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Theme

On Weak Lebesgue (Marcinkiewicz) Spaces with Variable Exponents

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Dedication

Nothing compares to the joy of graduation, as it is one of the most beautiful moments in our lives. The tiredness of the years, the sleepless nights and the prayers of the parents have been harvested, and the psychological pressures we have gone through have disappeared and we have forgotten them as soon as we feel the joy of graduation.

To those who are not matched by anyone in the universe, to whom God has commanded us to honor them, to those who have made a great deal, and have given what cannot be returned, to you these words, my dear mother and father, I dedicate this research to you. You have been my best supporter throughout my study career .

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To everyone who advised me, guided me, and contributed to helping me accomplish this research .

In the end I wish success to all my colleagues.

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List of Symbols

Everywhere in the sequel we use the following notations:

- $\mathbb{R}^n$: Euclidean, n-dimensional space.
- $d\mu$ or dx : Lebesgue measure n-dimensional .
- χ_E : Characteristic function of the set E

$$\chi_E = \begin{cases} 1, & x \in E \\ 0, & elsewhere, \end{cases}$$

- Ω: Open set in $\mathbb{R}^n$.
- D_f: The distribution function.
- $\rho_{p(.)}$: Modular function.

$$\rho_{p_{(.)}}(f) = \int_{\Omega \backslash \Omega_\infty} |f(x)|^{p(x)} dx < \infty$$

- p': The conjugate exponent $\dfrac{1}{p} + \dfrac{1}{p'} = 1.$
- $\mathbb{Z}$: The set of all integer numbers.
- p_+, p_-: Essential supremum and infimum of p,

$$p_-(\Omega) = ess_{\Omega_*} \inf p(x),$$
$$p_+(\Omega) = ess_{\Omega_*} \sup p(x)$$

- τ_h : Translation operator .
- $\|\|_p$: norm .
- $L^1_{loc}(\mathbb{R}^n)$: The collection of all locally integrable function on $\mathbb{R}^n$.
- $L_{p(.)}(\Omega)$: The variable exponent Lebesgue space.
- $L_{p)}(\Omega)$: The grand Lebesgue space.

- $f * g$: The convolution is defined by

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy. \quad f, g \in L^1(\mathbb{R}^n).$$

- $C_0^\infty(\Omega)$: the space of smooth functions with compact support in Ω.
- $C_+^0(\overline{\Omega}) = \left\{p \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} p(x) > 0\right\}$
- "i.e": Stands simply for "in other words".
- "a.e": Stands simply for "almost every were".

Introduction

This memory master is devoted to the study of some basic properties of Marcinkiewicz spaces (weak Lebesgue) with variable exponents which is necessary to treat some nonlinear partial differential equations. Its name comes from the Polish mathematician Józef Marcinkiewicz (1910–1940) who worked in real analysis and partial differential equations (look at [13]).

In the first chapter we study some definitions and elementary properties of Lebesgue spaces and weak Lebesgue spaces which includes the distribution function, convergence in measure, a first glimpse at interpolation. This chapter has been illustrated by simple examples to clarify the use these spaces in PDE. As already mentioned by [4] and [5] , these spaces has been already used to studying nonlinear partial differential equations (the reader can look at [14]).

Chapter 2 is devoted to study some definitions and elementary properties of variable exponent Lebesgue spaces involving weak Lebesgue spaces with variable exponents, Luxemburg-Nakano type norm, another version of the Luxemburg-Nakano norm, Hölder inequality, convergence, completeness, embeddings, and Dense Sets. In addiction, we give the main differences between spaces with variable exponent and constant exponent which has been studied in [3] [9], and [12]. In conclusion, we study the Hardy's inequality that introduced in [5]. Also, on a personal level, Marcinkiewicz spaces with variable exponents are one of the generalizations of the Lebesgue spaces with variable exponents.

CHAPTER 1

LEBESGUE SPACES AND WEAK LEBESGUE SPACES

In this chapter, we study some definition and properties of Lebesgue Spaces and Weak Lebesgue Spaces .

1.1 Lebesgue Spaces

This section is devoted to some definition and properties of L_p spaces.

1.1.1 Definition and elementary properties of L_p spaces

Definition 1.1.1. Let $p \in \mathbb{R}$ with $1 < p < \infty$; we set

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R}; f \quad is \quad measurable \quad and \quad |f|^p \in L^1(\Omega) \right\}$$

with

$$\|f\|_{L^p} = \|f\|_p = \left(\int_\Omega |f(x)|^p \, d\mu \right)^{\frac{1}{p}}$$

Definition 1.1.2. We set

$$L^\infty(\Omega) = \{ f : \Omega \to \mathbb{R}; \quad f \text{ is } \mu \text{ measurable and } \exists c : |f(x)| \le c \text{ for } \mu \text{ - a.e . } x \in \Omega, c > 0 \}$$

with

$$\|f\|_{L^\infty} = \|f\|_\infty = \inf \{ c; |f(x)| \le c \quad \text{for } \mu\text{-a.e. } x \in \Omega \}$$

Remark 1.1.3. see [4]

Theorem 1.1.4 (Hölder's inequality). *Assume that $f \in L^p$ and $g \in L^{p'}$ with $p' = \dfrac{p-1}{p}, 1 \le p \le \infty$. Then $fg \in L^1$ and*

$$\int |fg| \le \|f\|_p \, \|g\|_{p'} \, . \tag{1.1}$$

Theorem 1.1.5. *L^p is a vector space and $\|.\|_p$ is a norm for any $p, 1 \le p \le \infty$.*

Proof. For $1 < p < \infty$ and let $f, g \in L^p$. We have

$$|f(x) + g(x)|^p \leq (|f(x)| + |g(x)|)^p \leq 2^p (|f(x)|^p + |g(x)|^p).$$

So that, $f + g \in L^p$. Whereas,

$$\|f + g\|_p^p = \int |f + g|^{p-1} |f + g| \leq \int |f + g|^{p-1} |f| + \int |f + g|^{p-1} |g|.$$

But $|f + g|^{p-1} \in L^{p'}$, and by Hölder's inequality we obtain

$$\|f + g\|_p^p \leq \|f + g\|_p^{p-1} \left(\|f\|_p + \|g\|_p\right),$$

i.e, $\|f + g\|_p \leq \|f\|_p + \|g\|_p$. □

Theorem 1.1.6 (Fischer-Riesz). *[4] L^p is a Banach space for any $p, 1 \leq p \leq \infty$.*

Proof. See [4] □

1.2 Weak Lebesgue spaces

We start with a simple observation that will be used when defining the weak Lebesgue space.

Lemma 1.2.1. *[5] Let (E, A, μ) be a measure space and f be an a -measurable function that satisfies*

$$\mu(\{x \in E : |f(x)| > \lambda\}) \leq \left(\frac{c}{\lambda}\right)^p \tag{1.2}$$

for some $C > 0$. Then

$$\inf\left\{c > 0 : D_f(\lambda) \leq \left(\frac{c}{\lambda}\right)^p\right\} = \left(\sup_{\lambda>0} \lambda^p D_f(\lambda)\right)^{\frac{1}{p}} = \sup_{\lambda>0} \lambda (D_f(\lambda))^{\frac{1}{p}}$$

Proof. We set

$$\lambda = \inf\left\{c > 0 : D_f(\lambda) \leqslant \left(\frac{c}{\lambda}\right)^p\right\},$$

and

$$B = \left(\sup_{\lambda>0} \lambda^p D_f(\lambda)\right)^{\frac{1}{p}}.$$

Since f satisfies 1.2 then

$$D_f(\lambda) \leq \left(\frac{c}{\lambda}\right)^p$$

for some $C > 0$, then

$$\left\{c > 0 : D_f(\lambda) \leq \left(\frac{c}{\lambda}\right)^p, \forall \lambda > 0\right\} \neq 0.$$

On the other hand $\lambda^p D_f(\lambda) \leq B^p$, thus $\lambda^p D_f(\lambda) : \lambda > 0$ is bounded above by B^p and so $B \in \mathbb{R}$. Therefore

$$\alpha = \inf\left\{c > 0 : D_f(\lambda) \leq \left(\frac{c}{\lambda}\right)^p \quad \lambda > 0\right\} \leq B. \tag{1.3}$$

Now, let $\varepsilon > 0$, then there exists C such that

$$\lambda \leq C < \lambda + \varepsilon$$

and thus

$$D_f(\lambda) \leq \frac{c^p}{\lambda^p} < \frac{(\lambda+\varepsilon)^p}{\lambda^p}$$

from which we get

$$\sup_{\lambda>0} \lambda^p D_f(\lambda) < (\lambda+\varepsilon)^p.$$

By the arbitrariness of $\varepsilon > 0$, we obtain $B < \lambda$ which, together with 1.3, we obtain $B = \lambda$.

We now introduce the weak Lebesgue space. □

1.2.1 The distribution function

Definition 1.2.2. [10] For f a measurable function on E, the distribution function of f is the function D_f defined on $[0, \infty)$ as follows:

$$D_f(\lambda) = \mu(\{x \in E : |f(x)| > \lambda\}). \tag{1.4}$$

The distribution function D_f provides information about the size of f but not about the behavior of f it self near any given point. For instance, a function on $\mathbb{R}^n$ and each of its translates have the same distribution function. It follows from Definition 1.2.2 that D_f is a decreasing function of λ (not necessarily strictly).

Proposition 1.2.3. *[10] Let f and g be measurable functions on (E, μ). Then for all $\alpha, \beta > 0$ we have*

1. *$|g| \leqslant |f|$ μ -a.e. implies that $D_g \leqslant D_f$;*
2. *$D_{cf}(\alpha) = D_f(\alpha/|c|)$, for all $c \in \mathbb{R} \setminus \{0\}$;*
3. *$D_{f+g}(\alpha+\beta) \leqslant D_f(\alpha) + D_g(\beta)$;*
4. *$D_{fg}(\alpha\beta) \leqslant D_f(\alpha) + D_g(\beta)$.*

Proposition 1.2.4. *[10] Let (X, μ) be a σ -finite measure space. Then for f in $L^p(X, \mu)$, $0 < p < \infty$, we have*

$$\|f\|_{L^p}^p = p \int_0^\infty \lambda^{p-1} D_f(\lambda) d\lambda. \tag{1.5}$$

Remark 1.2.5. For any increasing continuously differentiable function φ on $[0, \infty)$ with $\varphi(0) = 0$ and every measurable function f on E with $\varphi(|f|)$ integrable on X, we have

$$\int_X \varphi(|f|) d\mu = \int_0^\infty \varphi'(\lambda) D_f(\lambda) d\lambda. \tag{1.6}$$

Proof. we have

$$\begin{aligned} p\int_0^\infty \lambda^{p-1}D_f(\lambda)d\lambda &= p\int_0^\infty \lambda^{p-1}\int_X \chi_{\{x:|f(x)|>\lambda\}}d\mu(x)d\lambda \\ &= \int_X\int_0^{|f(x)|} p\lambda^{p-1}d\lambda d\mu(x) \\ &= \int_X |f(x)|^p d\mu(x) \\ &= \|f\|_{L^p}^p \end{aligned}$$

□

Definition 1.2.6. [10] For $0 < p < \infty$, the space weak $L^p(X,\mu)$ is defined as the set of all μ-measurable functions f such that

$$\| f \|_{L^{p.\infty}} = inf\left\{C > 0 : D_f(\lambda) \leqslant \frac{C^p}{\lambda^p}, \lambda > 0\right\} \tag{1.7}$$

$$= sup\left\{\gamma D_f(\gamma)^{\frac{1}{p}} : \gamma > 0\right\} \tag{1.8}$$

is finite.

Remark 1.2.7. Note that the assumptions 1.8 and 1.7 are in fact equal according to Lemma 1.2.1.

Remark 1.2.8. The weak L^p spaces are denoted by $L^{p,\infty}(X,\mu)$.
Two functions in $L^{p,\infty}(X,\mu)$ are considered equal if they are equal μ-a.e. The notation $L^{p,\infty}(\mathbb{R}^n)$ is reserved for $L^{p,\infty}(\mathbb{R}^n, |.|)$.

Using Proposition 1.2.3 (2), we can easily show that

$$\| kf \|_{L^{p,\infty}} = |k| \| f \|_{L^{p,\infty}} \tag{1.9}$$

for any complex constant k. The analogue of (1.2.3) is

$$\| f+g \|_{L^{p,\infty}} \leqslant c_p(\| f \|_{L^{p,\infty}} + \| g \|_{L^{p,\infty}}), \tag{1.10}$$

where $c_p = max(2, 2^{\frac{1}{p}})$, a fact that follows from Proposition (1.2.3) (3), taking both α and β equal to $\frac{\alpha}{2}$. We also have that

$$\| f \|_{L^{p,\infty}(X,\mu)} = 0 \Rightarrow f = 0 \quad \mu - a.e. \tag{1.11}$$

In view of 1.9, 1.10, and 1.11, $L^{p,\infty}$ is a quasi-normed linear space for $0 < p < \infty$. The weak L^p spaces are larger than the usual L^p spaces. We have the following:

Proposition 1.2.9. *For any $0 < p < \infty$ and any f in $L^p(X,\mu)$ we have $\| f \|_{L^{p,\infty}} \leqslant \| f \|_{L^p}$. Than the embedding $L^p(X,\mu) \subsetneqq L^{p,\infty}(X,\mu)$ holds.*

Proof. This is just a trivial consequence of Chebyshev's inequality:

$$\lambda^p D_f(\lambda) = \int_{\{x:|f(x)|>\lambda\}} \lambda^p d\mu(x) \leqslant \int_{\{x:|f(x)|>\lambda\}} |f(x)|^p d\mu(x) \leqslant \int_X |f(x)|^p d\mu(x) = \| f \|_{L^p}^p .$$

□

Exemple 1.1. We put

$$f(x) = x^{-\frac{1}{p}} \quad then \quad \int_{-\infty}^{+\infty} \left| \frac{1}{x^{\frac{1}{p}}} \right|^p dx = \int_{-\infty}^{+\infty} \frac{1}{|x|} dx.$$

Is divergent but for $\forall \alpha > 0$;

$$\begin{aligned} \alpha^p D_f(\alpha) &= \alpha^p \mu \left\{ x : \left| x^{-\frac{1}{p}} \right| > \alpha \right\} \\ &= \alpha^p \mu \left\{ x : |x| < \frac{1}{\alpha^p} \right\} \\ &= \alpha^p . \frac{2}{\alpha^p} = 2. \end{aligned}$$

1.2.2 Convergence in measure

We present some convergence notions. The following notion is important in probability theory.

Definition 1.2.10. Let $f, f_n; n = 1, 2, ...$ be measurable functions on the measure space (E, μ). The sequence f_n is said to converge in measure to f if for all $\varepsilon > 0$ there exists an $n_0 \in \mathbb{Z}^+$ such that

$$n > n_0 \Longrightarrow \mu(\{x \in E : |f_n(x) - f(x)| > \varepsilon\}) < \varepsilon. \tag{1.12}$$

Remark 1.2.11. The preceding definition is equivalent to the following statement, for all $\varepsilon > 0$:

$$\lim_{n\to\infty} \mu(\{x \in E : |f_n(x) - f(x)| > \varepsilon\}) = 0. \tag{1.13}$$

Clearly 1.13 implies 1.12. To see the converse given $\varepsilon > 0$, pick $0 < \delta < \varepsilon$ and apply 1.12 for this δ.

There exists an $n_0 \in \mathbb{Z}^+$ such that :

$$\mu(\{x \in E : |f_n(x) - f(x)| > \delta\}) < \delta$$

holds for $n > n_0$. Since

$$\mu(\{x \in E : |f_n(x) - f(x)| > \varepsilon\}) \leqslant \mu(\{x \in E : |f_n(x) - f(x)| > \delta\})$$

we short that

$$\mu(\{x \in E : |f_n(x) - f(x)| > \varepsilon\}) < \delta$$

for all $n > n_0$. Let $n \longrightarrow \infty$ to deduce that

$$\limsup_{n \to \infty} \mu(\{x \in E : |f_n(x) - f(x)| > \varepsilon\}) \leqslant \delta. \tag{1.14}$$

Due to 1.14 holds for all $0 < \delta < \varepsilon$, 1.13 follows by letting $\delta \longrightarrow 0$. Convergence in measure is a weaker notion than convergence in either L^p or $L^{p,\infty}$, $0 < p \leqslant \infty$, as the following proposition indicates:

Proposition 1.2.12. *Let $0 < p \leqslant \infty$ and f_n , f be in $L^{p,\infty}(E, \mu)$.*

(1) If f_n, f are in L^p and $f_n \longrightarrow f$ in L^p, then $f_n \longrightarrow f$ in $L^{p,\infty}$.

(2) If $f_n \longrightarrow f$ in $L^{p,\infty}$, then f_n converges to f in measure.

Proof. Fix $0 < p < \infty$. Proposition1.2.9 gives that for all $\varepsilon > 0$ we have

$$\mu(\{x \in E : |f_n(x) - f(x)| > \varepsilon\}) \leqslant \frac{1}{\varepsilon^p} \int_X |f_n - f|^p d\mu.$$

This shows that convergence in L^p implies convergence in weak L^p. The case $p = \infty$ is tautological. Given $\varepsilon > 0$ find an n_0 such that for $n > n_0$, we have

$$\| f_n - f \|_{L^{p,\infty}} = \sup \alpha \mu(\{x \in X : |f_n(x) - f(x)| > \alpha\})^{\frac{1}{p}} < \varepsilon^{\frac{1}{p}+1}.$$

Taking $\alpha = \varepsilon$, we conclude that convergence in $L^{p,\infty}$ implies convergence in measure. □

Theorem 1.2.13. *Let f_n and f be complex-valued measurable functions on a measure space (E, μ) and suppose that f_n converges to f in measure. Then some subse-quence of f_n converges to f μ-a.e.*

Definition 1.2.14. We say that a sequence of measurable functions f_n on the measure space (E, μ) is Cauchy in measure if for every $\varepsilon > 0$, there exists an $n_0 \in \mathbb{Z}^+$ such that for $n, m > n_0$ we have

$$\mu(\{x \in E : |f_m(x) - f_n(x)| > \varepsilon\}) < \varepsilon.$$

Theorem 1.2.15. *Let (E, μ) be a measure space and let f_n be a complex-valued sequence on X that is Cauchy in measure. Then some subsequence of f_n converges μ -a.e.*

Proof. The proof is very similar to that of Theorem1.2.13.Forall $k = 1, 2, ...$ choose n_k inductively such that

$$\mu(\{x \in E : |f_{n_k}(x) - f(x)| > 2^{-k}\}) < 2^{-k} \tag{1.15}$$

We define the sets

$$A_k = \{x \in E : |f_{n_k}(x) - f(x)| > 2^{-k}\}. \text{ For } n_1 < n_2 < ... < n_k < ...$$

As shown in the proof of Theorem 1.2.13 1.15 implies that

$$\mu(\cap_{m=1}^{\infty} \cup_{k=m}^{\infty} A_k) = 0 \tag{1.16}$$

For $x \notin \cup_{k=m}^{\infty} A_k$ and $i \geq j \geq j_0 \geq m$ (and j_0 large enough) we have $|f_{n_i}(x) - f_{n_j}(x)| \leq \sum_{l=j}^{i-1} |f_{n_l}(x) - f_{n_{l+1}}(x)| \leq \sum_{l=j}^{i-1} 2^{-l} \leq 2^{1-j} \leq 2^{1-j_0}$. This implies that the sequence $\{f_{n_i}(x)\}_i$ is Cauchy for every x in the set $(\bigcup_{k=m}^{\infty} A_k)^c$ and therefore converges for all such x. We define a function

$$f(x) = \begin{cases} \lim_{j \longrightarrow \infty} f_{n_j}(x) & x \notin \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} A_k; \\ 0 & x \in \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} A_k \end{cases}$$

Then $f_{n_j} \longrightarrow f$ almost everywhere. □

A First glimpse at interpolation

Remark 1.2.16. It is a useful fact that if a function f is in $L^p(E, \mu)$ and in $L^q(E, \mu)$, then it also lies in $L^r(E, \mu)$ for all $p < r < q$.

The usefulness of the spaces $L^{p,\infty}$ can be seen from the following sharpening of this statement:

Proposition 1.2.17. *Let $0 < p < q \leqslant \infty$ and let f in $L^{p,\infty}(E, \mu) \cap L^{q,\infty}(E, \mu)$, where X is a σ-finite measure space. Then f is in $L^r(X, \mu)$ for all $p < r < q$ and*

$$\| f \|_{L^r} \leqslant \left(\frac{r}{r-p} + \frac{r}{q-r} \right)^{\frac{1}{r}} \| f \|_{L^{p,\infty}}^{\frac{\frac{1}{r} - \frac{1}{q}}{\frac{1}{p} - \frac{1}{q}}} \| f \|_{L^{q,\infty}}^{\frac{\frac{1}{p} - \frac{1}{r}}{\frac{1}{p} - \frac{1}{q}}} \tag{1.17}$$

with the interpretation that $1/\infty = 0$.

Definition 1.2.18. [10] For $K \subset \mathbb{R}^n$, $0 < p < \infty$, the space $L^p_{loc}(\mathbb{R}^n, |.|)$ or simply $L^p_{loc}(\mathbb{R}^n)$ is the set of all Lebesgue-measurable functions f on $(\mathbb{R}^n)$ that satisfy

$$\int_K |f(x)|^p \, dx < \infty. \tag{1.18}$$

Functions that satisfy 1.18 with $p = 1$ are called locally integrable functions on $\mathbb{R}^n$. The union of all $L^p(\mathbb{R}^n)$ spaces for $1 \le p \le \infty$ is contained in $L^1_{loc}(\mathbb{R}^n)$

Proposition 1.2.19. *More generally, for $0 < p < q < \infty$ we have the following:*

$$L^q(\mathbb{R}^n) \subsetneqq L^q_{loc}(\mathbb{R}^n) \subsetneqq L^p_{loc}(\mathbb{R}^n)\,.$$

Remark 1.2.20. Functions in $L^p(\mathbb{R}^n)$ for $0 < p < 1$ may not be locally integrable.

Exemple 1.2. Take $f(x) = |x|^{-n-\alpha}\chi_{|x|\le 1}$ hich is in $L^p(\mathbb{R}^n)$ when $\alpha > 0$ and $p < n/(n+\alpha)$, and observe that f is not integrable over any open set in $\mathbb{R}^n$ containing the origin.

Theorem 1.2.21. *Let $\{a_j\}_{j\in\mathbb{N}}$ be a sequence of positives reals.*

1. $\left(\sum_{j=1}^{\infty} a_j\right)^{\theta} \le \sum_{j=1}^{\infty} a_j^{\theta}$*, for any* $0 \le \theta \le 1$.
2. $\sum_{j=1}^{\infty} a_j^{\theta} \le \left(\sum_{j=1}^{\infty} a_j\right)^{\theta}$*, for any* $1 \le \theta < \infty$.
3. $\left(\sum_{j=1}^{N} a_j\right)^{\theta} \le N^{\theta-1}\sum_{j=1}^{N} a_j^{\theta}$*, when* $1 \le \theta < \infty$.
4. $\left(\sum_{j=1}^{N} a_j^{\theta}\right) \le N^{1-\theta}\left(\sum_{j=1}^{N} a_j\right)^{\theta}$*, when* $0 \le \theta \le 1$.

CHAPTER 2

NONSTANDARD LEBESGUE SPACES

This chapter is devoted to study of variable exponent space and Weak Lebesgue space with variable exponent, Grand Lebesgue Spaces.

2.1 Variable exponent Lebesgue spaces

In this section we define the so-called variable exponent Lebesgue spaces $L_{p(.)}(\Omega)$, introduce an appropriate norm and study some fundamental properties of the space, for simplicity we will work only on a measurable subset Ω of $\mathbb{R}^n$ with the Lebesgue measure. By $P(\Omega)$ we denote the family of all measurable functions $p : \Omega \longrightarrow [1, \infty]$. For $p \in P(\Omega)$ we define the following sets
$\Omega_1(p) := \Omega_1 = \{x \in \Omega : p(x) = 1\}$,
$\Omega_\infty(p) := \Omega_\infty = \{x \in \Omega : p(x) = \infty\}$,
$\Omega_+(p) := \Omega_* = \{x \in \Omega : 1 < p(x) < \infty\}$.

Definition 2.1.1. The variable exponent Lebesgue space are note by $L_{p(.)}(\Omega)$, as the set of all measurable functions $f : \Omega \longrightarrow \mathbb{R}$ such that

$$\rho_{p(.)}(f) := \int_{\Omega\setminus\Omega_\infty} |f(x)|^{p(x)} dx < \infty \tag{2.1}$$

and

$$ess \max_{x\in\Omega_\infty} |f(x)| < \infty.$$

Where the measurable function $p : \Omega \longrightarrow (0, \infty]$ is called variable exponent. The functional $\rho_{p(.)}$ is known as a modular.

Remark 2.1.2. If $m(\Omega_*) > 0$, and $p_- = p_+ = 1$ if $m(\Omega_*) = 0$. For $p \in P(\Omega)$ we define the dual exponent or the conjugate exponent has

$$p'(x) = \begin{cases} \infty, & x \in \Omega_1 \\ \dfrac{p(x)}{p(x)-1}, & x \in \Omega_*, \\ 1, & x \in \Omega_\infty \end{cases}$$

which implies the pointwise inequality

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1.$$

If a measurable function $p : \mathbb{R}^n \longrightarrow [1, \infty)$ satisfies

$$1 < p_-, p_+ < \infty, \tag{2.2}$$

then the conjugate function

$$p'(x) := \frac{p(x)}{p(x) - 1}$$

is well defined and moreover it satisfies 2.2. Working with the definition of p_-, p_+ and the conjugate exponent, we have the following relations

1. $(p'(.))_+ = (p_-)'$;

2. $(p'(.))_- = (p_+)'$.

Lemma 2.1.3. *The space $L_{p(.)}(\Omega)$ is linear if and only if $p_+ < \infty$.*

Proof. Suppose that $p_+ = \infty$. We will show that there exists a function $f_0 \in L_{p(.)}(\Omega)$ such that $2f_0 \notin L_{p(.)}(\Omega)$. Let $A_m = x \in \Omega \setminus \Omega_\infty : m - 1 \leq p(x) \leq m$. Since $p_+ = \infty$, there exists a sequence $m_k \longrightarrow \infty, k \in \mathbb{N}$ such that $m(A_{m_k}) > 0$. We now construct a step function f_0 ; i.e, $f_0(x) = c_m$ for $x \in A_m$, where c_m is given by the relation

$$\int_{A_m} c_m^{p(x)} dx = m^{-2},$$

this defines c_m univocally if $m(A_m) \neq 0$. We then have

$$\rho_{p(.)}(f_0) = \sum_{m=1}^{\infty} \int_{A_m} c_m^{p(x)} dx = \sum_{m=1}^{\infty} m^{-2} < \infty.$$

which entails that $f_0 \in L_{p(.)}(\Omega)$. On the other hand,

$$\begin{aligned} \rho_{p(.)}(2f_0) &\geq \sum_{k=1}^{\infty} \int_{A_{m_k}} (2c_{m_k})^{p(x)} dx \\ &\geq \sum_{k=1}^{\infty} 2^{m_k - 1} \int_{A_{m_k}} c_{m_k}^{p(x)} dx \\ &= \sum_{k=1}^{\infty} 2^{m_k - 1} m_k^{-2} = \infty, \end{aligned}$$

which means that $2f_0 \notin L_{p(.)}(\Omega)$.
Let $p_+ < \infty$. We have

$$\rho_{p(.)}(cf) \leq \max\{|c|^{p_+}, 1\}\, \rho_{p(.)}(f)$$

and

$$\rho_{p(.)}(f+g) \leq 2^{p_+}[\rho_{p(.)}(f) + \rho_{p(.)}(g)]$$

for all function f and g in $L_{p(.)}(\Omega)$.
The next result tells us that the definition of the variable Lebesgue space is not void, in the sense that it always contains the set of step functions, whenever $p_+ < \infty$. □

2.1.1 Luxemburg-Nakano type norm

We already know from Lemma 2.1.3 that the space is linear if and only if p + . We now want to study a norm in the Lebesgue space with variable exponents.

Lemma 2.1.4. *Let $f \in L_{p(.)}(\Omega), 0 \leq p(x) \leq \infty$. The function*

$$F(\lambda) = \rho_{p(.)}(\frac{f}{\lambda}), \lambda > 0 \tag{2.3}$$

Take finite values for all $\lambda \geq 1$. Moreover, this function is continuous, decreasing, and $\lim_{\lambda \longrightarrow \infty} F(\lambda) = 0$. If $p_+ < \infty$, the same is true for all $\lambda > 0$.

Proof. By definition we have that $F(1) < \infty$. It is clear that the function 2.3 is decreasing, which immediately entails that $F(\lambda) < \infty$ for all $\lambda \geq 1$. The continuity follows from

$$\begin{aligned} \lim_{\lambda \to \lambda_0} |F(\lambda) - F(\lambda_0)| &\leq \lim_{\lambda \to \lambda_0} \int_{\Omega \setminus \Omega_\infty} |f(x)|^{p(x)} \left| \lambda^{-p(x)} - \lambda_0^{-p(x)} \right| dx \\ &\leq \int_{\Omega \setminus \Omega_\infty} \lim_{\lambda \longrightarrow \lambda_0} |f(x)|^{p(x)} \left| \lambda^{-p(x)} - \lambda_0^{-p(x)} \right| dx \end{aligned} \tag{2.4}$$

where we used the Lebesgue dominated convergence theorem since $\lambda^{p(x)} \leq 1$ for $\lambda \geq 1$. Using again the Lebesgue dominated convergence theorem we obtain $\lim_{\lambda \longrightarrow \infty} F(\lambda) = 0$.
When $p_+ < \infty$, for $\lambda < 1$ we have that $F(\lambda) \leq F(1)\lambda^{-p_+} < \infty$. The continuity follow, once again, from 2.4 since $\lambda^{-p(x)} \leq c\lambda_0^{-p_+}$ for λ near λ_0 . We now introduce a norm in the space $L_{p(.)}(\Omega)$ □

Theorem 2.1.5. *Let $0 \leq p(x) \leq \infty$, for any $f \in L_{p(.)}(\Omega)$ the functional*

$$\| f \|_{(p)} = \inf \left\{ \lambda > 0 : \int_{\Omega \setminus \Omega_\infty} \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\} \tag{2.5}$$

takes finite values and

$$\rho_{p(.)} \left(\frac{f}{\| f \|_{(p)}} \right) \leq 1, \quad \| f \|_{(p)} \neq 0. \tag{2.6}$$

If the exponent satisfies $p_+ < \infty$ or $\| f \|_{(p)} \geq 1$, then

$$\rho_{p(.)} \left(\frac{f}{\| f \|_{(p)}} \right) = 1, \quad \| f \|_{(p)} \neq 0. \tag{2.7}$$

Moreover, if $1 \leq p(x) \leq p_+ < \infty, x \in \Omega \setminus \Omega_\infty$, we have that

$$\| f \|_{L_{p(.)}(\Omega)} = \| f \|_{(p)} + ess \sup_{x \in \Omega_\infty} |f(x)| \tag{2.8}$$

is a norm in the space $L_{p(.)}(\Omega)$.

Proof. By Lemma 2.3we have that $\|f\|_{(p)}$ is finite whenever $f \in L_{p(.)}(\Omega)$ and2.6- 2.7 are consequences of the definition given in 2.5 and Lemma 2.3. To show that 2.8 is a norm, it suffices to show the triangle inequality for $\|f\|_{(p)}$, which follows from the inequality

$$|\lambda y_1 + (1-\lambda) y_2|^p \leq \lambda |y_1|^p + (1-\lambda) |y_2|^p , \tag{2.9}$$

for $0 \leq \lambda \leq 1$ and $p \geq 1$, since $t \mapsto t^p$ is a convex function.
We now obtain upper and lower bounds for the modular $\rho_{p(.)}$ via the functional $\|.\|_{(p)}$. □

Corollary 2.1.6. *The functional 2.5 and the modular $\rho_{p(.)}$ are related by the following estimates*

$$\left(\frac{\|f\|_{(p)}}{\lambda} \right)^{p_+} \leq \rho_{p(.)} \left(\frac{f}{\lambda} \right) \leq \left(\frac{\|f\|_{(p)}}{\lambda} \right)^{p_-} , \lambda \geq \|f\|_{(p)} , \tag{2.10}$$

$$\left(\frac{\|f\|_{(p)}}{\lambda} \right)^{p_-} \leq \rho_{p(.)} \left(\frac{f}{\lambda} \right) \leq \left(\frac{\|f\|_{(p)}}{\lambda} \right)^{p_+} , 0 < \lambda \leq \|f\|_{(p)} . \tag{2.11}$$

where the extreme cases $p_- = 0$ or $p_+ = \infty$ are admitted.

Proof. Let us rewrite 2.10 and 2.11 as

$$\lambda^{p_+} \leq \rho_{p(.)} \left(\frac{\lambda}{\|f\|_{(p)}} f \right) \leq \lambda^{p_-}, 0 < \lambda \leq 1, \tag{2.12}$$

and

$$\lambda^{p_-} \leq \rho_{p(.)} \left(\frac{\lambda}{\|f\|_{(p)}} f \right) \leq \lambda^{p_+}, \lambda \geq 1. \tag{2.13}$$

We now have that 2.12 and 2.13 are a consequence of 2.7 if $p_+ < \infty$ or $p_+ = \infty$ with $\|f\|_{(p)} \geq 1$. If $p_+ = \infty$ and $\|f\|_{(p)} \leq 1$, the right-hand side of the inequality in 2.12 is a consequence of 2.6, and the left-hand side of 2.13 holds since
$\|g\|_{(p)} = \lambda \geq 1$ for $g(x) = \lambda f(x)/\|f\|_{(p)}$. □

Corollary 2.1.7. *Let p be a measurable function, $0 \leq p_- \leq p(x) \leq p_+ < \infty, x \in \Omega \setminus \Omega_\infty$, we have the following estimates*

$$\|f\|_{(p)}^{p_+} \leq \rho_{p(.)}(f) \leq \|f\|_{(p)}^{p_-}, \|f\|_{(p)} \leq 1, \tag{2.14}$$

$$\|f\|_{(p)}^{p_-} \leq \rho_{p(.)}(f) \leq \|f\|_{(p)}^{p_+}, \|f\|_{(p)} \geq 1. \tag{2.15}$$

Remark 2.1.8. Corollary 2.1.7 states that in questions related to convergence, $\rho_{p(.)}(.)$ and $\|.\|_{(p)}$ are equivalent. This observation is quite useful due to the fact that the norm is given by a supremum and calculating explicitly the norm can be impossible, except in trivial cases. With these estimates at hand, we can get an upper and lower bound for the norm of an indicator function of a set.

Corollary 2.1.9. *Let E be a measurable set in $\Omega \setminus \Omega_\infty$. If $0 < p_- \leq p_+ < \infty$ we have the estimate*

$$m(E)^{1/p_-} \leq \|\chi_E\|_{(p)} \leq m(E)^{1/p_+},$$

when $m(E) \leq 1$. In the case $m(E) \geq 1$, the signs of the inequality are reversed. As a particular case, we have that $\|\chi_E\|_{(p)} = 1$ is equivalent to $m(E) = 1$.

Remark 2.1.10. The space $L_{p(.)}(\Omega)$ is ideal; i.e. , it is a complete space and the inequality
$|f(x)| \leq |g(x)|, g \in L_{p(.)}(\Omega)$ implies that $\|f\|_{L_{p(.)}(\Omega)} \leq \|g\|_{L_{p(.)}(\Omega)}$. Let $1 \leq p(x) \leq \infty$ be such that $p_+ < \infty$. The semi-norm $\|f\|_{(p)}$ can be represented in the form

$$\|f\|_{(p)} = \int_{\Omega\setminus\Omega_\infty} \phi(x) f(x) dx, \phi \in L_{p'(.)}(\Omega) \tag{2.16}$$

where $\phi(x) = \left|\frac{f(x)}{\|f\|_{(p)}}\right|^{p(x)-1} \frac{f(x)}{|f(x)|}, x \notin \Omega_\infty$ and $\|\phi\|_{(p')} \leq 1$. In reality 2.16 is simply 2.7, the inequality $\|\phi\|_{(p')} \leq 1$ is immediate.

Lemma 2.1.11. *Let $0 < p_- \leq p_+ \leq \infty$. If*

$$\rho_{p(.)}\left(\frac{f}{a}\right) \leq b, a > 0, b > 0, \tag{2.17}$$

then $\|f\|_{(p)} \leq ab^v$ with $v = 1/p_-$ if $b \geq 1$ and $v = 1/p_+$ if $b \leq 1$.

Proof. By 2.17 we have the inequality $\rho_{p(.)}(f/(ab^v)) \leq 1$, and now by the definition 2.5 we get that $\|f\|_{(p)} \leq ab^v$. The next result generalizes the property

$$\|f^\gamma\|_p = \|f\|_{\gamma p}^\gamma$$

for the variable setting. □

Lemma 2.1.12. *Let* $0 < \gamma(x) \leq p(x) \leq p_+ < \infty, x \in \Omega \backslash \Omega_\infty$. *Then*

$$\|f\|_{(p)}^{\gamma_-} \leq \|f^\gamma\|_{\left(\frac{p}{\gamma}\right)} \leq \|f\|_{(p)}^{\gamma_+}, \|f\|_{(p)} \geq 1, \tag{2.18}$$

$$\|f\|_{(p)}^{\gamma_+} \leq \|f^\gamma\|_{\left(\frac{p}{\gamma}\right)} \leq \|f\|_{(p)}^{\gamma_-}, \|f\|_{(p)} \leq 1. \tag{2.19}$$

where $f^\gamma = |f(x)|^{\gamma(x)}$. *If p and* γ *are continuous functions, there exists a point* $x_0 \in \Omega \backslash \Omega_\infty$ *such that*

$$\|f^\gamma\|_{\left(\frac{p}{\gamma}\right)} = \|f\|_{(p)}^{\gamma(x_0)} \tag{2.20}$$

Corollary 2.1.13. *Let* $0 \leq p_- \leq p(x) \leq p_+ < \infty, x \in \Omega \setminus \Omega_\infty$. *If p is a continuous function in* $\Omega \setminus \Omega_\infty$, *there exists a point* $x_0 \in \Omega \setminus \Omega_\infty$ *(which depends on f) such that*

$$\|f\|_{(p)} = \left\{ \int_{\Omega \backslash \Omega_\infty} |f|^{p(x)}\, dx \right\}^{\frac{1}{p(x_0)}}. \tag{2.21}$$

Proof. Taking $\gamma(x) = p(x)$ in the equality 2.20 we get 2.21. □

Definition 2.1.14. We define the sum space $L_p(\Omega) + L_q(\Omega)$ as
$L_p(\Omega) + L_q(\Omega) := \{f = g + h : g \in L_p(\Omega), h \in L_q(\Omega)\}$.
which is a Banach space with the norm

$$\|f\|_{L_p(\Omega)+L_q(\Omega)} = \inf_{f=g+h} \left\{ \|g\|_{L_p(\Omega)} + \|h\|_{L_q(\Omega)} \right\}$$

The intersection space $L_p(\Omega) \cap L_q(\Omega)$ is defined as

$$\|f\|_{L_p(\Omega)\cap L_q(\Omega)} = \max \left\{ \|f\|_{L_p(\Omega)}, \|f\|_{L_q(\Omega)} \right\}$$

which is a Banach space.

We now show that the variable exponent Lebesgue space is embedded between the sum and intersection spaces of the spaces L_{p_-} and L_{p_+}.

Lemma 2.1.15. *[5] Let* $1 \leq p_- \leq p(x) \leq p_+ \leq \infty, x \in \Omega, m(\Omega_\infty) = 0$. *Then*

$$L_{p(.)}(\Omega) \subseteq L_{p_-}(\Omega) + L_{p_+}(\Omega). \tag{2.22}$$

Moreover,

$$\|f\|_{L_{p(.)}(\Omega)} \leq \max\left\{\|f\|_{p_-}, \|f\|_{p_+}\right\}$$

The result follows from the splitting $f(x) = f_1(x) + f_2(x)$ *where* $f_1(x) = f(x)$ *if* $|f(x)| \leq 1$ *and* $f_1(x) = 0$ *otherwise. The Lemma 2.1.15 admits the following natural generalization.*

Lemma 2.1.16. *[5] Let* $1 \leq p_1(x) \leq p(x) \leq p_2(x) \leq \infty$ *and* $m(\Omega_\infty(p_2)) = 0$. *Then*

$$L_{p(.)}(\Omega) \subseteq L_{p_1(.)}(\Omega) + L_{p_2(.)}(\Omega).$$

In the previous lemmas, splitting the function in an appropriate way we were able to obtain embedding results. We now want to obtain embedding results where the splitting is applied to the underlying set Ω.

Lemma 2.1.17. *Let* $\Omega = \Omega_1 \bigcup \Omega_2$ *and let* p *be a function in* Ω, $p(x) \geq 1$ *with* $p_+ < \infty$. *Then*

$$\max\left\{\|f\|_{L_{p(.)}(\Omega_1)}, \|f\|_{L_{p(.)}(\Omega_2)}\right\} \leq \|f\|_{L_{p(.)}(\Omega)} \leq \|f\|_{L_{p(.)}(\Omega_1)} + \|f\|_{L_{p(.)}(\Omega_2)} \tag{2.23}$$

for all functions $f \in L_{p(.)}(\Omega)$.

Proof. Let us take $m(\Omega_\infty) = 0$ for simplicity. Without loss of generality, let
$a = \|f\|_{L_{p(.)}(\Omega_1)}, b = \|f\|_{L_{p(.)}(\Omega_2)}$ with $a \geq b$. We have

$$\int_\Omega \left|\frac{f(x)}{\max\{a,b\}}\right|^{p(x)} dx \geq \int_{\Omega_1} \left|\frac{f(x)}{a}\right|^{p(x)} dx = 1.$$

Therefore $\|f\|_{L_{p(.)}(\Omega)} \geq \max\{a,b\}$. To show the right-hand side inequality, we write

$$\frac{f(x)}{a+b} = \frac{a}{a+b}\frac{X_1(x)f(x)}{a} + \frac{b}{a+b}\frac{X_2(x)f(x)}{b}$$

where $X_i(x)$ are the characteristic functions of the sets $\Omega_i, i = 1, 2$. Using 2.9 we get

$$\int_\Omega \left|\frac{f(x)}{a+b}\right|^{p(x)} dx \leq 1,$$

which shows the right-hand side inequality in 2.23. For the case $m(\Omega_\infty) > 0$, the arguments are similar if we take into account the fact that the lemma was already proved for the case $\Omega \setminus \Omega_\infty = \Omega_1^* \bigcup \Omega_2^*$ where $\Omega_i^* = \Omega_i \setminus \Omega_\infty, i = 1, 2$. □

Another version of the Luxemburg-Nakano norm

The Luxemburg-Nakano type norm can be study directly with respect to all the set Ω in the following form

$$\|f\|_p^1 = \inf \left\{ \lambda > 0 : \rho_{p(.)} \left(\frac{f}{\lambda} \right) + ess\sup_{x \in \Omega_\infty} \left| \frac{f(x)}{\lambda} \right| \leq 1 \right\}, \tag{2.24}$$

which is well defined for $f \in L_{p(.)}(\Omega)$ and any variable exponent p with $0 \leq p(x) \leq \infty$. It is a norm if $1 \leq p(x) \leq \infty$, which can be shown in the same way as Theorem 2.1.5. In an analogous way to 2.7 it is possible to show that

$$\int_{\Omega \setminus \Omega_\infty} \left| \frac{f(x)}{\|f\|_p^1} \right|^{p(x)} dx + \frac{\|f\|_{L_\infty(\Omega_\infty)}}{\|f\|_p^1} = 1. \tag{2.25}$$

if $p_+ < \infty$ or $p_+ = \infty$, but $\|f\|_p^1 \geq 1$.

Theorem 2.1.18. *The norms 2.8 and 2.25 are equivalent, i.e.*

$$\frac{1}{2} \|f\|_{L_{p(.)}(\Omega)} \leq \|f\|_p^1 \leq \|f\|_{L_{p(.)}(\Omega)} \tag{2.26}$$

where $f \in L_{p(.)}(\Omega)$, $1 \leq p(x) \leq \infty, p_+ < \infty$.

Proof. The right-hand side inequality in 2.26 is equivalent to

$$\inf \{\lambda > 0 : F(\lambda) + c/\lambda \leq 1\} \leq \lambda_0 + c,$$

where $F(\lambda)$ is defined by 2.3 and

$$c = \|f\|_{L_\infty(\Omega_\infty)}, \lambda_0 = \|f\|_{(p)}.$$

From the above, it is sufficient to show that $F(\lambda_0 + c) + \dfrac{c}{\lambda_0 + c} \leq 1$, or in other words: $F(\lambda_0 + c) \leq \dfrac{\lambda_0}{\lambda_0 + c}$. Since $F(\lambda_0 + c) = \rho_{p(.)} \left(\dfrac{f}{\|f\|_{(p)} + c} \right)$, by 2.10 we obtain that $F(\lambda_0 + c) \leq \dfrac{\|f\|_{(p)}}{\|f\|_{(p)} + c} = \dfrac{\lambda_0}{\lambda_0 + c}$.
The left-hand side in 2.26 is a consequence of the inequalities

$$\inf \left\{ \lambda > 0 : F(\lambda) + \frac{c}{\lambda} \leq 1 \right\} \geq \inf \{\lambda > 0 : F(\lambda) \leq 1\} = \lambda_0,$$

and

$$\inf \left\{ \lambda > 0 : F(\lambda) + \frac{c}{\lambda} \leq 1 \right\} \geq \inf \left\{ \lambda > 0 : \frac{c}{\lambda} \leq 1 \right\} = c$$

since the left-hand side inequality is not less that $\dfrac{\lambda_0 + c}{2}$. □

2.1.2 Hölder inequality

We now proceed to get Hölder's inequality and after that we will get the Minkowski inequality using F. Riesz construction via Hölder's inequality.

Theorem 2.1.19 (Hölder's inequality). *Let $f \in L_{p(.)}(\Omega), \varphi \in L_{p'(.)}(\Omega)$ and $1 \le p(x) \le \infty$. Then*

$$\int_{\Omega} |f(x)\varphi(x)|\, dx \le k \, \|f\|_{L_{p(.)}(\Omega)} \, \|\varphi\|_{L_{p'(.)}(\Omega)} \tag{2.27}$$

with $k = \frac{1}{p_-} + \frac{1}{(p')_-} = \sup \frac{1}{p(x)} + \sup \frac{1}{p'(x)}$

Proof. Let us note that, under the conditions of the theorem, the functionals $\|f\|_{L_{p(.)}(\Omega)}$ and $\|\varphi\|_{p'(.)}$ are not necessarily norms and the classes $L_{p(.)}$ and $L_{p'(.)}$ are not necessarily linear, but they always exist by Theorem 2.1.5. To show 2.27, we use the Young inequality

$$ab \le \frac{a^p}{p} + \frac{b^{p'}}{p'} \tag{2.28}$$

with $a > 0, b > 0, \frac{1}{p} + \frac{1}{p'} = 1$ and $1 < p < \infty$. The inequality (2.28) is valid for $p = 1$ in the form $ab \le \frac{a^p}{p}$ if $b \le 1$ and for $p = \infty$ in the form $ab \le \frac{b^{p'}}{p'}$ if $a \le 1$. Therefore,

$$\left| \frac{f(x)\varphi(x)}{\|f\|_{p(.)} \|\varphi\|_{p'(.)}} \right| \le \frac{1}{p(x)} \left| \frac{f(x)}{\|f\|_{p(.)}} \right|^{p(x)} + \frac{1}{p'(x)} \left| \frac{\varphi(x)}{\|\varphi\|_{p'(.)}} \right|^{p'(x)}$$

where $x \in \Omega \setminus \Omega_\infty(p) \bigcup \Omega_\infty(p')$, meanwhile for $x \in \Omega_\infty(p)$ and $x \in \Omega_\infty(p')$ we have to omit the first and second terms respectively in the right-hand side, since $\left| \frac{f(x)}{\|f\|_{p(.)}} \right| \le 1$ for $x \in \Omega_\infty(p)$ and $\left| \frac{\varphi(x)}{\|\varphi\|_{p'(.)}} \right| \le 1$ for $x \in \Omega_\infty(p')$. Integrating over Ω and estimating p and p', we arrive at 2.27. In the constant exponent case $p(x) \equiv p$, the Hölder's inequality has a generalization of the form

$$\|uv\|_r \le \|u\|_p \, \|v\|_q \, , \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r},$$

which is an immediate consequence of the Hölder's inequality and the relation

$$\| |u|^r \|_p = \|u\|_{pr}^r \, . \tag{2.29}$$

In the variable exponent Lebesgue space the relation 2.29 is no more valid in general, cf. Lemma 2.1.12. Nonetheless, the inequality is valid. □

Lemma 2.1.20. *Let* $\frac{1}{p(x)} + \frac{1}{q(x)} \equiv \frac{1}{r(x)}, p(x) \geq 1, r(x) \geq 1$ *and let* $R = \sup_{x \in \Omega \setminus \Omega_\infty(r)} r(x) < \infty$. *then*

$$\|uv\|_{L_{r(.)}(\Omega)} \leq c \|u\|_{L_{p(.)}(\Omega)} \|v\|_{L_{q(.)}(\Omega)} \tag{2.30}$$

for all functions $u \in L_{p(.)}$ *and* $v \in L_{q(.)}$ *with* $c = c_1 + c_2, c_1 = \sup_{x \in \Omega \setminus \Omega_\infty(r)} \frac{r(x)}{p(x)}$ *and* $c_2 = \sup_{x \in \Omega \setminus \Omega_\infty(r)} \frac{r(x)}{q(x)}$.

Proof. To show 2.30 we use the inequality

$$(AB)^r \leq \frac{r}{p} A^p + \frac{r}{q} B^q$$

with $A > 0, B > 0, p > 0, q > 0$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Integrating the inequality

$$|u(x)v(x)|^{r(x)} \leq \frac{r(x)}{p(x)} |u(x)|^{p(x)} + \frac{r(x)}{q(x)} |v(x)|^{q(x)}$$

we get

$$\int_{\Omega \setminus \Omega_\infty(r)} |u(x)v(x)|^{r(x)} \, dx \leq c_1 \int_{\Omega \setminus \Omega_\infty(p)} |u(x)|^{p(x)} \, dx + c_2 \int_{\Omega \setminus \Omega_\infty(q)} |v(x)|^{q(x)} \, dx \tag{2.31}$$

since $\Omega_\infty(r) = \Omega_\infty(p) \cap \Omega_\infty(q)$. From (2.31) and (2.6) it follows

$$\int_{\Omega \setminus \Omega_\infty(r)} \left| \frac{u(x)v(x)}{\|u\|_{(p)} \|v\|_{(q)}} \right|^{r(x)} dx \leq c_1 \int_{\Omega \setminus \Omega_\infty(p)} \left| \frac{u(x)}{\|u\|_p} \right|^{p(x)} dx + c_2 \int_{\Omega \setminus \Omega_\infty(q)} \left| \frac{v(x)}{\|v\|_{(q)}} \right|^{q(x)} dx \leq c_1 + c_2.$$

From Lemma (2.1.11) we now get $\|uv\|(r) \leq (c1 + c2) \|u\|_{(p)} \|v\|_{(q)}$, since $c_1 + c_2 \geq 1$.

The inequality (2.30) is also valid in the form

$$\rho_{r(.)}(uv) \leq c \|u\|_{L_{p(.)}(\Omega)} \|v\|_{L_{q(.)}(\Omega)}$$

if $\|u\|_{L_{p(.)}(\Omega)} \leq 1$ and $\|v\|_{L_{q(.)}(\Omega)} \leq 1$, which follows from the Hölder inequality (2.27) and the estimate (2.19). □

Convergence and completeness

Theorem 2.1.21. *Let* $1 \leq p(x) \leq p_+ < \infty$. *The space* $L_{p(.)}(\Omega)$ *is complete.*

Proof. The space $L_{p(.)}(\Omega)$ is the sum of $L_{p(.)}(\Omega_*) + L_\infty(\Omega_\infty)$ where each space is understood as the space of functions which are 0 outside the sets Ω_* and Ω_∞, respectively. Therefore, we only need to show the completeness of the space $L_{p(.)}(\Omega_*)$.

Let f_k be a Cauchy sequence in $L_{p(.)}(\Omega_*)$ such that for any positive number s exists $N_s (N_1 < N_2 < \ldots)$ such that

$$\left\|f_{N_{s+1}} - f_{N_s}\right\|_{L_{p(.)}(\Omega_*)} < 2^{-s}, \quad s = 1, 2, 3.....$$

then

$$\sum_{s=1}^{\infty} \left\|f_{N_{s+1}} - f_{N_s}\right\|_{L_{p(.)}(\Omega_*)} < \infty.$$

Let $\Omega_r = \{x \in \Omega_* : |x| < r\}, r > 0$. By Hölder's inequality (2.27) we obtain

$$\sum_{s=1}^{\infty} \int_{\Omega_r} \left|f_{N_{s+1}} - f_{N_s}\right| dx \leq c_r \sum_{s=1}^{\infty} \left\|f_{N_{s+1}} - f_{N_s}\right\|_{L_{p(.)}(\Omega_*)} < \infty \tag{2.32}$$

where $c_r = \left(\frac{1}{p_-} + \frac{1}{(p')_-}\right) \|X_{\Omega_r}\|_{L_{p'(.)}(\Omega_*)} < \infty$. By (2.32),$\{f_{N_s}(x)\}$ is a Cauchy sequence in $L_1(\Omega_r)$. Therefore, there exists the limit $f(x) = \lim_{s \longrightarrow \infty} f_{N_s}(x)$ for almost all $x \in \Omega_r$, which entails that the same happens for almost all $x \in \Omega_*$ since $r > 0$ is arbitrary. Now we only need to show that

$$\lim_{k \longrightarrow \infty} \|f_k - f\|_{L_{p(.)}(\Omega_*)} = 0.$$

Since $\{f_k\}$ is a Cauchy sequence, we have that $\|f_k - f_{N_s}\|_{L_{p(.)}(\Omega_*)} < \varepsilon$ whenever k and s are sufficiently large. Now by 2.14 we get

$$\int_{\Omega_*} |f_k(x) - f_{N_s}(x)|^{p(x)} dx \leq \varepsilon^{p_-} \leq \varepsilon.$$

Invoking Fatou's Lemma we obtain

$$\begin{aligned} \int_{\Omega_*} |f_k(x) - f(x)|^{p(x)} dx &\leq \lim_{s \longrightarrow \infty} \inf \int_{\Omega_*} |f_k(x) - f_{N_s}(x)|^{p(x)} dx \\ &\leq \sup_s \int_{\Omega_*} |f_k(x) - f_{N_s}(x)|^{p(x)} dx \leq \varepsilon \end{aligned}$$

□

Lemma 2.1.22. *Let $0 < p_- \leq p(x) \leq p_+ < \infty, x \in \Omega \setminus \Omega_\infty$. The convergence*

$$\int_{\Omega \setminus \Omega_\infty} |f_m(x) - f(x)|^{p(x)} dx + ess \sup_{x \in \Omega_\infty} |f(x) - f_m(x)| < \varepsilon$$

is equivalent to the norm convergence

$$\|f - f_m\|_{(p)} + ess \sup_{x \in \Omega_\infty} |f(x) - f_m(x)| < \varepsilon$$

Proof. Follows from Corollary 2.6. □

2.1.3 Embeddings and dense sets

Theorem 2.1.23. *Let $0 \le r(x) \le p(x) \le \infty$ and let $m(\Omega \setminus \Omega_\infty(r)) < \infty$ if $\Omega_\infty(r) \subseteq \Omega_\infty(p)$ and*

$$R := \sup_{x \in \Omega_\infty(p) \setminus \Omega_\infty(r)} r(x),$$

then $L_{p(.)}(\Omega) \subseteq L_{r(.)}(\Omega)$ and

$$\rho_{r(.)}(f) \le \rho_{p(.)}(f) + m(\Omega_\infty(p) \setminus \Omega_\infty(r)) \, \|f\|^R_{L_\infty(\Omega_\infty(p)\setminus\Omega_\infty(r))} + m(\Omega \setminus \Omega_\infty(r)) \tag{2.33}$$

for any $f \in L_{p(.)}(\Omega)$. (In the case $\Omega_\infty(p) = \Omega_\infty(r)$, the second term in the right-hand side should be omitted and R can be infinite). If, moreover, $1 \le r(x) \le p(x)$ and $\Omega_\infty(p) = \Omega_\infty(r)$, the inequality for norms is also valid:

$$\|f\|_r \le c_0^v \, \|f\|_{(p)} \tag{2.34}$$

where $c_0 = c_2 + (1 - c_1) m((\Omega \setminus \Omega_\infty(r))$, $\quad c_1 = \inf_{x \in \Omega \setminus \Omega_\infty(p)} \dfrac{r(x)}{p(x)}$, $\quad c_2 = \sup_{x \in \Omega \setminus \Omega_\infty(p)} \dfrac{r(x)}{p(x)}$, $v = \dfrac{1}{r_0}$ *if* $\quad c_0 \ge 1$ *and* $\quad v = \dfrac{1}{R}$ *if* $\quad c_0 \le 1$.

Proof. The estimate (2.33) is derived from the equality $\rho_{r(.)}(f) = \int_{\Omega_1} + \int_{\Omega_2} + \int_{\Omega_3}$ with
$\Omega_1 = \{x \in \Omega \setminus \Omega_\infty(p) : |f(x)| \ge 1\}$, $\Omega_2 = \{x \in \Omega_\infty(p) \setminus \Omega_\infty(r) : |f(x)| \ge 1\}$,
$\Omega_3 = \{x \in \Omega \setminus \Omega_\infty(r) : |f(x)| \le 1\}$.
The classical technique to show the inequality (2.34) for norms is based on the Hölder inequality with the exponents $p_1(x) = \dfrac{p(x)}{r(x)}$ and $p_2(x) = \dfrac{r(x)}{p(x) - r(x)}$ which is no more appropriate for the variable setting since we can have $p(x) = r(x)$ in some arbitrary set. Using the inequality $(AB)^r \le \dfrac{r}{p} A^p + \dfrac{r}{q} B^q$ and taking $A = |f(x)| \,/\, \|f\|_{(p)}$ and$B = 1$, we get, via (2.6), that

$$\int_{\Omega \setminus \Omega_\infty} \left| \frac{f(x)}{\|f\|_{(p)}} \right|^{r(x)} dx \le c_0.$$

Therefore, by Lemma 2.1.11 we get (2.34). We now show the denseness of the bounded functions with compact support. □

Lemma 2.1.24. *Let $m(\Omega_\infty(p)) = 0, 1 \le p(x) \le p_+ < \infty$. The set of bounded functions with compact support is dense in $L_{p(.)}(\Omega)$.*

Proof. For $f \in L_{p(.)}(\Omega)$ we define $f_{N,m}$ as

$$f_{N,m} = \begin{cases} f(x), & \textit{when } |f(x)| \le N \text{ and } |x| \le m; \\ 0 & \text{otherwise,} \end{cases}$$

By Lemma 2.1.22, we have

$$\int_{\Omega} |f(x) - f_{N,m}(x)|^{p(x)}\, dx \leq \int_{\omega_m} |g(x)|\, dx + \int_{\Omega_N} |g(x)|\, dx \longrightarrow 0$$

when $m \longrightarrow \infty, N \longrightarrow \infty$, with $\omega_m = \{x \in \Omega : |x| \geq m\}, \Omega_N = \{x \in \Omega : f(x) \geq N\}$ and $g(x) = |f(x)|^{p(x)} \in L_1(\Omega)$. ☐

Theorem 2.1.25. *Let $p \in \rho(\Omega) \bigcap L_\infty(\Omega)$. Then the set $C(\Omega) \bigcap L_{p(.)}(\Omega)$ is dense in $L_{p(.)}(\Omega)$. Moreover, if Ω is open, then the set of all functions infinitely differentiable with compact support $C_c^\infty(\Omega)$ is dense in $L_{p(.)}(\Omega)$.*

Proof. Let $f \in L_{p(.)}(\Omega)$ and $\varepsilon > 0$. From Lemma 2.1.24 there exists a bounded function $g \in L_{p(.)}(\Omega)$ such that

$$\|f - g\|_{L_{p(.)}(\Omega)} < \varepsilon. \tag{2.35}$$

By Luzin's Theorem, there exists a function $h \in C(\Omega)$ and an open set U such that

$$m(U) < \min\left\{1, \left(\frac{\varepsilon}{2\|g\|_\infty}\right)^{p_+}\right\},$$

$g(x) = h(x)$ for all $x \in \Omega \setminus U$ and $\sup|h(x)| = \sup_{\Omega\setminus U}|g(x)| \leq \|g\|_\infty$. Then,

$$\rho_{p(.)}\left(\frac{g-h}{\varepsilon}\right) \leq \max\left\{1, \left(\frac{2\|g\|_\infty}{\varepsilon}\right)^{p_+}\right\} m(U) \leq 1$$

i.e., $\|g - h\|_{L_{p(.)}(\Omega)} \leq \varepsilon$, which together with (2.35) implies that

$$\|f - h\|_{L_{p(.)}(\Omega)} \leq 2\varepsilon \tag{2.36}$$

On the other hand, let us assume that Ω is open. Since $p \in L_\infty(\Omega)$, we have that $C_c^\infty(\Omega) \subset L_{p(.)}(\Omega)$ and $\rho_{p(.)}\left(\frac{h_{\chi_{\Omega\setminus G}}}{\varepsilon}\right) \leq 1$. In other words,

$$\|h - h_{\chi_G}\|_{L_{p(.)}(\Omega)} \leq \varepsilon \tag{2.37}$$

By the weierstrass approximation theorem, let m be a polynomial which satisfies the condition $\sup|h(x) - m(x)| \leq \varepsilon \min\left\{1, |G|^{-1}\right\}$ Therefore $\rho_{p(.)}\left(\frac{h_{\chi_G} - m_{\chi_G}}{\varepsilon}\right) \leq \min\left\{1, |G|^{-1}\right\}|G| \leq 1$, from which

$$\|h_{\chi_G} - m_{\chi_G}\|_{L_{p(.)}(\Omega)} \leq \varepsilon \tag{2.38}$$

Finally, similar considerations to the ones that were used to get(2.37) permit to conclude that for a sufficient small number a.

The compact set $K_a = \{x \in G : dist(x, \delta G) \geq a\}$ satisfies that $\|m_{\chi G} - m_{\chi Ka}\|_{L_{p(.)}(\Omega)} \leq \varepsilon$. Taking $\varphi \in C_c^\infty(G)$ such that $0 \leq \varphi(x) \leq 1$ for $x \in G$ and
$\varphi(x) = 1$ for $x \in K_a$ we obtain

$$\|m_{\chi G} - m_\varphi\|_{L_{p(.)}(\Omega)} \leq \|m_{\chi G} - m_{\chi Ka}\|_{L_{p(.)}(\Omega)} \leq \varepsilon,$$

from which, together with (2.36) and (2.38), we conclude that

$$\|f - m\varphi\|_{L_{p(.)}(\Omega)} \leq 4\varepsilon.$$

Clearly $m\varphi \in C_c^\infty(\Omega)$, which concludes the proof. By $L_c^\infty(\mathbb{R}^n)$ we denote the class of all bounded functions in $\mathbb{R}^n$ with compact support. From Theorem 2.1.25 we get the result. □

Lemma 2.1.26. *[5] Let $p : \mathbb{R}^n \longrightarrow [0, \infty)$ be a measurable function such that $1 < p_- \leq p_+ < \infty$. Then $L_c^\infty(\mathbb{R}^n)$ is dense in $L_{p(.)}(\mathbb{R}^n)$ and in $L_{p(.)}(\mathbb{R}^n)$. We now show that the set of step functions is dense in the framework of variable exponent spaces with finite exponent.*

Theorem 2.1.27. *[5] Let $p : \mathbb{R}^n \longrightarrow [0, \infty)$ be a measurable function such that $1 < p_- \leq p_+ < \infty$. The set S of step functions is dense in $L_{p(.)}(\Omega)$.*

Proof. It follows from Lemma **??** and from Theorem 2.1.25 together with the fact that continuous functions in compact sets are uniformly approximated by step functions. □

Theorem 2.1.28. *Under the conditions of Lemma 2.1.24 the space $L_{p(.)}(\Omega)$ is separable.*

Proof. By Theorem 2.1.25 it is sufficient to show that any continuous function f with compact support $F \subset \Omega$ can be approximated by functions in some enumerable set. We know that such functions can be approximated uniformly by polynomials $r_m(x)$ with rational coefficients. Taking $f_m(x) = r_m(x)$ for $x \in F$ and $f_m(x) = 0$ for $x \notin F$, we see that the functions $f_m(x)$ approximate uniformly the function $f(x)$. □

2.1.4 Duality

We study the dual space of variable exponent Lebesgue spaces, which is similar to the classical Lebesgue space, viz. the dual space of L_p is $L_{p'}$, where p' is the conjugate exponent. For simplicity, we will work with $m(\Omega) < \infty$. For $m(\Omega) = \infty$ see Cruz-Uribe and Fiorenza [5].

Theorem 2.1.29. *Let $1 < p_- \leq p(x) \leq p_+ < \infty$ and $m(\Omega) < \infty$. Then*

$$\left[L_{p(.)}(\Omega)\right]^* = L_{p'(.)}(\Omega).$$

Proof. The inclusion $Lp'(.)(\Omega) \subseteq \left[L_{p(.)}(\Omega)\right]^*$ is an immediate consequence of the Hölder inequality (2.27). We now show the opposite inclusion $\left[L_{p(.)}(\Omega)\right]^* \subseteq L_{p'(.)}(\Omega)$. Let $\Phi \in \left[L_{p(.)}(\Omega)\right]^*$, then we define the set function μ as $\mu(E) = \phi(\chi_E)$ for all measurable sets E such that $E \subset \Omega$. Since $\chi_{E\cup F} = \chi_E + \chi_F - \chi_{E\cap F}$ we have that μ is an additive function. In fact it is σ -additive. To show that, let

$$E = \cup_{j=1}^{\infty} E_j$$

where $E_j \subset \Omega$ are pairwise disjoint set, and let

$$F_K = \cup_{j=1}^{k} E_j.$$

Then

$$\|\chi_E - \chi_{F_k}\|_{L_{p(.)}(\Omega)} \leq \|\chi_E - \chi_{F_k}\|_{p_+} = C.m(E \setminus F_k)^{1/p_+}.$$

Since $m(E) < \infty, m(E \setminus F_k)$ tends to 0 when $k \longrightarrow \infty$, therefore $\chi_{F_k} \mapsto \chi_E$ in norm. From the continuity of ϕ we have that $\phi(\chi_{F_k}) \longrightarrow \phi(\chi_E)$, which is equivalent to

$$\sum_{j=1}^{\infty} \mu(E_j) = \mu(E)$$

and from this we get that μ is σ -additive. The function μ is a measure in Ω and, moreover, is absolutely continuous: if $E \subset \Omega$ and $m(E) = 0$, therefore $\mu(E) = \phi(\chi_E) = 0$, since $|\phi(f)| \leq \|\Phi\| \, \|f\|_{L_{p(.)}(\Omega)}$. By the Radon-Nikodym Theorem, there exists $g \in L_1(\Omega)$ such that

$$\phi(\chi_E) = \mu(E) = \int_\Omega \chi_E(x) g(x) dx$$

By the linearity of ϕ, for a step function $f = \sum_{i=1}^{n} a_i \chi_{E_i}, E_i \subset \Omega$, we get

$$\phi(f) = \int_\Omega f(x) g(x) dx.$$

Using a density argument, similar to the constant case, we get the result. □

Corollary 2.1.30. *Let $1 < p_- \leq p(x) \leq p_+ < \infty$ and $m(\Omega) < \infty$. Then the space $L_{p(.)}(\Omega)$ is reflexive.*

2.1.5 Associate norm

We now study a norm inspired by the Riesz representation theorem for linear functionals in $L^{p^{\cdot}}$ Let

$$S_{p(.)(\Omega)} := \left\{ f \in S(\Omega, L); \left| \int_\Omega f(x)\varphi(x) dx \right| < \infty, \forall \varphi \in L_{p'(.)}(\Omega) \right\} \tag{2.39}$$

with $1 \le p(x) \le \infty$. This space coincides with the space $L_{p(.)}(\Omega)$ under certain natural conditions in the variable exponent p and it is in fact the associate space of $L_{p'(.)}(\Omega)$ (see Definition (2.3.5) for the notion of associate space in the context of Banach Function Spaces).

The inclusion

$$L_{p(.)}(\Omega) \subseteq S_{p(.)}(\Omega), 1 \le p(x) \le \infty \tag{2.40}$$

is an immediate consequence of the Hölder inequality (2.27). Observe that the space defined in (2.39) is always linear. From Lemma 2.1.3, we have that this space cannot coincide with the space $L_{p(.)}(\Omega)$ if $p_+ = \infty$.

Let us study the following notation

$$p_-^1 = ess \inf_{x \in \Omega \setminus \Omega_1(p)} p(x); \quad (p')_-^1 = ess \inf_{x \in \Omega \setminus \Omega_1(p')} p'(x)$$

We have

$$\Omega_1(p) = \Omega_\infty(p'), \Omega_1(p') = \Omega_\infty(p), (p')_+ = \frac{p_-^1}{p_-^1 - 1}, (p')_-^1 = \frac{p_+}{p_+ - 1}$$

The space introduced in (2.39) can be equipped with the next natural norms

$$\|f\|_p^* = \sup_{\delta_{p'(.)}(\varphi) \le 1} \left| \int_\Omega f(x)\varphi(x)dx \right|, \tag{2.41}$$

and

$$\|f\|_p^{**} = \sup_{\|\varphi\|_{p'(.)} \le 1} \left| \int_\Omega f(x)\varphi(x)dx \right|, \tag{2.42}$$

where we take $\delta p(.)(\varphi)$ as

$$\delta_{p(.)}(\varphi) = \left(\int_{\Omega \setminus \Omega_\infty} |\varphi(x)|^{p(x)} dx \right)^{\frac{1}{p_+}} + ess \sup_{x \in \Omega_\infty} |\varphi(x)|$$

and we assume that $(p')_+ < \infty$ (i.e, $p_-^1 > 1$) in (2.41), while $p(x)$ can be taken arbitrary $(1 \le p(x) \le \infty)$ in the case (2.42). Sometimes the norm (2.42) is called Orlicz type norm.

Note that by (2.10) we have

$$\|f\|_{L_{p(.)}(\Omega)} \le \|f\|_p^{**}$$

in the case $1 \le p(x) \le p_+ < \infty$ and $m(\Omega_\infty) = 0$.

Lemma 2.1.31. *Let $f \in S_{p(.)}(\Omega), (p')_-^1 > 1$.then $\|f\|_p^* < \infty$ and*

$$\int_\Omega |f(x)\varphi(x)|\, dx \le \|f\|_p^* \|\varphi\|_{p'}^1 \le \|f\|_p^* \|\varphi\|_{L_{p'(.)}(\Omega)} \tag{2.43}$$

for all $\varphi \in L_{p'(.)}(\Omega)$, where $\|\varphi\|_{p'}^1$, is the norm(2.24). Moreover, the functional (2.41) is a norm in $S_{p(.)}(\Omega)$.

Proof. Suppose that $\|f\|_p^* = \infty$. Then there exists a function $f_0(x) \in S_{p(.)}(\Omega)$ and a sequence $\varphi_k \in L_{p'(.)}(\Omega)$ such that $\delta_{p'(.)}(\varphi_k) \leq 1$ and

$$\int_\Omega f_0(x)\varphi_k(x)dx \geq 2^{\mathrm{Q}k}, k = 1, 2, 3, ...$$

$(f_0 \geq 0, \varphi_k \geq 0)$. Therefore $j_m = \sum_{k=1}^{m} 2^{-\mathrm{Q}k}\varphi_k(x)$ is an increasing sequence. Direct calculations show that $\delta_{p'(.)}(j_m) \leq 1$ and

$$\int_\Omega f_0(x)j_m(x)dx = \sum_{k=1}^{m} 2^{-\mathrm{Q}k} \int_\Omega f_0(x)\varphi_k(x)dx \geq m. \tag{2.44}$$

The sequence $j_m(x)$ converges monotonically to the function

$$j(x) = \sum_{k=1}^{\infty} 2^{-\mathrm{Q}k}\varphi_k(x).$$

Also,

$$\int_{\Omega\setminus\Omega_\infty(p')} |j(x)|^{p'(x)}\,dx = \lim_{m\to\infty} \int_{\Omega\setminus\Omega_\infty(p')} |j_m(x)|^{p'(x)}\,dx \leq 1$$

by the Lebesgue monotone convergence theorem and, since

$$\sup_{x\in\Omega_\infty(p')} j(x) = \sum_{k=1}^{\infty} 2^{-\mathrm{Q}k} < \infty$$

we get that $j \in L_{p'(.)}(\Omega)$. By the Lebesgue monotone convergence theorem and by (2.44) we obtain that$\int_\Omega f_0(x)j(x)dx = \infty$ which is a contradiction due to the fact that $f_0(x) \in S_{p(.)}(\Omega)$. Therefore, $\|f\|_p^* < \infty$ and by the definition (2.41) we get

$$\left|\int_\Omega f(x)\varphi(x)dx\right| \leq A\,\|f\|_p^*$$

where $A > 0$ and $\delta_{p'(.)}(\varphi \mid A) \leq 1$. Taking infimum with respect to A, we get the left-hand side of (2.43) due to the definition (2.24). The right-hand side of the inequality follows from (2.26). We only need to verify the norm axioms. The homogeneity and the triangle inequality are evident. Taking $\|f\|_p^* = 0$, then $\int_\Omega f(x)\varphi(x)dx = 0$ for all $\varphi \in L_{p'(.)}(\Omega)$ which entails that all function $\varphi(x) \in L$ by the Lemma **??**. Therefore $f(x) \equiv 0$ We now show that the norms (2.41) and (2.42) are equivalent. □

Lemma 2.1.32. *[5] Let* $1 \leq p(x) \leq \infty, p_-^1 > 1$ *, and* $p_+ < \infty$ *The norms (2.41) and (2.42) are equivalent in functions* $f \in S_{p(.)}(\Omega)$ *:*

$$2^{1-(p')+/(p')_-^1}\,\|f\|_p^{**} \leq \|f\|_p^* \leq \|f\|_p^{**}. \tag{2.45}$$

The norms coincide in the cases:

(1) $m(\Omega_1(p)) = 0$,

(2) $p(x) = const$ *for* $x \in \Omega \setminus (\Omega_\infty \cup \Omega_1)$.

Proof. To obtain the right-hand side inequality, we show that

$$\{\varphi : \delta_{p'(.)}(\varphi) \leq 1\} \subseteq \left\{\varphi : \|\varphi\|_{L_{p'(.)}(\Omega)} \leq 1\right\} \tag{2.46}$$

for $\varphi \in L_{p'(.)}(\Omega)$. Let $\delta_{p'(.)}(\varphi) \leq 1$. We have that $\rho_{p'(.)}(\varphi) \leq 1$ whenever $\|\varphi\|_{(p')} \leq 1$ by (2.14)-(2.15). Then, by (2.14) we have that $\|\varphi\|_{(p')} \leq \left(\rho_{p'(.)}(\varphi)\right)^{1/(p')_+} \leq 1$
which implies the inequality

$$\|\varphi\|_{L_{p'(.)}(\Omega)} \leq \left[\rho_{p'(.)}(\varphi)\right]^{1/(p')_+} + ess \sup_{x\in\Omega_\infty(p')} |\varphi(x)| = \delta_{p'(.)}(\varphi) \leq 1$$

whence (2.45) is proved.

Furthermore, let $c = 2^{1-(p')_+/(p')_-} \leq 1$. We will show that

$$\left\{\varphi : \|\varphi\|_{L_{p'(.)}(\Omega)} \leq 1\right\} \subseteq \{\varphi : \delta_{p'(.)}(c\varphi) \leq 1\},$$

which shows the left-hand side inequality in (2.45). We have $\|\varphi\|_{L_{p'(.)}(\Omega)} \leq 1$
therefore $\|c\varphi\|_{(p')} \leq 1$ and we get $\left(\rho_{p(.)}(c\varphi)\right)^{1/(p')_+} \leq \|c\varphi\|_{(p')}^{(p')^1_-/(p')_+}$ by (2.14). This entails that

$$\rho_{p(.)}(c\varphi) \leq \|c\varphi\|_{(p')}^{(p')^1_-/(p')_+} + \|c\varphi\|_{L_\infty(\Omega_\infty(p'))}.$$

Since $A^\lambda + B \leq 2^{1-\lambda}(A+B)^\lambda, 0 \leq \lambda \leq 1, A \geq 0, 0 \leq B \leq 1$, we get that $\delta_{p'(.)}(c\varphi) < 1$ and (2.46) is proved as the left-hand side inequality of (2.45).

To finish, if $m(\Omega_1(p)) = 0$ or $p(x) = const$ for $x \in \Omega\backslash(\Omega_\infty \cup \Omega_1)$, then we have $\|\varphi\|_{L_\infty(\Omega_\infty(p'))} = 0$ or $(p')^1_-/(p')_+ = 1$, respectively, and we obtain (2.46) with $c = 1$, which implies the coincidence of norms.

The Luxemburg-Nakano norm is equivalent to the norm given in (2.41) in the following way. ◻

Theorem 2.1.33. *Let $p^1_- > 1$. The spaces $L_{p(.)}(\Omega)$ and $S_{p(.)}(\Omega)$ coincide modulo norm convergence:*

$$\frac{1}{3}\|f\|_{L_{p(.)}(\Omega)} \leq \|f\|^*_p \leq \left(\frac{1}{p_-} + \frac{1}{(p')_-}\right)\|f\|_{L_{p(.)}(\Omega)} \tag{2.47}$$

where $1/3$ can be replaced by 1 if $m(\Omega_1) = m(\Omega_\infty) = 0$.

Proof. From the inclusion in (2.40) it suffices to show

$$S_{p(.)}(\Omega) \subseteq L_{p(.)}(\Omega). \tag{2.48}$$

Let $f \in S_{p(.)}(\Omega)$ and let us take first the case $\|f\|^*_p \leq 1$. Take $\varphi_0(x) = |f(x)|^{p(x)-1}$
if $x \in \Omega \setminus (\Omega_1 \cup \Omega_\infty)$ and $\varphi_0(x) = 0$ otherwise. We now show that

$$\varphi_0 \in L_{p'(.)}(\Omega) \quad and \quad \rho_{p'(.)}(\varphi_0) \leq 1. \tag{2.49}$$

Suppose that $\rho_{p'(.)}(\varphi_0) > 1$. Then

$$\rho_{p(.)}(f) \geq \int_{\Omega\setminus\Omega_\infty(p')} |\varphi_0(x)|^{p'(x)}\,dx > 1. \tag{2.50}$$

Let

$$f_{N,k}(x) = \begin{cases} f(x), & \text{when } |f(x)| \leq N \text{ and } |X| \leq K \\ 0 & \text{otherwise,} \end{cases}$$

then $\varphi_{N,k}(x) = |f_{N,k}|^{p(x)-1} \in L_{p'(.)}(\Omega)$ From (2.50) we derive the existence of an $N_0 \longrightarrow \infty$ and $k_0 \longrightarrow \infty$ such that

$$\int_{\Omega\setminus\Omega_\infty(p)} |f_{N_0,k_0}|^{p(x)}\,dx > 1 \tag{2.51}$$

In consequence, from (2.43) we obtain

$$1 < \rho_{p(.)}\left(f_{N_0,k_0}\right) \leq \|f_{N_0,k_0}\|_p^* \left\| f_{N_0,k_0}^{p(.)-1} \right\|_{L_{p'(.)}(\Omega)} .$$

Henceforth, in virtue of (2.14)-(2.15)

$$1 < \|f_{N_0,k_0}\|_p^* \max\left\{ \left[\rho_{p(.)}\left(f_{N_0,k_0}\right)\right]^{\frac{1}{(p')_+}}, \left[\rho_{p(.)}\left(f_{N_0,k_0}\right)\right]^{\frac{1}{(p')_-}} \right\}. \tag{2.52}$$

then

$$\min\left\{ \left[\rho_{p(.)}\left(f_{N_0,k_0}\right)\right]^{1-\frac{1}{(p')_+}}, \left[\rho_{p(.)}\left(f_{N_0,k_0}\right)\right]^{1-\frac{1}{(p')_-}} \right\} \leq \|f_{N_0,k_0}\|_p^*$$

which, from inequality (2.51) we conclude that $1 < \|f_{N_0,k_0}\|_p^*$. This means that

$$\sup_{\rho_{p'(.)}(\varphi)\leq 1} \left| \int_\Omega f(x)\varphi^{N,k}(x)dx \right| > 1$$

where

$$\varphi^{N,k}(x) = \begin{cases} \varphi(x), & \textit{when } |f(x)| \leq N \textit{ and } |x| \leq K; \\ 0 & \textit{otherwise.} \end{cases}$$

Nevertheless, since $\rho_{p'(.)}(\varphi^{N,k}) \leq \rho_{p'(.)}(\varphi)$ this contradicts the supposition that $\|f\|_p^* \leq 1$, from which we get (2.48).

As a result

$$\int_{\Omega\setminus(\Omega_1(p)\cup\Omega_\infty(p))} |f(x)|^{p(x)}\,dx \leq 1$$

and to get the embedding (2.48) it is only necessary to show that $\int_{\Omega_1(p)} |f(x)|\,dx < \infty$ and moreover that $\sup_{x\in\Omega_\infty(p)} |f(x)| < \infty$, which follows from the inequality

$$\int_{\Omega_i} |f(x)\varphi(x)|\,dx \leq c\,\|\varphi\|_{L_{p'(.)}(\Omega_i)}, i = 1, 2,$$

(see (2.43)), where $\Omega_1 = \Omega_1(p), \Omega_2 = \Omega_\infty(p)$ and $f \in L_1, \varphi \in L_\infty(q = 1)$ in the first case and $f \in L_\infty, \varphi \in L_1(q = \infty)$ in the second one.

We now take $\|f\|_p^* > 1$. Then $f(x)/\|f\|_p^* \in L_{p(.)(\Omega)}$, as was previously proved. Therefore, $f \in L_{p(.)(\Omega)}$ by the linearity of the space $L_{p(.)(\Omega)}$ under the condition $p_+ < \infty$. The embedding (2.48) is then proved. It is only necessary to show the inequality(2.47) for the norms. The right-hand side inequality is a consequence of the Hölder inequality (2.27) and from the definition of the norm (2.42). To show the left-hand side of the inequality we write $f(x) = f_1(x) + f_2(x) + f_3(x)$ with $f_2(x) = f(x), x \in \Omega_1$ and $f_2(x) = 0, x \in \Omega \setminus \Omega_1$ and $f_3(x) = f(x), x \in \Omega_\infty$, and $f_3(x) = 0, x \in \Omega \setminus \Omega_\infty$. Let us show that

$$\|f_1\|_{L_{p(.)}(\Omega)} \leq \|f\|^*_{p(\Omega \setminus (\Omega_1 \cup \Omega_\infty))} \tag{2.53}$$

We have that

$$\rho_{p(.)}\left(\frac{f_1}{\lambda}\right) = \frac{1}{\lambda}\int_{\Omega\setminus\Omega_\infty} |f_1(x)|\,\varphi_\lambda(x)dx, \lambda > 0, \tag{2.54}$$

with $\varphi_\lambda(x) = \left|\frac{f_1(x)}{\lambda}\right|^{p(x)-1}$. Choosing $\lambda = \|f_1\|_{(p)}$, due to (2.43) and (2.53) we obtain

$$1 = \frac{1}{\|f_1\|_{(p)}}\int_{\Omega\setminus\Omega_\infty} |f_1(x)|\,\varphi_\lambda(x)dx \leq \frac{\|f\|_p^*}{\|f\|_{(p)}}\|\varphi_\lambda\|_{L_{p'(.)}(\Omega)}.$$

Since $\rho_{p'(.)}(\varphi_\lambda) \leq \rho_{p(.)}\left(\frac{f_1}{\lambda}\right) = 1$ we also conclude that $\|\varphi_\lambda\|_{L_{p'(.)}(\Omega)} \leq 1$ due to (2.14)-(2.15) and we obtain the coincidence $\|\varphi_\lambda\|_{L_{p'(.)}(\Omega)} = \|\varphi_\lambda\|_{(p')}$. Therefore(2.54)implies(2.52).
Since $\|f_2\|_{L_{p(.)}(\Omega)} = \|f\|^*_{L_1(\Omega_1)}$ and $\|f_3\|_{L_{p(.)}(\Omega)} = \|f\|^*_{L_\infty(\Omega_\infty)}$ we obtain the left-hand side inequality.
□

Corollary 2.1.34. *Let $f \in L_{p(.)}(\Omega), \varphi \in L_{p'(.)}(\Omega), 1 \leq p(x) \leq \infty$. Regarding the norms (2.41)-(2.42) the Hölder inequality is valid with constant 1:*

$$\int_\Omega |f(x)\varphi(x)|\,dx \leq \|f\|_p^*\,\|\varphi\|_{L_{p'(.)}(\Omega)}, p_-^1 > 1, \tag{2.55}$$

and

$$\int_\Omega |f(x)\varphi(x)|\,dx \leq \|f\|_p^{**}\,\|\varphi\|_{L_{p'(.)}(\Omega)} \tag{2.56}$$

The inequality

$$\int_\Omega |f(x)\varphi(x)|\,dx \leq \|f\|_p^*\,\|\varphi\|_{p'}^* \tag{2.57}$$

is valid in the case

$$p_-^1 > 1, p_+ < \infty, m(\Omega_\infty(p)) = m(\Omega_1(p)) = 0. \tag{2.58}$$

In reality, the inequality (2.55) was already given in (2.43); meanwhile the inequality (2.56) follows directly from the definition (2.42). The inequality (2.57) is a consequence of (2.55) since $\|\varphi\|_{L_{p'(.)}(\Omega)} \leq \|\varphi\|_{p'}^*$, under the condition (2.58) by Theorem 2.31.

More on the space $L_{p(.)}(\Omega)$ in the case $p_+ = \infty$

The definition given in(2.39)is one of the possible ways to define the space $L_{p(.)}(\Omega)$ in order to be linear in the case $p_+ = \infty$. It is also possible to define the spaces from the beginning as the convex hull of the space $L_{p(.)}(\Omega)$ or as

$$L_{p(.)}(\Omega) := \left\{ f \in S(\Omega, l) : \exists \lambda > 0 \quad \text{such that} \int_{\Omega \setminus \Omega_\infty} \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx + \|f\|_{L_\infty(\Omega_\infty)} < \infty \right\}. \tag{2.59}$$

This space is always linear for $0 \leq p(x) \leq \infty$. The homogeneity is obvious, mean- while the additivity is evident in the set $\{x \in \Omega : p(x) \leq 1\}$ due to the inequality $(a+b)^p \leq a^p + b^p, p \leq 1$ meanwhile in the set $x \in \Omega : p(x) > 1$ it is verified by the convexity (2.9) of the function
$t \mapsto t^p, p > 1$.

Therefore, in the case $p_+ = \infty$ we can use the three different versions of the definition, i.e, $span(L_{p(.)}), L_{p(.)}$, or $S_{p(.)}$. We can see that

$$span\left(L_{p(.)}\right) = L_{p(.)} \subseteq S_{p(.)}. \tag{2.60}$$

The norm in the space $L_{p(.)}$ is given by (2.42) whereas the norm is given by (2.5) in the spaces $span(L_{p(.)}) = L_{p(.)}$.

2.1.6 Minkowski integral inequality

Theorem 2.1.35. *Let $1 \leq p(x) \leq p_+ < \infty$ and $p_-^1 > 1$. Then we have the Minkowski integral inequality in the variable exponent Lebesgue space*

$$\left\| \int_\Omega f(.,y)dy \right\|_p^{**} \leq \int_\Omega \|f(.,y)\|_p^{**} dy. \tag{2.61}$$

Proof. Let J be the expression in the left-hand side. We get

$$J \leq \sup_{\|\varphi\|_{L_{p'(.)}(\Omega)} \leq 1} \int_\Omega \left(\int_\Omega |\varphi(x) f(x,y)| \, dx \right) dy.$$

Using the definition of norm given in (2.42), we obtain the desired inequality. □

Corollary 2.1.36. *Let $1 \leq p(x) \leq p_+ < \infty$ and p_-^1 . Then*

$$\left\| \int_\Omega f(.,y)dy \right\|_p^{*} \leq c_1 \int_\Omega \|f(.,y)\|_p^{*} dy, \tag{2.62}$$

and

$$\left\| \int_\Omega f(.,y)dy \right\|_{L_{p(.)}(\Omega)} \leq c_2 \int_\Omega \|f(.,y)\|_{L_{p(.)}(\Omega)} dy, \tag{2.63}$$

where $c_1 = 1$ if $m(\Omega_1) = 0$ and $c_1 = 2^{-1+(p')_+}/(p')_-^1$ in the other case. The constant $c_2 = kc_1$ if $m(\Omega_\infty) = m(\Omega_1) = 0$ and $c_2 = 3kc_1$ in the other case, where $k = \dfrac{1}{p_-} + \dfrac{1}{(p')_-}$.

Proof. The inequality (2.62) with the constant $c_1 = 2^{-1+(p')_+}/(p')_-^{\frac{1}{-}}$ is a consequence of(2.61)due to(2.45). In the same way(2.63)follows from(2.61)by virtue of(2.47) and (2.45). To prove that $c_1 = 0$ in (2.62) in the case $m(\Omega_1) = 0$, note that

$$\left\| \int_\Omega f(.,y)dy \right\|_p^* \leq \sup_{\delta(p'(.))(\varphi)\leq 1} \int_\Omega \|\varphi\|_{L_{p'(.)}(\Omega)} \|f(.,y)\|_p^* \, dy.$$

To finish the proof it is only necessary to see that the conditions $\delta_{p'(.)}(\varphi) \leq 1$ and $\|\varphi\|_{L_{p'(.)}(\Omega)} \leq 1$ are equivalent in the case $m(\Omega_1) = 0$, as a result of (2.14)-(2.15). ☐

2.1.7 Some differences between spaces with variable exponent and constant exponent

In this case, let us take the space $L_{p(.)}(\Omega)$ given in (2.59). Let $\Omega = [1, \infty), p(x) = x$ and $f(x) \equiv a$ where $a > 0$. We have that $f \in L_x(\Omega)$ since taking some $\lambda > a$ the integral $\int_1^\infty |f(x)/\lambda|^x \, dx$ is finite but $f \notin L_p(\Omega)$ for any constant p .

We now show two more differences between the constant and the variable frame- work, namely in regards to the invariance under translation and the Young convolution.

Invariance under translations

An important result in the classical theory of Lebesgue spaces has to do with the boundedness of the translation operator , i.e, if $f \in L_p(\mathbb{R}^n)$ then we have that
$\tau_h f \in L_p(\mathbb{R}^n)$, where $\tau_h f(x) := f(x-h)$. This result stems from the fact that the classical Lebesgue space is isotropic with respect to the exponent, since the power p is the same in any direction.On the other hand, the variable exponent Lebesgue space is, in general, anisotropic regarding the exponent. This anisotropy of the space generates problems for the translation operator.

Example 2.1.37. $f(x) = |x|^{-\frac{1}{3}}$. This function $f \in L_{p(.)}((-1,1))$ taking the following exponent

$$p(x) = \begin{cases} 2, & x \in |x| < \varepsilon; \\ 5 & x \in |x| \geq \varepsilon, \end{cases} \tag{2.64}$$

but $\tau_\delta f \notin L_{p(.)}((-1,1))$, when $\delta > \varepsilon$, since we translated the singularity from 0 to δ but the exponent was not shifted. ($\tau_\delta f$ is understood as the zero extension whenever necessary). One could argue that the problem in this example is the non- smoothness nature of the exponent. From (2.64) we can construct a smooth function (for example, via urysohn construction) and we will end up with the same problem. Our example is not an isolated incident, since Diening proved that this phenomenon is persistent, i.e, if $p_+ > p_-$, then there exists a $h \in R\backslash\{0\}$ such that the translation operator τ_h is not continuous, cf. Diening, Harjulehto, $Hästö$, and $Růžička$ [4].

Young convolution inequality in variable exponent Lebesgue spaces

Let

$$Kf(x) = (k * f)(x) = \int_{\mathbb{R}^n} k(x-y)f(y)dy = \int_{\mathbb{R}^n} k(y)f(x-y)dy \tag{2.65}$$

where $*$ is called convolution. The Young's inequality for convolutions states that

$$\|k * f\|_{L_r(\mathbb{R}^n)} \leq \|k\|_{L_q(\mathbb{R}^n)} \|f\|_{L_p(\mathbb{R}^n)}, \frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r},$$

which can be proved, among other means, using the following decomposition $|f(x-y)k(y)| = |f(x-y)|^{1-s} |k(y)| |f(x-y)|^s$, for $s = 1 - p/r$, the Hölder inequality and the integral Minkowski inequality. Since the convolution depends on the translation operator, which is not continuous, the natural question is: does the Young inequality for convolutions holds in general in the case of variable Lebesgue spaces ? The answer is no, in general, although there are some particular cases where it is possible to have some version of the inequality. Let us start with a counter-example.

Theorem 2.1.38. *Let p and q be variable exponents such that* $\frac{1}{p(x)} + \frac{1}{q(x)} \equiv 1 + \frac{1}{r}$ *where* $r = const \geq 1$. *If* $k \in L_{q_-}(\mathbb{R}^n) \cap L_{(p')_+}(\mathbb{R}^n)$ *then the convolution operator* (2.65)

$$k * . : L_{p(.)}(\mathbb{R}^n) \longrightarrow L_r(\mathbb{R}^n)$$

is bounded.

Proof. Let us take f such that $\|f\|_{L_{p(.)}(\Omega)} \leq 1$. Then $|(k*f)(x)| \leq \int_{\mathbb{R}^n} A^{1-\mu(y)} |f(y)|^{\frac{p(y)}{r}} |k(x-y)|^{\mu(y)} |f(y)|^{1-\frac{p(y)}{r}} \left|\frac{k(x-y)}{A}\right|^{1-\mu(y)} dy$ where the constant $A > 0$ and the function $\mu(y), 0 < \mu(y) < 1$, will be chosen later. Using the generalized Hölder inequality with the exponents

$$p_1(y) = r, p_2(y) = \frac{rp(y)}{r - p(y)}, p_3(y) = p'(y) = \frac{p(y)}{p(y)-1}$$

we obtain

$|(k * f)(x)|$

$$\leq c \left\{ \int_{\mathbb{R}^n} A^{r - r\mu(y)} |f(y)|^{p(y)} |k(x-y)|^{r\mu(y)} dy \right\}^{\frac{1}{r}} \times \left\| |f(y)|^{1-\frac{p(y)}{r}} \right\|_{p_2(y)} \left\| \left| \frac{k(x-y)}{A} \right|^{1-\mu(y)} \right\|_{p'(y)} \tag{2.66}$$

By the estimate (2.19) we get

$$\left\| |f(y)|^{1-\frac{p(y)}{r}} \right\|_{p_2(y)} \leq \|f\|_{p(y)}^{ess\inf\left[1-\frac{p(y)}{r}\right]} \leq 1 \tag{2.67}$$

since $\|f\|_{L_{p(.)}(\Omega)} \leq 1$ and the fact that $p < r$. To estimate the third factor in (2.66) it is natural to choose $\mu(y)$ in such a way that $[1-\mu(y)]\, p'(y) = q(y)$, i.e.

$$\mu(y) = \frac{q(y)}{r}\,.$$

We now want to use the inequality (2.19) in the third factor. We are now interested in

$$\left\| \frac{k(x-y)}{A} \right\|_{q(y)} = \frac{1}{A}\, \|k(x-y)\|_{q(y)} \leq 1. \tag{2.68}$$

To get (2.68) we choose

$$A = \|k\|_{q_-} + \|k\|_{(p')_+}\,.$$

In this way (2.68) is valid by Lemma 2.1.15. We can now apply (2.19) and obtain

$$\left\| \left| \frac{k(x-y)}{A} \right|^{1-\mu(y)} \right\|_{p'(y)} \leq 1. \tag{2.69}$$

From the inequalities (2.67) and (2.69) we get, via (2.66), the estimate

$$\begin{aligned} \|k*f\|_r &\leq cA^v \left(\int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} |f(y)|^{p(y)}\, |k(x-y)|^{q(y)}\, dy \right)^{\frac{1}{r}} \\ &= cA^v \left(\int_{\mathbb{R}^n} |f(y)|^{p(y)}\, dy \int_{\mathbb{R}^n} |k(x)|^{q(x+y)}\, dx \right)^{\frac{1}{r}} \end{aligned}$$

where $v = 1 - q_+/r$ if $A \leq 1$ and $v = 1 - q_-/r$ if $A \geq 1$. Therefore

$$\|k*f\|_r \leq cA^v \left(\|k\|_{q_-}^{\frac{q_-}{r}} + \|k\|_{(p')_+}^{\frac{(p')_+}{r}} \right) \int_{\mathbb{R}^n} |f(y)|^{p(y)}\, dy$$

To finish the proof, we only need to take into account that the integral is bounded by 1 due to (2.14). □

2.2 Weak Lebesgue spaces with variable exponent

We give the definition of the Marcinkiewicz (weak Lebesgue) spaces with variable exponent and we study their relation with variable exponent Lebesgue spaces.

Definition 2.2.1. ([1]) Let $p \in C_+^0(\overline{\Omega})$, Ω an open bounded in $\mathbb{R}^n$. We say that a measurable function $u : \Omega \longrightarrow \mathbb{R}$ belongs to the Marcinkiewicz space $L^{p(.),\infty}(\Omega, \mu)$ if

$$\|u\|_{L^{p(.),\infty}(\Omega,\mu)} = \sup_{\lambda>0} \lambda \, \|D_u(\lambda)\|_{L^{p(.)}(\Omega)} < \infty.$$

The inequalities (2.14)-(2.15) imply that the requirement in Definition (2.2.1) is equivalent to say that, there exists a positive constant M such that

$$\int_{\{|u|>\lambda\}} \lambda^{p(x)} dx \leq M, \quad for \quad all \quad \lambda > 0. \tag{2.70}$$

If $p, q \in C_+^0(\overline{\Omega})$ with $q \leq p$, then we have the following two inclusions:

$$L^{p(.)}(\Omega) \subset L^{p(.),\infty}(\Omega, \mu) \subset L^{q(.),\infty}(\Omega, \mu).$$

Remark 2.2.2. If $u \in L^{q(.),\infty}(\Omega, \mu)$ with $q^- > 0$, then $\mu|u| > K \leq \dfrac{M + |\Omega|}{K^{q^-}}$, for all $K > 0$, where M is the constant appeared in (2.70). A direct result is that $\mu\,\{|u| > k\} \longrightarrow 0$, as $k \longrightarrow +\infty$.

Remark 2.2.3. Let $p, q \in C_+^0(\overline{\Omega})$. If $(p - q)^- > 0$, then

$$L^{p(.),\infty}(\Omega, \mu) \subset L^{q(.)}(\Omega, \mu)$$

2.3 Grand Lebesgue spaces

In this section we will introduce the so-called grand Lebesgue spaces.

2.3.1 Banach function spaces

In the following, we give the definitions and list some results regarding Banach Function Spaces. see Bennett and Sharpley [2] and Pick, Kufner, John, and $Fu\check{c}\acute{i}k$ [15] for the proofs.

In the sequel, Ω denotes an open subset Ω in $\mathbb{R}^n$. Let M_0 be the set of all measurable functions whose values lie in $[-\infty, \infty]$ and are finite a.e. in Ω. Also, let M_0^+ be the class of functions in M_0 whose values lie in $(0, \infty)$.

Definition 2.3.1. A mapping $\rho : M_0^+ \longrightarrow [0, \infty]$ is called a Banach function norm if for all f, g, f_n in $M_0^+, n \in \mathbb{N}$, for all constants $a \geq 0$ and all measurable subsets $E \subset \Omega$, the following properties hold:

(P7) $\rho(f) = 0$, if and only if $f = 0$ a.e. in Ω ;

(P7) $\rho(af) = a\rho(f)$;

(P7) $\rho(f + g) \leq \rho(f) + \rho(g)$;

(P7) $0 \leq g \leq f$ a.e. in Ω implies that $\rho(g) \leq \rho(f)$ (lattice property);

(P7) $0 \leq f_n \uparrow f$ a.e. in Ω implies that $\rho(f_n) \uparrow \rho(f)$ (Fatou's property);

(P7) $m(E) < +\infty$ implies that $\rho(X_E) < \infty$;

(P7) $m(E) < +\infty$ implies that $\int_E f dx \leq C_E \rho(f)$ (for some constant $C_E, 0 < C_E < \infty$, depending on E and ρ but independent of f).

It is noteworthy to mention that the lattice property is a consequence of the Fatou property, see Problem 2.72.

Based upon the notion of Banach function norm, we introduce the Banach function space X_ρ .

Definition 2.3.2. If ρ is a Banach function norm, the Banach space

$$X(\rho) = X_\rho = X = \{f \in M_0 : \rho(|f|) < +\infty\} \tag{2.71}$$

is called a Banach Function Space. For each $f \in X$ define

$$\|f\|_X = \rho(|f|). \tag{2.72}$$

There is also a notion of rearrangement invariant Banach function space, namely:

Definition 2.3.3. Let ρ be a Banach function norm. We say that the norm is rearrangement invariant if

$$\rho(f) = \rho(g)$$

for all equimeasurable functions f and g. In this case the Banach function space $X(\rho)$ is said to be a rearrangement invariant Banach function space.

A very important property of the Lebesgue space is its dual characterization, for example, in $L_p[(0,1)]$ we have

$$\|f\|_{L_p[(0,1)]} = \sup_{L_{p'}[(0,1)]} \int_0^1 f(x)g(x)dx$$

where p and p' are conjugate exponents. This characterization gives us immediately one of the key inequalities in the theory of Lebesgue spaces, namely the Hölder inequality which gives an upper bound for the integral of the product of two functions based upon their norms. The following notion is introduced to capture this "duality" in the framework of Banach function spaces.

Definition 2.3.4. If ρ is a Banach function norm, its associative function normρ' defined on M_0^+ is given by

$$\rho'(g) = \sup\left\{\int_\Omega fgdx : f \in M_0^+, \rho(f) \leq 1\right\}. \tag{2.73}$$

As in the case of Banach function space, we can introduce the associate Banach function space based upon the concept of Banach associative function norm.

Definition 2.3.5. Let ρ be a function norm and let $X = X(\rho)$ be the Banach function space determined by ρ. Let ρ' be the Banach associate function norm of ρ. The Banach function space $X' = X'(\rho')$ determined by ρ' is called the associate space of X. In particular from the definition of $\|f\|_X$ it follows that the norm of a function g in the associate space X' is given by

$$\|g\|_{X'} = \sup\left\{\int_\Omega fgdx : f \in M^+, \|f\|_X \leq 1\right\}.$$

Theorem 2.3.6. *Every Banach function space X coincides with its second associate space X''. This proposition tells us, in particular, that the notion of associate space is different from the notion of dual space, but under certain conditions both notions coincide, cf. Theorem 2.3.13.*

Theorem 2.3.7. *If X and Y are Banach function spaces and $X \hookrightarrow Y$, then $Y' \hookrightarrow X'$.*

Definition 2.3.8. *A* function f in a Banach function space X is said to have absolutely continuous norm on X if

$$\lim_{n\to\infty} \|f\chi_{E_n}\|_X = 0$$

for every sequence $\{En\}_{n=1}^{\infty}$ satisfying $E_n \downarrow 0$.

Definition 2.3.9. The subspace of functions in X with absolutely continuous norm is denoted by X a. If $X = X_a$, then the space X it self is said to have absolutely continuous norm.

Definition 2.3.10. Let X be a Banach function space. The closure in X of the set of bounded functions is denoted by X_b .

Theorem 2.3.11. *Let X be a Banach function space. Then $X_a \subseteq X_b \subseteq X$.*

Corollary 2.3.12. *If $X_a = X$, then $X_b = X$.*

Theorem 2.3.13. *The dual space X^* of a Banach function space X is canonically isometric to the associate space X' if and only if X has absolutely continuous norm.*

Theorem 2.3.14. *A Banach function space is reflexive if and only if both X and its associate space X' have absolutely continuous norm.*

2.3.2 Grand Lebesgue spaces

Definition 2.3.15. The grand Lebesgue space $L_{p)}(\Omega)$ is defined as the set of measurable functions on Ω for which

$$\|f\|_{p)} = \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{m(\Omega)} \int_\Omega |f|^{p-\varepsilon}\, dx \right)^{\dfrac{1}{p-\varepsilon}}$$

is finite, i.e.

$$L_{p)}(\Omega) = \left\{ f \in S(\Omega, L) : \|f\|_{p)} < \infty \right\},$$

where $1 < p < +\infty$. We stress that $m(\Omega) < \infty$. The following theorem justifies the nomenclature of grand Lebesgue space.

Theorem 2.3.16. *[5] For $p > 1$ we have*

$$L_p(\Omega) \subsetneq L_{p)}(\Omega).$$

Theorem 2.3.17. *[5] Let $1 < p < \infty$. We have the inclusion*

$$L_{(p,\infty)}(\Omega) \subset L_{p)}(\Omega).$$

Proof. Let $f \in L_{(p,\infty)}$, then

$$\frac{\varepsilon}{m(\Omega)} \int_\Omega |f|^{p-\varepsilon}\, dx = \frac{\varepsilon(p-\varepsilon)}{m(\Omega)} \int_0^\infty \lambda^{p-\varepsilon-1} D_f(\lambda) d\lambda \tag{2.74}$$

$$= \frac{\varepsilon(p-\varepsilon)}{m(\Omega)} \left[\int_0^a \lambda^{p-\varepsilon-1} D_f(\lambda) d\lambda + \int_a^\infty \lambda^{p-\varepsilon-1} D_f(\lambda) d\lambda \right]$$

We have that $\lambda^p D_f(\lambda) \le \|f\|^p_{L_{(p,\infty)}}$, then $D_f(\lambda) \le \lambda^{-p} \|f\|^p_{L_{(p,\infty)}}$, therefore from(2.74) we get

$$\frac{\varepsilon}{m(\Omega)} \int_\Omega |f|^{p-\varepsilon}\, dx \le \frac{\varepsilon(p-\varepsilon)}{m(\Omega)} \left[\frac{m(\Omega) a^{p-\varepsilon}}{p-\varepsilon} + \frac{a^{-\varepsilon}}{\varepsilon} \|f\|^p_{L_{(p,\infty)}} \right] \tag{2.75}$$

$$= \varepsilon a^{p-\varepsilon} + \frac{a^{-\varepsilon}}{\varepsilon} \frac{\varepsilon(p-\varepsilon)}{m(\Omega)} \|f\|^p_{L_{(p,\infty)}}.$$

Let $a = \|f\|_{L_{(p,\infty)}}$, replacing a in (2.75) we have

$$\frac{\varepsilon}{m(\Omega)} \int_\Omega |f|^{p-\varepsilon}\, dx \le \varepsilon \|f\|^{p-\varepsilon}_{L_{(p,\infty)}} + \frac{p-\varepsilon}{m(\Omega)} \|f\|^{p-\varepsilon}_{L_{(p,\infty)}}.$$

$$= \left(\varepsilon + \frac{p-\varepsilon}{m(\Omega)} \right) \|f\|^{p-\varepsilon}_{L_{(p,\infty)}}.$$

and thus

$$\sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{m(\Omega)} \int_\Omega |f|^{p-\varepsilon}\, dx \right)^{\frac{1}{p-\varepsilon}} \le C \|f\|_{L_{(p,\infty)}}$$

where $C = \sup_{0<\varepsilon<p-1} \left(\varepsilon + \dfrac{p-\varepsilon}{m(\Omega)} \right)^{\frac{1}{p-\varepsilon}}$, hence $L_{(p,\infty)} \subset L_{p)}$. We now show that the grand Lebesgue space is a Banach space under natural restrictions. □

Theorem 2.3.18. *Let $1 < p < \infty$. The grand Lebesgue space $L_{p)}(\Omega)$ is a Banach space.*

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $L_{p)}(\Omega)$, i.e.

$$\lim_{m\to\infty n\to\infty} \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{m(\Omega)} \int_\Omega |f_m - f_n|^{p-\varepsilon}\, dx \right)^{\frac{1}{p-\varepsilon}} = 0.$$

Hence for an arbitrary $\eta > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\left(\frac{\varepsilon}{m(\Omega)} \int_\Omega |f_m - f_n|^{p-\varepsilon}\, dx \right)^{\frac{1}{p-\varepsilon}} < \frac{\eta}{3}$$

for an arbitrary $\varepsilon, 0 < \varepsilon < p-1$, when $m > n_0, n > n_0$. Consequently $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L_{p-\varepsilon}(\Omega)$ for an arbitrary $\varepsilon, 0 < \varepsilon < p-1$, and let f be its limit in $L_{p-\varepsilon}(\Omega)$. Let $n > n_0$. According to the definition of the supremum there exists an ε_0 (de pending generally speaking on n), $0 < \varepsilon_0(n) < p-1$, such that

$$\begin{aligned} \|f - f_n\|_{p)} &= \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{|\Omega|} \int_\Omega |f - f_n|^{p-\varepsilon}\, dx \right)^{\frac{1}{p-\varepsilon}} \\ &\le \left(\frac{\varepsilon_0(n)}{m(\Omega)} \int_\Omega |f - f_n|^{p-\varepsilon_0(n)}\, dx \right)^{\frac{1}{p-\varepsilon_0(n)}} + \frac{\eta}{3} \end{aligned}$$

Furthermore, there exists $n_1 \in \mathbb{N}$ such that $m > n_1$

$$\left(\frac{\varepsilon_0(n)}{m(\Omega)} \int_\Omega |f_m - f_n|^{p-\varepsilon_0(n)}\, dx \right)^{\frac{1}{p-\varepsilon_0(n)}} < \frac{\eta}{3},$$

therefore

$$\begin{aligned} \|f - f_n\|_{p)} &\le \left(\frac{\varepsilon_0(n)}{m(\Omega)} \int_\Omega |f_m - f_n|^{p-\varepsilon_0(n)}\, dx \right)^{\frac{1}{p-\varepsilon_0(n)}} \\ &\quad + \left(\frac{\varepsilon_0(n)}{m(\Omega)} \int_\Omega |f_m - f|^{p-\varepsilon_0(n)}\, dx \right)^{\frac{1}{p-\varepsilon_0(n)}} + \frac{\eta}{3} \\ &< \frac{\eta}{3} + \frac{\eta}{3} + \frac{\eta}{3} = \eta \end{aligned}$$

whenever $n > n_1$ and $m > n_1$.

One of the drawbacks of grand Lebesgue spaces is the fact that the set of C_0^∞ functions is not a dense set. Fortunately we have a characterization of the closure of C_0^∞ functions in the grand Lebesgue norm given in a somewhat manageable way. □

Theorem 2.3.19. *The set$C_0^\infty(\Omega)$ is not dense in $L_{p)}(\Omega)$. Its closure $\overline{C_0^\infty}\ |_{L_{p)}(\Omega)}$ consists of functions $f \in L_{p)}(\Omega)$ such that*

$$\lim_{\varepsilon\longrightarrow 0} \varepsilon \int_\Omega |f|^{p-\varepsilon}\, dx = 0. \tag{2.76}$$

Proof. Let $f \in \overline{C_0^\infty}\,|_{L_{p)}(\Omega)}$, then there is a sequence of functions $fn \in C_0^\infty$ such that

$$\|f - f_n\|_{p)} \longrightarrow 0$$

as $n \longrightarrow \infty$.
Let us take $\delta > 0$. Choose n_0 such that

$$\|f - f_{n_0}\|_{p)} < \frac{\delta}{2} \quad and \quad f_{n_0} \in C_0^\infty.$$

Now observe that for f_{n_0}, by Hölder's inequality, we have

$$\left(\frac{\varepsilon}{m(\Omega)}\int_\Omega |f_{n_0}|^{p-\varepsilon}\,dx\right)^{\frac{1}{p-\varepsilon}} \leq \varepsilon^{\frac{1}{p-\varepsilon}}\left(\frac{1}{m(\Omega)}\int_\Omega |f_{n_0}|^{p}\,dx\right)^{\frac{1}{p}} \longrightarrow 0$$

as $\varepsilon \to 0$. Hence there is an $\varepsilon_0 > 0$ such that when $\varepsilon < \varepsilon_0$, we have the bound

$$\left(\frac{\varepsilon}{m(\Omega)}\int_\Omega |f_{n_0}|^{p-\varepsilon}\,dx\right)^{\frac{1}{p-\varepsilon}} < \frac{\delta}{2}$$

Finally

$$\begin{aligned}\left(\frac{\varepsilon}{m(\Omega)}\int_\Omega |f_{n_0}|^{p-\varepsilon}\,dx\right)^{\frac{1}{p-\varepsilon}} &\leq \left(\frac{\varepsilon}{m(\Omega)}\int_\Omega |f - f_{n_0}|^{p-\varepsilon}\,dx\right)^{\frac{1}{p-\varepsilon}} \\ &\quad + \left(\frac{\varepsilon}{m(\Omega)}\int_\Omega |f_{n_0}|^{p-\varepsilon}\,dx\right)^{\frac{1}{p-\varepsilon}} \\ &\leq \|f - f_{n_0}\|_{p)} + \frac{\delta}{2} \\ &\leq \frac{\delta}{2} + \frac{\delta}{2}\end{aligned}$$

when $\varepsilon < \varepsilon_0$. □

We now use the Theorem 2.3.14 which gives information regarding reflexivity of the space based upon the absolute continuity of the norm.

Theorem 2.3.20. *The spaces $L_{p)}(\Omega)$ is not reflexive.*

Proof. The non-reflexivity follows from the fact that there exists a function Φ for which the norm $\|\Phi\|_{p)}$ is not absolute continuous. Indeed taking the function Φ as

$$\Phi(x) = x^{-\frac{1}{p}}, x \in (0, 1),$$

we obtain

$$\lim_{a \longrightarrow 0} \sup_{\varepsilon > 0}\left(\varepsilon\int_0^a x^{-\frac{p-\varepsilon}{p}}\,dx\right)^{\frac{1}{p-\varepsilon}} \neq 0.$$

□

From Fiorenza and Karadzhov [5], We give the following characterization of the grand Lebesgue spaces (in the case $\mu(\Omega) = 1$, for simplicity):

$$\|f\|_{L_{p)}(\Omega)} = \sup_{0<t<1} (1 - \log t)^{-\frac{1}{p}} \left(\int_t^1 |f^*(s)|^p \, ds \right)^{\frac{1}{p}},$$

where f^* is a decreasing rearrangement of f defined as

$$f^*(t) = \sup_{m(E)=t} \inf_E f$$

with $t \in (0, 1)$.

We can introduce a generalization of the grand Lebesgue spaces, namely the spaces $L_{p),\theta}(\Omega), \theta > 0$, defined by

$$\|f\|_{p),\theta} = \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon^\theta}{m(\Omega)} \int_\Omega |f|^{p-\varepsilon} \, dx \right)^{\frac{1}{p-\varepsilon}}.$$

For $\theta = 0$ we have $\|f\|_{p),\theta} = \|f\|_p$ and for $\theta = 1$ such spaces reduce obviously to the spaces $L_{p)}(\Omega)$.

Many results of grand Lebesgue spaces are also valid for generalized grand Lebesgue spaces, we will just mention the following:

Theorem 2.3.21. *The subspace $C_0^\infty(\Omega)$ is not dense in $f \in L_{p),\theta)}(\Omega)$. Its closure consists of functions $f \in L_{p),\theta}(\Omega)$ such that*

$$\lim_{\varepsilon \to 0} \varepsilon^{\frac{\theta}{p}} \|f\|_{p-\varepsilon} = 0.$$

2.3.3 Hardy's inequality

We recall the classical Hardy inequality for Lebesgue spaces

$$\left(\int_0^1 \left(\frac{1}{x} \int_0^x f(y) dy \right)^p dx \right)^{\frac{1}{p}} \leq \frac{p}{p-1} \left(\int_0^1 f^p(x) dx \right)^{\frac{1}{p}}.$$

Here we discuss the Hardy inequality in grand Lebesgue spaces to show some common techniques used in the aforementioned spaces.

Theorem 2.3.22. *Let $1 < p < \infty$. There exists a constant $C(p) > 1$ such that*

$$\left\| \frac{1}{x} \int_0^x f(y) dy \right\|_{L_{p)}([0,1])} \leq C(p) \|f\|_{L_{p)}([0,1])}$$

for non negative measurable functions f on $[0, 1]$.

Conclusion

In this memory, we have studied the so-called weak Lebesgue spaces with variable exponents (Marcinkiewicz) that are one of the most important spaces used to solve some boundary problems. This work raises a number of questions that deserve to be addressed. For instance, it would be wise to think in perspective of following:

- Is the next lemma true if we assume $p : \overline{\Omega} \longrightarrow (1, +\infty)$ is a continuous fonction instead of p where Ω an open bounded in $\mathbb{R}^n$?

  Assume that $(\mathbb{R}^n, A, \mu)$ is a measure space and f is a measurable function that satisfies

  $$\mu\left(\{x \in \mathbb{R}^n : |f(x)| > \lambda\}\right) \leq \left(\frac{c}{\lambda}\right)^p$$

  for some $C > 0$. Then

  $$\inf\left\{c > 0 : D_f(\lambda) \leq \left(\frac{c}{\lambda}\right)^p\right\} = \left(\sup_{\lambda>0} \lambda^p D_f(\lambda)\right)^{\frac{1}{p}} = \sup_{\lambda>0} \lambda \left(D_f(\lambda)\right)^{\frac{1}{p}},$$

- Does the next injections hold for $\Omega = \mathbb{R}^n$?

  $$L^{p(.)}(\Omega) \subset L^{p(.),\infty}(\Omega, \mu) \subset L^{q(.),\infty}(\Omega, \mu)$$

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ملخص:

في هذه المذكرة ، قمنا بدراسة ما يسمى بفضاءات مارسينكويتش(فضاءات ليباق الضعيفة) ذات الاسس المتغيرة والتي تعد من اولى التعميمات لمساحات ليباق.
في اطار عمل فضاءات مارسينكويتش ،سوف ندرس ، من بين موضوعات أخرى ،نتائج التضمين ،والتقارب في القياس ،ونتائج التوليد ،ومسألة قابلية القياس للفضاء .
نعرض أيضا نوع فاتو لمساحات مارسينكويتش بالإضافة الى اكتمال شبه –معيار . يظهر المساواة في ليابونوف والمساواة في هولدر.

كلمات مفتاحية : فضاءات ليباق الضعيفة ، دالة التوزيع ، القياس ، فضاءات ليباق متغيرة الاساس ، المعايير ، مساوات هولدر، دالة فضاءات بناخ.

Abstract :

In this memory, we study the so-called Marcinkiewicz spaces (weak Lebesgue spaces) with variable exponents which are one of the first generalizations of the Lebesgue spaces. In the framework of Marcinkiewicz spaces we will study, among other topics, embedding results, convergence in measure, interpolation results, and the question of normability of the space. We also show a Fatou type lemma for Marcinkiewicz spaces as well as the completeness of the quasi-norm. The Lyapunov inequality and the Hölder inequality are shown to hold.

Keywords : Weak Lebesgue spaces,The Distribution Function,measure ,the variable exponent Lebesgue space , The norms, Hölder Inequality,Banach Function Spaces.

Résumé :

Dans ce mémoire, nous étudions les espaces dits de Marcinkiewicz (espaces de Lebesgue faibles) avec exposants variables qui sont une des premières généralisations des espaces de Lebesgue. Dans le cadre des espaces de Marcinkiewicz, nous étudierons, entre autres sujets, l'intégration des résultats, la convergence en mesure, les résultats d'interpolation et la question de la normabilité de l'espace. Nous montrent également un lemme de type Fatou pour les espaces de Marcinkiewicz ainsi que la complétude des quasi-norme. L'inégalité de Lyapunov et l'inégalité de Hölder se vérifient.

Mots clés : espaces de Lebesgue faibles, fonction de distribution, mesure, espace de Lebesgue à exposant variable, normes, inégalité de Hölder, espaces de fonctions de Banach.