République Algérienne Démocratique et Populaire
Ministère de l'Enseignement Supérieur et de la Recherche Scientifique
Université Mohamed Boudiaf de M'sila

جامعة محمد بوضياف - المسيلة
Université Mohamed Boudiaf - M'sila

Modélisation Mathématique pour la Biologie et la Santé ; MMBS-2023

Attestation de Participation

*Le Comité d'Organisation de la 2ème Rencontre « **Modélisation Mathématique pour la Biologie et la Santé** » qui s'est tenue le 13 Décembre 2023 à l'Université Mohamed BOUDIAF de M'sila- Algérie, atteste que :*

M. Bilal BASTI

*A participé avec succès aux travaux de la rencontre en présentant la communication **orale** suivante :*

Mathematical Modeling and Analysis for Studying the Behavior of Infectious Diseases

Président de la Rencontre

Pr. Noureddine BENHAMIDOUCHE

L.M.P.A

Doyen de la Faculté

عميد كلية الرياضيات
و الإعلام الآلي
نويري إبراهيم

Mathematical Modeling and Analysis for Studying the Behavior of Infectious Diseases

Bilal Basti

Department of Mathematics, Ziane Achour University of Djelfa, Algeria.
E-mail: bilalbasti@gmail.com

Abstract: In our present work, we thoroughly examine and provide analytical insights into a mathematical models for infectious diseases as COVID-19, employing the Caputo-Katugampola derivative. Numerous stability results are meticulously computed based on well-defined parameters, ensuring compliance with conditions that effectively impede pandemic occurrences. Furthermore, the paper rigorously investigates the critical aspect of the existence and uniqueness of solutions for the SIRD model, leveraging the robust properties inherent in Schauder's and Banach's fixed point theorems. This multifaceted analysis not only enhances our understanding of the intricate dynamics of COVID-19 but also contributes valuable knowledge to the broader field of mathematical epidemiology.
Keywords: Pandemic; mathematical model; fractional derivative; stability; existence and uniqueness.

1 Introduction

A better understanding and evaluation of the existence, stability and control of infectious diseases can be acquired through modeling them mathematically. However, mathematical models' classical approaches are not highly accurate in modeling such diseases; hence, the introduction of fractional differential equations for handling these problems.
Recently, [Hasan et al.(2020)] proposed a new compartmental SIRD model of COVID-19, which is represented as follow:

$$\left\{\begin{array}{lcl} \frac{d}{dt}\mathfrak{S}(t) & = & -\beta\frac{\mathfrak{S}(t)\mathfrak{I}(t)}{\mathcal{N}}, \\ \frac{d}{dt}\mathfrak{I}(t) & = & \beta\frac{\mathfrak{I}(t)\mathfrak{S}(t)}{\mathcal{N}} - (\gamma+\kappa)\,\mathfrak{I}(t), \\ \frac{d}{dt}\mathfrak{R}(t) & = & \gamma\mathfrak{I}(t), \\ \frac{d}{dt}\mathfrak{D}(t) & = & \kappa\mathfrak{I}(t). \end{array}\right. \quad t \geq 0. \tag{1.1}$$

This model requires the division of the total population $\mathcal{N}$ into four epidemiological classes:

$\mathfrak{S}$: Susceptible class, $\mathfrak{I}$: Infected class, $\mathfrak{R}$: Recovered class, and $\mathfrak{D}$: Death class.

The parameters could be described as follows:
∘ β is the average number of contacts per person per time t,
∘ γ is the recovery rate,
∘ κ is the death rate.
For $0 < \alpha < 1$ and $\rho > 0$, and by considering the positive vaccination factor υ, we put

$$\left\{\begin{array}{lcl} {}^{C}\mathcal{D}_{0^+}^{\alpha,\rho}\mathfrak{S}(t) & = & -\upsilon\mathfrak{S}(t) - \beta\frac{\mathfrak{I}(t)\mathfrak{S}(t)}{\mathcal{N}_0}, \\ {}^{C}\mathcal{D}_{0^+}^{\alpha,\rho}\mathfrak{I}(t) & = & \beta\frac{\mathfrak{I}(t)\mathfrak{S}(t)}{\mathcal{N}_0} - (\gamma+\kappa)\,\mathfrak{I}(t), \\ {}^{C}\mathcal{D}_{0^+}^{\alpha,\rho}\mathfrak{R}(t) & = & \upsilon\mathfrak{S}(t) + \gamma\mathfrak{I}(t), \\ {}^{C}\mathcal{D}_{0^+}^{\alpha,\rho}\mathfrak{D}(t) & = & \kappa\mathfrak{I}(t). \end{array}\right. \tag{1.2}$$

along with the positive initial conditions

$$\mathfrak{S}(0) = \mathfrak{S}_0,\ \mathfrak{I}(0) = \mathfrak{I}_0,\ \mathfrak{R}(0) = \mathfrak{R}_0,\ \mathfrak{D}(0) = \mathfrak{D}_0, \tag{1.3}$$

where β, γ and κ are positive parameters, $\mathcal{N}_0 > 0$ is the initial total population at the moment $t = 0$, with $0 \leq t \leq T < \infty$.

2 Definitions and preliminary results

The Banach space of continuous functions from $[0,T]$ into $\mathbb{R}$ is denoted by $C([0,T],\mathbb{R})$, with the norm:

$$\|\varphi\|_{\infty}=\sup_{t\in[0,T]}|\varphi(t)|.$$

Recently, Katugampola proposed a generalized derivative. It reads

$$^{C}\mathcal{D}_{0^{+}}^{\alpha,\rho}\varphi(t)=\mathcal{I}_{0^{+}}^{1-\alpha,\rho}\left(\tau^{1-\rho}\frac{d}{d\tau}\varphi\right)(t)=\frac{\rho^{\alpha}}{\Gamma(1-\alpha)}\int_{0}^{t}\frac{\varphi'(\tau)}{(t^{\rho}-\tau^{\rho})^{\alpha}}d\tau, \tag{2.1}$$

where $\alpha\in(0,1)$, $\rho>0$ and

$$\mathcal{I}_{0^{+}}^{\alpha,\rho}\varphi(t)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{0}^{t}\frac{\tau^{\rho-1}}{(t^{\rho}-\tau^{\rho})^{1-\alpha}}\varphi(\tau)\,d\tau,\ \text{with }\varphi\in C([0,T],\mathbb{R}). \tag{2.2}$$

A decomposition formula for Caputo-Katugampola operators could be obtained as follows:

$$\mathcal{I}_{0^{+}}^{\alpha,\rho}\,{}^{C}\mathcal{D}_{0^{+}}^{\alpha,\rho}\varphi(t)=\varphi(t)-\varphi(0),\ \text{for }\varphi\in C([0,T],\mathbb{R}). \tag{2.3}$$

3 Main results

Here, we present the feasibility region's discussion and equilibrium points' analysis.

Lemma 3.1. *The solution of the model under consideration is restricted to the feasible region given by*

$$u=\left\{(\mathfrak{S},\mathfrak{I},\mathfrak{R},\mathfrak{D})\in\mathbb{R}_{+}^{4},\ 0\leq\mathcal{N}(t)\leq\mathcal{N}_{0}\right\},$$

and the pandemic will occur if $\mathfrak{S}_0>\frac{\gamma+\kappa}{\beta}\mathcal{N}_0$, where $\frac{\gamma+\kappa}{\beta}$ is called threshold phenomenon or critical community size for the pandemic.

Theorem 3.1. *The disease-free equilibrium point of (1.2) is*

$$u^{*}=\left(\frac{\gamma+\kappa}{\beta}\mathcal{N}_{0},0,\mathfrak{R}_{0},0\right).$$

Theorem 3.2. *If the susceptible class $\mathfrak{S}(t)<\mathfrak{S}^{*}$, the pandemic free equilibrium point of (1.2) is locally asymptotically stable and is unstable if $\mathfrak{S}(t)>\mathfrak{S}^{*}$.*

Let $u=(\mathfrak{S},\mathfrak{I},\mathfrak{R},\mathfrak{D})\in E$, where $E=[C([0,T],\mathbb{R}_{+})]^{4}$ is a Banach space equipped with the norm

$$\|u\|_{E}=\|\mathfrak{S}\|_{\infty}+\|\mathfrak{I}\|_{\infty}+\|\mathfrak{R}\|_{\infty}+\|\mathfrak{D}\|_{\infty}$$

and let $f=(f_1,f_2,f_3,f_4)$, be such that

$$\left\{\begin{array}{lll} f_1(t,u(t)) & = & -\upsilon\mathfrak{S}(t)-\beta\frac{\mathfrak{I}(t)\mathfrak{S}(t)}{\mathcal{N}_0}, \\ f_2(t,u(t)) & = & \beta\frac{\mathfrak{I}(t)\mathfrak{S}(t)}{\mathcal{N}_0}-(\gamma+\kappa)\mathfrak{I}(t), \\ f_3(t,u(t)) & = & \upsilon\mathfrak{S}(t)+\gamma\mathfrak{I}(t), \\ f_4(t,u(t)) & = & \kappa\mathfrak{I}(t), \end{array}\right.$$

it is clear that the function $f\in([0,T]\times E)^{4}$ is continuous.
By applying the fractional integral to both sides of the system (1.2), we get

$$\left\{\begin{array}{lll} \mathfrak{S}(t) & = & \mathfrak{S}_0+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t\frac{\tau^{\rho-1}}{(t^{\rho}-\tau^{\rho})^{1-\alpha}}f_1(\tau,u(\tau))\,d\tau, \\ \mathfrak{I}(t) & = & \mathfrak{I}_0+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t\frac{\tau^{\rho-1}}{(t^{\rho}-\tau^{\rho})^{1-\alpha}}f_2(\tau,u(\tau))\,d\tau, \\ \mathfrak{R}(t) & = & \mathfrak{R}_0+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t\frac{\tau^{\rho-1}}{(t^{\rho}-\tau^{\rho})^{1-\alpha}}f_3(\tau,u(\tau))\,d\tau, \\ \mathfrak{D}(t) & = & \mathfrak{D}_0+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t\frac{\tau^{\rho-1}}{(t^{\rho}-\tau^{\rho})^{1-\alpha}}f_4(\tau,u(\tau))\,d\tau, \end{array}\right.$$

By choosing $u_0 = (u_1, u_2, u_3, u_4) = (\mathfrak{S}_0, \mathfrak{I}_0, \mathfrak{R}_0, \mathfrak{D}_0)$, we get

$$u(t) = u_0 + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} f(\tau, u(\tau))\, d\tau. \tag{3.1}$$

In what follows, we present the principal theorems:

Theorem 3.3. *Let $\beta, \upsilon, \gamma, \kappa, \alpha, \rho, T \in \mathbb{R}_+$, be such that $\alpha \in (0,1)$ and*

$$T < \left(\frac{\rho^\alpha \Gamma(\alpha+1)}{4(\beta + \upsilon + \gamma + \kappa + 3)} \right)^{\frac{1}{\rho\alpha}}, \tag{3.2}$$

then, there is at least one solution of the problem (1.2)–(1.3) on $[0, T]$.

Theorem 3.4. *Let $\alpha \in (0,1)$ and $\beta, \gamma, \kappa, \rho, \mu \in \mathbb{R}_+$, be such that*

$$\mu = \max\{\beta + \upsilon + 2, \beta + \gamma + \kappa + 2, \upsilon + \gamma + 3, \kappa + 3\}.$$

If

$$\frac{4\mu T^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} < 1, \tag{3.3}$$

then the problem (1.2)–(1.3) admits a unique solution on $[0, T]$.

4 Conclusion

Our work have discussed some analytical studies for a modified fractional-order SIRD mathematical model of the COVID-19 disease, with Caputo-Katugampola's fractional derivative being used as the differential operator, which unifies the Hadamard and Caputo fractional derivatives into a single form. Using real data, the feasibility region of the proposed solution and the equilibrium points' stability analysis were derived. The behavior of these solutions depends on some symmetrical parameters that satisfy some conditions which prevent the pandemic from occurring. The existence of one solution, at least, in addition to its uniqueness, requires some essential conditions derived from the Banach contraction principle and Schauder's fixed point theorem.

References

[Basti et al.(2020)] B. Basti, Y. Arioua and N. Benhamidouche, *Existence results for nonlinear Katugampola fractional differential equations with an integral condition*, Acta Math. Univ. Comenianae, **89**(2) (2020), 243–260.

[Basti et al.(2021)] B. Basti, N. Hammami, I. Berrabah, F. Nouioua, R. Djemiat, and N. Benhamidouche, *Stability analysis and existence of solutions for a modified SIRD model of COVID-19 with fractional derivatives*, Symmetry, **13**(8) (2021), 1431.

[Biazar et al.(2006)] J. Biazar, *Solution of the epidemic model by Adomian decomposition method*, Appl. Math. Comput., **173**(2) (2006), 1101–1106.

[Brauer et al.(2019)] F. Brauer, C. Castillo-Chavez, Z. Feng, *Mathematical models in epidemiology*, Springer-Verlag, 2019.

[Djordjevic et al.(2018)] J. Djordjevic, C. J. Silva, D. F. M. Torres, *A stochastic SICA epidemic model for HIV transmission*, Appl. Math. Lett., **84** (2018), 168–175.

[Hasan et al.(2020)] A. Hasan, H. Susanto, V. R. Tjahjono, R. Kusdiantara, E. R. M. Putri, P. Hadisoemarto and N. Nuraini, 2003. *A new estimation method for COVID-19 time-varying reproduction number using active cases*, 2020.

[Kilbas et al.(2006)] Kilbas A. A.; Srivastava H. H.; Trujillo J. J., 2006. *Theory and Applications of Fractional Differential Equations*, Elsevier Science B.V, Amsterdam.

Université Mohamed BOUDIAF – M'sila

Le laboratoire des mathématiques pures et appliquées organise :

Mercredi 13 décembre 2023

2ème RENCONTRE

« Modélisation Mathématique pour la Biologie et la Santé; MMBS-2023 »

En collaboration :

Institut Pasteur d'Algérie – M'sila et Annexe de Médecine – M'sila

Comité d'honneur

1. Pr. Amar BOUDELAA. Recteur de l'Université de M'sila
2. Pr. Nouiri BRAHIM. Doyen de la faculté des Mathématiques et Informatique - U. M'sila
3. Pr. Ettayib BENSACI. Doyen de la faculté des Sciences - U. M'sila

Comité d'Organisation / Scientifique

Président : Noureddine BENHAMIDOUCHE, Directeur du LMPA U. M'sila

1. Mohamed Lamine FREIDJA ; Annexe de Médecine - M'sila
2. Nabil BENAZI ; Institut Pasteur d'Algérie – Annexe M'sila
3. Fatima GHERBAOUI ; Faculté des Sciences - U. M'sila

Programme

08h00 – 08h30 : Accueil des participants

08h30 – 09h00 : Allocutions d'ouverture de la 2ème Rencontre « MMBS -2023 »

09h00 – 10h30 : Communications orales ; Session 01 :

1- Dr. Khelaf SAIDANI ; Université de Blida-1

2- Dr. Bilal BASTI ; Université de Djelfa

3- Dr. Nabil BENAZI ; Institut Pasteur d'Algérie – M'sila

10h30 – 11h00 : Pause-café / Communications affichées « Poster »

11h00 – 12h30 : Communications orales ; Session 02 :

1- Mme Fatima Ezahra BENTATA ; Université de Tebessa

2- Dr. Toufik MANSOURI ; Université de Mostaganem

3- Dr. Somia GUECHI ; Université de M'sila

12h30 – 13h00 : - Distribution des attestations de participation

- Clôture de la Rencontre

Adresse : ***Pôle universitaire des sciences et technologies – Université Mohamed BOUDIAF – M'sila***

****