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Nonlinear two conformable fractional differential equation with integral boundary condition

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Abstract. This paper deals with a boundary value problem for a nonlinear differential equation with two conformable fractional derivatives and integral boundary conditions. The results of existence, uniqueness and stability of positive solutions are proved by using the Banach contraction principle, Guo-Krasnoselskii's fixed point theorem and Hyers-Ulam type stability. Two concrete examples are given to illustrate the main results.

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1. Introduction

The subject of fractional as a definition has attracted increasing interest researchers since L'Hospital's letter in 1695. Later on, many definitions are made (the most popular ones are the Riemann-Liouville fractional derivative and Caputo's fractional derivative) and increasingly used in a variety of fields witch prove that the subject of fractional derivative is as important as calculus; see ([11, 17, 15, 6]). Moreover, Khalil et al. in ([10]) introduced new fractional derivative, namely "the conformable fractional derivative", since then, the basic concepts of conformable fractional calculus has been greatly development due to the nature of definition witch is satisfy all the requirements of the standard derivative.

Integral boundary conditions of fractional differential equations is recently approached by various researchers by applying different fixed point theorems, also, there are a few papers concerning conformable fractional differential equations with integral boundary conditions, see ([8, 13, 14, 19]), for example; the authors in ([19]) discussed

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the existence of positive solutions for

$$D_\alpha x(t) = f(t, x(t)), \ t \in [0,1], \ \alpha \in (1,2],$$
$$x(0) = 0, \ x(1) = \lambda \int_0^1 x(t)\, dt,$$

where $f \in C([0,1] \times \mathbb{R}_+, \mathbb{R}_+)$. By using the fixed point theorem in a cone.

Another aspect has increasingly attracted the attentions of researchers known as stability analysis. Different kinds of stability have been studied for fractional differential equations including exponential, Mittag-Leffler, Lyapunov stability, the Ulam-Hyers-Rassias stability, etc; for instance, M. Houas et all. in ([9]) studied the existence, uniqueness and stability of solutions to the following fractional boundary value problem with two Caputo fractional derivatives involving nonlocal boundary conditions:

$$D^\alpha \left(D^\beta + \lambda\right) x(t) = f(t, x(t)) + \int_0^t \frac{(t-s)^{\sigma-1}}{\Gamma(\sigma)} f(s, x(s))\, ds, \ t \in [0,T],$$
$$x(0) = x_0 + g(x), \ x(T) = \theta \int_0^\eta \frac{(\eta - s)^{p-1}}{\Gamma(p)} x(s)\, ds, \ \eta \in (0,T),$$

where $D^\alpha, \ D^\beta$ denote the Caputo fractional derivatives, with

$$0 < \alpha, \beta \leq 1, \ 1 < \alpha + \beta \leq 2, \ f : [0,T] \times \mathbb{R} \to \mathbb{R}$$

and $g : C([0,T], \mathbb{R}) \to \mathbb{R}$ are given continuous functions, and $\sigma, p > 0, \lambda, x_0, \theta$ are real constants, $g(x)$ may be regarded as

$$g(x) = \sum_{j=0}^{m} k_j x(t_j),$$

where $k_j, j = 1, \ldots, m$ are given constants and $0 < t_0 < \ldots < t_m \leq 1$. The existence, uniqueness and Ulam's stability for conformable fractional differential equations was studied as well; see ([4, 18, 12]).

On the other hand, Avery et all. in ([3]) investigated the existence of positive solution of the following conformable fractional boundary value problem with Sturm-Liouville boundary conditions

$$-D_\beta D_\alpha u(t) = f(t, u(t)), \ t \in (0,1),$$
$$\gamma u(0) - \delta D_\alpha u(0) = 0 = \eta u(1) + \zeta D_\alpha u(1),$$

where $0 < \alpha, \beta \leq 1, \ \gamma, \delta, \eta, \zeta \geq 0$ and $d = \eta\delta + \gamma\zeta + \gamma\eta/\alpha > 0$. By employing a functional compression expansion fixed point theorem.

In this paper, we concern by study the existence, uniqueness and Ulam stability of positive solutions to the following fractional boundary value problem with two conformable fractional derivatives involving integral boundary condition (for short CFBVP)

$$D_\beta D_\alpha x(t) + \lambda f(t, x(t)) = 0, \ t \in [0,1], \tag{1.1}$$

$$D_\alpha x(0) = 0, \ x(1) = \gamma \int_0^1 x(t)\, dt, \tag{1.2}$$

where $0 < \alpha, \beta \leq 1$, $\lambda > 0$, $\gamma \geq 0$, the derivatives are conformable fractional derivatives and the function $f : [0, 1] \times [0, +\infty) \to [0, +\infty)$ is continuous.

This paper is organized as follows. In Section 2, we give some basic concepts and properties results that will be used to prove our main results. In Section 3, we obtain the existence and uniqueness of the positive solutions for CFBVP (1.1)-(1.2), by the use of Gou-Krasnosel'skii fixed point theorem and Banach contraction mapping principle. Furthermore, we study different types of Ulam stability: Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability, and generalized Ulam-Hyers-Rassias stability for CFBVP considered.

2. Preliminaries

In this section, we recall some useful definitions, lemmas and theorems. It is always assumed that $0 < \alpha, \beta \leq 1$ throughout this paper.

Definition 2.1. ([10]). The conformable fractional derivative of a function $x : [0, \infty) \to \mathbb{R}$ of order α is defined by

$$D_\alpha x(t) = \lim_{\epsilon \to 0} \frac{x\left(t + \epsilon t^{1-\alpha}\right) - x(t)}{\epsilon}, \ \textit{for all } t > 0.$$

If $D_\alpha x(t)$ exists on $(0, b), b > 0$, then $D_\alpha x(0) = \lim_{t \to 0} D_\alpha x(t)$.

Definition 2.2. ([10, 1]). The fractional integral of a function $x : [0, \infty) \to \mathbb{R}$ of order α and of order $\alpha\beta$ are defined respectively by

$$\begin{aligned} I_\alpha x(t) &= \int_0^t s^{\alpha-1} x(s)\, ds, \\ I_\alpha I_\beta x(t) &= \frac{1}{\beta} \int_0^t s^{\alpha-1} \left(t^\beta - s^\beta\right) x(s)\, ds. \end{aligned}$$

Lemma 2.3. ([10, 1]).

(i). *If x is a continuous function on $[0, \infty)$, then $D_\alpha (I_\alpha x(t)) = x(t)$.*

(ii). *If $D_\alpha x(t)$ is continuous function on $[0, \infty)$, then $I_\alpha (D_\alpha x(t)) = x(t) - x(0)$.*

Theorem 2.4. ([10, 1]).

(i). *If x is differentiable on $(0, \infty)$, then $D_\alpha x(t) = t^{1-\alpha} x'(t)$.*

(ii). *If x is twice differentiable on $(0, \infty)$, then*

$$D_\beta D_\alpha x(t) = t^{1-\beta} \left[t^{1-\alpha} x'(t)\right]' = (1 - \alpha) t^{1-\beta-\alpha} x'(t) + t^{2-\beta-\alpha} x''(t).$$

Remark 2.5. Note that $D_\beta D_\alpha \neq D_\alpha D_\beta$.

Further, we present the following fixed point theorems which will be used in studying of our main results.

Theorem 2.6. (Guo-Krasnoselskii fixed point theorem [7]). *Let E be a Banach space, $P \subset E$ be a cone and Ω_1, Ω_2 are two bounded open subsets of E with $\overline{\Omega}_1 \subset \Omega_2$. Assume that $\mathcal{T} : P \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right)$ is a completely continuous operator such that either*

$$\begin{aligned} \|\mathcal{T}x\| &\geq \|x\|, \ x \in P \cap \partial\Omega_1 \text{ and } \|\mathcal{T}x\| \leq \|x\|, \ x \in P \cap \partial\Omega_2 \text{ or,} \\ \|\mathcal{T}x\| &\leq \|x\|, \ x \in P \cap \partial\Omega_1 \text{ and } \|\mathcal{T}x\| \geq \|x\|, \ x \in P \cap \partial\Omega_2. \end{aligned}$$

Then $\mathcal{T}$ has at least one fixed point in $P \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right)$.

Theorem 2.7. *(The Banach contraction principle theorem [5]). Let E be a Banach space, $P \subseteq E$ a nonempty closed subset. If $\mathcal{T} : P \to P$ is a contraction mapping, then $\mathcal{T}$ has a unique fixed point in P.*

To facilitate the use of Theorem 2.6, we provide the following definitions and theorem:

Definition 2.8. ([16]). Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is called a cone if for all $x \in P$ and $\lambda \geq 0$, $\lambda x \in P$ and if $x, -x \in P$ then $x = 0$.

Definition 2.9. ([16]). An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

Theorem 2.10. *(Ascoli-Arzelà [2]). Let E be a compact space. If $\mathcal{T}$ is an equicontinuous, bounded subset of $C(E)$, then $\mathcal{T}$ is relatively compact.*

Next, we present an integral presentation of the solution for the linearized equation related to the equation (1.1)

$$D_\beta D_\alpha x(t) + \lambda g(t) = 0, \tag{2.1}$$

with the boundary conditions (1.2).

Lemma 2.11. *Let $g \in C[0,1]$, then the CFBVP (2.1)-(1.2) has a unique solution x given by*

$$x(t) = \lambda \int_0^1 G(t,s) g(s) ds,$$

where

$$G(t,s) = \frac{1}{\beta} \begin{cases} \left[\frac{\beta+1-\gamma}{(\beta+1)(1-\gamma)} \left(1 - s^\beta\right) - \left(t^\beta - s^\beta\right)\right] s^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\ \frac{\beta+1-\gamma}{(\beta+1)(1-\gamma)} \left(1 - s^\beta\right) s^{\alpha-1}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.2}$$

Proof. By the continuity of g and Lemma 2.3, it follows from (2.1) that

$$x(t) = x(0) + I_\alpha D_\alpha x(0) - \lambda I_\alpha I_\beta g(t), \ t \in [0,1].$$

This, together the boundary conditions, implies

$$x(t) = \gamma \int_0^1 x(t) dt + \lambda I_\alpha I_\beta g(1) - \lambda I_\alpha I_\beta g(t), \ t \in [0,1]. \tag{2.3}$$

Now, we integrate (2.3) from 0 to 1 in both sides and by using the Fubini theorem, we get

$$\begin{aligned}\int_0^1 x(t)dt &= \gamma \int_0^1 x(t)\,dt + \frac{\lambda}{\beta}\int_0^1 s^{\alpha-1}\left(1-s^{\beta}\right)g(s)\,ds \\ &\quad - \frac{\lambda}{\beta(\beta+1)}\int_0^1 s^{\alpha-1}\left(1-s^{\beta}\right)g(s)\,ds,\end{aligned}$$

which implies

$$\int_0^1 x(t)dt = \frac{\lambda}{(\beta+1)(1-\gamma)}\int_0^1 s^{\alpha-1}\left(1-s^{\beta}\right)g(s)\,ds. \tag{2.4}$$

Substituting (2.4) into (2.3), which yields

$$\begin{aligned}x(t) &= \frac{\lambda\gamma}{(\beta+1)(1-\gamma)}\int_0^1 s^{\alpha-1}\left(1-s^{\beta}\right)g(s)\,ds \\ &\quad + \frac{\lambda}{\beta}\int_0^1 s^{\alpha-1}\left(1-s^{\beta}\right)g(s)\,ds - \frac{\lambda}{\beta}\int_0^t s^{\alpha-1}\left(t^{\beta}-s^{\beta}\right)g(s)\,ds.\end{aligned}$$

□

The Green function G in (2.2) has several important properties given as follows:

Lemma 2.12. *For any* (t,s) *in* $[0,1]\times[0,1]$ *and* $\gamma\in[0,1)$*:*

(G1). $0 \le G(t,s)$ *and continuous,*

(G2). $G(1,s) \le G(t,s) \le G(0,s)$*,*

(G3). $G(0,s) = G(s,s) = \frac{\beta+1-\gamma}{\gamma\beta}G(1,s)$*.*

Proof. Obviously that G is positive, continuous and $\frac{\partial G(t,s)}{\partial t} \le 0$, for $0 \le t,s \le 1$, then $G(t,s)$ is decreasing with respect to $t\in[0,1]$, and therefore

$$G(1,s) \le G(t,s) \le G(0,s)\,,\ \ for\ 0 \le t,s \le 1.$$

A simple calculation shows that

$$\begin{aligned}G(0,s) &= \frac{\beta+1-\gamma}{\beta(\beta+1)(1-\gamma)}\left(1-s^{\beta}\right)s^{\alpha-1} = G(s,s)\,, \\ G(1,s) &= \frac{\gamma}{(\beta+1)(1-\gamma)}\left(1-s^{\beta}\right)s^{\alpha-1} = \frac{\gamma\beta}{\beta+1-\gamma}G(0,s)\,.\end{aligned}$$

□

3. Main results

For investigating the existence, uniqueness and stability of positive solutions for the CFBVP (1.1)-(1.2), we define the Banach space $E = C[0,1]$ with the norm $\|x\| = \max_{t\in[0,1]}|x(t)|$ and the bounded subset Ω_r of E, with $\Omega_r = \{x\in E,\ \|x\| \le r,\ r>0\}$. As well, define the cone P in E by

$$P = \left\{x\in E,\ x(t) \ge \frac{\gamma\beta}{\beta+1-\gamma}\|x\|\,,\ t\in[0,1]\,,\ \gamma\in[0,1)\right\}$$

Furthermore, define

$$\Lambda_1 = \int_0^1 G(0,s)\,ds,\ \ \Lambda_2 = \frac{\gamma\beta}{\beta+1-\gamma}\int_0^1 G(0,s)\,ds.$$

Also, define the operators $\mathcal{T}: E \to E$ as

$$\mathcal{T}x(t) = \lambda\int_0^1 G(t,s)\,f(s,x(s))\,ds,$$

under the properties of G in Lemma 2.12 and our assumptions on f, the operator is well-defined, continuous, positive and has the following properties.

Lemma 3.1. **(i).** *$\mathcal{T}(P) \subset P$.*
(ii). *The operator $\mathcal{T}: P \to P$ is completely continuous.*

Proof. (i) From Lemma 2.12 and the definition of the cone P, we have

$$\begin{aligned}
\mathcal{T}x(t) &= \lambda\int_0^1 G(t,s)\,f(s,x(s))\,ds\\
&\geq \frac{\lambda\gamma\beta}{\beta+1-\gamma}\int_0^1 G(0,s)\,f(s,x(s))\,ds\\
&\geq \frac{\lambda\gamma\beta}{\beta+1-\gamma}\max_{t\in[0,1]}\int_0^1 G(t,s)\,f(s,x(s))ds\\
&\geq \frac{\gamma\beta}{\beta+1-\gamma}\left\|\mathcal{T}x\right\|,\ \ \textit{for all}\ t\in[0,1].
\end{aligned}$$

Hence $\mathcal{T}x \in P$.

(ii) Let $x \in \Omega_r$, then there exists a positive constant L_0 such that

$$\sup_{\|x\|\leq r}\max_{t\in[0,1]} f(t,x) \leq L_0,$$

then, it holds that

$$\left\|\mathcal{T}x(t)\right\| = \max_{t\in[0,1]}\lambda\int_0^1 G(t,s)\,f(s,x(s))\,ds \leq \lambda L_0\int_0^1 G(0,s)\,ds,$$

which implies that $\mathcal{T}(\Omega_r)$ is bounded. Hence, for all $t_1, t_2 \in [0,1]$, $t_1 < t_2$ and by Lemma 2.12, we have

$$\begin{aligned}
\left\|\mathcal{T}x(t_2) - \mathcal{T}x(t_1)\right\| &\leq \max_{t\in[0,1]}\int_{t_1}^{t_2} G(t,s)\,f(s,x(s))\,ds\\
&\leq L_0\int_{t_1}^{t_2} G(0,s)\,ds\\
&= \frac{L_0\lambda(\beta+1-\gamma)}{\beta(\beta+1)(1-\gamma)}\int_{t_1}^{t_2}\left(1-s^{\beta}\right)s^{\alpha-1}ds\\
&\leq \frac{L_0\lambda(\beta+1-\gamma)}{\alpha\beta(\beta+1)(1-\gamma)}\left(t_2^{\alpha}-t_1^{\alpha}\right),
\end{aligned}$$

$\|\mathcal{T}x(t_2) - \mathcal{T}x(t_1)\| \to 0$ as $t_1 \to t_2$ which implies that the set $\mathcal{T}(\Omega_r)$ is equicontinuous. By the Arzelà-Ascoli theorem $\mathcal{T} : \Omega_r \to \Omega_r$ is compact. We thus complete the proof. □

Lemma 3.2. *The CFBVP* (1.1)-(1.2) *has a positive solution* $x \in E$ *if and only if it is a fixed point of* $\mathcal{T}$ *in* P.

Proof. Let x be a fixed point of $\mathcal{T}$ in P, then

$$\begin{aligned} x(t) &= \lambda \int_0^1 G(t,s) f(s, x(s))\, ds, \ t \in [0,1], \\ &= \gamma \int_0^1 x(t)\, dt + \lambda I_\alpha I_\beta f(t, x(t)), \end{aligned} \tag{3.1}$$

and thus, by the continuity of f and Lemma 2.3, we obtain

$$D_\beta D_\alpha x(t) = \lambda f(t, x(t)).$$

Furthermore, the equality (3.1) directly implies

$$x(1) = \gamma \int_0^1 x(t)\, dt \text{ and } D_\alpha x(0) = 0.$$

Therefore, x is a positive solution of the CFBVP (1.1)-(1.2).
Moreover, the Lemmas 2.11 and 3.1 imply that x is a fixed point of $\mathcal{T}$ in P. □

3.1. The existence of positive solutions of the CFBVP

Before presenting our results, we present some important notations as follows:

$$f^0 = \lim_{x\to 0} \max_{t\in[0,1]} \frac{f(t,x)}{x}, \ f^\infty = \lim_{x\to\infty} \max_{t\in[0,1]} \frac{f(t,x)}{x},$$
$$f_0 = \lim_{x\to 0} \min_{t\in[0,1]} \frac{f(t,x)}{x}, \ f_\infty = \lim_{x\to\infty} \min_{t\in[0,1]} \frac{f(t,x)}{x}.$$

Theorem 3.3. *Assume there exists* $r_2 > r_1 > 0$, *such that*

$$f(t,x) \le \frac{r_2}{\lambda\Lambda_1}, \ x \in [0, r_2], \ t \in [0,1],$$
$$f(t,x) \ge \frac{r_1}{\lambda\Lambda_2}, \ x \in [0, r_1], \ t \in [0,1],$$

then the CFBVP (1.1)-(1.2) *has at least one positive solution.*

Proof. By Lemma 2.12, for $x \in P \cap \partial\Omega_{r_1}$, we have

$$\|\mathcal{T}x\| \ge \mathcal{T}x(t) \ge \frac{\gamma\beta}{\beta+1-\gamma} \int_0^1 G(0,s) \frac{r_1}{\Lambda_2} ds = r_1.$$

For $x \in P \cap \partial\Omega_{r_2}$, we get

$$\|\mathcal{T}x\| = \int_0^1 G(0,s) f(s, x(s))\, ds \le \int_0^1 G(0,s) \frac{r_2}{\Lambda_1} ds = r_2.$$

Applying Theorem 2.6 yields that $\mathcal{T}$ has at least one fixed point $x \in P \cap \left(\overline{\Omega}_{r_2} \backslash \Omega_{r_1}\right)$ with $r_1 \le \|x\| \le r_2$. It follows from Lemma 3.2 that the CFBVP (1.1)-(1.2) has at least one positive solution x. The proof is complete. □

Theorem 3.4. *Let* $f_\infty \frac{\gamma\beta}{\beta+1-\gamma} \geq 1$ *and* $f^0 \leq \frac{\gamma\beta}{\beta+1-\gamma}$ *are satisfied, then for each* $\lambda \in \left(\frac{1}{\Lambda_1}, \frac{1}{\Lambda_2}\right)$ *the CFBVP* (1.1)-(1.2) *has at least one positive solution.*

Proof. From the definition of f^0, there exists $r_1 > 0$, such that

$$f(t,x) \leq f^0 x, \ \textit{for all } t \in [0,1], \ 0 < x \leq r_1.$$

For $x \in P \cap \partial\Omega_{r_1}$, we have

$$\begin{aligned}\|\mathcal{T}x\| &= \lambda \int_0^1 G(0,s) f(s,x(s)) \, ds \\ &\leq \lambda \int_0^1 G(0,s) f^0 x(s) \, ds \\ &\leq \lambda f^0 \|x\| \Lambda_1 \\ &\leq \|x\|.\end{aligned}$$

Consequently

$$\|\mathcal{T}x\| \leq \|x\|, \ x \in P \cap \partial\Omega_{r_1}. \tag{3.2}$$

By the definition of f_∞, there exists $r_3 > 0$, such that

$$f(t,x) \geq f_\infty x, \ \textit{for all } t \in [0,1], \ x \geq r_3.$$

If $x \in P \cap \partial\Omega_{r_2}$ with $r_2 = \max\{2r_1, r_3\}$, then by the definition of cone P, we have

$$\begin{aligned}\|\mathcal{T}x\| &= \lambda \int_0^1 G(0,s) f(s,x(s)) \, ds \\ &\geq \lambda f_\infty \int_0^1 G(0,s) x(s) \, ds \\ &\geq \lambda \frac{\gamma\beta}{\beta+1-\gamma} f_\infty \|x\| \int_0^1 G(0,s) \, ds \\ &\geq \|x\|.\end{aligned}$$

Hence

$$\|\mathcal{T}x\| \geq \|x\|, \ x \in P \cap \partial\Omega_{r_2}. \tag{3.3}$$

From (3.2)-(3.3) and Theorem 2.6 we assurance that the operator $\mathcal{T}$ has at least one fixed point $x \in P \cap \left(\overline{\Omega}_{r_2} \backslash \Omega_{r_1}\right)$ with $r_1 \leq \|x\| \leq r_2$. It follows from Lemma 3.2 that the CFBVP (1.1)-(1.2) has at least one positive solution x. □

Theorem 3.5. *If* $\frac{\gamma\beta}{\beta+1-\gamma} f_0 \geq 1$ *and* $f^\infty \leq \frac{\gamma\beta}{2(\beta+1-\gamma)}$ *are satisfied, then for each* $\lambda \in \left(\frac{1}{\Lambda_1}, \frac{1}{\Lambda_2}\right)$ *the CFBVP* (1.1)-(1.2) *has at least one positive solution.*

Proof. From the definition of f_0, there exists $r_1 > 0$, such that

$$f(t,x) \geq f_0 x, \ \textit{for all } t \in [0,1], \ 0 < x \leq r_1.$$

Further, for $x \in P$ with $\|x\| = r_1$, then as previously

$$\begin{aligned} \|\mathcal{T}x\| &\geq \lambda \int_0^1 G(0,s) f_0 x(s)\, ds \\ &\geq \lambda \frac{\gamma\beta}{\beta+1-\gamma} f_0 \|x\| \int_0^1 G(0,s)\, ds \\ &\geq \|x\|. \end{aligned}$$

Hence

$$\|\mathcal{T}x\| \geq \|x\|,\ x \in P \cap \partial\Omega_{r_1}.$$

By the definition of f^{∞}, there exists $L > 0$, such that

$$f(t,x) \leq f^{\infty} x,\ \ for\ all\ t \in [0,1],\ x \geq r_4,$$

it follows that there exists $\delta > 0$, such that

$$\delta = \max_{t\in[0,1]} f(t, r_4),\ \ for\ all\ t \in [0,1],\ 0 < x \leq r_4.$$

Then

$$f(t,x) \leq f^{\infty} x + \delta,\ \ for\ all\ t \in [0,1],\ x \geq 0.$$

If $x \in P \cap \partial\Omega_{r_2}$, with $r_2 = \max\left\{2r_1,\ \frac{2\gamma\beta\delta}{\beta+1-\gamma}\right\}$, we get

$$\begin{aligned} \|\mathcal{T}x\| &= \lambda \int_0^1 G(0,s) f(s, x(s))\, ds \\ &\leq \lambda \int_0^1 G(0,s) \left(f^{\infty} x(s) + \delta\right) ds \\ &\leq \lambda \left(f^{\infty} \|x\| + \delta\right) \Lambda_1 \\ &\leq \|x\|. \end{aligned}$$

Thus

$$\|\mathcal{T}x\| \leq \|x\|,\ x \in P \cap \partial\Omega_{r_2}.$$

Applying Theorem 2.6 yields that $\mathcal{T}$ has at least one fixed point $x \in P \cap \left(\overline{\Omega}_{r_2} \backslash \Omega_{r_1}\right)$ and Lemma 3.2 ensure that the CFBVP (1.1)-(1.2) has at least one positive solution x. □

Example 3.6. Consider the CFBVP (1.1)-(1.2) with $\beta = 1,\ \alpha = \frac{1}{2},\ \gamma = \frac{3}{4}$ and

$$f(t,x) = \begin{cases} (t+1)\, x^2, & (t,x) \in [0,1] \times (0,2], \\ 2(t+1)\, x, & (t,x) \in [0,1] \times (2,\infty), \end{cases}$$

$$F(t,x) = (2t+1)\left(\sin x + e^{-x}\right),$$

the functions f, F are continuous for any $t \in [0,1]$ and any $x > 0$, we have

$$\begin{aligned} f^0 &= \lim_{x\to 0} \max_{t\in[0,1]} \frac{f(t,x)}{x} = 0,\ f_{\infty} = \lim_{x\to\infty} \min_{t\in[0,1]} \frac{f(t,x)}{x} = 2, \\ F_0 &= \lim_{x\to 0} \min_{t\in[0,1]} \frac{f(t,x)}{x} = \infty,\ F^{\infty} = \lim_{x\to\infty} \max_{t\in[0,1]} \frac{F(t,x)}{x} = 0. \end{aligned}$$

By simple calculations we obtain $\frac{\gamma\beta}{\beta+1-\gamma} = \frac{3}{5}$. On the other hand, we get

$$\begin{aligned}\Lambda_1 &= \int_0^1 G(0,s)\,ds = \frac{\beta+1-\gamma}{\beta(\beta+1)(1-\gamma)} \int_0^1 \left(1-s^\beta\right) s^\alpha ds = \frac{2}{3},\\ \Lambda_2 &= \frac{\gamma\beta}{\beta+1-\gamma} \int_0^1 G(0,s)\,ds = \frac{2}{5}.\end{aligned}$$

For $\lambda \in \left(\frac{3}{2}, \frac{5}{2}\right)$, for specified function f the Theorem 3.4 (or for function F the Theorem 3.5) gives that the CFBVP (1.1)-(1.2) has at least one positive solution x defined on $[0,1]$.

3.2. The uniqueness and Ulam-Hyers stability of positive solution of the CFBVP

In this subsection, we present four types of Ulam stability definition, namely Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias, and generalized Ulam-Hyers-Rassias:

Definition 3.7. The CFBVP (1.1)-(1.2) is Ulam-Hyers stable if there exists $c_f \in \mathbb{R}_+$ such that for each $\varepsilon > 0$ and for every solution $y \in C^2([0,1],[0,\infty))$ of the inequality

$$|D_\beta D_\alpha y(t) + \lambda f(t, y(t))| \leq \varepsilon,\ t \in [0,1], \tag{3.4}$$

there exists a unique solution $x \in C^2([0,1],[0,\infty))$ of the CFBVP (1.1)-(1.2) with

$$\|y - x\| \leq c_f \varepsilon,\ t \in [0,1].$$

Definition 3.8. The CFBVP (1.1)-(1.2) is generalized Ulam-Hyers stable if there exists $\theta_f \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\theta_f(0) = 0$ such that for each $\varepsilon > 0$ and for every solution $y \in C^2([0,1],[0,\infty))$ of the inequality (3.4), there exists a unique solution $x \in C^2([0,1],[0,\infty))$ of the CFBVP (1.1)-(1.2) with

$$\|y - x\| \leq \theta_f(\varepsilon),\ t \in [0,1].$$

Definition 3.9. The CFBVP (1.1)-(1.2) is Ulam-Hyers-Rassias stable with respect to $\varphi \in C([0,1], \mathbb{R}_+)$ if there exists $c_f \in \mathbb{R}_+$ such that for each $\varepsilon > 0$ and for every solution $y \in C^2([0,1],[0,\infty))$ of the inequality

$$|D_\beta D_\alpha y(t) + \lambda f(t, y(t))| \leq \varepsilon\varphi(t),\ t \in [0,1], \tag{3.5}$$

there exists a unique solution $x \in C^2([0,1],[0,\infty))$ of the equations (1.1)-(1.2) with

$$\|y - x\| \leq c_f \varepsilon\varphi(t),\ t \in [0,1].$$

Definition 3.10. The CFBVP (1.1)-(1.2) is generalized Ulam-Hyers-Rassias stable with respect to $\varphi \in C([0,1], \mathbb{R}_+)$, if there exists $c_{f,\varphi} \in \mathbb{R}_+$, such that for every solution $y \in C^2([0,1],[0,\infty))$ of the inequality

$$|D_\beta D_\alpha y(t) + \lambda f(t, y(t))| \leq \varphi(t),\ t \in [0,1], \tag{3.6}$$

there exists a unique solution $x \in C^2([0,1],[0,\infty))$ of the equations (1.1)-(1.2) with

$$\|y - x\| \leq c_{f,\varphi}\varphi(t),\ t \in [0,1].$$

Remark 3.11. Clearly,

(i). Definition 3.7 $\Rightarrow$ Definition 3.8.

(ii). Definition 3.9 $\Rightarrow$ Definition 3.10.

Theorem 3.12. *Assume there exists $L > 0$ such that*

$$|f(t,x) - f(t,y)| \leq L|x-y|, \ \textit{for almost every } t \in [0,1], \ \textit{and all } x, y \in E.$$

Then, if

$$\Delta = \lambda L \Lambda_1 < 1, \tag{3.7}$$

the CFBVP (1.1)-(1.2) *has exactly one positive solution defined on* $[0,1]$.

Proof. Using Lemma 2.3, we have

$$\begin{aligned} \|\mathcal{T}x(t) - \mathcal{T}y(t)\| &\leq \lambda \int_0^1 G(0,s) |(f(s,x(s)) - f(s,y(s)))| \, ds \\ &\leq \lambda L \|x-y\| \int_0^1 G(0,s) \, ds \\ &= \Delta \|x-y\|. \end{aligned}$$

Then, Theorem 2.7 and Lemma 3.2 ensure that there is a unique and positive x in E with $x = \mathcal{T}x$. ☐

Theorem 3.13. *Let* (3.7) *holds, then the CFBVP* (1.1)-(1.2) *is Ulam-Hyers stable and consequently generalized Ulam-Hyers stable.*

Proof. Let $y \in C^2([0,1],[0,\infty))$ be any solution of the inequality (3.4), Thank to Lemma 2.11, we obtain

$$y(t) = \lambda \int_0^1 G(t,s) f(s,y(s)) \, ds,$$

which yields

$$\begin{aligned} \left| y(t) - \lambda \int_0^1 G(t,s) f(s,y(s)) \, ds \right| &\leq \frac{\varepsilon}{\beta} \int_0^t \left(t - s^\beta\right) s^{\alpha-1} ds \\ &\leq \frac{\varepsilon}{\beta} \int_0^1 \left(1 - s^\beta\right) s^{\alpha-1} ds \\ &\leq \varepsilon \Lambda_1. \end{aligned}$$

Let $x \in C^2([0,1],[0,\infty))$ be the unique solution of the CFBVP (1.1)-(1.2), we have for any $t \in [0,1]$

$$\begin{aligned} |y(t) - x(t)| &= \left| y(t) - \lambda \int_0^1 G(t,s) f(s,x(s)) \, ds \right| \\ &= \left| y(t) - \lambda \int_0^1 G(t,s) f(s,y(s)) \, ds + \lambda \int_0^1 G(t,s) f(s,y(s)) \, ds \right. \\ &\quad \left. -\lambda \int_0^1 G(t,s) f(s,x(s)) \, ds \right| \end{aligned}$$

$$\begin{aligned} &\leq \left| y(t) - \lambda \int_0^1 G(t,s) f(t, y(s))\, ds \right| \\ &\quad + \lambda \left| \int_0^1 G(t,s) \left(f(s, y(s)) - f(s, x(s)) \right) ds \right| \\ &\leq \varepsilon \Lambda_1 + \lambda L \int_0^1 G(0,s) \left| (y(s) - x(s)) \right| ds, \end{aligned}$$

which implies

$$\| y - x \| \leq \varepsilon \Lambda_1 + \lambda L \Lambda_1 \| y - x \|,$$

on simplification it gives

$$\| y - x \| \leq \varepsilon c_f, \ \textit{where } c_f = \frac{\Lambda_1}{1 - \lambda L \Lambda_1},$$

which completes the proof. By putting $\theta_f(\varepsilon) = \varepsilon c_f, \theta_f(0) = 0$, then the CFBVP (1.1)-(1.2) is generalized Ulam-Hyers stable. □

Theorem 3.14. *Let* (3.7) *holds. Assume that, there exists an increasing function* $\varphi \in C([0,1], \mathbb{R}_+) \in E$ *and there exists* $\sigma_\varphi \in \mathbb{R}_+$ *such that for any* $t \in [0,1]$

$$I_\alpha I_\beta \varphi(t) \leq \sigma_\varphi \varphi(t),$$

is satisfied, then the solutions of the CFBVP (1.1)-(1.2) *are Ulam-Hyers-Rassias stable. Further the solutions of the considered CFBVP* (1.1)-(1.2) *are generalized Ulam-Hyers-Rassias stable.*

Proof. Similar to the proof of Theorem 3.13, let $y \in C^2([0,1], [0,\infty))$ be any solution of the inequality (3.5), Thank to Lemma 2.11, we obtain

$$\begin{aligned} \left| y(t) - \lambda \int_0^1 G(t,s) f(s, y(s))\, ds \right| &\leq \varepsilon I_\alpha I_\beta \varphi(t) \\ &\leq \varepsilon \sigma_\varphi \varphi(t). \end{aligned}$$

Let $x \in C^2([0,1], [0,\infty))$ be the unique solution of the CFBVP (1.1)-(1.2), we have for any $t \in [0,1]$

$$\begin{aligned} |y(t) - x(t)| &\leq \left| y(t) - \lambda \int_0^1 G(t,s) f(t, y(s))\, ds \right| \\ &\quad + \lambda \left| \int_0^1 G(t,s) \left(f(s, y(s)) - f(s, x(s)) \right) ds \right| \\ &\leq \varepsilon \sigma_\varphi \varphi(t) + \lambda L \int_0^1 G(0,s) \left| (y(s) - x(s)) \right| ds, \end{aligned}$$

which implies that

$$\| y - x \| \leq c_f \varepsilon \sigma_\varphi \varphi(t), \ \textit{where } c_f = \frac{1}{1 - \lambda L \Lambda_1},$$

which completes the proof of the theorem. Moreover, if we set $\varphi(\varepsilon) = \varepsilon \varphi(t)$, then $\varphi(0) = 0$. Analogously one can easily prove that the solutions of CFBVP (1.1)-(1.2) are generalized Ulam-Hyers-Rassias stable. □

Example 3.15. Consider the CFBVP (1.1)-(1.2) with $\beta = \frac{1}{2}, \alpha = 1, \gamma = \frac{3}{4}$ and

$$f(t,x) = \frac{1}{t+2}\sin x,$$

the function f is continuous for any $t \in [0,1]$ and any $x > 0$, by simple calculations we obtain

$$|f(t,x) - f(t,y)| \leq \frac{1}{2}|x-y| \ \ and \ \Lambda_1 = \frac{2}{5}.$$

For $\lambda \in (0,5)$, Theorem 3.12 give that the CFBVP (1.1)-(1.2) has exactly one positive solution x defined on $[0,1]$. Now, let

$$\left| D_{\frac{1}{2}} y'(t) + \frac{3}{5(t+2)}\sin x \right| \leq \varepsilon, \ \ t \in [0,1],$$

then, by Theorem 3.13 the CFBVP (1.1)-(1.2) is Ulma-Hyers stable with $c_f = \frac{5}{11}$. On the other hand, Consider the inequality

$$\left| D_{\frac{1}{2}} y'(t) + \frac{3}{5(t+2)}\sin x \right| \leq \varepsilon t, \ \ t \in [0,1],$$

by Theorem 3.14 the CFBVP (1.1)-(1.2) is Ulam-Hyers-Rassias stable with

$$c_f = \frac{1}{1-\lambda L \Lambda_1} = \frac{25}{22}, \ \ \sigma_t = \frac{1}{(\alpha+1)\beta} = 1.$$

4. Conclusion

By using the Banach contraction principle, Guo-Krasnoselskii's fixed point theorem and Hyers-Ulam type stability, we discuss problem (1.1)-(1.2), a two conformable fractional differential equation with integral boundary conditions. We present our results of the existence, uniqueness of positive solution and Hyers-Ulam type stability. Two concrete examples are given to better demonstrate our main results.

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