Moroccan J. of Pure and Appl. Anal. (MJPAA)
Volume 7(2), 2021, Pages 227–247
ISSN: Online 2351-8227 - Print 2605-6364
DOI: 10.2478/mjpaa-2021-0016

A new class of mixed fractional differential equations with integral boundary conditions

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Abstract. This paper deals with a new class of mixed fractional differential equations with integral boundary conditions. We show an important equivalence result between our problem and nonlinear integral Fredholm equation of the second kind. The existence and uniqueness of a positive solution are proved using Guo-Krasnoselskii's fixed point theorem and Banach's contraction mapping principle. Different types of Ulam-Hyers stability are discussed. Three examples are also given to show the applicability of our results.

Mathematics Subject Classification (2020). 26A33, 34K37, 47H10, 34B18.
Key words and phrases. Caputo fractional derivative, Conformable fractional derivative, Fixed point theorems, Positive solution, Ulam-Hyers stability.

1. Introduction

The fractional calculus has grown rapidly in recent years and increasingly used in a variety of fields of science and engineering (mathematical physics, biology, bioengineering, control theory, hydrology, thermodynamics, mechanic and finance; see the books [17, 27, 25, 13]). Most of them used an integral form for the fractional derivative. The most popular ones are "Riemann-Liouville fractional derivative" or "Caputo fractional derivative". However, a new

Received : August 31, 2020 - Accepted: December 05, 2020.

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fractional derivative has been introduced by Khalil et al. in [16] namely "the conformable fractional derivative" which is satisfy all the requirements of the standard derivative.

Integral boundary conditions of fractional problems are approached by various researchers by applying different fixed point theorems, iterative technique, and upper and lower solutions method. We refer the reader to a series of articles [22, 23, 37, 29, 28, 35, 6, 33, 3, 10, 9, 32, 20, 34, 31, 5]. Another aspect of stability analysis has been taken up by a number of mathematicians called "Ulam-Hyers stability" this concept was introduced in the mid of 19^{th} century and recently it is a one of the most important subjects in the mathematical analysis area. For details, see [15, 21, 7, 36, 19, 8, 5] and the references cited therein.

Recently, fractional differential equations including both left and right fractional derivatives are also attracting much attention , there are many results on boundary value problems concerning mixed fractional derivatives of different types. For instance Ntouyas et al. [24] investigated the existence and uniqueness of solutions of single and multi-valued boundary value problems involving both Riemann-Liouville and Caputo fractional derivatives, and nonlocal fractional integro-differential boundary conditions. In [18] the authors proved the existence of solutions for a boundary value problem involving both left Riemann-Liouville and right Caputo fractional derivatives by applying Krasnoselskii's fixed point theorem. Bashir et al. [4] studied the existence theory of nonlocal three-point boundary value problems for differential equations and inclusions involving both left Caputo and right Riemann-Liouville fractional derivatives by using the Banach and Krasnoselskii fixed point theorems and the Leray-Schauder nonlinear alternative. The existence of solutions for the multivalued problem concerning the upper semicontinuous and Lipschitz cases is proved by applying nonlinear alternative for Kakutani maps and Covitz and Nadler fixed point theorem. Chatzarakis et al. [11] has been presented the first work on conformable and Riemann-Liouville left sided fractional derivatives appearing in the oscillation problem of mixed fractional order nonlinear differential equations by using the generalized Riccati technique and the integral averaging method to establish new oscillation criteria.

In this paper, we study the existence, uniqueness of positive solution and Ulam-Hyers stability to the following mixed fractional boundary value problem (for short MFBVP) with integral boundary conditions:

$$\mathcal{D}_{1^-}^{(\beta)}\left({}^c\mathcal{D}_{0^+}^{\alpha}x\right)(t) = f\left(t, x\left(t\right)\right),\ 0 < t < 1, \tag{1.1a}$$

$$x\left(0\right) = \gamma \int_0^1 x\left(t\right) dt, \tag{1.1b}$$

$${}^c\mathcal{D}_{0^+}^{\alpha}x\left(1\right) = 0, \tag{1.1c}$$

where $\alpha, \beta \in \left]0, 1\right]$, $\gamma \in \left]0, 1\right[$, $\mathcal{D}_{1^-}^{(\beta)}$ denote the right conformable fractional derivative, ${}^c\mathcal{D}_{0^+}^{\alpha}$ denote the left Caputo fractional derivative, x is the unknown function and $f : [0, 1] \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function.

This paper is organized as follows. In Section 2, we recall some useful definitions, lemmas and theorems on fractional calculus. In Section 3, we prove an important equivalence result between the problem considered and nonlinear integral Fredholm equation of the second kind. This result plays a key role in the forthcoming analysis in this paper. Then, we prove the existence and uniqueness of the positive solution for MFBVP (1.1a)-(1.1c) by using Guo-Krasnoselskii's fixed point theorem and Banach's contraction mapping principle. Furthermore, we discus different types of Ulam-Hyers stability for MFBVP considered. Three examples are also given to illustrate our results. Finally, we end this paper with a conclusion.

2. Preliminaries

In this section, we recall some useful definitions, lemmas and theorems on fractional calculus.

Definition 2.1 ([1, 16])**.** The left conformable fractional derivative starting from a of a function $x:[a,+\infty)\to\mathbb{R}$ of order $0<\beta\leq 1$ is defined by

$$\mathcal{D}_{a^+}^{(\beta)}x(t)=\lim_{\varepsilon\to 0}\frac{x\left(t+\varepsilon(t-a)^{1-\beta}\right)-x(t)}{\varepsilon},\ for\ all\ t>0.$$

If $\mathcal{D}_{a^+}^{(\beta)}x$ exists on $]\,a,b\,[$ then $\mathcal{D}_{a^+}^{(\beta)}x(a)=\lim\limits_{t\to a^+}\mathcal{D}_{a^+}^{(\beta)}x(t)$.

The right conformable fractional derivative of order $0<\beta\leq 1$ terminating at b of x is defined by

$$\mathcal{D}_{b^-}^{(\beta)}x(t)=-\lim_{\varepsilon\to 0}\frac{x\left(t+\varepsilon(b-t)^{1-\beta}\right)-x(t)}{\varepsilon},\ for\ all\ t>0.$$

If $\mathcal{D}_{b^-}^{(\beta)}x(t)$ exists on $]\,a,b\,[$ then $\mathcal{D}_{b^-}^{(\beta)}x(b)=\lim\limits_{t\to b^-}\mathcal{D}_{b^-}^{(\beta)}x(t)$.

The left and right conformable fractional integral of a function x of order $0<\beta\leq 1$ is defined respectively by

$$I_{a^+}^{\beta}x(t)=\int_a^t(s-a)^{\beta-1}x(s)\,ds, \tag{2.1a}$$

$$I_{b^-}^{\beta}x(t)=\int_t^b(b-s)^{\beta-1}x(s)\,ds. \tag{2.1b}$$

Definition 2.2 ([17, Theorem 2.1])**.** The left Caputo fractional derivative of order $0<\alpha\leq 1$ of an absolutely continuous function $x:[a,b]\to\mathbb{R}$ is given by

$${}^c\mathcal{D}_{a^+}^{\alpha}x(t)=\frac{1}{\Gamma(1-\alpha)}\int_a^t(t-s)^{-\alpha}x'(s)\,ds. \tag{2.2}$$

The right Caputo fractional derivative of order $0<\alpha\leq 1$ terminating at b of x is defined by

$${}^c\mathcal{D}_{b^-}^{\alpha}x(t)=-\frac{1}{\Gamma(1-\alpha)}\int_t^b(s-t)^{-\alpha}x'(s)\,ds.$$

The left and right Riemann-Liouville fractional integral of order $0<\alpha\leq 1$ of a function x are defined respectively by

$$J_{a^+}^{\alpha}x(t)=\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}x(s)\,ds, \tag{2.3a}$$

$$J_{b^-}^{\alpha}x(t)=\frac{1}{\Gamma(\alpha)}\int_t^b (s-t)^{\alpha-1}x(s)\,ds, \tag{2.3b}$$

where $\Gamma(\cdot)$ is the Euler gamma function, see [17, p.24].

Lemma 2.1 ([1, 16, 17]). We have

(i): If x is a continuous function on $]a,b[$ then

$$\mathcal{D}_{b^-}^{(\beta)}\left(I_{b^-}^{\beta}x(t)\right)=\mathcal{D}_{a^+}^{(\beta)}\left(I_{a^+}^{\beta}x(t)\right)={}^{c}\mathcal{D}_{b^-}^{\alpha}\left(J_{b^-}^{\alpha}x(t)\right)={}^{c}\mathcal{D}_{a^+}^{\alpha}\left(J_{a^+}^{\alpha}x(t)\right)=x(t). \tag{2.4}$$

(ii): If $\mathcal{D}_{a^+}^{(\beta)}x$, $\mathcal{D}_{b^-}^{(\beta)}x$, ${}^{c}\mathcal{D}_{a^+}^{\alpha}x$, ${}^{c}\mathcal{D}_{b^-}^{\alpha}x$ are continuous on $]a,b[$ then

$$I_{a^+}^{\beta}\left(\mathcal{D}_{a^+}^{(\beta)}x(t)\right)=J_{a^+}^{\alpha}\left({}^{c}\mathcal{D}_{a^+}^{\alpha}x(t)\right)=x(t)-x(a), \tag{2.5a}$$

$$I_{b^-}^{\beta}\left(\mathcal{D}_{b^-}^{(\beta)}x(t)\right)=J_{b^-}^{\alpha}\left({}^{c}\mathcal{D}_{b^-}^{\alpha}x(t)\right)=x(t)-x(b). \tag{2.5b}$$

(iii): If x is differentiable on $]a,b[$ then

$$\mathcal{D}_{a^+}^{(\beta)}x(t)=(t-a)^{1-\beta}x'(t), \tag{2.6a}$$

$$\mathcal{D}_{b^-}^{(\beta)}x(t)=-(b-t)^{1-\beta}x'(t). \tag{2.6b}$$

Further, we present the following fixed point theorems which will be used in studying of our main results.

Theorem 2.1 (Contraction Mapping Principle [2, 12]). Let $(E,\|\cdot\|)$ be a Banach space, $\mathcal{P}\subseteq E$ a nonempty closed subset. If $\mathcal{T}:\mathcal{P}\to\mathcal{P}$ is a strictly contraction, i.e.,

$$\exists L\in\,]0,1[\,,\forall x,y\in\mathcal{P}:\|\mathcal{T}x-\mathcal{T}y\|\leq L\|x-y\|,$$

then $\mathcal{T}$ has a unique fixed point in $\mathcal{P}$.

Theorem 2.2 (Guo-Krasnoselskii's fixed point theorem [14]). Let $(E,\|\cdot\|)$ be a Banach space, $\mathcal{P}\subset E$ be a cone and Ω_1,Ω_2 are two bounded open subsets in E such that $0\in\Omega_1\subset\overline{\Omega}_1\subset\Omega_2$. Let operator $\mathcal{T}:\mathcal{P}\cap\left(\overline{\Omega}_2\backslash\Omega_1\right)\to\mathcal{P}$ be completely continuous. Suppose that one of the two conditions

$$(H_1)\quad \|\mathcal{T}x\|\leq\|x\|,\ \forall x\in\mathcal{P}\cap\partial\Omega_1\ \text{ and }\ \|\mathcal{T}x\|\geq\|x\|,\ \forall x\in\mathcal{P}\cap\partial\Omega_2$$

and

$$(H_2)\quad \|\mathcal{T}x\|\geq\|x\|,\ \forall x\in\mathcal{P}\cap\partial\Omega_1\ \text{ and }\ \|\mathcal{T}x\|\leq\|x\|,\ \forall x\in\mathcal{P}\cap\partial\Omega_2$$

is satisfied. Then $\mathcal{T}$ has at least one fixed point in $\mathcal{P}\cap\left(\overline{\Omega}_2\backslash\Omega_1\right)$.

Let ε be a positive real number, $f:[0,1]\times\mathbb{R}_+\to\mathbb{R}_+$ be a continuous function and $\varphi:[0,1]\to\mathbb{R}_+$ be a continuous function.

For the MFBVP (1.1a)-(1.1c), we focus on the following inequalities:

$$\left|\mathcal{D}_{1^-}^{(\beta)}\left({}^{c}\mathcal{D}_{0^+}^{\alpha}y\right)(t)-f(t,y(t))\right|\leq\varepsilon,\ t\in[0,1]. \tag{2.7}$$

$$\left|\mathcal{D}_{1^-}^{(\beta)}\left({}^{c}\mathcal{D}_{0^+}^{\alpha}y\right)(t)-f(t,y(t))\right|\leq\varphi(t),\ t\in[0,1]. \tag{2.8}$$

$$\left|\mathcal{D}_{1^-}^{(\beta)}\left({}^c\mathcal{D}_{0+}^{\alpha}y\right)(t)-f(t,y(t))\right|\leq\varepsilon\varphi(t),\ t\in[0,1]. \tag{2.9}$$

Definition 2.3 ([26, 30])**.** The MFBVP (1.1a)-(1.1c) is Ulam-Hyers stable if there exist constants $\lambda>0$ such that for each $\varepsilon>0$ and for each solution $y\in E$ of the inequality (2.7) there exists a solution $x\in E$ of the MFBVP (1.1a)-(1.1c) such that

$$|y(t)-x(t)|\leq\lambda\varepsilon,\ t\in[0,1].$$

Definition 2.4 ([26, 30])**.** The MFBVP (1.1a)-(1.1c) is generalized Ulam-Hyers stable if there exists $\theta\in\mathcal{C}(\mathbb{R}_+,\mathbb{R}_+)$, $\theta(0)=0$ such that for each $\varepsilon>0$ and for each solution $y\in E$ of the inequality (2.7), there exists a solution $x\in E$ of the MFBVP (1.1a)-(1.1c) such that

$$|y(t)-x(t)|\leq\theta(\varepsilon),\ t\in[0,1].$$

Definition 2.5 ([26, 30])**.** The MFBVP (1.1a)-(1.1c) is Ulam-Hyers-Rassias stable with respect to φ if there exists a real number $c>0$ such that for each $\varepsilon>0$ and for each solution $y\in E$ of the inequality (2.9) there exists a solution $x\in E$ of the MFBVP (1.1a)-(1.1c) such that

$$|y(t)-x(t)|\leq c\varepsilon\varphi(t),\ t\in[0,1].$$

Definition 2.6 ([26, 30])**.** The MFBVP (1.1a)-(1.1c) is generalized Ulam-Hyers-Rassias stable with respect to φ if there exists a real number $c>0$ such that for each solution $y\in E$ of the inequality (2.8) there exists a solution $x\in E$ of the MFBVP (1.1a)-(1.1c) such that

$$|y(t)-x(t)|\leq c\varphi(t),\ t\in[0,1].$$

Remark 1 ([30])**.** *It is clear that:*

(i) Definition 2.3 $\Rightarrow$ Definition 2.4.
(ii) Definition 2.5 $\Rightarrow$ Definition 2.6.
(iii) Definition 2.5 for $\varphi(\cdot)=1\Rightarrow$ Definition 2.3.

Remark 2 ([26, 30])**.** *(1) A function $y\in E$ is a solution of inequality (2.7) if, and only if, there exists a function $\omega\in\mathcal{C}([0,1],\mathbb{R})$ such that*
(a) $|\omega(t)|\leq\varepsilon$, $t\in[0,1]$,
(b) $\mathcal{D}_{1^-}^{(\beta)}\left({}^c\mathcal{D}_{0+}^{\alpha}y\right)(t)=f(t,y(t))+\omega(t)$, $t\in[0,1]$.
(2) Also a function $y\in E$ is a solution of the inequality (2.8) if, and only if, there exist $h\in\mathcal{C}([0,1],\mathbb{R})$ such that
(a) $|h(t)|\leq\varphi(t)$, $t\in[0,1]$,
(b) $\mathcal{D}_{1^-}^{(\beta)}\left({}^c\mathcal{D}_{0+}^{\alpha}y\right)(t)=f(t,y(t))+h(t)$, $t\in[0,1]$.
(3) Similarly for (2.9) there exist a function $\Phi\in\mathcal{C}([0,1],\mathbb{R})$ such that
(a) $|\Phi(t)|\leq\varepsilon\varphi(t)$, $t\in[0,1]$,
(b) $\mathcal{D}_{1^-}^{(\beta)}\left({}^c\mathcal{D}_{0+}^{\alpha}y\right)(t)=f(t,y(t))+\Phi(t)$, $t\in[0,1]$.

3. Main results

This section is devoted to give an existence, uniqueness and stability results of a positive solution for the MFBVP (1.1a)-(1.1c). To this end, we define the Banach space $E=\mathcal{C}([0,1],\mathbb{R}_+)$ with the norm $\|x\|=\max_{t\in[0,1]}|x(t)|$.

3.1. Equivalence problem. In this subsection, we give the first main result which concerning the equivalence problem of the MFBVP (1.1a)-(1.1c).

Theorem 3.1. Let $\alpha, \beta \in \left]0,1\right]$, $\gamma \in \left]0,1\right[$ and $f : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous function. Then, $x \in E$ is solution of the MFBVP (1.1a)-(1.1c) if, and only if, $x \in E$ satisfies the nonlinear and homogeneous Fredholm integral equation of the second kind,

$$x(t) = \int_0^1 G(t,s) f(s, x(s))\, ds, \tag{3.1}$$

where G is the Green function given by

$$G(t,s) = \begin{cases} \left[\frac{\gamma}{(1-\gamma)(\alpha+1)}\left(1-(1-s)^{\alpha+1}\right) + t^{\alpha} - (t-s)^{\alpha}\right] \frac{(1-s)^{\beta-1}}{\Gamma(\alpha+1)} & \text{if } 0 \le s \le t \le 1, \\ \left[\frac{\gamma}{(1-\gamma)(\alpha+1)}\left(1-(1-s)^{\alpha+1}\right) + t^{\alpha}\right] \frac{(1-s)^{\beta-1}}{\Gamma(\alpha+1)} & \text{if } 0 \le t \le s \le 1. \end{cases} \tag{3.2}$$

Proof. (i) First we prove the necessity. We apply the right fractional integral $I_{1^-}^{\beta}$ defined by (2.1b) on equation (1.1a), using (2.5b) and (1.1c), we obtain

$$^{c}\mathcal{D}_{0^+}^{\alpha} x(t) = I_{1^-}^{\beta} f(t, x(t)), \tag{3.3}$$

and we apply the left fractional integral $J_{0^+}^{\alpha}$ defined by (2.3a) on (3.3), using (2.5a) and (1.1b) we get

$$x(t) = \gamma \int_0^1 x(t)\, dt + J_{0^+}^{\alpha} I_{1^-}^{\beta} f(t, x(t)).$$

On the other hand, using (2.1b) and (2.3a), we obtain

$$\begin{aligned} J_{0^+}^{\alpha} I_{1^-}^{\beta} f(t,x(t)) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} I_{1^-}^{\beta} f(r, x(r))\, dr \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} \left[\int_r^1 (1-s)^{\beta-1} f(s,x(s))\, ds\right] dr \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} \left[\int_r^t (1-s)^{\beta-1} f(s,x(s))\, ds + \int_t^1 (1-s)^{\beta-1} f(s,x(s))\, ds\right] dr \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} \left[\int_r^t (1-s)^{\beta-1} f(s,x(s))\, ds\right] dr \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} \left[\int_t^1 (1-s)^{\beta-1} f(s,x(s))\, ds\right] dr, \end{aligned}$$

and with Fubini theorem, we get

$$\begin{aligned} J_{0^+}^{\alpha} I_{1^-}^{\beta} f(t,x(t)) &= \frac{1}{\Gamma(\alpha)} \int_0^t \left(\int_0^s (t-r)^{\alpha-1}\, dr\right) (1-s)^{\beta-1} f(s,x(s))\, ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_t^1 \left(\int_0^t (t-r)^{\alpha-1}\, dr\right) (1-s)^{\beta-1} f(s,x(s))\, ds \end{aligned}$$

$$= \frac{1}{\Gamma(\alpha+1)} \int_0^t \left(t^\alpha - (t-s)^\alpha\right)(1-s)^{\beta-1} f(s, x(s))\, ds$$
$$+ \frac{1}{\Gamma(\alpha+1)} \int_t^1 t^\alpha (1-s)^{\beta-1} f(s, x(s))\, ds$$
$$= \frac{1}{\Gamma(\alpha+1)} \left[\int_0^1 t^\alpha (1-s)^{\beta-1} f(s, x(s))\, ds - \int_0^t (t-s)^\alpha (1-s)^{\beta-1} f(s, x(s))\, ds\right],$$

wich yields

$$x(t) = \gamma \int_0^1 x(t)\, dt + \frac{1}{\Gamma(\alpha+1)} \left[\int_0^1 t^\alpha (1-s)^{\beta-1} f(s, x(s))\, ds - \int_0^t (t-s)^\alpha (1-s)^{\beta-1} f(s, x(s))\, ds\right]. \tag{3.4}$$

Now, we integrate (3.4) on $[0, 1]$ in both sides and using the Fubini theorem, we get

$$\begin{aligned}
(1-\gamma) \int_0^1 x(t)\, dt &= \frac{1}{\Gamma(\alpha+1)} \int_0^1 \left[\int_0^1 t^\alpha (1-s)^{\beta-1} f(s, x(s))\, ds\right] dt \\
&\quad - \frac{1}{\Gamma(\alpha+1)} \int_0^1 \left[\int_0^t (t-s)^\alpha (1-s)^{\beta-1} f(s, x(s))\, ds\right] dt \\
&= \frac{1}{\Gamma(\alpha+1)} \int_0^1 t^\alpha dt \int_0^1 (1-s)^{\beta-1} f(s, x(s))\, ds \\
&\quad - \frac{1}{\Gamma(\alpha+1)} \int_0^1 \left[\int_s^1 (t-s)^\alpha\, dt\right] (1-s)^{\beta-1} f(s, x(s))\, ds \\
&= \frac{1}{\Gamma(\alpha+2)} \int_0^1 \left[(1-s)^{\beta-1} - (1-s)^{\alpha+\beta}\right] f(s, x(s))\, ds.
\end{aligned} \tag{3.5}$$

Substituting (3.5) into (3.4) yields

$$\begin{aligned}
x(t) &= \int_0^t \left[\frac{\gamma}{(1-\gamma)(\alpha+1)} \left(1 - (1-s)^{\alpha+1}\right) + t^\alpha - (t-s)^\alpha\right] \frac{(1-s)^{\beta-1}}{\Gamma(\alpha+1)} f(s, x(s))\, ds \\
&\quad + \int_t^1 \left[\frac{\gamma}{(1-\gamma)(\alpha+1)} \left(1 - (1-s)^{\alpha+1}\right) + t^\alpha\right] \frac{(1-s)^{\beta-1}}{\Gamma(\alpha+1)} f(s, x(s))\, ds \\
&= \int_0^1 G(t, s) f(s, x(s))\, ds,
\end{aligned}$$

and thus the necessity is proved.

(ii) Now, let $x \in E$ be the solution to the nonlinear homogeneous Fredholm integral equation (3.1). Let us first show that x satisfies the boundary condition (1.1b). From (3.1), (3.2) and Fubini theorem, we obtain

$$\begin{aligned} x(0)-\gamma\int_0^1 x(t)\,dt &= \int_0^1 G(0,s)\,f(s,x(s))\,ds \\ &\quad -\gamma\int_0^1\left[\int_0^t G(t,s)\,f(s,x(s))\,ds+\int_t^1 G(t,s)\,f(s,x(s))\,ds\right]dt \\ &= \int_0^1 G(0,s)\,f(s,x(s))\,ds \\ &\quad -\gamma\int_0^1\left[\int_s^1 G(t,s)\,dt+\int_0^s G(t,s)\,dt\right]f(s,x(s))\,ds. \end{aligned} \tag{3.6}$$

On the other hand and from (3.2), we have

$$\begin{aligned} \int_s^1 G(t,s)\,dt &= \frac{\gamma(1-s)}{(1-\gamma)\Gamma(\alpha+2)}\left[(1-s)^{\beta-1}-(1-s)^{\alpha+\beta}\right] \\ &\quad+\frac{1}{\Gamma(\alpha+2)}\left[1-s^{\alpha+1}-(1-s)^{\alpha+1}\right](1-s)^{\beta-1}, \end{aligned} \tag{3.7}$$

and

$$\int_0^s G(t,s)\,dt = \frac{\gamma s}{(1-\gamma)\Gamma(\alpha+2)}\left[(1-s)^{\beta-1}-(1-s)^{\alpha+\beta}\right]+\frac{1}{\Gamma(\alpha+2)}s^{\alpha+1}(1-s)^{\beta-1}. \tag{3.8}$$

Substituting (3.7) and (3.8) into (3.6), using (3.2) we obtain

$$\begin{aligned} x(0)-\gamma\int_0^1 x(t)\,dt &= \int_0^1 G(0,s)\,f(s,x(s))\,ds \\ &\quad-\frac{\gamma}{(1-\gamma)\Gamma(\alpha+2)}\int_0^1\left[1-(1-s)^{\alpha+1}\right](1-s)^{\beta-1}f(s,x(s))\,ds \\ &= \int_0^1 G(0,s)\,f(s,x(s))\,ds-\int_0^1 G(0,s)\,f(s,x(s))\,ds=0. \end{aligned}$$

Now, we show that $x\in E$ satisfies the boundary condition (1.1c). Using (3.1), (3.2) and (2.2) we get

$$\begin{aligned} {}^{c}\mathcal{D}_{0^+}^{\alpha}x(1) &= \frac{1}{\Gamma(1-\alpha)}\int_0^1(1-s)^{-\alpha}x'(s)\,ds \\ &= \frac{1}{\Gamma(1-\alpha)}\int_0^1(1-s)^{-\alpha}\int_0^s\frac{\partial G}{\partial s}(s,r)\,f(r,x(r))\,drds \\ &\quad+\frac{1}{\Gamma(1-\alpha)}\int_0^1(1-s)^{-\alpha}\int_s^1\frac{\partial G}{\partial s}(s,r)\,f(r,x(r))\,drds \\ &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^1\int_0^s(1-s)^{-\alpha}\left[s^{\alpha-1}-(s-r)^{\alpha-1}\right](1-r)^{\beta-1}f(r,x(r))\,drds \\ &\quad+\frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^1\int_s^1(1-s)^{\alpha-1}s^{\alpha-1}(1-r)^{\beta-1}f(r,x(r))\,drds \end{aligned}$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^1 \int_0^1 (1-s)^{-\alpha} s^{\alpha-1} (1-r)^{\beta-1} f(r, x(r))\, drds$$
$$- \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^1 \int_0^s (1-s)^{-\alpha} (s-r)^{\alpha-1} (1-r)^{\beta-1} f(r, x(r))\, drds.$$

Using Fubini theorem, we obtain

$$ {}^c\mathcal{D}_{0^+}^{\alpha} x(1) = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \left[\int_0^1 (1-s)^{-\alpha} s^{\alpha-1} ds\right] \left[\int_0^1 (1-r)^{\beta-1} f(r, x(r))\, dr\right]$$
$$- \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^1 \left[\int_r^1 (1-s)^{-\alpha} (s-r)^{\alpha-1} ds\right] (1-r)^{\beta-1} f(r, x(r))\, dr. \tag{3.9}$$

Using the relation between the beta and Euler gamma functions, see [17, p.26], we have

$$B(\alpha, \beta) = \int_0^1 (1-s)^{\alpha-1} s^{\beta-1} ds = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{3.10}$$

and using the change of variable $\mu = \frac{s-r}{1-r}$, we obtain

$$\int_r^1 (1-s)^{-\alpha} (s-r)^{\alpha-1} ds = B(\alpha, 1-\alpha). \tag{3.11}$$

From (3.9), (3.10) and (3.11) we obtain (1.1c).

It remains to show that $x \in E$ satisfies the equation (1.1a). From (2.2), (3.1), (3.2) and Fubini theorem we get

$$ {}^c\mathcal{D}_{0^+}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^t \int_0^s (t-s)^{-\alpha} \left[s^{\alpha-1} - (s-r)^{\alpha-1}\right] (1-r)^{\beta-1} f(r, x(r))\, drds$$
$$+ \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^t \int_s^1 (t-s)^{-\alpha} s^{\alpha-1} (1-r)^{\beta-1} f(r, x(r))\, drds$$
$$= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^t \int_0^1 (t-s)^{-\alpha} s^{\alpha-1} (1-r)^{\beta-1} f(r, x(r))\, drds$$
$$- \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^t \int_0^s (t-s)^{-\alpha} (s-r)^{\alpha-1} (1-r)^{\beta-1} f(r, x(r))\, drds$$
$$= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^1 \left[\int_0^t (t-s)^{-\alpha} s^{\alpha-1} ds\right] (1-r)^{\beta-1} f(r, x(r))\, dr$$
$$- \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^t \left[\int_r^t (t-s)^{-\alpha} (s-r)^{\alpha-1} ds\right] (1-r)^{\beta-1} f(r, x(r))\, dr. \tag{3.12}$$

Using the change of variables $\mu = \frac{s}{t}$ and $\mu = \frac{s-r}{t-r}$, we obtain

$$\int_0^t (t-s)^{-\alpha} s^{\alpha-1} ds = \int_r^t (t-s)^{-\alpha} (s-r)^{\alpha-1} ds = B(\alpha, 1-\alpha). \tag{3.13}$$

From (3.12), (3.13) and (3.10) we obtain

$$ {}^c\mathcal{D}_{0^+}^{\alpha} x(t) = \int_0^1 (1-r)^{\beta-1} f(r, x(r))\, dr - \int_0^t (1-r)^{\beta-1} f(r, x(r))\, dr. \tag{3.14}$$

By applying the right conformable derivative defined by (2.6b) in both sides of (3.14), we obtain (1.1a). This completes the proof.

Now, we prove a several important properties of the Green's function G in (3.2).

Lemma 3.1. For all $t \in \,]0,1\,]$ and $s \in [0,1\,[$, we have

(1) $G(t,s) > 0$.

(2) $t^{\alpha} G(1,s) \leq G(t,s) \leq G(1,s)$.

Proof. (1) For all $t \in \,]0,1\,[$ and from (3.2), we have:

$$\frac{\partial G(t,s)}{\partial t} = \frac{1}{\Gamma(\alpha)} \begin{cases} \left(t^{\alpha-1} - (t-s)^{\alpha-1}\right)(1-s)^{\beta-1} & \text{if } \; 0 \leq s < t < 1, \\ t^{\alpha-1}(1-s)^{\beta-1} & \text{if } \; 0 < t < s < 1. \end{cases}$$

Clearly that $\frac{\partial G(t,s)}{\partial t} \geq 0$ for all $t \in \,]0,1\,]$ and $s \in [0,1\,[$, then $G(t,s)$ is increasing with respect to $t \in \,]0,1\,]$. Therefore, for all $t \in \,]0,1\,]$ and $s \in [0,1\,[$, we have

$$G(t,s) \geq G(0,s) = \left[\frac{\gamma}{(1-\gamma)\Gamma(\alpha+2)}\left(1-(1-s)^{\alpha+1}\right)\right](1-s)^{\beta-1} > 0.$$

(2) Using the increasing of the Green's function $G(t,s)$ with respect to t, we obtain for all $t \in \,]0,1\,]$ and $s \in [0,1\,[$,

$$G(t,s) \leq G(1,s) = \frac{1}{\Gamma(\alpha+1)}\left[\frac{\gamma}{(1-\gamma)(\alpha+1)}\left(1-(1-s)^{\alpha+1}\right) + 1 - (1-s)^{\alpha}\right](1-s)^{\beta-1}.$$

On the other hand, From (3.2) we have

$$G(t,s) - t^{\alpha}G(1,s) = \begin{cases} \frac{\gamma(1-t^{\alpha})(1-s)^{\beta-1}}{(1-\gamma)\Gamma(\alpha+2)}\left[1-(1-s)^{\alpha+1}\right] + \frac{t^{\alpha}(1-s)^{\beta-1}}{\Gamma(\alpha+1)}\left[(1-s)^{\alpha} - \left(1-\frac{s}{t}\right)^{\alpha}\right] & \text{if } s \leq t, \\ \frac{\gamma(1-t^{\alpha})(1-s)^{\beta-1}}{(1-\gamma)\Gamma(\alpha+2)}\left[1-(1-s)^{\alpha+1}\right] + \frac{t^{\alpha}(1-s)^{\alpha}(1-s)^{\beta-1}}{\Gamma(\alpha+1)} & \text{if } s \geq t, \end{cases}$$

consequently, $t^{\alpha}G(1,s) \leq G(t,s) \leq G(1,s)$.

3.2. Existence of positive solution. In this subsection, we prove the existence of positive solution of the MFBVP (1.1a)-(1.1c). To this end, let $\Omega_r \in E$ is the bounded open subset defined by

$$\Omega_r := \{x \in E, \; \|x\| < r, \; r > 0\}$$

and $\mathcal{P}$ is the cone defined by

$$\mathcal{P} := \{x \in E, \; x(t) \geq t^{\alpha}\|x\|, \; t \in [0,1]\}.$$

Furthermore, define the operator $\mathcal{T} : E \to E$ such that

$$\mathcal{T}x(t) := \int_0^1 G(t,s) f(s, x(s))\, ds, \tag{3.15}$$

where G defined in (3.2). The operator $\mathcal{T}$ has the following properties.

Lemma 3.2. We have

(1) $\mathcal{T}(\mathcal{P}) \subset \mathcal{P}$.

(2) The operator $\mathcal{T} : \mathcal{P} \to \mathcal{P}$ is completely continuous.

Proof. (1) Let $x \in \mathcal{P}$. From Lemma 3.1, we have

$$\begin{aligned} \mathcal{T}x(t) &= \int_0^1 G(t,s) f(s,x(s))\,ds \\ &\geq t^\alpha \int_0^1 G(1,s) f(s,x(s))\,ds \\ &\geq t^\alpha \int_0^1 G(t,s) f(s,x(s))\,ds. \end{aligned}$$

Then, for all $t \in [0,1]$ we have

$$\mathcal{T}x(t) \geq t^\alpha \max_{t\in[0,1]} \int_0^1 G(t,s) f(s,x(s))ds = t^\alpha \|\mathcal{T}x\|.$$

Hence $\mathcal{T}x \in \mathcal{P}$.

(2) Let $\Omega \subset \mathcal{P}$ be bounded defined by

$$\Omega := \{x \in \mathcal{P} : \|x\| \leq M,\ M > 0\}.$$

Define now

$$L_M := \max_{t\in[0,1],x\in\Omega} f(t,x).$$

Then, for all $x \in \Omega$ we have

$$\|\mathcal{T}x\| = \max_{t\in[0,1]} \int_0^1 G(t,s) f(s,x(s))\,ds \leq L_M \int_0^1 G(1,s)\,ds,$$

which implies that $\mathcal{T}(\Omega)$ is bounded in $\mathcal{P}$.

For each $x \in \Omega$, we have

$$\begin{aligned} \left|(\mathcal{T}x)'(t)\right| &= \left|\int_0^1 \frac{\partial G}{\partial t}(t,s) f(s,x(s))\,ds\right| \\ &= \left|\frac{1}{\Gamma(\alpha)} \int_0^t \left[t^{\alpha-1} - (t-s)^{\alpha-1}\right] (1-s)^{\beta-1} f(s,x(s))\,ds \right. \\ &\quad \left. + \frac{1}{\Gamma(\alpha)} \int_t^1 t^{\alpha-1} (1-s)^{\beta-1} f(s,x(s))\,ds\right| \\ &= \left|\frac{1}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1} (1-s)^{\beta-1} f(s,x(s))\,ds \right. \\ &\quad \left. - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (1-s)^{\beta-1} f(s,x(s))\,ds\right| \\ &\leq \frac{L_M}{\Gamma(\alpha)} \int_0^1 (1-s)^{\beta-1}\,ds + \frac{L_M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\,ds \end{aligned}$$

$$\leq \frac{(\alpha+\beta)\, L_M}{\beta\Gamma(\alpha+1)}.$$

As consequence, for all $t_1, t_2 \in [0,1]$, $t_1 < t_2$, we have

$$|(\mathcal{T}x)(t_2) - (\mathcal{T}x)(t_1)| = \left|\int_{t_1}^{t_2} (\mathcal{T}x)'(s)\, ds\right| \leq \int_{t_1}^{t_2} \left|(\mathcal{T}x)'(s)\right| ds \leq \frac{(\alpha+\beta)\, L_M\, |t_2 - t_1|}{\beta\Gamma(\alpha+1)}.$$

Then, $|\mathcal{T}x(t_2) - \mathcal{T}x(t_1)| \to 0$ as $t_1 \to t_2$ which implies that the set $\mathcal{T}(\Omega)$ is equicontinuous.

Now, from Arzelà-Ascoli theorem, see [2], we conclude that $\overline{T(\Omega)}$ is compact, i.e., $\mathcal{T} : \mathcal{P} \to \mathcal{P}$ is a completely continuous operator.

We give some important notations as follows:

$$f^0 = \lim_{x\to 0} \max_{t\in[0,1]} \frac{f(t,x)}{x},\ f^\infty = \lim_{x\to +\infty} \max_{t\in[0,1]} \frac{f(t,x)}{x},$$

$$f_0 = \lim_{x\to 0} \min_{t\in[0,1]} \frac{f(t,x)}{x},\ f_\infty = \lim_{x\to +\infty} \min_{t\in[0,1]} \frac{f(t,x)}{x},$$

$$\Lambda_1 = \int_0^1 G(1,s)\, ds,\ \Lambda_2 = \rho^\alpha \int_\rho^{1-\rho} G(1,s)\, ds.$$

where $\rho \in \left]0, \frac{1}{2}\right[$ and the function G is defined as in (3.2). Now, we give the second main result.

Theorem 3.2. Assume that one of the three following conditions

(i) There exists $r_2 > r_1 > 0$, such that $\forall x \in [r_1, r_2], \forall t \in [0,1] : \frac{r_1}{\Lambda_2} \leq f(t,x) \leq \frac{r_2}{\Lambda_1}$.

(ii) $\Lambda_1 f^0 \leq \frac{1}{2}$ and $\Lambda_2 f_\infty \geq 2$.

(iii) $\Lambda_2 f_0 \geq 2$ and $\Lambda_1 f^\infty \leq \frac{1}{4}$.

is fulfilled. Then, the MFBVP (1.1a)-(1.1c) has at least one positive solution.

Proof. (i) Let $x \in \mathcal{P} \cap \partial\Omega_{r_1}$, i.e., $x \in \mathcal{P}$ and $\|x\| = r_1$. Using Lemma 3.1, we get

$$\begin{aligned}
\|\mathcal{T}x\| = \max_{t\in[0,1]} \int_0^1 G(t,s)\, f(s,x(s))\, ds &\geq \int_0^1 G(t,s)\, f(s,x(s))\, ds \\
&\geq t^\alpha \int_0^1 G(1,s)\, f(s,x(s))\, ds \\
&\geq \frac{r_1}{\Lambda_2} t^\alpha \int_0^1 G(1,s)\, ds \\
&\geq \frac{r_1}{\Lambda_2} t^\alpha \left[\int_0^\rho G(1,s)\, ds + \int_\rho^{1-\rho} G(1,s)\, ds + \int_{1-\rho}^1 G(1,s)\, ds\right] \\
&\geq \frac{r_1}{\Lambda_2} t^\alpha \int_\rho^{1-\rho} G(1,s)\, ds
\end{aligned}$$

$$\geq \frac{r_1}{\Lambda_2}\left(\frac{t}{\rho}\right)^{\alpha}\rho^{\alpha}\int_{\rho}^{1-\rho} G\left(1,s\right)ds,\ t\in[\rho,1-\rho]$$
$$\geq \frac{r_1}{\Lambda_2}\rho^{\alpha}\int_{\rho}^{1-\rho} G\left(1,s\right)ds = r_1.$$

then $\|\mathcal{T}x\| \geq \|x\|$.

For $x \in \mathcal{P}\cap\partial\Omega_{r_2}$, i.e., $x\in\mathcal{P}$ and $\|x\| = r_2$, using Lemma 3.1, we get

$$\mathcal{T}x\left(t\right) = \int_0^1 G\left(t,s\right)f\left(s,x\left(s\right)\right)ds \leq \int_0^1 G\left(1,s\right)f\left(s,x\left(s\right)\right)ds \leq \frac{r_2}{\Lambda_1}\int_0^1 G\left(1,s\right)ds = r_2,$$

then, $\|\mathcal{T}x\| \leq \|x\|$. Applying Theorem 2.2 yields that $\mathcal{T}$ has at least one fixed point $x\in\mathcal{P}\cap\left(\overline{\Omega}_{r_2}\backslash\Omega_{r_1}\right)$ with $r_1\leq\|x\|\leq r_2$. It follows from Theorem 3.1 that the MFBVP (1.1a)-(1.1c) has at least one positive solution x.

(ii) From the definition of f^0, there exists $r_1>0$, such that

$$f\left(t,x\right)\leq\left(f^0+\varepsilon\right)x,\quad \text{for all }\ t\in[0,1],\ 0<x\leq r_1,\ \text{where } \varepsilon>0 \text{ satisfies } \Lambda_1\varepsilon\leq\frac{1}{2}.$$

Let $x\in\mathcal{P}\cap\partial\Omega_{r_1}$, i.e., $x\in\mathcal{P}$ and $\|x\|=r_1$. Using Lemma 3.1 we obtain

$$\begin{aligned}\mathcal{T}x\left(t\right) = \int_0^1 G\left(t,s\right)f\left(s,x\left(s\right)\right)ds &\leq \int_0^1 G\left(1,s\right)f\left(s,x\left(s\right)\right)ds\\ &\leq\left(f^0+\varepsilon\right)\int_0^1 G\left(1,s\right)x\left(s\right)ds\\ &\leq\left(f^0+\varepsilon\right)\|x\|\int_0^1 G\left(1,s\right)ds\\ &\leq\Lambda_1\left(f^0+\varepsilon\right)\|x\|\\ &\leq\|x\|.\end{aligned}$$

Consequently, $\|\mathcal{T}x\|\leq\|x\|$.

By the definition of f_∞, there exists $r_3>0$, such that

$$f\left(t,x\right)\geq\left(f_\infty-\varepsilon\right)x,\quad\text{for all }\ t\in[0,1],\ x\geq r_3,\ \text{where } \varepsilon>0 \text{ satisfies } \Lambda_2\varepsilon\leq 1. \tag{3.16}$$

Let $x\in\mathcal{P}\cap\partial\Omega_{r_2}$, i.e., $x\in\mathcal{P}$ and $\|x\|=r_2$ with $r_2=\max\{2r_1,\rho^{-\alpha}r_3\}$. We have

$$x\left(t\right)\geq t^{\alpha}\|x\|\geq\rho^{\alpha}r_2\geq r_3,\quad\text{for } t\in[\rho,1-\rho],$$

and hence, by the inequality (3.16)

$$f\left(t,x\right)\geq\left(f_\infty-\varepsilon\right)x,\quad\text{for } t\in[\rho,1-\rho],\ x\in\mathcal{P}\cap\partial\Omega_{r_2}\text{ and }\Lambda_2\varepsilon\leq 1.$$

Using Lemma 3.1, we have

$$\begin{aligned}\|\mathcal{T}x\| &= \max_{t\in[0,1]}\int_0^1 G\left(t,s\right)f\left(s,x\left(s\right)\right)ds\\ &\geq\int_0^1 G\left(t,s\right)f\left(s,x\left(s\right)\right)ds\end{aligned}$$

$$
\begin{aligned}
&\geq t^{\alpha} \int_{0}^{1} G(1,s) f(s,x(s))\, ds \\
&\geq t^{\alpha} \int_{\rho^2}^{1-\rho^2} G(1,s) f(s,x(s))\, ds, \ 0<\rho<1/2 \\
&\geq t^{\alpha} (f_{\infty}-\varepsilon) \int_{\rho^2}^{1-\rho^2} G(1,s)\, x(s)\, ds \\
&\geq \frac{t^{2\alpha}}{\rho^{2\alpha}} (f_{\infty}-\varepsilon) \|x\| \rho^{2\alpha} \int_{\rho^2}^{1-\rho^2} G(1,s)\, ds \\
&\geq \Lambda_2 (f_{\infty}-\varepsilon) \|x\| \\
&\geq \|x\| .
\end{aligned}
$$

From Theorem 2.2 the operator $\mathcal{T}$ has at least one fixed point $x \in \mathcal{P} \cap \left(\overline{\Omega}_{r_2} \backslash \Omega_{r_1}\right)$ with $r_1 \leq \|x\| \leq r_2$. It follows from Theorem 3.1 that the MFBVP (1.1a)-(1.1c) has at least one positive solution x.

(iii) From the definition of f_0, there exists $r_1>0$, such that

$f(t,x) \geq (f_0-\varepsilon)\, x,$ for all $t \in [0,1],\ 0<x\leq r_1,$ where $\varepsilon>0$ satisfies $\Lambda_2\varepsilon \leq 1$.

Let $x \in \mathcal{P} \cap \partial\Omega_{r_1}$, i.e., $x \in \mathcal{P}$ and $\|x\| = r_1$. Using Lemma 3.1, we obtain

$$
\begin{aligned}
\|\mathcal{T}x\| &\geq \int_{0}^{1} G(t,s) f(s,x(s))\, ds \\
&\geq t^{\alpha} \int_{0}^{1} G(1,s) f(s,x(s))\, ds \\
&\geq t^{\alpha} (f_0-\varepsilon) \int_{0}^{1} G(1,s)\, x(s)\, ds \\
&\geq t^{2\alpha} (f_0-\varepsilon) \|x\| \int_{0}^{1} G(1,s)\, ds \\
&\geq \frac{t^{2\alpha}}{\rho^{2\alpha}} (f_0-\varepsilon) \|x\| \rho^{2\alpha} \int_{\rho^2}^{1-\rho^2} G(1,s)\, ds, \ t \in [\rho, 1-\rho] \\
&\geq \Lambda_2 (f_0-\varepsilon) \|x\| \\
&\geq \|x\| .
\end{aligned}
$$

By the definition of f^{∞}, there exists $r_4>0$, such that

$f(t,x) \leq (f^{\infty}+\varepsilon)\, x,$ for all $t \in [0,1],\ x \geq r_4,$ where $\varepsilon>0$ satisfies $\Lambda_1\varepsilon \leq 1/4$.

it follows that there exists $\delta>0$, such that

$$\delta = \max_{t\in[0,1]} f(t,r_4), \quad \text{for all } t \in [0,1].$$

Then

$$f(t,x) \leq (f^{\infty}+\varepsilon)\, x + \delta, \quad \text{for all } t \in [0,1],\ x \geq r_4.$$

Let $x \in \mathcal{P} \cap \partial\Omega_{r_2}$, i.e., $x \in \mathcal{P}$ and $\|x\| = r_2$ with $r_2 = \max\{2r_1, 2\delta\Lambda_1\}$. Using Lemma 3.1, we get

$$\begin{aligned}
\mathcal{T}x(t) &= \int_0^1 G(t,s) f(s,x(s))\,ds \\
&\leq \int_0^1 G(1,s) f(s,x(s))\,ds \\
&\leq \int_0^1 G(1,s)\left[\left(f^\infty+\varepsilon\right)x(s)+\delta\right] ds \\
&\leq \left(f^\infty+\varepsilon\right)\int_0^1 G(1,s)\,x(s)\,ds + \delta\int_0^1 G(1,s)\,ds \\
&\leq \Lambda_1\left(f^\infty+\varepsilon\right)\|x\| + \delta\Lambda_1 \\
&\leq \frac{\|x\|}{2} + \delta\Lambda_1 \\
&\leq \frac{\|x\|}{2} + \frac{r_2}{2} \\
&\leq \|x\|.
\end{aligned}$$

Consequently, $\|\mathcal{T}x\| \leq \|x\|$. Applying Theorem 2.2 yields that $\mathcal{T}$ has at least one fixed point $x \in \mathcal{P} \cap \left(\overline{\Omega}_{r_2} \backslash \Omega_{r_1}\right)$ and Theorem 3.1 ensure that the MFBVP (1.1a)-(1.1c) has at least one positive solution x.

By the two and three parts of Theorem 3.2, we directly obtain the following corollary.

Corollary 3. *Assume that one of the two following conditions*

- $f^0 = 0$ *and* $f_\infty = +\infty$
- $f_0 = +\infty$ *and* $f^\infty = 0$.

is fulfilled. Then, the MFBVP (1.1a)-(1.1c) *has at least one positive solution.*

Example 1. Consider the following MFBVP

$$\begin{cases} \mathcal{D}_{1^-}^{(\beta)}\left({}^c\mathcal{D}_{0^+}^{\alpha}x\right)(t) = f(t,x(t)),\ 0<t<1, \\ x(0) = \gamma\int_0^1 x(t)\,dt,\ {}^c\mathcal{D}_{0^+}^{\beta}x(1) = 0, \end{cases} \tag{3.17}$$

where $\alpha = \beta = \gamma = 1/2$.

- If $f(t,x) = (1+t)\,x\ln(1+x)$, we have
$$f^0 = \lim_{x\to 0}\max_{t\in[0,1]}\frac{f(t,x)}{x} = 0 \ \text{ and }\ f_\infty = \lim_{x\to+\infty}\min_{t\in[0,1]}\frac{f(t,x)}{x} = +\infty.$$
Thus, by the first part of Corollary 3, we can get that the problem (3.17) has at least one positive solution.
- If $f(t,x) = (2t+1)\,e^{-x}\cos x$, we have
$$f_0 = \lim_{x\to 0}\min_{t\in[0,1]}\frac{f(t,x)}{x} = +\infty \ \text{ and }\ f^\infty = \lim_{x\to+\infty}\max_{t\in[0,1]}\frac{f(t,x)}{x} = 0,$$

then by the second part of Corollary 3, we can get that the problem (3.17) has at least one positive solution.

3.3. Uniqueness of positive solution. In this subsection, we give the third main result.

Theorem 3.3. Assume there exists $L > 0$ such that

$$|f(t,x) - f(t,y)| \leq L|x-y|, \text{ for almost every } t \in [0,1] \text{ and all } x,y \in \mathcal{P}. \tag{3.18}$$

If

$$0 < L\Lambda_1 < 1, \tag{3.19}$$

then, the MFBVP (1.1a)-(1.1c) has exactly one positive solution in $\mathcal{P}$.

Proof. Let $x,y : [0,1] \to \mathbb{R}_+$, $x \neq y$, two positive solutions of the MFBVP (1.1a)-(1.1c). Using (3.18) and Lemma 3.1, we get

$$\begin{aligned} |\mathcal{T}x(t) - \mathcal{T}y(t)| &\leq \int_0^1 G(t,s)\,|(f(s,x(s)) - f(s,y(s)))|\,ds \\ &\leq L\int_0^1 G(1,s)\,|x(s) - y(s)|\,ds \\ &\leq L\Lambda_1 \|x-y\|. \end{aligned}$$

Consequently, $\|\mathcal{T}x - \mathcal{T}y\| \leq L\Lambda_1 \|x-y\|$. By the condition (3.19), the operator $\mathcal{T}$ is a strictly contraction. From Theorem 2.1 and Theorem 3.1, the MFBVP (1.1a)-(1.1c) has exactly one positive solution in $\mathcal{P}$.

3.4. Ulam-Hyers stability. In this subsection, we give the fourth main result which concerning the various types of Ulam-Hyers stability for the MFBVP (1.1a)-(1.1c).

Theorem 3.4. Suppose $f : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and satisfying (3.18) and (3.19). Then, we have

(i) The MFBVP (1.1a)-(1.1c) is Ulam-Hyers stable and consequently generalized Ulam-Hyers stable.

(ii) If $\varphi : [0,1] \to \mathbb{R}_+$ is differentiable and increasing function such that $\varphi(0) \neq 0$, then the MFBVP (1.1a)-(1.1c) is Ulam-Hyers-Rassias stable. Further the MFBVP (1.1a)-(1.1c) is generalized Ulam-Hyers-Rassias stable.

Proof. (i) Let $y \in E$ be any solution of the inequality (2.7), then by Remark 2, we have

$$\mathcal{D}_{1^-}^{(\beta)}\left({}^c\mathcal{D}_{0^+}^{\alpha}y\right)(t) = f(t,y(t)) + \omega(t),\ t \in [0,1].$$

Using Theorem 3.1, we can write

$$y(t) = \int_0^1 G(t,s)\,f(s,y(s))\,ds + \int_0^1 G(t,s)\,\omega(s)\,ds,$$

which gives

$$\left| y(t) - \int_0^1 G(t,s)\,f(s,y(s))\,ds \right| \leq \Lambda_1 \varepsilon. \tag{3.20}$$

Now, let $x \in E$ be a unique solution of the MFBVP (1.1a)-(1.1c), from Lemma 3.1 we have for any $t \in [0,1]$

$$\begin{aligned} |y(t) - x(t)| &= \left| y(t) - \int_0^1 G(t,s) f(s, x(s))\, ds \right| \\ &\leq \left| y(t) - \int_0^1 G(t,s) f(t, y(s))\, ds \right| \\ &+ \left| \int_0^1 G(t,s) \left(f(s, y(s)) - f(s, x(s)) \right) ds \right|. \end{aligned}$$

From (3.20), (3.18) and Lemma 3.1 we have

$$\|y - x\| \leq \varepsilon \Lambda_1 + L\Lambda_1 \|y - x\|,$$

which further implies

$$\|y - x\| \leq \lambda \varepsilon,$$

where $\lambda = \frac{\Lambda_1}{1 - L\Lambda_1} > 0$. Then, the MFBVP (1.1a)-(1.1c) is Ulam-Hyers stable. Moreover, if we set $\theta(\varepsilon) = \lambda\varepsilon$, then the MFBVP (1.1a)-(1.1c) is generalized Ulam-Hyers stable.

(ii) Let $y \in E$ be any solution of the inequality (2.9), then by Remark 2, we have

$$\mathcal{D}_{1^-}^{(\beta)} \left({}^c\mathcal{D}_{0^+}^{\alpha} y \right)(t) = f(t, y(t)) + \Phi(t), \ t \in [0,1].$$

Using Theorem 3.1, we obtain

$$y(t) = \int_0^1 G(t,s) f(s, y(s))\, ds + \int_0^1 G(t,s) \Phi(s)\, ds.$$

From Remark 2, we have

$$\begin{aligned} \left| y(t) - \int_0^1 G(t,s) f(s, y(s))\, ds \right| &\leq \int_0^1 G(t,s) |\Phi(s)|\, ds \\ &\leq \varepsilon \int_0^1 G(t,s) \varphi(s)\, ds \\ &\leq \varepsilon \left[\int_0^t G(t,s) \varphi(s)\, ds + \int_t^1 G(t,s) \varphi(s)\, ds \right]. \end{aligned} \tag{3.21}$$

$\varphi : [0,1] \to \mathbb{R}_+$ is increasing function, then by Lemma 3.1 we obtain

$$\left. \begin{array}{r} s \leq t \Rightarrow \varphi(s) \leq \varphi(t) \\ G(t,s) \leq G(1,s) \end{array} \right\} \Rightarrow \int_0^t G(t,s) \varphi(s)\, ds \leq \varphi(t) \int_0^t G(1,s)\, ds \tag{3.22}$$

and

$$\left. \begin{array}{r} s \leq 1 \Rightarrow \varphi(s) \leq \varphi(1) \\ G(t,s) \leq G(1,s) \end{array} \right\} \Rightarrow \int_t^1 G(t,s) \varphi(s)\, ds \leq \varphi(1) \int_t^1 G(1,s)\, ds. \tag{3.23}$$

From (3.21),(3.22) and (3.23), we obtain

$$\left| y(t) - \int_0^1 G(t,s) f(s, y(s))\, ds \right| \leq \varepsilon \left[\varphi(t) \int_0^t G(1,s)\, ds + \varphi(1) \int_t^1 G(1,s)\, ds \right]. \tag{3.24}$$

Let $\mu : [0,1] \to \mathbb{R}_+$ be a function defined by:

$$\mu(t) = \varphi(t) \int_0^t G(1,s)\, ds + \varphi(1) \int_t^1 G(1,s)\, ds - \frac{\varphi(1)}{\varphi(0)} \Lambda_1 \varphi(t).$$

The function μ is differentiable on $]0,1[$ and for all $t \in]0,1[$, we have

$$\mu'(t) = \varphi'(t) \left[\int_0^t G(1,s)\, ds - \frac{\varphi(1)}{\varphi(0)} \Lambda_1 \right] + (\varphi(t) - \varphi(1))\, G(1,t).$$

φ is differentiable and increasing, then $\mu'(t) \leq 0$. On other hand, we have $\mu(0) = 0$. Then, from (3.24) we obtain

$$\left| y(t) - \int_0^1 G(t,s) f(s, y(s))\, ds \right| \leq \frac{\varphi(1)}{\varphi(0)} \varepsilon \Lambda_1 \varphi(t). \tag{3.25}$$

Let $x \in E$ be a unique solution of the MFBVP (1.1a)-(1.1c), from Lemma 3.1 we have

$$\begin{aligned} |y(t) - x(t)| &= \left| y(t) - \int_0^1 G(t,s) f(s, x(s))\, ds \right| \\ &\leq \left| y(t) - \int_0^1 G(t,s) f(t, y(s))\, ds \right| \\ &+ \left| \int_0^1 G(t,s) \left(f(s, y(s)) - f(s, x(s)) \right) ds \right|. \end{aligned}$$

From (3.25), (3.18) and Lemma 3.1 we have

$$\|y - x\| \leq c\varepsilon\varphi(t),$$

where $c = \frac{\Lambda_1 \varphi(1)}{(1 - L\Lambda_1)\varphi(0)} > 0$. Then, the MFBVP (1.1a)-(1.1c) is Ulam-Hyers-Rassias stable. Consequently, From Remark 1 the MFBVP (1.1a)-(1.1c) is generalized Ulam-Hyers-Rassias stable, which completes the proof.

3.5. Examples. In this subsection, we present two examples to explain the applicability of the stability results.

Example 2. Consider the following MFBVP

$$\begin{cases} \mathcal{D}_{1^-}^{(1/2)} \left({}^c\mathcal{D}_{0^+}^{1/2} x \right)(t) = \frac{\cos(x(t))}{t+3}, \; 0 < t < 1, \\ x(0) = \frac{1}{2} \int_0^1 x(t)\, dt, \; {}^c\mathcal{D}_{0^+}^{1/2} x(1) = 0. \end{cases} \tag{3.26}$$

The function $f(t, x(t)) = \frac{\cos(x(t))}{t+3}$ is continuous for any $t \in [0,1]$ and any $x > 0$. Then, we have

$$|f(t,x) - f(t,y)| \leq \frac{1}{3} |x - y|, \; L = \frac{1}{3}, \; \Lambda_1 = \int_0^1 G(1,s)\, ds = \frac{28 - 3\sqrt{\pi}}{6\sqrt{\pi}} \text{ and } L\Lambda_1 \approx 0.71 < 1.$$

From Theorem 3.3, the MFBVP (3.26) has exactly one positive solution x on $[0,1]$.

Now, let $y \in E$ be a solution of inequality

$$\left| \mathcal{D}_{1^-}^{(1/2)} \left({}^cD_{0^+}^{1/2} y \right) (t) - \frac{\cos (y(t))}{t+3} \right| \leq \varepsilon, \ t \in [0,1],$$

then, by Theorem 3.4 the MFBVP (3.26) is Ulma-Hyers stable with

$$\lambda = \frac{\Lambda_1}{1 - L\Lambda_1} = \frac{84 - 9\sqrt{\pi}}{21\sqrt{\pi} - 28} > 0.$$

On the other hand, consider the inequality

$$\left| \mathcal{D}_{1^-}^{(1/2)} \left({}^cD_{0^+}^{1/2} y \right) (t) - \frac{\cos (y(t))}{t+3} \right| \leq \varepsilon \varphi (t), \ t \in [0,1],$$

where $\varphi(t) = e^t$. By Theorem 3.4 the MFBVP (3.26) is Ulam-Hyers-Rassias stable with

$$c = \frac{\varphi(1)\Lambda_1}{(1 - L\Lambda_1)\varphi(0)} = \frac{\left(84 - 9\sqrt{\pi}\right)e}{21\sqrt{\pi} - 28} > 0.$$

Example 3. Consider the following MFBVP

$$\begin{cases} \mathcal{D}_{1^-}^{(0.8)} \left({}^c\mathcal{D}_{0^+}^{0.2} x \right)(t) = \frac{x(t)}{11\pi e^t + x(t)}, \ 0 < t < 1, \\ x(0) = \frac{98}{100} \displaystyle\int_0^1 x(t)\,dt, \ {}^c\mathcal{D}_{0^+}^{0.2} x(1) = 0. \end{cases} \tag{3.27}$$

The function $f(t, x(t)) = \frac{x(t)}{11\pi e^t + x(t)}$ is continuous for any $t \in [0,1]$ and any $x > 0$. Then, we have

$$|f(t,x) - f(t,y)| \leq \frac{1}{11\pi} |x - y|, \ L = 1/11\pi \text{ and } \Lambda_1 = \int_0^1 G(1,s)\,ds \approx 33.63 \text{ and } L\Lambda_1 \approx 0.97.$$

From Theorem 3.3, the MFBVP (3.27) has exactly one positive solution x on $[0,1]$.

Let $y \in E$ be a solution of the inequality

$$\left| \mathcal{D}_{1^-}^{(0.8)} \left({}^c\mathcal{D}_{0^+}^{0.2} y \right)(t) - \frac{y(t)}{11\pi e^t + y(t)} \right| \leq \varepsilon, \ t \in [0,1].$$

Using Theorem 3.4, the MFBVP (3.27) is Ulma-Hyers stable with

$$\lambda = \frac{\Lambda_1}{1 - L\Lambda_1} \approx 1121.$$

Let $y \in E$ be a solution of the inequality

$$\left| \mathcal{D}_{1^-}^{(0.8)} \left({}^c\mathcal{D}_{0^+}^{0.2} y \right)(t) - \frac{y(t)}{11\pi e^t + y(t)} \right| \leq \varepsilon \varphi(t), \ t \in [0,1],$$

where $\varphi(t) = e^t$. By Theorem 3.4, the MFBVP (3.27) is Ulam-Hyers-Rassias stable with

$$c = \lambda \frac{\varphi(1)}{\varphi(0)} = \lambda e > 0.$$

4. Conclusion

In this paper, we present some results about existence, uniqueness and Ulam-Hyers stability of positive solution of nonlinear mixed fractional differential equations with integral boundary conditions by using Guo-Krasnoselskii's fixed point theorem and Banach's contraction mapping principle. We discuss various types of Ulam-Hyers stability. This study may be provide a new way for the researchers to discuss interesting problems in the mathematical analysis area.

Acknowledgments

The authors would like to thank the referee for his/her careful reading and valuable comments, which greatly improved the paper. This research work is supported by the General Direction of Scientific Research and Technological Development (DGRSDT)-Algeria.

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