Numerical Algebra, Control and Optimization
Vol. 12, No. 3, September 2022, pp. 621–636
doi:10.3934/naco.2021026

THE NUMERICAL SOLUTION OF THE SPACE-TIME FRACTIONAL DIFFUSION EQUATION INVOLVING THE CAPUTO-KATUGAMPOLA FRACTIONAL DERIVATIVE.

Kaouther Bouchama

Laboratory for Pure and Applied Mathematics
University of M'sila, Bp 166 M'sila, 28000, Algeria

Yacine Arioua* and Abdelkrim Merzougui

Department of Mathematics
Laboratory for Pure and Applied Mathematics
University of M'sila, Bp 166 M'sila, 28000, Algeria

(Communicated by Bülent Karasözen)

Abstract. In this paper, a numerical approximation solution of a space-time fractional diffusion equation (FDE), involving Caputo-Katugampola fractional derivative is considered. Stability and convergence of the proposed scheme are discussed using mathematical induction. Finally, the proposed method is validated through numerical simulation results of different examples.

1. Introduction.

In recent years, fractional calculus and fractional analysis have received increasing attention from research communities and have been studied extensively. It is found that fractional differential equations and partial fractional equations have a broad spectrum of applications in the areas of physics and engineering (physics [6],[22], chaos theory [15], viscoelasticity [12], control system engineering [14], fractional signal processing techniques [20] and many others areas (see e.g. [17])). Analytical methods have been proposed to provide better solutions to some of fractional differential equations or partial fractional equations, but these solutions are expressed in terms of special functions which are even inaccessible for some of fractional nonlinear equations and they are usually difficult for numerical evaluation. Recently, considerable research has been devoted to the study of numerical methods lead to a rapid increase development of numerical methods for fractional differential and partial equations. Many methods have been presented ([1], [3], [5], [19], [21], [23]) to overcome above mentioned problems. Among them yields only an approximate solution which can be derived using perturbation method, adomian decomposition method, generalized differential transform method or finite difference method .

2020 *Mathematics Subject Classification.* Primary: 35R11, 65M06, 65M12.

Key words and phrases. Fractional diffusion equation, Caputo-Katugampola fractional derivative, Finite difference scheme, Convergence, Stability.

This work was financially supported by the General Direction of Scientific Research and Technological Development (DGRSDT)-Algeria, PRFU(Grant No. C00L03UN280120180010).

*Corresponding author: Yacine Arioua.

A fundamental transport process in the mechanics of environmental fluids is diffusion. Diffusion is a microscopic phenomenon which designates the natural tendency of a system to make the concentrations homogeneous chemical species within it, and it is characterized by the equation:

$$\partial_t u(x,t) = \partial_x^2 u(x,t) + f(x,t).$$

Fractional diffusion equation in time-space is obtained from the classical diffusion equation by replacing the first order derivative in time with Caputo-Katugampola derivative of order α, $0 < \alpha \leq 1$ and the second-order space derivative with a Riesz-Caputo-Katugampola derivative of order β, $1 \leq \beta \leq 2$, which is given by the following equation:

$$^C\partial_t^{\alpha,\rho} u(x,t) = \frac{\partial^{\beta,\rho} u(x,t)}{\partial |x|^\beta} + f(x,t), \;\; (x,t) \in]x_0, L[\times]t_0, T[\,, \tag{1}$$

where $^C\partial_t^{\alpha,\rho}$, $\dfrac{\partial^{\beta,\rho}}{\partial |x|^\beta}$ denotes the Caputo-Katugampola fractional derivative and Riesz-Caputo-Katugampola fractional derivative of order α and β respectively, with $\rho > 1$, $t_0, x_0 > 0$ and $f(x,t)$ is the source term. The fractional derivative $^C\partial_t^{\alpha,\rho}$ in (1) is in the Caputo–Katugampola fractional derivative type of order $\alpha \in \,]0,1[$ defined by

$$^C\partial_t^{\alpha,\rho} u(x,t) = \frac{\rho^\alpha}{\Gamma(1-\alpha)} \int_{t_0}^{t} (t^\rho - \tau^\rho)^{-\alpha} \frac{\partial u(x,\tau)}{\partial \tau} d\tau, \tag{2}$$

The fractional derivative $\dfrac{\partial^{\beta,\rho}}{\partial |x|^\beta}$ in (1) is the Riesz-Caputo-Katugampola fractional derivative of order $\beta \in [1,2]$ defined by

$$\frac{\partial^{\beta,\rho} u(x,t)}{\partial |x|^\beta} = \frac{1}{2}\left({}^C\partial_{x_0,x}^{\beta,\rho} + {}^C\partial_{x,L}^{\beta,\rho} \right) u(x,t)\,,$$

where $^C\partial_{x_0,x}^{\beta,\rho}$ and $^C\partial_{x,L}^{\beta,\rho}$ stand for the left and right Caputo-Katugampola fractional derivative respectively.

In this work we appreciated the finite difference method, and hence unconditionally stable and convergent, for giving the numerical solution of the fractional diffusion problem defined by

$$\begin{cases} ^C\partial_t^{\alpha,\rho} u(x,t) = \dfrac{\partial^{\beta,\rho} u(x,t)}{\partial |x|^\beta} + f(x,t), \;\; (x,t) \in]x_0, L[\times]t_0, T[\,, \\ u(x,t_0) = u_0(x), \;\; x \in [x_0, L]\,, \\ \dfrac{\partial u}{\partial x}(x_0,t) = \psi(t), u(x_0,t) = \phi(t), \;\; u(L,t) = \varphi(t), \quad t \in [t_0, T]\,. \end{cases} \tag{3}$$

Where $u_0(x)$, ψ, ϕ, φ are continuous functions. The rest of the paper is organized as follows: Section 2 presents some properties of the Katugampola fractional integrals and fractional derivatives of various types. Section 3 finite difference methods (FDM) for the space-time fractional diffusion problem is presented. Sections 4 and 5, discuss the stability and the convergence analysis of the finite difference scheme. Typical examples are presented in Sections 6 (Example 1 and 2) to demonstrate the validity of the method, numerical simulation results are also included. Conclusion close the paper in Sections 6.

2. **Preliminaries.** In this section, we recall some concepts on fractional calculus and present additional properties that will be used later.
As be introduced in [10], let us denote $X_c^p[a,\, b]$, $(c\in\mathbb{R},\, 1\le p\le\infty)$ which means the space of Lebesgue measurable functions φ on $[a,\, b]$ for which $\|\varphi\|_{X_c^p}<\infty$, is defined by

$$\|\varphi\|_{X_c^p}=\left(\int_a^b |s^c\varphi(s)|^p\,\frac{ds}{s}\right)^{\frac{1}{p}}<\infty,$$

for $1\le p<\infty,\ c\in\mathbb{R}$, and

$$\|\varphi\|_{X_c^\infty}=ess\sup_{a\le t\le b}\left[t^c\,|\varphi(t)|\right],\ (c\in\mathbb{R}).$$

Definition 2.1 (Katugampola fractional integral)**.** (see [9]). The left-sided Katugampola fractional integral $\mathcal{I}_{a^+}^{\alpha;\rho}u(t)$ of order $\alpha\in\mathbb{C}\,(Re(\alpha)>0)$ of $u(t)\in X_c^p[a,\, b]$ is defined by

$$\left({}^\rho\mathcal{I}_{a+}^{\alpha}u\right)(t)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t\frac{\tau^{\rho-1}u(\tau)}{(t^\rho-\tau^\rho)^{1-\alpha}}d\tau,\text{ for } t>a, \tag{4}$$

Similarly we can define the right-siderd Katugampola fractional integral ${}^\rho\mathcal{I}_{b-}^{\alpha}u$ by

$$\left({}^\rho\mathcal{I}_{b-}^{\alpha}u\right)(t)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_t^b\frac{\tau^{\rho-1}u(\tau)}{(\tau^\rho-t^\rho)^{1-\alpha}}d\tau,\text{ for } t<b. \tag{5}$$

Definition 2.2 (Katugampola fractional derivative)**.** (see [9]). Let $\alpha\in\mathbb{C}$, $Re(\alpha)\ge 0$, $m=[Re(\alpha)]+1$ and $\rho>0$. The Katugampola fractional derivative corresponding to the Katugampola fractional integral (4) and (5) are defined respectively as

$$\begin{aligned}{}^\rho\mathcal{D}_{a+}^{\alpha}u(t)&=\left(t^{1-\rho}\frac{d}{dt}\right)^m\left({}^\rho\mathcal{I}_{a^+}^{m-\alpha}u\right)(t)\\&=\frac{\rho^{\alpha-m+1}}{\Gamma(m-\alpha)}\left(t^{1-\rho}\frac{d}{dt}\right)^m\int_a^t\frac{\tau^{(\rho-1)}u(\tau)}{(t^\rho-\tau^\rho)^{\alpha-m+1}}d\tau,\end{aligned}$$

and

$$\begin{aligned}{}^\rho\mathcal{D}_{b-}^{\alpha}u(t)&=\left(-t^{1-\rho}\frac{d}{dt}\right)^m\left({}^\rho\mathcal{I}_{b^-}^{m-\alpha}u\right)(t)\\&=\frac{\rho^{\alpha-m+1}}{\Gamma(m-\alpha)}\left(-t^{1-\rho}\frac{d}{dt}\right)^m\int_t^b\frac{\tau^{(\rho-1)}u(\tau)}{(\tau^\rho-t^\rho)^{\alpha-m+1}}d\tau,\end{aligned}$$

if the integrals exist.

A recent generalization introduced in [2]. The authors define the generalization of the Katugampola fractional derivatives and present properties of such derivatives. This new generalization is now know as the Caputo-Katugampola fractional derivatives and is given by the following definition:

Definition 2.3 (Caputo-Katugampola fractional derivative)**.** ([2]). Let m be the smallest integer greater than α. Then, the left and right Caputo-Katugampola fractional derivatives of order $\alpha>0$ of a function u are defined respectively as

$${}^C\mathcal{D}_{a^+}^{\alpha,\rho}u(t)=\frac{\rho^{\alpha-m+1}}{\Gamma(m-\alpha)}\int_a^t\frac{\tau^{(\rho-1)(1-m)}}{(t^\rho-\tau^\rho)^{\alpha-m+1}}u^{(m)}(\tau)d\tau, \tag{6}$$

and

$$ {}^{C}\mathcal{D}_{b^-}^{\alpha,\rho}u\left(t\right)=\frac{\left(-1\right)^{m}\rho^{\alpha-m+1}}{\Gamma\left(m-\alpha\right)}\int_{t}^{b}\frac{\tau^{\left(\rho-1\right)\left(1-m\right)}}{\left(\tau^{\rho}-t^{\rho}\right)^{\alpha-m+1}}u^{^{\left(m\right)}}(\tau)d\tau. \tag{7} $$

From the Riesz-Caputo fractional derivative for a function $u(x)$, $(a \leq x \leq b)$ of order $\beta > 0$, ([16]),

$$ \begin{aligned} {}_{a}^{RC}\mathcal{D}_{b}^{\beta}u\left(x\right) &= \frac{1}{\Gamma\left(m-\beta\right)}\int_{a}^{b}\frac{u^{^{\left(m\right)}}\left(s\right)}{\left|x-s\right|^{\beta-m+1}}ds \\ &= \frac{1}{2}\left({}_{a}^{C}\mathcal{D}_{x}^{\beta}+\left(-1\right)^{m}{}_{x}^{C}\mathcal{D}_{b}^{\beta}\right)u\left(x\right), \end{aligned} $$

where ${}_{a}^{C}\mathcal{D}_{x}^{\beta}$ and ${}_{x}^{C}\mathcal{D}_{b}^{\beta}$ is the left and right Caputo derivative respectively. We define the Riesz-Caputo-Katugampola as follows:

Definition 2.4. The Riesz-Caputo-Katugampola fractional derivative for a function $u(x)$, $(a \leq x \leq b)$, of order $\beta > 0$ is

$$ \begin{aligned} {}_{a}^{RC}\mathcal{D}_{b}^{\beta,\rho}u\left(x\right) &= \frac{\rho^{\beta-m+1}}{\Gamma\left(m-\beta\right)}\int_{a}^{b}\frac{s^{\left(\rho-1\right)\left(1-m\right)}}{\left|x^{\rho}-s^{\rho}\right|^{\beta-m+1}}u^{^{\left(m\right)}}(s)ds \\ &= \frac{1}{2}\left({}^{C}\mathcal{D}_{a,x}^{\beta,\rho}+\left(-1\right)^{m}{}^{C}\mathcal{D}_{x,b}^{\beta,\rho}\right)u\left(x\right), \end{aligned} $$

with ${}^{C}\mathcal{D}_{a,x}^{\beta,\rho}$ and ${}^{C}\mathcal{D}_{x,b}^{\beta,\rho}$ are the left and right Caputo-Katugampola fractional derivative defined in (6)-(7).

3. **The finite difference scheme.** In this section, for the finite difference approximation, from [23] we sub-divide the intervals $[x_0, L]$ and $[t_0, T]$ with

$$ \begin{aligned} x_i &= \left(x_0^{\rho}+ik\right)^{\frac{1}{\rho}}, \ i=0,1,...,M, \\ t_j &= \left(t_0^{\rho}+jh\right)^{\frac{1}{\rho}}, \ j=0,1,...,N, \end{aligned} $$

where $k = \dfrac{L^{\rho}-x_0^{\rho}}{M}$ and $h = \dfrac{T^{\rho}-t_0^{\rho}}{N}$ are the spatial and temporal step sizes, respectively. We denote u_i^{n+1} the numerical approximation to $u(x_i, t_{n+1})$ and $f_i^{n+1} = f(x_i, t_{n+1})$.

1. The initial boundary conditions of (3), are discretized as

$$ \begin{cases} u(x_i,t_0)=u_i^0, \\ u_x(x_0,t_{n+1})=\psi^{n+1}, \\ u(x_0,t_{n+1})=\phi^{n+1}, \\ u(L,t_{n+1})=\varphi^{n+1}. \end{cases} $$

2. The space fractional derivative term $\dfrac{\partial^{\beta,\rho}u\left(x_i,t_{n+1}\right)}{\partial\left|x\right|^{\beta}}$ can be approximated by the following

$$ \frac{\partial^{\beta,\rho}u\left(x_i,t_{n+1}\right)}{\partial\left|x\right|^{\beta}}=\frac{1}{2}\left({}^{C}\partial_{x_0,x_m}^{\beta,\rho}+{}^{C}\partial_{x_m,x_M}^{\beta,\rho}\right)u\left(x_i,t_{n+1}\right), $$

where

$$
\begin{aligned}
{}^{C}\partial^{\beta,\rho}_{x_0,x_m} u\left(x_i,t_{n+1}\right) &= \frac{\rho^{\beta-1}}{\Gamma\left(2-\beta\right)} \int\limits_{x_0}^{x_m} \frac{s^{\rho-1}}{\left(x_m^{\rho}-s^{\rho}\right)^{\beta-1}} s^{2(1-\rho)} \frac{\partial^2 u(s,t_{n+1})}{\partial s^2} ds \\
&= \frac{\rho^{\beta-1}}{\Gamma\left(2-\beta\right)} \sum_{i=0}^{m-1} \int\limits_{x_i}^{x_{i+1}} \frac{s^{\rho-1}}{\left(x_m^{\rho}-s^{\rho}\right)^{\beta-1}} \\
&\times x_i^{2(1-\rho)} \frac{u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1}}{(x_{i+1}-x_i)^2} ds \\
&= \frac{k^{2-\beta}\rho^{\beta-2}}{\Gamma\left(3-\beta\right)} \sum_{i=0}^{m-1} x_i^{2(1-\rho)} \frac{u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1}}{(x_{i+1}-x_i)^2} \\
&\times \left[\left(m-i\right)^{2-\beta}-\left(m-i-1\right)^{2-\beta}\right],
\end{aligned}
$$

with $m \in \{1, 2, \ldots, M-1\}$.

3. Set $E_{n+1} = \left|{}^{C}\partial^{\beta,\rho}_{x_0,x_m} u\left(x_i,t_{n+1}\right) - {}^{C}\partial^{\beta,\rho}_{x_0,x_m} u_i^{n+1}\right|$ and $M_i = \max\limits_{x\in[x_0,L]} \left|\dfrac{\partial^i u}{\partial x^i}\right|$, with $i = 1, ...4$, hence, we obtain

$$
\begin{aligned}
E_{n+1} \leq & \frac{\rho^{\beta-1}}{\Gamma\left(2-\beta\right)} \sum_{i=0}^{m-1} \int\limits_{x_i}^{x_{i+1}} \frac{s^{\rho-1}}{\left(x_m^{\rho}-s^{\rho}\right)^{\beta-1}} \\
& \times \left| s^{2(1-\rho)} \frac{\partial^2 u}{\partial s^2} - x_i^{2(1-\rho)} \frac{u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1}}{(x_{i+1}-x_i)^2} \right| ds.
\end{aligned}
$$

It follows from Taylor's theorem, one has for each $i \in \{0, 1, ..., m-1\}$ and $s \in [x_i, x_{i+1}]$ and because $\rho > 1$, we have

$$
\begin{aligned}
& \left| s^{2(1-\rho)} \frac{\partial^2 u}{\partial s} - x_i^{2(1-\rho)} \frac{u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1}}{(x_{i+1}-x_i)^2} \right| \\
&= \left| s^{2(1-\rho)} \frac{\partial^2 u}{\partial s^2} - x_i^{2(1-\rho)} \left(\frac{\partial^2 u}{\partial x^2}(x_i,t_{n+1}) + \frac{\partial^{(4)} u}{\partial x^4}(\eta_1,t_{n+1}) \frac{(x_{i+1}-x_i)^2}{12} \right) \right| \\
&\leq \left| s^{2(1-\rho)} \left(\frac{\partial^2 u}{\partial s^2}(x_i,t_{n+1}) + \frac{\partial^{(3)} u}{\partial s^3}(x_i,t_{n+1})(s-x_i) + \frac{\partial^{(4)} u}{\partial s^4}(\eta_2,t_{n+1}) \frac{(s-x_i)^2}{2} \right) \right. \\
&- \left. x_i^{2(1-\rho)} \frac{\partial^2 u}{\partial x^2}(x_i,t_{n+1}) \right| + M_4 x_i^{2(1-\rho)} \frac{(x_{i+1}-x_i)^2}{12} \\
&\leq x_i^{2(1-\rho)} M_2 + x_i^{2(1-\rho)} (x_{i+1}-x_i) M_3 + x_i^{2(1-\rho)} (x_{i+1}-x_i)^2 \frac{M_4}{2} \\
&+ x_i^{2(1-\rho)} (x_{i+1}-x_i)^2 \frac{M_4}{12},
\end{aligned}
$$

finally, we conclude

$$
\begin{aligned}
& \left| s^{2(1-\rho)} \frac{\partial^2 u}{\partial s} - x_i^{2(1-\rho)} \frac{u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1}}{(x_{i+1}-x_i)^2} \right| \\
& \leq \frac{x_0^{2(1-\rho)} \rho^{\beta-1}}{\Gamma\left(2-\alpha\right)} \times \left(M_2 + x_m M_3 + \frac{7}{12} x_m^2 M_4 \right),
\end{aligned} \tag{8}
$$

where $\eta_2 \in [x_i, s]$ and $\eta_1 \in [x_i, x_{i+1}]$. Furthermore, for any $\beta \in [1,2]$, $i \in \{0,1,...,m-1\}$, $\rho > 1$ and $s \in [x_i, x_{i+1}]$ we have

$$0 \leq \left(x_m^\rho - s^\rho\right)^{1-\beta} \leq \left(x_{i+1}^\rho - s^\rho\right)^{1-\beta},$$

Therefore, one yields

$$\begin{aligned} 0 &\leq \sum_{i=0}^{m-1} \int_{x_i}^{x_{i+1}} \frac{s^{\rho-1}}{\left(x_m^\rho - s^\rho\right)^{\beta-1}} \leq \sum_{i=0}^{m-1} \int_{x_i}^{x_{i+1}} \frac{s^{\rho-1}}{\left(x_{i+1}^\rho - s^\rho\right)^{\beta-1}} \\ &= \sum_{i=0}^{m-1} \frac{\left(x_{i+1}^\rho - x_i^\rho\right)^{2-\beta}}{\rho\left(2-\beta\right)}. \end{aligned} \tag{9}$$

According to (8) and (9), we imply

$$\begin{aligned} E_{n+1} &\leq \frac{x_0^{2(1-\rho)}\rho^{\beta-1}}{\Gamma(2-\alpha)} \left(M_2 + x_m M_3 + \frac{7}{12} x_m^2 M_4\right) \sum_{i=0}^{m-1} \frac{\left(x_{i+1}^\rho - x_i^\rho\right)^{2-\beta}}{\rho\left(2-\beta\right)} \\ &\leq \frac{x_0^{2(1-\rho)}\rho^{\beta-2}}{\Gamma(3-\alpha)} \left(M_2 + x_m M_3 + \frac{7}{12} x_m^2 M_4\right) k^{2-\beta} \sum_{i=0}^{m-1} \left(x_{i+1}^\rho - x_i^\rho\right). \\ &\leq \frac{x_0^{2(1-\rho)}\rho^{\beta-2}}{\Gamma(3-\alpha)} \left(M_2 + x_m M_3 + \frac{7}{12} x_m^2 M_4\right) \left(x_m^\rho - x_0^\rho\right) k^{2-\beta}. \end{aligned}$$

This means that

$${}^C\partial_{x_0,x_m}^{\beta,\rho} u\left(x_i, t_{n+1}\right) := {}^C\partial_{x_0,x_m}^{\beta,\rho} u_i^{n+1} + c_{m,\beta,\rho} k^{2-\beta}.$$

Similarly, we find

$$\begin{aligned} {}^C\partial_{x_m,x_M}^{\beta,\rho} u\left(x_i, t_{n+1}\right) = \frac{k^{2-\beta}\rho^{\beta-2}}{\Gamma(3-\beta)} \sum_{i=m}^{M-1} x_i^{2(1-\rho)} \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\left(x_{i+1} - x_i\right)^2} \\ \times \left[\left(i+1-m\right)^{2-\beta} - \left(i-m\right)^{2-\beta}\right], \end{aligned}$$

and

$${}^C\partial_{x_m,x_M}^{\beta,\rho} u\left(x_i, t_{n+1}\right) := {}^C\partial_{x_m,x_M}^{\beta,\rho} u_i^{n+1} + c_{M,\beta,\rho} k^{2-\beta}.$$

The following result is obtained.

Lemma 3.1. *Let* $0 < \alpha \leq 1$, $1 \leq \beta \leq 2$, *and* $\rho > 1$ *we have*

(a) *The approximation of the Riesz-Caputo-Katugampola fractional derivative* $\dfrac{\partial^{\beta,\rho} u(x,t)}{\partial |x|^\beta}$ *in (1) is given by the following scheme*

$$\frac{\partial^{\beta,\rho} u\left(x_i, t_{n+1}\right)}{\partial |x|^\beta} = \frac{k^{2-\beta}\rho^{\beta-2}}{2\Gamma(3-\beta)} \begin{pmatrix} \displaystyle\sum_{i=0}^{m-1} a_{i,m}^{\beta,\rho} \left(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right) \\ \displaystyle + \sum_{i=m}^{M-1} z_{i,m}^{\beta,\rho} \left(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right) \end{pmatrix}, \tag{10}$$

for $m \in \{1, 2, \ldots, M-1\}$, *where*

$$\begin{cases} a_{i,m}^{\beta,\rho} = \dfrac{x_i^{2(1-\rho)}}{(x_{i+1}-x_i)^2}\left[(m-i)^{2-\beta} - (m-i-1)^{2-\beta}\right], i = 0, \ldots, m-1, \\ z_{i,m}^{\beta,\rho} = \dfrac{x_i^{2(1-\rho)}}{(x_{i+1}-x_i)^2}\left[(i+1-m)^{2-\beta} - (i-m)^{2-\beta}\right], i = m, \ldots, M-1, \end{cases} \tag{11}$$

and

$$\frac{\partial^{\beta,\rho} u(x_i, t_{n+1})}{\partial |x|^\beta} := \frac{\partial^{\beta,\rho}}{\partial |x|^\beta} u_i^{n+1} + c_{\beta,\rho} k^{2-\beta}.$$

(b) *The approximation of the Caputo-Katugampola fractional derivative* ${}^{C}\partial_t^{\alpha,\rho}$ *in* (1) *is given by the following scheme* [23]*:*

$${}^{C}\partial_t^{\alpha,\rho} u(x_i, t_{n+1}) = \frac{h^{1-\alpha}\rho^{\alpha-1}}{\Gamma(2-\alpha)} \sum_{j=0}^{n} b_j^{\alpha,\rho}\left(u_i^{j+1} - u_i^j\right), \tag{12}$$

where

$$b_j^{\alpha,\rho} = \frac{t_j^{(1-\rho)}}{t_{j+1}-t_j}\left((n-j+1)^{1-\alpha} - (n-j)^{1-\alpha}\right), j = 0, ..., n. \tag{13}$$

and

$${}^{C}\partial_t^{\alpha,\rho} u(x_i, t_{n+1}) := {}^{C}\partial_t^{\alpha,\rho} u_i^{n+1} + c_{\alpha,\rho} h^{1-\alpha}.$$

Now, By using the space-time fractional approximation (10) and (12) we obtain the following numerical approximation to equation (1),

$$\frac{k^{2-\beta}\rho^{\beta-2}}{2\Gamma(3-\beta)}\left(\sum_{i=0}^{m-1} a_i^{\beta,\rho}\left(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right) + \sum_{i=m}^{M-1} z_{i,m}^{\beta,\rho}\left(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right)\right)$$
$$= \frac{h^{1-\alpha}\rho^{\alpha-1}}{\Gamma(2-\alpha)} \sum_{j=0}^{n} b_j^{\alpha,\rho}\left(u_i^{j+1} - u_i^j\right) + f_i^{n+1},$$

Then, for each $n = 0, 1, ..., N-1$, and $m = 1, ..., M-1$, setting

$$\lambda = \frac{\Gamma(2-\alpha)\, k^{2-\beta} h^{\alpha-1}}{2\Gamma(3-\beta)\,\rho^{\alpha-\beta+1}},$$

we obtain the following difference approximation for $l \in \{1, 2, \ldots, M-1\}$

$$\sum_{i=1}^{M-1} \omega_{i,m} u_i^{n+1} + b_n u_l^{n+1} = b_0^{\alpha,\rho} u_l^0 + \sum_{j=1}^{n} G_j u_l^j + V_l^{n+1}, \tag{14}$$

with

$$V_l^{n+1} = \lambda\left(\left(a_1^{\beta,\rho} - a_0^{\beta,\rho}\right)\phi^{n+1} + z_{M-1,m}^{\beta,\rho}\varphi^{n+1} - k a_0^{\beta,\rho}\psi^{n+1}\right) + \frac{h^{\alpha-1}\Gamma(2-\alpha)}{\rho^{\alpha-1}} f_l^{n+1},$$

$G_j = \left(b_j^{\alpha,\rho} - b_{j-1}^{\alpha,\rho}\right)$ and

$$w_{i,m} = \begin{cases} \lambda\left(-a_{i+1,m}^{\beta,\rho} + 2a_{i,m}^{\beta,\rho} - a_{i-1,m}^{\beta,\rho}\right), \text{ if } 1 \leq i \leq m-2, \\ \lambda\left(-z_{m,m}^{\beta,\rho} + 2a_{m-1,m}^{\beta,\rho} - a_{m-2,m}^{\beta,\rho}\right), \text{ if } i = m-1, \\ \lambda\left(-z_{m+1,m}^{\beta,\rho} + 2z_{m,m}^{\beta,\rho} - a_{m-1,m}^{\beta,\rho}\right), \text{ if } i = m, \\ \lambda\left(-z_{i+1,m}^{\beta,\rho} + 2z_{i,m}^{\beta,\rho} - z_{i-1,m}^{\beta,\rho}\right), \text{ if } m+1 \leq i \leq M-2, \\ \lambda\left(2z_{M-1,m}^{\beta,\rho} - z_{M-2,m}^{\beta,\rho}\right), \text{ if } i = M-1. \end{cases}$$

So, for $n = 0$ and $l \in \{1, 2, \ldots, M-1\}$ we have

$$\sum_{i=1}^{M-1} \omega_{i,m} u_i^1 + b_0^{\alpha,\rho} u_l^1 = b_0^{\alpha,\rho} u_l^0 + V_l^1, \tag{15}$$

then, with $n > 0$ and $l \in \{1, 2, \ldots, M-1\}$ we obtain

$$\sum_{i=1}^{M-1} \omega_{i,m} u_i^{n+1} + b_n^{\alpha,\rho} u_l^{n+1} = b_0^{\alpha,\rho} u_l^0 + \sum_{j=1}^{n} G_j u_l^j + V_l^{n+1}, \tag{16}$$

Thus, we have the difference scheme in the matrix form

$$\begin{cases} \mathbf{U}^0 = u_i^0, \text{ for } i = 1, ..., M-1, \\ \mathbf{A}^1 \mathbf{U}^1 = b_0 \mathbf{U}^0 + \mathbf{V}^1, \\ \mathbf{A}^n \mathbf{U}^{n+1} = b_0 \mathbf{U}^0 + G_1 \mathbf{U}^1 + G_2 \mathbf{U}^2 + \ldots + G_n \mathbf{U}^n + \mathbf{V}^{n+1}, \end{cases}$$

with

$$\begin{cases} \mathbf{U}^0 = \left[u_1^0, u_2^0, \ldots, u_{M-1}^0\right]^T, \\ \mathbf{U}^n = \left[u_1^n, u_2^n, \ldots, u_{M-1}^n\right]^T, \\ \mathbf{V}^{n+1} = \left[V_1^{n+1}, V_2^{n+1}, \ldots, V_{M-1}^{n+1}\right]^T, \end{cases}$$

and $\mathbf{A}^n$ is square matrix of dimension $(M-1) \times (M-1)$ of coefficients:

$$\mathbf{A}_{(i,j)}^n = \begin{cases} \omega_{j,m}, & \text{if } \ i \neq j, \\ \omega_{i,m} + b_n^{\alpha,\rho}, & \text{if } \ i = j, \end{cases}$$

Lemma 3.2. *The coefficients* $a_i^{\beta,\rho}$, $b_j^{\alpha,\rho}$, *and* $z_{i,m}^{\beta,\rho}$ *in (11),(13) satisfy :*

1. $a_{i,m}^{\beta,\rho} > 0$, $z_{i,m}^{\beta,\rho} > 0$, *and* $b_j^{\alpha,\rho} > 0$, *for* $i = 0, \ldots, m-1$, $i = m, \ldots, M-1$ *and* $j = 0, \ldots, n$.
2. $a_i^{\beta,\rho} > a_{i-1}^{\beta,\rho}$ *and* $b_j^{\alpha,\rho} > b_{j-1}^{\alpha,\rho}$, *for* $i = 1, \ldots, m-1$ *and* $j = 1, \ldots, n$.
3. $z_{i+1,m}^{\beta,\rho} < z_{i,m}^{\beta,\rho}$, *for* $i = m, \ldots, M-1$.

4. Stability analysis of finite difference scheme for FDE.

In this section, we suppose that $\widetilde{u}_l^n$ is the approximate solution of (15) and (16), the error $\varepsilon_l^n = \widetilde{u}_l^n - u_l^n$, for $l \in \{1, 2, \ldots, M-1\}$ and $n \in \{1, 2, \ldots, M-1\}$ satisfies

$$\begin{aligned} &\sum_{i=1}^{M-1} \omega_{i,m}\varepsilon_i^1 + b_0^{\alpha,\rho}\varepsilon_l^1 = b_0^{\alpha,\rho}\varepsilon_l^0, \\ &\sum_{i=1}^{M-1} \omega_{i,m}\varepsilon_i^{n+1} + b_n^{\alpha,\rho}\varepsilon_l^{n+1} = b_0^{\alpha,\rho}\varepsilon_l^0 + \sum_{j=1}^{n} G_j\varepsilon_l^j + V_l^{n+1}. \end{aligned} \tag{17}$$

So, $n = 1, 2, \ldots, N-1$, the above formula can be written in the matrix form as:

$$\begin{cases} \mathbf{A}^0\mathbf{E}^1 = b_0^{\alpha,\rho}\mathbf{E}^0, \\ \mathbf{A}^n\mathbf{E}^{n+1} = b_0^{\alpha,\rho}\mathbf{E}^0 + G_1\mathbf{E}^1 + G_2\mathbf{E}^2 + \ldots + G_n\mathbf{E}^n, \\ \mathbf{E}^0 = 0, \end{cases}$$

where $\mathbf{E}^n = \left(\varepsilon_1^n, \varepsilon_2^n, \ldots, \varepsilon_{M-1}^n\right)^T$. Hence, the stability analysis of the difference approximation is studied via mathematical induction method.

Let $\left\|\mathbf{E}^1\right\|_\infty = \left|\varepsilon_l^1\right| = \max\limits_{1\le i\le M-1}\left|\varepsilon_i^1\right|$ and

$$\gamma_{M,m} = \lambda\left(2a_{m-1,m}^{\beta,\rho} - a_{0,m}^{\beta,\rho} - a_{1,m}^{\beta,\rho} + 2\left(z_{m,m}^{\beta,\rho} - z_{M-2,m}^{\beta,\rho}\right)\right).$$

Then, for $n = 0$, note that $a_{i,m}^{\beta,\rho}$ is increasing and $z_{i,m}^{\beta,\rho}$ is decreasing (Lemma 3.2), we have

$$\begin{aligned} \left(\gamma_{M,m} + b_0^{\alpha,\rho}\right)\left|\varepsilon_l^1\right| &\le \left(\gamma_{m,M} + \lambda z_{M-1,m}^{\beta,\rho} + b_0^{\alpha,\rho}\right)\left|\varepsilon_l^1\right| \\ &= \left|\sum_{i=1}^{M-1}\omega_{i,m}\varepsilon_l^1 + b_0^{\alpha,\rho}\varepsilon_l^1\right| \le b_0^{\alpha,\rho}\left|\varepsilon_l^0\right|, \end{aligned}$$

hence, $\left|\varepsilon_l^1\right| \le \dfrac{b_0^{\alpha,\rho}}{\left(\gamma_{M,m} + b_0^{\alpha,\rho}\right)}\left|\varepsilon_l^0\right|$. Its follows

$$\left\|\mathbf{E}^1\right\|_\infty \le \left\|\mathbf{E}^0\right\|_\infty.$$

Let $\left\|\mathbf{E}^{n+1}\right\|_\infty = \left|\varepsilon_l^{n+1}\right| = \max\limits_{1\le i\le M-1}\left|\varepsilon_i^{n+1}\right|$, we assum that $\left\|\mathbf{E}^j\right\|_\infty \le \left\|\mathbf{E}^0\right\|_\infty$, $(j = 1, 2, \ldots, n)$, using Lemma 3.2, we also have

$$\begin{aligned} \left(\gamma_{m,M} + b_n^{\alpha,\rho}\right)\left|\varepsilon_l^{n+1}\right| &\le \left|\sum_{i=1}^{M-1} w_{i,m}\varepsilon_l^{n+1} + b_n^{\alpha,\rho}\varepsilon_l^{n+1}\right| \le \left|b_0^{\alpha,\rho}\varepsilon_l^0 + \sum_{j=1}^{n} G_j\varepsilon_l^j\right| \\ &\le b_0^{\alpha,\rho}\left|\varepsilon_l^0\right| + \left|\sum_{j=1}^{n} G_j\right|\left|\varepsilon_l^j\right| \le b_0^{\alpha,\rho}\left|\varepsilon_l^0\right| + \left|\sum_{j=1}^{n} b_j^{\alpha,\rho} - b_{j-1}^{\alpha,\rho}\right|\left|\varepsilon_l^j\right| \\ &\le b_0^{\alpha,\rho}\left|\varepsilon_l^0\right| + \left(b_n^{\alpha,\rho} - b_0^{\alpha,\rho}\right)\left|\varepsilon_l^0\right| \end{aligned}$$

finally, we find

$$\left|\varepsilon_l^{n+1}\right| \le \frac{b_n^{\alpha,\rho}}{\left(\gamma_{M,m} + b_n^{\alpha,\rho}\right)}\left\|E^0\right\|_\infty,$$

imply,

$$\left\|\mathbf{E}^{n+1}\right\|_\infty \le \left\|\mathbf{E}^0\right\|_\infty.$$

The following theorem holds.

Theorem 4.1. *The finite difference schemes (15) and (16) for the FDE (1) are unconditionally stable.*

5. **Convergence analysis of the approximate scheme for FDE.** In this section, we discuss the convergence of the approximate scheme (15) and (16). Let $u(x_i, t_n)$ be the exact solution of the fractional diffusion equation (1) at mesh points (x_i, t_n) where $i = 0, 1, 2, \ldots, M$ and $n = 0, 1, 2, \ldots, N$.

Define $e_i^n = u(x_i, t_n) - u_i^n$ and $\mathbf{e}^n = \left(e_1^n, e_2^n, \ldots, e_{M-1}^n\right)^T$. Using $\mathbf{e}^0 = 0$, substituting $u_i^n = u(x_i, t_n) - e_i^n$ into (15) and (16) leads to:

1. For $n = 0$, and $l \in \{1, 2, \ldots, M-1\}$, we have

$$\begin{aligned}
&\sum_{i=1}^{M-1} \omega_{i,m} e_i^1 + b_0^{\alpha,\rho} e_l^1 \\
&= \sum_{i=1}^{M-1} \omega_{i,m} u(x_i, t_1) + b_0^{\alpha,\rho} u(x_l, t_1) - b_0^{\alpha,\rho}\left(u(x_l, t_0) - e_l^0\right) - V_l^1 \\
&= R_l^1.
\end{aligned}$$

2. For $n > 0$, and $l \in \{1, 2, \ldots, M-1\}$, the approximate scheme becomes

$$\begin{aligned}
&\sum_{i=1}^{M-1} \omega_{i,m} e_i^{n+1} + b_n^{\alpha,\rho} e_l^{n+1} \\
&= \sum_{i=1}^{M-1} \omega_{i,m} u(x_i, t_{n+1}) + b_n^{\alpha,\rho} u(x_l, t_{n+1}) \\
&- b_0^{\alpha,\rho}\left(u(x_l, t_0) - e_l^0\right) - \sum_{j=1}^{n} G_j \left(u(x_l, t_j) - e_l^j\right) - V_l^{n+1} \\
&= \sum_{j=1}^{n} G_j e_l^j + R_l^{n+1},
\end{aligned}$$

where

$$\begin{aligned}
R_l^{n+1} &= \sum_{j=0}^{n} b_j^{\alpha,\rho}\left(u(x_l, t_{j+1}) - u(x_l, t_j)\right) \\
&-\lambda \left(\begin{array}{l} \displaystyle\sum_{i=0}^{m-1} a_{i,m}^{\beta,\rho}\left(u(x_{i+1}, t_{n+1}) - 2u(x_i, t_{n+1}) + u(x_{i-1}, t_{n+1})\right) \\ \displaystyle+ \sum_{i=m}^{M-1} z_{i,m}^{\beta,\rho}\left(u(x_{i+1}, t_{n+1}) - 2u(x_i, t_{n+1}) + u(x_{i-1}, t_{n+1})\right) \end{array}\right) \\
&- \frac{h^{\alpha-1}\Gamma(2-\alpha)}{\rho^{\alpha-1}} f_l^{n+1}.
\end{aligned}$$

From (1), we have

$$\begin{aligned}
R_l^{n+1} &= \frac{h^{\alpha-1}\Gamma(2-\alpha)}{\rho^{\alpha-1}} \\
&\times \left({}^{C}\partial_t^{\alpha,\rho}(x_l, t_{n+1}) - \frac{\partial^{\beta,\rho} u(x_l, t_{n+1})}{\partial |x|^{\beta}} - f_l^{n+1} + c_{\alpha,\rho} h^{1-\alpha} - c_{\beta,\rho} k^{2-\beta}\right) \quad (18) \\
&= \frac{h^{\alpha-1}\Gamma(2-\alpha)}{\rho^{\alpha-1}}\left(c_{\alpha,\rho} h^{1-\alpha} - c_{\beta,\rho} k^{2-\beta}\right).
\end{aligned}$$

Hence, there exist $c_{\alpha,\beta,\rho} > 0$, such that

$$\left|R_i^{n+1}\right| \le c_{\alpha,\beta,\rho}\left(1 + h^{\alpha-1}k^{2-\beta}\right),\ i = 1,2,\dots,M-1,\ n = 0,1,\dots,N-1.$$

Consequently, using mathematical induction, we prove

$$\left\|\mathbf{e}^{n+1}\right\|_\infty \le \left(b_n^{\alpha,\rho}\right)^{-1} C_{\alpha,\beta,\rho}\left(1 + h^{\alpha-1}k^{2-\beta}\right).$$

Let $\left\|\mathbf{e}^{n+1}\right\|_\infty = \left|e_l^{n+1}\right| = \max\limits_{1\le i\le M-1}\left|e_i^{n+1}\right|$, then

1. For $n = 0$, and $i \in \{1,2,\dots,M-1\}$ we get

$$\begin{aligned}\left(\gamma_{M,m} + b_0^{\alpha,\rho}\right)\left|e_l^1\right| &\le \left(\gamma_{M,m} + \lambda z_{M-1,m}^{\beta,\rho} + b_0^{\alpha,\rho}\right)\left|e_l^1\right| \\ &= \left|\sum_{i=1}^{M-1}\omega_{i,m}e_l^1 + b_0^{\alpha,\rho}e_l^1\right| \le \left|R_i^1\right|,\end{aligned}$$

imply,

$$\begin{aligned}\left|e_l^1\right| &\le \left(\gamma_{M,m} + b_0^{\alpha,\rho}\right)^{-1}\left|R_i^1\right| \\ &\le \left(b_0^{\alpha,\rho}\right)^{-1} c_{\alpha,\beta,\rho}\left(1 + h^{\alpha-1}k^{2-\beta}\right).\end{aligned}$$

2. For $n > 0$ and $i \in \{1,2,\dots,M-1\}$, suppose that

$$\left|e_l^j\right| \le \left(b_{j-1}^{\alpha,\rho}\right)^{-1} c_{\alpha,\beta,\rho}\left(1 + h^{\alpha-1}k^{2-\beta}\right),$$

for $j = 1,\dots,n$, we have $\left(b_{j-1}^{\alpha,\rho}\right)^{-1} \le \left(b_0^{\alpha,\rho}\right)^{-1}$ (Lemma 3.2), we get

$$\begin{aligned}\left|e_l^{n+1}\right| &\le \left(\gamma_{M,m} + b_n^{\alpha,\rho}\right)^{-1}\left|\sum_{j=1}^{n}\left(b_j^{\alpha,\rho} - b_{j-1}^{\alpha,\rho}\right)\right|\left|e_i^j\right| + \left(\gamma_{M,m} + b_n^{\alpha,\rho}\right)^{-1}\left|R_i^{n+1}\right| \\ &\le \left(b_n^{\alpha,\rho}\right)^{-1} C'_{\alpha,\beta,\rho}\left(1 + h^{\alpha-1}k^{2-\beta}\right).\end{aligned}$$

We can prove that

$$\lim_{n\to\infty}\frac{1}{b_n^{\alpha,\rho}\left(\frac{t_0^\rho}{h} + n\right)^{\alpha-1}} = 0,$$

therefor, there exist a constant $\zeta > 0$ suth that

$$\left\|e_l^{n+1}\right\|_\infty \le \zeta\left(\frac{t_0^\rho}{h} + n\right)^{\alpha-1} C'_{\alpha,\beta,\rho}\left(1 + h^{\alpha-1}k^{2-\beta}\right),$$

then

$$\begin{aligned}\left|e_l^{n+1}\right| &\le C'_{\alpha,\beta,\rho}\zeta\left(\frac{t_0^\rho}{h} + n\right)^{(\alpha-1)} h^{\alpha-1}\left(h^{1-\alpha} + k^{2-\beta}\right) \\ &\le C'_{\alpha,\beta,\rho}\zeta t_n^{\rho(\alpha-1)}\left(h^{1-\alpha} + k^{2-\beta}\right) \\ &\le C'_{\alpha,\beta,\rho}\zeta T^{\rho(\alpha-1)}\left(h^{1-\alpha} + k^{2-\beta}\right),\end{aligned}$$

is finite, we have

$$\left\|\mathbf{e}^{n+1}\right\|_\infty \le C_{\alpha,\beta,\rho}\left(h^{1-\alpha} + k^{2-\beta}\right).$$

Then, the convergence of the finite difference scheme is given by the following theorem:

Theorem 5.1. *Let u_i^n be the approximate value of $u(x_i, t_n)$, then there is a positive constant $C_{\alpha,\beta,\rho}$, such that*

$$\left|u_i^n - u\left(x_i, t_n\right)\right| \leq C_{\alpha,\beta,\rho}\left(h^{1-\alpha} + k^{2-\beta}\right), \ i = 1, 2, \ldots, M-1, \ n = 1, 2, \ldots, N.$$

6. **Illustrative examples.** In this section, we present some examples to illustrate the usefulness of our main results.

Example 1. Let $(x,t) \in [1,2] \times [1,2]$, $\rho = 2$ and

$$f(x,t) = \left(\frac{x^{2\rho} - 3}{2\rho}\right) \frac{\rho^{\alpha}}{\Gamma(2-\alpha)} \left(t^{\rho} - 1\right)^{1-\alpha} - t^{\rho} \frac{(2\rho - 1)\rho^{\beta-2}}{2\Gamma(3-\beta)} \left(\left(x^{\rho} - 1\right)^{2-\beta} + \left(2^{\rho} - x^{\rho}\right)^{2-\beta}\right).$$

Consider the following space-time fractional diffusion equation

$$\begin{cases} {}^{C}\partial_t^{\alpha,\rho} u(x,t) = \dfrac{\partial^{\beta,\rho} u(x,t)}{\partial |x|^{\beta}} + f(x,t), \\ u(x,1) = \left(\dfrac{x^{2\rho} - 3}{2\rho}\right), \\ \partial_x u(1,t) = t^{\rho}, u(1,t) = t^{\rho}\left(\dfrac{-1}{\rho}\right), \ u(2,t) = t^{\rho}\left(\dfrac{2^{2\rho} - 3}{2\rho}\right). \end{cases} \tag{19}$$

The exact solution for this problem is

$$u(x,t) = t^{\rho}\left(\frac{x^{2\rho} - 3}{2\rho}\right).$$

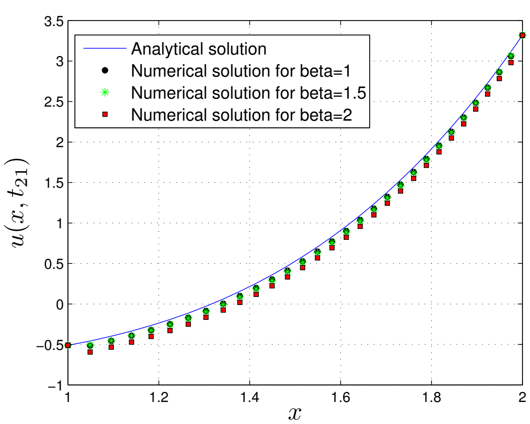

FIGURE 1. Graphical comparison of the numerical and the exact solution with $h = 0.001$, $k = 0.1$, $\rho = 2$, $\alpha = 0.7$, $n = 20$ and $m = 25$.

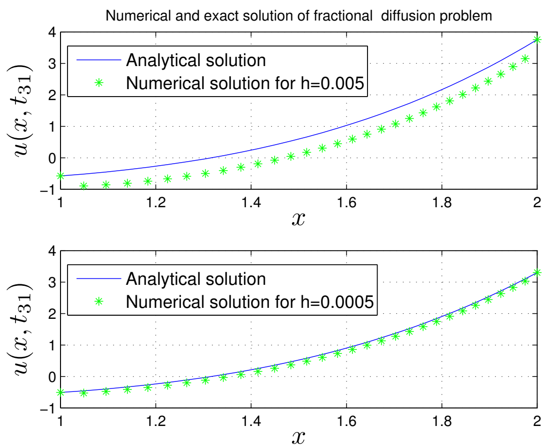

FIGURE 2. Graphical comparison of the numerical and the exact solution with $k = 0.1$, $\rho = 2$, $\alpha = 0.6$, $\beta = 1.8$, $n = 30$ and $m = 25$.

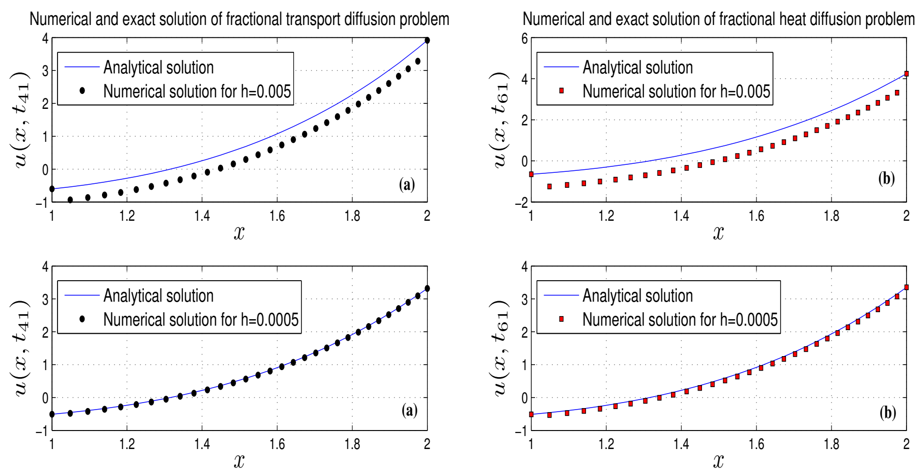

FIGURE 3. Graphical comparison of the numerical and the exact solution with $k = 0.1$, $\rho = 2$, $\alpha = 0.9$, (a) $\beta = 1$, (b) $\beta = 2$ and $m = 25$.

Example 2. Let $(x, t) \in [1, 2] \times [1, 2]$, $\rho = 3$ and

$$f(x,t) = \left(\frac{-x^{2\rho} + x}{2\rho}\right) \frac{\rho^{\alpha}}{\Gamma(2-\alpha)} (t^{\rho} - 1)^{1-\alpha} + \frac{(2\rho - 1)\rho^{\beta - 2}}{2\Gamma(3-\beta)} (t^{\rho} + 1) \left((x^{\rho} - 1)^{2-\beta} + (2^{\rho} - x^{\rho})^{2-\beta}\right).$$

Consider the following space-time fractional diffusion equation

$$\begin{cases} {}^{C}\partial_t^{\alpha,\rho} u(x,t) = \dfrac{\partial^{\beta,\rho} u(x,t)}{\partial |x|^{\beta}} + f(x,t), \\ u(x,1) = 2\left(\dfrac{-x^{2\rho}+x}{2\rho}\right), \\ \partial_x u(1,t) = \left(\dfrac{-2\rho+1}{2\rho}\right)(t^{\rho}+1), u(1,t) = 0, u(2,t) = (t^{\rho}+1)\left(\dfrac{-2^{2\rho-1}+1}{\rho}\right). \end{cases} \tag{20}$$

The exact solution of the given problem is given by

$$u(x,t) = (t^{\rho}+1)\left(\frac{-x^{2\rho}+x}{2\rho}\right).$$

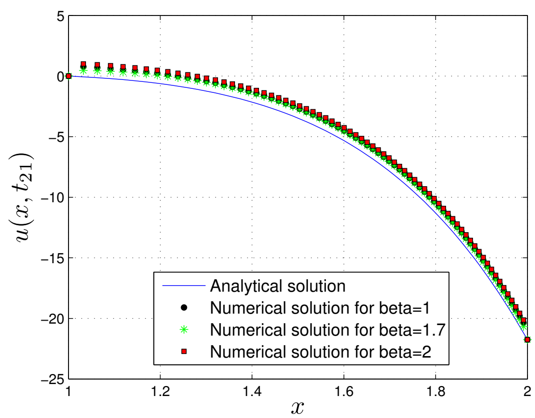

FIGURE 4. Graphical comparison of the numerical and the exact solution with $h = 0.005$, $k = 0.1$, $\rho = 3$, $\alpha = 0.7$, $n = 20$ and $m = 15$.

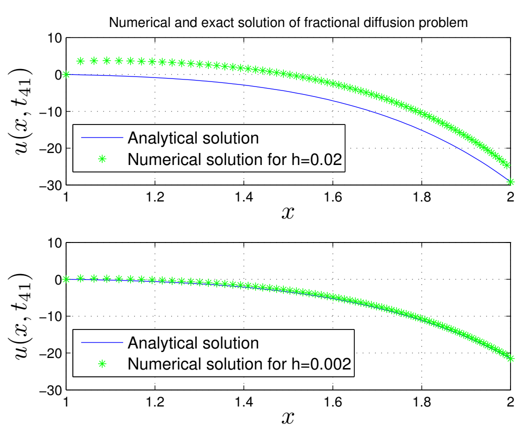

FIGURE 5. Graphical comparison of the numerical and the exact solution with $k = 0.1$, $\rho = 3$, $\alpha = 0.8$, $\beta = 1.8$, $n = 40$ and $m = 15$.

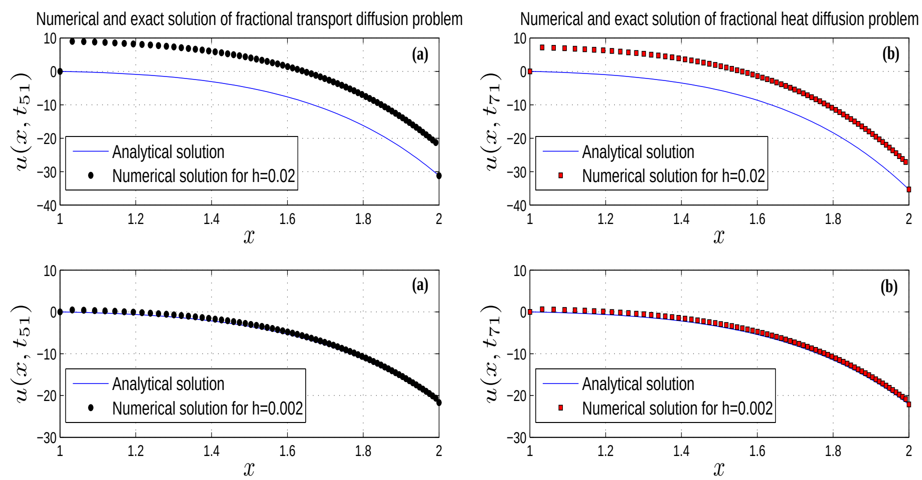

FIGURE 6. Graphical comparison of the numerical and the exact solution with $k = 0.1$, $\rho = 3$, $\alpha = 0.9$, (a) $\beta = 1$, (b) $\beta = 2$ and $m = 15$.

7. **Conclusion.** This work has, in addition to an academic interest, a practical interest: to apprehend and solve fractional problems related to biology, ecology, physics and economics, these problems generally admit rather complicated solutions, this is the reason we introduced the numerical solution of the space-time fractional diffusion. Fractional diffusion equation (1) with $\rho > 1$, $0 < \alpha < 1$ and $1 \leq \beta \leq 2$, is a generalization of classical diffusion equation. We have developed a new numerical method for solving fractional partial differential equations, this method is based on finite difference approximation, it is found that the approximate solutions produced by our method are in complete agreement with the corresponding exact solutions. Moreover, various results were obtained for different values of the parameters β, α and ρ. So, in the case of $\beta = 1$, we obtain the numerical solution of the fractional transport equation, (Figure 3 (a), Figure 6 (a)). In addition, if $\beta = 2$, we obtain numerically solution of fractional diffusion equation (Figure 3 (b), Figure 6 (b)). Eventually, different values for h and k have been tested on example 1 and 2 to evaluate the validity of the approach, the results obtained show a good global approximation and an improved convergence with an error $C_{\alpha,\beta,\rho}(h^{1-\alpha} + k^{2-\beta})$ reaching to zero.

Acknowledgments. The authors are deeply grateful to the anonymous referees for their acceptance to read this paper.

REFERENCES

[1] R. B Albadarneha, I. M Batihab and M Zurigatb, Numerical solutions for linear fractional differential equations of order $1 < \alpha < 2$ using finite difference method (FFDM), *Int. J. Math.*, **16** (2016), 103–111.

[2] R. Almeida, T. Odzijewicz and Agnieszka B. Malinowska, Fractional differential Equations with dependence on the Caputo-Katugampola derivative, *J. Comput. Nonlinear Dyn*, **11** (2016), 11.

[3] J. F Cheng and Y. M Chu, Solution to the linear fractional differential equation using Adomian decomposition method, *Math. Probl. Eng.*, **2011** (2011), 14.

[4] K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer Science Berlin, New York, 2010.

[5] N. Faraz, Y. Khan and D. S. Sankar, Decomposition-transform method for fractional differential equations, *Int. J. Nonl. Sci. Num. Sim.*, **11** (2010), 305–310.
[6] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.
[7] M. Janaki, K. Kanagarajan and D. Vivek, Analytic study on fractional implicit differential equations with impulses via Katugampola fractional derivative, *Int. J. Math. And Appl.*, **55** (2018), 7.
[8] U. N. Katugampola, New approach to a generalized fractional integral, *Appl. Math. Comput.*, **218** (2011), 860–865.
[9] U. N. Katugampola, A new approach to generalized fractional derivatives, *J. Math. Anal. Appl.*, **6** (2014), 1–15.
[10] A. A. Kilbas, H. H. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science B.V, Amsterdam, 2006.
[11] R. L. Magin, *Fractional Calculus in Bioengineering*, Begell House Publishers, 2006.
[12] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models Imperial*, College Press, Singapore, 2010.
[13] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, 1993.
[14] C. A. Monje, Y. Q. Chen, B. M. Vinagre, D. Xue and V. Feliu, *Fractional-Order Systems and Controls, Advances in Industrial Control*, Springer, 2010.
[15] I. Petras, *Fractional-Order Nonlinear Systems*, Springer, New York, 2011.
[16] I. Podlubny, *Fractional Differential Equations, Mathematics in Science and Engineering*, Academic Press, New York, 1999.
[17] J. Sabatier, O. P. Agrawal and J. A. Tenreiro Machado (Eds.), *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering*, Springer, 2007.
[18] S. G. Samko, A. A. Kilbas and O.I. Marichev, *Fractional Integral and Derivatives (Theory and Applications)*, Gordon and Breach, Switzerland, 1993.
[19] S. Shen, F. Liu and V. Anh, Numerical approximations and solution techniques for the space-time Riesz-Caputo fractional advection-diffusion equation, *Numer. Algorithms*, **56** (2011), 383–403.
[20] H. Sheng, Y. Q. Chen and T. S. Qiu, *Fractional Processes and Fractional-order Signal Processing*, Springer, London, 2012.
[21] X. C. Shi, L. L. Huang and Y. Zeng, Fast Adomian decomposition method for the Cauchy problem of the time–fractional reaction diffusion equation, *Adv. Mech. Eng*, **8** (2016), 1–5.
[22] B.J. West, M. Bologna and P. Grigolini, *Physics of Fractal Operators*, 1^{st} edition, Springer, New York, 2003.
[23] S. Zeng, D. Baleanu, Y. Bai and G. Wu , Fractional differential equations of Caputo–Katugampola type and numerical solutions, *Appl. Math. Comput.*, **315** (2017), 549–554.

Received May 2021; 1^{st} revision June 2021; Final revision June 2021; Early access July 2021.

E-mail address: yacine.arioua@univ-msila.dz
E-mail address: kaouther.bouchama@univ-msila.dz
E-mail address: abdelkrim.merzougui@univ-msila.dz