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THEORETICAL STUDIES ON THE EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR A MULTIDIMENSIONAL NONLINEAR TIME AND SPACE-FRACTIONAL REACTION-DIFFUSION/WAVE EQUATION

Abstract. This paper discusses and theoretically studies the existence and uniqueness of radially symmetric solutions for a multidimensional nonlinear time and space-fractional reaction-diffusion/wave equation that enables treating vibration and control, signal and image processing, and modeling earthquakes, among other physical phenomena. Additionally, application of Schauder's and Banach's fixed point theorems facilitates identifying the existence and uniqueness of solutions for the selected equation. The applicability of our main results is demonstrated through examples and explicit solutions.

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Key words and phrases. Multidimensional nonlinear equation, reaction-diffusion/wave, time and space-fractional order, radially symmetric solutions, existence and uniqueness.

რეზიუმე. სტატია განიხილავს და თეორიულად იკვლევს მრავალგანზომილებიანი არაწრფივი დროისა და სივრცის ფრაქციული რეაქციულ-დიფუზიური/ტალღის განტოლების რადიალურად სიმეტრიული ამონახსნების არსებობასა და ერთადერთობას, რაც საშუალებას იძლევა სხვა ფიზიკურ მოვლენებთან ერთად განიხილულ იქნას რხევა, სიგნალისა და გამოსახულების დამუშავების კონტროლი და მიწისძვრების მოდელირება. გარდა ამისა, შაუდერისა და ბანახის უძრავი წერტილის თეორემების გამოყენება ხელს უწყობს ამონახსნების არსებობისა და ერთადერთობის დადგენას არჩეული განტოლებისთვის. ძირითადი შედეგების გამოყენებადობა დემონსტრირებულია მაგალითებითა და ცხადი ამონახსნებით.

1 Introduction and statement of results

Partial differential equations (PDEs) with fractional order have recently become a valuable tool for modeling numerous tangible incidents that science attempts to explain and have approached more frequently in recent years. Their application spans studies of vibration and control, signal and image processing, and modeling earthquakes, among others (Samko *et al.* 1993 [36], Podlubny 1999 [34], Kilbas *et al.* 2006 [23], Diethelm 2010 [17]).

Exact solutions of fractional-order PDEs are crucial for rendering many qualitative features of natural science processes and phenomena fathomable, that become obtainable by using various methods including the residual power series, symmetry, spectral, Fourier transform, similarity, *etc.* (for more details, see [1–13, 16, 18, 19, 22, 25–29, 31, 33, 35, 37–40]).

In this work, we give an example of a class of fractional-order PDEs, which helps to describe various complex phenomena; it is a multidimensional nonlinear time and space-fractional reaction-diffusion/wave equation which is written as follows:

$$\partial_t^\alpha u - \kappa^2 \Delta u = F\big(t, x, u, \partial_t^\beta u, (-\Delta)^s u\big) \text{ for } 0 < s \le 1 < \beta \le \alpha \le 2, \tag{1.1}$$

where $u = u(t,x)$ is a scalar function of the time $t \ge 0$ and space variables $x \in \mathbb{R}^m$, with $m \in \mathbb{N}$. Also, $F : [0,\infty) \times \mathbb{R}^m \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a nonlinear function, $\kappa \in \mathbb{R}^*$ is a real constant and

$$\partial_t^\alpha u(t,x) = \begin{cases} \dfrac{\partial^n u}{\partial t^n}, & \alpha = n \in \mathbb{N}, \\ \mathcal{I}_{0^+}^{n-\alpha} \partial_t^n u = \displaystyle\int_0^t \frac{(t-\tau)^{n-\alpha-1}}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial \tau^n} u(\tau, x)\, d\tau, & n-1 < \alpha < n. \end{cases}$$

The symbol $(-\Delta)^s$ defines the fractional Laplacian operator [24]

$$(-\Delta)^s u = C_{m,s}\, \text{P.V.} \int_{\mathbb{R}^m} \frac{u(t,x) - u(t,y)}{|x-y|^{m+2s}}\, dy \text{ for } 0 < s < 1,$$

where P.V. stands for the Cauchy principal value, and the constant $C_{m,s}$ is given by

$$C_{m,s} = \frac{2^{2s} s \Gamma(\frac{m+2s}{2})}{\pi^{m/2} \Gamma(1-s)}.$$

We take the fractional power of $(-\Delta)$ to obtain a positive operator. As a result, our definition of the fractional Laplacian $(-\Delta)^s$ is the negative generator of the standard isotropic s-stable Lévy process [24], which is reduced to $-\Delta = -\partial^2/\partial^2 x_1 - \partial^2/\partial^2 x_2 - \cdots - \partial^2/\partial^2 x_m$ when $s = 1$.

1.1 The significance of the equation

Equation (1.1) is a representation of a large class of linear and nonlinear equations. Note that for $F \equiv 0$ and $\alpha = 1$ (resp. $\alpha = 2$), the PDE (1.1) represents the standard heat equation (resp. the wave equation). In addition, it becomes the Klein–Gordon equation when we choose $F = \kappa u$, $|\kappa| = 1$ and $\alpha = 2$. All these equations fall under the name of the fractional reaction-diffusion/wave equation (see Table 1).

Obviously, the development of accurate mathematical models for the description of complex anomalous systems depends on swapping the fractional Laplacian with integer-order Laplacian.

Fractional equation (1.1) is an equation that arises in relativistic quantum mechanics and quantum field theory, which is also crucial for high energy particle physics and is used to model many types of phenomena, including the propagation of dislocations in crystals and the behavior of elementary particles.

In [32], the authors investigated the first-order derivatives in space and half-order derivative in time contained in a time-fractional derivative in relation to a diffusion equation. The relationship that lays between the fractional diffusion equation proposed in their work and the classical diffusion

Table 1: Significant equations involving fractional Laplacian

Fractional equation	Formula
Reaction-diffusion/wave [8, 9, 11, 18, 19, 22, 28, 31, 33, 40]	$\partial_t^\alpha u + \kappa^2(-\Delta)^s u + c(t,x)u = 0$
Quasi-geostrophic [13]	$\partial_t v + \theta \cdot \nabla v + \kappa(-\Delta)^s v = f$
Cahn–Hilliard [1–3]	$\partial_t w + (-\Delta)^s(-\varepsilon^2 \Delta w + f(w)) = 0$
Porous medium [1–3, 16]	$\partial_t u + (-\Delta)^s(\lvert u\rvert^{m-1} \operatorname{sign} u) = 0$
Schrödinger [25]	$i\hbar\partial_t \psi = \partial_t^\alpha(-\hbar^2\Delta)^s\psi + c(t,x)\psi$
Ultrasound [12, 37]	$\frac{1}{c_0^2}\,\partial_t^2\theta = \nabla^2\theta - \left\{\tau\partial_t(-\Delta)^s + \eta(-\Delta)^{s+\frac{1}{2}}\right\}\theta$

equation is also considered. Nigmatullin [30] noticed the possibility of the accurate modeling of several universal electromagnetic, acoustic, and mechanical responses; according to him, such modeling can be achieved by using diffusion-wave equations with time-fractional derivatives.

Additionally, usages of (1.1) include denoising and edge stabilizing in image processing. This has been approached to examine diffusion processes and variational principles (heat equation and energy method, respectively). Authors of [15] proposed the first approach to image processing (see also [14, 20]) by means of a simple two-dimensional fractional integrodifferential equation given by the linear equation

$$\begin{cases} \partial_t u(t,x,y) = \dfrac{1}{\Gamma(\alpha-1)} \displaystyle\int_0^t (t-\tau)^{\alpha-2}\Delta u(\tau,x,y)\,d\tau \quad \text{for } 1<\alpha<2, \\ u(0,x,y) = u_0(x,y), \end{cases}$$

or, equivalently, $\frac{\partial^\alpha u}{\partial t^\alpha}(t,x,y) = \Delta u(t,x,y)$, with u_0 being the initial data representing the original image. This linear integrodifferential equation preserves object boundaries and enhances the interior regions in a stable and reliable way, even for grey-level images [14, 15, 20].

1.2 Problem statement and main results

Let $0 < s \le 1$, $1 < \beta \le \alpha \le 2$, $\varepsilon, \ell > 0$, and $T_\varepsilon = \ell\varepsilon^{\frac{2}{\alpha}}$ be such that $\Omega = [0, T_\varepsilon] \times [\varepsilon/\sqrt{m}, +\infty)^m$. We consider

$$\begin{cases} \partial_t^\alpha u - \kappa^2\Delta u = F\big(t,x,u,\partial_t^\beta u,(-\Delta)^s u\big), & (t,x)\in\Omega, \quad \kappa\in\mathbb{R}^*, \\ u(0,x) = \lvert x\rvert^\delta u_0, \quad \dfrac{\partial u}{\partial t}(0,x) = 0, & \delta, u_0 \in \mathbb{C}, \end{cases} \tag{1.2}$$

where $F : \Omega \times \mathbb{C}\times\mathbb{C}\times\mathbb{C} \to \mathbb{C}$ is a nonlinear function.

This paper's contribution regards determining the existence, uniqueness, and main properties of the general solution of stability problems obtained through replacing classical rules with fractional quadrature rules of the radially symmetric solution (see [8, 9, 11, 18, 19, 22, 26, 31, 35, 38, 39])

$$u(t,x) = \lvert x\rvert^\delta f(\lvert x\rvert^{-\frac{2}{\alpha}}t) \text{ for } \lvert x\rvert = \sqrt{x_1^2+\cdots+x_m^2} \text{ and } \delta\in\mathbb{C}, \tag{1.3}$$

the basic profile f is not known in advance and is to be identified.

Taking into consideration the regularization processes, our major aim is employing of the solutions' intermediate properties for the fractional order PDEs problem (1.2). We consider the intermediacy of the multidimensional nonlinear reaction-diffusion equation and the wave equation.

We illustrate that using analytical techniques to obtain the existence and uniqueness of weak solutions via the use of form (1.3) is promising and can also bring new results for other applications in fractional-order PDEs. It permits us to reduce the fractional-order PDE (1.1) to a fractional differential equation; the idea is well illustrated in this paper through selected examples and explicit solutions.

For the forthcoming analysis, we impose the following hypotheses:

($Hyp.$ 1) $F : \Omega \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a continuous function that is invariant by the change of scale (1.3). It gives us

$$F\big(t, x, u, \partial_t^\beta u, (-\Delta)^s u\big) = |x|^{\delta-2}\Big(J(\eta, f(\eta), f'(\eta), {}^C\mathcal{D}_{0+}^\beta f(\eta)) - \frac{4\kappa^2}{\alpha^2}\,\eta^2 f''(\eta)\Big), \tag{1.4}$$

where $\eta = |x|^{-\frac{2}{\alpha}} t$ and $J : [0, \ell] \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a continuous function.

($Hyp.$ 2) There exist three positive constants $\omega_1, \omega_2, \omega_3 > 0$ such that the continuous function J given by (1.4) satisfies

$$\big|J(\eta, f, g, h) - J(\eta, \widetilde{f}, \widetilde{g}, \widetilde{h})\big| \le \omega_1|f - \widetilde{f}| + \omega_2|g - \widetilde{g}| + \omega_3|h - \widetilde{h}|$$

for any $f, g, h, \widetilde{f}, \widetilde{g}, \widetilde{h} \in \mathbb{C}$.

($Hyp.$ 3) There exist four positive functions $a, b, c, d \in C([0, \ell], \mathbb{R}_+)$ such that the continuous function J given by (1.4) satisfies

$$|J(\eta, f, g, h)| \le a(\eta) + b(\eta)|f| + c(\eta)|g| + d(\eta)|h|$$

for any $f, g, h \in \mathbb{C}$ and $\eta \in [0, \ell]$.

λ denotes the positive constant defined by

$$\lambda = \sup\left\{\frac{\alpha\ell^{\beta-1}(|q| + c^*) + d^*}{\ell^{\beta-\alpha}\Gamma(\alpha-\beta+1)}, \frac{\alpha\ell^{\beta-1}(|q| + \omega_2) + \omega_3}{\ell^{\beta-\alpha}\Gamma(\alpha-\beta+1)}\right\},$$

where

$$q = -\frac{2\kappa^2}{\alpha^2}\big(\alpha(2\delta + m + 2) + 2\big)$$

and

$$a^* = \sup_{\eta\in[0,\ell]} a(\eta), \quad b^* = \sup_{\eta\in[0,\ell]} b(\eta), \quad c^* = \sup_{\eta\in[0,\ell]} c(\eta), \quad d^* = \sup_{\eta\in[0,\ell]} d(\eta).$$

Now, we present the main theorems of this work.

Theorem 1.1. *Assume the hypotheses* ($Hyp.$ 1)–($Hyp.$ 3) *hold. If we put* $\lambda \in (0, 1)$ *and*

$$\frac{T_\varepsilon^\alpha(|\delta\kappa^2(\delta + m - 2)| + b^*)}{\Gamma(\alpha + 1)(1 - \lambda)} < \varepsilon^2, \tag{1.5}$$

then there is at least one solution to problem (1.2) *on* Ω *in the radially symmetric form* (1.3).

Theorem 1.2. *Assume the hypotheses* ($Hyp.$ 1) *and* ($Hyp.$ 2) *hold. We give* $\lambda \in (0, 1)$ *and*

$$\mathcal{K} = \Big(\frac{\Gamma(\alpha + 1)(1 - \lambda)}{|\delta\kappa^2(\delta + m - 2)| + \omega_1}\Big)^{\frac{1}{\alpha}}.$$

If we put

$$T_\varepsilon < \varepsilon^{\frac{2}{\alpha}}\mathcal{K}, \tag{1.6}$$

then problem (1.2) *admits a unique solution in the radially symmetric form* (1.3) *on* Ω.

2 Preliminaries and necessary definitions

In this section, we present the necessary definitions from the fractional calculus theory. By $C([0,\ell],\mathbb{C})$ we denote the Banach space of continuous functions from $[0,\ell]$ into $\mathbb{C}$ with the norm

$$\|f\|_\infty = \sup_{\eta\in[0,\ell]} |f(\eta)|.$$

We start with the definitions introduced in [23] with a slight modification in the notation.

Definition 2.1 ([23])**.** The left-sided (arbitrary) fractional integral of order $\alpha > 0$ of a continuous function $f : [0,\ell] \to \mathbb{C}$ is given by

$$\mathcal{I}_{0^+}^{\alpha} f(\eta) = \frac{1}{\Gamma(\alpha)} \int_0^\eta (\eta-\xi)^{\alpha-1} f(\xi)\, d\xi, \quad \eta \in [0,\ell].$$

$\Gamma(\alpha) = \int_0^\infty \xi^{\alpha-1} \exp(-\xi)\, d\xi$ is the Euler gamma function.

Definition 2.2 (Caputo's fractional derivative [23])**.** The left-sided Caputo's fractional derivative of order $\alpha > 0$ of a function $f : [0,\ell] \to \mathbb{C}$ is given by

$$^{C}\mathcal{D}_{0^+}^{\alpha} f(\eta) = \begin{cases} \dfrac{d^n f(\eta)}{d\eta^n} & \text{for } \alpha = n \in \mathbb{N}_0, \\ \mathcal{I}_{0^+}^{n-\alpha} \dfrac{d^n f(\eta)}{d\eta^n} = \displaystyle\int_0^\eta \frac{(\eta-\xi)^{n-\alpha-1}}{\Gamma(n-\alpha)} \frac{d^n f(\xi)}{d\xi^n}\, d\xi & \text{for } n-1 < \alpha < n \in \mathbb{N}. \end{cases}$$

Lemma 2.1 ([23])**.** *Assume that* $^{C}\mathcal{D}_{0^+}^{\alpha} f \in C([0,\ell],\mathbb{C})$ *for all* $\alpha > 0$*, then*

$$\mathcal{I}_{0^+}^{\alpha}\, {}^{C}\mathcal{D}_{0^+}^{\alpha} f(\eta) = f(\eta) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!}\, \eta^k, \quad n-1 < \alpha \le n \in \mathbb{N}.$$

3 Basic-profile's existence and uniqueness results

Our initial aim is to infer that the function f in (1.3) satisfies an equation that is employed in the definition of radially symmetric solutions.

Theorem 3.1. *Let* $\delta, u_0 \in \mathbb{C}$, $\alpha, \beta \in \mathbb{R}$ *be such that* $1 < \beta \le \alpha \le 2$ *and* $p = \delta\kappa^2(\delta+m-2)$ *with* $\kappa \in \mathbb{R}^*$*. If the hypothesis* $(Hyp.\ 1)$ *holds, the problem of time and space-fractional order* (1.2) *is reduced by transformation* (1.3) *to the fractional differential equation of the form*

$$^{C}\mathcal{D}_{0^+}^{\alpha} f(\eta) = \varphi(\eta), \quad \eta \in [0,\ell], \tag{3.1}$$

where

$$\varphi(\eta) = pf(\eta) + q\eta f'(\eta) + J\big(\eta, f(\eta), f'(\eta), {}^{C}\mathcal{D}_{0^+}^{\beta} f(\eta)\big),$$

with the conditions

$$f(0) = u_0 \ \text{ and } \ f'(0) = 0. \tag{3.2}$$

Proof. Substituting expression (1.3) into the original PDE of fractional order (1.1) results in a fractional equation that needs to be narrowed down to the standard bilinear functional equation (check [8, 9, 11, 18, 19, 22, 26, 31, 35, 38, 39]). First, for $\eta = |x|^{-\frac{2}{\alpha}} t$, we get $\eta \in [0,\ell]$ and

$$\Delta u(t,x) = |x|^{\delta-2}\Big(\delta(\delta+m-2) f(\eta) - \frac{2}{\alpha^2}\big[\alpha(2\delta+m+2)+2\big]\eta f'(\eta) + \frac{4}{\alpha^2}\eta^2 f''(\eta)\Big). \tag{3.3}$$

On the other hand, for $\xi = |x|^{-\frac{2}{\alpha}}\tau$, we get

$$\begin{aligned}\frac{\partial^\alpha u}{\partial t^\alpha} &= \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-\tau)^{1-\alpha} \frac{\partial^2 u(\tau,x)}{\partial \tau^2}\, d\tau = \frac{|x|^\delta}{\Gamma(2-\alpha)} \int_0^t (t-\tau)^{1-\alpha} \frac{d^2}{d\tau^2} f(|x|^{-\frac{2}{\alpha}}\tau)\, d\tau \\ &= \frac{|x|^{\delta-2}}{\Gamma(2-\alpha)} \int_0^\eta (\eta-\xi)^{1-\alpha} \frac{d^2}{d\xi^2} f(\xi)\, d\xi = |x|^{\delta-2}\, {}^C\mathcal{D}^\alpha_{0+} f(\eta). \end{aligned} \quad (3.4)$$

If we replace (1.4), (3.3) and (3.4) in the first equation of (1.2), we obtain

$${}^C\mathcal{D}^\alpha_{0+} f(\eta) = \varphi(\eta),$$

From the conditions in (1.2), we find

$$u(t,x) = |x|^\delta f(0) \ \text{ and } \ \frac{\partial u}{\partial t}(0,x) = |x|^{\delta-\frac{2}{\alpha}} f'(0),$$

which implies that

$$f(0) = u_0 \ \text{ and } \ f'(0) = 0.$$

The proof is complete. □

Lemma 3.1. *Assume that $J : [0,\ell]\times\mathbb{C}\times\mathbb{C}\times\mathbb{C} \to \mathbb{C}$ is a continuous function, then problem (3.1), (3.2) is equivalent to the integral equation*

$$f(\eta) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^\eta (\eta-\xi)^{\alpha-1} \varphi(\xi)\, d\xi \ \ \forall\, \eta \in [0,\ell],$$

where $\varphi \in C([0,\ell],\mathbb{C})$ satisfies the functional equation

$$\varphi(\eta) = p(u_0 + \mathcal{I}^\alpha_{0+}\varphi(\eta)) + \psi(\eta,\varphi(\eta)),$$

where $\psi : [0,\ell]\times\mathbb{C} \to \mathbb{C}$ is a function satisfying

$$\psi(\eta,\varphi(\eta)) = q\eta \mathcal{I}^{\alpha-1}_{0+}\varphi(\eta) + J\Big(\eta, u_0 + \mathcal{I}^\alpha_{0+}\varphi(\eta), \mathcal{I}^{\alpha-1}_{0+}\varphi(\eta), \mathcal{I}^{\alpha-\beta}_{0+}\varphi(\eta)\Big).$$

Proof. Using Theorem 3.1, and applying $\mathcal{I}^\alpha_{0+}$ to equation (3.1), we obtain $\mathcal{I}^\alpha_{0+}\, {}^C\mathcal{D}^\alpha_{0+} f(\eta) = \mathcal{I}^\alpha_{0+}\varphi(\eta)$. From Lemma 2.1, we simply find $\mathcal{I}^\alpha_{0+}\, {}^C\mathcal{D}^\alpha_{0+} f(\eta) = f(\eta) - f(0) - \eta f'(0)$. Substituting (3.2) gives us

$$f(\eta) = u_0 + \mathcal{I}^\alpha_{0+}\varphi(\eta). \quad (3.5)$$

As

$$f'(\eta) = \frac{d}{d\eta}\left[u_0 + \mathcal{I}^\alpha_{0+}\varphi(\eta)\right] = \mathcal{I}^{\alpha-1}_{0+}\varphi(\eta)$$

and

$${}^C\mathcal{D}^\beta_{0+} f(\eta) = {}^C\mathcal{D}^\beta_{0+}\left[u_0 + \mathcal{I}^\alpha_{0+}\varphi(\eta)\right] = \mathcal{I}^{\alpha-\beta}_{0+}\varphi(\eta),$$

then

$$\begin{aligned}\varphi(\eta) &= p f(\eta) + q\eta f'(\eta) + J\big(\eta, f(\eta), f'(\eta), {}^C\mathcal{D}^\beta_{0+} f(\eta)\big) \\ &= p(u_0 + \mathcal{I}^\alpha_{0+}\varphi(\eta)) + q\eta\mathcal{I}^{\alpha-1}_{0+}\varphi(\eta) + J\Big(\eta, u_0 + \mathcal{I}^\alpha_{0+}\varphi(\eta), \mathcal{I}^{\alpha-1}_{0+}\varphi(\eta), \mathcal{I}^{\alpha-\beta}_{0+}\varphi(\eta)\Big) \\ &= p(u_0 + \mathcal{I}^\alpha_{0+}\varphi(\eta)) + \psi(\eta,\varphi(\eta)).\end{aligned}$$

Otherwise, starting by applying ${}^C\mathcal{D}^\alpha_{0+}$ on both sides of equation (3.5) and using the linearity of Caputo's derivative and the fact that ${}^C\mathcal{D}^\alpha_{0+} u_0 = 0$, we easily find (3.1). Furthermore,

$$\begin{aligned} f(0) &= (u_0 + \mathcal{I}^\alpha_{0+}\varphi)(0) = u_0, \\ f'(0) &= \mathcal{I}^{\alpha-1}_{0+}\varphi(0) = 0.\end{aligned}$$

The proof is complete. □

Theorem 3.2. *Assume the hypotheses* $(Hyp.\ 2)$, $(Hyp.\ 3)$ *hold. If we put* $\lambda \in (0,1)$ *and*

$$\frac{\ell^{\alpha}(|p| + b^{*})}{\Gamma(\alpha+1)(1-\lambda)} < 1, \tag{3.6}$$

then problem (3.1), (3.2) *has at least one solution on* $[0,\ell]$.

Proof. To begin the proof, we will transform problem (3.1), (3.2) into a fixed point problem. Let us define

$$\mathcal{A}g(\eta) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^{\eta} (\eta - \xi)^{\alpha-1} \varphi(\xi)\, d\xi, \tag{3.7}$$

where

$$\varphi(\eta) = pg(\eta) + \psi(\eta, \varphi(\eta)), \quad \eta \in [0,\ell],$$

with

$$\psi(\eta,\varphi(\eta)) = q\eta \mathcal{I}_{0^+}^{\alpha-1}\varphi(\eta) + J\Big(\eta, u_0 + \mathcal{I}_{0^+}^{\alpha}\varphi(\eta), \mathcal{I}_{0^+}^{\alpha-1}\varphi(\eta), \mathcal{I}_{0^+}^{\alpha-\beta}\varphi(\eta)\Big).$$

Since the hypotheses $(Hyp.\ 2)$, $(Hyp.\ 3)$ hold, we notice that if $\varphi \in C([0,\ell],\mathbb{C})$, then $\mathcal{A}g$ is indeed continuous (see the step 1 in this proof); therefore, it is an element of $C([0,\ell],\mathbb{C})$ and is equipped with the standard norm

$$\|\mathcal{A}g\|_{\infty} = \sup_{\eta \in [0,\ell]} |\mathcal{A}g(\eta)|.$$

Clearly, the fixed points of $\mathcal{A}$ are solutions of problem (3.1), (3.2).

We demonstrate that $\mathcal{A}$ satisfies the assumption of Schauder's fixed point theorem (see [21]). This could be proved through three steps.

Step 1: $\mathcal{A}$ is a continuous operator. Let $(g_n)_{n\in\mathbb{N}_0}$ be a real sequence such that $\lim_{n\to\infty} g_n = g$ in $C([0,\ell],\mathbb{C})$. Then $\forall\, \eta \in [0,\ell]$,

$$|\mathcal{A}g_n(\eta) - \mathcal{A}g(\eta)| \leq \frac{1}{\Gamma(\alpha)} \int_0^{\eta} (\eta-\xi)^{\alpha-1} |\varphi_n(\xi) - \varphi(\xi)|\, d\xi,$$

where

$$\varphi_n(\eta) = pg_n(\eta) + \psi(\eta, \varphi_n(\eta)), \quad \varphi(\eta) = pg(\eta) + \psi(\eta,\varphi(\eta)).$$

Consequently,

$$\begin{aligned} |\varphi_n(\eta) - \varphi(\eta)| &= \Big| p(g_n(\eta) - g(\eta)) + \big(\psi(\eta,\varphi_n(\eta)) - \psi(\eta,\varphi(\eta))\big)\Big| \\ &\leq (|p| + \omega_1)\, |g_n(\eta) - g(\eta)| + (|q| + \omega_2)\, \big|\mathcal{I}_{0^+}^{\alpha-1}(\varphi_n(\eta) - \varphi(\eta))\big| \\ &\quad + \omega_3 \big|\mathcal{I}_{0^+}^{\alpha-\beta}(\varphi_n(\eta) - \varphi(\eta))\big|. \end{aligned}$$

We have

$$\big|\mathcal{I}_{0^+}^{\alpha-1}(\varphi_n(\eta) - \varphi(\eta))\big| \leq \frac{\ell^{\alpha-1}}{\Gamma(\alpha)} \|\varphi_n - \varphi\|_{\infty}.$$

As $\Gamma(\alpha+1) > \Gamma(\alpha-\beta+1)$ for any $1 < \beta \leq \alpha \leq 2$, then

$$\big|\mathcal{I}_{0^+}^{\alpha-1}(\varphi_n(\eta) - \varphi(\eta))\big| \leq \frac{\alpha\ell^{\alpha-1}}{\Gamma(\alpha-\beta+1)} \|\varphi_n - \varphi\|_{\infty}.$$

In another way, we have

$$\big|\mathcal{I}_{0^+}^{\alpha-\beta}(\varphi_n(\eta) - \varphi(\eta))\big| \leq \frac{\ell^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \|\varphi_n - \varphi\|_{\infty}.$$

Then we get

$$\begin{aligned}\|\varphi_n-\varphi\|_\infty &\le (|p|+\omega_1)\,\|g_n-g\|_\infty+\frac{\alpha\ell^{\beta-1}(|q|+\omega_2)+\omega_3}{\ell^{\beta-\alpha}\Gamma(\alpha-\beta+1)}\,\|\varphi_n-\varphi\|_\infty\\ &\le (|p|+\omega_1)\,\|g_n-g\|_\infty+\lambda\|\varphi_n-\varphi\|_\infty.\end{aligned}$$

As $\lambda\in(0,1)$, thus we have

$$\|\varphi_n-\varphi\|_\infty\le\frac{|p|+\omega_1}{1-\lambda}\,\|g_n-g\|_\infty.$$

Since $g_n\to g$, we get $\varphi_n\to\varphi$ when $n\to\infty$.

Now, let $\mu>0$ be such that for each $\eta\in[0,\ell]$, we get $|\varphi_n(\eta)|\le\mu$, $|\varphi(\eta)|\le\mu$. Then, we have

$$\frac{(\eta-\xi)^{\alpha-1}}{\Gamma(\alpha)}\,|\varphi_n(\eta)-\varphi(\eta)|\le\frac{(\eta-\xi)^{\alpha-1}}{\Gamma(\alpha)}\,\big[\,|\varphi_n(\eta)|+|\varphi(\eta)|\,\big]\le\frac{2\mu}{\Gamma(\alpha)}\,(\eta-\xi)^{\alpha-1}.$$

The function $\xi\to\frac{2\mu}{\Gamma(\alpha)}(\eta-\xi)^{\alpha-1}$ is integrable on $[0,\eta]$, $\forall\,\eta\in[0,\ell]$; thus, what the dominated convergence theorem of Lebesgue implies is

$$|\mathcal{A}g_n(\eta)-\mathcal{A}g(\eta)|\to 0\ \text{ as }\ n\to\infty,$$

and hence

$$\lim_{n\to\infty}\|\mathcal{A}g_n-\mathcal{A}g\|_\infty=0.$$

This indicates the continuity of $\mathcal{A}$.

Step 2: Using (3.6), we put the positive real

$$r\ge\Big(|u_0|+\frac{a^*\ell^\alpha}{(1-\lambda)\Gamma(\alpha+1)}\Big)\,\frac{(1-\lambda)\Gamma(\alpha+1)}{(1-\lambda)\Gamma(\alpha+1)-\ell^\alpha(|p|+b^*)}\,,$$

and define the subset H as follows: $H=\big\{g\in C([0,\ell],\mathbb{C}):\ \|g\|_\infty\le r\big\}$. It is clear that H is bounded, closed and convex subset of $C([0,\ell],\mathbb{C})$.

Let $\mathcal{A}:H\to C([0,\ell],\mathbb{C})$ be the integral operator defined by (3.7), then $\mathcal{A}(H)\subset H$.

Indeed, for each $\eta\in[0,\ell]$ we have

$$|\varphi(\eta)|=\big|pg(\eta)+\psi(\eta,\varphi(\eta))\big|\le a^*+(|p|+b^*)\,|g(\eta)|+\lambda\|\varphi\|_\infty.$$

Then we get

$$\|\varphi\|_\infty\le\frac{a^*+(|p|+b^*)r}{1-\lambda}\,.$$

Thus

$$\begin{aligned}|\mathcal{A}g(\eta)| &\le |u_0|+\frac{1}{\Gamma(\alpha)}\int_0^\eta(\eta-\xi)^{\alpha-1}|\varphi(\xi)|\,d\xi\\ &\le\frac{\big(|u_0|+\frac{a^*\ell^\alpha}{(1-\lambda)\Gamma(\alpha+1)}\big)\,\frac{(1-\lambda)\Gamma(\alpha+1)}{(1-\lambda)\Gamma(\alpha+1)-\ell^\alpha(|p|+b^*)}}{\frac{(1-\lambda)\Gamma(\alpha+1)}{(1-\lambda)\Gamma(\alpha+1)-\ell^\alpha(|p|+b^*)}}+\frac{\ell^\alpha(|p|+b^*)r}{(1-\lambda)\Gamma(\alpha+1)}\le r.\end{aligned}$$

Then $\mathcal{A}(H)\subset H$.

Step 3: $\mathcal{A}(H)$ is equicontinuous.

Let $\eta_1, \eta_2 \in [0,\ell]$, $\eta_1 < \eta_2$, and $g \in H$. Then

$$
\begin{aligned}
|\mathcal{A}g(\eta_2) - \mathcal{A}g(\eta_1)| &= \frac{1}{\Gamma(\alpha)} \left| \int_0^{\eta_2} (\eta_2-\xi)^{\alpha-1}\varphi(\xi)\,d\xi - \int_0^{\eta_1} (\eta_1-\xi)^{\alpha-1}\varphi(\xi)\,d\xi \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^{\eta_1} \left|((\eta_2-\xi)^{\alpha-1} - (\eta_1-\xi)^{\alpha-1})\varphi(\xi)\right| d\xi + \frac{1}{\Gamma(\alpha)} \int_{\eta_1}^{\eta_2} (\eta_2-\xi)^{\alpha-1}|\varphi(\xi)|\,d\xi \\
&\leq \frac{a^* + (|p|+b^*)r}{\Gamma(\alpha)(1-\lambda)} \left[\int_0^{\eta_1} |(\eta_2-\xi)^{\alpha-1} - (\eta_1-\xi)^{\alpha-1}|\,d\xi + \int_{\eta_1}^{\eta_2} (\eta_2-\xi)^{\alpha-1}\,d\xi \right]. \qquad (3.8)
\end{aligned}
$$

We have

$$(\eta_2-\xi)^{\alpha-1} - (\eta_1-\xi)^{\alpha-1} = -\frac{1}{\alpha}\frac{d}{d\xi}\left[(\eta_2-\xi)^{\alpha} - (\eta_1-\xi)^{\alpha}\right],$$

then

$$\int_0^{\eta_1} \left|(\eta_2-\xi)^{\alpha-1} - (\eta_1-\xi)^{\alpha-1}\right| d\xi \leq \frac{1}{\alpha}\left[(\eta_2-\eta_1)^{\alpha} + (\eta_2^{\alpha} - \eta_1^{\alpha})\right],$$

we also have

$$\int_{\eta_1}^{\eta_2} (\eta_2-\xi)^{\alpha-1}\,d\xi = -\frac{1}{\alpha}\left[(\eta_2-\xi)^{\alpha}\right]_{\eta_1}^{\eta_2} \leq \frac{1}{\alpha}(\eta_2-\eta_1)^{\alpha}.$$

Thus (3.8) gives

$$|\mathcal{A}g(\eta_2) - \mathcal{A}g(\eta_1)| \leq \frac{2(\eta_2-\eta_1)^{\alpha} + (\eta_2^{\alpha} - \eta_1^{\alpha})}{\Gamma(\alpha+1)(1-\lambda)}\left(a^* + (|p|+b^*)r\right).$$

The right-hand side of the latter inequality tends to zero when $\eta_1 \to \eta_2$.

As a consequence of steps 1 to 3, and through the Ascoli–Arzelà theorem, we infer the continuity of $\mathcal{A} : H \to H$, its compact nature and its satisfaction of the assumption of Schauder's fixed point theorem [21]. Therefore, $\mathcal{A}$ has a fixed point which solves problem (3.1), (3.2) on $[0,\ell]$. □

Theorem 3.3. *Assume the hypothesis (Hyp. 2) holds. If we put $\lambda \in (0,1)$ and*

$$\ell < \left(\frac{\Gamma(\alpha+1)(1-\lambda)}{|p| + \omega_1}\right)^{\frac{1}{\alpha}}, \qquad (3.9)$$

then problem (3.1), (3.2) admits a unique solution on $[0,\ell]$.

Proof. Theorem 3.2 states that (3.1), (3.2) can be rendered a problem of a fixed point (3.7).

Let $g_1, g_2 \in C([0,\ell],\mathbb{C})$, then we get

$$\mathcal{A}g_1(\eta) - \mathcal{A}g_2(\eta) = \frac{1}{\Gamma(\alpha)} \int_0^{\eta} (\eta-\xi)^{\alpha-1}(\varphi_1(\xi) - \varphi_2(\xi))\,d\xi,$$

where $\varphi_i \in C([0,\ell],\mathbb{C})$ are such that

$$
\begin{aligned}
\varphi_i(\eta) &= p(c_0 + \mathcal{I}_{0+}^{\alpha}\varphi_i(\eta)) + \psi(\eta,\varphi_i(\eta)) \quad \text{for} \quad i = 1,2, \\
\psi(\eta,\varphi_i(\eta)) &= q\eta\mathcal{I}_{0+}^{\alpha-1}\varphi_i(\eta) + J\Big(\eta, c_0 + \mathcal{I}_{0+}^{\alpha}\varphi_i(\eta), \mathcal{I}_{0+}^{\alpha-1}\varphi_i(\eta), \mathcal{I}_{0+}^{\alpha-\beta}\varphi_i(\eta)\Big).
\end{aligned}
$$

Also,

$$|\mathcal{A}g_1(\eta) - \mathcal{A}g_2(\eta)| \leq \frac{1}{\Gamma(\alpha)} \int_0^{\eta} (\eta-\xi)^{\alpha-1}|\varphi_1(\xi) - \varphi_2(\xi)|\,d\xi. \qquad (3.10)$$

We have

$$\|\varphi_1-\varphi_2\|_\infty \le \frac{|p|+\omega_1}{1-\lambda}\,\|g_1-g_2\|_\infty.$$

From (3.10) we find

$$\|\mathcal{A}g_1-\mathcal{A}g_2\|_\infty \le \frac{\ell^\alpha(|p|+\omega_1)}{\Gamma(\alpha+1)(1-\lambda)}\,\|g_1-g_2\|_\infty.$$

Thus, according to (3.9), $\mathcal{A}$ is considered as a contraction operator.

The Banach contraction principle (see [21]) helps us to infer that $\mathcal{A}$ has only one fixed point which is the unique solution of problem (3.1), (3.2) on $[0,\ell]$. □

4 Proofs of main theorems and illustrative examples

This section demonstrates the proof of the existence and uniqueness of solutions of the given problem for a multidimensional nonlinear time and space-fractional reaction-diffusion/wave equation, which is

$$\begin{cases}\partial_t^\alpha u-\kappa^2\Delta u = F(t,x,u,\partial_t^\beta u,(-\Delta)^s u), & (t,x)\in\Omega,\quad \kappa\in\mathbb{R}^*,\\ u(0,x)=|x|^\delta u_0,\quad \dfrac{\partial u}{\partial t}(0,x)=0, & \delta,u_0\in\mathbb{C},\end{cases}\tag{4.1}$$

under the radially symmetric form

$$u(t,x)=|x|^\delta f(\eta),\quad \text{with}\quad \eta=|x|^{-\frac{2}{\alpha}}t.\tag{4.2}$$

Proof of Theorem 1.1. Assume that the hypotheses ($Hyp.$ 1)–($Hyp.$ 3) hold. Given Theorem 3.1, using transformation (4.2), problem (4.1) is reduced to the fractional order ordinary differential equation of the form

$${}^C\mathcal{D}^\alpha_{0+}f(\eta)=\varphi(\eta),\tag{4.3}$$

where

$$\varphi(\eta)=pf(\eta)+q\eta f'(\eta)+J\big(\eta,f(\eta),f'(\eta),{}^C\mathcal{D}^\beta_{0+}f(\eta)\big)$$

with

$$p=\delta\kappa^2(\delta+m-2)\quad\text{and}\quad q=-\frac{2\kappa^2}{\alpha^2}\big(\alpha(2\delta+m+2)+2\big),\tag{4.4}$$

along with the conditions

$$f(0)=u_0\quad\text{and}\quad f'(0)=0.\tag{4.5}$$

By using (4.4), condition (1.5) is equivalent to (3.6), which is

$$\frac{\ell^\alpha(|p|+b^*)}{\Gamma(\alpha+1)(1-\lambda)}<1\quad\text{with}\quad\lambda\in(0,1).$$

Therefore, after proving that problem (4.3), (4.5) has a solution as in Theorem 3.2 when (3.6) holds, we can similarly prove the existence of at least a solution of the problem for the multidimensional nonlinear time and space-fractional reaction-diffusion/wave equation (4.1) under the radially symmetric form (4.2). This can be achieved if (1.5) holds. □

Example 4.1. If we choose $s=1$, $\beta=\frac{3}{2}$, $\alpha=\frac{7}{4}$, $\delta=1$, $m=2$, $\varepsilon=1$, $\kappa=\sqrt{\frac{7}{96}}$ and $\ell=\frac{6}{25}$, we obtain $\Omega=[0,\frac{6}{25}]\times[\frac{1}{\sqrt{2}},+\infty)^2$. Consequently, the considered problem will be stated as follows:

$$\begin{cases}\partial_t^{\frac{7}{4}}u-\dfrac{7}{96}\,\Delta u=F(t,x,u,\partial_t^{\frac{3}{2}}u,\Delta u), & (x,y)\in\Omega,\\ u(0,x,y)=\sqrt{x^2+y^2},\quad \dfrac{\partial u}{\partial t}(0,x,y)=0,\end{cases}\tag{4.6}$$

where

$$F(t,x,u,\partial_t^{\frac{3}{2}}u,\Delta u) = \frac{|x|^{-1}\exp(-|x|^{-\frac{8}{7}}t)[2|x|+|u|+|x|^2|\partial_t^{\frac{3}{2}}u|]}{(|x|^{-\frac{8}{7}}t+2\ln(|x|^{-\frac{8}{7}}t+e))[|x|+|u|+|x|^2|\partial_t^{\frac{3}{2}}u|]} - \frac{7}{96}\,\Delta u$$
$$= |x|^{-1}\Big[J\big(\eta,f,f',{}^{C}\mathcal{D}_{0^+}^{\frac{3}{2}}f(\eta)\big) - \frac{2}{21}\,\eta^2 f''(\eta)\Big]$$

with $\eta \in [0, \frac{6}{25}]$ and

$$J(\eta,f,g,h) = \frac{\exp(-\eta)[2+|f|+|h|]}{(\eta+2\ln(\eta+e))[1+|f|+|h|]} - \frac{7}{96}\,f + \frac{25}{42}\,\eta g.$$

Clearly, the function J is jointly continuous. For any $f,g,h,\widetilde{f},\widetilde{g},\widetilde{h} \in \mathbb{C}$ and $\eta \in [0, \frac{6}{25}]$, we get

$$\big|J(\eta,f,g,h) - J(\eta,\widetilde{f},\widetilde{g},\widetilde{h})\big| \le \frac{55}{96}\,|f-\widetilde{f}| + \frac{1}{7}\,|g-\widetilde{g}| + \frac{1}{2}\,|h-\widetilde{h}|.$$

Therefore, the hypothesis ($Hyp.$ 2) is satisfied with

$$\omega_1 = \frac{55}{96}\,,\quad \omega_2 = \frac{1}{7}\ \text{ and }\ \omega_3 = \frac{1}{2}\,.$$

Also, we have

$$|J(\eta,f,g,h)| \le \frac{\exp(-\eta)}{\eta+2\ln(\eta+e)}\,\big(2+|f|+|h|\big) + \frac{7}{96}\,|f| + \frac{25}{42}\,\eta|g|.$$

Thus, the hypothesis ($Hyp.$ 3) is satisfied with

$$a(\eta) = \frac{2\exp(-\eta)}{\eta+2\ln(\eta+e)}\,,\quad b(\eta) = \frac{\exp(-\eta)}{\eta+2\ln(\eta+e)} + \frac{7}{96}\,,\quad c(\eta) = \frac{25}{42}\,\eta\,,\quad d(\eta) = \frac{\exp(-\eta)}{\eta+2\ln(\eta+e)}\,.$$

Then

$$a^* = 1,\ \ b^* = \frac{55}{96}\,,\ \ c^* = \frac{1}{7}\,,\ \ d^* = \frac{1}{2}\,,$$

and

$$\lambda = \sup\left\{\frac{\alpha\ell^{\beta-1}(|q|+c^*)+d^*}{\ell^{\beta-\alpha}\Gamma(\alpha-\beta+1)}, \frac{\alpha\ell^{\beta-1}(|q|+\omega_2)+\omega_3}{\ell^{\beta-\alpha}\Gamma(\alpha-\beta+1)}\right\} \simeq 0.87474 < 1.$$

Condition (1.5) gives

$$\frac{T_\varepsilon^\alpha(|\delta\kappa^2(\delta+m-2)+b^*)}{\Gamma(\alpha+1)(1-\lambda)} \simeq 0.26381 < \varepsilon^2 = 1.$$

It follows from Theorem 1.1 that problem (4.6) has at least one solution on Ω.

Proof of Theorem 1.2. Similarly to the steps that we followed during the proof of Theorem 1.1, the existence and uniqueness of a radically symmetric solution to problem (4.1) is demonstrated by using Theorem 3.3, provided that condition (1.6) holds true. The proof is complete. □

Example 4.2. If we put $s=1$, $\beta = \frac{5}{4}$, $\alpha = \frac{3}{2}$, $\delta = 2$, $m = 4$, $\varepsilon = \sqrt[\frac{4}{3}]{\frac{1}{\pi}}$, $\kappa = -\sqrt{\frac{9}{272}}$ and $\ell = \frac{\pi}{8}$, we get $\Omega = [0,\frac{1}{8}] \times [\frac{1}{2}\sqrt[\frac{4}{3}]{\frac{1}{\pi}}, \infty)^4$. Thus, the studied problem will be written as follows:

$$\begin{cases} \partial_t^{\frac{3}{2}}u - \dfrac{9}{272}\,\Delta u = F(t,x,u,\partial_t^{\frac{5}{4}}u,\Delta u), & (t,x_1,\dots,x_4) \in \Omega, \\ u(0,x_1,\dots,x_4) = 2(x_1^2+\dots+x_4^2), & \dfrac{\partial u}{\partial t}(0,x_1,\dots,x_4) = 0, \end{cases} \tag{4.7}$$

where

$$F(t,x,u,\partial_t^{\frac{5}{4}}u,\Delta u) = \frac{\pi|x|^2\cos(|x|^{-\frac{4}{3}}t)}{(4\pi^2+\tan(|x|^{-\frac{4}{3}}t))[|x|^2+|u|+|x|^2|\partial_t^{\frac{5}{4}}u|]} - \frac{9}{272}\,\Delta u$$
$$= J\big(\eta,f,f',{}^{C}\mathcal{D}_{0^+}^{\frac{5}{4}}f(\eta)\big) - \frac{1}{17}\,\eta^2 f''(\eta),$$

with $\eta \in [0, \frac{\pi}{8}]$ and

$$J(\eta, f, g, h) = \frac{\pi\cos(\eta)}{(4\pi^2 + \tan(\eta))[1 + |f| + |h|]} - \frac{9}{34} f + \frac{1}{2}\eta g.$$

As $\tan(\eta)$, $\cos(\eta)$ are positive continuous functions for $\eta \in [0, \frac{\pi}{8}]$, the function f is jointly continuous. For any $f, g, h, \widetilde{f}, \widetilde{g}, \widetilde{h} \in \mathbb{C}$ and $\eta \in [0, \frac{\pi}{8}]$, we have $\frac{1}{2}(\sqrt{2}+2)^{\frac{1}{2}} \leq \cos(\eta) \leq 1$, and $0 \leq \tan(\eta) \leq \sqrt{2} - 1$, also

$$\big|J(\eta, f, g, h) - J(\eta, \widetilde{f}, \widetilde{g}, \widetilde{h})\big| \leq \Big(\frac{9}{34} + \frac{1}{4\pi}\Big)|f - \widetilde{f}| + \frac{\pi}{16}|g - \widetilde{g}| + \frac{1}{4\pi}|h - \widetilde{h}|.$$

Hence the hypothesis $(Hyp.\ 2)$ is satisfied with

$$\omega_1 = \frac{9}{34} + \frac{1}{4\pi}, \quad \omega_2 = \frac{\pi}{16}, \quad \omega_3 = \frac{1}{4\pi}$$

and

$$\lambda = \frac{\alpha\ell^{\beta-1}(|q| + \omega_2) + \omega_3}{\ell^{\beta-\alpha}\Gamma(\alpha - \beta + 1)} \simeq 0.79165 < 1.$$

It remains to show that condition (1.6) is satisfied. Indeed,

$$T_\varepsilon = \frac{1}{8} < \varepsilon^{\frac{2}{\alpha}}\Big(\frac{\Gamma(\alpha+1)(1-\lambda)}{|\delta\kappa^2(\delta + m - 2)| + \omega_1}\Big)^{\frac{1}{\alpha}} \simeq 0.18825.$$

It follows from Theorem 1.2 that problem (4.7) has a unique solution on Ω.

5 Explicit solutions

Now, we present some explicit solutions of the radially symmetric form of problem (4.1).

Solution 5.1. Let $p, q, \gamma \in \mathbb{C}$ for $s = 1$ and $1 < \beta \leq \alpha \leq 2$, we get that

$$f(\eta) = \eta^\gamma \ \text{ with } \ \mathrm{Re}(\gamma) > 1$$

is a solution of (4.3), 4.5, where

$$J\big(\eta, f(\eta), f'(\eta), {}^C\mathcal{D}^\beta_{0^+} f(\eta)\big) = \frac{\eta^{\beta-\alpha}\Gamma(\gamma - \beta + 1)}{\Gamma(\gamma - \alpha + 1)}\, {}^C\mathcal{D}^\beta_{0^+} f(\eta) - pf(\eta) - q\eta f'(\eta).$$

Then the radially symmetric solution of problem (4.1) is presented as follows:

$$u(t, x) = |x|^{\delta - \frac{2\gamma}{\alpha}} t^\gamma,$$

where

$$F\big(t, x, u, \partial^\beta_t u, (-\Delta)^s u\big) = \frac{\Gamma(\gamma - \beta + 1)u(t, x)}{t^{\alpha-\beta+\gamma}\Gamma(\gamma - \alpha + 1)}\, |x|^{\frac{2\gamma}{\alpha} - \delta}\partial^\beta_t u(t, x) - \kappa^2\Delta u(t, x).$$

Solution 5.2. Let $p, q, \gamma \in \mathbb{C}$ for $s = 1$ and $1 < \beta \leq \alpha \leq 2$, we have

$$f(\eta) = \exp(\gamma\eta) - \gamma\eta,$$

which is a solution of (4.3), (4.5), where

$$J\big(\eta, f(\eta), f'(\eta), {}^C\mathcal{D}^\beta_{0^+} f(\eta)\big) = \frac{\eta^{\beta-\alpha}E_{1,3-\alpha}(\gamma\eta)}{E_{1,3-\beta}(\gamma\eta)}\, {}^C\mathcal{D}^\beta_{0^+} f(\eta) - pf(\eta) - q\eta f'(\eta).$$

Here, $E_{\alpha,\beta}(\eta)$ is the Mittag–Leffler function. Then the solution of problem (4.1) is presented as follows:

$$u(t, x) = |x|^\delta\big(e^{\gamma|x|^{-\frac{2}{\alpha}} t} - \gamma|x|^{-\frac{2}{\alpha}} t\big),$$

where

$$F\big(t, x, u, \partial^\beta_t u, (-\Delta)^s u\big) = \frac{|x|^{-\delta} t^{\beta-\alpha} E_{1,3-\alpha}(\gamma|x|^{-\frac{2}{\alpha}} t) u(t, x)}{(e^{\gamma|x|^{-\frac{2}{\alpha}} t} - \gamma|x|^{-\frac{2}{\alpha}} t) E_{1,3-\beta}(\gamma|x|^{-\frac{2}{\alpha}} t)}\, \partial^\beta_t u(t, x) - \kappa^2\Delta u(t, x).$$

Solution 5.3. Let $p, q, \gamma \in \mathbb{C}$ for $s = 1$ and $1 < \beta \leq \alpha \leq 2$, we get that

$$f(\eta) = \sin(\gamma\eta) + \cos(\gamma\eta) - \gamma\eta$$

is a solution of problem (4.3), (4.5), where

$$\begin{aligned} J\big(\eta, f(\eta), f'(\eta), {}^C\mathcal{D}_{0^+}^{\beta} f(\eta)\big) \\ = \frac{\eta^{\beta-\alpha}[(i-1)E_{1,3-\alpha}(i\gamma\eta) - (1+i)E_{1,3-\alpha}(-i\gamma\eta)]}{(i-1)E_{1,3-\beta}(i\gamma\eta) - (1+i)E_{1,3-\beta}(-i\gamma\eta)} {}^C\mathcal{D}_{0^+}^{\beta} f(\eta) - pf(\eta) - q\eta f'(\eta). \end{aligned}$$

Then the solution of problem (4.1) is presented as follows:

$$u(t,x) = |x|^{\delta}\Big(\sin(\gamma|x|^{-\frac{2}{\alpha}}t) + \cos(\gamma|x|^{-\frac{2}{\alpha}}t) - \gamma|x|^{-\frac{2}{\alpha}}t\Big),$$

where

$$\begin{aligned} F(t,x,u,\partial_t^{\beta}u,(-\Delta)^s u) = -\kappa^2\Delta u(t,x) + \frac{|x|^{-\delta}t^{\beta-\alpha}u(t,x)\partial_t^{\beta}u(t,x)}{(\sin(\gamma|x|^{-\frac{2}{\alpha}}t) + \cos(\gamma|x|^{-\frac{2}{\alpha}}t) - \gamma|x|^{-\frac{2}{\alpha}}t)} \\ \times \frac{(i-1)E_{1,3-\alpha}(i\gamma|x|^{-\frac{2}{\alpha}}t) - (1+i)E_{1,3-\alpha}(-i\gamma|x|^{-\frac{2}{\alpha}}t)}{(i-1)E_{1,3-\beta}(i\gamma|x|^{-\frac{2}{\alpha}}t) - (1+i)E_{1,3-\beta}(-i\gamma|x|^{-\frac{2}{\alpha}}t)}. \end{aligned}$$

6 Conclusion

Using Schauder's fixed point theorem and Banach contraction principle, this paper explored the main properties and the existence of at least a radially symmetric solution and its uniqueness for a class of multidimensional nonlinear time and space-fractional reaction-diffusion/wave equation with mixed conditions, while Caputo's fractional derivative was used as the differential operator. The behavior of radially symmetric solutions for the mentioned equation enables treating several physical phenomena.

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