Du_n D(T_k(u_n)-\varphi) + \int_\Omega f_n^- \frac{u_n}{\frac{1}{n}+u_n}(T_k(u_n)-\varphi) = \int_\Omega f_n^+ (T_k(u_n)-\varphi)$$

$$\int_\Omega a(x,T_n(u_n))DT_k(u_n) - D\varphi = \int_\Omega f_n^+ (T_k(u_n)-\varphi) - $$
$$- \int_\Omega f_n^- \frac{u_n}{\frac{1}{n}+u_n}(T_k(u_n)-\varphi)$$

$$\int_\Omega a(x,T_n(u_n))DT_k(u_n) - \int_\Omega a(x,T_n(u_n))D\varphi \le \int_\Omega f_n (T_k(u_n)-\varphi) +$$
$$+ \frac{1}{n}\int_\Omega f_n^- (T_k(u_n)-\varphi)$$

We obtain

$$\begin{aligned}\int_\Omega a(x,T_n(u_n)))\,Du_n DT_k(u_n) - \int_\Omega a(x,T_n(u_n)))\,Du_n D\varphi \le \\ \le \int_\Omega f_n\left(T_k(u_n)-\varphi\right) + \frac{1}{n}\left\|f_n^-\right\|_{L^1(\Omega)}.\end{aligned} \tag{3.35}$$

The right hand side easily passes to the limit as n tends to infinity. As for the left hand side, we note that condition (3.33) holds; moreover $a(x,T_n u_n)\,D\varphi$ converges to $a(x,u)\,D\varphi$ in any $L^p(\Omega)$. Thus, it is possible to pass to the limit in (3.35) to obtain

$$\int_\Omega a(x,u)\,|DT_k u|^2 - \int_\Omega a(x,u)\,DuD\varphi \le \int_\Omega f\left(T_k u - \varphi\right).$$

A further limits on $k \to +\infty$ yelds

$$\int_\Omega a(x,u)\,Du\left(Du - D\varphi\right) \le \int_\Omega f\left(u-\varphi\right), \tag{3.36}$$

$\forall \varphi \in C_0^\infty(\Omega), \varphi(x) \ge 0$ a.e. $x \in \Omega$. At least, by standard density argument we can prove that 3.36 holds also for non negative test functions in $H_0^1(\Omega)$.

The proof of the inequality (3.4) follows as in Theorems 3.1.1 and 3.1.2. □

Proof. of Theorem 3.1.4. Let $\{f_n\}$ be a sequence of functions satisfying (3.12) and (3.13), with m as in the statement of Theorem 3.1.4, and let $\{u_n\}$ be a sequence of solutions of problem (3.14). As in the proof of Theorem 3.1.3, $\{u_n\}$ has a subsequence, still denoted by sati $\{u_n\}$satisfying (3.31). Moreover, $u(x) \geq 0$ a.e. $x \in \Omega$.
Let $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $v(x) \geq 0$ a.e.$x \in \Omega$.
Taking $T_k(u_n - v)$ as test function in (3.18)

$$\int_\Omega a(x, T_n(u_n)) Du_n Dv + \int_\Omega f_n^- \frac{u_n}{\frac{1}{n} + u_n} v = \int_\Omega f_n^+ v$$

$$\int_\Omega a(x, T_n(u_n)) Du_n DT_k(u_n - v) + \int_\Omega f_n^- \frac{u_n}{\frac{1}{n} + u_n} T_k(u_n - v) = \int_\Omega f_n^+ T_k(u_n - v)$$

$$\int_\Omega a(x, T_n(u_n)) Du_n DT_k(u_n - v) \leq \int_\Omega f_n T_k(u_n - v) + \frac{1}{n} \int_\Omega f_n^- T_k(u_n - v)$$

We obtain

$$\int_\Omega a(x, T_n u_n) Du_n DT_k(u_n - v) \leq \int_\Omega f_n T_k(u_n - v) + \frac{1}{n} \|f_n^-\|_{L^1(\Omega)}. \tag{3.37}$$

The left hand side of the previous inequality can be rewritten as follows

$$\int_\Omega a(x, T_n(u_n)) |DT_k(u_n - v)|^2 - \int_\Omega a(x, T_n(u_n)) D_v DT_k(u_n - v).$$

Since the sequence $\{T_k(u_n - v)\}$ is weakly convergent to $T_k(u - v)$ in H_0^1 by Lemma (3.2.4) with $v_n = T_k(u_n - v)$ we have

$$\int_\Omega a(x, u) |DT_k(u - v)|^2 \leq \liminf_{n \to +\infty} \int_\Omega a(x, T_n(u_n)) |DT_k(u_n - v)|^2.$$

Also, due to the boundedness and continuity of a we obtain

$$\int_\Omega a(x, u) D_v DT_k(u - v) = \lim_{n \to +\infty} \int_\Omega a(x, T_n(u_n)) D_v DT_k(u_n - v).$$

Then the first member of (3.37) passes to the limit, as well as the second member.
Hence u satisfies

$$\int_\Omega a(x, u) D_u DT_k(u - v) \leq \int_\Omega f T_k(u - v),$$

For every v in $H_0^1 \cap L^\infty(\Omega)$, $v(x) \geq 0$ a.e. $x \in \Omega$, that is u is a solution of the unilateral problem (3.11). □

As for as the inequality (3.4) is concerned, we can prove it as in Theorems (3.1.1) and (3.1.2).

Conclusion and Further Prospects

In this memory, we have studied some results on a class of nonlinear elliptic equations with degenerate coercivity of the form

$$\begin{cases} Au = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.38}$$

Where Ω be an open bounded subset of $\mathbb{R}^N$, $N \geq 2$, A is defined as a nonlinear operator

$$Au = -div\ a(x, Du)$$

Where Ω is a bounded Lipschitz domain in $\mathbb{R}^N$ with $N > 2$, $f \in L^m$, $m > 1$ and $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Caratheodory function such that for $a.e.$ $x \in \Omega$ and $\forall \xi, \eta \in \mathbb{R}^N, (\xi \neq \eta)$ the following assumptions hold:

$$a(x,\xi)\xi \geq \alpha|\xi|^p$$

$$|a(x,\xi)| \leq \beta\left(h(x) + |\xi|^{p-1}\right)$$

$$\left(a(x,\xi) - a(x,\eta)\right)\left(\xi - \eta\right) > 0$$

With $\alpha, \beta > 0$ and $h(x)$ is a non-negative function in $L^{p'}(\Omega)$(here p' denotes the conjugate exponent of p).

- If $f \in L^m(\Omega), m > \frac{N}{2}$, then $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

- If $f \in L^m(\Omega), \frac{2N}{N+2-\theta(N-2)} \leq m < \frac{N}{2}$, then $u \in H_0^1(\Omega) \cap L^r(\Omega)$,with $r = \frac{Nm(1-\theta)}{N-2m}$.

- If $f \in L^m(\Omega), \frac{N(2-\theta)}{N+2-N\theta} \leq m < \frac{2N}{N+2-\theta(N-2)}$, then $u \in W_0^{1,q}(\Omega)$, with $q = \frac{Nm(1-\theta)}{N-m(1+\theta)}$

- If $f \in L^m(\Omega), \frac{N}{N+1-\theta(N-1)} < m < \frac{N(2-\theta)}{N+2-N\theta}$, then $u \in W_0^{1,q}(\Omega)$, with $q = \frac{Nm(1-\theta)}{N-m(1+\theta)}$

This thesis memory contributes to the understanding of nonlinear elliptic equations with degenerate coercivity, providing important theoretical insights and establishing conditions for the existence and regularity of solutions. The results obtained in this study lay the groundwork for further investigations in this field and can serve as a valuable resource for researchers and practitioners working in related areas.

This study raises some questions for researchers to explore in further studies: We suggest to study the following problem

$$\begin{cases} Au = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Where Ω is a bounded Lipschitz domain in $\mathbb{R}^N$ with $N > 2$, A is defined as a nonlinear operator

$$Au = -div\ a(x, Du)$$

With $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Caratheodory function such that for $a.e.$ $x \in \Omega$ and $\forall \xi, \eta \in \mathbb{R}^N, (\xi \neq \eta)$ the following assumptions hold:

$$a(x,\xi)\xi \geq \alpha|\xi|^p$$

$$|a(x,\xi)| \leq \beta\left(h(x) + |\xi|^{p-1}\right)$$

$$\left(a(x,\xi) - a(x,\eta)\right)(\xi - \eta) > 0$$

With $\alpha, \beta > 0$, $h(x)$ is a non-negative function in $L^{p'}(\Omega)$(here p' denotes the conjugate exponent of p) ,and $f \in L_\rho(\Omega)$

Let $\rho : [0, +\infty) \longrightarrow [0, +\infty)$ be an N-function, i.e, a convex function such that

$$\lim_{\eta \longrightarrow 0^+} \frac{\rho(\eta)}{\eta} = 0 \text{ and } \lim_{\eta \longrightarrow +\infty} \frac{\rho(\eta)}{\eta} = +\infty.$$

Then it is possible to define the Orlicz space

$$L_\rho(\Omega) = \left\{ f \text{ measurable on } \Omega \mid \exists M > 0 : \int_\Omega \rho(\frac{|f|}{M}) < +\infty \right\}.$$

where $L_\rho(\Omega)$ is a Banach space under the norm

$$\|f\|_{L_\rho(\Omega)} = \inf\left\{ M > 0 : \int_\Omega \rho(\frac{|f|}{M}) \leq 1 \right\}$$

(more details about it look at [7])

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