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Some Sobolev embeddings of fractional type and applications

Presented by:
Benlatrache Kenza

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Mr Mokhtari Abdelhak	MCA,	University of M'sila	**President**.
Mr Saadi Abderachid	MCA,	University of M'sila	**Supervisor**.
Mr Bougharara Brahim	MCA,	University of M'sila	**Examiner**.

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God will see your work, and his Me
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you. Lastly, I would like to dedicate my graduation to my
and striving throughout the
my graduation marks the beginning of a promising future, where knowledge and
understanding play a significant role in my life and in serving the community.

Benlatrache Kenza

DEDICATIONS

I dedicate this mode

To my parent

To my brothers billel,mohamed,wail,

To my sisters Houda ,Aya,Fadwa and Ibtissem,

To my uncle

To all my friends and my familly from the mathematic

Benlatrache Kenza

CONTENTS

NOTATIONS

- $\mathbb{N}$:Set of positive integer numbers.
- $\mathbb{R}$: Set of real numbers .
- $\mathscr{C}^0(a;b) = \mathscr{C}(a,b)$: Space of continuous functions on the interval.
- $L^p(a;b)$: Space of power functions $p \in [1, +\infty[$ integrable on $(a;b)$.
- $L^\infty(a;b)$: Space of functions essentially bounded on $(a;b)$.
- $\mathscr{C}^n(a;b)$: Space of functions n times differentiable with continuity on $(a;b)$.
- $\mathscr{C}^\infty(a;b)$: Space of infinitely continously differentiable function on $(a;b)$.
- $\text{supp}\ f$: Support of f.
- $\mathscr{C}_c^\infty$: Space of infinitely differentiable functions with compact support .
- p' : conjugate of holder of p ($p' = \frac{p}{p-1}$).
- $AC^n(a,b)$: Space of n absolute continuous function of order n on (a,b).
- $W^{1;p}(a,b)$: The usual Sobolev space on (a,b).
- $\|.\|_{L^p}$: Norm in L^p.
- $\|.\|_{L^\infty}$: Norm in L^∞.
- $\|.\|_{w^{1;p}}$: Norm in $w^{1;p}$.
- $\Gamma(.)$: Gamma function.
- $\beta(.,.)$: Beta function .
- I_{a+}^α: The fractional integral on the left of order α in the sense of Riemann -Liouville.
- I_{b-}^α: The fractional integral on the right of order α in the sense of Riemann -Liouville.
- $\frac{d}{dx}$: The usual derivative .

- $D^{\alpha}_{a^+}$: The fractional derivative on the left of order α in the sense of Riemann - Liouville.
- $D^{\alpha}_{b^-}$: The fractional derivative on the right of order α in the sense of Riemann - Liouville.

INTRODUCTION

Between 1832 and 1837, Liouville published a series of articles on fractional-order integral and differential equations. In one of his articles, he defined the integral of a complex number α, with a positive real part:

$$I^{\alpha} f(x) = D^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} f(x+t) t^{\alpha-1} dt \quad x \in \mathbb{R}.$$

($\Gamma(\alpha)$ is the Euler gamma function).

This result is significant. In 1847, Riemann studied it, and his research, published in 1876, provided the current definition of fractional integration known as the Riemann-Liouville integral:

$$I^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^{x} f(t)(x-t)^{\alpha-1} dt \quad x > 0.$$

This was a precursor to fractional-order differentiation of the Riemann-Liouville type.

Research in this field has proliferated, making it difficult for researchers to keep up with the latest studies. Many differential and partial differential equations have been modified to fractional-order type, leading to numerous applications based on these modified models.

Some researchers may not favor this work, as it involves hypothetical equations that may not apply to physical, biological, and other sciences. However, these studies are original, with correct results based on previous research. This is the essence of research in pure and applied mathematics.

Nevertheless, fractional-order differential equations have practical applications in various fields, including electronics, hydrodynamics, fluid mechanics, dynamic systems, geophysics, soil science, biochemistry, economics, and finance.

In 1935, Sobolev introduced a theory for the general solutions of the wave equation, defining them as limits in the L^1 space of $\mathscr{C}^2$ solutions. He introduced the concept of continuous functionals on the set of continuously differentiable functions up to a certain order (later known as ”distributions of bounded order”). This led to his theory on the existence of solutions for a wide range of hyperbolic equations.

In 1938, Sobolev defined weak derivatives and Sobolev spaces, denoted by L_p^ν, which later evolved into W_p^m and eventually to the current notation $W^{m,p}$. Research in these spaces developed rapidly from the 1950s onward.

One of the most famous books in this field is R. A. Adams’ ”Sobolev Spaces,” [1] which provided definitions and properties of these spaces. Another well-known book for specialists in functional analysis is H. Brezis’ ”Analyse Fonctionnelle,” [4] which includes an overview of Sobolev spaces and some related boundary value problems.

Among the key concepts associated with Sobolev spaces is embedding into broader spaces, particularly Lebesgue spaces, which has significant implications for boundary problems in differential and partial differential equations. This research highlights key embeddings in fractional-order Sobolev spaces related to Riemann-Liouville calculus.

This involves introducing fractional-order Sobolev spaces of the Riemann-Liouville type, which have been actively studied for about ten years. We will discuss their algebraic and topological structure, followed by the embedding of these spaces into broader spaces, especially Lebesgue spaces and spaces of continuous functions. This extends classical concepts to fractional-order calculus, emphasizing their importance and wide applications.

This work is divided into four chapters:

Chapter One: Fundamental concepts related to basic functional spaces and the Riemann-Liouville fractional calculus.

Chapter Two: Spaces of absolutely continuous functions of fractional order, Sobolev spaces, their relationship, and topological properties.

Chapter Three: Various embeddings of Sobolev spaces into broader spaces and the relationship between fractional-order and first-order Sobolev spaces.

Chapter Four: Application of the previous concepts to a nonlinear boundary problem using the Faedo-Galerkin method and the fixed-point theorem, both involving fractional-order Sobolev spaces.

CHAPTER 1

PRELIMENARIES

In this chapter, we will review fundamental concepts related to Lebesgue spaces and spaces of smooth functions. Additionally, we will provide brief definitions and properties of Riemann-Liouville fractional calculus.

In all of the chapter, (a, b) design a bounded interval of $\mathbb{R}, 1 \leq p \leq \infty$. and α be a real number such that $0 < \alpha < 1$.

1.1 Functional Spaces

L^p spaces

Definition 1.1 *[1, 4]*

1. *The Lebesgue space $L^p(a, b)$ with $1 \leq p < \infty$ is defined by:*

$$L^p(a,b) = \left\{ f : (a,b) \to \mathbb{R} \quad f \text{ is measurable and } \int_a^b |f(x)|^p \, dx < \infty \right\},$$

equipped with the norm

$$\|f\|_{L^p} = \left(\int_a^b |f(x)|^p \, dx \right)^{\frac{1}{p}}, p \geq 1.$$

2. *The Lebesgue space $L^\infty(a, b)$ is defined by:*

$$L^\infty(a,b) = \left\{ f : (a,b) \to \mathbb{R} \quad \middle| \quad \begin{array}{l} f \text{ is measurable, there exists a constant } C \\ \text{such that } |f(x)| \leq C \text{ a.e on } (a,b) \end{array} \right\}$$

equipped with the norm

$$\|f\|_{L^\infty} = \inf\{C, |f(x)| \leq C \text{ a.e on } (a,b)\}$$

Theorem 1.1 *[1, 4] The space $(L^p(a,b), \|.\|_{L^p})$ is*

- *a Banach space for $1 \leq p \leq \infty$,*
- *a Separable space for $1 \leq p < \infty$,*
- *a reflexive space for $1 < p < \infty$.*

Proposition 1.1 (Hölder inequality) *[1, 4] Let $f \in L^p(a,b)$ and $g \in L^q(a,b)$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, $f, g \in L^1(a,b)$ and we have*

$$\|f.g\|_{L^1} \leq \|f\|_{L^p}\|g\|_{L^q}$$

Theorem 1.2 (Lebesgue) *[4] Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in $L^p(a,b)$. Assume that*

1. *$f_n(x) \to f(x)$ a.e on (a,b)*
2. *There exists a function $g \in L^p(a,b)$ such that for each $n \in \mathbb{N}$ we have $|f_n(x)| \leq g(x)$ a.e on (a,b).*

Then, $f \in L^p(a,b)$ and $\lim\limits_{n\to\infty} \|f_n - f\|_{L^p} = 0$

Space of regular functions on (a,b)

Let $f : (a,b) \to \mathbb{R}$ be a function and $n \in \mathbb{N}$.

Definition 1.2 *[10] We call $\operatorname{supp} f$ the support of f, defined by:*

$$\operatorname{supp} f = \overline{\{x \in (a,b) : f(x) \neq 0\}},$$

that is to say the smallest closed subset outside of which f is identically zero.

Definition 1.3 *We say that*

1. *$f \in \mathscr{C}(a,b)$ if f is continuous on (a,b).*
2. *$f \in \mathscr{C}^n(a,b)$ if f is n time continuous and differentiable on (a,b).*
3. *$f \in \mathscr{C}^\infty(a,b)$ if f is infinitely differentiable on (a,b).*

Definition 1.4 *We say that*

1. *$f \in \mathscr{C}([a,b])$ if f is continuous functions on $[a,b]$.*
2. *$f \in \mathscr{C}^n([a,b])$ if f is a restriction of a function that belongs to the space $\mathscr{C}^n(\mathbb{R})$.*
3. *$f \in \mathscr{C}^\infty([a,b])$ if f is a restriction of a function that belongs to the space $\mathscr{C}^\infty(\mathbb{R})$.*

Definition 1.5 *[10] We say that $f \in C_c^\infty(a,b)$ if f is indefinitely differentiable, with compact support in (a,b). In other words,*

$$C_c^\infty(a,b) = \left\{f : (a,b) \longrightarrow \mathbb{R}, f \in C^\infty(a,b) : supp\varphi \subset (a,b)\right\}.$$

Space of absolutly continuous functions

Definition 1.6 *[8] A function $f : [a,b] \to \mathbb{R}$ is called absolutely continuous on $[a,b]$, if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $\{x_i, y_i\}_{i=1}^{n} \subset [a,b]$ satisfying $\sum_{i=1}^{n} |x_i - y_i| < \delta$ we have $\sum_{i=1}^{n} |f(y_i) - f(x_i)| < \varepsilon$.*
We denote by $AC(a,b)$ the space of absolutely continuous on $[a,b]$.

Theorem 1.3 *[8] f is absolutely continuous on $[a,b]$ if and only if there exists $\varphi \in L^1(a,b)$ such that*

$$f(x) = c + \int_a^x \varphi(t)dt$$

Definition 1.7 *[7, 11] Let $n \in \mathbb{N}$, $AC^n(a,b)$ is the space of $(n-1)$ times continuously differentiable, such that $f^{(n-1)} \in AC(a,b)$, i.e.*

$$AC^n(a,b) = \left\{ f : [a,b] \to \mathbb{R}, f^{(n-1)} \in AC(a,b) \right\}.$$

Remark 1.1 *On the same method, we introduce the notions $AC^p(a,b)$, $AC^{n,p}(a,b)$ by replacing $L^1(a,b)$ by $L^p(a,b)$.*

Sobolev Spaces

Definition 1.8 *[1, 4] The space $W^{1,p}(a,b)$ is the space given by :*

$$W^{1,p}(a,b) = \left\{ u \in L^p \ \exists g \in L^p, \forall \varphi \in C_c^\infty(a,b) \ \int_a^b u(t)\varphi'(t)dt = -\int_a^b g(t)\varphi(t)dt \right\},$$

equipped with the norm

$$\|u\|_{W^{1,p}} = \|u\|_{L^p} + \|u'\|_{L^p}.$$

Theorem 1.4 *[1, 4] The space $(W^{1,p}(a,b), \|.\|_{W^{1,p}})$ is*

- *a Banach space for $1 \leq p \leq \infty$,*
- *a Separable space for $1 \leq p < \infty$,*
- *a reflexive space for $1 < p < \infty$.*

Theorem 1.5 *[4] We have the following embeddings*

- *$W^{1,p}(a,b) \hookrightarrow L^\infty(a,b)$, for all $1 \leq p \leq \infty$.*
- *$W^{1,p}(a,b) \hookrightarrow \mathscr{C}([a,b])$ with compactness.*
- *The embeddings $W^{1,1}(a,b) \hookrightarrow \mathscr{C}([a,b])$ is continuous but not pas compact.*
- *The embedding $W^{1,1}(a,b) \hookrightarrow L^q(a,b)$ is compact for any for all $1 \leq p \leq \infty$.*

1.2 Special Functions

Gamma Function

Definition 1.9 *[11] The Gamma function* (Γ) *is defined for a complex number* z *with* $Re(z) > 0$, *as follows:*

$$\Gamma(z) = \int_0^{\infty} e^{-x} x^{z-1} dx.$$

Proposition 1.2 *[11] We have the following properties:*

1. $\Gamma(1) = 1$.
2. $\Gamma(z+1) = z\Gamma(z)$.
3. *If* n *is a non-negative integer then,* $\Gamma(n+1) = n!$.

Beta Function

Definition 1.10 *[11] The function* β *is defined for a couple of complex numbers* (z,t) *with* $Re(z) > 0,\ Re(t) > 0$ *to* $\mathbb{R}$ *and given by :*

$$\beta(z,t) = \int_0^1 x^{z-1}(1-x)^{t-1} dx.$$

Proposition 1.3 *[11] We have the following properties:*

1. *Symmetry of Beta function ; i.e* $\beta(z,t) = \beta(t,z)$.
2. $\beta(z,t) = \beta(z+1,t) + \beta(z,t+1)$.
3. *If* n, m *are two non-negative integers then,* $\beta(n,m) = \frac{(n-1)!(m-1)!}{(n+m+1)!}$.
4. *In general:* $\beta(z,t) = \dfrac{\Gamma(z)\Gamma(t)}{\Gamma(z+t)}$.

1.3 Riemann -Liouville Fractional integral

Let $a, b \in \mathbb{R}, \alpha > 0$ and $1 \leq p \leq +\infty$.

Definition 1.11 *[7, 11] The Riemann -Liouville Fractional integral* $I^{\alpha}_{a+}f$ *and* $I^{\alpha}_{b-}f$ *of order* α *and a function* $f \in L^p(a,b)$ *are defined by:*

$$(I^{\alpha}_{a+}f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt \ \ (a < x \leq b),$$

$$(I^{\alpha}_{b-}f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (x-t)^{\alpha-1} f(t) dt \ \ (a \leq x < b).$$

Theorem 1.6 *The Riemann-Liouville integral $I^\alpha_{a+}f$ and $I^\alpha_{b-}f$ are well defined for all $f \in L^p(a,b)$. Moreover, we have:*

$$\|I^\alpha_{a+}f\|_{L^p} \leq \frac{(b-a)^\alpha}{|\Gamma(\alpha+1)}\|f\|_{L^p}. \tag{1.1}$$

$$\|I^\alpha_{b-}f\|_{L^p} \leq \frac{(b-a)^\alpha}{\Gamma(\alpha+1)}\|f\|_{L^p}. \tag{1.2}$$

Demonstration. Let $f \in L^p(a,b)$. We discus the following cases

- **case 1:** $p = +\infty$

$$\begin{aligned}
|I^\alpha_{a+}f(x)| &\leq \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}|f(t)|dt,\\
&\leq \frac{\|f\|_{L^\infty}}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}dt,\\
&\leq \frac{\|f\|_{L^\infty}}{\Gamma(\alpha)}\frac{(x-a)^\alpha}{\alpha} < \infty,\\
&\leq \frac{(b-a)^\alpha}{\Gamma(\alpha+1)}\|f\|_{L^\infty}.
\end{aligned}$$

Hence, $I^\alpha_{a+}f \in L^\infty(a,b)$ and we have $\|I^\alpha_{a+}f\|_{L^\infty} \leq \frac{(b-a)^\alpha}{\Gamma(\alpha+1)}\|f\|_{L^\infty}$.

- **case 2:** $p = 1$

$$\begin{aligned}
\int_a^b |I^\alpha_{a+}f(t)| &\leq \int_a^b \left|\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt\right| dx,\\
&\leq \frac{1}{\Gamma(\alpha)}\int_a^b |f(t)|dt \int_t^b (x-t)^{\alpha-1}dx,\\
&\leq \frac{(b-a)^\alpha}{\alpha\Gamma(\alpha)}\int_a^b |f(t)|dt < \infty,\\
&= \frac{(b-a)^\alpha}{\Gamma(\alpha+1)}\|f\|_{L^1}.
\end{aligned}$$

- **case 3**: $1 < p < +\infty$

$$\begin{aligned}
|I^\alpha_{a+}f(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}|f(t)|dt,\\
&= \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{(\alpha-1)\frac{p-1}{p}}.(x-t)^{(\alpha-1)p}|f(t)|dt,\\
&\leq \frac{1}{\Gamma(\alpha)}\left(\int_a^x (x-t)^{\alpha-1}\right)^{\frac{p-1}{p}}\left(\int_a^x (x-t)^{\alpha-1}|f(t)|^p\right)^{\frac{1}{p}},\\
&= \frac{(b-a)^{\frac{\alpha(p-1)}{p}}}{\Gamma(\alpha)\alpha^{\frac{p-1}{p}}}\left(\int_a^x (x-t)^{\alpha-1}|f(t)|^p\right)^{\frac{1}{p}}.
\end{aligned}$$

So,

$$\begin{aligned}\int_a^b |I_{a+}^{\alpha} f|^p &\leq \frac{1}{(\Gamma(\alpha))^p}\frac{(b-a)^{\alpha(p-1)}}{\alpha(p-1)}\frac{(b-a)^{\alpha}}{\alpha}\|f\|_{L^p}^p,\\ &= \frac{(b-a)^{\alpha p}}{\Gamma^p(\alpha+1)}\|f\|_{L^p}^p.\end{aligned}$$

Hence, $\|I_{a+}^{\alpha} f\|_{L^p} \leq \dfrac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\|f\|_{L^p}$.

$I_{b^-}^{\alpha} f$ is given by the same proof . ■

Proposition 1.4 *Let $\alpha > 0$, $\gamma > 0$ and $f \in L^p(a,b)$. Then,*

$$I_{a+}^{\alpha} I_{a+}^{\gamma}\varphi = I_{a+}^{\alpha+\gamma}\varphi.$$

Proof.

$$\begin{aligned}I_{a+}^{\alpha} I_{a+}^{\gamma}\varphi(x) &= \frac{1}{\Gamma\alpha}\int_a^x (x-t)^{\alpha-1}\left(\frac{1}{\Gamma(\gamma)}\int_a^t (t-s)^{\gamma-1}\varphi(s)ds\right)dt,\\ &= \frac{1}{\Gamma(\alpha)\Gamma(\gamma)}\int_a^x\int_a^t (x-t)^{\alpha-1}(t-s)^{\gamma-1}\varphi(s)dsdt,\\ &= \frac{1}{\Gamma(\alpha)\Gamma(\gamma)}\int_a^x \varphi(s)\int_s^t (x-t)^{\alpha-1}(t-s)^{\gamma-1}dtds.\end{aligned}$$

Setting the variable change $u = \frac{t-s}{x-s}$, so

$$\begin{aligned}I_{a+}^{\alpha} I_{a+}^{\gamma}\varphi(t) &= \frac{1}{\Gamma(\alpha)\Gamma(\gamma)}\int_a^t \varphi(s)(t-s)^{\alpha+\gamma-1}\int_s^t (1-u)^{\alpha-1}u^{\gamma-1}duds,\\ &= \frac{1}{\Gamma(\alpha)\Gamma(\gamma)}\int_a^t \varphi(s)(t-s)^{\alpha+\gamma-1}B(\alpha,\gamma)duds \mid B(\alpha,\gamma) = \frac{\Gamma(\alpha)\Gamma(\gamma)}{\Gamma(\alpha+\gamma)},\\ &= \frac{1}{\Gamma(\alpha)\Gamma(\gamma)}\frac{\Gamma(\alpha)\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\int_a^t \varphi(s)(t-s)^{\alpha+\gamma-1}ds,\\ &= \frac{1}{\Gamma(\alpha+\gamma)}\int_a^t (t-s)^{\alpha+\gamma-1}\varphi(s)ds,\\ &= I_{a^+}^{\alpha+\gamma}\varphi(t).\end{aligned}$$

■

Definition 1.12 *[7, 11] Let $\alpha > 0, 1 \leq p \leq \infty$. The spaces of functions $I_{a^+}^{\alpha}(L^p)$ and $I_{b^-}^{\alpha}(L^p)$ are introduced by:*

$$\begin{aligned}I_{a^+}^{\alpha}(L^p) &= \left\{f : f = I_{a^+}^{\alpha}\varphi,\ \ \varphi \in L^p(a,b)\right\},\\ I_{b^-}^{\alpha}(L^p) &= \left\{f : f = I_{b^-}^{\alpha}\phi,\ \ \phi \in L^p(a,b)\right\}.\end{aligned}$$

1.4 Riemann-Liouville Fractional derivative

Let $a, b \in \mathbb{R}, \alpha > 0$ and $1 \leq p \leq +\infty$. Set $n = [\alpha] + 1$.

Definition 1.13 *[7, 11] The Riemann -Liouville Fractional derivatives $D^{\alpha}_{a+}f$ and $D^{\alpha}_{b-}f$ of order α are defined by:*

$$\begin{aligned}\left(D^{\alpha}_{a+}f\right)(x) &= \left(\frac{d}{dx}\right)^{n}(I^{n-\alpha}_{a+}f)(x) \quad (f \in AC^{n,p}_{a^+}(a,b), x > a),\\ &= \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^{n}\int_{a}^{x}(x-t)^{n-\alpha-1}f(t)dt, \qquad (1.3)\\ (D^{\alpha}_{b-}f)(x) &= \left(-\frac{d}{dx}\right)^{n}(I^{n-\alpha}_{b-}f)(x) \quad (f \in AC^{n,p}_{b^-}(a,b), x < b),\\ &= \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dx}\right)^{n}\int_{x}^{b}(x-t)^{n-\alpha-1}f(t)dt. \qquad (1.4)\end{aligned}$$

Remark 1.2 *For $\alpha = n \in \mathbb{N}$ we have $D^{n}_{a^+} = (\frac{d}{dx})^{n}$ and $D^{n}_{b^-} = (-\frac{d}{dx})^{n}$.*

Proposition 1.5 *[7, 11] Let $\alpha > \gamma > 0$. Set $m = [\gamma] + 1$.*

1. *For all $f \in L^{p}(a,b)$ we have*

$$I^{\alpha}_{a^+}(I^{\gamma}_{a^+}f(x)) = I^{\alpha+\gamma}_{a^+}f(x), \quad x \in [a,b], \qquad (1.5)$$
$$I^{\alpha}_{b^-}(I^{\gamma}_{b^-}f(t)) = I^{\alpha+\gamma}_{b^-}f(t), \quad x \in [a,b]. \qquad (1.6)$$

2. *For all $f \in L^{p}(a,b)$ we have*

$$D^{\alpha}_{a^+}(I^{\alpha}_{a^+}f(x)) = f(x), \quad x \in [a,b], \qquad (1.7)$$
$$D^{\alpha}_{b^-}(I^{\alpha}_{b^-}f(t)) = f(t), \quad x \in [a,b]. \qquad (1.8)$$

3. *For all $f \in L^{p}(a,b)$ we have*

$$(D^{\gamma}_{a^+}I^{\alpha}_{a^+})f(x) = I^{\alpha-\gamma}_{a^+}f(x), \qquad (1.9)$$
$$(D^{\gamma}_{b^-}I^{\alpha}_{b^-})f(x) = I^{\alpha-\gamma}_{b^-}f(x). \qquad (1.10)$$

4. *If $D^{\alpha}_{a^+}f, D^{m+\alpha}_{a^+}f, D^{\alpha}_{b^-}f$ and $D^{m+\alpha}_{b^-}f$ exist then,*

$$D^{m}_{a^+}(D^{\alpha}_{a^+}f(x)) = D^{m+\alpha}_{a^+}f(x), \qquad (1.11)$$
$$D^{m}_{b^-}(D^{\alpha}_{b^-}f(x)) = D^{m+\alpha}_{b^-}f(x). \qquad (1.12)$$

5. *If $f \in AC^{n}(a,b)$ then,*

$$I^{\alpha}_{a^+}D^{\alpha}_{a^+}f(x) = f(x) - \frac{(I^{1-\alpha}_{a^+}f)(a)}{\Gamma(\alpha)}(x-a)^{\alpha-1}. \qquad (1.13)$$

1.5 Integration by parts

Theorem 1.7 *Let $f \in L^p(a,b)$ and $g \in L^q(a,b)$ such that $\frac{1}{p}+\frac{1}{q} \leq 1+\alpha$. Then, we have :*

$$\int_a^b f(x) I_{b^-}^{\alpha} g(x)dx = \int_a^b g(x) I_{a^+}^{\alpha} f(x)dx. \tag{1.14}$$

Demonstration. We have

$$\begin{aligned}\int_a^b f(x) I_{b^-}^{\alpha} g(x)dx &= \frac{1}{\Gamma(\alpha)} \int_a^b f(x) \int_x^b (t-x)^{\alpha-1} g(t)dtdx,\\ &= \frac{1}{\Gamma(\alpha)} \int_a^b \int_x^b (t-x)^{\alpha-1} f(x)g(t)dtdx.\end{aligned}$$

Using the Fubinis' theorem we obtain

$$\begin{aligned}\int_a^b f(x) I_{b^-}^{\alpha} g(x)dx &= \frac{1}{\Gamma(\alpha)} \int_a^b \int_a^t (t-x)^{\alpha-1} f(x)g(t)dtdx,\\ &= \frac{1}{\Gamma(\alpha)} \int_a^b g(t) \int_a^t (t-x)^{\alpha-1} f(x)dxdt,\\ &= \int_a^b g(t) \left(\frac{1}{\Gamma(\alpha)} \int_a^t (t-x)^{\alpha-1} f(x)dx\right) dt,\\ &= \int_a^b g(t) I_{a^+}^{\alpha} f(t)dt.\end{aligned}$$

Hence, $\int_a^b f(x) I_{b^-}^{\alpha} g(x)dx = \int_a^b g(x) I_{a^+}^{\alpha} f(x)dx.$ ■

Theorem 1.8 *Let $f \in I_{a^+}^{\alpha}(L^p), g \in I_{a^+}^{\alpha}(L^p)$ such that $\frac{1}{p}+\frac{1}{q} \leq 1+\alpha$. Then,*

$$\int_a^b f(x) D_{b^-}^{\alpha} g(x)dx = \int_a^b g(x) D_{a^+}^{\alpha} f(x)dx. \tag{1.15}$$

Demonstration. Since $f \in I_{a^+}^{\alpha}(L^p), g \in I_{a^+}^{\alpha}(L^p)$, there exist $\varphi \in L^p(a,b), \psi \in L^q(a,b)$ such that: $f(x) = I_{a^+}^{\alpha}\varphi$ and $g(x) = I_{b^-}^{\alpha}\psi$.
Taking into consideration that $\varphi = D_{a^+}^{\alpha} f$ and $\psi = D_{b^-}^{\alpha} g$, we obtain

$$\begin{aligned}\int_a^b f(x) D_{b^-}^{\alpha} g(x)dx &= \int_a^b I_{a^+}^{\alpha}\varphi(x).D_{b^-}^{\alpha} g(x)dx,\\ &= \int_a^b I_{a^+}^{\alpha}\varphi(x).\psi(x)dx,\\ &= \int_a^b \varphi(x) I_{b^-}^{\alpha}\psi(x)dx,\\ &= \int_a^b g(x).D_{a^+}^{\alpha} f(x).\end{aligned}$$

■

CHAPTER 2

FRACTIONAL SOBELEV SPACES

In this chapter, we introduce the concepts of fractional absolutely continuous functions and fractional Sobolev spaces, along with the relationship between them. We will establish some topological and metric properties of fractional Sobolev spaces.

Let $0 < \alpha < 1, 1 \leq p \leq \infty$ and $a, b \in \mathbb{R}$.

2.1 Spaces $AC^{\alpha,p}_{a^+}(a,b)$ and $AC^{\alpha,p}_{b^-}(a,b)$

Definition 2.1 *[6] We introduce the following spaces*

i) $AC^{\alpha,p}_{a^+}(a,b)$, *the set of all functions* $f : [a,b] \to \mathbb{R}$ *such that :*

$$f(x) = \frac{c}{\Gamma(\alpha)}(x-a)^{\alpha-1} + I^{\alpha}_{a^+}\varphi(x), \quad x \in [a,b], \tag{2.1}$$

where $c \in \mathbb{R}$ *and* $\varphi \in L^p(a,b)$.

ii) $AC^{\alpha,p}_{b^-}(a,b)$ *the set of all functions* $g : [a,b] \to \mathbb{R}$ *such that:*

$$g(x) = \frac{d}{\Gamma(\alpha)}(b-x)^{\alpha-1} + I^{\alpha}_{b^-}\psi(x), \quad t \in [a,b], \tag{2.2}$$

where, $d \in \mathbb{R}$ *and* $\psi \in L^p(a,b)$.

Theorem 2.1 *[3] Let* $0 < \alpha, \gamma < 1$*.Then,*

$$AC^{\alpha,p}_{a^+}(a,b) \subseteq AC^{\gamma,p}_{a^+}(a,b) \text{ if only if } \gamma \leq \alpha.$$

Theorem 2.2 *Let* $0 < \alpha < 1, 0 < \gamma < \alpha$ *and* $f \in AC^{\alpha,p}_{a^+}(a,b)$ *written in the form* (2.1). *Then, we can be written*

$$f(x) = I^{\beta}_{a^+}\psi(x), \quad \text{where } \psi(t) = \frac{c}{\Gamma(\alpha-\beta)}(x-a)^{\alpha-\beta} + I^{\alpha-\beta}_{a^+}\varphi(x).$$

Demonstration. We have

$$\begin{aligned} f(x) &= \frac{c}{\Gamma(\alpha)}(x-a)^{\alpha-1} + I^{\alpha}_{a^+}\varphi(x), \\ &= \frac{c}{\Gamma(\alpha)}(x-a)^{\alpha-1} + I^{\gamma}_{a^+}I^{\alpha-\gamma}_{a^+}\varphi(x). \end{aligned}$$

Taking into account that

$$I^{\gamma}_{a^+}\left(\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma)}(x-a)^{\alpha-\gamma}\right) = (x-a)^{\alpha-1},$$

we obtain

$$\begin{aligned} f(x) &= I^{\beta}_{a^+}\left(\frac{c}{\Gamma(\alpha-\beta)}(x-a)^{\alpha-\beta-1} + I^{\alpha-\beta}_{a^+}\varphi(x)\right), \\ f(x) &= I^{\beta}_{a^+}\psi(x), \end{aligned}$$

where

$$\psi(t) = \frac{c}{\Gamma(\alpha-\beta)}(x-a)^{\alpha-\beta} + I^{\alpha-\beta}_{a^+}\varphi(x).$$

■

Remark 2.1 *From the above definition, we have the following properties*

- *. If* $f \in AC^{\alpha,p}_{a^+}(a,b)$, *writing written in the form* (2.1) *then,* $c = I^{1-\alpha}_{a^+}u(a)$ *and* $\varphi = D^{\alpha}_{a^+}f$. *Therefore,*

$$f(x) = \frac{I^{1-\alpha}_{a^+}u(a)}{\Gamma(\alpha)}(x-a)^{\alpha-1} + I^{\alpha}_{a^+}D^{\alpha}_{a^+}f(x), \quad x \in [a,b], \tag{2.3}$$

- *ii): If* $g \in AC^{\alpha,p}_{b^-}(a,b)$, *writing written in the form* (2.2) *then,* $d = I^{n-\alpha}_{b^-}u(b)$ *and* $\psi = D^{\alpha}_{b^-}g$. *Therefore,*

$$g(x) = \frac{I^{n-\alpha}_{b^-}u(b)}{\Gamma(\alpha)}(b-x)^{\alpha-1} + I^{\alpha}_{b^-}D^{\alpha}_{b^-}g(x), \quad t \in [a,b], \tag{2.4}$$

2.2 Integration by Parts

Theorem 2.3 *Let* $f \in AC^{\alpha,p}_{a^+}(a,b)$ *and* $g \in \mathscr{C}^1([a,b])$ *such that* $g(a) = g(b) = 0$. *Then,*

$$\int_a^b f(x)(D^{\alpha}_{b^-}g)(x)dx = \int_a^b g(x)(D^{\alpha}_{a^+}f)(x)dt. \tag{2.5}$$

Demonstration. We have

$$\begin{aligned}
\int_a^b f(x)(D_{b^-}^{\alpha} g)(x)dx &= \int_a^b f(x) I_{b^-}^{1-\alpha}(D_{b^-}^1 g)(x)dx, \\
&= \int_a^b \left(\frac{c}{\Gamma(\alpha)}(x-a)^{\alpha-1} + I_{a^+}^{\alpha}\varphi(x) \right) I_{b^-}^{1-\alpha}(D_{b^-}^1 g)(x)dx, \\
&= \int_a^b \frac{c}{\Gamma(\alpha)}(x-a)^{\alpha-1} I_{b^-}^{1-\alpha}(D_{b^-}^1 g)(x)dx + \int_a^b I_{a^+}^{\alpha}\varphi(x) I_{b^-}^{1-\alpha}(D_{b^-}^1 g)(x)dx, \\
&= \frac{c}{\Gamma(\alpha)} \int_a^b (x-a)^{\alpha-1} I_{b^-}^{1-\alpha}(D_{b^-}^1 g)(x)dx + \int_a^b I_{a^+}^{\alpha}\varphi(x) I_{b^-}^{-\alpha} I_{b^-}^1 D_{b^-}^1 g(x)dx, \\
&= c I_{b^-}^{\alpha} I_{b^-}^{1-\alpha} D_{b^-}^1 g(a) + \int_a^b I_{a^+}^{\alpha}\varphi(x) I_{b^-}^{-\alpha} g(x)dx, \\
&= c I_{b^-}^{\alpha} I_{b^-}^{1} I_{b^-}^{-\alpha} D_{b^-}^1 g(a) + \int_a^b g(x) I_{a^+}^{-\alpha} I_{a^+}^{\alpha}\varphi(x)dx, \\
&= c I_{b^-}^{1} D_{b^-}^1 g(a) + \int_a^b D_{a^+}^{\alpha} I_{a^+}^{\alpha}\varphi(x) g(x)dx, \\
&= c g(a) + \int_a^b D_{a^+}^{\alpha} f(x) g(x)dx, \\
&= \int_a^b D_{a^+}^{\alpha} f(x) g(x)dx.
\end{aligned}$$

■

Theorem 2.4 *Let $p > \frac{1}{\alpha}$ and $q > \frac{1}{\alpha}$. Then, for all $f \in AC_{a^+}^{\alpha,p}(a,b)$ and $g \in AC_{b^-}^{\alpha,q}(a,b)$ we have*

$$\int_a^b f(x)(D_{b^-}^{\alpha} g)(x)dx = (I_{a^+}^{1-\alpha} f)(a) g(a) - f(b)(I_{b^-}^{1-\alpha} g)(b) + \int_a^b (D_{a^+}^{\alpha} f)(x).g(x)dx \tag{2.6}$$

Demonstration. From (2.1), there exist $\varphi \in L^p(a,b)$ and $\psi \in L^q(a,b)$ such that

$$\begin{aligned}
f(x) &= \frac{I_{a^+}^{1-\alpha} f(a)}{\Gamma(\alpha)}(x-a)^{\alpha-1} + I_{a^+}^{\alpha}\varphi(x), \\
g(x) &= \frac{I_{b^-}^{1-\alpha} g(b)}{\Gamma(\alpha)}(b-x)^{\alpha-1} + I_{b^-}^{\alpha}\psi(x).
\end{aligned}$$

On one hand

$$\begin{aligned}
\int_a^b f(x)(D_{b^-}^{\alpha} g)(x)dx &= \int_a^b f(t)\psi(x)dx, \\
&= \int_a^b \left(\frac{I_{a^+}^{1-\alpha} f(a)}{\Gamma(\alpha)}(x-a)^{\alpha-1} + I_{a^+}^{\alpha}\varphi(x) \right)\psi(x)dx, \\
&= \frac{I_{a^+}^{1-\alpha} f(a)}{\Gamma(\alpha)} \int_a^b (x-a)^{\alpha-1}\psi(x)dx + \int_a^b I_{a^+}^{\alpha}\varphi(x)\psi(x)dx, \\
&= I_{a^+}^{1-\alpha} f(a)(I_{b^-}^{\alpha}\psi)(a) + \int_a^b I_{a^+}^{\alpha}\varphi(x)\psi(x)dx, \\
&= I_{a^+}^{1-\alpha} f(a).g(a) + \int_a^b I_{a^+}^{\alpha}\varphi(x)\psi(x)dx.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\int_a^b (D_{a+}^{\alpha} f)(x)g(x)dx &= \int_a^b \varphi(x)g(x)dx, \\
&= \int_a^b \varphi(x)\left(\frac{I_{b^-}^{1-\alpha}g(b)}{\Gamma(\alpha)}(b-x)^{\alpha-1} + I_{b^-}^{\alpha}\psi(x)\right)dx, \\
&= \frac{I_{b^-}^{1-\alpha}g(b)}{\Gamma(\alpha)}\int_a^b \varphi(x)(b-x)^{\alpha-1}dx + \int_a^b \varphi(x)I_{b^-}^{\alpha}\psi(x)dx, \\
&= I_{b^-}^{1-\alpha}g(b)(I_{a+}^{\alpha}\varphi)(b) + \int_a^b I_{a+}^{\alpha}\varphi(x)\psi(x)dx, \\
&= I_{b^-}^{1-\alpha}g(b).f(b) + \int_a^b I_{a+}^{\alpha}\varphi(x\psi(x)dx.
\end{aligned}$$

Consequently:

$$\begin{aligned}
\int_a^b (f(x)(D_{b^-}^{\alpha}g)(x) - (D_{a+}^{\alpha}f)(x)g(x))dx &= I_{a^+}^{1-\alpha}f(a).g(a) + \int_a^b I_{a+}^{\alpha}\varphi(x)\psi(x)dx \\
&\quad - f(b).I_{b^-}^{1-\alpha}g(b) - \int_a^b I_{a+}^{\alpha}\varphi(x)\psi(x)dx, \\
&= (I_{a^+}^{1-\alpha}f)(a)g(a) - (I_{b^-}^{1-\alpha}g)(b)f(b).
\end{aligned}$$

Thus, we get the result. ■

2.3 Fractional Sobolev Spaces

Definition 2.2 *[6] The space $W_{a^+}^{\alpha,p}(a,b)$ is the space given by :*

$$W_{a^+}^{\alpha,p}(a,b) = \left\{u \in L^p(a,b)/\ \exists g \in L^p(a,b), \forall \varphi \in C_c^{\infty}(a,b) : \int_a^b u(x)D_{b^-}^{\alpha}\varphi(x)dx = \int_a^b g(x)\varphi(x)dx\right\}.$$

A function g given above will be called the weak left fractional derivative of order α of u, let us denote it by $D_{a+}^{\alpha}u$.
We denote by H_{a+}^{α} the space $W_{a^+}^{\alpha,2}$.

Remark 2.2 *If $\alpha = 1$ we have $D_{b^-}^1\varphi = -D^1\varphi = -\varphi'$. The weak left fractional derivative $D_{a+}^1 u$ of u coincides with the classical weak derivative u'.*
Consequently: $W_{a^+}^{1,p} = W^{1,p} = AC^{1,p} = AC_{a^+}^{1,p}$.

Theorem 2.5 *For $1 < p < \infty$ we have :*

$$W_{a^+}^{\alpha,p} = AC_{a^+}^{\alpha,p} \cap L^p$$

Demonstration. If $u \in AC_{a^+}^{\alpha,p} \cap L^p$ then, from (2.1) it follows that u has the derivative $D_{a+}^{\alpha}u \in L^p$. So, the theorem 2.3 implies that

$$\int_a^b u(t)D_{b^-}^{\alpha}\varphi(x)dx = \int_a^b D_{a+}^{\alpha}u(x)\varphi(x)dx, \ \ \forall\varphi \in C_c^{\infty}(a,b).$$

So, $u \in W^{\alpha,p}_{a^+}(a,b)$ and $u^{\alpha}_{a^+} = g = D^{\alpha}_{a^+}u \in L^p(a,b)$.
Now, let us assume that $u \in W^{\alpha,p}_{a^+}(a,b)$. Then, $u \in L^p(a,b)$ and there exists $g \in L^p(a,b)$ such that:

$$\int_a^b u(t)D^{\alpha}_{b^-}\varphi(t)dt = \int_a^b g(t)\varphi(t)dt \ \ , \forall \varphi \in C^{\infty}_c(a,b).$$

To show that $u \in AC^{\alpha,p}_{a^+} \cap L^p$ it is sufficient to check that u possesses $D^{\alpha}_{a^+}u \in L^p$, we prove that $I^{1-\alpha}_{a^+}u$ is absolutely continuous on $[a,b]$ and its classical derivative of the first order belongs to $L^p(a,b)$. We have

$$\begin{aligned} \int_a^b u(x)(D^{\alpha}_{b^-}\varphi)(x)dx &= \int_a^b u(x)I^{1-\alpha}_{b^-}(D^1_{b^-}\varphi)(x)dx, \\ &= \int_a^b u(x)I^{1-\alpha}_{b^-}(-D^1\varphi)(x)dx, \\ &= \int_a^b (I^{1-\alpha}_{a^+}u)(x)(-D^1\varphi)(x)dx, \\ &= -\int_a^b (I^{1-\alpha}_{a^+}u)(x)(D^1\varphi)(x)dx. \end{aligned}$$

Consequently,

$$\int_a^b u(t)(D^{\alpha}_{b^-}\varphi)(t)dt = \int_a^b g(t)\varphi(t)dt, \quad \forall \varphi \in C^{\infty}_c(a,b).$$

So, $I^{1-\alpha}_{a^+} \in W^{1,p}_{a^+}$,Consequently, $I^{1-\alpha}_{a^+}u$ is absolutely continuous and its classical derivative of the first order belongs to $L^p(a,b)$. ■

Theorem 2.6 *If $u \in W^{\alpha,p}_{a^+}(a,b)$ then,*

$$u(x) = \frac{I^{1-\alpha}_{a^+}u(a)}{\Gamma(\alpha)}(x-a)^{\alpha-1} + I^{\alpha}_{a^+}D^{\alpha}_{a^+}u(x). \tag{2.7}$$

Demonstration. Note that for $x \in (a,b)$ we can write $u(x) = \dfrac{d}{dx}I^{\alpha}_{a^+}I^{1-\alpha}_{a^+}u(x)$.
Putting $v = I^{1-\alpha}_{a^+}u$, then $v' = D^{\alpha}_{a^+}u \in L^p(a,b)$ and we have $u(x) = \dfrac{d}{dx}I^{\alpha}_{a^+}v(x)$.
We have

$$\begin{aligned} I^{\alpha}_{a^+}v(x) &= \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha}v(t)dt, \\ &= \frac{1}{\Gamma(\alpha)}\left[-\frac{(x-t)^{\alpha+1}}{\alpha+1}v(t)\right]_a^x + \frac{1}{\Gamma(\alpha)}\int_a^x \frac{(x-t)^{\alpha+1}}{\alpha+1}v'(t)dt, \\ &= \frac{(x-a)^{\alpha+1}}{(\alpha+1)\Gamma(\alpha)}v(a) + \frac{1}{(\alpha+1)\Gamma(\alpha)}\int_a^x (x-t)^{\alpha}v'(t)dt. \end{aligned}$$

Using the following formula

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)dt = b'(x)f(x,b(x)) - a'(x)f(x,a(x)) + \int_{a(x)}^{b(x)} \frac{\partial f}{\partial x}(x,t)dt,$$

we obtain

$$\begin{aligned}\frac{d}{dx}I_{a^+}^{\alpha}v &= \frac{(x-a)^{\alpha}}{\Gamma(\alpha)}v(a)+\frac{1}{(\alpha+1)\Gamma(\alpha)}\frac{d}{dx}\int_a^x (x-t)^{\alpha+1}v'(t)dt,\\ &= \frac{(x-a)^{\alpha}}{\Gamma(\alpha)}v(a)+\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha}v'(t)dt,\\ &= \frac{I_{a^+}^{\alpha}u(a)}{\Gamma(\alpha)}(x-a)^{\alpha-1}+I_{a^+}^{\alpha}D_{a^+}^{\alpha}u(x).\end{aligned}$$

■

Remark 2.3 *It follows from the preceding theorem that*

1. *If $p<\frac{1}{1-\alpha}$ then, $AC_{a^+}^{\alpha,p}\subset L^p$.*
   So, $W_{a^+}^{\alpha,p}(a,b)=AC_{a^+}^{\alpha,p}(a,b)$.
2. *If $p\geq\frac{1}{1-\alpha}$ then, $W_{a^+}^{\alpha,p}(a,b)$ is the set of all functions belonging to $AC_{a^+}^{\alpha,p}(a,b)$, satisfy the condition $I_{a^+}^{1-\alpha}u(a)=0$.*

Indeed, if $u\in AC_{a^+}^{\alpha,p}(a,b)$ then, $D_{a^+}^{\alpha}u\in L^p(a,b)$ and we have

$$u(x)=\frac{I_{a^+}^{1-\alpha}u(a)}{\Gamma(\alpha)}(x-a)^{\alpha-1}+I_{a^+}^{\alpha}D_{a^+}^{\alpha}u(x).$$

Then,

1. If $p<\frac{1}{1-\alpha}$ then, $(x-a)^{\alpha-1}\in L^p(a,b)$, and since $D_{a^+}^{\alpha}u\in L^p(a,b)$ we obtain $I_{a^+}^{\alpha}D_{a^+}^{\alpha}u\in L^p(a,b)$. So, $u\in L^p(a,b)$.
2. If $p\geq\frac{1}{1-\alpha}$ then, $(x-a)^{\alpha-1}\notin L^p(a,b)$. However $I_{a^+}^{\alpha}D_{a^+}^{\alpha}u\in L^p(a,b)$. So, $u\in L^p(a,b)$ if and only if $(I_{a^+}^{\alpha}u)(a)=0$.

Theorem 2.7 (Poincaré inequality) *Let $u\in D_{a^+}^{\alpha,p}(a,b)$. Then,*

$$\|u-\frac{I_{a^+}^{1-\alpha}u(a)}{\Gamma(\alpha)}(x-a)^{\alpha-1}\|_{L^p}\leq\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\|D_{a^+}^{\alpha}u\|_{L^p}. \tag{2.8}$$

In particular, If $I_{a^+}^{1-\alpha}u(a)=0$ we get

$$\|u\|_{L^p}\leq\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\|D_{a^+}^{\alpha}u\|_{L^p}. \tag{2.9}$$

Demonstration. From (2.7), we have

$$u(x)-\frac{(x-a)^{\alpha-1}I_{a^+}^{\alpha}u(a)}{\Gamma(\alpha)}=I_{a^+}^{\alpha}D_{a^+}^{\alpha}u$$

So, from (1.1) we obtain

$$\begin{aligned}\|u-\frac{I_{a^+}^{1-\alpha}u(a)}{\Gamma(\alpha)}(x-a)^{\alpha-1}\|_{L^p} &= \|I_{a^+}^{\alpha}D_{a^+}^{\alpha}u\|,\\ &\leq \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\|D_{a^+}^{\alpha}u\|_{L^p}.\end{aligned}$$

■

2.4 Norms in $W^{\alpha,p}_{a^+}(a,b)$

Definition 2.3 *[6] We consider in the space $W^{\alpha,p}_{a^+}$ two norms ${}^1\|.\|_{W^{\alpha,p}_{a^+}}$ and ${}^2\|.\|_{W^{\alpha,p}_{a^+}}$ given by:*

$$ {}^1\|u\|_{W^{\alpha,p}_{a^+}} = (\|u\|^p_{L^p} + \|D^\alpha_{a^+}u\|^p_{L^p})^{\frac{1}{p}}, \tag{2.10}$$

$$ {}^2\|u\|_{W^{1,p}_{a^+}} = (|I^{1-\alpha}_{a^+}u(a)|^p + \|D^\alpha_{a^+}u\|^p_{L^p})^{\frac{1}{p}}. \tag{2.11}$$

Theorem 2.8 *The norm ${}^1\|.\|_{W^{1,p}_{a^+}}$ is equivalent to the norm ${}^2\|u\|_{W^{1,p}_{a^+}}$.*

Demonstration. We discus two cases:

Case 1: $p < \dfrac{1}{1-\alpha}$

For $u \in W^{\alpha,p}_{a^+}$, using (2.7) we can write

$$u(x) = \frac{I^{1-\alpha}_{a^+}u(a)}{\Gamma(\alpha)}(x-a)^{\alpha-1} + I^\alpha_{a^+}D^\alpha_{a^+}u(x).$$

Then, we have

$$\|u\|^p_{L^p} = \int_a^b \left| \frac{I^{1-\alpha}_{a^+}u(a)}{\Gamma(\alpha)}(x-a)^{\alpha-1} + I^\alpha_{a^+}D^\alpha_{a^+}u(x)\right|^p dx.$$

Using the following lemma

Lemma 2.1 *For all $x, y \geq 0$ and $p \geq 1$ we have :*

$$(x+y)^p \leq 2^{p-1}(x^p + y^p), \tag{2.12}$$

we obtain

$$\begin{aligned}
\|u\|^p_{L^p} &\leq 2^{p-1}\left(\frac{|I^{1-\alpha}_{a^+}u(a)|^p}{\Gamma^p(\alpha)}\int_a^b (x-a)^{(\alpha-1)p}dx + \int_a^b |I^\alpha_{a^+}D^\alpha_{a^+}u(x)|^p dx\right),\\
&= 2^{p-1}\left(\frac{|I^{1-\alpha}_{a^+}u(a)|^p}{\Gamma^p(\alpha)}\int_a^b (t-a)^{(\alpha-1)p}dt + \|I^\alpha_{a^+}D^\alpha_{a^+}u\|^p_{L^p}\right),\\
&= 2^{p-1}\left(\frac{(b-a)^{(\alpha-1)p+1}}{[(\alpha-1)p+1]\Gamma^p(\alpha)}.|I^{1-\alpha}_{a^+}u(a)|^p + \|I^\alpha_{a^+}D^\alpha_{a^+}u\|^p_{L^p}\right).
\end{aligned}$$

According to (2.12), we get

$$\begin{aligned}
\|u\|^p_{L^p} &\leq 2^{p-1}\left(\frac{(b-a)^{(\alpha-1)p+1}}{[(\alpha-1)p+1]\Gamma^p(\alpha)}.|I^{1-\alpha}_{a^+}u(a)|^p + \frac{(b-a)^{\alpha p}}{\Gamma^p(\alpha+1)}\|D^\alpha_{a^+}u\|^p_{L^p}\right),\\
&\leq M_0\left(|I^{1-\alpha}_{a^+}u(a)|^p + \|D^\alpha_{a^+}u\|^p_{L^p}\right).
\end{aligned}$$

consequently:

$$\|u\|^p_{W^{\alpha,p}_{a^+}} = \|u\|^p_{L^p} + \|D^\alpha_{a^+}u\|^p_{L^p} \leq M_1{}^2\|u\|^p_{W^{\alpha,p}_{a^+}}.$$

Reciprocally: the mean value theorem implies the existence of $x_0 \in (a,b)$ such that

$$(I^{1-\alpha}_{a^+})(x_0) = \frac{1}{b-a}\int_a^b (I^{1-\alpha}_{a^+}u)(x)dx.$$

From the absolute continuity of $(I^{1-\alpha}_{a^+}u)$, we have

$$(I^{1-\alpha}_{a^+}u)(x) = (I^{1-\alpha}_{a^+}u)(x_0) + \int_{x_0}^x D^1(I^{1-\alpha}_{a^+}u)(t)dt \ \ ,\forall x \in (a,b).$$

Consequently

$$\begin{aligned} |(I^{1-\alpha}_{a^+})u(x)| &\leq \frac{1}{b-a}\int_a^b |(I^{1-\alpha}_{a^+}u(t)|dt + \int_{x_0}^x |D^\alpha_{a^+}u(t)|dt, \\ &\leq \frac{1}{b-a}\int_a^b |(I^{1-\alpha}_{a^+}u(x)|dx + \int_a^b |D^\alpha_{a^+}u(x)|dx. \end{aligned}$$

According to Hölder inequality, we obtain

$$\begin{aligned} |(I^{1-\alpha}_{a^+}u(x)| &\leq \frac{1}{b-a}.(b-a)^{1-\frac{1}{p}}\|I^{1-\alpha}_{a^+}u\|_{L^p} + (b-a)^{1-\frac{1}{p}}\|D^\alpha_{a^+}u\|_{L^p}, \\ &= (b-a)^{-\frac{1}{p}}\|I^{1-\alpha}_{a^+}u\|_{L^p} + (b-a)^{1-\frac{1}{p}}\|D^\alpha_{a^+}u\|_{L^p}. \end{aligned}$$

Using (1.1), we obtain

$$\|I^{1-\alpha}_{a^+}u\|_{L^p} \leq \frac{(b-a)^{1-\alpha}}{\Gamma(2-\alpha)}\|u\|_{L^p}.$$

Hence,

$$\begin{aligned} |(I^{1-\alpha}_{a^+}u(x)| &\leq \frac{(b-a)^{1-\frac{1}{p}-\alpha}}{\Gamma(2-\alpha)}\|u\|_{L^p} + (b-a)^{1-\frac{1}{p}}\|D^\alpha_{a^+}u\|_{L^p}, \\ &\leq M_1\left(\|u\|_{L^p} + \|D^\alpha_{a^+}u\|_{L^p}\right). \end{aligned}$$

In particular

$$|(I^{1-\alpha}_{a^+}u)(a)| \ \leq \ M_1(\|u\|_{L^p} + \|D^\alpha_{a^+}u\|_{L^p}).$$

So,

$$\begin{aligned} {}^2\|u\|^p_{W^{\alpha,p}_{a^+}} &= |(I^{1-\alpha}_{a^+}u)(a)|^p + \|D^\alpha_{a^+}u\|^p_{L^p}, \\ &\leq M_1(\|u\|_{L^p} + \|D^\alpha_{a^+}u\|_{L^p})^p + \|D^\alpha_{a^+}u\|^p_{L^p}, \\ &\leq 2^{p-1}M_1(\|u\|^p_{L^p} + \|D^\alpha_{a^+}u\|^p_{L^p}) + \|D^\alpha_{a^+}u\|^p_{L^p}, \\ &\leq M_2(\|u\|^p_{L^p} + \|D^\alpha_{a^+}u\|^p_{L^p}), \\ &= M_2{}^1\|u\|^p_{W^{\alpha,p}_{a^+}}. \end{aligned}$$

Thus, the equivalent of the norms ${}^1\|u\|_{W^{\alpha,p}_{a^+}}$ and ${}^2\|u\|_{W^{\alpha,p}_{a^+}}$.

Case 2: $p \geq \dfrac{1}{1-\alpha}$

In this case we have $I^{1-\alpha}_{a^+}u(a) = 0$, then

$$\begin{aligned}\|u\|^p_{L^p} &= \|I^{\alpha}_{a^+}D^{\alpha}_{a^+}u\|^p_{L^p},\\ &\leq \frac{(b-a)^{\alpha p}}{\Gamma^p(\alpha+1)}\|D^{\alpha}_{a^+}u\|^p_{L^p}.\end{aligned}$$

So,

$$\begin{aligned}{}^1\|u\|^p_{W^{\alpha,p}_{a^+}} &= \|u\|^p_{L^p} + \|D^{\alpha}_{a^+}u\|^p_{L^p},\\ &\leq \left(1 + \frac{(b-a)^{\alpha p}}{\Gamma^p(\alpha+1)}\right)\|D^{\alpha}_{a^+}u\|^p_{L^p},\\ &= \left(1 + \frac{(b-a)^{\alpha p}}{\Gamma^p(\alpha+1)}\right){}^2\|u\|^p_{W^{\alpha,p}_{a^+}}.\end{aligned}$$

Reciprocally:

$$\begin{aligned}{}^2\|u\|^p_{W^{\alpha,p}_{a^+}} &= \|D^{\alpha}_{a^+}u\|^p_{L^p},\\ &\leq \|u\|_{L^p} + \|D^{\alpha}_{a^+}u\|_{L^p},\\ &= {}^2\|u\|^p_{W^{\alpha,p}_{a^+}}.\end{aligned}$$

Thus, the equivalent of the norms ${}^1\|u\|_{W^{\alpha,p}_{a^+}}$ and ${}^2\|u\|_{W^{\alpha,p}_{a^+}}$.

■

Theorem 2.9 *The space $W^{\alpha,p}_{a^+}(a,b)$ equipped with the norm ${}^1\|.\|_{W^{\alpha,p}_{a^+}}$ or ${}^2\|.\|_{W^{\alpha,p}_{a^+}}$ is a Banach space, reflexive for $1 < p < \infty$ and separable for $1 \leq p < \infty$.*

Demonstration. We equipped the product space $E = L^p(a,b) \times L^p(a,b)$ by the norm $\|(u_1,u_2)\|_E = \|u_1\|_{L^p} + \|(u_2\|_{L^p}$, and considering the space $W^{\alpha,p}_{a^+}(a,b)$ equipped with the norm ${}^1\|.\|_{W^{\alpha,p}_{a^+}}$. We introduce the following operator:

$$\begin{aligned}T : W^{\alpha,p}_{a^+} &\rightarrow E\\ u &\rightarrow (u, D^{\alpha}_{a^+}u).\end{aligned}$$

We have $\|T(u)\|_E = (\|u\|^p_{L^p} + \|D^{\alpha}_{a^+}u\|^p_{L^p})^{\frac{1}{p}} = \|u\|_{W^{\alpha,p}_{a^+}}$.
Then, T is an isometry.
which deduce that $T(W^{\alpha,p}_{a^+})$ is a closed subspace in E .
Since E is reflexive for $1 < p < \infty$ and separable for $1 \leq p < \infty$, we get the result. ■

Remark 2.4 *The space $H^{\alpha}_{a^+}(a,b)$ is a reflexive and separable Hilbert space, with the inner product*

$$\langle u,v\rangle = \int_a^b u(x)v(x)dx + \int_a^b D^{\alpha,p}_{a^+}u(x).D^{\alpha,p}_{a^+}v(x)dx \;\; u,v \in H^{\alpha}_{a^+}(a,b).$$

CHAPTER 3

EMBEDDINGS

In this chapter, we present results on embeddings in fractional Sobolev spaces. First, we introduce some auxiliary results. Next, we discuss the relationship between fractional and integer Sobolev spaces. Finally, we present continuous and compact embeddings.

3.1 Some auxiliary results

Lemma 3.1 *Let x, y be two positive real numbers such that $y \leq x$. Then, for all $q \geq 1$ we have*

$$(x-y)^q \leq x^q - y^q.$$

Proof. Let $h : [0,1] \to \mathbb{R}$ defined by:

$$h(s) = (1-s)^q + s^q - 1.$$

We have: $h'(s) = -q[(1-s)^{q-1} - s^{q-1}]$. So,

$$h'(s) \leq 0 \text{ if } s \leq \frac{1}{2} \qquad f'(s) \geq 0 \text{ if } s \geq \frac{1}{2}.$$

Since $f(0) = f(1) = 0$, we deduce that $f \leq 0$.
Putting $s = \frac{y}{x}$ we get $(1 - \frac{y}{x})^q + s^{\frac{y}{x}} - 1 \leq 0$.
So, $(x-y)^q + y^q - x^q \leq 0$. Hence, $(x-y)^q \leq x^q - y^q$. ■

Lemma 3.2 *Let x, y be two positive real numbers such that $y \leq x$. Then, for all $0 < q \leq 1$ we have*

$$y^q - x^q \leq (x-y)^q.$$

Proof. Let $h : [0,1] \to \mathbb{R}$ defined by:

$$h(s) = (1-s)^q - s^q + 1.$$

We have: $h'(s) = -q[(1-s)^{q-1} + s^{q-1}] \leq 0$. So, $f(s) \geq f(1) = 0 \geq 0$.
Putting $s = \frac{y}{x}$ we get $(1-\frac{y}{x})^q - s^{\frac{y}{x}} + 1 \leq 0$.
So, $(x-y)^q - y^q + x^q \geq 0$. Hence, $y^q - x^q \leq (x-y)^q$. ■

Theorem 3.1 *Assume that $p > \frac{1}{\alpha}$, then for all $u \in L^p(a,b)$ the function $I^\alpha_{a^+}u$ is continuous on $(a,b]$ and the function $I^\alpha_{b^-}u$ is continuous on $[a,b)$.*

Demonstration. Let $u \in L^p(a,b)$,with $p > \frac{1}{\alpha}$and $a < y < x \leq b$. Putting,

$$|G(x,y)| = |I^\alpha_{a^+}u(x) - I^\alpha_{a^+}u(y)| = \frac{1}{\Gamma(\alpha)}\left|\int_a^x (x-t)^{\alpha-1}u(t)dt - \int_a^y (y-t)^{\alpha-1}u(t)dt\right|.$$

So,

$$\begin{aligned}
|G(x,y)| &\leq \frac{1}{\Gamma(\alpha)}|\int_a^y [(x-t)^{\alpha-1} - (y-t)^{\alpha-1}]u(t)dt| + \frac{1}{\Gamma(\alpha)}|\int_y^x (x-t)^{\alpha-1}u(t)dt|,\\
&\leq \frac{\|u\|_{L^p}}{\Gamma(\alpha)}\left(\int_a^y |(x-t)^{\alpha-1} - (y-t)^{\alpha-1}|^{\frac{p}{p-1}}dt\right)^{\frac{p-1}{p}} + \frac{\|u\|_{L^p}}{\Gamma(\alpha)}\left(\int_y^x |(x-t)^{\frac{(\alpha-1)p}{p-1}}dt\right)^{\frac{p-1}{p}}.
\end{aligned}$$

Using Lemma 3.1, we get

$$\begin{aligned}
|G(x,y)| &\leq \frac{\|u\|_{L^p}}{\Gamma(\alpha)}\left(\int_a^y |(y-t)^{\frac{(\alpha-1)p}{p-1}} - (x-t)^{\frac{p-1}{p}}|dt\right)^{\frac{p-1}{p}} + \frac{\|u\|_{L^p}}{\Gamma(\alpha)}\left(\int_y^x |(x-t)^{\frac{(\alpha-1)p}{p-1}}|dt\right)^{\frac{p-1}{p}},\\
&\leq \frac{\|u\|_{L^p}}{\Gamma(\alpha)}\left(\frac{p-1}{\alpha p-1}\right)^{\frac{p-1}{p}}\left((x-y)^{\frac{\alpha p-1}{p-1}} + (y-a)^{\frac{\alpha p-1}{p-1}} - (x-a)^{\frac{\alpha p-1}{p-1}}\right),\\
&\quad + \frac{\|u\|_{L^p}}{\Gamma(\alpha)}\left(\frac{p-1}{\alpha p-1}\right)^{\frac{p-1}{p}}(x-y)^{\frac{\alpha p-1}{p}},\\
&= \frac{\|u\|_{L^p}}{\Gamma(\alpha)}\left(\frac{p-1}{\alpha p-1}\right)^{\frac{p-1}{p}}\left(2(x-y)^{\frac{\alpha p-1}{p-1}} + (y-a)^{\frac{\alpha p-1}{p-1}} - (x-a)^{\frac{\alpha p-1}{p-1}}\right).
\end{aligned}$$

Using Lemma 3.2, we get

$$|G(x,y)| \leq \frac{3\|u\|_{L^p}}{\Gamma(\alpha)}\left(\frac{p-1}{\alpha p-1}\right)^{\frac{p-1}{p}}(x-y)^{\frac{\alpha p-1}{p-1}}.$$

Hence, $\lim_{y\to x}|I^\alpha_{a^+}u(x) - I^\alpha_{a^+}u(y)| = 0$ Therefore $I^\alpha_{a^+}u \in \mathscr{C}((a,b])$.
Using the same reasoning, we can demonstrate that $I^\alpha_{b^-}u \in \mathscr{C}([a,b))$. ■

3.2 A relationship between $W^{\alpha,p}_{a^+}(a,b)$ and $W^{1,p}(a,b)$

We introduce the following operator

$$\begin{array}{rcl} T : W^{\alpha,p}_{a^+}(a,b) & \longrightarrow & W^{1,p}(a,b) \\ u & \longmapsto & v = T(u) = I^{1-\alpha}_{a^+}u, \end{array}$$

where $W^{1,p}(a,b)$ is the usual Sobolev space on (a,b).

We have the following propositions

Proposition 3.1 *The operator T is well defined and injective.*

Proof. Let $u \in W^{\alpha,p}(a,b)$, set $v(x) = I^{1-\alpha}_{a^+}u(x)$. Then,

$$\begin{aligned} \|v\|_{L^p(a,b)} + \|v'\|_{L^p(a,b)} &= \|I^{1-\alpha}_{a^+}u\|_{L^p} + \|D^{\alpha}_{a^+}u\|_{L^p}, \\ &\leq \frac{(b-a)^{1-\alpha}}{\Gamma(2-\alpha)}\|u\|_{L^p(a,b)} + \|D^{\alpha}_{a^+}u\|_{L^p(a,b)}, \\ &\leq C.^1\|u\|_{W^{\alpha,p}_{a^+}} < \infty. \end{aligned}$$

So, $v \in W^{1,p}_{a^+}(a,b)$.
Moreover, $u \in KerT$ if and only if $I^{1-\alpha}_{a^+}u = 0$, i.e. $\int_a^x u = I^{\alpha}_{a^+}0 = 0$, which leads to $u = 0$.
Then, $I^{1-\alpha}_{a^+}$ injective. ■

Proposition 3.2 *The operator T is surjective:*

i) *from $W^{\alpha,p}_{a^+}(a,b)$ to $W^{1,p}(a,b)$ if $p < \frac{1}{1-\alpha}$,*

ii) *from $W^{\alpha,p}_{a^+}(a,b)$ to $\{v \in W^{1,p}(a,b) : v(a) = 0\}$ if $p \geq \frac{1}{1-\alpha}$.*

Proof. Let $u \in W^{\alpha,p}_{a^+}(a,b)$. Then, $v = I^{1-\alpha}_{a^+}u$ if and only if $u = \frac{d}{dx}I^{\alpha}_{a^+}v = D^{1-\alpha}_{a^+}v$.
Note that

$$\begin{aligned} I^{\alpha}_{a^+}v &= \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}v(t)dt, \\ &= \frac{1}{\Gamma(\alpha)}\left(\left[\frac{-(x-t)^{\alpha}}{\alpha}v(t)\right]_a^x + \int_a^x \frac{(x-t)^{\alpha}}{\alpha}v'(t)dt\right), \\ &= \frac{(x-a)^{\alpha}}{\alpha\Gamma(\alpha)}v(a) + \frac{1}{\alpha\Gamma(\alpha)}\int_a^x (x-t)^{\alpha}v'(t)dt. \end{aligned}$$

So,

$$\begin{aligned} u(x) &= \frac{d}{dx}I^{\alpha}_{a^+}v, \\ &= \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)}v(a) + \frac{1}{\alpha\Gamma(\alpha)}\left[(x-t)^{\alpha}v'(t)dt\right]_{\{x=t\}} + \frac{1}{\alpha\Gamma(\alpha)}\int_a^x \frac{\partial}{\partial x}\left[(x-t)^{\alpha}v'(t)\right]dt, \\ &= \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)}v(a) + \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}v'(t)dt, \\ &= \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)}v(a) + I^{\alpha}_{a^+}v'(x). \end{aligned}$$

We debusses two cases

1. if $p < \frac{1}{1-\alpha}$ then, $\dfrac{(x-a)^{\alpha-1}}{\Gamma(\alpha)}v(a) \in L^p$ *and* $I^\alpha_{a^+}v' \in L^p(a,b)$.
   So, $u \in W^{\alpha,p}_{a^+}(a,b)$. Therefore, $T : W^{\alpha,p}_{a^+}(a,b) \longrightarrow W^{1,p}_{a^+}(a,b)$ is surjective.
2. if $p \geq \frac{1}{1-\alpha}$ then, $v(a) = I^{1-\alpha}_{a^+}u(a) = 0, u = I^\alpha_{a^+}v' \in L^p(a,b)$ and $I^\alpha_{a^+}u = v' \in L^p(a,b)$.
   So, $T : W^{\alpha,p}_{a^+} \longrightarrow \left\{v \in W^{1,p}_{a^+}(a,b) : v(a) = 0\right\}$ is surjective.

■

Proposition 3.3 *The operator T is an isomorphism.*

Proof. Let $u \in W^{\alpha,p}_{a^+}(a,b)$. From Proposition 3.1, we have

$$\|Tu\|_{W^{\alpha,p}_{a^+}} \leq C.^1\|u\|_{W^{\alpha,p}_{a^+}}.$$

Then, T is continuous.
Now, let $v \in W^{1,p}(a,b)$.

1. if $p < \frac{1}{1-\alpha}$ then,

$$\begin{aligned}
\|T^{-1}v\|_{W^{\alpha,p}_{a^+}} &= \|T^{-1}v\|_{L^p} + \|D^\alpha_{a^+}T^{-1}v\|_{L^p} \\
&= \left\|\frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)}v(a) + I^\alpha_{a^+}v'(x)\right\|_{L^p} + \|D^\alpha_{a^+}D^{1-\alpha}_{a^+}v\|_{L^p}, \\
&\leq \frac{1}{\Gamma(\alpha)}|v(a)|.\|(x-a)^{\alpha-1}\|_{L^p} + \|I^\alpha_{a^+}v'(x)\|_{L^p} + \|D^\alpha_{a^+}D^{1-\alpha}_{a^+}v\|_{L^p}, \\
&= \frac{(x-a)^{(\alpha-1)+\frac{1}{p}}}{[(\alpha-1)p+1]^{\frac{1}{p}}\Gamma(\alpha)}|v(a)| + \|I^\alpha_{a^+}v'(x)\|_{L^p} + \|v'\|_{L^p}
\end{aligned}$$

From the Sobolev embedding in $W^{1,p}(a,b)$ we have:

$$\frac{(x-a)^{(\alpha-1)+\frac{1}{p}}}{[(\alpha-1)p+1]^{\frac{1}{p}}\Gamma(\alpha)}|v(a)| \leq \frac{(x-a)^{(\alpha-1)+\frac{1}{p}}}{[(\alpha-1)p+1]^{\frac{1}{p}}\Gamma(\alpha)}\|v\|_{L^\infty} \leq c_1\|v\|_{W^{1,p}}.$$

So,

$$\begin{aligned}
\|T^{-1}v\|_{W^{\alpha,p}_{a^+}} &\leq c_1\|v\|_{W^{1,p}} + \frac{(b-a)^\alpha}{\Gamma(1+\alpha)}\|v'\|_{L^p} + \|v'\|_{L^p}, \\
&\leq C\|v\|_{W^{1,p}}.
\end{aligned}$$

2. If $p \geq \frac{1}{1-\alpha}$ we have $v(a) = 0$. Then,

$$\begin{aligned}
\|T^{-1}v\|_{W^{\alpha,p}_{a^+}} &= \|I^\alpha_{a^+}v'\|_{L^p} + \|D^\alpha_{a^+}I^\alpha_{a^+}v'\|_{L^p} \\
&= \|I^\alpha_{a^+}v'\|_{L^p} + \|v'\|_{L^p} \\
&\leq \frac{(b-a)^\alpha}{\Gamma(1+\alpha)}\|v'\|_{L^p} + \|v'\|_{L^p} \\
&\leq M\|v\|_{W^{1,p}}.
\end{aligned}$$

Therefore, T^{-1} is an isomorphism. ■

3.3 Continuous embeddings

Theorem 3.2 *Let* $0 < \beta < 1$.

1. *If* $\beta < \alpha$ *then,*

$$AC_{a^+}^{\alpha,p} \subset I_{a^+}^{\beta}(L^q) \subset AC_{a^+}^{\beta,p}, \text{ for all } 1 \leq p < \infty \text{ and } \leq q < \frac{1}{1-\alpha+\beta},$$

which deduce that $W^{\alpha,p} \subseteq W^{\beta,q}$

2. *if* $\alpha = \beta$ *then,*

$$AC^{\alpha,p} \subset AC^{\beta,q}, \text{ for all } 1 \leq q \leq p < \infty,$$

which deduce that $W_{a^+}^{\alpha,p}(a,b) \subset W_{a^+}^{\alpha,q}(a,b)$

Demonstration.

1. Let us $p \in [1, +\infty[$. Theorem 2.1 imply that

$$AC_{a^+}^{\alpha,p} \subset AC_{a^+}^{\alpha,1} \subset I_{a^+}^{\beta}(L^q).$$

On the other hand, from Theorem 2.2 , if $f \in AC_{a^+}^{\alpha,p}$ then, $f = I_{a^+}^{\beta}\varphi$ with $\varphi \in L^q$. Hence, $f \in AC_{a^+}^{\beta,p}$ with $c = 0$. Consequently,

$$AC_{a^+}^{\alpha,p} \subset I_{a^+}^{\beta}(L^q) \subset AC_{a^+}^{\beta,p}$$

Now, we have

$$W_{a^+}^{\alpha,p}(a,b) = AC_{a^+}^{\alpha,p}(a,b) \cap L^p(a,b) \subset L^q(a,b) \cap AC_{a^+}^{\beta,q}(a,b) = W_{a^+}^{\beta,q}(a,b).$$

Then, $W_{a^+}^{\alpha,p}(a,b) \subset W_{a^+}^{\alpha,q}(a,b)$.

2. By using same method.

■

Theorem 3.3 *Assume that* $\alpha < \frac{1}{2}$. *Then, we have the following embeddings*

1. *If* $1 \leq p < \frac{1}{1-\alpha} < \frac{1}{\alpha}$ *then,* $W_{a^+}^{\alpha,p}(a,b) \hookrightarrow L^q(a,b)$ *for all* $q \in [1, \frac{1}{1-\alpha})$.

2. *If* $\frac{1}{1-\alpha} < p < \frac{1}{\alpha}$ *then,* $W_{a^+}^{\alpha,p}(a,b) \hookrightarrow L^q(a,b)$ *for all* $q \in [1, \frac{p}{1-\alpha p}]$.

3. *If* $p = \frac{1}{\alpha}$ *then,* $W_{a^+}^{\alpha,p}(a,b) \hookrightarrow L^q(a,b)$ *for all* $q \in [1, +\infty)$.

4. *If* $p > \frac{1}{\alpha}$ *then,* $W_{a^+}^{\alpha,p}(a,b) \hookrightarrow L^q(a,b)$ *for all* $q \in [1, +\infty]$.
In particular, $W_{a^+}^{\alpha,p}(a,b) \hookrightarrow \mathscr{C}([a,b])$.

Demonstration. Let $u \in W_{a^+}^{\alpha,p}(a,b)$. We know according to (2.7) that

$$u(x) = \frac{I_{a^+}^{1-\alpha}u(a)}{\Gamma(\alpha)}(x-a)^{\alpha-1} + I_{a^+}^{\alpha}D_{a^+}^{\alpha}u(x).$$

Note that $\frac{I_{a^+}^{1-\alpha}u(a)}{\Gamma(\alpha)}(x-a)^{\alpha-1} \in L^p(a,b)$ if only if $p < \frac{1}{1-\alpha}$ or $I_{a^+}^{1-\alpha}u(a) = 0$.
So, for $q \geq 1$ we have

$$\begin{aligned} \|u\|_{L^q} &= \left\| \frac{I_{a^+}^{1-\alpha}u(a)}{\Gamma(\alpha)}(x-a)^{\alpha-1} + I_{a^+}^{\alpha}D_{a^+}^{\alpha}u \right\|_{L^q} \\ &\leq \frac{|I_{a^+}^{1-\alpha}u(a)|}{\Gamma(\alpha)} \left\|(x-a)^{\alpha-1}\right\|_{L^q} + \|I_{a^+}^{\alpha}D_{a^+}^{\alpha}u\|_{L^q} \end{aligned}$$

1. If $1 \leq p < \frac{1}{1-\alpha} < \frac{1}{\alpha}$, since $D_{a^+}^{\alpha}u \in L^p(a,b)$, from [9, Theorem 0.2] there exists $c > 0$ such that $\left\|I_{a^+}^{\alpha}D_{a^+}^{\alpha}u\right\|_{L^q} \leq c. \left\|D_{a^+}^{\alpha}u\right\|_{L^p}$, for all $q \in [1, \frac{p}{1-\alpha p}]$.
   On the other hand, $(x-a)^{\alpha-1} \in L^q(a,b)$ if only if $q < \frac{1}{1-\alpha}$. In this case we get

   $$\frac{|I_{a^+}^{1-\alpha}u(a)|}{\Gamma(\alpha)} \left\|(x-a)^{\alpha-1}\right\|_{L^q} \leq \frac{(b-a)^{1-\alpha+\frac{1}{q}}}{\Gamma(\alpha).[(1-\alpha)q+1]^{\frac{1}{q}}}|I_{a^+}^{1-\alpha}u(a)|.$$

   Hence, for $q \in [1, \frac{1}{1-\alpha}) \cap [1, \frac{p}{1-\alpha p}] = [1, \frac{1}{1-\alpha})$ there exists $M > 0$ such that

   $$\|u\|_{L^q} \leq M \left[|I_{a^+}^{1-\alpha}u(a)|^p + \|D_{a^+}^{\alpha}u\|_{L^p}^p\right]^{\frac{1}{p}} = M \|u\|_{{}^2W_{a^+}^{\alpha,p}}.$$

   So, $W_{a^+}^{\alpha,p} \hookrightarrow L^q(a,b)$ for all $q \in [1, \frac{1}{1-\alpha})$.

2. If $\frac{1}{1-\alpha} < p < \frac{1}{\alpha}$ then, $I_{a^+}^{1-\alpha}u(a) = 0$. Therefore, from [9, Theorem 0.2] there exists $c > 0$ such that for all $q \in [1, \frac{p}{1-\alpha p}]$ we have

   $$\begin{aligned} \|u\|_{L^q} &= \|I_{a^+}^{\alpha}D_{a^+}^{\alpha}u\|_{L^q} \\ &\leq c\,\|D_{a^+}^{\alpha}u\|_{L^p} \\ &= c\,\|u\|_{W_{a^+}^{\alpha,p}}. \end{aligned}$$

   Then , $W_{a^+}^{\alpha,p} \hookrightarrow L^q(a,b)$ for all $q \in [1, \frac{p}{1-\alpha p}]$.

3. If $p = \frac{1}{\alpha}$ then, $I_{a^+}^{1-\alpha}u(a) = 0$. So, from [9, Theorem 0.3] there exists $c > 0$ such that for all $q \in [1, \infty)$ we have

   $$\begin{aligned} \|u\|_{L^q} &= \|I_{a^+}^{\alpha}D_{a^+}^{\alpha}u\|_{L^q} \\ &\leq c\,\|D_{a^+}^{\alpha}u\|_{L^p} \\ &= c\,\|u\|_{W_{a^+}^{\alpha,p}}. \end{aligned}$$

   So , $W_{a^+}^{\alpha,p} \hookrightarrow L^q(a,b)$ for all $q \in [1, \infty)$.

4. If $p > \frac{1}{\alpha}$ then, $I_{a^+}^{1-\alpha}u(a) = 0$. So, from [9, Theorem 0.4] there exists $c > 0$ such that for all $q \in [p, \infty]$ we have
$$\|u\|_{L^q} \leq c\,\|u\|_{W_{a^+}^{\alpha,p}}.$$
So , $W_{a^+}^{\alpha,p} \hookrightarrow L^q(a,b)$ for all $q \in [p, \infty]$.
In particular, since $p > \frac{1}{\alpha}$, using same arguments as in Theorem 3.1, we deduce that $u \in \mathscr{C}([a,b])$. So,
$$\|u\|_{\mathscr{C}([a,b])} = \|u\|_{L^\infty} \leq c\,\|u\|_{W_{a^+}^{\alpha,p}}$$
Hence, $W_{a^+}^{\alpha,p} \hookrightarrow \mathscr{C}([a,b])$.

■

In the same context, we ca, prove the following theorems

Theorem 3.4 *Assume that $\alpha > \frac{1}{2}$. Then, we have the following embeddings*

1. *If $1 \leq p \leq \frac{1}{\alpha}$ then, $W_{a^+}^{\alpha,p}(a,b) \hookrightarrow L^q(a,b)$ for all $q \in [1, \frac{1}{1-\alpha})$.*
2. *If $\frac{1}{\alpha} < p < \frac{1}{1-\alpha}$ then, $W_{a^+}^{\alpha,p}(a,b) \hookrightarrow L^q(a,b)$ for all $q \in [1, \frac{1}{1-\alpha})$.*
3. *If $p \geq \frac{1}{1-\alpha}$ then, $W_{a^+,0}^{\alpha,p}(a,b) \hookrightarrow L^q(a,b)$ for all $q \in [p, +\infty]$.*
   In particular, $W_{a^+,0}^{\alpha,p}(a,b) \hookrightarrow \mathscr{C}([a,b])$.

Theorem 3.5 *Assume that $\alpha = \frac{1}{2}$. Then, we have the following embeddings*

1. *If $1 \leq p \leq 2$ then, $W_{a^+}^{\frac{1}{2},p}(a,b) \hookrightarrow L^q(a,b)$ for all $q \in [1,2)$.*
2. *If $p = 2$ then, $H_{a^+}^{\frac{1}{2}}(a,b) \hookrightarrow L^q(a,b)$ for all $q \in [1, +\infty)$.*
3. *If $p > 2$ then, $W_{a^+,0}^{\frac{1}{2},p}(a,b) \hookrightarrow L^q(a,b)$ for all $q \in [p, +\infty]$.*
   In particular, $W_{a^+,0}^{\frac{1}{2},p}(a,b) \hookrightarrow \mathscr{C}([a,b])$.

3.4 Compact embedding

Theorem 3.6 *If the embedding $W_{a^+}^{\alpha,p}(a,b) \hookrightarrow L^q(a,b)$ $(q < +\infty)$ is satisfied, then it is compact.*

Demonstration. Let (u_n) ba a bounded sequence in $W_{a^+}^{\alpha,p}(a,b)$. Then, $(v_n) = (Tu_n)$ bounded in $W^{1,p}(a,b)$. So, we can extract a subsequence $(v_{n\ell})$ weakly convergence to $v = Tu$ in $W^{1,p}(a,b)$.
From usually Sobolev embeddings, we can extract a subsequence (v_{nk}) convergence to Tu in $L^q(a,b)$, i.e, $\|v_{nk} - v\|_{L^q} \to 0$.
Now, we have
$$\begin{aligned}\|u_{nk} - u\|_{L^q} &= \|T^{-1}(v_{nk} - v)\|_{L^q}, \\ &= \left\|\frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)}(v_{nk}(a) - v(a)) + I_{a^+}^{\alpha}(v'_{nk} - v')\right\|_{L^q}, \\ &\leq \frac{|v_{nk}(a) - v(a)|}{\Gamma(\alpha)}\|(x-a)^{\alpha-1}\|_{L^q} + \|I_{a^+}^{\alpha}(v'_{nk} - v')\|_{L^q}\end{aligned}$$

From (1.1), we obtain

$$\begin{aligned} \|I^{\alpha}_{a^+}(v'_{nk}-v')\|_{L^q} &\leq \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\|v'_{nk}-v'\|_{L^q}, \\ &\leq M\|v'_{nk}-v'\|_{L^p}\to 0. \end{aligned}$$

- If $I^{1-\alpha}_{a^+}u(a)=0$, then we obtain directly the convergence of (u_{nk}) to u in $L^q(a,b)$.
- If $I^{1-\alpha}_{a^+}u(a)\neq 0$ and $q<\frac{1}{1-\alpha}$ then, we have

$$\begin{aligned} \|u_{nk}-u\|_{L^q} &\leq C_1\|v_{nk}-v\|_{L^\infty}+C_2\|I^{\alpha}_{a^+}(v'_{nk}-v')\|_{L^q}, \\ &\leq C\left(\|v_{nk}-v\|_{W^{1,p}}+\|I^{\alpha}_{a^+}(v'_{nk}-v')\|_{L^q}\right)\to 0. \end{aligned}$$

  So, the convergence of (u_{nk}) to u in $L^q(a,b)$.

Thus, the compactness of the embedding. ■

Theorem 3.7 *If $\max\{\frac{1}{\alpha},\frac{1}{1-\alpha}\}<p<\infty$ then, the embedding $W^{\alpha,p}_{a^+}(a,b)\hookrightarrow \mathscr{C}([a,b])$ is compact.*

Demonstration. Since $W^{\alpha,p}_{a^+}(a,b)$ is reflexive, we we only have to prove that for all sequence $(u_n)\subset W^{\alpha,p}_{a^+}(a,b)$, weakly converges to u in $W^{\alpha,p}_{a^+}(a,b)$, we obtain that (u_n) is strongly converge to u in $\mathscr{C}([a,b])$, i.e $\|u_n-u\|_{L^\infty}\to 0$.

Let $(u_n)\subset W^{\alpha,p}_{a^+}(a,b)$, be a sequence weakly converge to u in $W^{\alpha,p}_{a^+}(a,b)$. Since $W^{\alpha,p}_{a^+}(a,b)\hookrightarrow \mathscr{C}([a,b])$, (u_n) weakly converges to u in $\mathscr{C}([a,b])$. Moreover, (u_n) is bounded in $W^{\alpha,p}_{a^+}(a,b)$. Hence, there exists a constant $C>0$ such that $\|D^{\alpha}_{a^+}u_n\|_{L^p}\leq C$.
Since $p>\frac{1}{1-\alpha}$, we obtain $I^{1-\alpha}_{a^+}u(a)=0$. So, $u=I^{\alpha}_{a^+}D^{\alpha}_{a^+}u$. Hence, from Theorem 3.1 we get for all $x,y\in[a,b]$:

$$\begin{aligned} |u(x)-u(y)| &= |I^{\alpha}_{a^+}D^{\alpha}_{a^+}u(x)-I^{\alpha}_{a^+}D^{\alpha}_{a^+}u(y)|, \\ &\leq \frac{3\|D^{\alpha}_{a^+}u\|_{L^p}}{\Gamma(\alpha)}\left(\frac{p-1}{\alpha p-1}\right)^{\frac{p-1}{p}}|x-y|^{\frac{\alpha p-1}{p-1}}. \\ &\leq \frac{3C}{\Gamma(\alpha)}\left(\frac{p-1}{\alpha p-1}\right)^{\frac{p-1}{p}}|x-y|^{\frac{\alpha p-1}{p-1}}, \\ &= M|x-y|^{\frac{\alpha p-1}{p-1}}. \end{aligned}$$

Hence, u is uniformly Lipschitz on $[a,b]$. From Ascoli's theorem, (u_n) is relatively compact in $\mathscr{C}([a,b])$. Consequently, there exists a subsequence (u_{nk}) of (u_n) converging strongly in $\mathscr{C}([a,b])$ to u by uniqueness of the weak limit. ■

3.5 Embedding of the subspace $W^{\alpha,p}_{a^+,0}(a,b)$

Definition 3.1 *The subspace $W^{\alpha,p}_{a^+,0}(a,b)$ is the set defined by*

$$W^{\alpha,p}_{a^+,0}(a,b)=\{u\in W^{\alpha,p}_{a^+}(a,b): I^{1-\alpha}_{a^+}u(a)=u(b)=0\}.$$

Setting: $H^{\alpha}_{a^+,0}(a,b)=W^{\alpha,2}_{a^+,0}(a,b)$.

Remark 3.1 *According to Poincaré-inequality (2.9), the quantity* $\|D^{\alpha}_{a^+}u\|_{L^p(a,b)}$ *defines a norm on* $W^{\alpha,p}_{a^+,0}(a,b)$, *equivalent to norms* ${}^1\|.\|_{W^{\alpha,p}_{a^+}(a,b)}$ *and* ${}^2\|.\|_{W^{\alpha,p}_{a^+}(a,b)}$. *This norm is denoted by* $\|.\|_{W^{\alpha,p}_{a^+,0}}$.

Theorem 3.8 *We have the following embeddings*

1. *If* $1 \le p < \frac{1}{\alpha}$ *then,* $W^{\alpha,p}_{a^+,0}(a,b) \hookrightarrow L^q(a,b)$ *for all* $q \in [1, \frac{p}{1-\alpha p}]$.
2. *If* $p = \frac{1}{\alpha}$ *then,* $W^{\alpha,p}_{a^+,0}(a,b) \hookrightarrow L^q(a,b)$ *for all* $q \in [1,+\infty)$.
3. *If* $1 \le p > \frac{1}{\alpha}$ *then,* $W^{\alpha,p}_{a^+,0}(a,b) \hookrightarrow L^q(a,b)$ *for all* $q \in [1,+\infty]$.
   In particular, $W^{\alpha,p}_{a^+,0}(a,b) \hookrightarrow \mathscr{C}([a,b])$.

Demonstration. Let $u \in W^{\alpha,p}_{a^+}(a,b)$. According to (2.7) and the definition 3.1 we have

$$u(x) = I^{\alpha}_{a^+}D^{\alpha}_{a^+}u(x).$$

So, for $q \ge 1$ we have

$$\|u\|_{L^q} = \|I^{\alpha}_{a^+}D^{\alpha}_{a^+}u\|_{L^q}.$$

1. If $1 \le p < \frac{1}{\alpha}$, from [, Theorem 0.2] there exists $c > 0$ such that for all $q \in \left[1, \frac{p}{1-\alpha p}\right]$ we have
$$\|u\|_{L^q} = \|I^{\alpha}_{a^+}D^{\alpha}_{a^+}u\|_{L^q} \le c.\|D^{\alpha}_{a^+}u\|_{L^p} = c.\|D^{\alpha}_{a^+}u\|_{W^{\alpha,p}_{a^+,0}}.$$
   So, $W^{\alpha,p}_{a^+,0} \hookrightarrow L^q(a,b)$ for all $q \in [1, \frac{p}{1-\alpha p}]$.

2. If $p = \frac{1}{\alpha}$ then, from [, Theorem 0.3] there exists $c > 0$ such that for all $q \in [1,\infty)$ we have
$$\begin{aligned}\|u\|_{L^q} &= \|I^{\alpha}_{a^+}D^{\alpha}_{a^+}u\|_{L^q}, \\ &\le c\,\|D^{\alpha}_{a^+}u\|_{L^p}, \\ &= c\,\|u\|_{W^{\alpha,p}_{a^+,0}}.\end{aligned}$$
   So , $W^{\alpha,p}_{a^+,0} \hookrightarrow L^q(a,b)$ for all $q \in [1,\infty)$.

3. If $p > \frac{1}{\alpha}$ then, from [, Theorem 0.4] there exists $c > 0$ such that for all $q \in [p,\infty]$ we have
$$\|u\|_{L^q} \le c\,\|u\|_{W^{\alpha,p}_{a^+}}.$$
   So , $W^{\alpha,p}_{a^+,0} \hookrightarrow L^q(a,b)$ for all $q \in [p,\infty]$.
   In particular, since $p > \frac{1}{\alpha}$, using same arguments as in Theorem 3.1, we deduce that $u \in \mathscr{C}([a,b])$. So,
$$\|u\|_{\mathscr{C}([a,b])} = \|u\|_{L^\infty} \le c\,\|u\|_{W^{\alpha,p}_{a^+}}.$$
   Hence, $W^{\alpha,p}_{a^+,0} \hookrightarrow \mathscr{C}([a,b])$.

■

Arguing as the previous section, we can prove the following compact embeddings

Theorem 3.9 *If the embedding* $W^{\alpha,p}_{a^+,0}(a,b) \hookrightarrow L^q(a,b)$ $(q < +\infty)$ *is satisfied, then it is compact.*

Theorem 3.10 *If* $p > \max\{\frac{1}{\alpha}, \frac{1}{1-\alpha}\}$ *then, the embedding* $W^{\alpha,p}_{a^+,0}(a,b) \hookrightarrow \mathscr{C}([a,b])$ *is compact.*

CHAPTER 4

APPLICATIONS

In this chapter, we will study a non-linear (semi-linear) problem, using faedo-Galerkin method and Schauder fixed point method.

Assume that $0 < \alpha < 1$ and let $f : (a,b) \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function, i.e

$$\begin{cases} x \mapsto f(x,u) \text{ is measurable on } (a,b), \text{for all } u \in \mathbb{R}, \\ u \mapsto f(x,u) \text{ is continuous on } \mathbb{R}, \text{a.e } x \in (a,b). \end{cases}$$

Consider the following problem

$$\begin{cases} D^{\alpha}_{b^-} D^{\alpha}_{a^+} u(x) = f(x,u) \quad : \quad \text{in } (a,b), \\ \\ I^{1-\alpha}_{a^+} u(a) = u(b) = 0. \end{cases} \tag{4.1}$$

Before studying the problem, it is necessary to introduce these two theorems

Theorem 4.1 *[5] Let $(E, \langle .,. \rangle)$ a be finite-dimensional Hilbert space and $p : E \to E$ be a continuous mapping such that there exists $r > 0$ for which very point x on the sphere of radius r satisfies $\langle p(x), x \rangle \geq 0$. Then, there exists a point x_0 with $\|x_0\| \leq r$ such that $p(x_0) = 0$*

Theorem 4.2 (Schauder) *[5] Let E be a Banach space, K a closed convex subset of E, and T a continuous mapping from K into K such that $T(K)$ is relatively compact. Then, T has a fixed point.*

Theorem 4.3 (Carathéodory) *[5] Let Ω a bounded open set of $\mathbb{R}^d$ and let f be a continuous function from $\Omega \times \mathbb{R}$ to $\mathbb{R}$ satisfying*

$$|f(x,u)| \leq \mu(x) + \lambda(x)|u|, \quad a.e\ x \in \Omega, \quad \mu \in L^2(\Omega), \lambda \in L^\infty(\Omega). \tag{4.2}$$

Then, the operator $f(.,u)$ from $L^2(\Omega)$ into $L^2(\Omega)$ is continuous.

4.1 Statement of the problem

Taking into consideration that each weak solution of (4.1) belongs to $H^{\alpha}_{a^+,0}(a,b)$.
To find the variational formulation it is necessary to follow the following steps :

- We multiply the first equation of (4.1) by a test function v smooth enough, we get

$$D^{\alpha}_{b^-}D^{\alpha}_{a^+}u(x)v(x) = f(x,u)v(x).$$

- We apply the integration by parts (2.6), we obtain

$$\int_a^b D^{\alpha}_{a^+}u(x)D^{\alpha}_{a^+}v(x)dx + I^{1-\alpha}_{a^+}v(a).D^{\alpha}_{a^+}u(a) - v(b)I^{1-\alpha}_{a^+}D^{\alpha}_{a^+}u(b) = \int_a^b f(x,u)v(x)dx.$$

- Assume that $v \in H^{\alpha}_{a^+,0}(a,b)$ we obtain the variational formulation of (4.1)

$$\int_a^b D^{\alpha}_{a^+}u(x)D^{\alpha}_{a^+}v(x)dx = \int_a^b f(x,u)vdx. \tag{4.3}$$

We need to make sure that the above formulation (4.3) is well defined.
First, for $u,v \in H^{\alpha}_{a^+,0}(a,b)$ we have

$$\left|\int_a^b D^{\alpha}_{a^+}uD^{\alpha}_{a^+}vdx\right| \leq \|D^{\alpha}_{a^+}u\|_{L^2}\|D^{\alpha}_{a^+}v\|_{L^2} < \infty.$$

Then, the right side of (4.3) is well defined.
To prove that the left side of (4.3) be well defined, we introduce the following theorems.

Theorem 4.4 *Assume that f is a Carathéodory function, satisfying the condition* (4.2). *Then, the problem* (4.3) *is well defined.*

Demonstration. Let $u,v \in W^{\alpha,2}_{a^+,0}(a,b)$.Then, we have

$$\left|\int_a^b f(x,u)v(x)dx\right| \leq \|\mu\|_{L^2}\|v\|_{L^2} + \|\lambda\|_{L^\infty}\|u\|_{L^2}\|v\|_{L^2} < \infty.$$

■

The following theorem ensure the existence of a solution for the problem (4.1)

Theorem 4.5 *Assume that f is a Carathéodory function, satisfying the condition* (4.2). *If*

$$\Gamma^2(\alpha+1) - \|\lambda\|_{L^\infty}(b-a)^{2\alpha} > 0.$$

Then, the problem (4.3) *admits at least one solution.*

We prove this theorem using the following two methods:

4.2 Compactness method

In this section, we use the Faedo - Galerkin's method to prove the Theorem 4.5

We will demonstrate this through the following steps

- **Approximation of the space $H^{\alpha}_{a^+,0}(a,b)$:**
  Since $H^{\alpha}_{a^+,0}(a,b)$ is a separable Hilbert space, there exists a countable basis $\{V_m\}_{m=1}^{\infty}$ such that $V_m = Vect\,\{v_j\}_{j=1}^{m}$.
  Using the dot product
  $$\langle v_i, v_j\rangle = \int_a^b v_i v_j dx \;\; v_i, v_j \in V_m \subseteq V_{m+1}.$$
  Note that :$H^{\alpha}_{a^+,0}(a,b) = \overline{\bigcup_{m=1}^{+\infty} V_m}$

- **Approximate problem:** For $u_m \in V_m$ we consider the following approximate problem
  $$\int_a^b D^{\alpha}_{a^+} u_m D^{\alpha}_{a^+} v = \int_a^b f(x,u_m) v \quad \forall v \in V_m \tag{4.4}$$
  Let $P_m(u_m)$ be the function from V_m to V_m, given by
  $$\langle P_m(u_m), v\rangle = \int_a^b D^{\alpha}_{a^+} u_m D^{\alpha}_{a^+} v - \int_a^b f(x,u_m) v \quad \forall v \in V_m$$
  So, if u_m is a solution of (4.4) then, $P_m(u_m) = 0$.
  From previous, P is continuous and we have
  $$\begin{aligned}\langle P_m(u_m), u_m\rangle &= \int_a^b |D^{\alpha}_{a^+} u_m|^2 - \int_a^b f(x,u_m) u_m,\\ &= \|D^{\alpha}_{a^+} u_m\|^2_{L^2} - \int_a^b f(x,u_m) u_m,\\ &\geq \|D^{\alpha}_{a^+} u_m\|^2_{L^2} - \|\mu\|_{L^2}\|u_m\|_{L^2} - \|\lambda\|_{L^\infty}\|u_m\|^2_{L^2}\end{aligned}$$
  Using the Poincaré inequality (2.9), we obtain
  $$\begin{aligned}\langle P_m(u_m), u_m\rangle &\geq \|D^{\alpha}_{a^+} u_m\|^2_{L^2} - \frac{\|\mu\|_{L^2}(b-a)^{\alpha}}{\Gamma(\alpha+1)}\|D^{\alpha}_{a^+} u_m\|_{L^2} - \frac{\|\lambda\|_{L^\infty}(b-a)^{2\alpha}}{\Gamma^2(\alpha+1)}\|D^{\alpha}_{a^+} u_m\|^2_{L^2},\\ &= M\|D^{\alpha}_{a^+} u_m\|_{L^2}\left(\|D^{\alpha}_{a^+} u_m\|_{L^2} - r\right),\end{aligned}$$
  where
  $$M = \frac{\Gamma^2(\alpha+1) - (b-a)^{2\alpha}\|\lambda\|_{L^\infty}}{\Gamma^2(\alpha+1)}, \quad r = \frac{\Gamma(\alpha+1)(b-a)^{\alpha}\|\mu\|_{L^2}}{\Gamma^2(\alpha+1) - (b-a)^{\alpha}\|\lambda\|_{L^\infty}}.$$
  So, for u belongs to the sphere of radius r, we get $\langle P_m(u_m), u_m\rangle \geq 0$.
  From Theorem 4.1, there exists $u \in W^{\alpha}_{a^+,0}$ such that $\|u_m\|_{H^{\alpha}_{a^+,0}} \leq r$ and $P_m(u_m) = 0$, i.e u_m is a solution of the problem (4.2).

- **Prior estimate** We have

$$\begin{aligned}
\|u_m\|^2_{H^\alpha_{a^+,0}} &= \|D^\alpha_{a^+}u_m\|^2_{L^2},\\
&= \int_a^b |D^\alpha_{a^+}u_m|^2,\\
&= \int_a^b f(x,u_m)u_m,\\
&\leq \|\mu\|_{L^2}\|u_m\|_{L^2} + \|\lambda\|_{L^\infty}\|u_m\|^2_{L^2},\\
&\leq \frac{(b-a)^\alpha\|\mu\|_{L^2}}{\Gamma(\alpha+1)}\|D^\alpha_{a^+}u_m\|_{L^2} + \frac{(b-a)^{2\alpha}\|\lambda\|_{L^\infty}}{\Gamma^2(\alpha+1)}\|D^\alpha_{a^+}u_m\|_{L^2}.
\end{aligned}$$

So, $M\|u_m\|^2_{H^\alpha_{a^+,0}} \leq \dfrac{(b-a)^\alpha\|\mu\|_{L^2}}{\Gamma(\alpha+1)}\|u_m\|_{H^\alpha_{a^+,0}}$.
Hence, $\|u_m\|_{H^\alpha_{a^+,0}} \leq r$.
Therefore, (u_m) is bounded in $H^\alpha_{a^+,0}(a,b)$

- **Passage to limit**
Since (u_m) is bounded in $H^\alpha_{a^+,0}(a,b)$, there exists a subsequence (u_{mk}) such that

$$u_{mk} \rightharpoonup u \text{ in } H^\alpha_{a^+}(a,b), \qquad \text{and} \qquad D^\alpha_{a^+}u_{mk} \rightharpoonup D^\alpha_{a^+}u \text{ in } L^2(a,b).$$

Therefore, for $m \geq j$ we obtain

$$\forall v_j : \int_a^b D^\alpha_{a^+}u_{mk}D^\alpha_{a^+}v_j \longrightarrow \int_a^b D^\alpha_{a^+}uD^\alpha_{a^+}v_j$$

Using the fact that $W^{\alpha,p}_{a^+}(a,b) \hookrightarrow L^2(a,b)$ with compactness, we get

$$u_{mk} \longrightarrow u \qquad \text{in } L^2(a,b).$$

Hence, from Theorem 4.3, we have

$$f(x,u_{mk}) \longrightarrow f(x,u) \qquad \text{in } L^2(a,b).$$

So,

$$f(x,u_{mk}) \rightharpoonup f(x,u) \qquad \text{in } L^2(a,b),$$

which lead to

$$\int_a^b f(u_{mk})v_j \to \int_a^b f(x,u)v_j.$$

Hence,

$$\int_a^b D^\alpha_{a^+}uD^\alpha_{a^+}v_j = \int_a^b f(x,u)v_j, \quad \forall v_j.$$

Setting $W = \bigcup_{m=1}^{\infty} v_j$, then each $w \in W$ can be written $w = \sum_{m=1}^{\infty}\alpha_j v_j$.
Therefore,

$$\int_a^b D^\alpha_{a^+}uD^\alpha_{a^+}w = \int_a^b f(x,u)w, \quad \forall w \in \bigcup_{m=1}^{\infty} v_j$$

Taking into account that $\overline{\bigcup_{m=1}^{+\infty} V_m} = H^{\alpha}_{a^+,0}(a,b)$ we obtain

$$\int_a^b D^{\alpha}_{a^+}u D^{\alpha}_{a^+}v = \int_a^b f(x,u)v, \quad \forall v \in H^{\alpha}_{a^+}(a,b).$$

4.3 Fixed point method

In this section, we use the Schauder's fixed point method to prove Theorem 4.5.

We will demonstrate this through the following steps

- **Linearization of the problem:**
  Let $w \in H^{\alpha}_{a^+,0}(a,b)$. Consider the following linear problem

$$\int_a^b D^{\alpha}_{a^+}u D^{\alpha}_{a^+}v dx = \int_a^b f(x,w)v dx \quad \forall v \in H^{\alpha}_{a^+,0}(a,b). \tag{4.5}$$

  Putting:

$$A(u,v) = \int_a^b D^{\alpha}_{a^+}u D^{\alpha}_{a^+}v dx, \qquad \ell(v) = \int_a^b f(x,w)v dx.$$

  - A **is continuous** : Let $u, v \in H^{\alpha}_{a^+,0}(a,b)$. Then,

$$\begin{aligned} |A(u,v)| &= \left| \int_a^b D^{\alpha}_{a^+}u(x) D^{\alpha}_{a^+}v(x) dx \right|, \\ &\leq \int_a^b |D^{\alpha}_{a^+}u||D^{\alpha}_{a^+}v|, \\ &\leq \left(\int_a^b |D^{\alpha}_{a^+}u|^2 \right)^{\frac{1}{2}} \left(\int_a^b |D^{\alpha}_{a^+}v|^2 \right)^{\frac{1}{2}}, \\ &= \|u\|_{H^{\alpha}_{a^+,0}} \|v\|_{H^{\alpha}_{a^+,0}}. \end{aligned}$$

    So, A is continuous.

  - A **is coercive:** Let $u \in H^{\alpha}_{a^+,0}$. Then,

$$\begin{aligned} A(u,u) &= \int_a^b |D^{\alpha}_{a^+}u(x)|^2 dx, \\ &= \|u\|^2_{H^{\alpha}_{a^+,0}}. \end{aligned}$$

    So, A coercive.

  - ℓ **is continuous:** Let $v \in H^{\alpha}_{a^+,0}(a,b)$. Then, from (4.3) we get

$$\begin{aligned} |\ell(v)| &= \left| \int_a^b f(x,w)v(x) dx \right|, \\ &\leq M\|\mu\|_{L^\theta}\|v\|_{L^2} + \|\lambda\|_{L^\infty}\|u\|_{L^2}\|v\|_{L^2}, \\ &\leq \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} (M\|\mu\|_{L^\theta} + \|\lambda\|_{L^\infty}\|w\|_{L^2})\|v\|_{H^{\alpha}_{a^+,0}} \end{aligned}$$

So, ℓ is continuous.
Consequently, from lax-Milgram theorem the linear problem (4.5) admits a unique solution in $H^{\alpha}_{a^+,0}$

- Let T operator, given as

$$T : L^2(a,b) \longrightarrow H^{\alpha}_{a^+,0}(a,b), \quad w \longmapsto u,$$

where u is a unique solution of linear problem (4.5).
Let $K = \bar{B}(0,R)$ be a ball from $H^{\alpha}_{a^+,0}(a,b)$. For $w \in K$ we have

$$\begin{aligned} \|T(w)\|_{H^{\alpha}_{a^+,0}} &= \|D^{\alpha}_{a^+}T(w)\|_{L^2}, \\ &= \|D^{\alpha}_{a^+}u\|_{L^2}, \\ &= \int_a^b f(x,T(w))u dx. \end{aligned}$$

Using the inequalities (1.1) and (4.3), we obtain

$$\begin{aligned} \|T(w)\|_{H^{\alpha}_{a^+,0}} &\leq M\|\mu\|_{L^\theta}\|T(w)\|_{L^2} + \|\lambda\|_{L^\infty}\|T(w)\|^2_{L^2}, \\ &\leq \frac{M\|\mu\|_{L^\theta}(b-a)^\alpha}{\Gamma(\alpha+1)}\|T(w)\|_{H^{\alpha}_{a^+,0}} + \frac{\|\lambda\|_{L^\infty}(b-a)^{2\alpha}}{\Gamma^2(\alpha+1)}\|T(w)\|^2_{H^{\alpha}_{a^+,0}}. \end{aligned}$$

So,

$$\left(1 - \frac{\|\lambda\|_{L^\infty}(b-a)^{2\alpha}}{\Gamma^2(\alpha+1)}\right)\|T(w)\|^2_{H^{\alpha}_{a^+,0}} \leq \frac{M\|\mu\|_{L^2}(b-a)^\alpha}{\Gamma(\alpha+1)}\|T(w)\|_{H^{\alpha}_{a^+,0}}$$

which can be written

$$\left(\Gamma^2(\alpha+1) - \|\lambda\|_{L^\infty}(b-a)^{2\alpha}\right)\|T(w)\|^2_{H^{\alpha}_{a^+,0}} \leq M\|\mu\|_{L^2}(b-a)^\alpha.\Gamma(\alpha+1)\|T(w)\|_{H^{\alpha}_{a^+,0}}$$

Thus, we obtain

$$\|T(w)\|_{H^{\alpha}_{a^+,0}} \leq \frac{M\|\mu\|_{L^2}(b-a)^\alpha}{\Gamma(\alpha+1) - \|\lambda\|_{L^\infty}(b-a)^{2\alpha}}.$$

So, for $R = \dfrac{M\|\mu\|_{L^2}(b-a)^\alpha}{\Gamma(\alpha+1) - \|\lambda\|_{L^\infty}(b-a)^{2\alpha}}$, we can write

$$T : \bar{B}(0,R) \longrightarrow \bar{B}(0,R),$$

where $B(\bar{0},R) = \left\{w \in H^{\alpha}_{a^+,0} : \|w\|_{H^{\alpha}_{a^+,0}} \leq R\right\}$.

- K **is convex** (Ball).
- K **is closed in** $L^2(a,b)$**:**
  Let $(w_n) \subseteq K$ converge to w in $L^2(a,b)$, we will prove that $v \in K$.
  Since (w_n) is a bounded sequence then, from the compactness embedding of $H^{\alpha}_{a^+,0}(a,b)$ into $L^2(a,b)$, we can extract a subsequence (w_nk) weakly convergence to v. Hence,

$$\|v\|_{H^{\alpha}_{a^+,0}(a,b)} \leq \liminf_{nk\to+\infty}\|v_{nk}\|_{H^{\alpha}_{a^+,0}} \leq R.$$

  Hence, $v \in K$.

- **T is continuous:**
  Consider the sequence $(w_n) \subset K$, converge to w in $L^2(a,b)$. We denote $u_n = T(w_n)$. So,
  $$\|u_n\| = \|T(w_n)\|_{H^\alpha_{a^+,0}(a,b)} \leq R.$$
  Therefore, (u_n) is bounded in $H^\alpha_{a^+,0}(a,b)$, which is reflexive space. Then, we can extract a subsequence $u_{nk} \rightharpoonup u$. From the compactness embedding of $H^\alpha_{a^+,0}(a,b)$ into $L^2(a,b)$, we have $u_{nk} \to u$ in $L^2(a,b)$.
  Hence, for all $v \in H^\alpha_{a^+,0}(a,b)$ we have
  $$\begin{array}{ccc} \int_a^b D^\alpha_{a^+} u_{nk}(x) D^\alpha_{a^+} v(x)dx & = & \int_a^b f(x, w_n)v(x)dx, \\ \text{weakly convergence} & & \text{Lebesgue theorem}, \\ \downarrow & & \downarrow \\ \int_a^b D^\alpha_{a^+} u D^\alpha_{a^+} v(x)dx & = & \int_a^b f(x,w)v(x)dx. \end{array}$$
  Then, $u = T(w)$, which deduce that $T(K)$ is relative compact.

From the all above, T admits a fixed point, a solution of the problem (4.2).

4.4 Uniqueness of solution

The following theorem give the condition which the problem (PV) has a unique solution.

Theorem 4.6 *In addition to the conditions of Theorem 4.5, we assume that f is decreasing. Then, the weak solution to problem (P) is unique.*

Demonstration. Let u_1 and u_2 be two solutions of (PV), then for $v \in H^\alpha_{a^+,0}(a,b)$ we have

$$\int_a^b (D^\alpha_{a^+} u_1(x) - D^\alpha_{a^+} u_2(x)) \cdot D^\alpha_{a^+} v(x)dx = \int_a^b [f(x,u_1) - f(x,u_2)]v(x)dx.$$

Setting $v = u_1 - u_2$, we get

$$\int_a^b \left(D^\alpha_{a^+} u_1(x) - D^\alpha_{a^+} u_2(x)\right)^2 (x)dx = \int_a^b [f(x,u_1) - f(x,u_2)](u_1 - u_2)(x)dx,$$

So,

$$\begin{aligned} \|u_1 - u_2\|^2_{H^\alpha_{a^+,0}} &= \int_a^b \left(D^\alpha_{a^+} u_1(x) - D^\alpha_{a^+} u_2(x)\right)^2 (x)dx, \\ &= \int_a^b [f(x,u_1) - f(x,u_2)](u_1 - u_2)(x)dx, \\ &\leq 0. \end{aligned}$$

Hence, $\|u_1 - u_2\|^2_{H^\alpha_{a^+,0}} \leq 0$, which deduce that $u_1 = u_2$. ■

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ملخص

نستعمل مشتقات ريمان-ليوفيل أساسا لتقديم فضاءات سوبولاف كسرية الرتبة بطريقة معمقة، حيث سنقدمها باستخدام المشتقات الضعيفةكسرية الرتبة. بعد ذلك نقدم التكافؤ بين نظيمين مقترحين لهذه الفضاءات، ندرس الانعكاسية وقابلية الفصل. نقدم أيضا بطريقة غير تقليدية بعض التباينات المستمرة والمتراصة المتعلقة بالفضاءات كسرية الرتبة من نمط ريمان-ليوفيل، مما يعمق معرفتنا لها. وأخيرا، نستعمل هذه المفاهيم لدراسة مسألة حدية معينة.

كلمات مفتاحية: فضاءات سوبولاف كسرية الرتبة، ريمان-لوفيل، تباينات سوبولاف.

Résumé

En utilisant les dérivées de Riemann-Liouville comme fondement, nous introduisons de manière approfondie des espaces de Sobolev fractionnaires, caractérisant leur nature distinctive. Nous définissons également des dérivées fractionnaires faibles et démontrons leur concordance avec les dérivées de Riemann-Liouville. Par la suite, nous établissions l'équivalence entre certaines normes au sein de ces espaces, déduisant ainsi leur exhaustivité, réflexivité, et séparabilité. De manière non conventionnelle, nous mettons en lumière certaines injections de Sobolev qui ne sont pas généralement classiques, enrichissant ainsi notre compréhension de ces espaces. Finalement, on applique ces notions sur un problème aux limites précisé.

Mots Clés : espaces de Sobolev d'ordre fractionnaire, Riemann-Liouville, injections de Sobolev.

Abstract

Using the Riemann-Liouville derivatives as a basis, we let us introduce in depth fractional Sobolev spaces, characterizing their distinctive nature. We also define derivatives weak fractional values and demonstrate their agreement with the derivatives of Riemann-Liouville. Subsequently, we established the equivalence between certain norms within these spaces, thus deducing their exhaustiveness, reflexivity, and separability. In an unconventional way, we highlight certain Sobolev embeddings which are not generally classical, thus enriching our understanding of these spaces. Finally, we apply these notions to a specified boundary problem.

Keywords: Sobolev spaces of fractional order, Riemann-Liouville, Sobolev injections.