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On a logarithmic wave equation with nonlinear dynamical boundary conditions: local existence and blow-up

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Abstract

This paper deals with a hyperbolic-type equation with a logarithmic source term and dynamic boundary condition. Given convenient initial data, we obtained the local existence of a weak solution. Local existence results of solutions are obtained using the Faedo-Galerkin method and the Schauder fixed-point theorem. Additionally, under suitable assumptions on initial data, the lower bound time of the blow-up result is investigated.

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1 Introduction

In this paper, we study the problem of wave equation with logarithmic nonlinearity and dynamic boundary condition

$$\begin{cases} u_{tt} - \triangle u = 0, & \text{in } (0,\infty) \times \Omega, \\ u(x,t) = 0 & \text{on } [0,\infty) \times \Gamma_0, \\ \frac{\partial}{\partial n} u(x,t) = -|u_t|^{k-2} u_t + |u|^{p-2} u \ln |u| & \text{on } [0,\infty) \times \Gamma_1, \\ u(x,0) = u_0(x), \qquad u_t(0,x) = u_1(x) & \text{on } \Omega, \end{cases} \tag{1}$$

where $\Omega \subset R^n$, $n \geq 1$ is a regular, bounded domain with a boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$, $\Gamma_0 \cap \Gamma_1 =$, where Γ_0 and Γ_1 are measurable over $\partial\Omega$, endowed with the $(n-1)$ dimensional Lebesgue measures $\lambda_{n-1}(\Gamma_0)$ and $\lambda_{n-1}(\Gamma_1)$. Additionally, $\lambda_{n-1}(\Gamma_0)$ and $\lambda_{n-1}(\Gamma_1)$ are assumed to be positive throughout paper. $k \geq 2$ and $p \geq 2$ are positive constants to be chosen later.

Dynamic boundary problems are widely applied in many mathematical models, such hydro logic filtration process, thermoelasticity, diffusion phenomenon, and hydrodynamics [2, 15, 25–27]. A dynamic boundary condition has been introduced by a group of physicists to underline the fact that the kinetics of the process, i.e. the term $\frac{\partial u}{\partial n}$ becomes more visible in some boundary conditions [18, 24]. This type of option is characterized by the interaction of the components of the system with the walls (i.e., within Γ) [7]. Since the paper by Lions [29] has been introduced in 1969, evolution equations with dynamical boundary conditions (first order equations in time) have been studied well. Later, mathematicians and physicists studied it for a long time and achieved creative success; see [3, 6, 9, 17, 19, 20, 22, 28, 31] and references therein.

In [31], the author considered the problem (1) without logarithmic source term for $\frac{\partial}{\partial n}u(x,t) = -|u_t|^{k-2}u_t + |u|^{p-2}u$ boundary condition and proved the local and global existence under suitable condition. When $2 < p < k$, the solutions exist globally for arbitrary initial data. For $k < p$, solutions blow up. Later, Zhang and Hu [36] considered the blow-up of the solution under the condition $E(0) < d$ when the initial data are in the unstable set. In [12], they established blow-up results of the solution for a finite time at a critical energy level or high-energy level for the same problem.

Let us go back and look at a wave equation with logarithmic nonlinearity associated with problem (1). In [8], Cazenave and Haraux considered the following equation for the Cauchy problem

$$u_{tt} - \triangle u = ku \ln |u|. \tag{2}$$

They studied deeply the existence and uniqueness of the solutions using different techniques. As far as is known, this type of problem has been employed in various areas of physics, such as geophysics, nuclear physics, and optics; see in Bialynicki-Birula and Mycielski [4, 5]. Moreover, there are many research points devoted to the given problem in different models of hyperbolic wave equation with logarithmic source term [10, 13, 14, 16, 21, 23, 33]. Ma and Fang [32] considered problem (2) with strong damping term. They proved decay estimates and blow-up result under the null Dirichlet boundary condition.

In [11], Cui and Chai considered the following equation

$$u_{tt} - \operatorname{div}\big(A(x)\nabla u\big) = |u|^p u \ln |u|$$

with acoustic boundary condition. They obtained local existence and uniqueness using the semigroup theory. As far as is known, not many works are related to the logarithmic wave equation with a dynamic boundary condition. According to the studies mentioned above, our work aims to expand the result of wave equation with logarithmic nonlinearity and dynamic boundary conditions. The rest of the work is arranged as follows: In Sect. 2 gives notations and lemmas to illustrate our paper path. Sections 3–4 state the local existence result and potential well of (1). In the last part, we established blow-up result for a lower bound time.

2 Preliminaries

First, we denote

$$\|\cdot\| = L^2(\Omega), \qquad \|\cdot\|_q = L^q(\Omega), \qquad \|\cdot\|_{q,\Gamma_1} = L^q(\Gamma_1), \quad 1 \le q \le \infty$$

and

$$H^1_{\Gamma_0}(\Omega) = \big\{u \in H^1(\Omega) : u|_{\Gamma_0} = 0\big\},$$

($u|_{\Gamma_0}$ is in the trace sense). Let $T > 0$ be a real number and X be a Banach space endowed with norm $\|\cdot\|_x$.$L^p(0,T;X)$ indicates the space of functions h, which are L^p over $(0,T)$ with values in X, which are measurable with $\|h\|_x \in L^p(0,T)$. We set the Banach space endowed with the norm

$$\|h\|_{L^p(0,T;X)} = \left(\int_0^T \|h\|_X^p\right)^{\frac{1}{p}}.$$

$L^\infty(0,T;X)$ denotes the space of functions $h:(0,T) \to X$, which are measurable with $\|h\|_x \in L^\infty(0,T)$. We set the Banach space endowed with the norm

$$\|h\|_{L^p(0,T;X)} = \left(\int_0^T \|h\|_X^p\right)^{\frac{1}{p}}.$$

We know that if X and Y are Banach spaces such that X is continuous embedding to Y, then $L^p(0,T;X) \hookrightarrow L^p(0,T;Y)$ for $1 \leq p \leq \infty$.

We define the total energy function as

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 + \frac{1}{p^2}\|u\|_{p,\Gamma_1}^p - \frac{1}{p}\int_{\Gamma_1} |u|^p \ln|u|\,dx. \tag{3}$$

By the definition of $E(t)$ on $H^1_{\Gamma_1}(\Omega)$, the initial energy can be considered

$$E(0) = \frac{1}{2}\|u_1\|^2 + \frac{1}{2}\|\nabla u_0\|^2 + \frac{1}{p^2}\|u_0\|_{p,\Gamma_1}^p - \frac{1}{p}\int_{\Gamma_1} |u_0|^p \ln|u_0|\,dx. \tag{4}$$

Lemma 1 [1] (*Trace-Sobolev Embedding inequality*). *Let $H^1_{\Gamma_0}(\Omega) \hookrightarrow L^p(\Gamma_1)$ for $2 \leq p < \varkappa$ hold, where*

$$\varkappa = \begin{cases} \frac{2(n-1)}{n-2}, & \text{if } n \geq 3, \\ \infty, & \text{if } n = 1,2. \end{cases}$$

So that, there is a constant C_p that is the smallest nonnegative number, satisfying

$$\|u\|_{p,\Gamma_1} \leq C_p\|\nabla u_0\|. \tag{5}$$

Proposition 2 *Suppose that Lemma 1 holds, we define*

$$\alpha^* = \begin{cases} \frac{2(n-1)}{n-2} - p, & \text{if } n \geq 3, \\ \infty, & \text{if } n = 1,2 \end{cases}$$

for any $\alpha \in [0,\alpha^*)$, then $H^1_{\Gamma_0}(\Omega) \hookrightarrow L^{p+\alpha}(\Gamma_1)$ continuously.

Lemma 3 *$E(t)$ is a nonincreasing function for $0 \leq s \leq t \leq T$ and*

$$E(t) + \int_s^t \left\|u_\tau(\tau)\right\|_{k,\Gamma_1}^k d\tau = E(s) \leq 0. \tag{6}$$

Proof By multiplying equation (1) by u_t and integrating on Ω, we have

$$\int_\Omega u_{tt}u_t\,dx - \int_\Omega \Delta u u_t\,dx = 0,$$
$$\frac{d}{dt}\left(\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2\Big|_s^t + \frac{1}{p^2}\|u\|_{p,\Gamma_1}^p - \frac{1}{p}\int_{\Gamma_1}|u|^p\ln|u|\,dx\right) = -\|u_t\|_{k,\Gamma_1}^k. \tag{7}$$

By integrating of (7) over (s,t), we have equality (6). □

Lemma 4 *Let ϑ be a positive number. Then, the inequality holds*

$$\big||s|^{p-2}\log|s|\big| \le A + s^{p-2+\vartheta}, \quad p>2 \tag{8}$$

for $A>0$.

Proof Notice that $\lim_{|s|\to\infty}\frac{\ln|s|}{s^\vartheta}=0$. Then, there is a positive constant $K>0$ such that

$$\frac{\log|s|}{s^\vartheta} < 1$$

for $\forall|s|>K$. Therefore,

$$\log|s| < s^\vartheta$$
$$|s|^{p-2}\log|s| < s^{p-2+\vartheta},$$

for $\forall|s|>K$. Since $p>2$, then $||s|^{p-2}\log|s|| \le A$, for some $A>0$ and for all $|s|\le K$.
Thus,

$$\big||s|^{p-2}\log|s|\big| \le A + s^{p-2+\vartheta}, \quad p>2.$$ □

3 Existence of local solution

We will apply the Faedo-Galerkin technique and the Schauder fixed-point theorem.

Theorem 5 *There exists $T>0$, such that problem (1) has a unique local weak solution u of (1) on $(0,T)\times\Omega$. Therefore,*

$$u \in C\big([0,T];H_{\Gamma_0}^1(\Omega)\big)\cap C^1\big([0,T);L^2(\Omega)\big),$$
$$u_t \in L^k\big((0,T)\times\Gamma_1\big)$$

and the energy identity

$$\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2\Big|_s^t + \int_s^t \|u_t\|_{k,\Gamma_1}^k = \int_s^t\int_{\Gamma_1}|u|^p\ln|u|u_t\,dx$$

holds for $0\le s\le t\le T$. Therefore, $T=T(\|u_0\|^2_{H^1_{\Gamma_0}(\Omega)}+\|u_1\|^2|_s^t,k,p,\Omega,\Gamma_1)$ is decreasing in the first variable.

Now, we will give some existence result and lemma used for the proof of Theorem 5.

To define the function and show that the fixed point exists, we introduce the following problem:

$$\begin{cases} v_{tt} - \triangle v = 0, & \text{in } (0,T) \times \Omega, \\ v = 0 & \text{on } [0,T) \times \Gamma_0, \\ \frac{\partial}{\partial n} v(x,t) = -|v_t|^{k-2} v_t + |u|^{p-2} u \ln |u| & \text{on } [0,T) \times \Gamma_1, \\ v(x) = u_0(x), v_t(x) = u_1(x) & \text{on } \Omega. \end{cases} \tag{9}$$

Let the solution v of problem (9) be $v = \zeta(u)$. We can see that v corresponds to u and $\zeta : X_T \to X_T$.

Lemma 6 *Let* $2 \leq p \leq \varkappa$ *and* $\frac{\varkappa}{\varkappa - p + 1} < k$. *Assume that* $u \in H^1_{\Gamma_0}(\Omega)$ *and* $u_1 \in L^k(\Omega)$ *hold. Then, there exists a unique weak solution u of* (9) *on* $(0,T) \times \Omega$. *Therefore,*

$$Y_T = \left\{ (v, v_t) \in C\left([0,T]; H^1_{\Gamma_0}(\Omega)\right) \cap C^1\left([0,T); L^2(\Omega)\right), v_t \in L^k\left((0,T) \times \Gamma_1\right) \right\} \tag{10}$$

endowed with the norm

$$\left\| (v, v_t) \right\|^2_{Y_T} = \max_{0 \leq t \leq T} \left[\|v_t\|^2 + \frac{1}{2} \|\nabla v\|^2 \right] + \|v_t\|^k_{L^k((0,T) \times \Gamma_1)}, \tag{11}$$

and the energy identity

$$\frac{1}{2} \|v_t\|^2 + \frac{1}{2} \|\nabla v\|^2 \bigg|_s^t + \int_s^t \|v_t\|^k_{k,\Gamma_1} = \int_s^t \int_{\Gamma_1} |u|^p \ln |u| u_t \, dx, \tag{12}$$

holds for $0 \leq s \leq t \leq T$.

To see the first step of the proof of Lemma 6, we will use the following proposition. The proposition was proved similar to [35]. We have some results in [35] as follows:

Proposition 7 *Let* $2 \leq p \leq \varkappa$ *and* $\frac{\varkappa}{\varkappa - p + 1} < k$. *Assume that* $u \in H^1_{\Gamma_0}(\Omega)$ *and* $u_1 \in L^k(\Omega)$ *hold. Then, there is* $T > 0$ *and a unique solution v for* (9) *problem on* $(0,T)$ *such that, i.e.*

$$u \in L^\infty\left([0,T]; H^1_{\Gamma_0}(\Omega)\right)$$

such that

$$u_t \in L^\infty\left(0,T; L^2(\Omega) \cap L^k\left((0,T) \times \Gamma_1\right)\right)$$

and

$$\int_0^T \Omega - u_t \varphi_t + \nabla u \nabla \varphi + \int_0^T \Gamma_1 |u_t|^{k-2} u_t \varphi - \int_0^T \Gamma_1 |u|^{p-2} u \ln |u| \varphi = 0$$

for all $\varphi \in C((0,T); H^1_{\Gamma_0}(\Omega)) \cap C^1((0,T); L^2(\Omega)) \cap L^k((0,T) \times \Gamma_1)$. *Then*

$$u \in C\left([0,T]; H^1_{\Gamma_0}(\Omega)\right) \cap C^1\left([0,T]; L^2(\Omega)\right),$$

and the energy identity

$$\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|\nabla u\|^2\bigg|_s^t+\int_s^t\|u_t\|_{k,\Gamma_1}^k=\int_s^t\int_{\Gamma_1}|u|^p\ln|u|u_t\,dx$$

holds for $0\le s\le t\le T$. *Now, we can state the proof of Lemma* 6.

Proof Let $\{w_j\}_{j=1}^{\infty}$ be a sequence of linearly independent vectors in $X=\{u\in H^1_{\Gamma_0}(\Omega):u|_{\Gamma_1}\in L^k(\Gamma_1)\}$ whose finite linear combinations are dense in X. In the event, using the Grahm-Schmidt orthogonalization method, we can conclude $\{w_j\}_{j=1}^{\infty}$ to be orthonormal in $L^2(\Omega)\cap L^2(\Gamma_1)$. Using some technical mathematical result, we can clearly see that $X(u\in H^1_{\Gamma_0}(\Omega)\cap L^k(\Gamma_1))$ is dense in $H^1_{\Gamma_0}(\Omega)$ and in $L^2(\Omega)$. Moreover, there exist $u_{0m},u_{1m}\in[w_1,w_2,\dots,w_m]$ where $w_1,w_2,\dots,w_m$ are the span of the vectors such that

$$\begin{aligned}
u_{0m}&=\sum_{i=1}^{m}\left(\int_\Omega u_0w_i\right)w_i\to u_0\quad\text{in }H^1_{\Gamma_0}(\Omega),\\
u_{1m}&=\sum_{i=1}^{m}\left(\int_\Omega u_1w_i\right)w_i\to u_1\quad\text{in }L^2(\Omega).
\end{aligned}\tag{13}$$

According to their multiplicity of

$$\Delta w_i+\lambda_iw_i=0$$

we denote by $\{\lambda_i\}$ the related eigenvalues to $w_1,w_2,\dots,w_m$. For all $m\ge1$, we will seek an approximate solution (m functions γ_{im}) such that

$$v_m(t)=\sum_{i=1}^{m}\gamma_i^m(t)w_i\tag{14}$$

satisfying the following Cauchy problem

$$\Big\{(v_{mtt},w_i)+(\nabla v_m,\nabla w_i)+\int_{\Gamma_1}|v_{mt}|^{k-2}v_{mt}w_i=\int_{\Gamma_1}|u|^{p-2}u\ln|u|w_i,\tag{15}$$

where $t\ge0$. In (15), for the first term, we obtain

$$\int_\Omega v_{mtt}(t)w_i\,dx=\int_\Omega\left(\sum_{j=1}^{m}\ddot{\gamma}_{jtt}^m(t)w_j\right)w_i\,dx=\ddot{\gamma}_i^m(t)\int_\Omega|w_i|^2\,dx=\ddot{\gamma}_i^m(t).\tag{16}$$

Similarly,

$$\begin{aligned}
\int_\Omega-\Delta v_mw_i\,dx&=-\int_\Omega\Delta\left(\sum_{j=1}^{m}\gamma_j^m(t)w_j\right)w_i\,dx\\
&=-\int_\Omega\left(\sum_{j=1}^{m}\gamma_j^m(t)\Delta w_j\right)w_i\,dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega \sum_{j=1}^m \gamma_j^m(t)\lambda_j w_j w_i \,dx \\
&= \gamma_i^m(t)\lambda_i \int_\Omega |w_i|^2 \,dx \\
&= \gamma_i^m(t)\lambda_i.
\end{aligned} \tag{17}
$$

For the fourth term, we get

$$
\begin{aligned}
\int_{\Gamma_1} \left|v_{mt}(t)\right|^{k-2} v_{mt}(t) w_i \,dx &= \int_{\Gamma_1} \left(\sum_{j=1}^m \left|\dot{\gamma}_j^m(t)\right|^{k-2} \dot{\gamma}_j^m(t) w_j \right) w_i \,dx \\
&= \left|\dot{\gamma}_j^m(t)\right|^{k-2} \dot{\gamma}_j^m(t) \int_\Omega |w_i|^2 \,dx \\
&= \left|\dot{\gamma}_j^m(t)\right|^{k-2} \dot{\gamma}_j^m(t).
\end{aligned} \tag{18}
$$

Then, we insert (16)–(18) in (15) so that (15) yields the following Cauchy problem for a linear ordinary differential equation for unknown functions $\gamma_i^m(t)$ for $i = 1, 2, \ldots, m$;

$$
\begin{aligned}
&\ddot{\gamma}_i^m(t) + \gamma_i^m(t)\lambda_i + \nabla\dot{\gamma}_j^m(t) + \left|\dot{\gamma}_j^m(t)\right|^{k-2} \dot{\gamma}_j^m(t) = G_i(t), \\
&\gamma_i^m(0) = \int_\Omega u_0 w_i \,dx, \qquad \dot{\gamma}_i^m(0) = \int_\Omega u_1 w_i \,dx,
\end{aligned} \tag{19}
$$

where

$$
G_i(t) = \int_{\Gamma_1} |u|^{p-2} u \ln|u| w_i, \quad i = 1, 2, \ldots, m, \tag{20}
$$

for $t \in [0, T]$. Then the problem above has a unique local solution $\gamma_i^m \in C^2[0, T]$ for all i, which satisfies a unique v_m defined by (14) and satisfies (15).

Now, taking $w_i = v_{mt}$ in equation (15) and then integrating over $[0, t], 0 < t < t_m$ and by parts,

$$
\begin{aligned}
&\left\|v_{mt}(t)\right\|^2 + \left\|\nabla v_m(t)\right\|^2 + 2\int_0^t \left\|v_{mt}(\tau)\right\|_{k,\Gamma_1}^k \,d\tau \\
&\quad = \left\|v_{1m}(t)\right\|^2 + \|\nabla v_{0m}\|^2 + 2\int_0^t \int_{\Gamma_1} |u|^{p-2} u \ln|u| v_{mt} \,dx\,ds
\end{aligned} \tag{21}
$$

for each $m \geq 1$.

To estimate the last term on the right-hand side of (21), set $v_m \in H^1(0, t_m; H^1_{\Gamma_0}(\Omega))$ and by the trace theorem; $v_m \in H^1(0, t_m; L^k(\Gamma_1))$. Applying the Young and the trace Sobolev inequalities, we conclude that

$$
\begin{aligned}
&2\int_0^t \int_{\Gamma_1} |u|^{p-2} u \ln|u| v_{mt} \,dx\,ds \\
&\quad \leq 2\int_0^t \int_{\Gamma_1} \left||u|^{p-1} \ln|u|\right| \left|v_{mt}(s)\right| \,dx\,ds \\
&\quad \leq \int_0^t \int_{\Gamma_1} ||u|^{p-1} \ln|u|^{\frac{k}{k-1}} \,dx\,ds + \int_0^t \left\|v_{mt}(s)\right\|_{k,\Gamma_1} ds,
\end{aligned} \tag{22}
$$

since Γ_1 is bounded. To estimate (22), we focus on the first term

$$\int_0^t \int_{\Gamma_1} \left||u|^{p-1} \ln |u|\right|^{\frac{k}{k-1}} \,dx\,ds.$$

We define

$$\Gamma_1^- = \left\{x \in \Omega; |u(x)| < 1\right\} \quad \text{and} \quad \Gamma_1^+ = \left\{x \in \Omega; |u(x)| \geq 1\right\},$$

where $\Gamma_1 = \Gamma_1^- \cup \Gamma_1^+$. Because of that, $\int_0^t \int_\Omega ||u|^{p-1} \ln |u||^{\frac{k}{k-1}} \,dx\,ds$ can be recalled as follows

$$\begin{aligned} &\int_{\Gamma_1} \left|\left|u(s)\right|^{p-1} \ln\left|u(s)\right|\right|^{\frac{k}{k-1}} \,dx \\ &\quad = \int_{\Gamma_1^-} \left|\left|u(s)\right|^{p-1} \ln\left|u(s)\right|\right|^{\frac{k}{k-1}} \,dx + \int_{\Gamma_1^+} \left|\left|u(s)\right|^{p-1} \ln\left|u(s)\right|\right|^{\frac{k}{k-1}} \,dx. \end{aligned} \tag{23}$$

Then, the use of Lemma 4 gives

$$\Gamma_1^- \left|\left|u(s)\right|^{p-1} \ln\left|u(s)\right|\right|^{\frac{k}{k-1}} \,dx \leq \left[e(p-1)\right]^{-\frac{k}{k-1}} |\Gamma_1| = C, \tag{24}$$

where

$$\inf_{s\in(0,1)} s^{p-1} \ln s = \left[e(p-1)\right]^{-1}.$$

Let

$$\theta = \frac{2(n-1)}{n-2} \cdot \frac{k}{k-1} - p + 1 > 0 \quad \text{for } n \geq 3; \text{each positive } \theta \text{ for } n = 1, 2.$$

By the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{\frac{2(n-1)}{n-2}}(\Gamma_1)$, recalling $u \in F = C([0,T]; H_0^1(\Omega))$, we obtain

$$\begin{aligned} \int_{\Gamma_1^+} \left|\left|u(s)\right|^{p-1} \ln\left|u(s)\right|\right|^{\frac{k}{k-1}} \,dx &\leq \int_{\Gamma_1^+} \theta^{-\frac{k}{k-1}} \theta^{\frac{k}{k-1}} \left(\left|u(s)\right|^{p-1} \ln\left|u(s)\right|\right)^{\frac{k}{k-1}} \,dx \\ &\leq \theta^{-\frac{k}{k-1}} \int \Gamma_1^+ \left(\left|u(s)\right|^{p-1} \ln\left|u(s)\right|^{\theta}\right)^{\frac{k}{k-1}} \,dx \\ &\leq \theta^{-\frac{k}{k-1}} \int_{\Gamma_1^+} \left(\left|u(s)\right|^{p-1+\theta}\right)^{\frac{k}{k-1}} \,dx \\ &\leq \theta^{-\frac{k}{k-1}} \int_{\Gamma_1^+} \left|u(s)\right|^{\frac{2(n-1)}{n-2}} \,dx \\ &\leq \theta^{-\frac{k}{k-1}} \int_{\Gamma_1 = \Gamma_1^- \cup \Gamma_1^+} \left|u(s)\right|^{\frac{2(n-1)}{n-2}} \,dx \\ &= \theta^{-\frac{k}{k-1}} \|u\|_{\frac{2(n-1)}{n-2}}^{\frac{2(n-1)}{n-2}} \\ &\leq C\|u\|_F^{\frac{2(n-1)}{n-2}} \leq C. \end{aligned} \tag{25}$$

Case $n = 1, 2$ proof is similar. So that, for taking $t = T$, we conclude that $|u(s)|^{p-1}\ln|u(s)|$ is bounded in $L^{\frac{k}{k-1}}((0,T)\times\Gamma_1)$.

Writing (24), (25) into (22), we conclude that

$$2\int_0^t\int_{\Gamma_1}|u(s)|^{p-2}u(s)\ln|u(s)|v_{mt}(s)\,dx\,ds \le CT + \int_0^t \left\|v_{mt}(s)\right\|_{k,\Gamma_1} ds. \tag{26}$$

Replacing (26) into (21), we can write

$$\begin{aligned}&\left\|v_{mt}(t)\right\|^2 + \left\|\nabla v_m(t)\right\|^2 + 2\int_0^t \left\|v_{mt}(\tau)\right\|_{k,\Gamma_1}^k d\tau \\ &\quad\le \left\|v_{1m}(t)\right\|^2 + \|\nabla v_{0m}\|^2 + CT + \int_0^t \left\|v_{mt}(s)\right\|_{k,\Gamma_1} ds,\end{aligned} \tag{27}$$

where C is a positive constant independent of m. Since the elementary estimate

$$x^a \le C_1 + C_2 x \quad\Rightarrow\quad x \le (1 + C_1 + C_2)^{\frac{1}{a-1}} \tag{28}$$

for $C_1, C_2 \ge 0$ and $a > 1$, (27) can be written as

$$\begin{aligned}&\left\|v_{mt}(t)\right\|^2 + \left\|\nabla v_m(t)\right\|^2 + 2\int_0^t \left\|v_{mt}(\tau)\right\|_{k,\Gamma_1}^k d\tau \\ &\quad\le C_4\end{aligned} \tag{29}$$

where $C_4 = \|v_{1m}(t)\|^2 + \|\nabla v_{0m}\|^2 + CT + (1 + C_1 + C_2)^{\frac{1}{k-1}}$. Since

$$\left\|v_m(t)\right\| \le \left\|v_m(0)\right\| + T\|v_{mt}\|_{L^\infty(0,T;L^2(\Omega))} \tag{30}$$

we have that $v_m(t)$ is bounded in $L^\infty(0,T;H^1_{\Gamma_0}(\Omega))$. Consequently, it follows from (29) and (30) that

$$\begin{cases} v_m, & \text{is bounded in } \quad L^\infty(0,T;H^1_{\Gamma_0}(\Omega)), \\ v_{mt}, & \text{is bounded in } L^\infty(0,T;L^2(\Omega)) \cap L^k(0,T;L^k(\Gamma_1)). \end{cases} \tag{31}$$

Using a standard procedure of the Aubin-Lions lemma [30, 34], we deduce that

$$\begin{cases} v_m \overset{z^*}{\rightarrow} v & \text{in } L^\infty(0,T;H^1_{\Gamma_0}(\Omega)), \\ v_{mt} \overset{z^*}{\rightarrow} \eta_1 & \text{in } L^\infty(0,T;L^2(\Omega)), \\ v_{mt} \overset{z}{\rightarrow} \eta_2 & \text{in } L^k((0,T)\times\Gamma_1), \\ |v_{mt}|^{k-2}v_{mt} \overset{z}{\rightarrow} |v_{mt}|^{k-2}v_{mt} & \text{in } L^{\frac{k}{k-1}}((0,T)\times\Gamma_1), \end{cases}$$

where $\eta_1 = v_t$ and $v(0) = v_0$. Now, we suppose that $\eta_2 = v_t$ a.e. in $(0,T)\times\Gamma_1$. It is clear that, since the weak limit of v_{mt} on $(0,T)\times\partial\Omega$ is equal to η_2 on $(0,T)\times\Gamma_1$ and to 0 on $(0,T)\times\Gamma_0$, and since $u = 0$ on $(0,T)\times\Gamma_0$, the assumption is that the weal limit of v_{mt} on $(0,T)\times\partial\Omega$ is the distribution time derivative of v on $(0,T)\times\partial\Omega$. Therefore, up to subsequence, we can pass to limit in (15) and find a weak solution (9) applying argument similar to that given in [35] (see Proposition 1).

Uniqueness proof is given by contradiction, claiming two distinct solutions exist. Say w and v have the same initial data. Subtracting both two equations and testing result by $w_t - v_t$, we conclude that

$$\begin{aligned} &\|w_t - v_t\|^2 + \|\nabla w - \nabla v\|^2 \\ &\quad + 2\int_0^T \int_{\Gamma_1} \left(|w_\tau|^{k-2} w_\tau - |v_\tau|^{k-2} v_\tau\right)(w_\tau - v_\tau)\, d\tau \\ &= 0. \end{aligned} \tag{32}$$

From the following inequality

$$\left(|f|^{k-2} f - |g|^{k-2} g\right)(f - g) \geq C|f - g|^k \quad \text{for } k \geq 2, \forall f, g \in R, \exists C > 0$$

equation (9) yields

$$\begin{aligned} &\|w_t - v_t\|^2 + \|\nabla w - \nabla v\|^2 \\ &\quad + c\left(+ \int_0^T \|\nabla w_\tau - \nabla v_\tau\|_{k,\Gamma_1}^k \right) \\ &\leq 0 \end{aligned}$$

which satisfies $w - v = 0$. Therefore, (9) satisfies a unique weak solution. □

Now, we can deal with the proof of Theorem 5.

Proof To obtain the proof, we apply the contraction mapping theorem. For $T > 0$, we denote the convex closed subset of Y_T as

$$X_T = \left\{(v, v_t) \in Y_T : v(0, x) = u_0(x), v_t(0, x) = u_1(x)\right\}.$$

We define

$$B_r(X_T) = \left\{v \in X_T : \left\|(v)\right\|_{X_T}^2 \leq r^2\right\},$$

where $r^2 = \frac{1}{2}(\|u_1\|^2 + \|\nabla u_0\|^2)$. Thanks to Lemma 6, for any $u \in B_r(X_T)$, we can introduce $v = \zeta(u)$, which is the unique solution of (9). We can see that v corresponds to u and $\zeta : X_T \to X_T$. Our aim is to get that ζ is a contraction map, which implies $\zeta(B_r(X_T)) \subset B_r(X_T)$ for any $T > 0$. Using energy identity for all $t \in (0, T]$, we have

$$\begin{aligned} &\frac{1}{2}\left(\|v_t\|^2 + \|\nabla v\|^2\right) + \int_0^t \left\|v_t(\tau)\right\|_{k,\Gamma_1}^k d\tau \\ &\quad \leq \frac{1}{2}\left(\|u_1\|^2 + \|\nabla u_0\|^2\right) \\ &\qquad + \int_0^t \int_{\Gamma_1} |u(s)|^{p-2} u(s) \ln|u(s)| v_t(s)\, dx\, ds. \end{aligned} \tag{33}$$

Then by

$$\int_{\Gamma_1} |u|^{p-2}u \ln|u| \leq \int_{\Gamma_1} |u|^p$$

(33) yields that

$$\begin{aligned} &\frac{1}{2}\left(\|v_t\|^2 + \|\nabla v\|^2\right) + \int_0^t \left\|v_t(\tau)\right\|_{k,\Gamma_1}^k d\tau \\ &\quad \leq \frac{1}{2}\left(\|u_1\|^2 + \|\nabla u_0\|^2\right) + \int_0^t \int_{\Gamma_1} |u|^p v_t \, dx \, ds. \end{aligned} \tag{34}$$

The last term on the right-hand side of inequality (34) can be estimated using the Holder inequality and similar calculations as for (23) and (25),

$$\begin{aligned} &\frac{1}{2}\left(\|v_t\|^2 + \|\nabla v\|^2\right) + \int_0^t \left\|v_t(\tau)\right\|_{k,\Gamma_1}^k d\tau \\ &\quad \leq \frac{1}{2}\left(\|u_1\|^2 + \|\nabla u_0\|^2\right) + Cr^{\frac{2(n-1)}{n-2}} T^{\frac{k-1}{k}} \|v_t\|_{L^k((0,T)\times\Gamma_1)}. \end{aligned} \tag{35}$$

By taking $t = T$ and using the inequality (28), we have

$$\|v_t\|_{L^k((0,T)\times\Gamma_1)} \leq C_1\left(1 + \frac{1}{2}r_0^2 + Cr^{\frac{2(n-1)}{n-2}} T^{\frac{k-1}{k}}\right)^{\frac{1}{k-1}}. \tag{36}$$

Because of the inequality for $X, Y \geq 0$,

$$(X+Y)^a \leq 2^{a-1}\left(X^a + Y^a\right), \tag{37}$$

where a is a positive constant, (36) yields that

$$\|v_t\|_{L^k((0,T)\times\Gamma_1)} \leq C_1\left(1 + \frac{1}{2}r_0^{\frac{2}{k-1}} + Cr^{\frac{2(n-1)}{(n-2)(k-1)}} T^{\frac{1}{k}}\right). \tag{38}$$

Now, we insert (38) into (35) and obtain the following inequality

$$\begin{aligned} &\frac{1}{2}\left(\|v_t\|^2 + \|\nabla v\|^2\right) \\ &\quad \leq \frac{1}{2}r_0^2 + C_5 r^{\frac{2(n-1)}{n-2}} T^{\frac{k-1}{k}}\left(1 + \frac{1}{2}r_0^{\frac{2}{k-1}} + C^{\frac{2(n-1)}{(n-2)(k-1)}} T^{\frac{1}{k}}\right). \end{aligned} \tag{39}$$

So that, we have

$$\|v_t\|_{L^\infty(0,T;L^2(\Omega))} \leq r_0^2 + C_6 r^{\frac{2(n-1)}{n-2}} T^{\frac{k-1}{k}}\left(1 + \frac{1}{2}r_0^{\frac{2}{k-1}} + Cr^{\frac{2(n-1)}{(n-2)(k-1)}} T^{\frac{1}{k}}\right). \tag{40}$$

Using inequality (37) and (40), we have

$$\begin{aligned}\|v\|_2^2 &\le \big(\|u_0\|_2 + {}_0^t\|v_t\|_2\big)^2 \\ &\le 2\|u_0\|_2^2 + 2T^2\|v_t\|_{L^\infty(0,T;L^2(\Omega))} \\ &\le 2r_0^2 + 2T^2\|v_t\|_{L^\infty(0,T;L^2(\Omega))} \\ &\le 2r_0^2 + 2T^2\left[r_0^2 + C_6 r^{\frac{2(n-1)}{n-2}} T^{\frac{k-1}{k}}\left(1 + \frac{1}{2}r_0^{\frac{2}{k-1}} + Cr^{\frac{2(n-1)}{(n-2)(k-1)}} T^{\frac{1}{k}}\right)\right] \\ &\le 2\big(1+T^2\big)r_0^2 + 2C_6 T^{\frac{3k-1}{k}} r^{\frac{2(n-1)}{n-2}}\left(1 + \frac{1}{2}r_0^{\frac{2}{k-1}} + \mathrm{C}r^{\frac{2(n-1)}{(n-2)(k-1)}} T^{\frac{1}{k}}\right).\end{aligned} \tag{41}$$

Combining (39) and (41), we have

$$\|v_t\|_{L^\infty(0,T;H^1_{\Gamma_0}(\Omega))} \le \big(3+T^2\big)r_0^2 + C_7 T^{\frac{k-1}{k}} r^{\frac{2(n-1)}{n-2}}\big(1+T^2\big)\left(1 + \frac{1}{2}r_0^{\frac{2}{k-1}} + Cr^{\frac{2(n-1)}{(n-2)(k-1)}} T^{\frac{1}{k}}\right).$$

By choosing T small enough and r large enough, we derive that $\zeta(u) \in B_r(X_T)$ and $T = T(r_0^2, k, p, \Omega, \Gamma_1)$ is a decreasing with respect to the first variable.

Next, we will verify that ζ is a contraction mapping continuous on $B_r(X_T)$ and ζ is compact in Y_T. Let $u_1, u_2 \in X_{r_0,T}$. We define $v_1 = \zeta(u_1)$, $v_2 = \zeta(u_2)$ with $u_1, u_2 \in B_r(X_T)$, and $z = v_1 - v_2$, then, clearly z is a solution of the problem

$$\begin{cases} z_{tt} - \Delta z = 0 & \text{in } (0,T)\times\Omega, \\ z = 0 & \text{on } [0,T)\times\Gamma_0, \\ \frac{\partial}{\partial n}z(x,t) = -|v_{1t}|^{k-2}v_{1t} + |v_{2t}|^{k-2}v_{2t} & \text{on } [0,T)\times\Gamma_1, \\ \qquad + |u_1|^{p-2}u_1\ln|u_1| - |u_2|^{p-2}u_2\ln|u_2| & \text{on } \Omega, \\ z(0,x) = z_t(0,x) = 0. \end{cases} \tag{42}$$

Since $v_{1t}, v_{2t} \in L^m((0,T)\times\Gamma_1)$, it is clearly that $|v_{1t}|^{k-2}v_{1t}$ and $|v_{2t}|^{k-2}v_{2t}$ belong to $L^{\frac{k}{k-1}}((0,T)\times\Gamma_1)$. Also, the functions $|u_1|^{p-2}u_1\ln|u_1|$ and $|u_2|^{p-2}u_2\ln|u_2|$ belong to $L^{\frac{k}{k-1}}((0,T)\times\Gamma_1)$. Then, by using Lemma 6, the energy functional can be written for problem (42) such that

$$\begin{aligned}&\frac{1}{2}\|z_t\|^2 + \frac{1}{2}\|\nabla z\|^2 + {}_0^t\int_{\Gamma_1}\big(|v_{1t}|^{k-2}v_{1t} - |v_{2t}|^{k-2}v_{2t}\big)(v_{1t} - v_{2t}) \\ &\quad = \int_0^t\int_{\Gamma_1}\big(|u_1|^{p-2}u_1\ln|u_1| - |u_2|^{p-2}u_2\ln|u_2|\big)(v_{1t} - v_{2t})\,dx\,ds\end{aligned} \tag{43}$$

for $0 \le t \le T$. We denote the basic inequality for $x \ge 2, a_1, a_2 \in R$ such that

$$\big(|a_1|^{b-2}a_1 - |a_2|^{b-2}a_2\big)(a_1 - a_2) \ge C^*|a_1 - a_2|^b.$$

For estimating the last integral on the left-hand side of (43), we apply the basic inequality by taking $b = k$ when $k \geq 2$ and $b = \frac{k}{k-1}$ when $1 < k < 2$. So that, (43) becomes

$$\begin{aligned}&\frac{1}{2}\|z_t\|^2 + \frac{1}{2}\|\nabla z\|^2 + C^*\|z_t\|^k_{L^k((0,T)\times\Gamma_1)}\\&\quad = \int_0^t \int_{\Gamma_1} \bigl(|u_1|^{p-2}u_1 \ln|u_1| - |u_2|^{p-2}u_2 \ln|u_2|\bigr)(v_{1t} - v_{2t})\,dx\,ds\end{aligned} \tag{44}$$

for $k \geq 2$, and

$$\begin{aligned}&\frac{1}{2}\|z_t\|^2 + \frac{1}{2}\|\nabla z\|^2 + C^*\bigl\||v_{1t}|^{k-2}v_{1t} - |v_{2t}|^{k-2}v_{2t}\bigr\|^{\frac{k}{k-1}}_{L^{\frac{k}{k-1}}((0,T)\times\Gamma_1)}\\&\quad = \int_0^t \int_{\Gamma_1} \bigl(|u_1|^{p-2}u_1 \ln|u_1| - |u_2|^{p-2}u_2 \ln|u_2|\bigr)(v_{1t} - v_{2t})\,dx\,ds\end{aligned} \tag{45}$$

for $1 < k < 2$.

Now, we need to estimate the logarithmic term in (45). If we set

$$G(s) = |s|^{p-2}s \ln|s|,$$

then

$$\begin{aligned}G'(s) &= (p-1)|s|^{p-2}\ln|s| + |s|^{p-2}\\&= \bigl(1 + (p-1)\ln|s|\bigr)|s|^{p-2}.\end{aligned}$$

From the mean value theorem, we have

$$\begin{aligned}&\bigl|G(u_1) - G(u_2)\bigr|\\&\quad = \bigl|G'\bigl(\vartheta u_1 + (1-\vartheta)u_2\bigr)(u_1 - u_2)\bigr|\\&\quad \leq \bigl[1 + (p-1)\ln\bigl|\bigl(\vartheta u_1 + (1-\vartheta)u_2\bigr)\bigr|\bigr]\bigl|\bigl(\vartheta u_1 + (1-\vartheta)u_2\bigr)\bigr|^{p-2}|u_1 - u_2|,\end{aligned}$$

where $0 < \vartheta < 1$. From Lemma 4, we conclude that

$$\begin{aligned}\bigl|G(u_1) - G(u_2)\bigr| &\leq \bigl|\bigl(\vartheta u_1 + (1-\vartheta)u_2\bigr)\bigr|^{p-2}|u_1 - u_2| + (p-1)A|u_1 - u_2|\\&\qquad + (p-1)|u_1 - u_2|\bigl|\bigl(\vartheta u_1 + (1-\vartheta)u_2\bigr)\bigr|^{p-2+\varepsilon}\\&\leq \bigl|(u_1 + u_2)\bigr|^{p-2}|u_1 - u_2| + (p-1)A|u_1 - u_2|\\&\qquad + (p-1)|u_1 - u_2||u_1 + u_2|^{p-2+\varepsilon}.\end{aligned} \tag{46}$$

Inserting (47) into (45), we obtain

$$\begin{aligned}&\frac{1}{2}\|z_t\|^2 + \frac{1}{2}\|\nabla z\|^2 + C^*\bigl\||v_{1t}|^{k-2}v_{1t} - |v_{2t}|^{k-2}v_{2t}\bigr\|^{\frac{k}{k-1}}_{L^{\frac{k}{k-1}}}\bigl((0,T)\times\Gamma_1\bigr)\\&\quad = \int_0^T \int_{\Gamma_1} \begin{pmatrix}|(u_1 + u_2)|^{p-2}|u_1 - u_2| + (p-1)A|u_1 - u_2|\\+(p-1)|u_1 - u_2||u_1 + u_2|^{p-2+\varepsilon}\end{pmatrix}(v_{1t} - v_{2t})\,dx\,ds.\end{aligned} \tag{47}$$

We choose $\varkappa_0 \in (p, \varkappa)$ such that

$$\frac{\varkappa}{\varkappa - p + 1} < \frac{\varkappa_0}{\varkappa_0 - p + 1} < k. \tag{48}$$

Using (49), we can define $l \in (0, 1)$ such that

$$\frac{1}{k} + \frac{1}{\varkappa} + \frac{1}{l} = 1, \tag{49}$$

where $l < \frac{\varkappa_0}{p-2}$.

Using (37) and the Holder inequality, we can write the first term of the integral term of (48) as

$$\begin{aligned} &\int_0^T \int_{\Gamma_1} \left|(u_1 + u_2)\right|^{p-2} |u_1 - u_2| (v_{1t} - v_{2t}) \\ &\leq 2^{l-1} \int_0^T \|u_1 - u_2\|_{\varkappa_0,\Gamma_1} \left(\|u_1\|_{l(p-2),\Gamma_1}^{(p-2)} + \|u_2\|_{l(p-2),\Gamma_1}^{(p-2)}\right) \|z_t\|_{k,\Gamma_1}. \end{aligned} \tag{50}$$

Since $l(p-2) < \varkappa_0$, by the trace Sobolev embedding and definition of r, we obtain

$$\begin{aligned} &\int_0^T \int_{\Gamma_1} \left|(u_1 + u_2)\right|^{p-2} |u_1 - u_2| (v_{1t} - v_{2t}) \\ &\leq 2^{l-1} \int_0^T \|u_1 - u_2\|_{\varkappa_0,\Gamma_1} \left(\|\nabla u_1\|_2^{(p-2)} + \|\nabla u_2\|_2^{(p-2)}\right) \|z_t\|_{k,\Gamma_1} \\ &\leq 2C_{10} r^{p-2} \int_0^T \|u_1 - u_2\|_{\varkappa_0,\Gamma_1} \|z_t\|_{k,\Gamma_1}. \end{aligned} \tag{51}$$

Applying the Holder inequality, we conclude that

$$\begin{aligned} &\int_0^T \int_{\Gamma_1} \left|(u_1 + u_2)\right|^{p-2} |u_1 - u_2| (v_{1t} - v_{2t}) \\ &\leq 2C_{10} r^{p-2} T^{\frac{k-1}{k}} \|u_1 - u_2\|_{L^\infty(0,T;L^{\varkappa}(\Gamma_1))} \|z_t\|_{L^k((0,T)\times(\Gamma_1))}. \end{aligned} \tag{52}$$

Thanks to (40) and $r_0 \leq r$, (52) yields

$$\begin{aligned} &\int_0^T \int_{\Gamma_1} \left|(u_1 + u_2)\right|^{p-2} |u_1 - u_2| (v_{1t} - v_{2t}) \\ &\leq C_{11} r^{p-2} \left[\left(1 + \frac{1}{2} r^{\frac{2}{k-1}}\right) T^{\frac{k-1}{k}} + \mathrm{C} r^{\frac{2(n-1)}{(n-2)(k-1)}} T\right] \\ &\quad \times \|u_1 - u_2\|_{L^\infty(0,T;L^{\varkappa_0}(\Gamma_1))}. \end{aligned} \tag{53}$$

If we choose $\varkappa_1 \in (p, \varkappa)$ such that

$$\frac{\varkappa}{\varkappa - (p+\varepsilon) + 1} < \frac{\varkappa_1}{\varkappa_1 - (p+\varepsilon) + 1} < k. \tag{54}$$

Using (54), we can define $l_1 \in (0, 1)$ such that

$$\frac{1}{k} + \frac{1}{\varkappa} + \frac{1}{l_1} = 1,$$

where $l_1 < \frac{\varkappa_1}{p-2+\varepsilon}$. Using calculations similar to (50)–(52), we obtain

$$\begin{aligned}
&\int_0^T \int_{\Gamma_1} |u_1 - u_2||u_1 + u_2|^{p-2+\varepsilon}(v_{1t} - v_{2t}) \\
&\leq C_{12} r^{p+\varepsilon-2}\left[\left(1 + \frac{1}{2} r^{\frac{2}{k-1}}\right) T^{\frac{k-1}{k}} + \mathrm{C} r^{\frac{2(n-1)}{(n-2)(k-1)}}\, T\right] \\
&\quad \times \|u_1 - u_2\|_{L^\infty(0,T;L^{\varkappa}(\Gamma_1))},
\end{aligned} \tag{55}$$

where $\varepsilon > 0$ constant.

Using the trace Sobolev embedding and the Holder inequality in time and (36), we have

$$\begin{aligned}
&\int_0^T \int_{\Gamma_1} |u_1 - u_2|(v_{1t} - v_{2t}) \\
&\leq \int_0^T \|u_1 - w_{u2}\|_{\varkappa_3,\Gamma_1} \|z_t\|_{k,\Gamma_1} \\
&\leq \|u_1 - u_2\|_{L^\infty(0,T;L^{\varkappa}(\Gamma_1))} \|z_t\|_{L^k((0,T)\times(\Gamma_1))} \\
&\leq C_1\left(1 + \frac{1}{2} r_0^{\frac{2}{k-1}} + \mathrm{C} r^{\frac{2(n-1)}{(n-2)(k-1)}}\, T^{\frac{1}{k}}\right) \|u_1 - u_2\|_{L^\infty(0,T;L^{\varkappa_3}(\Gamma_1))},
\end{aligned} \tag{56}$$

where $\varkappa_3 \in (p, \varkappa)$.

By combining (56),(55), and (53), we obtain

$$\begin{aligned}
&\int_0^T \int_{\Gamma_1} \begin{pmatrix} |(u_1 + u_2)|^{p-2}(v_{1t} - v_{2t}) + (p-1)A(v_{1t} - v_{2t}) \\ +(p-1)(v_{1t} - v_{2t})|u_1 + u_2|^{p-2+\varepsilon} \end{pmatrix} (v_{1t} - v_{2t})\, dx\, ds \\
&\leq K \|u_1 - u_2\|_{L^\infty(0,T;L^{\varkappa^*}(\Gamma_1))},
\end{aligned} \tag{57}$$

where

$$\varkappa^* = \max\{\varkappa_1, \varkappa_2, \varkappa_3\}$$

and

$$K = \max \begin{cases} C_1(1 + \frac{1}{2} r_0^{\frac{2}{k-1}} + C r^{\frac{2(n-1)}{(n-2)(k-1)}}\, T^{\frac{1}{k}}), \\ C_{12} r^{p+\varepsilon-2}[(1 + \frac{1}{2} r^{\frac{2}{k-1}}) T^{\frac{k-1}{k}} + C r^{\frac{2(n-1)}{(n-2)(k-1)}}\, T], \\ C_{11} r^{p-2}[(1 + \frac{1}{2} r^{\frac{2}{k-1}}) T^{\frac{k-1}{k}} + C r^{\frac{2(n-1)}{(n-2)(k-1)}}\, T]. \end{cases}$$

Consequently, by inserting (57) into (44) and (45), we get the following estimates

$$\|v_{1t} - v_{2t}\|^2_{L^\infty(0,T;L^2(\Omega))} \leq K_1 \|u_1 - u_2\|_{L^\infty(0,T;L^{\varkappa^*}(\Gamma_1))}, \tag{58}$$

$$\|\nabla v_1 - \nabla v_2\|^2_{L^\infty(0,T;L^2(\Omega))} \leq K_1 \|u_1 - u_2\|_{L^\infty(0,T;L^{\varkappa^*}(\Gamma_1))}, \tag{59}$$

and

$$\|v_{1t} - v_{2t}\|^k_{L^k((0,T)\times\Gamma_1)} \leq K_1 \|u_1 - u_2\|_{L^\infty(0,T;L^{\varkappa^*}(\Gamma_1))}, \tag{60}$$

where $k \geq 2$, while

$$\left\| |v_{1t}|^{k-2}v_{1t} - |v_{2t}|^{k-2}v_{2t} \right\|^{\frac{k}{k-1}}_{L^{\frac{k}{k-1}}((0,T)\times\Gamma_1)} \leq K_1 \|u_1 - u_2\|_{L^\infty(0,T;L^{\varkappa^*}(\Gamma_1))}, \tag{61}$$

where $1 < k < 2$ and $K_1 > 0$ is a constant, which depends on $(p, k, \Omega, \Gamma_1, T, r)$. Thanks to $v_1(0) = v_2(0) = 0$, we conclude that for $0 \leq t \leq T$,

$$\|v_1 - v_2\|_2 \leq \int_0^T \|v_{1t} - v_{2t}\|_2 \leq T\|v_{1t} - v_{2t}\|_{L^\infty(0,T;L^2(\Omega))}. \tag{62}$$

Plug (58) into (62) yields that

$$\|v_1 - v_2\|^2_{L^\infty(0,T;L^2(\Omega))} \leq K_1 T^2 \|u_1 - u_2\|_{L^\infty(0,T;L^{\varkappa^*}(\Gamma_1))}. \tag{63}$$

Thus, from estimates (58)–(63), we get contractiveness of ζ in $B_r(X_T)$. It follows that $v = \zeta(u)$ is a Cauchy sequence in Y_T. The proof is completed. □

4 Potential well

In this section, we will demonstrate the global existence of the proofs of solution (1).

We defined some useful functionals total energy function as

$$J(u) = \frac{1}{2}\|\nabla u\|^2 + \frac{1}{p^2}\|u\|^p_{p,\Gamma_1} - \frac{1}{p}\int_{\Gamma_1} u^p \ln|u|\,dx, \tag{64}$$

$$I(u) = \|\nabla u\|^2 - \int_{\Gamma_1} u^p \ln|u|\,dx. \tag{65}$$

Then, combining (64), (65), and definition of $E(u)$ gives

$$J(u) = \frac{1}{p}I(u) + \frac{1}{p^2}\|u\|^p_p \tag{66}$$

and

$$E(u) = \frac{1}{2}\|u_t\|^2 + J(u). \tag{67}$$

The potential well depth is defined as

$$W = \left\{(u_0, u_1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) : J(u) \leq d, I(u) > 0\right\} \cup \{0\}, \tag{68}$$

and the outer space of the potential well

$$V = \left\{(u_0, u_1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) : J(u) \leq d, I(u) < 0\right\}. \tag{69}$$

Lemma 8 *Let* $u_0 \in H^1_{\Gamma_0}(\Omega)\backslash\{0\}$, $\|u\|^p_{p,\Gamma_1} \neq 0$. *Then*

i) $\lim_{\lambda\to 0^+} J(\lambda u) = 0$, $\lim_{\lambda\to\infty} J(\lambda u) = -\infty$;

ii) *There exists* $\lambda^* > 0$ *satisfying* $\frac{d}{d\lambda}J(\lambda^* u) = 0$ *such that*

$$I(\lambda u) = \frac{d}{d\lambda}J(\lambda u)\begin{cases} > 0, & 0 \le \lambda < \lambda^*, \\ = 0, & \lambda = \lambda^*, \\ < 0, & \lambda < \lambda^* < \infty. \end{cases}$$

Proof i) Take $J(\lambda u)$,

$$\begin{aligned} J(\lambda u) &= \frac{1}{p}\|\lambda \nabla u\|^2 + \frac{1}{p^2}\|\lambda u\|^p_{p,\Gamma_1} - \frac{1}{p}\int_{\Gamma_1} \ln|\lambda u|(\lambda u)^p\,dx \\ &= \frac{\lambda^2}{2}\|\nabla u\|^2 + \frac{\lambda^p}{p^2}\|u\|^p_{p,\Gamma_1} - \frac{\lambda^p}{p}\int_{\Gamma_1}|u|^p \ln|u|\,dx - \frac{\lambda^p}{p}\ln|\lambda|\int_{\Gamma_1}|u|^p\,dx. \end{aligned}$$

By virtue of $\|u\|^p_{p,\Gamma_1}$, we see that $\lim_{\lambda\to 0} J(\lambda u) = 0$, $\lim_{\lambda\to\infty} J(\lambda u) = -\infty$.

ii) Now, taking the derivative of $J(\lambda u)$ with respect to λ, we have

$$\frac{d}{d\lambda}J(\lambda u) = \lambda\left(\|\nabla u\|^2 - \lambda^{p-2}\int_{\Gamma_1}|u|^p \ln|u|\,dx - \lambda^{p-2}\ln|\lambda|\|u\|^p_{p,\Gamma_1}\right). \tag{70}$$

Thanks to definition of $J(\lambda u)$, it is clearly from (70) that $\lambda^{-1}\frac{d}{d\lambda}J(\lambda u) = N(\lambda u)$. So, we obtain

$$\begin{aligned} \frac{d}{d\lambda}N(\lambda u) &= (2-p)\lambda^{p-3}\int_{\Gamma_1}|u|^p \ln|u|\,dx - \lambda^{p-3}\|u\|^p_{p,\Gamma_1} + (2-p)\lambda^{p-3}\ln|\lambda|\|u\|^p_{p,\Gamma_1} \\ &= \lambda^{p-3}\left((2-p)\int_{\Gamma_1}|u|^p \ln|u|\,dx - \|u\|^p_{p,\Gamma_1} + (2-p)\ln|\lambda|\|u\|^p_{p,\Gamma_1}\right). \end{aligned}$$

Therefore, there is a unique λ^1 such that $\frac{d}{d\lambda}N(\lambda u)|_{\lambda=\lambda^1} = 0$, by taking

$$\lambda^1 = \exp\left(\frac{\|u\|^p_{p,\Gamma_1} - (2-p)\int_{\Gamma_1}|u|^p \ln|u|\,dx}{(2-p)\|u\|^p_{p,\Gamma_1}}\right) > 0$$

such that $\frac{d}{d\lambda}N(\lambda u) > 0$ on $(0,\lambda^1)$ and $\frac{d}{d\lambda}N(\lambda u) < 0$ on (λ^1,∞). Because of $N(\lambda u)|_{\lambda=0} = \|\nabla u\|^2 > 0$ and $\lim_{\lambda\to\infty} N(\lambda u) = -\infty$, there is one $\lambda^* > 0$ such that $N(\lambda^* u) = 0$, i.e $\frac{d}{d\lambda}J(\lambda^* u) = 0$.

A simple corollary of the fact that

$$\frac{d}{d\lambda}J(\lambda u) = \lambda N(\lambda u)$$

which gives that $\frac{d}{d\lambda}J(\lambda u) > 0$ on $(0,\lambda^*)$ and $\frac{d}{d\lambda}J(\lambda u) < 0$ on (λ^*,∞). Thus, we have the desired results such that

$$I(\lambda u) = \lambda g'(\lambda)\begin{cases} > 0, & 0 \le \lambda < \lambda^*, \\ = 0, & \lambda = \lambda^*, \\ < 0, & \lambda < \lambda^* < \infty. \end{cases}$$

□

Lemma 9 i) *The depth of potential well depth defined by*

$$d = \inf_{u\in H^1_{\Gamma_0}(\Omega)\setminus\{0\}, u|_{\Gamma_1}\neq 0} \sup_{\lambda>0} J(\lambda u). \tag{71}$$

Then d is a positive function such that

$$0 < d = \inf_{u\in N} J(u) \tag{72}$$

where N is the Nehari manifold given by

$$N = \left\{ u \in H^1_{\Gamma_0}(\Omega)\setminus\{0\} : I(u) = 0 \right\},$$

and d has a positive lower bound, namely,

$$d \geq \frac{1}{2}\left(\frac{e\alpha}{C^*}\right)^{\frac{2}{p+\alpha-2}},$$

where C is defined as a positive constant.

Proof i) By (64), thanks to definitions of the Nehari manifold and d, it satisfies $d \geq 0$. So that, our purpose is to prove that there is a positive function such that $J(u) = d$. We define $\{u_i\}_{i=1}^{\infty} \subset N$ as a minimizing sequence for J. So that, we conclude that

$$\lim_{i\to\infty} J(u_i) = d.$$

It is clearly that, $\{|u_i|\}_{i=1}^{\infty} \subset N$ a minimizing sequence for J. Now, we suppose that $u_i > 0$ in Ω for all $i \in \mathbb{N}$.

We also obtain that J is coercive on $u \in N$ satisfying $\{u_i\}_{i=1}^{\infty}$ and is bounded in $H^1_{\Gamma_0}(\Omega)$. Since $H^1_{\Gamma_0}(\Omega) \hookrightarrow L^{p+\alpha}(\Gamma_1)$ is compact embedding, there is a function u and a subsequence of $\{|u_{i_n}|\}_{i=1}^{\infty}$ of $\{|u_i|\}_{i=1}^{\infty}$, such that

$$\begin{aligned} &u_{i_n} \to u \quad \text{weakly in } H^1_{\Gamma_0}(\Omega),\\ &u_{i_n} \to u \quad \text{strongly in } L^{p+\alpha}(\Gamma_1),\\ &u_{i_n} \to u \quad \text{a.e. in } \Omega, \end{aligned}$$

where $i_n \to \infty$.

Then, we get $u \geq 0$ a.e. in Ω. Moreover, using the dominated convergence theorem, weak lower semicontinuity and definition of $J(u)$, $I(u)$ and N gives

$$\begin{aligned} &J(u) \leq \lim_{i_n\to\infty} \inf J(u_{i_n}) = d,\\ &I(u) \leq \lim_{i_n\to\infty} \inf I(u_{i_n}) = 0. \end{aligned}$$

Since $x^{-y}\ln x \le \frac{1}{ey}$ for $x, y > 0$ and the trace Sobolev embbedding theorem, we have

$$\begin{aligned}
\|\nabla u_{i_n}\|^2 &= \int_{\Gamma_1} |u_{i_n}|^p \ln|u_{i_n}| \\
&\le \int_{\Gamma_1} \frac{1}{e\alpha} |u_{i_n}|^p \ln|u_{i_n}|^\alpha \\
&= \frac{1}{e\alpha} \|u_{i_n}\|_{p+\alpha,\Gamma_1}^{p+\alpha} \\
&\le \frac{C^*}{e\alpha} \|\nabla u_{i_n}\|^{p+\alpha},
\end{aligned}$$

where C^* is the best Sobolev constant, which means

$$\left(\frac{e\alpha}{C^*}\right)^{\frac{2}{p+\alpha-2}} < \|\nabla u_{i_n}\|^2.$$

Therefore, we conclude that

$$\int_{\Gamma_1} |u_{i_n}|^p \ln|u_{i_n}| \ge \left(\frac{e\alpha}{C^*}\right)^{\frac{2}{p+\alpha-2}}.$$

Using the dominated convergence theorem, we have

$$\int_{\Gamma_1} |u|^p \ln|u| \ge \left(\frac{e\alpha}{C^*}\right)^{\frac{2}{p+\alpha-2}} > 0$$

which means that $u \neq 0$.

Last, we show that $I(u) = 0$. Indeed, if it is not true, we get $I(u) < 0$. So, thanks to Lemma 8, we have a positive constant $\lambda^* < 1$ implying that $I(\lambda^* u) = 0$. Therefore, it follows that

$$\begin{aligned}
d \le J(\lambda^* u) &= \frac{(\lambda^*)^2}{2} \|\nabla u\|^2 + \frac{(\lambda^*)^p}{p^2} \|u\|_{p,\Gamma_1}^p \\
&= (\lambda^*)^2 \left[\frac{1}{2}\|\nabla u\|^2 + \frac{(\lambda^*)^{p-2}}{p^2} \|u\|_{p,\Gamma_1}^p\right] \\
&\le (\lambda^*)^2 \left[\frac{1}{2}\|\nabla u\|^2 + \frac{1}{p^2} \|u\|_{p,\Gamma_1}^p\right] \\
&\le (\lambda^*)^2 \lim_{i_n\to\infty} \inf\left[\frac{1}{2}\|\nabla u_{i_n}\|^2 + \frac{1}{p^2} \|u_{i_n}\|_{p,\Gamma_1}^p\right] \\
&\le (\lambda^*)^2 \lim_{k\to\infty} \inf J(u_{i_n}) \\
&= (\lambda^*)^2 d < d,
\end{aligned}$$

where $d = \frac{1}{2}(\frac{e\alpha}{C^*})^{\frac{2}{p+\alpha-2}}$. it is a contradiction. □

5 Lower bound for blow-up time

In this part, we prove a lower bound for blow-up time of problem (1). First, we give lemma, which will play a role of the proof of Theorem 11.

Lemma 10 *Suppose that $(u_0,u_1) \in V$, $2 \leq p < \varkappa$. So, we get $(u,u_t) \in V$ for all $t \geq 0$. Proof. By way of contradiction, suppose that (u_0,u_1) leaves V at time $t = t_0$, so there is a sequence $\{t_s\}, t_s \to t_0^-$ such that $I(u(t_s)) \leq 0$ and $E(u(t_s)) \leq d$. Thanks to weak lower semicontinuity $\|\cdot\|_{H^1_{\Gamma_0}}$, we obtain*

$$I\big(u(t_0)\big) \leq \lim_{n\to\infty} \inf I\big(u(t_s)\big) \leq 0, \tag{73}$$

and

$$E(t_0) \leq \lim_{n\to\infty} \inf E\big(u(t_s)\big) \leq d. \tag{74}$$

If we take $(u(t_0),u_t(t_0)) \notin V$, $I(u(t_0)) = 0$ or $E(u(t_0)) > d$. Because of (6), taking $E(t_0) > d$ is impossible, which is a contradiction with inequality (74). By the continuity of function $I(u(t))$ about time, if we take $I(u(t_0)) = 0$, by definition of d, (64) and (3), we arrive at

$$d \geq E(t_0) \geq J\big(u(t_0)\big) \geq \inf_{u\in N} J(u) = d.$$

Moreover, we have a contradiction. So, we get $(u,u_t) \in V$ for all $t \geq 0$.

Theorem 11 *Assume that $(u_0,u_1) \in V$, $2 \leq p < \varkappa$ and $2 < p < 1 + \frac{(2k-2)(n-1)}{k(n-2)}$. Then, the solutions u of problem (1) are bounded at finite time $t = T_1$ with*

$$\lim_{t\to T_1^-} \|u_t\|^2 + \|\nabla u\|^2 = \infty.$$

Therefore, we give lower bound for T_1 such that

$$T_1 \geq \frac{d\theta}{H(t) + e(p-1)^{-\frac{k}{k-1}}|\Gamma_1| + (e\alpha)^{-\frac{k}{k-1}}(K_2)^{\frac{k}{k-1}(p-1+\alpha)}\theta^{\frac{k}{k-1}(p-1+\alpha)}},$$

where $0 < \alpha < \frac{2n-2}{n-2} - p$, K_2 is the positive Sobolev constant and $H(0) = \|u_1\|^2 + \|\nabla u_0\|^2$.

Proof We define a function as

$$H(t) = \|u_t\|^2 + \|\nabla u\|^2. \tag{75}$$

By testing equation of problem (1) by $u_t(x,t)$ and using the Green formula, we obtain

$$(u_{tt},u_t) = -\int_\Omega \nabla u, \nabla u_t + \int_{\Gamma_1} \big[-|u_t|^{k-2}u_t + |u|^{p-2}u \ln|u|\big]u_t. \tag{76}$$

By differentiating (75) and using (76), we conclude that

$$\begin{aligned} H'(t) &= 2(u_{tt},u_t) + 2\int_\Omega \nabla u \nabla u_t \\ &= -2\int_\Omega \nabla u, \nabla u_t + 2\int_{\Gamma_1} \big[-|u_t|^{k-2}u_t + |u|^{p-2}u\ln|u|\big]u_t + 2\int_\Omega \nabla u\nabla u_t \\ &= -2\int_{\Gamma_1} |u_t|^{k-1}u_t + 2\int_{\Gamma_1} |u|^{p-2}u\ln|u|u_t. \end{aligned} \tag{77}$$

Since $2 < p < 1 + \frac{(2k-2)(n-1)}{k(n-2)}$, by applying the trace Sobolev embedding theorem where α is a positive constant such that $(p-1+\alpha) < \frac{(2k-2)(n-1)}{k(n-2)}$. Therefore, if we use the Young inequality and Sobolev theorems, (77) yields that

$$\begin{aligned}
&2\int_{\Gamma_1} |u|^{p-2}u\ln|u|u_t \\
&\leq \|u_t\|_{k,\Gamma_1}^k + \left(\int_{\Gamma_1} |u|^{p-2}u\ln|u|\right)^{\frac{k}{k-1}} \\
&\leq \|u_t\|_{k,\Gamma_1}^k \int_{\Gamma_1^-} \left(|u|^{p-2}u\ln|u|\right)^{\frac{k}{k-1}} + \int_{\Gamma_1^+} \left(|u|^{p-2}u\ln|u|\right)^{\frac{k}{k-1}} \\
&\leq \|u_t\|_{k,\Gamma_1}^k + e(p-1)^{-\frac{k}{k-1}}|\Gamma_1| + (e\alpha)^{-\frac{k}{k-1}}\int_{\Gamma_1^+} \left(|u|^{p-1+\alpha}\right)^{\frac{k}{k-1}} \\
&\leq \|u_t\|_{k,\Gamma_1}^k + e(p-1)^{-\frac{k}{k-1}}|\Gamma_1| + (e\alpha)^{-\frac{k}{k-1}}(K_2)^{\frac{k}{k-1}(p-1+\alpha)}\|\nabla u\|_2^{\frac{k}{k-1}(p-1+\alpha)} \\
&\leq H(t) + e(p-1)^{-\frac{k}{k-1}}|\Gamma_1| + (e\alpha)^{-\frac{k}{k-1}}(K_2)^{\frac{k}{k-1}(p-1+\alpha)}H(t)^{\frac{k}{k-1}(p-1+\alpha)},
\end{aligned} \tag{78}$$

where $|x^y\ln x| \leq \frac{1}{ey}$ for $0 < x < 1$ and $x^{-y}\ln x \leq \frac{1}{ey}$ for $x \geq 1$.

Inserting (78) into (77) gives

$$H'(t) \leq H(t) + e(p-1)^{-\frac{k}{k-1}}|\Gamma_1| + (e\alpha)^{-\frac{k}{k-1}}(K_2)^{\frac{k}{k-1}(p-1+\alpha)}H(t)^{\frac{k}{k-1}(p-1+\alpha)}. \tag{79}$$

Using integration of (79) over t, we conclude

$$\int_{H(0)}^{H(t)} \frac{d\theta}{H(t) + e(p-1)^{-\frac{k}{k-1}}|\Gamma_1| + (e\alpha)^{-\frac{k}{k-1}}(K_2)^{\frac{k}{k-1}(p-1+\alpha)}\theta^{\frac{k}{k-1}(p-1+\alpha)}}.$$

It is easy to see that there is a time T_1 such that the solution goes to the infinity with $\lim_{t\to T_1} H(t) = \infty$. Thus, we have a lower bound for T_1 given by

$$T_1 \geq \frac{d\theta}{H(t) + e(p-1)^{-\frac{k}{k-1}}|\Gamma_1| + (e\alpha)^{-\frac{k}{k-1}}(K_2)^{\frac{k}{k-1}(p-1+\alpha)}\theta^{\frac{k}{k-1}(p-1+\alpha)}}.$$

This completed the proof. □

6 Conclusion

This work proves the existence of the result for a hyperbolic-type equation with logarithmic nonlinearity and dynamical boundary condition. This result is modern for these types of problems, and it can be generalized to many problems in the coming literature.

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