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ORIGINAL PAPER

Finite Difference Approximation for the Space-Time Fractional Linear Diffusion Equation Involving the Caputo-Hadamard Fractional Derivative

Kaouther Bouchama[1] · Yacine Arioua[1] · Abdelkrim Merzougui[1]

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Abstract
In this paper, we provide an accurate numerical solution for space-time fractional linear diffusion equation involving the fractinal Caputo-Hadamard derivative. To do so, we have used a finite difference method. The Convergence and stability of the given finite difference scheme are studied using the mathematical induction technique. Moreover, Numerical examples are given to demonstrate the effectiveness of our results.

Keywords Fractional diffusion equation · Caputo-Hadamard fractional derivative · Fractional finite difference method · Convergence · Stability

Mathematics Subject Classification 35R11 · 65M12 · 65M06

Introduction

The theory of fractional calculus has attracted many researchers from different fields of knowledge. There has been a large and fast increasing literature on fractional differential equations and partial fractional equations. Readers can refer to Kilbas et al. [16] for more theoretical details. Fractional derivatives has found wide applications in many scientific and academic areas such as physics [14, 32], biology, chemistry, mechanics, engineering, elasticity, viscoelasticity [19], control theory, electronics, modeling, probability, economics, etc.

Nowadays, researchers focus on studying the solutions of fractional differential equations (FDEs) or fractional partial equations (FPEs) using various methods, such as the Laplace

✉ Yacine Arioua
yacine.arioua@univ-msila.dz

Kaouther Bouchama
kaoutherbouchama73@gmail.com

Abdelkrim Merzougui
abdelkrim.merzougui@univ-msila.dz

[1] Department of Mathematics, Laboratory of Pure and Applied Mathematics, Mohamed Boudiaf University of M'sila, Bp 166, 28000 M'sila, Algeria

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transform method, the Fourier transform method [16]. However, most of fractional derivatives definitions involve integral form and some of them have some nonlocal property, wich make the analytic methods for solving FDEs and FPEs not effective. As a result many numerical techniques are developed and employed to solve complex FDEs and FPEs, such as the Adomian decomposition method, variational iteration method, the Homotopy analysis method, generalized differential transform method or finite difference method (FDM) and so on. Several authors have done much work on this topic ([10, 12, 15, 22, 24, 26, 31]).

Fractional linear diffusion equation in time and space represent extensions of basic equations of mathematical physics. It is obtained from the classical diffusion equation by replacing the first derivative with respect to time by a Caputo-Hadamard fractional derivative of order α and the derivative with respect to the space by a Caputo-Hadamard fractional derivative of order β, which is given by the following equation

$$ {}^{CH}\partial_t^{\alpha} u\left(x,t\right) = {}^{CH}\partial_x^{\beta} u\left(x,t\right) + f(x,t),\ \ (x,t) \in]x_0, L[\times]t_0, T[\tag{1} $$

where ${}^{CH}\partial_t^{\alpha,\rho}$, ${}^{CH}\partial_x^{\beta,\rho}$ denotes the Caputo-Hadamard fractional derivative operator of order α and β respectively, with $0 < \alpha \leq 1$, $1 < \beta \leq 2$, $t_0, x_0 > 0$ and $f(x,t)$ is the source term. Recently, considerable research has been devoted to the study of numerical methods lead to a rapid increase development of numerical methods for fractional diffusion equation, references can be made to A. Bhardwaj et al. [2] used a radial basis function-based meshless approach to approximate the time-fractional nonlinear mixed diffusion and diffusion-wave equation. A. Bhardwaj et al. [3] developed an RBF-based meshfree method to solve numerically the multi-term time-fractional nonlinear two-dimensional diffusion-wave equation. A. Bhardwaj et al. [4] proposed a meshless method based on radial basis function (RBF) to solve the time-fractional mixed diffusion and diffusion-wave equation. A. Bhardwaj et al. [5] proposed an RBF based meshless method to investigate the time-fractional Tricomi-type equation which is used to overcome above mentioned problems.

In this study, we use the FDM to obtain the numerical solution of space-time fractional linear diffusion equation defined by (1) with initial conditions

$$ u\left(x,t_0\right) = u^0(x),\ \ (x,t) \in \left[x_0, L\right] \times \left[t_0, T\right]. \tag{2} $$

and Neumann conditions

$$ u\left(x_0,t\right) = \varphi(t),\ \frac{\partial u}{\partial x}\left(x_0,t\right) = \psi(t),\ \ t \in \left[t_0, T\right], \tag{3} $$

with u, φ, ψ are continuous functions.

The paper is organized as follows. In the next section, we give most important mathematical definitions of the Hadamard fractional integrals and fractional derivatives of various types. In Sect. 3, finite difference methods for the space-time fractional linear diffusion problem is presented. In Sects. 4 and 5, the stability and the convergence are analyzed respectively. In Sect. 6, to demonstrate the validity of the method we have provided some numerical examples.

Preliminaries

In this section, we recall some concepts on fractional calculus and present additional properties that will be used later.

Definition 1 (Hadamard fractional integral)(see [16])

The left-sided Hadamard fractional integral of order $\alpha > 0$ of a function $y : (a, b) \to \mathbb{R}$is given by

$$\mathcal{I}_{a+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} y(s) \frac{ds}{s}, \tag{4}$$

provided the right integral converges.

Similarly we can define right-sided integrals [16].

Definition 2 (Hadamard fractional derivative)(see [16]).
The left-sided Hadamard fractional derivative of order $\alpha \geq 0$ of a continuous function $y : (a, b) \to \mathbb{R}$is given by

$$\mathcal{D}_{a+}^{\alpha} y(t) = \delta^n \mathcal{I}_{a+}^{n-\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \left(t\frac{d}{dt}\right)^n \int_a^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} y(s) \frac{ds}{s}, \tag{5}$$

where $n = [\alpha] + 1$, and $[\alpha]$ denotes the integer part of the real number α and $\delta = \left(t\frac{d}{dt}\right)$, provided the right integral converges.

A recent generalization introduced by Jarad and al in [17]. The authors define the generalization of the Hadamard fractional derivatives and present properties of such derivatives. This new generalization is now know as the Caputo-Hadamard fractional derivatives and is given by the following definition:

Definition 3 (Caputo-Hadamard fractional derivative)(see [17]).
The left-sided Caputo-type modification of left- Hadamard fractional derivatives of order α is given by

$${}^{CH}\mathcal{D}_{a+}^{\alpha} y(t) = \mathcal{I}_{a+}^{n-\alpha} \delta^n y(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \delta^n y(s) \frac{ds}{s}. \tag{6}$$

The Finite Difference Scheme

In this section, For the finite difference approximation, we equally sub-divide the intervals $[x_0, L]$ with $x_i = (x_0 + ik)$, $i \in \{0, 1, ..., M\}$ and $[t_0, T]$ with $t_j = (t_0 + jh)$, $j \in \{0, 1, ..., N\}$, where $k = \frac{L - x_0}{M}$ and $h = \frac{T - t_0}{N}$ are the spatial and temporal step sizes, respectively. We denote u_i^{n+1} be the numerical approximation to $u(x_i, t_{n+1})$ and $f_i^{n+1} = f(x_i, t_{n+1})$.

1. The initial boundary conditions (2), (3) are discretized as

$$u(x_i, t_0) = u_i^0,$$
$$u(x_0, t_{n+1}) = \varphi^{n+1}, \ u_x(x_0, t_{n+1}) = \psi^{n+1}.$$

2. The approximation for the time fractional derivative ${}^{CH}\partial_t^{\alpha} u(xi, t_{n+1})$ and space fractional derivative term ${}^{CH}\partial_x^{\beta} u(x_i, t_{n+1})$:

Theorem 1 *Let $u : [x_0, L] \times [t_0, T] \to \mathbb{R}$ be such that $u \in C^4([x_0, L] \times [t_0, T], \mathbb{R})$, $\alpha \in]0, 1]$ and $\beta \in]1, 2]$. Then for each positive integer $N \in \mathbb{N}$ and $M \in \mathbb{N}$, we have for each $n \in \{0, 1, ..., N-1\}$*

(a)

$$ {}^{CH}\partial_x^{\beta} u\left(x_i, t_{n+1}\right) = {}^{CH}\partial_x^{\beta} u_i^{n+1} + c_{\beta} k^{2-\beta}, $$

where ${}^{CH}\partial_x^{\beta} u_i^{n+1}$ *is defined by:*

$$ \begin{aligned} {}^{CH}\partial_x^{\beta} u_i^{n+1} =& \frac{1}{k^2 \Gamma\left(3-\beta\right)} \\ & \times \sum_{i=0}^{M-1} a_i^{\beta} \left(x_{i+1} u_{i+1}^{n+1} - \left(x_i + x_{i+1}\right) u_i^{n+1} + x_i u_{i-1}^{n+1} \right), \end{aligned} \tag{7} $$

with

$$ a_i^{\beta} = x_i \left[\left(\log \frac{x_M}{x_i} \right)^{2-\beta} - \left(\log \frac{x_M}{x_{i+1}} \right)^{2-\beta} \right], \tag{8} $$

(b)

$$ {}^{CH}\partial_t^{\alpha} u\left(x_i, t_{n+1}\right) = {}^{CH}\partial_t^{\alpha} u_i^{n+1} + c_{\alpha} h^{1-\alpha}, $$

where ${}^{CH}\partial_t^{\alpha} u_i^{n+1}$ *is defined as follows:*

$$ {}^{CH}\partial_t^{\alpha} u_i^{n+1} = \frac{1}{h \Gamma\left(2-\alpha\right)} \sum_{j=0}^{n} b_j^{\alpha} \left(u_i^{j+1} - u_i^{j} \right), \tag{9} $$

with

$$ b_j^{\alpha} = t_j \left[\left(\log \frac{t_{n+1}}{t_j} \right)^{1-\alpha} - \left(\log \frac{t_{n+1}}{t_{j+1}} \right)^{1-\alpha} \right]. \tag{10} $$

Proof For any $M \in \mathbb{N}$ and $\beta \in (1, 2]$ fixed, we can calculate

$$ \begin{aligned} {}^{CH}\partial_x^{\beta} u\left(x_i, t_{n+1}\right) &= \frac{1}{\Gamma\left(2-\beta\right)} \int_{x_0}^{x_M} \left(\log \frac{x_M}{s} \right)^{1-\beta} \left(s \frac{d}{ds} \right)^2 u(s, t_{n+1}) \frac{ds}{s} \\ &= \frac{1}{\Gamma\left(2-\beta\right)} \sum_{i=0}^{M-1} \int_{x_i}^{x_{i+1}} \left(\log \frac{x_M}{s} \right)^{1-\beta} \\ &\quad \times \left(s \frac{\partial u(s, t_{n+1})}{\partial s} + s^2 \frac{\partial^2 u(s, t_{n+1})}{\partial s^2} \right) \frac{ds}{s} \\ &\approx \frac{1}{\Gamma\left(2-\beta\right)} \sum_{i=0}^{M-1} \int_{x_i}^{x_{i+1}} \left(\log \frac{x_M}{s} \right)^{1-\beta} \\ &\quad \times \left(x_i \frac{u_{i+1}^{n+1} - u_i^{n+1}}{k} + x_i^2 \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{k^2} \right) \frac{ds}{s} \\ &= \frac{1}{\Gamma\left(2-\beta\right)} \sum_{i=0}^{M-1} \left[- \frac{\left(\log \frac{x_M}{s} \right)^{2-\beta}}{\left(2-\beta\right)} \right]_{x_i}^{x_{i+1}} \\ &\quad \times \left(x_i \frac{k u_{i+1}^{n+1} - k u_i^{n+1}}{k^2} + x_i^2 \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{k^2} \right) \end{aligned} $$

$$= \frac{1}{k^2 \Gamma(3-\beta)} \sum_{i=0}^{M-1} x_i \left[\left(\log \frac{x_M}{x_i} \right)^{2-\beta} - \left(\log \frac{x_M}{x_{i+1}} \right)^{2-\beta} \right] \times \left(x_{i+1} u_{i+1}^{n+1} - (x_i + x_{i+1}) u_i^{n+1} + x_i u_{i-1}^{n+1} \right).$$

Set $E_{n+1} = \left| {}^{CH}\partial_x^\beta u(x_i, t_{n+1}) - {}^{CH}\partial_x^\beta u_i^{n+1} \right|$ and $M_i = \max_{x \in [x_0, L]} \left| \frac{\partial^i u}{\partial x^i} \right|$, $i = 1, ...4$, hence, we can obtain

$$E_{n+1} \le \frac{1}{\Gamma(2-\beta)} \sum_{i=0}^{M-1} \int_{x_i}^{x_{i+1}} \left(\log \frac{x_M}{s} \right)^{1-\beta} \times \left| s \frac{\partial u}{\partial s} + s^2 \frac{\partial^2 u}{\partial s} - \left(x_i \frac{u_{i+1}^{n+1} - u_i^{n+1}}{(x_{i+1} - x_i)} + x_i^2 \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(x_{i+1} - x_i)^2} \right) \right| \frac{ds}{s}.$$

It follows from Taylor's theorem, one has for each $i \in \{0, 1, ..., M\}$ and $s \in [x_i, x_{i+1}]$, we have

$$\begin{aligned}
&\left| s \frac{\partial u}{\partial s} + s^2 \frac{\partial^2 u}{\partial s} - \left(x_i \frac{u_{i+1}^{n+1} - u_i^{n+1}}{(x_{i+1} - x_i)} + x_i^2 \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(x_{i+1} - x_i)^2} \right) \right| \\
&\le \left| s \frac{\partial u}{\partial s} - x_i \frac{u_{i+1}^{n+1} - u_i^{n+1}}{(x_{i+1} - x_i)} \right| + \left| s^2 \frac{\partial^2 u}{\partial s^2} - \left(x_i^2 \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(x_{i+1} - x_i)^2} \right) \right| \\
&\le \left| \begin{array}{l} s \left(\frac{\partial u}{\partial s}(x_i, t_{n+1}) + \frac{\partial^2 u}{\partial s^2}(x_i, t_{n+1})(s - x_i) \right) \\ + s \left(\frac{\partial^3 u}{\partial s^3}(x_i, t_{n+1}) \frac{(s - x_i)^2}{2!} + \frac{\partial^4 u}{\partial s^4}(x_i, \eta_2) \frac{(s - x_i)^3}{3!} \right) \\ - x_i \left(\frac{\partial u}{\partial x_i}(x_i, t_{n+1}) + \frac{\partial^{(2)} u}{\partial x_i^2}(x_i, t_{n+1}) \frac{k}{2!} \right) \\ - x_i \left(\frac{\partial^{(3)} u}{\partial x_i^3}(x_i, t_{n+1}) \frac{k^2}{3!} + \frac{\partial^{(4)} u}{\partial x_i^4}(x_i, \eta_1) \frac{k^3}{4!} \right) \end{array} \right| \\
&\quad + \left| \begin{array}{l} s^2 \left(\frac{\partial^2 u}{\partial s^2}(x_i, t_{n+1}) + \frac{\partial^3 u}{\partial s^3}(x_i, t_{n+1})(s - x_i) + \frac{\partial^4 u}{\partial s^4}(x_i, \eta_2) \frac{(s - x_i)^2}{2!} \right) \\ - x_i^2 \left(\frac{\partial^2 u}{\partial x^2}(x_i, t_{n+1}) + \frac{\partial^{(4)} u}{\partial x_i^4}(x_i, \eta_1) \frac{k^2}{12} \right) \end{array} \right| \\
&\le (s - x_i) M_1 + \left(s(s - x_i) - x_i \frac{k}{2!} \right) M_2 + \left(s \frac{(s - x_i)^2}{2!} - x_i \frac{k^2}{3!} \right) M_3 \\
&\quad + s \frac{(s - x_i)^3}{6} M_4 + x_i \frac{k^3}{4!} M_4 \\
&\quad + \left(s^2 - x_i^2 \right) M_2 + s^2 (s - x_i) M_3 + s^2 M_4 \frac{(s - x_i)^2}{2} + x_i^2 M_4 \frac{k^2}{12} \\
&\le L M_1 + \frac{3}{2} L^2 M_2 + \frac{4}{3} L^3 M_3 + \frac{19}{24} L^4 M_4,
\end{aligned}$$

where $\eta_2 \in [x_i, s]$ and $\eta_1 \in \left[x_i, x_{i+1}\right]$. Therefore, we conclude

$$
\begin{aligned}
E_n &\leq \frac{L}{\Gamma(2-\beta)}\left(M_1+\frac{3}{2}LM_2+\frac{4}{3}L^2M_3+\frac{19}{24}L^3M_4\right)\sum_{i=0}^{M-1}\int_{x_i}^{x_{i+1}}\left(\log\frac{x_M}{s}\right)^{1-\beta}\frac{ds}{s}\\
&\leq \frac{L}{\Gamma(2-\beta)}\left(M_1+\frac{3}{2}LM_2+\frac{4}{3}L^2M_3+\frac{19}{24}L^3M_4\right)\sum_{i=0}^{M-1}\int_{x_i}^{x_{i+1}}\left(\log\frac{x_{i+1}}{s}\right)^{1-\beta}\frac{ds}{s}\\
&\leq \frac{L}{\Gamma(3-\beta)}\left(M_1+\frac{3}{2}LM_2+\frac{4}{3}L^2M_3+\frac{19}{24}L^3M_4\right)\sum_{i=0}^{M-1}\left(\log\frac{x_{i+1}}{x_i}\right)^{2-\beta}\\
&\leq \frac{L}{\Gamma(3-\beta)}\left(M_1+\frac{3}{2}LM_2+\frac{4}{3}L^2M_3+\frac{19}{24}L^3M_4\right)\sum_{j=0}^{M-1}L^{2-\beta}k^{2-\beta}\\
&\leq \frac{L^{3-\beta}(L-x_0)}{\Gamma(3-\beta)}\left(M_1+\frac{3}{2}LM_2+\frac{4}{3}L^2M_3+\frac{19}{24}L^3M_4\right)k^{2-\beta}.
\end{aligned}
$$

This means that

$$
{}^{CH}\partial_x^\beta u(x_i, t_{n+1}) = {}^{CH}\partial_x^\beta u_i^{n+1} + c_\beta k^{2-\beta}.
$$

Similarly, for any $N \in \mathbb{N}$, with $0 \leq n \leq N-1$ and $\alpha \in (0, 1]$ fixed, it is easy to get the approximation of the time fractional derivative term ${}^{CH}\partial_t^\alpha u(x_i, t_{n+1})$: For any $N \in \mathbb{N}$ and for each $n \in \{0, 1, \ldots, N-1\}$, we have

$$
\begin{aligned}
{}^{CH}\partial_t^\alpha u(x_i, t_{n+1}) =& \frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t_{n+1}}\left(\log\frac{t_{n+1}}{s}\right)^{-\alpha}\left(s\frac{\partial}{\partial s}\right)u(s)\frac{ds}{s}\\
\approx& \frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}\left(\log\frac{t_{n+1}}{s}\right)^{-\alpha}t_j\left(\frac{u_i^{j+1}-u_i^j}{t_{j+1}-t_j}\right)\frac{ds}{s}\\
=& \frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{n}t_j\left(\frac{u_i^{j+1}-u_i^j}{h}\right)\int_{t_j}^{t_{j+1}}\left(\log\frac{t_{n+1}}{s}\right)^{-\alpha}\frac{ds}{s}\\
=& \frac{1}{h\Gamma(1-\alpha)}\sum_{j=0}^{n}t_j\left(u_i^{j+1}-u_i^j\right)\left[-\frac{\left(\log\frac{t_{n+1}}{s}\right)^{1-\alpha}}{(1-\alpha)}\right]_{t_j}^{t_{j+1}}\\
=& \frac{1}{h\Gamma(2-\alpha)}\sum_{j=0}^{n}t_j\left[\left(\log\frac{t_{n+1}}{t_j}\right)^{1-\alpha}-\left(\log\frac{t_{n+1}}{t_{j+1}}\right)^{1-\alpha}\right]\left(u_i^{j+1}-u_i^j\right)\\
=& \frac{1}{h\Gamma(2-\alpha)}\sum_{j=0}^{n}b_j\left(u_i^{j+1}-u_i^j\right)\\
=& {}^{CH}\partial_t^\alpha u_i^{n+1}.
\end{aligned}
$$

Set $E_n = \left| {}^{CH}\partial_t^{\alpha} u(x_i, t_{n+1}) - {}^{CH}\partial_t^{\alpha} u_i^{n+1} \right|$ and $M_i = \max_{t\in[t_0,T]} \left|\frac{\partial^i u}{\partial t^i}\right|$, $i = 1, 2$, hence, we can obtain

$$E_n \leq \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \left(\log \frac{t_{n+1}}{s}\right)^{-\alpha} \left| s\frac{\partial u}{\partial s} - t_j \left(\frac{u_i^{j+1} - u_i^j}{t_{j+1} - t_j}\right)\right| \frac{ds}{s}.$$

It follows from Taylor's theorem, one has for each $j \in \{0, \ldots, N-1\}$, with $s \in [t_j, t_{j+1}]$ and $\eta_1 \in [t_j, t_{j+1}]$, $\eta_2 \in [t_j, s]$

$$\begin{aligned}
\left| s\frac{\partial u}{\partial s} - t_j \left(\frac{u_i^{j+1} - u_i^j}{t_{j+1} - t_j}\right)\right| &= \left| s\frac{\partial u}{\partial s} - t_j \left(\frac{\partial u(t_j)}{\partial s} - \frac{\partial^{(2)} u(\eta_1)}{\partial s^2}\frac{h}{2!}\right)\right| \\
&\leq \left| \left(s\frac{\partial u}{\partial s} - t_j \frac{\partial u(t_j)}{\partial s}\right)\right| + M_2 \frac{t_j h}{2} \\
&= \left| s\left(\frac{\partial u(t_j)}{\partial s} - t_j \frac{\partial u(t_j)}{\partial s} + \frac{\partial^{(2)} u(\eta_1)}{\partial s^2}(s - t_j)\right)\right| + M_2 \frac{t_j h}{2} \\
&\leq M_1 (t_{j+1} - t_j) + M_2 t_j \frac{3}{2} h \\
&\leq \left(M_1 + \frac{3T}{2} M_2\right) h.
\end{aligned}$$

Furthermore, for any $0 < \alpha \leq 1$ and $n \in \{0, \ldots, N-1\}$ with $j \leq n$, $s \in [t_j, t_{j+1}]$

$$0 \leq \left(\log \frac{t_{n+1}}{s}\right)^{-\alpha} \leq \left(\log \frac{t_{j+1}}{s}\right)^{-\alpha},$$

Therefore, we conclude

$$\begin{aligned}
E_n \leq& \frac{1}{\Gamma(1-\alpha)} \left(M_1 + \frac{3T}{2} M_2\right) h \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \left(\log \frac{t_{n+1}}{s}\right)^{-\alpha} \frac{ds}{s} \\
\leq& \frac{1}{\Gamma(1-\alpha)} \left(M_1 + \frac{3T}{2} M_2\right) h \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \left(\log \frac{t_{j+1}}{s}\right)^{-\alpha} \frac{ds}{s} \\
\leq& \frac{1}{\Gamma(2-\alpha)} \left(M_1 + \frac{3T}{2} M_2\right) h \sum_{j=0}^{n} \left(\log \frac{t_{j+1}}{t_j}\right)^{1-\alpha} \\
\leq& \frac{1}{\Gamma(2-\alpha)} \left(M_1 + \frac{3T}{2} M_2\right) h \sum_{j=0}^{n} \left(\frac{t_{j+1}}{t_j}\right)^{1-\alpha} \\
\leq& \frac{1}{\Gamma(2-\alpha)} \left(M_1 + \frac{3T}{2} M_2\right) h \sum_{i=0}^{N-1} h^{1-\alpha} T^{1-\alpha} \\
\leq& \frac{1}{\Gamma(2-\alpha)} \left(M_1 + \frac{3T}{2} M_2\right) T^{1-\alpha} h^{1-\alpha} \sum_{j=0}^{N-1} h \\
\leq& \frac{1}{\Gamma(2-\alpha)} \left(M_1 + \frac{3T}{2} M_2\right) (T - t_0) T^{1-\alpha} h^{1-\alpha},
\end{aligned}$$

which means

$$\begin{aligned} {}^{CH}\partial_t^{\alpha} u\left(x_i, t_{n+1}\right) &= {}^{CH}\partial_t^{\alpha} u_i^{n+1} + c_\alpha h^{1-\alpha}, \\ {}^{CH}\partial_t^{\alpha} u_i^{n+1} &= \frac{1}{h\Gamma(2-\alpha)} \sum_{j=0}^{n} t_j \left[\left(\log \frac{t_{n+1}}{t_j} \right)^{1-\alpha} - \left(\log \frac{t_{n+1}}{t_{j+1}} \right)^{1-\alpha} \right] \\ &\quad \times \left(u_i^{j+1} - u_i^j \right), \end{aligned}$$

and we obtain

$$\left| {}^{CH}\partial_t^{\alpha} u\left(x_i, t_{n+1}\right) - {}^{CH}\partial_t^{\alpha} u_i^{n+1} \right| \leq \frac{T^{1-\alpha}\left(T - t_0\right)}{\Gamma(2-\alpha)} \left(M_1 + \frac{3T}{2} M_2 \right) h^{1-\alpha}.$$

□

By using the space-time fractional approximation (7) and (9) we obtain the following numerical approximation to equation (1)

$$\begin{aligned} &\frac{-1}{k^2 \Gamma(3-\beta)} \sum_{i=0}^{M-1} a_i^{\beta} \left(x_{i+1} u_{i+1}^{n+1} - \left(x_i + x_{i+1}\right) u_i^{n+1} + x_i u_{i-1}^{n+1} \right), \\ &= \frac{-1}{h\Gamma(2-\alpha)} \sum_{j=0}^{n} b_j^{\alpha} \left(u_i^{j+1} - u_i^j \right) + f_i^{n+1}, \end{aligned}$$

then, for each $n = 0, 1, \ldots, N-1$, setting $\lambda = \dfrac{h\Gamma(2-\alpha)}{k^2\Gamma(3-\beta)}$, we obtain the following difference approximtion

$$\begin{aligned} &-\lambda \left(\sum_{i=0}^{M-1} a_i^{\beta} \left(x_{i+1} u_{i+1}^{n+1} - \left(x_i + x_{i+1}\right) u_i^{n+1} + x_i u_{i-1}^{n+1} \right) \right) \\ &= -b_n^{\alpha} u_i^{n+1} + b_n^{\alpha} u_i^n - \sum_{j=0}^{n-1} b_j^{\alpha} \left(u_i^{j+1} - u_i^j \right) + h\Gamma(2-\alpha) f_i^{n+1}, \end{aligned}$$

for $l \in \{1, \ldots, M\}$, the above equation can be simplified to

$$\sum_{i=1}^{M} \omega_i u_i^{n+1} + b_n^{\alpha} u_l^{n+1} = b_0^{\alpha} u_l^0 + \sum_{j=1}^{n} G_j u_l^j + V_l^{n+1},$$

where $G_j = \left(b_j^{\alpha} - b_{j-1}^{\alpha} \right)$,

$$V_l^{n+1} = \lambda \left(x_1 \left(a_1^{\beta} - a_0^{\beta} \right) \varphi^{n+1} - k a_0^{\beta} x_0 \psi^{n+1} \right) + h\Gamma(2-\alpha) f_l^{n+1},$$

and

$$\omega_i = \begin{cases} \lambda \left(-x_i a_{i-1}^{\beta} + a_i^{\beta} \left(x_i + x_{i+1}\right) - a_{i+1}^{\beta} x_{i+1} \right), \text{ for } 1 \leq i \leq M-2, \\ \lambda \left(a_{M-1}^{\beta} \left(x_{M-1} + x_M\right) - a_{M-2}^{\beta} x_{M-1} \right), \text{ for } i = M-1, \\ -\lambda x_M a_{M-1}^{\beta}, \text{ for } i = M, \end{cases}$$

So:

1. For $n = 0$ and $l \in \{1, ..., M\}$ we have

$$\sum_{i=1}^{M} \omega_i u_i^{n+1} + b_0^\alpha u_l^{n+1} = b_0^\alpha u_l^0 + V_l^1, \tag{11}$$

2. For $n > 0$ and $l \in \{1, ..., M\}$ we obtain

$$\sum_{i=1}^{M} \omega_i u_i^{n+1} + b_n^\alpha u_l^{n+1} = b_0^\alpha u_l^0 + \sum_{j=1}^{n} G_j u_l^j + V_l^{n+1}, \tag{12}$$

Thus, we have the difference scheme can be rewritten as the following matrix form

$$\begin{cases} U^0 = u_i^0, \text{ for } i = 1, \ldots, M, \\ A_n U^{n+1} = b_0^\alpha U^0 + G_1 U^1 + G_2 U^2 + \ldots + G_n U^n + V^{n+1}, \end{cases} \tag{13}$$

with A^n is square matrix of dimension $M \times M$ of coefficients:

$$A_{(i,j)}^n = \begin{cases} w_j, \text{ if } i \neq j, \\ w_i + b_n^\alpha, \text{ if } i = j, \end{cases}$$

and

$$\begin{cases} U^n = \left[u_1^n, u_2^n, \ldots, u_M^n\right]^T, \\ V^n = \left[V_1^n, V_2^n, \ldots, V_M^n\right]^T. \end{cases}$$

Stability of Finite Difference Scheme

In this section, we discuss the stability of the solution of space-time fractional finite difference scheme (11) and (12) for the STFLDE (1). To do so, we need the following lemma

Lemma 1 *The coeffcients a_i^β, b_j^α in (8) and (10) satisfy*

1. $a_i^\beta > 0$, $b_j^\alpha > 0$, *for* $i = 0, \ldots, M-1$ *and* $j = 0, \ldots, n$.
2. $a_i^\beta > a_{i-1}^\beta$, $b_j^\alpha > b_{j-1}^\alpha$, *for* $i = 1, \ldots, M-1$ *and* $j = 1, \ldots, n$.

We suppose that $\widetilde{u}_l^n$ is the approximate solution of (11) and (12), the error $\varepsilon_l^n = \widetilde{u}_l^n - u_l^n$, for $l \in \{1, ..., M\}$ and $n \in \{0, 1, ..., N-1\}$ satisfies

$$\sum_{i=1}^{M} \omega_i \varepsilon_i^{n+1} + b_n^\alpha \varepsilon_l^{n+1} = b_0^\alpha \varepsilon_l^0 + \sum_{j=1}^{n} G_j \varepsilon_l^j + V_l^{n+1},$$

which can be written as

$$A_n E^{n+1} = b_0^\alpha E^0 + G_1 E^1 + G_2 E^2 + \ldots + G_n E^n,$$

where $E^n = \left(\varepsilon_1^n, \varepsilon_2^n, \ldots, \varepsilon_{M-1}^n\right)^t$. Hence, the following result can be proved.

Lemma 2 *We have*

$$\left\| E^{n+1} \right\|_\infty \leq \left\| E^0 \right\|_\infty, \ (n = 0, 1, 2, \ldots, N-1).$$

Proof We will use mathematical induction to get the above result. For $n = 0$, let $\left|\varepsilon_l^1\right| = \max_{1\le i\le M}\left|\varepsilon_i^1\right|$ and $\gamma = \lambda\left(2x_{M-1}a_{M-1}^{\beta} - x_1\left(a_0^{\beta} - a_1^{\beta}\right)\right)$. Applying Lemma 1, we have

$$\begin{aligned} b_0^{\alpha}\left|\varepsilon_l^1\right| &\le \left(\gamma + b_0^{\alpha}\right)\left|\varepsilon_l^1\right| \\ &= \left|\sum_{i=1}^{M}\omega_i\varepsilon_l^1 + b_0^{\alpha}\varepsilon_l^1\right| \\ &\le b_0^{\alpha}\left|\varepsilon_l^0\right| \end{aligned}$$

It follows that

$$\left\|E^1\right\|_{\infty} \le \left\|E^0\right\|_{\infty}.$$

Suppose that $\left\|E^j\right\|_{\infty} \le \left\|E^0\right\|_{\infty}$, $j = 1, 2, \ldots, n$. Let $\left|\varepsilon_l^{n+1}\right| = \max_{1\le i\le M}\left|\varepsilon_i^{n+1}\right|$. Using Lemma 1, we also have

$$\begin{aligned} b_n^{\alpha}\left|\varepsilon_l^{n+1}\right| &\le \left(\gamma + b_0^{\alpha}\right)\left|\varepsilon_l^{n+1}\right| \\ &= \left|\sum_{i=1}^{M}\omega_i\varepsilon_l^{n+1} + b_n^{\alpha}\varepsilon_l^{n+1}\right| \\ &\le \left|b_0^{\alpha}\varepsilon_l^0 + \sum_{j=1}^{n}G_j\varepsilon_l^j\right| \\ &\le b_0^{\alpha}\left|\varepsilon_l^0\right| + \left|\sum_{j=1}^{n}G_j\right|\left|\varepsilon_l^j\right| \\ &\le b_0^{\alpha}\left|\varepsilon_l^0\right| + \left|\sum_{j=1}^{n}b_j^{\alpha} - b_{j-1}^{\alpha}\right|\left|\varepsilon_l^0\right| \\ &\le b_0^{\alpha}\left\|E^0\right\|_{\infty} + \left(b_n^{\alpha} - b_0^{\alpha}\right)\left\|E^0\right\|_{\infty} \\ &\le b_n^{\alpha}\left\|E^0\right\|_{\infty} \end{aligned}$$

finly, we find that

$$\left\|E^{n+1}\right\|_{\infty} \le \left\|E^0\right\|_{\infty}.$$

□

The following theorem holds.

Theorem 2 *The solution of the discretised scheme (11) and (12) for the space-time fractional diffusion equation (1) is stable.*

Convergence of the Approximate Scheme

In this section, the convergence analysis of the approximate scheme (11) and (12) is discussed

Theorem 3 *Let $u(x_i, t_n)$ be the exact solution of the space-time fractional linear diffusion equation (1) at mesh points (x_i, t_n), where $i = 0, 1, 2, \ldots, M,\ n = 0, 1, 2, \ldots, N$ and u_i^n the approximate value of $u(x_i, t_n)$ computed by use of the difference scheme (11) and (12). Then there is a positive constant $C_{\alpha,\beta}$, such that*

$$\left|u_i^n - u(x_i, t_n)\right| \le C_{\alpha,\beta}\left(h^{1-\alpha} + k^{2-\beta}\right).$$

Proof Define $e_i^n = u(x_i, t_n) - u_i^n$, ($e^n = \left(e_1^n, e_2^n, \ldots, e_M^n\right)^T$) using $e^0 = 0$. Substitution $u_i^n = u(x_i, t_n) - e_i^n$ into (11) and (12) leads to:

1. For $n = 0$ and $l \in \{1, ..., M\}$ we have

$$\sum_{i=1}^{M} \omega_i e_i^1 + b_0^\alpha e_l^1 = \sum_{i=1}^{M} \omega_i u(x_i, t_1) + b_0^\alpha u(x_l, t_1) - b_0^\alpha\left(u(x_l, t_0) - e_l^0\right) - V_l^1$$
$$= R_l^1.$$

2. For $n > 0$, and $l \in \{1, ..., M\}$, the approximate scheme becomes

$$\sum_{i=1}^{M} \omega_i e_i^{n+1} + b_n^\alpha e_l^{n+1} = \sum_{i=1}^{M} \omega_i u(x_i, t_{n+1}) + b_n^\alpha u(x_l, t_{n+1})$$
$$- b_0^\alpha\left(u(x_l, t_0) - e_l^0\right) - \sum_{j=1}^{n} G_j\left(u\left(x_l, t_j\right) - e_l^j\right) - V_l^{n+1}$$
$$= \sum_{j=1}^{n} G_j e_l^j + b_0^\alpha e_l^0 + R_l^{n+1},$$

where

$$R_l^{n+1} = \sum_{j=0}^{n} b_j^\alpha\left(u\left(x_l, t_{j+1}\right) - u\left(x_l, t_j\right)\right)$$
$$- \lambda \sum_{i=0}^{M} a_i^\beta\left(x_{i+1} u(x_{i+1}, t_{n+1}) - (x_i + x_{i+1}) u(x_i, t_{n+1}) + x_i u(x_{i-1}, t_{n+1})\right)$$
$$- h\Gamma(2-\alpha) f_l^{n+1}.$$

From (1), we have

$$R_l^{n+1} = h\Gamma(2-\alpha)\left({}^{CH}\partial_t^\alpha(x_l, t_{n+1}) - {}^{CH}\partial_x^\beta(x_l, t_{n+1}) - f_l^{n+1} - c_\alpha h^{1-\alpha} + c_\beta k^{2-\beta}\right)$$
$$= h\Gamma(2-\alpha)\left(c_\alpha h^{1-\alpha} - c_\beta k^{2-\beta}\right).$$

Hence, there exist $c_{\alpha,\beta}$, such that

$$\left|R_i^{n+1}\right| \le c_{\alpha,\beta}\left(h^{2-\alpha} + hk^{2-\beta}\right),\ i = 1, ..., M \text{ and } n = 0, ..., N-1.$$

Consequently, using mathematical induction, we will prove for $n = 1, 2, \ldots, N$,

$$\left|e^n\right|_\infty \le \left(b_{n-1}^\alpha\right)^{-1} C_{\alpha,\beta}\left(h^{2-\alpha} + hk^{2-\beta}\right).$$

we have

Let $\left|e_l^1\right| = \max_{1 \le i \le M} \left|e_i^1\right|$, for $n = 0$ and $i \in \{1, ..., M\}$. According to Lemma 1, we get

$$\begin{aligned} b_0^\alpha \left|e_l^1\right| &\le \left(\gamma + b_0^\alpha\right) \left|e_l^1\right| \\ &= \left|\sum_{i=1}^{M} \omega_i e_l^1 + b_0^\alpha e_l^1\right| \\ &\le \left|R_i^1\right|. \end{aligned}$$

which implies that

$$\left|e_l^1\right| \le \left(b_0^\alpha\right)^{-1} c_{\alpha,\beta} \left(h^{2-\alpha} + hk^{2-\beta}\right).$$

Suppose that $\left|e_l^j\right| \le \left(b_{j-1}^\alpha\right)^{-1} c_{\alpha,\beta} \left(h^{2-\alpha} + hk^{2-\beta}\right),\ j = 1, \ldots, n$. Let $\left|e_l^{n+1}\right| = \max_{1 \le i \le M} \left|e_i^{n+1}\right|$ and $i \in \{1, ..., M\}$. Using Lemma 1, we also have

$$\begin{aligned} b_n^\alpha \left|e_l^{n+1}\right| &\le \left|\sum_{j=1}^{n} G_j e_i^j + R_i^{n+1}\right| \\ &\le \left|\sum_{j=1}^{n} G_j\right| \left|e_i^j\right| + \left|R_i^{n+1}\right| \\ &\le \left|\sum_{j=1}^{n} \left(b_j^\alpha - b_{j-1}^\alpha\right)\right| \left|e_i^j\right| + \left|R_i^{n+1}\right|, \end{aligned}$$

Finally we find

$$\left|e_l^{n+1}\right| \le \left(b_n^\alpha\right)^{-1} C'_{\alpha,\beta} \left(h^{2-\alpha} + hk^{2-\beta}\right).$$

We can prove that $\lim_{n \to \infty} \dfrac{\left(b_n^\alpha\right)^{-1}}{\left(\frac{t_0}{h} + n\right)} = 0$. Therefore, there exist a constant $\zeta > 0$ such that

$$\frac{1}{b_n^\alpha \left(\frac{t_0}{h} + n\right)} \le \zeta,$$

then

$$\begin{aligned} \left|e_l^{n+1}\right| &\le C'_{\alpha,\beta} \zeta \left(\frac{t_0}{h} + n\right) h \left(h^{1-\alpha} + k^{2-\beta}\right) \\ &\le C'_{\alpha,\beta} \zeta t_n \left(h^{1-\alpha} + k^{2-\beta}\right) \\ &\le C'_{\alpha,\beta} \zeta T \left(h^{1-\alpha} + k^{2-\beta}\right), \end{aligned}$$

Finally we have

$$\left|e_l^{n+1}\right| \le C_{\alpha,\beta} \left(h^{1-\alpha} + k^{2-\beta}\right).$$

□

Illustrative Examples

In the present section, two numerical examples are presented to illustrate the usefulness of our main results.

Example 1 Let $(x,t) \in [1,2] \times [1,2]$, $\alpha = 0.9$ and

$$f(x,t) = \frac{1}{\Gamma(2-\alpha)} \left(\log\left(\frac{x}{3}\right)\right)^2 (\log t)^{1-\alpha} - \frac{2}{\Gamma(3-\beta)} \log\left(\frac{t}{2}\right) (\log x)^{(2-\beta)},$$

Consider the following space-time fractional linear diffusion equation

$$\begin{cases} {}^{CH}\partial_t^{\alpha} u(x,t) = {}^{CH}\partial_x^{\beta} u(x,t) + f(x,t), \\ u(x,1) = \log\left(\frac{1}{2}\right)\left(\log\left(\frac{x}{3}\right)\right)^2, \\ u(1,t) = \log\left(\frac{t}{2}\right)\left(\log\left(\frac{1}{3}\right)\right)^2, \ \partial_x u(1,t) = 2\log\left(\frac{t}{2}\right)\log\left(\frac{1}{3}\right). \end{cases} \tag{14}$$

Clearly, the exact solution of (14) is given by

$$u(x,t) = \log\left(\frac{t}{2}\right)\left(\log\left(\frac{x}{3}\right)\right)^2.$$

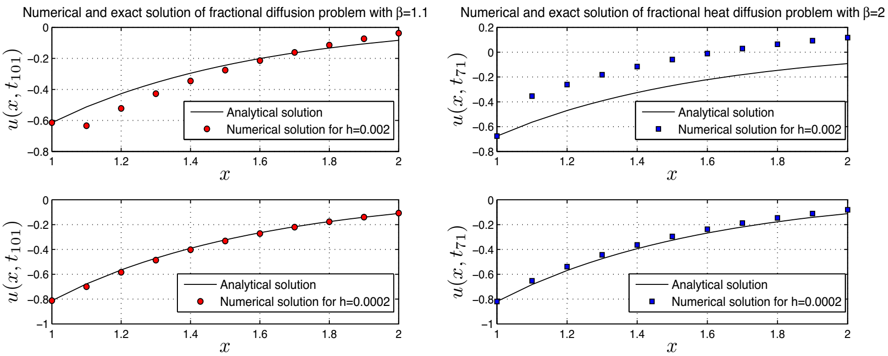

Fig. 1 Graphical comparison of the numerical and the exact solution with $k = 0.1$ and $\alpha = 0.9$

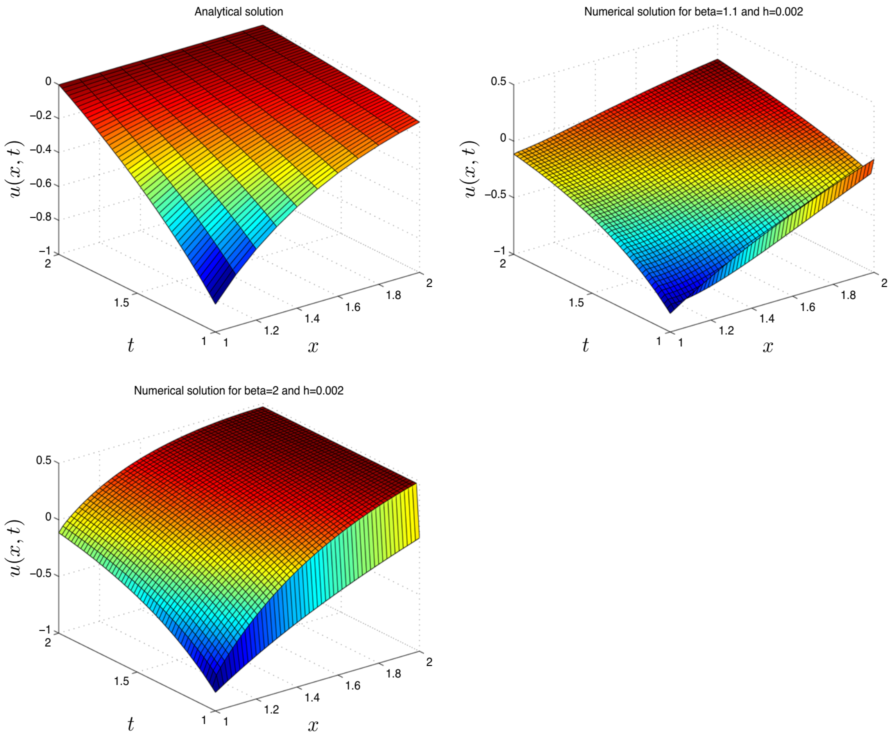

Fig. 2 Surface plot of numerical and exact solutions with $h = 0.002$, $\alpha = 0.9$, $\beta = 1.1$ and $\beta = 2$

Table 1 Comparison of the numerical and the exact solutions of fractinal diffusion problem with $h = 0.02$, $h = 0.002$, $\alpha = 0.9$, $\beta = 2$, $n = 70$, CPU time is 7.251834 seconds

x	Exact solution	Approx solution	Error for $h = 0.002$	x	Exact solution	Approx solution	Error for $h = 0.0002$
1.0	−0.67633	−0.67633	0.00000e+000	1.0	−0.81958	−0.81958	0.00000e+000
1.1	−0.56407	−0.35407	2.10003e−001	1.1	−0.68354	−0.65284	3.06981e−002
1.2	−0.47048	−0.26050	2.09982e−001	1.2	−0.57012	−0.53942	3.06979e−002
1.3	−0.39187	−0.18191	2.09964e−001	1.3	−0.47486	−0.44417	3.06977e−002
1.4	−0.32549	−0.11554	2.09950e−001	1.4	−0.39443	−0.36373	3.06975e−002
1.5	−0.26923	−0.05929	2.09937e−001	1.5	−0.32625	−0.29555	3.06974e−002
1.6	−0.22143	−0.01150	2.09927e−001	1.6	−0.26832	−0.23763	3.06973e−002
1.7	−0.18078	0.02914	2.09918e−001	1.7	−0.21906	−0.18837	3.06972e−002
1.8	−0.14622	0.06369	2.09910e−001	1.8	−0.17719	−0.14650	3.06971e−002
1.9	−0.11691	0.09300	2.09903e−001	1.9	−0.14167	−0.11097	3.06970e−002
2.0	−0.09213	0.11777	2.09898e−001	2.0	−0.11164	−0.08094	3.06969e−002

Example 2 Let $(x,t) \in [1,2] \times [1,2]$, $\alpha = 0.8$ and

$$f(x,t) = \frac{-1}{\Gamma(2-\alpha)} \left(\log\left(\frac{2}{x}\right)\right)^2 (\log t)^{1-\alpha} - \frac{2}{\Gamma(3-\beta)} \log\left(\frac{\sqrt{3}}{t}\right) (\log x)^{2-\beta},$$

consider the following space-time fractional linear diffusion equation

$$\begin{cases} {}^{CH}\partial_t^{\alpha} u(x,t) = {}^{CH}\partial_x^{\beta} u(x,t) + f(x,t), \\ u(x,1) = \log\left(\sqrt{3}\right)\left(\log\left(\frac{2}{x}\right)\right)^2, \\ u(1,t) = \log\left(\frac{\sqrt{3}}{t}\right)(\log(2))^2, \ \partial_x u(1,t) = 2\log\left(\frac{\sqrt{3}}{t}\right)\log 2. \end{cases} \tag{15}$$

Clearly the exact solution of (15) is given by

$$u(x,t) = \log\left(\frac{\sqrt{3}}{t}\right)\left(\log\left(\frac{2}{x}\right)\right)^2.$$

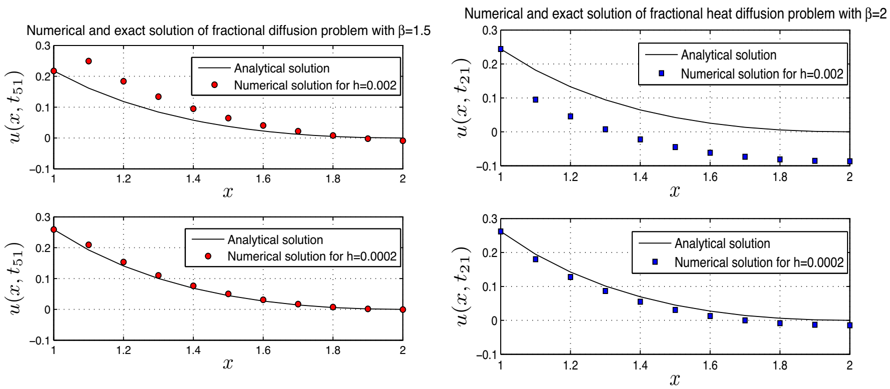

Fig. 3 Graphical comparison of the numerical and the exact solution with $k = 0.1$, $\alpha = 0.8$

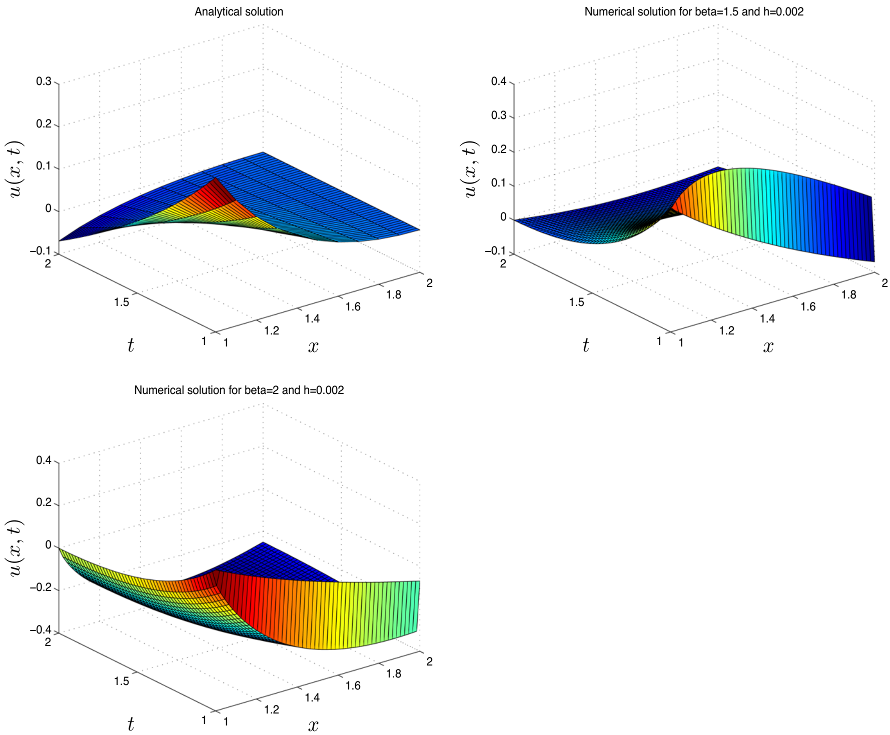

Fig. 4 Surface plot of numerical and exact solutions with $h = 0.002$, $\alpha = 0.8$, $\beta = 1.5$ and $\beta = 2$

Table 2 Comparison of the numerical and the exact solutions of fractinal heat diffusion problem with $h = 0.02$, $h = 0.002$, $\alpha = 0.8$, $\beta = 1.5$, $n = 50$, CPU time is 8.972140 seconds

x	Exact solution	Approx solution	Error for $h = 0.002$	x	Exact solution	Approx solution	Error for $h = 0.0002$
1.0	0.21725	0.21725	0.00000e+000	1.0	0.25904	0.25904	0.00000e+000
1.1	0.16161	0.24920	8.75830e−002	1.1	0.19270	0.20938	1.66783e−002
1.2	0.11799	0.18374	6.57508e−002	1.2	0.14069	0.15343	1.27413e−002
1.3	0.08391	0.13393	5.00191e−002	1.3	0.10005	0.10996	9.90440e−003
1.4	0.05752	0.09502	3.74948e−002	1.4	0.06859	0.07624	7.64590e−003
1.5	0.03742	0.06445	2.70280e−002	1.5	0.04462	0.05038	5.75848e−003
1.6	0.02252	0.04053	1.80176e−002	1.6	0.02685	0.03098	4.13379e−003
1.7	0.01194	0.02205	1.01020e−002	1.7	0.01424	0.01695	2.70657e−003
1.8	0.00502	0.00806	3.04324e−003	1.8	0.00599	0.00742	1.43395e−003
1.9	0.00119	−0.00213	3.32457e−003	1.9	0.00142	0.00170	2.85971e−004
2.0	0.00000	−0.00912	9.12254e−003	2.0	0.00000	−0.00076	7.59194e−004

Conclusion

The paper aims to provide a numerical solution of the space-time fractional linear diffusion equation (STFLDE) with Dirichlet-Neumann initial conditions. The differential operator was taken to be the Caputo-Hadamard one. Also, the convergence and stability of the scheme are proved. The main objective of this paper is to find accurate approximate solutions for (STFLDE) of order α and β with $0 < \alpha \leq 1$ and $1 < \beta \leq 2$. Hence, we have used the Finite Difference Method (FDM). The efficiency of (FDM) has been discussed and illustrated by solving two typical examples of (Examples 1 and 2). Moreover, various results were obtained for different values of the parameters α and β. In the case of $\beta = 2$, we obtain the numerical solution of the fractional heat equation. Eventually, different values for h and k have been tested on examples 1 and 2 to claim that the approach discussed in this paper is useful for solving (STFLDE) (Figs. 1, 2, 3, 4). The results obtained an improved convergence with an error $C_{\alpha,\beta}\left(h^{1-\alpha} + k^{2-\beta}\right)$ tends to zero (Table 1, Table 2).

Author Contributions In this paper we have discussed the numerical solutions of the space-time fractional linear diffusion equation (STFLDE) with Neumann and initial conditions. The differential operator was defined in Caputo-Hadamard sense. Also, the convergence and stability of the scheme are proved.

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Data Availibility Enquiries about data availability should be directed to the authors.

Declarations

Conflict of interest The authors declare no competing interests.

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