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NOTES ON THE EQUATION $d(n) = d(\varphi(n))$ AND RELATED INEQUALITIES

Djamel Bellaouar*, c — Abdelmadjid Boudaoud** — Rafael Jakimczuk***

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ABSTRACT. Let $d(n)$ denotes the number of positive integers dividing the positive integer n, and let $\varphi(n)$ denotes Euler's function representing the number of numbers less than and prime to n. In this paper, we present some notes on the equation $d(n) = d(\varphi(n))$. Several results on the related inequalities are also obtained.

1. Introduction

Solving equations involving arithmetic functions is an interesting problem. In 1966, Nicol [10] studied Diophantine equations of the form

$$c_1 f_1(n) + \cdots + c_k f_k(n) = c, \tag{1.1}$$

where $c, c_1, \ldots, c_k \in \mathbb{Z}$ and $f_1, \ldots, f_k$ are arithmetic functions. He observed that many such equations hold exactly when n is prime. As a continuation of this work, several authors have been treating particular cases of (1.1). For more details, see [3: pp. 78–80], [5: e.g., §B-13, §B-18], [6] and [13: pp. 108–118].

For any positive integer n let $d(n)$ and $\varphi(n)$ be the number of divisors of n and the Euler function, respectively. Recall that $d(n)$ varies with extreme irregularity as n tends to infinity, tending itself to infinity or remaining small depending on the multiplicative structure of n. By a well-known theorem, if n has the prime factorization $n = q_1^{a_1} q_2^{a_2} \cdots q_k^{a_k}$ with distinct primes $q_1, q_2, \ldots, q_k$ and positive integers $a_1, a_2, \ldots, a_k$, then

$$d(n) = (a_1 + 1)(a_2 + 1) \cdots (a_k + 1).$$

Thus the divisor function can be large, since by a classical result of Wigert (see [15]), one has

$$\log d(n) \leq \frac{\log 2 \log n}{\log \log n} + O\left(\frac{\log n}{(\log \log n)^2}\right),$$

with equality holding if $n = p_1 p_2 \cdots p_k$, where p_k is the k-th prime number.

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c Corresponding author.