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Theme

Burgers equation and it's applications

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Dedication

Allah Thank you for surrounding us with tenderness and love, may god the almighty keep you by our side so that you guide our steps affectionately.

I dedicate this thesis to:

My father "Acheur", you have imposed real sacrifices on yourself for our happiness, this work is the fruit of the education that you have cultivated. Today I have the opportunity to express all my affection to you and say thank you.

My mother "Djamila" I say thank you, maybe not in big words,but for those who have just come from deep. Thank you for surrounding us with tenderness and love, may god the almighty keep you by our side so that you guide our steps affectionately. .

Thanks to my director PF N.benhamidouch.

I dedicate this work and say thanks to all my true friends and I wish them a lot of success in the process of life.

Thanks

I thank my almighty god for giving me the strength to survive, as well as the audacity to overcome all difficulties.

I would like to start by thanking Mr.N.Ben hamidouche for his daily supervision and having smiled at this work and guiding me, his advice has always been valuable and our discussions important to evolve this project.His remarks on the manuscript were fundamental to make it more clear and understandable.

Thanks to my friends,their friendship,their moral support.

for the end with a flourish, finally, I thank my family.They will always support me. They are very proud of me and they push me to always do better. I can tell you that I have no intention of stopping. I will always try to improve, I express all my gratitude to my family for having supported me throughout my schooling.

Thanks all

Contents

Introduction

To understand the physical phenomena, we use in general, partial differential equations, the last are generally denoted in abbreviation PDEs.

In everyday life, the majority of mechanical engineers as well as in physics encounter problems related to the phenomenon of fluid dynamics and mass transport, and in particular, the biological phenomena are describe by PDEs models.

In general it is very difficult to solve these problems, but in some cases we can find analytical solution in particular forms.

In this work we shall study the Burgers equation with her different forms: the first type, the second type and the third type which called the forced Burgers equation.

These types of Burgers equation, are studied by many authors [20] ,[5],[3],[9], some class of methods are used by these authors:
-The reduction method, which transform the PDEs to EDOs, and finding a particular solutions, such as self similar solutions and travelling wave solutions.
- The Cole-Hopf transformation, is used particularly for all types of Burgers equations, which transform them to a linear equations and in particular the heat equation. Our objectif in this thesis, is to develop all these methods for resolving the three types of the Burgers equations.

This work is organized in three chapters:

The first chapter, we present some ideas about partial differential equations, heat equation, wave equation, definition of the self-similar solution, some notions about differential equations and reduction methods for PDEs. We also gave some biological models for the PDE.

In the second chapter, we give the definition of the Burgers equation and it types, Cole-Hopf transformation and we find by reduction methods ,the self similar solution, the travelling wave solution for the Burgers equation with the following form:

$$u_t + uu_x = 0. \tag{1}$$

We also find a self similar solution and travelling wave solution and a solution based on the Cole-Hopf transformation, for the Burgers equation for the second type with the form:

$$u_t + uu_x = u_{xx}. \tag{2}$$

In the third chapter, we introduce the forced Burgers equation in the form:

$$u_t - uu_x - u_{xx} = F_{x,t,u}, \tag{3}$$

we study some particular cases, when $F = C$, is constant, and $F = F_x$ which call stationary forcing, and finally, the case which $F = F_{x,t}$, which call transient forcing, and find an exact solution for it, using the Cole-Hopf transformation, and searching a travelling wave solution.

Chapter 1

Introduction to partial differential equations

1.1 General definitions

We first give some definitions on the partial differential equations.

1.1.1 Definition

Let u a variable (unknow) depends on n independent variables:

$$(x_1.....x_n) \in \mathbb{R}^n.$$

All relation between u, and $x_i, (i = 1....n)$ and the partial derivatives of u relative to x_i,

$$F\left(u, x_1....x_n, u_{x_1}, u_{x_2}...., u_{x_1 x_2}, u_{x_1 x_2}.....u_{x_1.....x_n}\right) = 0. \tag{1.1}$$

Constitutes an equation to partial differential (for short:PDE).

Definition

A PDE is said **linear** when it is compared to u and to all of it partial differential. If u is it partial differential appear separately in the PDE, than it is linear.

Example

The heat equation:

$$u_t - u_{xx} = 0.$$

Definition

The PDE is **non-linear**, means that the relation between the partial differential, is non-linear.

For example the Burgers equation :

$$u_t = uu_x.$$

1.2 The heat equation

In mathematics and theoretical physics, the heat equation is a parabolic partial differential equation, Originally introduced in 1811 by Fourier to describe the physical phenomenon of the heat conduction, and it is given by The following form:

$$u_t = cu_{xx} \quad x \in \mathbb{R} \quad t \succ 0, \tag{1.2}$$

where $c \succ 0$ is a given constant.

Here $u = u(x,t)$ is the temperature in a one-dimensional capacitor. The value of $u(x,t)$ is depends on the time $t \geq 0$, and the position x.

So we want to solve the following problem:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = c\dfrac{\partial^2 u}{\partial^2 x}(x,t), \quad x \in \mathbb{R} \\ \qquad\qquad u(x,0) = u_0, \quad x \in \mathbb{R}. \end{cases} \tag{1.3}$$

1.2.1 Fundamental solution of the heat equation

To find a function $u(x,t)$ satisfies problem (1.8), we apply the exponential Fourier transform.

We define :

$$U(t,\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(x,t)e^{-i\xi x}dx \quad \xi \in \mathbb{R}, \tag{1.4}$$

Step 1:(Transformation) After replacing (1.9) in the Eq (1.8), we get the following ordinary differential equation :

$$\begin{cases} U_t(t) + |\xi^2|\, U(t) = 0 \quad t \succ 0. \\ U_0 = F[u_0], \end{cases} \tag{1.5}$$

Step 2 :(solving the transformed problem) The solution for Eq.(1.10) is:

$$U(t,\xi) = F[u_0](\xi)e^{-|\xi|^2 t}.$$

Step 3: (Finding the inverse transform)
We have,

$$u(x,t) = F^{-1}[U(t,\xi)] = F^{-1}\left[F[u_0](\xi)e^{-|\xi|^2 t}\right],$$

than

$$u(x,t) = F^{-1}\left[F[u_0](\xi)\right] * F^{-1}\left[e^{-|\xi|^2 t}\right],$$

$$u(x,t) = u_0(x) * \left[\frac{1}{\sqrt{2t}} e^{-|x|^2/(4t)}\right],$$

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-|x-y|^2/(4t)} u_0(y)dy,$$

so the
Fundamental solution for the heat equation is written as:

$$\begin{cases} \phi(x,t) = \dfrac{1}{\sqrt{4\pi t}} \displaystyle\int_{\mathbb{R}} e^{-|x|^2/(4t)}. & x \in \mathbb{R}, \quad t \succ 0, \\ \qquad 0 & x \in \mathbb{R}, \quad t \prec 0, \end{cases} \tag{1.6}$$

1.3 The wave equation

The wave equation[3] is a second-order partial differential equation, used to describe the waves that happens in the classical physics. The form of the wave equation is:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \tag{1.7}$$

where c is a fixed non-negative real coefficient.

1.3.1 Travelling wave solution

Travelling wave[4] is a temporary wave, that creates a disturbance and moves along the transmission line at a constant speed. It is occurs for a short duration. Such waves are observed in fluid dynamics, solid mechanics we get it once we solve a PDEs.

These travelling waves solutions are presented as:

$$u(x,t) = v(\xi),$$

and

$$\xi = x - ct,$$

when the travelling wave approaches a constant state,

$$v(-\infty) = u_l,$$

and

$$v(\infty) = u_r,$$

with

$$u_l \neq u_r,$$

we call it a wave front. which mean we have the following transformations:

$$u_t = -cv',$$

and

$$u_x = v',$$

and

$$u_{xx} = v''.$$

1.3.2 Solution to the wave equation

Let consider the following Cauchy problem for the wave equation:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \\ u(x,0) = f(x), \\ \frac{\partial u}{\partial t}(x,0) = g(x). \end{cases} \tag{1.8}$$

The solution of the problem (1.13), is written as:

$$u(x,t) = \phi(\epsilon) + \varphi(\mu),$$

where

$$\epsilon = x + ct,$$

and

$$\mu = x - ct,$$

so,

$$u(x,0) = \phi(x) + \varphi(x) = f(x),$$

and
$$\frac{\partial u}{\partial t}(x,0) = c\phi'(x) - c\varphi'(x) = g(x),$$
integrating the last equation we get
$$c\phi(x) - c\varphi(x) = \int_{\mathbb{R}} g(x),$$
now we have these system:
$$\begin{cases} \phi(x) + \varphi(x) = f(x), \\ c\phi(x) - c\varphi(x) = \int g(x). \end{cases}$$
After resolving the system, we find
$$\phi(x) = \frac{cf(x) + \int g(x)}{2c},$$
$$\varphi(x) = \frac{-cf(x) + \int g(x)}{-2c},$$
than
$$u(\epsilon,\mu) = \frac{-2cf(\epsilon) - 2\int g(\mu)}{-2c},$$
since $\epsilon = x + ct$, and $\mu = x - ct$,.

Finally
$$u(x,t) = 1/2\left[f(x+ct) + f(x-ct)\right] + 1/2c\int_{x-ct}^{x+ct} g(s)ds,$$

1.4 Some reminders about differential equations

1.4.1 The Riccati equations

They are equations in the form:
$$y' = a(t)y^2 + b(t)y + c(t), \;\; t \in \mathbb{R}. \tag{1.9}$$
The general solution of eq (1.14), is
$$y = y_1(x) + \frac{1}{z(x)},$$
where y_1 is a particular solution to the eq.(1.14), which mean that
$$\frac{dy}{dx} = \frac{dy_1}{dx} - \frac{1}{z^2}\frac{dz}{dx},$$
replacing this equation in the Riccati equation, we get a linear differential equation for k. Ones we find k, we have the solution to the Riccati equation.

1.4.2 The Bernoulli equations

They are equations in the form:

$$y' = p(t)y + q(t)y^{\alpha}, \quad t \in \mathbb{R}, \tag{1.10}$$

where α is a real different from 1, to solve the Eq.(1.15)

first we divide the differential equation on y^{α} we get,

$$y'y^{-\alpha} = p(t)y^{1-\alpha} + q(x), \tag{1.11}$$

then we put

$$z = y^{1-\alpha},$$

which mean that

$$z' = (1-\alpha)y^{-\alpha}y'.$$

Now matching this equations with (1.16), we obtain:

$$\frac{1}{1-n}z' + p(x)z = q(x).$$

This is a linear differential equation that we can solve for z, then we can get the

solution to the original differential equation (1.16).

1.5 Exact solution for PDE

In order to find a solutions to the PDEs, we have some methods for solving them.

1.5.1 Some methods for resolving a PDE

To reduce a partial differential equation, to an ordinary differential equation ($PDE \hookrightarrow ODE$).

There are several methods , which are:

-The classical method [5] "Lie group of infinitesimal transformation".

-The direct method of Clarkson and Kruskal.

- The Cole-Hopf transformation.

We have also the self similar solutions and the travelling waves solutions.

1.5.2 The direct method

In 1989 Clarakson and Kruskal have developed the direct method [5], to obtain previously unknown reductions of the Burgers equation, and also they have found the similarity reduction of several non-linear partial differential equations.

The Principe of the method

It is enough to look for a solution of the general form:

$$u(x,t) = \alpha(x,t) + \beta(x,t)\omega(z), \;\; x \in \mathbb{R}, \;\; t \succ 0,$$

where α, β and $z = z(x,t)$ are functions to be determined subsequently, it is enough to replace the solution of this form in the equation to be solved.

To transform the obtained differential equation into an ordinary differential equation (ODE),

we take coefficients so that each coefficient is the product of $\prod_i(z)$,

$i = 1, ..., n$, with the coefficients of the derivative function (of the similarity function $\omega(z)$).

The result obtained (the product) will be compared with the rest of the coefficients of the previous differential equation.

The comparison generates a set of partial differential equations that must be solved.

To solve this system ,special cases are used which finally gives **the similar solution** .

1.5.3 Self-similar solution

A solution for an PDE is said self-similar[1], if it is invariant by scale, which mean

we apply this change of variables:

$$u = a^{\lambda}u \ \ , x = a^{s}x, \ \ t = a^{\gamma}t,$$

so:

$$u_t = \frac{\partial \left(a^{\lambda}u\right)}{\partial \left(a^{\gamma}t\right)} = a^{\lambda-\gamma}\frac{\partial u}{\partial t},$$

$$u_{xx} = \frac{\partial^2 (a^{\lambda u})}{\partial \left(a^{s}x\right)^2} = a^{\lambda-2s}\frac{\partial^2 u}{\partial x^2},$$

we get

$$a^{\lambda\gamma}\frac{\partial u}{\partial t} = a^{\lambda-2s}\frac{\partial^2 u}{\partial x^2},$$

so we have

$$a^{\lambda-\gamma} = a^{\lambda-2s},$$

the similarity condition is:

$$\lambda - \gamma = \lambda - 2s,$$

imply

$\gamma = 2s$ if we pose $a^{\gamma} = 1 \Rightarrow a = t^{\frac{1}{\gamma}}$,

imply

$$u(x,t) \Rightarrow a^{\lambda}u\left(a^{s}x, a^{\gamma}t\right) \Rightarrow t^{\frac{-1}{\gamma}}u\left(a^{s}x, 1\right) = t^{\frac{-\lambda}{\gamma}}\phi(t^{\frac{-s}{\lambda}}x),$$

and because we have

$$\gamma = 2s,$$

we obtain the form

$$t^{\frac{-\lambda}{\gamma}}\phi(t^{\frac{-1}{2}}x),$$

this give us a particular form of self similar solution witch written as:

$$u(x,t) = t^{\alpha}\phi(\xi), \xi = xt^{-\beta},$$

1.5.4 Example

We will apply self similarity on the heat equation(1.8)

Let us search for a solution with the form:

$$u(x,t) = t^{\alpha}\phi(\xi), \ \ \xi = xt^{-\beta},$$

so :

$$\frac{\partial u}{\partial t} = \alpha t^{\alpha-1}\phi - \beta\xi t^{\alpha-1}\phi_{\xi},$$

$$\frac{\partial^2 u}{\partial x^2} = t^{\alpha-2\beta}\phi_{\xi\xi},$$

we get :

$$\alpha t^{\alpha-1}\phi - \beta\xi t^{\alpha-1}\phi_{\xi} = t^{\alpha-2\beta}\phi_{\xi\xi},$$

we divides by $t^{\alpha-1}$ we get:

$$\alpha\phi - \beta\xi\phi_{\xi} = t^{1-2\beta}\phi_{\xi\xi},$$

which imply

$$t^{1-2\beta} = 0,$$

means that $1 - 2\beta = 0$ so $\beta = 1/2$.

Then the differential equation corresponding is,

$$\alpha\phi - 1/2\xi\phi_{\xi} = \phi_{\xi\xi},$$

the goal is to find for each problem ϕ.

1.5.5 The Cole-Hopf transformation

In 50's century Hopf and Cole independently[22], showed that the Burgers equation is

equivalent to the linear diffusion equation, for a new variable $U(x,t)$,

$$U_t = U_{xx},$$

which is a heat equation, than we put the following transformation:

$$u(x,t) = \frac{U_x}{U}.$$

Generally the goal of Cole-Hopf transformation, is to change a non-linear differential equations into a linear differential equation.

We will see an application of this method in the next chapter.

1.6 Biological model of PDEs

Many PDEs and differential equations has application in different fields Mathematical models, these PDEs can help understanding the interactions inside and between cells, population dynamics, DNA molecules.

1.6.1 Exponential growth model

Exponential growth definition in math, is growth with a constant growth rate, such that the y-values are multiplied by a constant amount for any given change in x. When applied in biology, it is growth with constant growth rate.

Exponential growth can occur in biology if the population size is small,compared to the environment.

$$\frac{du}{dt} = cu, \tag{1.12}$$

where u is the number of cells, and c is the population growth rate. This model assumes that, the population rate of change, is proportional to the population size u.

By integration, we can obtain an analytic solution that describes the number of cells in the population, as a function of time t and growth rate,

$$u(t) = u_0 e^{ct}, \tag{1.13}$$

where u_0 is the initial number of cells in the population. A typical exponential growth curve is shown in figure. 1.1.

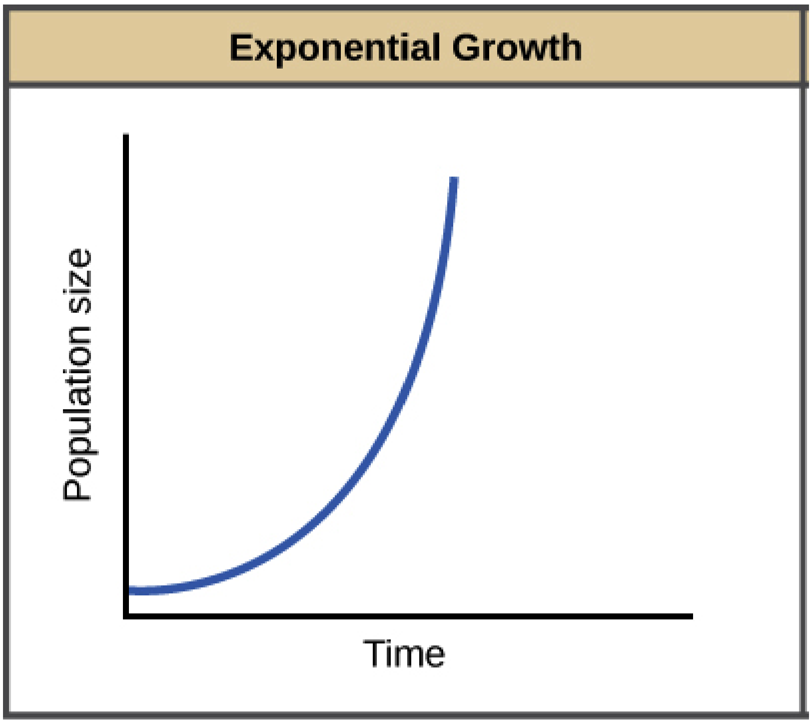

Figure 1.1

Chapter 2

The Burgers equations

2.1 Introduction

Burgers equation[12] or Bateman-equation is a model of partial differential equations non linear and convection diffusion equation, occurring in various areas of applied mathematics, such as fluid mechanics, gaz dynamics, and traffic flow. The equation was introduced by Harry Bateman in 1915, and later studied by Johannes martinus Burgers in 1948. In this chapter we will search some particular solutions for this equation.

2.1.1 Types of the Burgers equations

There are two types of Burgers equation.

-First type

It is equation which is written in this form

$$u_t = uu_x \quad x \in \mathbb{R} \;\; t \succ 0, \tag{2.1}$$

this equation is called also the non-linear conservation law.

-Second type

The equation is written in the following form:

$$u_t = uu_x + u_{xx} \qquad\qquad x \in \mathbb{R}, t \succ 0. \tag{2.2}$$

There exist also another type of Burgers equation, called the forced Burgers equation

which is written in this form:

$$u_t - uu_x - u_{xx} = F(x,t,u) \qquad x \in \mathbb{R},\ \ t \succ 0. \tag{2.3}$$

we will study it in the chapter 3.

In this chapter, we will see three methods to solve the Burgers equations of the

first type and second type.

Firstly, we search two solutions forms as follow:

1. The self similar solutions
2. Travelling wave solutions.

Secondly, we use the Cole-Hopf transformation to search a solutions which is related with

the heat equation.

2.2 Solutions to the first type of Burgers equation

We have the following equation,

$$u_t = uu_x \ \ x \in \mathbb{R},\ \ t \succ 0. \tag{2.4}$$

Firstly, we want to find a particular solution of this equation, in the self similar form.

2.2.1 Self similar solution

Consider the following transformation, where the exponents α and β must be

found. We replace the following form of the solution in eq (2.4), we obtain:

$$u = t^{\alpha}\phi(\xi), \quad \xi = \frac{x}{t^{\beta}}, \tag{2.5}$$

then :

$$u_t = \alpha t^{\alpha-1}\phi(\xi) - \beta t^{\alpha-1}\xi\phi_{\xi}, \tag{2.6}$$

and

$$uu_x = t^{2\alpha-\beta}\phi\phi_\xi,$$

so eq (2.4), become

$$\alpha t^{\alpha-1}\phi(\xi) - \beta t^{\alpha-1}\xi\phi_\xi = t^{2\alpha-\beta}\phi\phi_\xi,$$

we divide on $t^{\alpha-1}$, we get:

$$\alpha\phi(\xi) - \beta\xi\phi_\xi = t^{\alpha-\beta+1}\phi\phi_\xi,$$

means that

$$\beta = \alpha + 1,$$

we obtain

$$\alpha\phi - (\alpha+1)\xi\phi_\xi = \phi\phi_\xi,$$

In other hand we suppose

$$\int u(x,t)dx = 1 \quad (conservation\ \ of\ \ mass).$$

So

$$\int t^\alpha \phi(\xi)d\xi = 1,$$

imply

$$t^{\alpha+\beta}\int \phi(x)dx = 1,$$

so

$$\alpha + \beta = 0,$$

then

$$\alpha = -\beta.$$

Now we have:

$$\begin{cases} \alpha = \beta - 1, \\ \alpha = -\beta, \end{cases} \tag{2.7}$$

which mean that $\alpha = -\frac{1}{2}$, and $\beta = \frac{1}{2}$,

so

$$-\frac{1}{2}\phi - \frac{1}{2}\xi\phi_\xi = \phi\phi_\xi,$$

equal to

$$-\frac{1}{2}(\phi + \xi\phi_\xi) = \frac{2}{2}(\phi\phi_\xi),$$

we get:

$$-\frac{1}{2}(\xi\phi)' = \frac{1}{2}(2\phi\phi_\xi)$$

imply

$$-(\xi\phi)' = (\phi^2)',$$

after the integration we get:

$$-\xi\phi = \phi^2 + c,$$

if we put $c = 0$, ($\phi(0) = 0$)

we obtain

$$\phi(\xi) = -\xi.$$

Finally we have the self similar solution to the eq (2.4),

$$u(x,t) = -\frac{x}{\sqrt{t}}. \tag{2.8}$$

2.2.2 Travelling wave solution

Now we will try to find a travelling wave solution for Burgers equation first type,

$$u_t = uu_x \;\; x \in \mathbb{R}, \;\; t \succ 0, \tag{2.9}$$

we define the next transformation:

$$u(x,t) = v(\xi), \tag{2.10}$$

where $\xi = x - ct$ and c is the speed of the wave.

Now we replace (2.10) in the equation (2.9), we obtain an ordinary equation

$$-c\frac{dv}{d\xi} = v\frac{dv}{d\xi}, \tag{2.11}$$

this equal to

$$v(\xi) = -c, \tag{2.12}$$

so the travelling wave solution for eq (2.9) is:

$$u(x,t) = -c.$$

2.3 Solutions to the second type of Burgers equations

In this section, we will search for a particular solution to the Burgers equations

of second type.

$$u_t = uu_x + u_{xx} \qquad\qquad x \in \mathbb{R}, t \succ 0. \tag{2.13}$$

We find firstly self similar solution, secondly a travelling wave solution, and also a

solution, with the Cole-Hopf transformation method.

2.3.1 Self similar solution

To search a self similar solution for eq(2.13),

we do this transformation:

$$u(x,t) = t^{\alpha}\phi(\xi),$$

with $\xi = \frac{x}{t^{\beta}}$, where the exponents α and β must be

found. So

$$u_t = \alpha t^{\alpha-1}\phi - \beta t^{\alpha-1}\xi\phi_{\xi},$$

and

$$uu_x = (t^{2\alpha-\beta}\phi)\phi_{\xi},$$

and

$$u_{xx} = t^{\alpha-2\beta}\phi_{\xi\xi},$$

after we replace the previews changes in Eq.(2.13),

we obtain

$$\alpha t^{\alpha-1}\phi - \beta t^{\alpha-1}\xi\phi_{\xi} = (t^{2\alpha-\beta}\phi)\phi_{\xi} + t^{\alpha-2\beta}\phi_{\xi\xi},$$

then we divide on $t^{\alpha-1}$, we get:

$$\alpha\phi - \beta\xi\phi_{\xi} = t^{1-2\beta}\phi_{\xi\xi} + t^{1+\alpha-\beta}\phi\phi_{\xi},$$

then

$$\begin{cases} 1 - 2\beta = 0 \\ 1 + \alpha - \beta = 0, \end{cases}$$

imply
$\alpha = -1/2, \ \ and \ \ \beta = 1/2,$

so,

$$-1/2\,(\phi + \xi\phi_\xi) = \phi_{\xi\xi} + \phi\phi_\xi,$$

means that,

$$-1/2(\xi\phi)' = \phi'' + \left(\phi^2/2\right)',$$

after integrating we get:

$$\phi' + \left(\phi^2/2\right) + 1/2\,(\xi\phi) = 0. \tag{2.14}$$

Eq (2.14) is a Bernoulli equation, (see section (1.15)), with:

$$a(x) = 1, \ \ \gamma = 2, \ \ c(x) = 1/2\xi, \ \ b(x) = 1/2.$$

We divide the equation (2.14) on ϕ^2,

we find

$$\phi'\phi^{-2} + \frac{1}{2} + \frac{1}{2}\left(\xi\phi^{-1}\right) = 0, \tag{2.15}$$

we put:

$$z(x) = \phi^{-1},$$

imply

$$z'(x) = -\phi'\phi^{-2},$$

so eq (2.15) become

$$z' - \frac{1}{2}\xi z = -\frac{1}{2}. \tag{2.16}$$

This is a linear differential equation for z,

let

$$p(\xi) = \frac{1}{2}\xi, \ \ q(\xi) = -\frac{1}{2}.$$

To solve eq (2.16), we have

1. $\mu = e^{\int p(\xi)d\xi} \ \ = e^{\int \frac{1}{2}\xi d\xi} \ \ = e^{\frac{1}{4}\xi^2}.$

1. $\mu z = \int \mu q(\xi)d\xi.$

So,

$$z = -\frac{1}{2}\left(\int e^{\frac{1}{4}\xi^2}\right)e^{-\frac{1}{4}\xi^2},$$

now we replace z to get $\phi(\xi)$,

then

$$\phi(\xi) = \left(\frac{1}{2}\left(\int e^{\frac{1}{4}\xi^2}\right)e^{-\frac{1}{4}\xi^2}\right)^{-1}.$$

Finally, the self similar solution to the the Burgers equation is:

$$u(x,t) = \frac{1}{\sqrt{t}}\left(\frac{1}{2}\left(\int e^{\frac{1}{4}\frac{x^2}{t}}\right)e^{-\frac{1}{4}\frac{x^2}{t}}\right)^{-1}.$$

2.3.2 Travelling wave solution

Consider the second type of burgers equation in the form:

$$u_t + 2uu_x = u_{xx}, \tag{2.17}$$

we look for a travelling wave solution to eq (2.17), we put $u(x,t) = v(\xi)$

where $\xi = x - ct$, c being the speed of the wave.

We get:

$$u_t = c\frac{dv}{d\xi},$$

and

$$u_x = \frac{dv}{d\xi},$$

also

$$u_{xx} = \frac{d^2v}{d\xi^2},$$

our equation takes the form:

$$c\frac{dv}{d\xi} + 2v\frac{dv}{d\xi} + \frac{d^2v}{d\xi^2} = 0, \tag{2.18}$$

let

$$\frac{dv}{d\xi} = \varphi(v),$$

where $\varphi(v)$ is an unknown function to be determined so,

$$\frac{d^2v}{d\xi^2} = \frac{d}{d\xi}\varphi(v),$$

so

$$\frac{d\varphi}{dv}\frac{dv}{d\xi} = \varphi\frac{d\varphi}{dv},$$

then we obtain from (2.17):

$$\frac{d\varphi}{dv} + 2v + c = 0,$$

we integrate for v, we get

$$\varphi(v) = -v^2 - cv + C_0, \tag{2.19}$$

Let us consider the following conditions at infinity:

$$u(x \to -\infty, t) = u_-, \;\; u(x \to \infty, t) = u_+,$$

which mean

$$\frac{dv(\xi \to \pm\infty)}{d\xi} = 0,$$

so

$$\varphi(u_+) = \varphi(u_-) = 0,$$

imply that u_- *and* u_+ are the roots to the square

polynomial (2.19), then

$$\varphi(v) = -(v - u_-)(v - u_+), \tag{2.20}$$

so

$$-v^2 + v[u_- + u_+] - (u_- u_+), \tag{2.21}$$

From comparison between (2.19) and (2.21) we get,

$$c = -(u_- + u_+),$$

$$C_0 = -(u_- - u_+).$$

We know that $\varphi(v) = \frac{dv}{d\xi}$, then we can solve Eq (2.20) :

$$\frac{dv}{d\xi} = -(v - u_-)(v - u_+),$$

so

$$-\frac{dv}{(v-u_+)(v-u_-)} = d(\xi),$$

so,

$$-\int \frac{1}{(v-u_+)} \frac{1}{(v-u_-)} = \int (d\xi),$$

we obtain

$$\log \left| \frac{v-u_+}{v-u_-} \right| = (\xi - \xi_0), \tag{2.22}$$

where ξ_0 is the integration constant.

Now, we can find $v(\xi)$,

we have from eq (2.22),

$$\frac{v-u_+}{v-u_-} = e^{\xi-\xi_0},$$

then

$$v[1-e^{\xi-\xi_0}] = u_+ - u_- e^{\xi-\xi_0},$$

we get

$$v = \frac{u_+ - u_- e^{\xi-\xi_0}}{1-e^{\xi-\xi_0}},$$

we suppose $u_- = 0$, and $u_+ = 1$.

Finally the travelling wave for the second type Burgers equation is:

$$u(x,t) = v(\xi) = 1 + \left(1 + \exp^{(\xi-\xi_0)}\right)^{-1}. \tag{2.23}$$

while the wave speed $c = -1$.

.

Figure 2.1 this is a figure of the exact solutions to the Burgers equation

second type.

Curves 1 to 4 show the wave profiles at $t = 0$, $t = 20$,

$t = 40$ and $t = 60$, respectively.

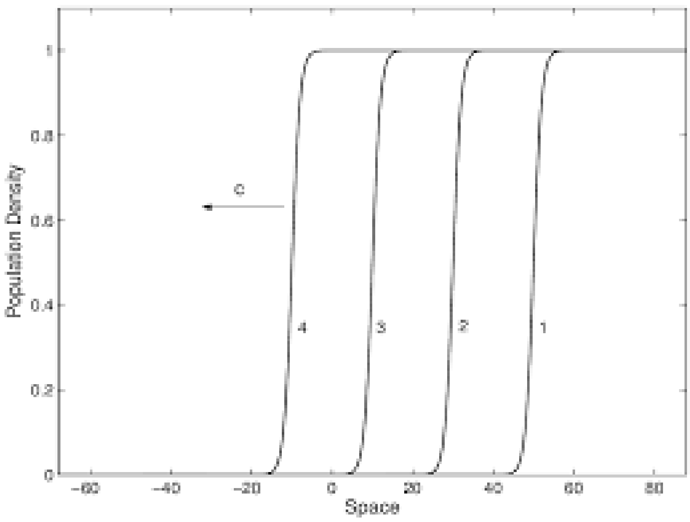

Figure 2.1

2.3.3 The Cole-Hopf transformation

Now, we are going to apply the Cole-Hopf transformation on the second type of Burgers

equation ,

$$u_t + 2uu_x = u_{xx}, \tag{2.24}$$

we put the following transformation:

$$u(x,t) = \frac{U_x}{U}, \tag{2.25}$$

we obtain

$$u_t = \frac{U_{xt}U - U_xU_t}{U_2},$$

and

$$2uu_x = 2\frac{U_x}{U}\left(\frac{U_{xx}U - U_x^2}{U^2}\right),$$

also

$$u_{xx} = \frac{U_{xxx}U^2 - 3U_{xx}U_xU + 2U_x^3}{U^3},$$

replace the changes in $(2,24)$, we obtain

$$\left(\frac{U_{xt}}{U} - \frac{U_xU_t}{U^2}\right) + 2\frac{U_x}{U}\left(\frac{U_{xx}}{U} - \frac{U_x^2}{U^2}\right) = \left(\frac{U_{xxx}}{U}3\frac{U_{xx}U_x}{U^2} + 2\frac{U_x^3}{U^3}\right), \tag{2.26}$$

then

$$\frac{U_{xt}}{U}-\frac{U_xU_t}{U^2}+2\frac{U_xU_{xx}}{U^2}-2\frac{U_xU_x^2}{U^3}=\frac{U_{xxx}}{U}-3\frac{U_{xx}U_x}{U^2}+2\frac{U_x^3}{U^3},$$

so

$$\frac{(U_t-U_{xx})_x}{U}+\frac{U_x(U_t-U_{xx})}{U^2}=0, \tag{2.27}$$

imply

$$\frac{UU_{xt}-U_{xxx}U+U_xU_t-U_xU_{xx}}{U^2}=0.$$

then

$$UU_{xt}-U_{xxx}U+U_x[U_t-U_{xx}]=0,$$

equals to

$$U_x[U_t-U_{xx}]=-UU_{xt}+U_{xxx}U,$$

equals to

$$U_x[U_t-U_{xx}]=-U[U_t-U_{xx}]_x.$$

If U is a solution to the heat equation $U_t-U_{xx}=0$,

it means that $u(x,t)$, giving by transformation (2.25), solve our Burgers equation.

In other hand, eq (2.26) can be written as:

$$u=\log(U)_x,$$

we get,

$$U(x,t)=exp\left(\int u(x,t)dx\right). \tag{2.28}$$

Let us consider the initial value for Eq (2.24) in an infinite domain,

$-\infty \prec x \prec \infty$, with the initial conditions being described by a

certain function $u_0(x)$,

it means that

$$U(x,0)=U_0(x)=\exp\left(\int_{-\infty}^{\infty}u_0(\zeta)d\zeta\right), \tag{2.29}$$

so we have changed the eq (2.24) to this problem:

$$\begin{cases} U_t - U_{xx} = 0, \;\; x \in \mathbb{R}, \;\; t \succ 0, \\ U(x,0) = U_0(x) = \exp\left(\int_{-\infty}^{\infty} u_0(\zeta) d\zeta\right), \;\; x \in \mathbb{R}. \end{cases} \tag{2.30}$$

2.3.4 Heat equation

Now we will solve problem (2.30), applying the Fourier transformation respect to x (section (1.3)), on both the heat equation and the initial condition $U_0(x)$, we obtain

$$\begin{cases} \hat{U}_t = \xi^2 \hat{U}, \;\; \xi \in \mathbb{R} \;\; t \succ 0 \\ \hat{U}(\xi, 0) = \hat{U}_0(\xi) \;\; \xi \in \mathbb{R}, \end{cases}$$

where
$\hat{U}(\xi, t) = \int_{-\infty}^{\infty} U(x,t) e^{i\xi x} dx$.

The solution of this problem is:

$$\hat{U}(\xi, t) = \hat{0}(\xi) e^{\xi^2 t}.$$

To find $U(x,t)$, we use the inverse Fourier transformation F^{-1},

$U(x,t) = F^{-1}(\hat{U}(\xi,t)) = F^{-1}(\hat{U_0}(\xi) e^{\xi^2 t} = U_0(x) * F^{-1} e^{\xi^2 t}$,

we have

$$F^{-1} e^{\xi^2 t} = \frac{1}{\sqrt{4\pi t}} e^{-(x-\xi)^2/4t},$$

we get

$$U(x,t) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} U_0(\xi) e^{-(x\xi)^2/4} d\xi.$$

Finally, from the transformation (2.25) ,the solution of second type Burgers equation is:

$$u(x,t) = \frac{\int_{-\infty}^{\infty} \frac{x-\xi}{t} U_0(\xi) e^{\frac{(x-\xi)^2}{4\pi t}} d\xi}{\int_{-\infty}^{\infty} \frac{x-\xi}{t} U_0(y) e^{\frac{(x-\xi)^2}{4\pi t}} d\xi}. \tag{2.31}$$

Chapter 3

A forced Burger equation

3.1 Introduction

We shall now study the third type of Burgers equation, which called the forced Burgers equation, abbreviated (FBE).

The forced Burgers equation is a Burgers equation with a non trivial right-hand side, and F is an external force.

The general form of the forced Burgers equation is:

$$u_t - 2uu_x - u_{xx} = F(x,t,u). \tag{3.1}$$

Eq. (3.1) is applied to population dynamics.

The general case $F = F(x,t,u)$ is however very difficult to treat analytically.

instead, we will consider separately three cases: The first case, where $F(x) = C$ which called Constant forcing.

The second case, $F = F(x)$, which called stationary forcing , And the third case, $F = F(x,t)$ which called the transient forcing.

3.2 Exact Solutions for the forced Burgers equation

In the next section, we will find a particular solutions to the forced Burgers equation.

We begin with the first case.

3.2.1 Constant forcing

Let us consider the Burgers equation with a constant forcing:

$$u_t - 2uu_x - u_{xx} = C, \;\; C \in \mathbb{R}. \tag{3.2}$$

We will use two methods to find exact solutions to this equation.

The first method:

we search a traveling wave solution to this equation, we put the next transformations

$$u(x,t) = v(\xi),$$

where $\xi = x - ct$, so :

$$u_t = \frac{dv}{d\xi},$$

and

$$uu_x = v\frac{dv}{d\xi},$$

and

$$u_{xx} = \frac{d^2v}{dv^2}.$$

Replacing this changes into eq (3.2), we get

$$-\frac{d^2v}{d\xi^2} - 2v\frac{dv}{d\xi} + \frac{dv}{d\xi} = C,$$

equals to

$$-\frac{d^2v}{d\xi^2} - (v^2)' + \frac{dv}{d\xi} = C,$$

because $2(v\frac{dv}{d\xi}) = (v^2)'$.

Now we integrate over ξ, we get:

$$-v' - (v^2) + v = C\xi. \tag{3.3}$$

Equation (3.3) is a Riccati equation (see (1.4.1))

with, $a(t) = -1$, $b(t) = 1$, $c(t) = C\xi$,

to solve equation (3.3), we need to know the particular solution v_p, so we can put

$$v = v_p + \frac{1}{z(x)},$$

imply

$$v' = \left(\frac{dv_p}{d\xi}\right) - \left(\frac{1}{z^2}\frac{dz}{d\xi}\right).$$

Substituting the previews equations in eq (3.3), we get

$$-\left(\frac{dv_p}{d\xi} - \frac{1}{z^2}\frac{dz}{d\xi}\right) - \left(v_p + \frac{1}{z}\right)^2 + \left(v_p + \frac{1}{z}\right) = C\xi,$$

these equals to

$$\frac{1}{z^2}\frac{dz}{d\xi} - \frac{dvp}{d\xi} - v_p^2 - 2\frac{v_p}{z} + v_p + \frac{1}{z} - C\xi = 0,$$

multiplying by z^2:

$$\frac{dz}{d\xi} = z^2\left[\frac{dv_p}{d\xi} - v_p^2 + v_p - C\xi\right] - z\left[2v_p - 1\right].$$

since v_p is a solution to eq (3.3), then

$$\frac{dv_p}{d\xi} - v_p^2 + v_p = 0,$$

we obtain

$$\frac{dz}{d\xi} + z\left[2v_p - 1\right] = 0.$$

This is a homogenous linear differential equation order one,

so that $z_h = e^{-\int (2v_p+1)\xi d\xi}$,

imply $ze^{\int -(2v_p+1)\xi} = \int e_{-(2v_p+1)\xi} d\xi + C$,

then $z = \left[\int e^{-(2v_p+1)\xi} + e^{\int -(2v_p+1)\xi}\right]$, going back to $v = v_p + \frac{1}{z} \Rightarrow z = \frac{1}{v - v_p}$,

imply

$$v(\xi) = \left(\int e^{-(2v_p+1)\xi} + C \right)^{-1} \left(e^{-\int (2v_p+1)\xi} \right) + v_p.$$

We have already put $u(x,t) = v(\xi)$.

Finally, the exact solution to the constant forcing Burgers equation is:

$$u(x,t) = \left(\int e^{-(2v_p+1)(x-ct)} + C \right)^{-1} \left(e^{-\int (2v_p+1)(x-ct)} \right) + v_p.$$

The second method:

to solve eq (3.2), we will use the Cole -Hopf transformation,

let us consider a new modification:

$$u = \frac{U_x}{U} + k, \tag{3.4}$$

where k is a constant, while

$$u_t = \frac{U_{xt}U - U_x U_t}{U_2},$$

and

$$2uu_x = 2\left(\frac{U_x}{U} + k \right) \left(\frac{U_{xx}U - U_x^2}{U^2} \right),$$

and

$$u_{xx} = \frac{U_{xxx}U^2 - 3U_{xx}U_xU + 2U_x^3}{U^3},$$

replacing the above changes into (3.2), we get:

$$\frac{U_{tx}U - U_xU_tUU^2 - 2\left[\frac{U_x}{U} + k\right]\left[\frac{U_{xx}U - U_x^2}{U^2}\right] - 3UU_{xx}U_x - U^2U_{xxx}}{U^3} = C,$$

imply

$$\frac{U^2U_{xt} - UU_xU_t - 2UU_xU_{xx} + 2U_xU_x^2 - 2kU^2U_{xx} + 2kU_x^2U - U_{xxx}U^2 + 3U_{xx}U_xU - 2U_xU^3}{U^3} = C,$$

so

$$\left(U_t - 2kU_x - U_{xx} \right)_x = (k + C)_x,$$

we integrate over x, we get:

$$U_t - 2kU_x - U_{xx} = (k + C)\,.$$

Since the right-hand side of constant forcing Burgers equation, means that $t = 0$

$u_t = 0$, we obtain a linear equation:

$$-2kU_x = U_{xx},$$

and $k = -C$, we will now solve the linear equation.

Let us look for travelling wave solution to eq: $-2kU_x = U_{xx}$,

we put $U(x,t) = v(\xi)$, with $\xi = x - y(t)$, replacing these transformations into the linear equation we get :

$$-\left(\frac{dy}{dt} + 2k\right)\frac{dv}{d\xi} = \frac{d^2v}{d\xi^2}. \tag{3.5}$$

These imply that $dy/dt + 2k = \phi(\xi)$, where $\phi(\xi)$ is a certain function, these

move is possible only if $\phi(\xi) = \beta\xi + \gamma$, we obtain:

$$\frac{dy}{dt} + 2k = \beta\xi + \gamma,$$

which can be written as

$$\frac{dy}{dt} + 2C = \beta(x - y(t)),$$

so

$$y(t) = e^{\beta t(\gamma - 2c)}.$$

We know $y(t)$, and k, we can find $v(\xi)$,

then:

$$-(2C\xi + \gamma)\frac{dv}{d\xi} = \frac{d^2v}{d\xi^2}.$$

Putting $q(\xi) = \frac{dv}{d\xi}$, the above equation become

$$q(-2C + \gamma) = \frac{dq}{d\xi},$$

which lead to

$$q = e^{(\beta\xi^2 + \gamma\xi)},$$

then we have $v(\xi) = (1 + erf\,[\beta(\xi + \gamma)])$, where $erf(z)$, is

the error function.

Coming back to $U(x,t) = v(\xi)$, we obtain:

$$U(x,t) = \left(1 + erf\left[\beta(x - e^{\beta t(\gamma - 2C)} + \gamma)\right]\right).$$

Finally the exact solution to the Burgers equation with constant forcing is:

$$u(x,t) = C + \frac{exp\left[-C(x + e^{\beta t(\gamma - 2C))}\right]}{2[\pi/(4C)]erf[C(x - \gamma e^{-2Ct}}))].$$

3.2.2 Stationary forcing

Now consider the following stationary forcing Burgers equation:

$$u_t - 2uu_x - u_{xx} = f(x), \tag{3.6}$$

to solve eq (3.2), we start with the Cole -Hopf transformation, we consider new modification:

$$u = \frac{U_x}{U} + k(x), \tag{3.7}$$

where $k(x)$ is a function we must find,

with

$$u_t = \frac{U_{xt}U - U_xU_t}{U_2},$$

and

$$2uu_x = 2\left(\frac{U_x}{U} + k(x)\right)\left(\frac{U_{xx}U - U_x^2}{U^2} + k'k\right),$$

and

$$u_{xx} = \frac{U_{xxx}U^2 - 3U_{xx}U_xU + 2U_x^3}{U^3} + (k'k'' + kk'),$$

replacing the above changes in (3.6), we get:

$$\frac{U_tU_x - U_{xt}U}{U^2} - 2\left[\frac{U_x}{U} + k(x)\right]\left[\frac{U_{xx}U - U_x^2}{U^2}\right] + (k'k) - \frac{2U_x^3 - 3UU_{xx}U_x - U^2U_{xxx}}{U^3}(k'k + k'k'') = f(x),$$

imply

$$\frac{U\left(-U_tU_x+UU_{xt}\right)-2U\left[U_xU+K\right]\left[U_{xx}U-U^2\right]+U^3kk'}{U^3}$$
$$-\frac{U_{xxx}U^2-3U_{xx}U_xU+2U_xU^3+U^3\left(k'k''+kk'\right)}{U^3}-f(x)=0,$$

so

$$\left(\frac{U_t-2kU_x-U_{xx}}{U}\right)_x=\left(\frac{dk}{dx}+k^2+f(x)-C(t)\right)_x,$$

$C(t)$ is a function of time.

Now we integrate over x, we obtain:

$$U_t-2kU_x-U_{xx}=\left(\frac{dk}{dx}+k^2+\psi(x)-C(x)\right),$$

$\psi(x)$, is the primitive of $f(x)$, which means that

$$\frac{d\psi}{dx}=f(x).$$

The equation (3.6) is transformed into the above equation, where the left-hand side which is under the form:

$$U_t-2kU_x=U_{xx}. \tag{3.8}$$

And the right-hand side which is written as:

$$\frac{dk}{dx}+k^2=-\psi(x)+C(t), \tag{3.9}$$

is a Riccati equation.

If $k(x)$ is a solution to the Riccati equation (3.9), then $U(x,t)$ is a solution to eq (3.8) .

Equation (3.9) has the time only as a parameter, which mean that we can put $C(t)=0$, so

the Riccati equation equivalent to the Burgers equation in stationary space where

$u_t \equiv 0$. This mean that k is a stationary solution to eq (3.6),

and the transformation we've put (3.7) has a meaning. Then the linear equation (3.8) become

$$-2kU_x = U_{xx}. \tag{3.10}$$

It is very difficult to solve the Riccati equation with a right hand side (3.9),

for that we will try to search a travelling wave solution to the equation (3.8),

we put

$$U(x,t) = v(\xi),$$

where

$$\xi = x - y(t),$$

replacing this change in eq (3.8),

we obtain

$$-\left(\frac{dy}{dt} + 2k(x)\right)\frac{dv}{d\xi} = \frac{d^2}{v}d\xi^2. \tag{3.11}$$

this move is correct, only if the eq (3.11) contains x, and t for the new variable ξ.

Now we put

$$\frac{dy}{dt} + 2k(x) = \phi(\xi), \tag{3.12}$$

since x and ξ are related in a linear equation ($\xi = x - y(t)$),

the change we have put on eq (3.12), is possible only when k and ϕ are linear functions, which means that

$$k(x) = Bx + B_1,$$

and

$$\phi(\xi) = \beta\xi + \gamma,$$

where B, β, B_1, γ are parameters.

Now, we replace $k(x) = Bx + B_1$ in our Riccati equation (3.9), it become:

$$B + (Bx + B_1)^2 = -\psi(x) + C,$$

then we derive it over x, we obtain

$$\frac{d(B + (Bx + B_1)^2}{dx} = -\frac{d\psi}{dx}.$$

so

$$f(x) = -2B(Bx + B_1),$$

because

$$(f(x) = \frac{d\psi}{dx}).$$

We can put $B_1 = 0$, the forced Burgers equation has a travelling wave solution only when the force is linear to x, $f(x) = -2B^2x$,

so $B + (Bx + B_1)^2 = -\int f + C$, then $B + (Bx + B_1)^2 = B^2x^2$,

matching the two above equation we get, $B = C$, and $k(x) = Bx$, we obtain:

$$\frac{dy}{dt} + 2Bx = \beta\xi + \gamma = \beta(x - y(t)) + \gamma,$$

equal to

$$\frac{dy}{dt} + \beta y - \gamma = (\beta - 2B)x,$$

the previous equation is possible in case both of it sides are equal to zero which mean $\beta = 2B$, and

$$\frac{dy}{dt} + \beta y - \gamma = 0. \tag{3.13}$$

For $t = 0$ we get $\xi = x$, so $y(0) = 0$, then

$$\frac{dy}{dt} = -\beta y + \gamma,$$

so

$$\frac{dy}{y} = -(\beta + \gamma)dt,$$

then

$$y(t) = \delta(1 - e^{-2Bt}),$$

with $\delta = \gamma/(2B)$,

after finding $y(t)$ and $k(x)$, we get:

$$-(2B\xi + \gamma)\frac{dv}{d\xi} = \frac{d^2v}{d\xi^2},$$

we put new variable $p(\xi) = dv/d\xi$ the above equation change to :

$$\frac{dp}{d\xi} = -(2B\xi + \gamma)p,$$

we can solve this equation :

$$\frac{dp}{p} = (-2B\xi - \gamma)d\xi.$$

So $p = C.exp(-B\xi^2 - \gamma\xi)$,

equivalent to

$$\frac{dv}{d\xi} = C.exp(-B\xi^2 - \gamma\xi). \tag{3.14}$$

There are two cases of B :

for $B \prec 0$ has no mean in biology because $p(\xi)$ will increase non stop when x goes to infinity.

When $B \succ 0$, we get the solution to (3.14):

$$v(\xi) = a + b\left(1 + erf\left[B^{1/2}(\xi + \gamma\xi)\right]\right),$$

where a and b are paremeters can be found by the initial conditions, $erf(z)$ is the error function.

Now replacing the value of $\xi = x - y(t)$ and $y(t) = \delta(1 - e^{-2Bt})$,

getting :

$$U(x,t) = a + b\left(1 + erf\left[B^{1/2}\left(x + \delta e^{-2Bt}\right)\right]\right).$$

Finally, going back to eq (3.6) we obtain the exact solution of the Burgers equation with stationary forcing, giving by

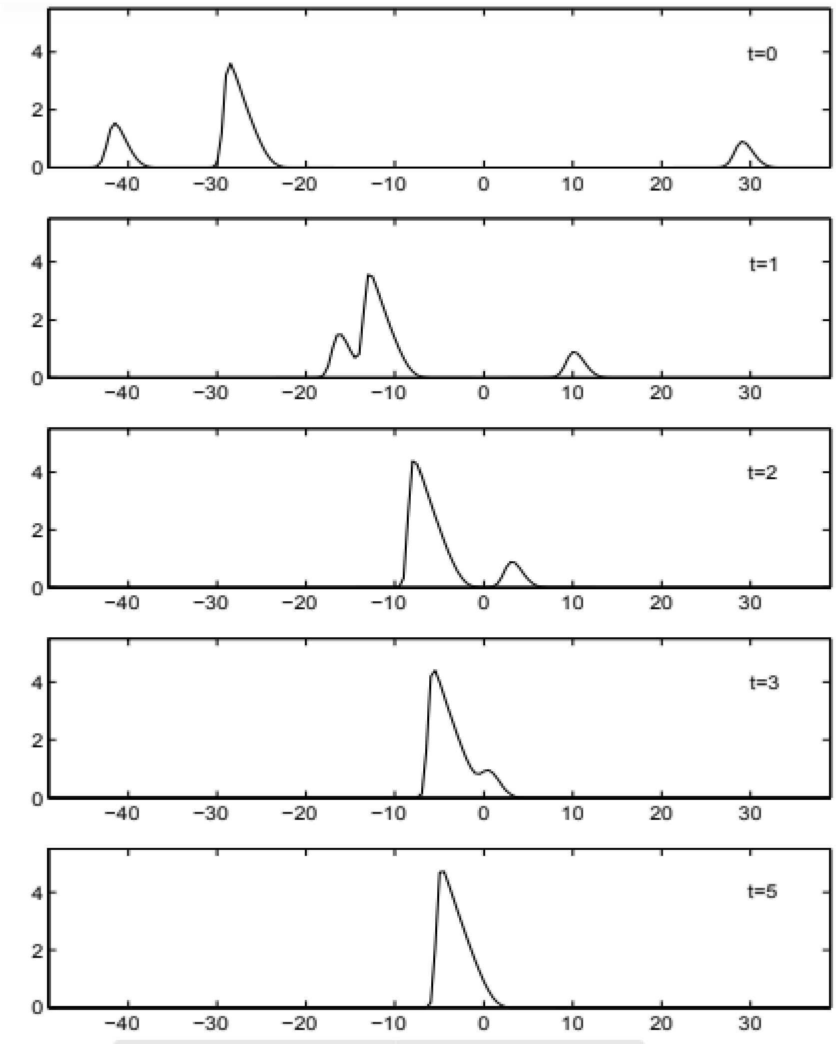

Figure 3.1

$$u(x,t) = Bx + \frac{exp\left[-B\left(x+\delta e^{-2Bt}\right)^2\right]}{\mu + [\pi/(4B)]^{1/2} erf B^{1/2}(x+\delta e^{-2Bt})}.$$

FIGURE 3.1 This is a figure of the exact solution to the stationary forcing Burgers equation with $k(x) = Bx$. Parameters are :

$\delta_1 = 40, \delta_2 = 25\, \delta_3 = -30, x = 0.000001, x_2 = 0.1, x_3 = 0.899999, B = 0.5$ and $\mu = 1.2532957$,

3.2.3 Transient forcing

The previews case (stationary forcing), the right-hand side of the Burgers equation depended on x, here the force depended both on t and x. The general form of the transient forcing Burgers equation is:

$$u_t - 2uu_x - u_{xx} = f(x,t). \tag{3.15}$$

It is clear to see that in this case transformation (3.7),

also leads to linear equation (3.8), where k now depends on x and t.

However ,the coupling equation is no longer a Riccati equation, but coincides with the original equation , in the case of transient forcing.

Substitution (3.7) describes an "self transformation" of the solution so that, if $k(x,t)$ is a solution of the forced Burgers equation F, then

$$u(x,t) = \frac{U_x}{U} + k(x,t), \tag{3.16}$$

is another solution (corresponding to different initial conditions), provided that $U(x,t)$ is a solution of Eq (3.8).

Relation(3.16) can be used to construct exact solution of the forced Burgers equation, when F depends on time.

It is very difficult to resolve this case, but we can give an example which takes a particular form.

Special case

As an example, let us consider the special case when forcing include time:

$$u_t - 2uu_x - u_{xx} = -\frac{ax}{(t+t_0)^2}, \tag{3.17}$$

where a and t_0 are parameters. in order to avoid singularities for $t \succ 0$, we assume that $t_0 \succ 0$ it is readily seen that the function

$$k(x,t) = \frac{bx}{t+t_0}, \tag{3.18}$$

is a solution of Eq (3.17), if $(b+2b)^2 = a$, the solutions of Eq.(3.17) may have different properties depending on the sign of a and b.

Since our goal here is, more to show how the self transformation (3.16) can be used to generate exact solutions of the Burgers equation with transient forcing, rather than to investigate it in all details, we restrict our consideration to the case $a \succ 0$. The solution (3.18) by it self is unlikely to be of mush interest because its behavior is too simple. However, it can be used to construct other solution with more interesting properties. By replacing (3.17), the function $u = k + (U_x/U)$, is also a solution in case $U(x,t)$ is a solution of the following equation:

$$U_t - \frac{2bx}{t+t_0} U_x = U_{xx}, \tag{3.19}$$

the combination of x and t, in which they appear in Eq. (3.19), gives us a hint that it may be possible to look for a self-similar solution, in the form $u(x,t) = w(\theta)$, where $\theta = x\phi(t)$, and functions w and ϕ are to be determined. having substituted it into Eq.(3.19) we obtain:

$$\left(x\phi^{-2}\frac{\partial \phi}{\partial t} - \frac{2bx}{(t+t_0)\phi} \right) \frac{\partial w}{\partial \theta} = \frac{\partial^2 w}{\partial \theta^2},$$

the transition to self-similar variables is mathematically correct only in case the expression in parentheses is a function of θ.

In order to satisfy this condition, we require that

$$\phi^{-2}\frac{\partial \phi}{\partial t} = \lambda\phi, \frac{1}{(t+t_0)\phi} = \eta^{-2}\phi, \tag{3.20}$$

where λ and η are certain constants, we immediately arrive to

$$\phi(t) = \eta(t + t_0)^{-1/2},$$

$\lambda = -0.5\eta^{-2}$.

Letting $\theta(x, 0) = x$, we obtain $\eta = t_0^{1/2}$.

Eq.(3.20) then takes the following form:

$$-2\alpha^2\theta\frac{\partial w}{\partial\theta} = \frac{\partial^2 w}{\partial w\theta^2},$$

where $\alpha^2 = (b + 0.25)/t_0$, The last equation is solved easily to the

following solution:

$$w(\theta) = A_1\text{erf}(\alpha\theta) + A_2, \tag{3.21}$$

where A_1 and A_2 are parameters determined by the initial conditions.

Taking into account (3.15), we arrive at the following exact solution of Eq.(3.16):

$$u(x,t) = \frac{bx}{t + t_0} + \frac{2\alpha}{\sqrt{\pi}}\left(\frac{t_0}{t + t_0}\right)^{1/2}\frac{exp(-\alpha^2\theta^2)}{k + erf(\alpha\theta)}. \tag{3.22}$$

where $k = (A_2/A_1)\text{and}\theta = x[t_0/(t + t_0)]^{1/2}$ For $|k| \succ 1$,

the function given by (3.21) is continuous at all x and $t \succ 0$, exact solution

(3.19) describes self-similar diffusion, and decay of a dome-shaped initial disturbance

of the linear distribution. The simple solution (3.15), was used (3.13), to generate

a more interesting solution (3.19).

Conclusion

In this work we have searched for solutions to the three types of Burgers equations, we have studied the first type, and fond two solutions and for the second type we fond three solutions, also the third type which called the forced Burgers equation, we resolve it in her three cases, constant forcing, stationary forcing and the transient forcing.

We have used many methods, the first method is the self-similar method, the travelling wave solutions and the Cole-Hopf transformation.

We have detailed all forms of these methods and finding an analytical solutions for all the three type of Burgers equation. The only case where it is not possible to calculate analytical solution is in the third type, where the right side depend on x,t and u.

The last case will be treated in futur work.

Résumé

Ce travail a pour objet l'étude des méthods de réduction pour l'EDPs non-linear, particuliérement, l'étude de l'équation de Burgers sous ses trois formes et en servant de trois méthode de auto similarité solutions,l'équation des ondes et on utilisé le transformation de Cole-Hopf pour obtenie une equation de Burgers en EDO, de plus trouver la solution auto similaire et travelling a travers des cas particuliers.

Mots clés

Equation de Burgers, solution auto similaire, le onde progressive, Cole-Hopf transformation.

Summary

The object of this work is the study of reduction methods for non-linear EDPs, in particular the study of the Burgers equation in its three forms and using my method of self-similarity, and we used the Cole-Hopf transformation to obtain an equation of Burgers in EDO, moreover find the similar auto solution and traveling through special cases.

Key words

Burgers equation, similar auto solution, the progressive wave, Cole-Hopf transformation.

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