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EXISTENCE, UNIQUENESS AND MULTIPLICITY OF SOLUTIONS FOR A CLASS OF FRACTIONAL p-KIRCHHOFF PROBLEMS

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Abstract. This study investigates the existence and uniqueness of solutions for a class of Kirchhoff problems involving the fractional p-Laplacian operator and a singular nonlinearities. We also study the existence and infinitely many solutions but without the singular term and with a presence of a nonlinear term satisfies a growth assumption. Our proofs are based on variational methods and genus theory.

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1. Introduction

The first part of this paper is devoted to study the existence and uniqueness of positive solution for the following fractional p-Kirchhoff problem involving a singular nonlinearity:

$$(P_1)\begin{cases} \left(a+b\int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dxdy\right)(-\Delta)_p^s u = \frac{k(x)}{u^\gamma} - \lambda u^q, & \text{in } \Omega; \\ u>0, & \text{in } \Omega; \\ u=0, & \text{on } \mathbb{R}^N\setminus\Omega, \end{cases}$$

where $s\in(0,1)$, $\lambda\geq 0$, $0<\gamma<1$ and $\Omega\subset\mathbb{R}^N$ is a smooth bounded domain with $N>sp$, $p>1$ and $0\leq q\leq p_s^*-1$, such that $p_s^*=\frac{Np}{N-sp}$ is the fractional critical Sobolev exponent, $a,b\geq 0$, $a+b>0$ are parameters and $k:\Omega\to\mathbb{R}$ is a positive function.

The new nonlocal and nonlinear operator $(-\Delta)_p^s$ is defined, for $p>1$ and $s\in(0,1)$ and u smooth enough, by

$$(-\Delta)_p^s u(x) = 2\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}dy, \quad x\in\mathbb{R}^N,$$

where $B_\varepsilon(x)=\{y\in\mathbb{R}^N : |y-x|\leq\varepsilon\}$, this operator is known as the fractional p-Laplacian. For the basic properties of this operator, see [9, 25].

Since the emergence of the original model of the Kirchhoff problems [14], researchers have been working on this type of problems, especially after the great development that took place in the tools of functional analysis, it is evident through the intensity of research and the generalization of the usual differential operators to different operators such as the fractional Laplacian $(-\Delta)^s u$, the fractional p-Laplacian $(-\Delta)_p^s$, the $p(x)$-Laplacian... etc. In the last few years, many researchers were interested in problems involving fractional operators with a singular nonlinear term, in which one encounter many difficulties resulting in nonclassical operator and the loss of regularity of some quantities, which lead to adopting unusual ways to solve them. Among the pioneering works in this field, we refer the reader to [10, 12, 13, 20]. In [20], the authors discussed a class of fractional p-Laplacian problems with weights which are possibly singular on the boundary of the domain and they provided the existence and multiplicity results as well as characterizations of critical groups and related applications. In [4], Canino et al discussed the following nonlocal quasilinear singular problem

$$\begin{cases} (-\Delta)_p^s u = \frac{f(x)}{u^\gamma}, & \text{in } \Omega; \\ u > 0, & \text{in } \Omega; \\ u = 0, & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases}$$

with a slightly different weak notion of solution is considered, the authors studied the existence of solution by considering first the simplest case $0 < \gamma \leq 1$ and then the case $\gamma > 1$. The proof was by proving a solution to a regularized problem by using Shauder fixed point theorem and then they performed apriori uniform estimates to pass to the limit. By giving a distinct definition of the Dirichlet boundary condition, they proved the uniqueness result for all $\gamma > 0$ by using a more general comparison principle. In [10], the authors treat a class of Kirchhoff type problems driven by a nonlocal fractional operator and involving a singular term and a critical nonlinearity. In particular, they covered the delicate degenerate case, that is, when the Kirchhoff function is zero at zero, by combining variational methods with an appropriate truncation argument, they provided the existence of two solutions. Problem (P_1) has been also studied with different elliptic operators. We refer the reader to [19, 24]. In [19], the authors established, by using the minimax method and some analysis techniques, the uniqueness of positive solutions for a class of Kirchhoff type problems with singularity.

In the second part, we keep almost the same problem but with absence of the singular term; we consider

$$(P_2) \begin{cases} \left(a + b \int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\, dxdy\right)(-\Delta)_p^s u = g(x,u) - \lambda |u|^q, & \text{in } \Omega; \\ u > 0, & \text{in } \Omega; \\ u = 0, & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $s \in (0,1)$, $\lambda \geq 0$, $0 < \gamma < 1$ and $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain with $N > sp$, $p > 1$ and $0 \leq q < p_s^* - 1$, let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfies certain conditions which we will present later. The first part of our work is a generalization of the results of [19] to the fractional p-Laplacian case and the second part is a generalization of the results of the existence and multiplicity of the solutions presented in [8]. We believe that the lack of works that dealt with this matter motivated us to shed more light on it.

2. Preliminaries

In this paragraph, we will provide a suitable functional space to study the problem (P_1) and (P_2) with some preliminary results. In the following we will assume that $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain with $N \geq 2$. We consider, for any $p > 1$ and $s \in (0,1)$, the space

$$W^{s,p}(\mathbb{R}^N) = \{u \in L^p(\mathbb{R}^N), \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \in L^1(\mathbb{R}^{2N})\}$$

endowed with the Gagliardo norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} = \Big(\int_{\mathbb{R}^N} |u(x)|^p \, dx + \int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \, dxdy\Big)^{\frac{1}{p}}.$$

Let us introduce the next space

$$W_0^{s,p}(\Omega) = \{u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ in } \mathbb{R}^N \backslash \Omega\},$$

endowed with the standard Gagliardo norm

$$\|u\| = \Big(\int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \, dxdy\Big)^{\frac{1}{p}}.$$

Thanks to the Poincaré inequality holds in $W_0^{s,p}(\Omega)$, we observe that this norm is equivalent to the full norm $\|\cdot\|_{W^{s,p}(\mathbb{R}^N)}$. Let

$$p_s^* = \begin{cases} \frac{Np}{N-sp}, & \text{if } N > sp; \\ \infty, & \text{if } N \leq sp. \end{cases}$$

The space $W_0^{s,p}(\Omega)$ is a reflexive Banach space, continuously embedded in $L^\alpha(\Omega)$ for any $\alpha \in [1, p_s^*]$ if $N > sp$, for any $1 \leq \alpha < \infty$ if $N = sp$ and in $L^\infty(\Omega)$ for any $N < sp$. It is also compactly embedded in $L^\alpha(\Omega)$ for any $\alpha \in [1, p_s^*)$ if $N \geq sp$ and into $L^\infty(\Omega)$ for $N < sp$. Furthermore, $C_0^\infty(\Omega)$ is a dense subspace of $W_0^{s,p}(\Omega)$ with respect to the norm $\|\cdot\|$. In particular, restrictions to Ω of functions in $W_0^{s,p}(\Omega)$ belong to the closure of $C_0^\infty(\Omega)$ in $W^{s,p}(\Omega)$, i.e. with respect to the localized norm

$$\Big(\int_{\Omega} |u(x)|^p \, dx + \int\int_{\Omega^2} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \, dxdy\Big)^{\frac{1}{p}}.$$

This closure is often denoted with the same symbol $W_0^{s,p}(\Omega)$. For more details about the theory of these spaces see [2, 9, 25]. The symbol $\|\cdot\|_p$ stands for the standard norm for the $L^p(\Omega)$ space.

We now recall the Krasnoselskii genus, more information on this subject may be found in [1, 5, 15]. Let E be a real Banach space. Let us denote by Σ the class of all closed subsets $A \subset E - \{0\}$ that are symmetric with respect to the origin, that is, $u \in A$ implies $-u \in A$.

Definition 2.1. *Let $A \in \Sigma$. The Krasnoselskii genus $\gamma(A)$ is defined as being the least positive integer n such that there is an odd mapping $\varphi \in C(A, \mathbb{R}^n - \{0\})$. If such an n does not exist we set $\gamma(A) = +\infty$. Furthermore, by definition, $\gamma(\emptyset) = 0$.*

Theorem 2.2. *Let $E = \mathbb{R}^N$ and $\partial\Omega$ be the boundary of an open, symmetric and bounded subset $\Omega \subset \mathbb{R}^N$ with $0 \in \Omega$. Then $\gamma(\partial\Omega) = N$.*

Note $\gamma(S^{N-1}) = N$. If E is of infinite dimension and separable and S is the unit sphere in E, then $\gamma(S) = +\infty$.

Proposition 2.3. *Let $A, B \in \Sigma$. Then if there exists an odd map $f \in C(A,B)$, then $\gamma(A) \leq \gamma(B)$. Consequently, if there exists an odd homeomorphism $f : A \to B$, then $\gamma(A) = \gamma(B)$.*

We now state a theorem due to Clarke.

Theorem 2.4. *Let $J \in C^1(E,\mathbb{R})$ be a functional satisfying the Palais-Smale condition. Also suppose that:*

- *J is bounded from below and even,*
- *there is a compact set $K \in \Sigma$ such that $\gamma(K) = k$ and $\sup_{x\in K} J(x) < J(0)$.*

Then J possesses at least k pairs of distinct critical points and their corresponding critical values are less than $J(0)$.

In what follows, we denote by C_i, $i \in \mathbb{N}$, general positive numbers whose value may change from line to line. We denote by $u^+ := \max\{u, 0\}$ and $u^- := \max\{-u, 0\}$ respectively the positive and negative part of a function u. To simplify, we put $\xi(t) = |t|^{p-2}t$, for $t \in \mathbb{R}$.

3. Main Results

3.1. **Existence and uniquness for Problem** (P_1)**.** We say that u is a weak solution of problem (P_1) if $u \in W_0^{s,p}(\Omega)$ with $u > 0$ and

$$\Big(a + b\int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dxdy\Big)\int\int_{\mathbb{R}^{2N}} \frac{\xi\big(u(x)-u(y)\big)(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dxdy$$

$$= \int_\Omega \frac{k\varphi}{u^\gamma}\,dx + \int_\Omega u^q\varphi\,dx. \tag{1}$$

The functional $J : W_0^{s,p}(\Omega) \to \mathbb{R}$ associated with problem (P_1) is defined by

$$J_\lambda(u) = \frac{a}{p}\|u\|^p + \frac{b}{2p}\|u\|^{2p} - \frac{1}{1-\gamma}\int_\Omega k(x)|u|^{1-\gamma}\,dx + \frac{\lambda}{q+1}\int_\Omega |u|^{q+1}\,dx$$

Obviously, this energy functional is not differentiable because of the appearance of a singular term. Thus, the usual critical point theory is no longer applicable to this kind of functional. We formulate the main result as follows

Theorem 3.1. *Assume that $k \in L^{\frac{p_s^*}{p_s^*+\gamma-1}}(\Omega)$. Then problem (P_1) has a unique positive solution.*

In order to prove our main result, we need to study the geometry conditions.

Lemma 3.2. *Under the assumptions of Theorem 3.1, the functional J_λ is bounded from below and attains the global minimizer in $W_0^{s,p}(\Omega)$; that is, there exists $u_0 \in W_0^{s,p}(\Omega)$ such that $m = \inf_{u\in W_0^{s,p}(\Omega)} J_\lambda(u) = J_\lambda(u_0)$. Furthermore, $J_\lambda(u_0) < 0$.*

Proof. By the Hölder inequality and Sobolev inequality, for any $u \in W_0^{s,p}(\Omega)$, we have

$$\int_\Omega k(x)|u|^{1-\gamma}\,dx \leq \|k\|_{\frac{p_s^*}{p_s^*+\gamma-1}}\|u\|_{p_s^*}^{1-\gamma} \leq C^{1-\gamma}\|k\|_{\frac{p_s^*}{p_s^*+\gamma-1}}\|u\|^{1-\gamma}. \tag{2}$$

where $C = C(N, s, p)$ is a positive constant. Hence, since $\lambda \geq 0$, we have

$$\begin{aligned} J_\lambda(u) &= \frac{a}{p}\|u\|^p + \frac{b}{2p}\|u\|^{2p} - \frac{1}{1-\gamma}\int_\Omega f(x)u^{1-\gamma}\,dx + \frac{\lambda}{q+1}\int_\Omega u^{q+1}\,dx \\ &\geq \frac{a}{p}\|u\|^p + \frac{b}{2p}\|u\|^{2p} - \frac{C^{1-\gamma}}{1-\gamma}\|k\|_{\frac{p_s^*}{p_s^*+\gamma-1}}\|u\|^{1-\gamma}, \end{aligned}$$

which implies that J_λ is coercive and bounded from below on $W_0^{s,p}(\Omega)$. Now, we show that J_λ is weakly lower semi continuous; let (u_n) be a sequence that converges weakly to u in $W_0^{s,p}(\Omega)$, we deduce that

$$u_n \to u \quad \text{strongly in } L^\alpha(\Omega) \text{ for any } \alpha \in [1, p_s^*) \tag{3}$$

$$u_n(x) \to u(x) \quad \text{a. e. } x \in \Omega, \tag{4}$$

as $n \to +\infty$. Letting $v_n = u_n - u$, by the Brezis-Lieb lemma (see Theorem 1 of [3]), we have

$$\|u_n\|^p = \|v_n\|^p + \|u\|^p + o_n(1). \tag{5}$$

$$\|u_n\|_{p_s^*}^{p_s^*} = \|v_n\|_{p_s^*}^{p_s^*} + \|u\|_{p_s^*}^{p_s^*} + o_n(1). \tag{6}$$

where $o_n(1)$ is an infinitesimal as $n \to +\infty$. On other hand, by the Vitali convergence theorem, we get

$$\lim_{n\to+\infty}\int_\Omega k(x)|u_n|^{1-\gamma}\,dx = \int_\Omega k(x)|u|^{1-\gamma}\,dx. \tag{7}$$

If $0 < q < p_s^* - 1$, thanks to (3), (5) and (7), we have that

$$\begin{aligned} J_\lambda(u_n) &= \frac{a}{p}\|u_n\|^p + \frac{b}{2p}\|u_n\|^{2p} - \int_\Omega f(x)|u_n|^{1-\gamma}\,dx + \frac{\lambda}{q+1}\int_\Omega |u_n|^{q+1}\,dx \\ &= J_\lambda(u) + \frac{a}{p}\|v_n\|^p + \frac{b}{2p}\|v_n\|^{2p} + \frac{b}{p}\|v_n\|^p\|u\|^p + o_n(1) \\ &\geq J_\lambda(u). \end{aligned}$$

If $q = p_s^* - 1$, by (5), (6) and (7), we have that

$$\begin{aligned} J_\lambda(u_n) &= J_\lambda(u) + \frac{a}{p}\|v_n\|^p + \frac{b}{2p}\|v_n\|^{2p} + \frac{b}{p}\|v_n\|^p\|u\|^p + \frac{\lambda}{p_s^*}\|v_n\|_{p_s^*}^{p_s^*}. \\ &\geq J_\lambda(u). \end{aligned}$$

Thus, for any $0 < q \leq p_s^* - 1$, the functional J_λ is weakly lower semi continuous. So, there exists $u_0 \in W_0^{s,p}(\Omega)$ such that $m = \inf_{u\in W_0^{s,p}(\Omega)} J_\lambda(u) = J_\lambda(u_0)$. Moreover, since $0 < \gamma < 1$ and $k(x) > 0$ almost every $x \in \Omega$, we have $J_\lambda(t\rho) < 0$ for all $\rho \neq 0$ and small $t > 0$; then $m < 0$. The assertion is proved. □

Lemma 3.3. *u_0 is a positive function.*

Proof. The previous lemma shows that u_0 is a nontrivial local minimizer of the functional J_λ, then for any $\varphi \in W_0^{s,p}(\Omega)$ be a nonnegative function and for any $t > 0$ sufficiently small such that $\|u_0 + t\varphi\| \leq \rho$, we have

$$\frac{J_\lambda(u_0 + t\varphi) - J_\lambda(u_0)}{t} \geq 0. \tag{8}$$

We know that,

(9)
$$\lim_{t\to 0}\frac{\|u_0+t\varphi\|^p-\|u_0\|^p}{t}=p\int\int_{\mathbb{R}^{2N}}\frac{\xi\big(u_0(x)-u_0(y)\big)(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dxdy,$$

(10)
$$\lim_{t\to 0}\frac{\|u_0+t\varphi\|^{2p}-\|u_0\|^{2p}}{t}=2p\|u_0\|^p\int\int_{\mathbb{R}^{2N}}\frac{\xi\big(u_0(x)-u_0(y)\big)(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}dxdy.$$

By the dominated convergence theorem, we have

$$\lim_{t\to 0}\frac{1}{q+1}\int_\Omega\frac{(u_0+t\varphi)^{q+1}-u_0^q}{t}=\int_\Omega u_0^q\varphi\,dx. \tag{11}$$

We observe that

$$\frac{1}{1-\gamma}\frac{(u_0+t\varphi)^{1-\gamma}-u_0^{1-\gamma}}{t}\to u_0^{-\gamma}\varphi\quad\text{a.e. in }\ \Omega\ \text{ as }\ t\to 0. \tag{12}$$

Thus, using the Monotone Convergence Theorem (Beppo-Levi), we obtain, with possibility that the right side can be equal to $+\infty$ when $u_0(x)=0$,

$$\liminf_{t\to 0}\int_\Omega\frac{k(x)}{1-\gamma}\frac{(u_0+t\varphi)^{1-\gamma}-u_0^{1-\gamma}}{t}=\int_\Omega k(x)u_0^{-\gamma}\varphi\,dx, \tag{13}$$

Hence, combining (9-13) and using (8), we deduce that

$$\begin{aligned}(a+b\|u_0\|^p)\int\int_{\mathbb{R}^{2N}}&\frac{\xi\big(u_0(x)-u_0(y)\big)(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dxdy\\ &-\int_\Omega k(x)u_0^{-\gamma}\varphi\,dx+\int_\Omega u_0^q\varphi\,dx\quad\geq\quad 0.\end{aligned} \tag{14}$$

Then, for any $\varphi\in W_0^{s,p}(\Omega)$ with $\varphi>0$, we have

$$\begin{aligned}\int_\Omega k(x)u_0^{-\gamma}\varphi\,dx\quad\leq\quad&(a+b\|u_0\|^p)\int\int_{\mathbb{R}^{2N}}\frac{\xi\big(u_0(x)-u_0(y)\big)(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dxdy\\ &+\int_\Omega u_0^q\varphi\,dx.\end{aligned} \tag{15}$$

Let $v_1\in W_0^{s,p}(\Omega)$ be the first eigenfunction of the operator $(-\Delta)_p^s$ with $v_1>0$ and $\|v_1\|=1$, taking $\varphi=v_1$ in (15) and by applying the Hölder inequality and the Sobolev inequality, we get

$$\begin{aligned}\int_\Omega k(x)u_0^{-\gamma}v_1\,dx\quad&\leq\quad(a+b\|u_0\|^p)\int\int_{\mathbb{R}^{2N}}\frac{\xi\big(u_0(x)-u_0(y)\big)(v_1(x)-v_1(y))}{|x-y|^{N+sp}}\,dxdy\\ &\qquad+\int_\Omega u_0^q v_1\,dx.\\ &\leq\quad C_\Omega(a+b\|u_0\|^p)\|u_0\|^{p-1}+\lambda\|u_0\|\|v_1\|<\infty.\end{aligned}$$

wich implies that $u_*(x)>0$ for almost every $x\in\Omega$.

□

Proof of Theorem 3.1. We show that inequality (15) is true all $\varphi\in W_0^{s,p}(\Omega)$, For this, we note that there exists $t_0\in(0,1)$ such that $u_0+tu_0\in W_0^{s,p}(\Omega)$

for all $|t| \leq t_0$, we define the function

$$\begin{array}{rcl} h : [-t_0, t_0] & \to & \mathbb{R} \\ t & \mapsto & h(t) = J_\lambda(u_* + tu_*). \end{array}$$

Thus, h attains it's minimum at $t = 0$, wich implies that $h'(0) = 0$, that is

$$\lim_{t\to 0} \frac{J_\lambda(u_0 + tu_0) - J_\lambda(u_0)}{t} = 0,$$

i.e.

$$a\|u_0\|^p + b\|u_0\|^{2p} + \lambda \int_\Omega u_0^{q+1}\, dx - \int_\Omega k(x) u_0^{1-\gamma}\, dx = 0. \tag{16}$$

Define $\psi_\varepsilon = (u_0 + \varepsilon\varphi)$ and $(u_0 + \varepsilon\varphi)^+ = \max\{0, u_* + \varepsilon\varphi\}$. We denote by $\{u \leq 0\} = \{x \in \mathbb{R}^N : u(x) \leq 0\}$ and $\{u \leq 0\}^c$ is its complement in $\mathbb{R}^N$. Putting $\varphi = \psi_\varepsilon^+ \geq 0$ in (14), we obtain

$$\begin{aligned} 0 \leq\ & (a + b\|u_0\|^p) \int\int_{\mathbb{R}^{2N}} \frac{\xi\big(u_0(x) - u_0(y)\big)(\psi_\varepsilon^+(x) - \psi_\varepsilon^+(y))}{|x-y|^{N+sp}}\, dxdy \\ & - \int_\Omega k(x) u_0^{-\gamma} \psi_\varepsilon\, dx + \lambda \int_\Omega u_0^q \psi_\varepsilon\, dx. \end{aligned}$$

Noting that $(u_0 + \varepsilon\varphi)^+ = (u_0 + \varepsilon\varphi) + (u_0 + \varepsilon\varphi)^-$, by (16), we can see that

$$\begin{aligned} 0 \leq\ & \varepsilon\Big[(a + b\|u_0\|^p) \int\int_{\mathbb{R}^{2N}} \frac{\xi\big(u_0(x) - u_0(y)\big)(\varphi(x) - \varphi(y))}{|x-y|^{N+sp}}\, dxdy \\ & - \int_\Omega k(x) u_0^{-\gamma} \varphi\, dx + \lambda \int_\Omega u_0^q \varphi\, dx\Big] \\ & + (a + b\|u_0\|^p) \int\int_{\mathbb{R}^{2N}} \frac{\xi\big(u_0(x) - u_0(y)\big)(\psi_\varepsilon^-(x) - \psi_\varepsilon^-(y))}{|x-y|^{N+sp}}\, dxdy \\ & - \int_\Omega k(x) u_0^{-\gamma} \psi_\varepsilon^-\, dx + \lambda \int_\Omega u_0^q \psi_\varepsilon^-\, dx. \end{aligned} \tag{17}$$

We note that

$$- \int_\Omega k(x) u_0^{-\gamma} \psi_\varepsilon^-\, dx \leq 0, \tag{18}$$

and
(19)

$$\lambda \int_\Omega u_0^q \psi_\varepsilon^-\, dx = -\lambda \int_{\{\psi_\varepsilon \leq 0\}} u_0^{q+1}\, dx - \lambda\varepsilon \int_{\{\psi_\varepsilon \leq 0\}} u_0^q \varphi\, dx \leq -\lambda\varepsilon \int_{\{\psi_\varepsilon \leq 0\}} u_0^q \varphi\, dx.$$

We pose

$$I(u_0, \varphi) = (a + b\|u_0\|^p) \int\int_{\mathbb{R}^{2N}} \frac{\xi\big(u_0(x) - u_0(y)\big)(\varphi(x) - \varphi(y))}{|x-y|^{N+sp}}\, dxdy.$$

By direct calculation, we have

$$\begin{aligned}
I(u_0,\psi_\varepsilon^-) &= (a+b\|u\|^p)\int\int_{\mathbb{R}^{2N}} \frac{\xi\big(u_0(x)-u_0(y)\big)(\psi_\varepsilon^-(x)-\psi_\varepsilon^-(y))}{|x-y|^{N+sp}}\,dxdy\\
&= -(a+b\|u\|^p)\int\int_{\{\psi_\varepsilon\le 0\}^2} \frac{|u_0(x)-u_0(y)|^p}{|x-y|^{N+sp}}\,dxdy\\
&\quad -\varepsilon(a+b\|u\|^p)\int\int_{\{\psi_\varepsilon\le 0\}^2} \frac{\xi\big(u_0(x)-u_0(y)\big)(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dxdy\\
&\quad +2(a+b\|u\|^p)\int\int_{\{\psi_\varepsilon\le 0\}\times\{\psi_\varepsilon\le 0\}^c} \frac{\xi\big(u_0(x)-u_0(y)\big)(\psi_\varepsilon^-(x)-\psi_\varepsilon^-(y))}{|x-y|^{N+sp}}\,dxdy\\
&\le -\varepsilon(a+b\|u\|^p)\int\int_{\{\psi_\varepsilon\le 0\}^2} \frac{\xi\big(u_0(x)-u_0(y)\big)(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dxdy\\
&\quad -2\varepsilon(a+b\|u\|^p)\int\int_{\{\psi_\varepsilon\le 0\}\times\{\psi_\varepsilon\le 0\}^c} \frac{\xi\big(u_0(x)-u_0(y)\big)(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dxdy
\end{aligned}$$

(20)

$$\le 2\varepsilon(a+b\|u\|^p)\int\int_{\{\psi_\varepsilon\le 0\}\times\mathbb{R}^N} \frac{|u_0(x)-u_0(y)|^{p-1}|\varphi(x)-\varphi(y)|}{|x-y|^{N+sp}}\,dxdy.$$

By (17), (18) and (19), we get

$$\begin{aligned}
0 &\le \varepsilon\Big[I(u_0,\varphi)-\int_\Omega k(x)u_0^{-\gamma}\varphi\,dx+\lambda\int_\Omega u_0^q\varphi\,dx\Big]\\
&\quad +2\varepsilon(a+b\|u\|^p)\int\int_{\{\psi_\varepsilon\le 0\}\times\mathbb{R}^N} \frac{|u_0(x)-u_0(y)|^{p-1}|\varphi(x)-\varphi(y)|}{|x-y|^{N+sp}}\,dxdy\\
&\quad -\lambda\varepsilon\int_{\{\psi_\varepsilon\le 0\}} u_0^q\varphi\,dx. \qquad (21)
\end{aligned}$$

Since $meas\{\psi_\varepsilon\le 0\}\to 0$ as $\varepsilon\to 0$, we have

$$(a+b\|u\|^p)\int\int_{\{\psi_\varepsilon\le 0\}\times\mathbb{R}^N} \frac{|u_0(x)-u_0(y)|^{p-1}|\varphi(x)-\varphi(y)|}{|x-y|^{N+sp}}\,dxdy-\lambda\int_{\{\psi_\varepsilon\le 0\}} u_0^q\varphi\,dx\to 0,$$

as $\varepsilon\to 0$. Consequently, dividing (21) by ε and letting $\varepsilon\to 0$, we have

$$\varepsilon I(u_0,\varphi)-\int_\Omega k(x)u_0^{-\gamma}\varphi\,dx+\lambda\int_\Omega u_0^q\varphi\,dx\ge 0,$$

for any $\varphi\in W_0^{s,p}(\Omega)$. This inequality also holds equally well for $-\varphi$, it follows that u_0 is a positive solution of Problem (P_1).

Now, we show the uniqueness of solution, for this, we suppose that u_0, u_1 are solutions of problem (P_1), substituting into (1) with u_0 and u_1, put $w=u_0-u_1$. Subtracting the resulting two equations, we find that:

$$(22)\quad a\int\int_{\mathbb{R}^{2N}} \frac{\Big[\xi\big(u_0(x)-u_0(y)\big)-\xi\big(u_1(x)-u_1(y)\big)\Big](w(x)-w(y))}{|x-y|^{N+sp}}\,dxdy$$

$$+ba(u_0,u_1)+\lambda\int_\Omega(u_0^q-u_1^q)w(x)\,dx-\int_\Omega k(x)(u_0^{-\gamma}-u_1^{-\gamma})w(x)\,dx=0,$$

where

$$a(u_0,u_1) = \|u_0\|^{2p} - \|u_0\|^p \int\int_{\mathbb{R}^{2N}} \frac{\xi\big(u_0(x)-u_0(y)\big)(u_1(x)-u_1(y))}{|x-y|^{N+sp}}\,dxdy$$
$$+\|u_1\|^{2p} - \|u_1\|^p \int\int_{\mathbb{R}^{2N}} \frac{\xi\big(u_1(x)-u_1(y)\big)(u_0(x)-u_0(y))}{|x-y|^{N+sp}}\,dxdy$$

It is clear that we have

$$(23) \qquad \int_\Omega (u_0^q - u_1^q)w(x)\,dx \geq 0, \qquad \int_\Omega k(x)(u_0^{-\gamma} - u_1^{-\gamma})w(x)\,dx \leq 0.$$

Using the elementary inequalities (see [21])

$$(24) \qquad (|x|^{p-2}x - |y|^{p-2}y)(x-y) \geq \begin{cases} c_p|x-y|^p, & \text{if } p\geq 2 \\ c_p\dfrac{|x-y|^2}{(|x|+|y|)^{2-p}}, & \text{if } 1<p<2. \end{cases}$$

where c_p is a positif constant, we get

$$(25) \qquad \int\int_{\mathbb{R}^{2N}} \frac{\Big[\xi\big(u_0(x)-u_0(y)\big) - \xi\big(u_1(x)-u_1(y)\big)\Big](w(x)-w(y))}{|x-y|^{N+sp}}\,dxdy$$

$$\geq \begin{cases} c_p\|w\|^p, & \text{if } p\geq 2 \\ c_p \displaystyle\int\int_{\mathbb{R}^{2N}} \frac{|w(x)-w(y)|^2}{\big(|u_0(x)-u_0(y)|+|u_1(x)-u_1(y)|\big)^{2-p}|x-y|^{N+sp}}\,dxdy, & \text{if } 1<p<2. \end{cases}$$

By the Hölder inequality, we have:

$$\begin{aligned} a(u_0,u_1) &\geq \|u_0\|^{2p} - \|u_0\|^{2p-1}\|u_1\| - \|u_1\|^{2p-1}\|u_0\| + \|u_1\|^{2p} \\ &= (\|u_0\|-\|u_1\|)(\|u_0\|^{2p-1} - \|u_1\|^{2p-1}) \\ &\geq 0. \end{aligned} \qquad (26)$$

From (22)-(26), we deduce that:

$$\begin{cases} a\|w\|^p \leq 0, & \text{if } p\geq 2 \\ a\displaystyle\int\int_{\mathbb{R}^{2N}} \frac{|w(x)-w(y)|^2}{\big(|u_0(x)-u_0(y)|+|u_1(x)-u_1(y)|\big)^{2-p}|x-y|^{N+sp}}\,dxdy \leq 0, & \text{if } 1<p<2. \end{cases}$$

If $a>0$, it follows that

$$\begin{cases} \|w\|^p = 0, & \text{if } p\geq 2 \\ a\displaystyle\int\int_{\mathbb{R}^{2N}} \frac{|w(x)-w(y)|^2}{\big(|u_0(x)-u_0(y)|+|u_1(x)-u_1(y)|\big)^{2-p}|x-y|^{N+sp}}\,dxdy = 0, & \text{if } 1<p<2. \end{cases}$$

Then $w(x)=0$ a.e. in $\mathbb{R}^N$ so $w(x)=0$ a.e. in Ω and consequently $u_0=u_1$.
If $a=0$, from (22) we have

$$\int_\Omega (u_0^q - u_1^q)w(x)\,dx = 0$$

this implies that $u_0(x)=u_1(x)$ a.e. in Ω. We deduce that u_0 is the unique positive solution of problem (P_1) which completes the proof of Theorem 3.1.

□

3.2. **Infinitely many different solutions without singularity.** In the problem (P_2), assume that g satisfies the following conditions

(H_1) There exist positive constants $c_1, c_2, \alpha > 0$ with $1 < \alpha < \min\{p, q+1\}$ such that:

$$c_1 t^{\alpha-1} \leq g(x,t) \leq c_2 t^{\alpha-1}, \text{for all } \ t \geq 0 \ \text{ and for a.e.} x \in \Omega.$$

(H_2) $g(x,-t) = -g(x,t)$, for all $t \geq 0$ and for a.e. $x \in \Omega$.

we say that $u \in W_0^{s,p}(\Omega)$ is a weak solution of problem (P_2) if and only if

$$\begin{aligned} \Big(a + b\|u\|^p\Big) \int\int_{\mathbb{R}^{2N}} \frac{\xi\big(u(x)-u(y)\big)(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\, dxdy \\ = \int_\Omega f(x,u)\varphi\, dx - \lambda \int_\Omega |u|^{q-1}u\varphi\, dx. \end{aligned} \tag{27}$$

for all $\varphi \in W_0^{s,p}(\Omega)$.

Naturally, we associate the problem (P_2) by the functional $A_\lambda : W_0^{s,p}(\Omega) \to \mathbb{R}$ defined by

$$A_\lambda(u) = \frac{a}{p}\|u\|^p + \frac{b}{2p}\|u\|^{2p} - \int_\Omega G(x,u)\, dx + \frac{\lambda}{q+1}\int_\Omega |u|^{q+1}\, dx,$$

where $G(x,t) = \int_0^t f(x,s)\, ds$. Under hypothesis $(H1)$, it is easy to see that $A_\lambda \in C^1(W_0^{s,p}(\Omega), \mathbb{R})$ and for all $u, \varphi \in W_0^{s,p}(\Omega)$, we have

$$\langle A'_\lambda(u), \varphi\rangle = \Big(a + b\|u\|^p\Big)\langle E'(u), \varphi\rangle - \int_\Omega f(x,u)\varphi\, dx + \lambda\int_\Omega |u|^{q-1}u\varphi\, dx,$$

where E' is the differential of E with $E(u) = \frac{1}{p}\|u\|^p$. Thus the critical points of A_λ are the weak solutions of (P_2).

Lemma 3.4. *The functional A_λ is bounded from below and satisfies the Palais-Smale condition.*

Proof. Using (H_1), for any $u \in W_0^{s,p}(\Omega)$, we have

$$A_\lambda(u) \geq \frac{a}{p}\|u\|^p + \frac{b}{2p}\|u\|^{2p} - \frac{c_2}{\alpha}\int_\Omega |u|^\alpha\, dx.$$

Since $1 < \alpha < p_s^*$, By the Sobolev embeddings, we have

$$A_\lambda(u) \geq \frac{a}{p}\|u\|^p + \frac{b}{2p}\|u\|^{2p} - \frac{c_3}{\alpha}\|u\|^\alpha, \tag{28}$$

for some $c_3 > 0$. Hence, since $\alpha < 2p$, then A_λ is bounded from blow. Now, we prove the second part of the lemma, let (u_n) be a Palais-Smale sequence for A_λ; that is

$$A_\lambda(u_n) \to c, \quad A'_\lambda(u_n) \to 0 \ \text{ in } \big(W_0^{s,p}(\Omega)\big)' \text{ as } n \to +\infty. \tag{29}$$

Thus $(A_\lambda(u_n))$ is a real bounded sequence. From (28) we get that (u_n) is bounded in $W_0^{s,p}(\Omega)$, thus passing to a subsequence, if necessary, still denoted by (u_n) and $u \in W_0^{s,p}(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{s,p}(\Omega)$ and and (u_n) converges strongly to u in $L^r(\Omega)$ for $1 < r < p_s^*$ and $u_n(x) \to u(x)$ a.e. $x \in \Omega$ as $n \to +\infty$.

We have

$$\langle A'_\lambda(u_n), u_n - u\rangle = (a + b\|u_n\|^p)\langle E'(u_n), u_n - u\rangle$$
$$+\lambda \int_\Omega |u_n|^{q-1}u_n(u_n - u)\, dx - \int_\Omega g(x, u_n)(u_n - u)\, dx. \quad (30)$$

Now, we prove that

$$\int_\Omega |u_n|^{q-1}u_n(u_n - u)\, dx \to 0, \text{ as } n \to +\infty.$$

Indeed, since $1 < q < p_s^* - 1$, we have $u_n \to u$ in $L^{q+1}(\Omega)$ and from the Hölder inequality we get

$$\Big|\int_\Omega |u_n|^{q-1}u_n(u_n - u)\, dx\Big| \le \|u_n\|_{q+1}^q \|u_n - u\|_{q+1} \to 0, \text{ as } n \to +\infty. \quad (31)$$

On the other hand, by the continuity of g we have $g(x, u_n)u_n \to g(x, u)u$ and $g(x, u_n)u \to g(x, u)u$ a.e. $x \in \Omega$, and the fact that the sequence (u_n) converges in $L^\alpha(\Omega)$ then there exists a dominated function h in $L^\alpha(\Omega)$, using (H_1) we get

$$|g(x, u_n)u_n| \le k|u_n|^\alpha \le kh^\alpha \in L^1(\Omega), \text{ and } |g(x, u_n)u| \le k|u_n|^{\alpha-1}|u| \le kh^\alpha \in L^1(\Omega).$$

for some $k > 0$. Hence, the dominated convergence theorem implies that

$$\int_\Omega g(x, u_n)(u_n - u)\, dx \to 0, \quad (32)$$

as $n \to +\infty$. The relations (29)-(32) implies that,

$$(a + b\|u_n\|^p)\langle E'(u_n), u_n - u\rangle \to 0, \text{ as } n \to +\infty.$$

we consider

$$L_n = (a + b\|u_n\|^p) \int\int_{\mathbb{R}^{2N}} \frac{\Big(\xi\big(u_n(x) - u_n(y)\big) - \xi\big(u(x) - u(y)\big)\Big)\Big(\big(u_n(x) - u_n(y)\big) - \big(u(x) - u(y)\big)\Big)}{|x - y|^{N+sp}}\, dxdy.$$

By a direct calculation we get

$$L_n = \langle (a + b\|u_n\|^p)\langle E'(u_n), u_n - u\rangle$$
$$+(a + b\|u_n\|^p)\Big[\|u\|^p - \int\int_{\mathbb{R}^{2N}} \frac{\xi\big(u(x) - u(y)\big(u_n(x) - u_n(y)\big)}{|x - y|^{N+sp}}\, dxdy\Big].$$

Easily to see that the weak convergence of (u_n) to u in $W_0^{s,p}(\Omega)$ implies that

$$\int\int_{\mathbb{R}^{2N}} \frac{\xi\big(u(x) - u(y)\big)\big(u_n(x) - u_n(y)\big)}{|x - y|^{N+sp}}\, dxdy \to \|u\|^p$$

so, we deduce that

$$L_n \to 0, \text{ as } n \to +\infty. \quad (33)$$

If $p \ge 2$, using the first elementary inequality in (24), obviously we get $\|u_n - u\| \to 0$ as $n \to +\infty$. If $1 < p < 2$, we put

$$H_n(x, y) = \big(\xi(u_n(x) - u_n(y)) - \xi(u(x) - u(y))\big)\big((u_n(x) - u_n(y)) - (u(x) - u(y))\big),$$
$$S_n(x, y) = |u_n(x) - u_n(y)|^p + |u(x) - u(y)|^p.$$

Using the second inequality in (24) and Hölder's inequality, we get

$$\int\int_{\mathbb{R}^{2N}} \frac{\Big|\big(u_n(x)-u_n(y)\big)-\big(u(x)-u(y)\big)\Big|^p}{|x-y|^{N+sp}}\,dxdy \;\le\; C\int\int_{\mathbb{R}^{2N}} \frac{\big(H_n(x,y)\big)^{\frac{p}{2}}(S_n(x,y))^{\frac{2-p}{2}}}{|x-y|^{N+sp}}\,dxdy$$
$$\le C\int\int_{\mathbb{R}^{2N}} \frac{H_n(x,y)}{|x-y|^{N+sp}}\,dxdy \int\int_{\mathbb{R}^{2N}} \frac{S_n(x,y)}{|x-y|^{N+sp}}\,dxdy.$$

From (33), we have

$$\int\int_{\mathbb{R}^{2N}} \frac{H_n(x,y)}{|x-y|^{N+sp}}\,dxdy \to 0, \tag{34}$$

Now, since $\int\int_{\mathbb{R}^{2N}} \frac{S_n(x,y)}{|x-y|^{N+sp}}\,dxdy$ is bounded, we deduce that

$$\|u_n-u\| = \int\int_{\mathbb{R}^{2N}} \frac{\Big|\big(u_n(x)-u_n(y)\big)-\big(u(x)-u(y)\big)\Big|^p}{|x-y|^{N+sp}}\,dxdy \to 0 \quad \text{as } n\to+\infty. \tag{35}$$

consequently, $u_n \to u$ in $W_0^{s,p}(\Omega)$, so A_λ satisfies the Palais-Smale condition. □

Theorem 3.5. *Assume (H_1) and (H_2) are satisfied, then Problem (P_2) has infinitely many distinct pairs solutions.*

Proof. We notice that in every Banach space of infinite dimension, there is an infinite-dimensional subspace with Schauder basis (background information can be found in e.g. [6, 26, 27]). Thus, we consider $\{e_1, e_2, ...\}$ a Schauder basis of a subspace X of $W_0^{s,p}(\Omega)$, and for each $k \in \mathbb{N}$, consider X_k, the subspace of X generated by k vectors $\{e_1, e_2, ..., e_k\}$. Obviously, X_k is a subspace of $W_0^{s,p}(\Omega)$. So we have $X_k \subset L^\alpha$. Then, the norms $\|\cdot\|_{W_0}$, $\|\cdot\|_\alpha$ are equivalent on X_k. Hence, there exists positive constants $C_k > 0$ such that

$$-\|u\|_{L^{\alpha(\cdot)}} \le -C_k\|u\|_{W_0}, \quad \text{for all } u \in X_k. \tag{36}$$

Let $u \in X_k$, using (36), (H_1) and the Sobolev embedding, we have

$$\begin{aligned} A_\lambda(u) &\le \frac{a}{p}\|u\|^p + \frac{b}{2p}\|u\|^{2p} + \frac{\lambda c_4}{1+q}\|u\|^{1+q} - \frac{c_1}{\alpha}\|u\|_\alpha^\alpha \\ &\le \frac{a}{p}\|u\|^p + \frac{b}{2p}\|u\|^{2p} + \frac{\lambda c_4}{1+q}\|u\|^{1+q} - \frac{c_1 C_k^\alpha}{\alpha}\|u\|^\alpha \\ &= \|u\|^\alpha\Big(\frac{a}{p}\|u\|^{p-\alpha} + \frac{b}{2p}\|u\|^{2p-\alpha} + \frac{\lambda c_4}{1+q}\|u\|^{1+q-\alpha} - \frac{c_1 C_k^\alpha}{\alpha}\Big). \end{aligned}$$

Since $\alpha < p$ and $\alpha < q+1$, so we can choose $0 < R < 1$ small enough such that

$$\frac{a}{p}R^{p-\alpha} + \frac{b}{2p}R^{2p-\alpha} + \frac{\lambda c_4}{1+q}R^{1+q-\alpha} - \frac{c_1 c_k^\alpha}{\alpha} < 0$$

Thus, for $0 < r < R$, we consider the set $K = \{u \in X_k : \|u\|_{W_0} = r\}$. For all $u \in K$, we have

$$\begin{aligned} A_\lambda(u) &\leq r^\alpha\big(\frac{a}{p}r^{p-\alpha} + \frac{b}{2p}r^{2p-\alpha} + \frac{\lambda c_4}{1+q}r^{1+q-\alpha} - \frac{c_1 c_k^\alpha}{\alpha}\big) \\ &< R^\alpha\big(\frac{a}{p}R^{p-\alpha} + \frac{b}{2p}R^{2p-\alpha} + \frac{\lambda c_4}{1+q}R^{1+q-\alpha} - \frac{c_1 c_k^\alpha}{\alpha}\big) \\ &< 0 = A_\lambda(0). \end{aligned}$$

We can consider the odd homeomorphism $h : K \to S^{k-1}$ defined by $u = \sum_{i=1}^{k} \alpha_i e_i \mapsto h(u) = (\alpha_1, \alpha_2, ..., \alpha_k)$, where S^{k-1} is the sphere in $\mathbb{R}^k$. From Theorem 2.2 and proposition 2.3 we have that $\gamma(K) = \gamma(S^{k-1}) = k$. By Theorem 2.4, A_λ has at least k pairs of different critical points. Since k is arbitrary, we obtain infinitely many critical points of A_λ, consequently, Problem (P) has at infinitely many distinct pairs of nontrivial solutions. □

4. Conclusion

Using the direct method of minimization combined with the theory of fractional Sobolev spaces, we have showed the existence and uniqueness of positive solution for the fractional p-Kirchhoff problem (P1) involving a singular nonlinearity. In addition, we have proved the existence of infinity of pairs of solutions for the fractional p-Kirchhoff problem (P2) with a regular nonlinear second member (without singularity). The approach that has been used is the critical points theory and krasnoselskii's genus. In future studies, we could consider the variable exponent cases of this kind of problems. So this will benefit future studies in this direction.

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