Tensor characterizations of summing polynomials

Dahmane Achour, Ahlem Alouani, Pilar Rueda and Enrique A. Sánchez Pérez

Abstract. Operators T that belong to some summing operator ideal, can be characterized by means of the continuity of an associated tensor operator $\overline{T}$ that is defined between tensor products of sequences spaces. In this paper we provide a unifying treatment of these tensor product characterizations of summing operators. We work in the more general frame provided by homogeneous polynomials, where an associated "tensor" polynomial —which plays the role of $\overline{T}$—, needs to be determined first. Examples of applications are shown.

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1. Introduction

Tensor products have proved to be a useful tool for the theory of operator ideals. Indeed, the excellent monograph [16] deals with the tensor product point of view of the theory and provides many applications to the study of the structure of several spaces of summing linear operators. In the last decades this linear theory has spreaded to non-linear contexts that include multilinear mappings, polynomials, holomorphic functions or Lipschitz mappings among others. Transferring summability properties to non linear mappings is not an obvious task as shows the variety of different generalizations of several classes of summing operators to the multilinear case, and to hit the multilinear class that is closest, in some sense, to the original linear class is not trivial (see e.g. [12, 10, 21, 22]). Even more complicated is working with polynomials, where important lacks of basic results, as a Pietsch type factorization theorem for dominated polynomials, proved deep differences between the linear and the polynomial theories (e.g. [9, 11, 24]). The way that summing linear and multilinear mappings transform vector-valued sequences is the essence of the theory of summing operators. Botelho and Campos [7] show how these transformations can be treated from an unified point of view, and recover in detail some former inaccuracies that have appeared in the literature. If $S(X)$ denotes a X-valued sequence space, the summability with respect to S of an operator $T : X \to Y$ (being X and Y Banach spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) is related to the associated operator $\widehat{T} : S(X) \to S(Y)$ given by $\widehat{T}((x_i)_i) := (T(x_i))_i$. For instance, we will work with the following vector-valued sequence spaces ($1 \le p \le \infty$):

- $\ell_p(X)$, the space of all *absolutely p-summable sequences* in X; that is, sequences $(x_i)_i$ in X such that
$$\|(x_i)_i\|_{\ell_p(X)} := (\sum_i \|x_i\|^p)^{1/p} < \infty,$$
if $1 \le p < \infty$ or,
$$\|(x_i)_i\|_{\ell_\infty(X)} := \sup_i \|x_i\|,$$
if $p = \infty$,
- $\ell_p^w(X)$, the space of all *weakly p-sequences* in X; that is, sequences $(x_i)_i$ in X such that
$$\|(x_i)_i\|_{\ell_p^w(X)} := \sup_{x^* \in X^*, \|x^*\| \le 1} (\sum_i |x^*(x_i)|^p)^{1/p} < \infty,$$
if $1 \le p < \infty$ or,
$$\|(x_i)_i\|_{\ell_\infty^w(X)} := \sup_i \sup_{x' \in X^*, \|x'\| \le 1} |x'(x_i)| = \sup_i \|x_i\|,$$
if $p = \infty$,
- $\ell_p\langle X\rangle$, the space of all *strongly p-summable sequences* in X; that is, sequences $(x_i)_i$ in X such that
$$\|(x_i)_i\|_{\ell_p\langle X\rangle} := \sup_{(x_i^*)_i \in \ell_{p'}^w(X^*), \|x^*\|_{\ell_{p'}^w(X^*)} \le 1} \Big|\sum_i x_i^*(x_i)\Big| < \infty.$$

Several classes of linear operators that relate weakly/absolutely/strongly summable sequences have been extensively treated in the literature (e.g.[8, 16, 17, 23]).

- Let $1 \le p < \infty$. An operator $T : X \to Y$ is *absolutely p-summing* if there is a constant $C \ge 0$ such that
$$\Big\|\big(T(x_i)\big)_{i=1}^n\Big\|_{\ell_p(Y)} \le C\|(x_i)_{i=1}^n\|_{\ell_p^w(X)}$$
for all $x_1, \dots, x_n \in X$ and all $n \in \mathbb{N}$.
- Let $1 < p < \infty$. An operator $T : X \to Y$ is *Cohen p-nuclear* if there is a constant $C \ge 0$ such that
$$\Big\|\big(T(x_i)\big)_{i=1}^n\Big\|_{\ell_p\langle Y\rangle} \le C\|(x_i)_{i=1}^n\|_{\ell_p^w(X)}$$
for all $x_1, \dots, x_n \in X$ and all $n \in \mathbb{N}$.

- Let $1 < p \leq \infty$. An operator $T : X \to Y$ is *strongly p-summing* if there is a constant $C \geq 0$ such that
$$\left\|\big(T(x_i)\big)_{i=1}^n\right\|_{\ell_p\langle Y\rangle} \leq C\|(x_i)_{i=1}^n\|_{\ell_p(X)}$$
for all $x_1, \ldots, x_n \in X$ and all $n \in \mathbb{N}$.

These classes of linear operators T can be characterized by means of the associated operator $\widehat{T}$ (see [15]):

- T is absolutely p-summing if and only if $\widehat{T} : \ell_p^w(X) \to \ell_p(Y)$ is continuous.
- T is Cohen p-nuclear if and only if $\widehat{T} : \ell_p^w(X) \to \ell_p\langle Y\rangle$ is continuous.
- T is strongly p-summing if and only if $\widehat{T} : \ell_p(X) \to \ell_p\langle Y\rangle$ is continuous.

Indeed, Botelho and Campos [7] have unified these results by analyzing the transformations of vector-valued sequences by multilinear operators. In this paper, we provide the tensor product counterpart of this unifying approach in the context of polynomials.

The tensor product $\ell_p \otimes X$ can be seen as a subspace of $X^{\mathbb{N}}$ via the algebraic isomorphism into $s_{p,X} : \ell_p \otimes X \to X^{\mathbb{N}}$ given by $s_{p,X}(\sum_{i=1}^n (a_{ij})_j \otimes x_i) := (\sum_{i=1}^n a_{ij}x_i)_j$. The following facts are well-known and can be found in e.g. [15]:

- Let $1 \leq p \leq \infty$ and let p' be the conjugate of p, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. The space $\ell_p^w(X)$ induces the injective norm ε on $\ell_p \otimes X$, defined as $\varepsilon(u) := \sup_{\|x'\|_{\ell_{p'}} \leq 1, \|y'\| \leq 1} \left|\sum_{i=1}^n x'(x_i)y'(y_i)\right|$, for any $u = \sum_{i=1}^n x_i \otimes y_i \in \ell_p \otimes X$.
- Let $1 \leq p \leq \infty$. The space $\ell_p(X)$ induces the Δ_p norm on $\ell_p \otimes X$, defined as $\Delta_p(\sum_{i=1}^n e_i \otimes x_i) = (\sum_{i=1}^n \|x_i\|^p)^{1/p}$.
- Let $1 \leq p < \infty$. The space $\ell_p\langle X\rangle$ induces the projective norm π on $\ell_p \otimes X$, defined as $\pi(u) := \inf \sum_{i=1}^n \|x_i\|_{\ell_p}\|y_i\|$ where the infimum is taken over all representations of $u = \sum_{i=1}^n x_i \otimes y_i \in \ell_p \otimes X$.

The following characterizations (see [15]) provide nice examples of how tensor products come into the theory of summing operators:

- An operator $T : X \to Y$ is absolutely p-summing if and only if $I \otimes T : \ell_p \otimes_\varepsilon X \to \ell_p \otimes_{\Delta_p} Y$ is continuous.
- An operator $T : X \to Y$ is strongly p-summing if and only if $I \otimes T : \ell_p \otimes_{\Delta_p} X \to \ell_p \otimes_\pi Y$ is continuous.
- Let $1 < p < \infty$. An operator $T : X \to Y$ is Cohen p-nuclear if and only if $I \otimes T : \ell_p \otimes_\varepsilon X \to \ell_p \otimes_\pi Y$ is continuous.

A m-homogeneous polynomial is a mapping $P : X \to Y$ such that $P(x) = A(x, \ldots, x)$, $x \in X$, for some m-linear operator. Among all m-linear operators whose restriction to the diagonal is P, there is only one that is symmetric (i.e. $A(x_1, \ldots, x_m) = A(x_{\sigma(1)}, \ldots, x_{\sigma(m)})$ for any permutation σ of $\{1, 2, \ldots, m\}$). This unique symmetric m-linear operator associated to P will be denoted by $\overset{\circ}{P}$. We write $\mathcal{P}(^mX; Y)$ for the set of all continuous m-homogeneous polynomials from X to Y. Every continuous m-homogeneous polynomial is bounded on bounded sets and so $\mathcal{P}(^m X;Y)$ becomes a Banach space when endowed with the supremum norm $\|P\| := \sup_{\|x\|\leq 1}\|P(x)\|$, $P\in\mathcal{P}(^m X;Y)$.

The interplay between tensor products and summing polynomials has been explored for a long time, for example in [19, 18, 14, 11, 13]. In this paper we unify and characterize particular classes of summing polynomials by introducing an associated polynomial defined between tensor product subspaces of vector-valued sequences spaces. This approach can be applied to several classes of summing polynomials, as p-dominated polynomials, strongly Cohen p-summing polynomials or Cohen p-nuclear polynomials. Note that $\mathcal{P}(^1 X;Y)$ coincides with the space of all continuous linear operators from X to Y endowed with the usual norm, and so, to unify the linear case it just suffices to take $m=1$. In that case, $P=T$ is a continuous linear operator and the associated polynomial defined between tensor product spaces is nothing but the associated tensor product operator (see the next section for definitions). The concept of finitely determined sequence classes introduced in [7], that was the key of that study, also plays a fundamental role when dealing with the transformation of tensor product spaces by homogeneous polynomials.

2. Associated polynomials

Given a linear operator $T: X\to Y$, its associated tensor product operator $I\otimes T:\ell_p\otimes X\to\ell_p\otimes Y$ is defined by

$$I\otimes T\Big(\sum_{i=1}^n (c_{ij})_j\otimes x_i\Big) := \sum_{i=1}^n (c_{ij})_j\otimes T(x_i),$$

and this map is clearly linear. When dealing with a m-homogeneous polynomial $P: X\to Y$ one can be tempted naïvely to replace the T by the P in the above definition with the hope to have an associated polynomial. However, a quick look makes us to refuse such an approach as the resulting map is not even well defined when $m\geq 2$. So, some extra work is required to introduce an associated tensor polynomial that plays the role of $I\otimes T$.

This kind of tensor product of homogeneous polynomials has already been considered in the literature (see, e.g., [6, Section 6] and [4]). For our purposes we will consider the vector space ℓ_p^0 of all sequences in ℓ_p with all entries 0 but finitely many. Let e_j denote the canonical unit vector of ℓ_p^0 with 1 in the jth coordinate and 0 otherwise. Note that if u belongs to $\ell_p^0\otimes X$ then there exist non-necessarily unique $(a_{ij})_{j=1}^{k_i}\in\ell_p^0$ and $x_i\in X$, $i=1,\ldots,n$, so that

$$u=\sum_{i=1}^n (a_{ij})_{j=1}^{k_i}\otimes x_i.$$

Adding 0 if necessary, we can assume that all the k_i are equal. Let us denote them by k. Now define the associated "tensor" polynomial $\overline{P}:\ell_{mp}^0\otimes X\to$

$\ell_p \otimes Y$ by

$$\overline{P}(\sum_{i=1}^{n}(a_{ij})_{j=1}^{k} \otimes x_i) := \sum_{j=1}^{k} e_j \otimes P(\sum_{i=1}^{n} a_{ij}x_i).$$

To check that the map $\overline{P}$ is well defined we do

$$u = \sum_{i=1}^{n}(a_{ij})_{j=1}^{k} \otimes x_i = \sum_{i=1}^{n}\sum_{j=1}^{k} a_{ij}e_j \otimes x_j = \sum_{j=1}^{k} e_j \otimes y_j$$

where $y_j := \sum_{i=1}^{n} a_{ij}x_i$, $j = 1, \ldots, k$. An easy calculation shows that the representation of an element $u \in \ell_p^0 \otimes X$ of the form $u = \sum_{j=1}^{k} e_j \otimes y_j$ with $y_j \in X$ is unique and so $\overline{P}$ is well defined.

When we take $k = n$ and $a_{ij} := 1$ if $i = j$ and 0 otherwise, in particular we get

$$\overline{P}(\sum_{i=1}^{n} e_i \otimes x_i) = \sum_{i=1}^{n} e_i \otimes P(x_i).$$

In [25] tensor products have been used to characterize summability properties of linear and multilinear operators by means of an "order reduction" procedure and the calculus of traced tensor norms. The map $\overline{P}$ is the restriction to the diagonal of the m-linear symmetric operator

$$\overline{T} : (\ell_{mp}^0 \otimes X) \times (\ell_{mp}^0 \otimes X) \times \cdots \times (\ell_{mp}^0 \otimes X) \to \ell_p^0 \otimes Y$$

defined as

$$\overline{T}((\sum_{i=1}^{n} e_i \otimes x_i^1, \ldots, \sum_{i=1}^{n} e_i \otimes x_i^m)) := \sum_{i=1}^{n} e_i \otimes T(x_i^1, \ldots, x_i^m),$$

where T is the unique symmetric m-linear operator such that $T(x, \ldots, x) = P(x)$. Therefore $\overline{P} : \ell_{mp}^0 \otimes X \to \ell_p^0 \otimes Y$ is a m-homogeneous polynomial and $\overset{\circ}{\overline{P}} = \overline{\overset{\circ}{P}}$.

The class of all Banach spaces over $\mathbb{K}$ is denoted by BAN and if $X, Y \in$ BAN then $X \overset{1}{\hookrightarrow} Y$ means that X is a linear subspace of Y and $\|x\|_Y \leq \|x\|_X$ for all $x \in X$. The set of all sequences in X with all entries 0 but finitely many is denoted by $c_{00}(X)$. We take from [7] the following definition, that will be of interest in our study.

Definition 2.1. A *class of vector-valued sequences* S, or simply a *sequence class* S, is a rule that assigns to each $X \in$ BAN a Banach space $S(X)$ of X-valued sequences such that

$$c_{00}(X) \subset S(X) \overset{1}{\hookrightarrow} \ell_\infty(X) \text{ and } \|e_j\|_{S(\mathbb{K})} = 1 \text{ for every } j.$$

A sequence class S is *finitely determined* if for every sequence $(x_j)_{j=1}^{\infty} \in X^{\mathbb{N}}$, $(x_j)_{j=1}^{\infty} \in S(X)$ if and only if $\sup_k \|(x_j)_{j=1}^{k}\|_{S(X)} < +\infty$ and, in this case,

$$\|(x_j)_{j=1}^{\infty}\|_{S(X)} = \sup_k \|(x_j)_{j=1}^{k}\|_{S(X)}.$$

The sequences classes $\ell_\infty(\cdot)$, $\ell_p(\cdot)$, $\ell_p^w(\cdot)$ and $\ell_p\langle\cdot\rangle$ are finitely determined [7, Remark 1.3].

Given a m-homogeneous polynomial $P : X \to Y$, let us consider the associated m-homogeneous polynomial $\widehat{P} : X^{\mathbb{N}} \to Y^{\mathbb{N}}$ naturally defined by $\widehat{P}((x_i)_i) := (P(x_i))_i$.

The linear space $\ell_p^0 \otimes X$ can be seen as a vector subspace of $X^{\mathbb{N}}$ by means of the map

$$s_{p,X}(\sum_{i=1}^n (a_{ij})_{j=1}^k \otimes x_i) := (\sum_{i=1}^n a_{ij}x_i)_{j=1}^k,$$

which is an algebraic isomorphism into.

Lemma 2.2. *If $P : X \to Y$ is a m-homogeneous polynomial then*

$$\widehat{P} \circ s_{mp,X} = s_{p,Y} \circ \overline{P}.$$

Proof. For $\sum_{i=1}^n (a_{ij})_{j=1}^k \otimes x_i \in \ell_p^0 \otimes X$ we have

$$\begin{aligned} \widehat{P} \circ s_{mp,X}\left(\sum_{i=1}^n (a_{ij})_{j=1}^k \otimes x_i\right) &= \widehat{P}\left((\sum_{i=1}^n a_{ij}x_i)_{j=1}^k\right) = \left(P(\sum_{i=1}^n a_{ij}x_i)\right)_{j=1}^k \\ &= s_{p,Y}\left(\sum_{j=1}^k e_j \otimes P(\sum_{i=1}^n a_{ij}x_i)\right) \\ &= s_{p,Y} \circ \overline{P}\left(\sum_{i=1}^n (a_{ij})_{j=1}^k \otimes x_i\right). \end{aligned}$$

□

Lemma 2.3. *Let S_1 and S_2 be two finitely determined sequence classes. Let $P \in \mathcal{P}(^m X; Y)$ be such that $\widehat{P}(S_1(X)) \subset S_2(Y)$. Then $c_{00}(X)$ is a norming set for $\widehat{P} : S_1(X) \to S_2(Y)$.*

Proof. Let us write $N(\widehat{P}) := \sup \|(P(x_i))_i\|_{S_2(Y)}$, where the supremum is taken over all $(x_i)_i \in c_{00}(X)$ with $\|(x_i)_i\|_{S_1(X)} \leq 1$. Clearly $N(\widehat{P}) \leq \|\widehat{P}\|$. If $N(\widehat{P}) = \infty$ there is nothing to be proved. If we assume that $N(\widehat{P}) < \|\widehat{P}\|$, there is $(x_i)_{i=1}^\infty \in S_1(X)$ with $\|(x_i)_{i=1}^\infty\|_{S_1(X)} \leq 1$ such that $N(\widehat{P}) < \|(P(x_i))_{i=1}^\infty\|_{S_2(Y)}$. Since $S_1(X)$ and $S_2(Y)$ are finitely determined,

$$\|(x_i)_{i=1}^N\|_{S_1(X)} \leq \|(x_i)_{i=1}^\infty\|_{S_1(X)} \leq 1$$

for every $N \in \mathbb{N}$ and

$$\|(P(x_i))_{i=1}^\infty\|_{S_2(Y)} = \sup_N \|(P(x_i))_{i=1}^N\|_{S_2(Y)} \leq N(\widehat{P}),$$

which is a contradiction. □

From now on we consider two classes of vector-valued sequences S_1 and S_2, and $P \in \mathcal{P}(^m X; Y)$ so that $\widehat{P}(S_1(X)) \subset S_2(Y)$. Note that

$$s_{mp,X}(\ell_{mp}^0 \otimes X) \subset c_{00}(X) \subset S_1(X) \text{ and } s_{p,Y}(\ell_p^0 \otimes Y) \subset c_{00}(Y) \subset S_2(Y).$$

Therefore, the following diagram arises

$$\begin{array}{ccc} S_1(X) & \xrightarrow{\widehat{P}} & S_2(Y) \\ {\scriptstyle s_{mp,X}} \uparrow & & \uparrow {\scriptstyle s_{p,Y}} \\ \ell_{mp}^0 \otimes X & \xrightarrow{\overline{P}} & \ell_p^0 \otimes Y \end{array}$$

that, in virtue of Lemma 2.2, commutes.

Theorem 2.4. *Let X and Y be Banach spaces and let S_1 and S_2 be sequence classes. Let $P \in \mathcal{P}(^m X; Y)$ be so that $\widehat{P}(S_1(X)) \subset S_2(Y)$. Let α and β be norms on $\ell_{mp}^0 \otimes X$ and $\ell_p^0 \otimes Y$ respectively so that $s_{mp,X} : \ell_{mp}^0 \otimes_\alpha X \to S_1(X)$ and $s_{p,Y} : \ell_p^0 \otimes_\beta Y \to S_2(Y)$ are continuous.*

1. *If $s_{p,Y}$ is an isometry into then $\overline{P} : \ell_{mp}^0 \otimes_\alpha X \to \ell_p^0 \otimes_\beta Y$ is continuous whenever $\widehat{P} : S_1(X) \to S_2(Y)$ is continuous. In this case $\|\overline{P}\| \leq \|\widehat{P}\| \|s_{mp,X}\|$.*
2. *If $S_1(X)$ and $S_2(Y)$ are finitely determined and $s_{mp,X}$ is an isometry into then $\widehat{P} : S_1(X) \to S_2(Y)$ is continuous whenever $\overline{P} : \ell_{mp}^0 \otimes_\alpha X \to \ell_p^0 \otimes_\beta Y$ is continuous. In this case $\|\widehat{P}\| \leq \|\overline{P}\| \|s_{p,Y}\|$.*

Proof. (1) It follows immediately from Lemma 2.2 and the hypothesis on $s_{p,Y}$ being an isometry.

(2) Since $S_1(X)$ and $S_2(Y)$ are finitely determined, by Lemma 2.3 $c_{00}(X)$ is a norming set for $\widehat{P} : S_1(X) \to S_2(Y)$. Take $(x_i)_{i=1}^n \in c_{00}(X)$ with $\|(x_i)_{i=1}^n\|_{S_1(X)} \leq 1$. Then,

$$\begin{aligned} \big\|\widehat{P}\big((x_i)_{i=1}^n\big)\big\|_{S_2(Y)} &= \big\|\big(P(x_i)\big)_{i=1}^n\big\|_{S_2(Y)} = \Big\|\Big(s_{p,Y}\big(\sum_{i=1}^n e_i \otimes P(x_i)\big)\Big)\Big\|_{S_2(Y)} \\ &= \Big\|s_{p,Y}\Big(\overline{P}\big(\sum_{i=1}^n e_i \otimes x_i\big)\Big)\Big\|_{S_2(Y)} \\ &\leq \|s_{p,Y}\| \beta\Big(\overline{P}\big(\sum_{i=1}^n e_i \otimes x_i\big)\Big) \\ &\leq \|s_{p,Y}\| \|\overline{P}\| \alpha\big(\sum_{i=1}^n e_i \otimes x_i\big)^m = \|s_{p,Y}\| \|\overline{P}\| \big\|(x_i)_{i=1}^n\big\|_{S_1(X)}^m. \end{aligned}$$

□

Corollary 2.5. *Let X and Y be Banach spaces and let S_1 and S_2 be sequence classes. Let $P \in \mathcal{P}(^m X; Y)$ so that $\widehat{P}(S_1(X)) \subset S_2(Y)$. Let α and β be norms on $\ell_{mp}^0 \otimes X$ and $\ell_p^0 \otimes Y$ respectively so that $s_{mp,X} : \ell_{mp}^0 \otimes_\alpha X \to S_1(X)$ and $s_{p,Y} : \ell_p^0 \otimes_\beta Y \to S_2(Y)$ are isometries into. If $S_1(X)$ and $S_2(Y)$ are* *finitely determined then* $\widehat{P} : S_1(X) \to S_2(Y)$ *is continuous if and only if* $\overline{P} : \ell^0_{mp} \otimes_\alpha X \to \ell^0_p \otimes_\beta Y$ *is continuous. In this case* $\|\widehat{P}\| = \|\overline{P}\|$.

The next result is the polynomial version of [7, Proposition 1.4]. Note that although the proof cannot be adapted straightforwardly to polynomials (because it uses the closed graph theorem for multilinear operators), it still remains true.

Proposition 2.6. *Let* $m \in \mathbb{N}$, $P \in ({}^m X; Y)$ *and let* S_1 *and* S_2 *be sequence classes. The following are equivalent:*

1. $(P(x_i))_{i=1}^\infty \in S_2(Y)$ *whenever* $(x_i)_{i=1}^\infty \in S_1(X)$.
2. *The induced map* $\widehat{P} : S_1(X) \to S_2(Y)$ *is a well-defined continuous* m*-homogeneous polynomial.*

The conditions above imply condition (3) below, and they are all equivalent if the sequence classes S_1 *and* S_2 *are finitely determined.*

(3) *There is a constant* $C > 0$ *such that* $\left\|\left(P(x_i)\right)_{i=1}^n\right\|_{S_2(Y)} \leq C\|(x_i)_{i=1}^n\|_{S_1(X)}$ *for all* $x_1, \ldots, x_n \in X$ *and all* $n \in \mathbb{N}$.

In this case, $\|\widehat{P}\| = \inf\{C : (1) \text{ holds } \}$.

Proof. (2) implies (1) clearly. Assuming (2), it is also clear that $\widehat{P}$ is well-defined and a m-homogeneous polynomial. Let us prove the continuity. It suffices to be proved that the associated m-linear symmetric operator $\overset{\circ}{\widehat{P}}$ is continuous. Consider the m-linear operator induced by $\overset{\circ}{P}$, that is, $\widehat{\overset{\circ}{P}} : S_1(X) \times \cdots \times S_1(X) \to S_2(Y)$ given by $\widehat{\overset{\circ}{P}}((x_i^1)_{i=1}^\infty, \ldots, (x_i^m)_{i=1}^\infty) := \left(\overset{\circ}{P}(x_i^1, \ldots, x_i^m)\right)_{i=1}^\infty$, $(x_i^j)_{i=1}^\infty \in S_1(X)$, $j = 1, \ldots, m$. By the polarization formula (see e.g. [20, Theorem 1.10]), for each $i \in \mathbb{N}$

$$\overset{\circ}{P}(x_i^1, \ldots, x_i^m) = \frac{1}{m!2^m} \sum_{\varepsilon_1, \ldots, \varepsilon_m = \pm 1} \varepsilon_1 \cdots \varepsilon_m P(\varepsilon_1 x_i^1 + \cdots + \varepsilon_m x_i^m). \quad (1)$$

Since $S_1(X)$ is a Banach space, the sequence $(\varepsilon_1 x_i^1 + \cdots + \varepsilon_m x_i^m)_{i=1}^\infty$ belongs to $S_1(X)$. Then, by (1) the sequence $(P(\varepsilon_1 x_i^1 + \cdots + \varepsilon_m x_i^m))_{i=1}^\infty \in S_2(Y)$. The equality (1) gives now that the sequence $\left(\overset{\circ}{P}(x_i^1, \ldots, x_i^m)\right)_{i=1}^\infty \in S_2(Y)$. From Proposition 1.4 in [7] we get that $\widehat{\overset{\circ}{P}}$ is well-defined and continuous. Since $\overset{\circ}{\widehat{P}} = \widehat{\overset{\circ}{P}}$, it follows that $\widehat{P}$ is continuous and condition (2) is proved. We have actually shown that if P satisfies (1) then $\overset{\circ}{P}$ satisfies [7, Proposition 1.4.(a)], part (c) of that result gives easily (3). The rest of the proof follows the lines of [7, Proposition 1.4]. Immediately one gets (2) implies (3) and that $\|\widehat{P}\| \geq \inf\{C : (1) \text{ holds } \}$. We now assume (3) and that $S_1(X)$ and $S_2(Y)$ are finitely determined. Taking the supremum over n in (3) we get its equivalence with (2) and $\|\widehat{P}\| \leq \inf\{C : (1) \text{ holds } \}$. □

3. Applications to classes of summing polynomials

We apply now Theorem 2.4 and Proposition 2.6 to some classes of summing polynomials to get new characterizations in terms of tensor product transformations and also to recover probably known characterizations of these classes in terms of transformations of vector-valued sequences. With our approach, all the results are straightforward applications of Corollary 2.5 and Proposition 2.6, just using that the sequences classes $\ell_\infty(\cdot)$, $\ell_p(\cdot)$, $\ell_p^w(\cdot)$ and $\ell_p\langle\cdot\rangle$ are finitely determined [7, Remark 1.3] and that the maps $s_{p,X} : \ell_p \otimes_\varepsilon X \to \ell_p^w(X)$, $s_{p,X} : \ell_p \otimes_{\Delta_p} X \to \ell_p(X)$ and $s_{p,X} : \ell_p \otimes_\pi X \to \ell_p\langle X\rangle$ are isometries into.

3.1. p-dominated polynomials

Let $m \in \mathbb{N}$, $m \le p < \infty$ and let X and Y be Banach spaces. A m-homogeneous polynomial $P \in \mathcal{P}(^mX;Y)$ is *p-dominated* if there is a constant $C \ge 0$ such that

$$\left\| \big(P(x_i)\big)_{i=1}^n \right\|_{\ell_{p/m}(Y)} \le C \|(x_i)_{i=1}^n\|_{\ell_{p,w}(X)}^m$$

for all $x_1, \ldots, x_n \in X$ and all $n \in \mathbb{N}$. The infimum of all such $C > 0$ defines a norm on the space $\mathcal{P}_{p,d}(^mX;Y)$ of all p-dominated m-homogeneous polynomials from X to Y, that we denote $\|P\|_{p,d}$. For more information on p-dominated polynomials we refer to [5] and the references therein.

Corollary 3.1. *Let $m \in \mathbb{N}$, $m \le p < \infty$ and $P \in \mathcal{P}(^mX;Y)$. The following are equivalent:*

1. *P is p-dominated.*
2. *$(P(x_i))_{i=1}^\infty \in \ell_{p/m}(Y)$ whenever $(x_i)_{i=1}^\infty \in \ell_{p,w}(X)$.*
3. *The induced map $\widehat{P} : \ell_{p,w}(X) \to \ell_{p/m}(Y)$ is a well-defined continuous m-homogeneous polynomial.*
4. *The induced m-homogeneous polynomial $\overline{P} : \ell_{p,w}^0 \otimes_\varepsilon X \to \ell_{p/m}^0 \otimes_{\Delta_{p/m}} Y$ is continuous.*

In this case $\|P\|_{p,d} = \|\widehat{P}\| = \|\overline{P}\|$.

3.2. Cohen strongly p-summing polynomials

Let $m \in \mathbb{N}$, $1 < p \le \infty$ and let X and Y be Banach spaces. An m-homogeneous polynomial $P \in \mathcal{P}(^mX;Y)$ is *Cohen strongly p-summing* if there is a constant $C \ge 0$ such that

$$\left\| \big(P(x_i)\big)_{i=1}^n \right\|_{\ell_p\langle Y\rangle} \le C \|(x_i)_{i=1}^n\|_{\ell_p(X)}^m$$

for all $x_1, \ldots, x_n \in X$ and all $n \in \mathbb{N}$. The infimum of all such $C > 0$ defines a norm on the space $\mathcal{P}_{p,S}^c(^mX;Y)$ of all strongly Cohen p-summing m-homogeneous polynomials from X to Y, that we denote $\|P\|_{p,S}$. For more information on Cohen strongly p-summing polynomials we refer to [3].

Corollary 3.2. *Let $1 < p \le \infty$, $m \in \mathbb{N}$ and $P \in \mathcal{P}(^mX;Y)$. The following are equivalent:*

1. *P is Cohen strongly p-summing.*
2. *$(P(x_i))_{i=1}^{\infty} \in \ell_p\langle Y\rangle$ whenever $(x_i)_{i=1}^{\infty} \in \ell_p(X)$.*
3. *The induced map $\widehat{P} : \ell_p(X) \to \ell_p\langle Y\rangle$ is a well-defined continuous m-homogeneous polynomial.*
4. *The induced m-homogeneous polynomial $\overline{P} : \ell_p^0 \otimes_{\Delta_p} X \to \ell_p^0 \otimes_\pi Y$ is continuous.*

In this case $\|P\|_{p,S} = \|\widehat{P}\| = \|\overline{P}\|$.

3.3. Cohen p-nuclear polynomials

Let $m \in \mathbb{N}$, $1 \leq p \leq \infty$ and let X and Y be Banach spaces. An m-homogeneous polynomial $P \in \mathcal{P}(^m X; Y)$ is *Cohen p-nuclear* if there is a constant $C \geq 0$ such that

$$\left\| \left(P(x_i)\right)_{i=1}^{n} \right\|_{\ell_p\langle Y\rangle} \leq C \|(x_i)_{i=1}^n\|_{\ell_{mp}^w(X)}^m$$

for all $x_1, \ldots, x_n \in X$ and all $n \in \mathbb{N}$. The infimum of all such $C > 0$ defines a norm on the space $\mathcal{P}_{p,N}^c(^m X; Y)$ of all Cohen p-nuclear m-homogeneous polynomials from X to Y, that we denote $\|P\|_{p,N}$.

Clearly, P is Cohen p-nuclear if and only if the (unique) symmetric m-linear operator A given by $A(x, \ldots, x) = P(x)$, is either Cohen p-nuclear in the sense of [2] or absolutely $(1; mp, \ldots, mp, p')$-summing in the sense of [1].

Corollary 3.3. *Let $1 < p < \infty$, $m \in \mathbb{N}$ and $P \in \mathcal{P}(^m X; Y)$. The following are equivalent:*

1. *P is Cohen p-nuclear.*
2. *$(P(x_i))_{i=1}^{\infty} \in \ell_p\langle Y\rangle$ whenever $(x_i)_{i=1}^{\infty} \in \ell_{mp}^w(X)$.*
3. *The induced map $\widehat{P} : \ell_{mp}^w(X) \to \ell_p\langle Y\rangle$ is a well-defined continuous m-homogeneous polynomial.*
4. *The induced m-homogeneous polynomial $\overline{P} : \ell_{mp}^0 \otimes_\varepsilon X \to \ell_p^0 \otimes_\pi Y$ is continuous.*

In this case $\|P\|_{p,N} = \|\widehat{P}\| = \|\overline{P}\|$.

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Dahmane Achour
Laboratoire d'Analyse Fonctionnelle et Géométrie des Espaces
University of M'sila
M'sila, 28000
Algeria
e-mail: dachourdz@yahoo.fr

Ahlem Alouani
Laboratoire d'Analyse Fonctionnelle et Géométrie des Espaces
University of M'sila
M'sila, 28000
Algeria
e-mail: Ah-alouani@hotmail.fr

Pilar Rueda
Departamento de Análisis Matemático
Universidad de Valencia
46100 Burjassot Valencia
Spain.
e-mail: pilar.rueda@uv.es

Enrique A. Sánchez Pérez
Instituto Universitario de Matemática Pura y Aplicada
Universitat Politècnica de València
Camino de Vera s/n
46022 Valencia
Spain.
e-mail: easancpe@mat.upv.es