RÉPUBLIQUE ALGÉRIENNE DÉMOCRATIQUE ET POPULAIRE
MINISTÈRE D'ENSEIGNEMENT SUPÉRIEUR ET DE RECHERCHE SCIENTIFIQUE
UNIVERSITÉ MOHAMED BOUDIAF - M'SILA
FACULTÉ DE MATHÉMATIQUES ET DE L'INFORMATIQUE
DÉPARTEMENT DE MATHÉMATIQUES

N° d'ordre: ....

THÈSE

Présenté pour l'obtention du diplôme
de Doctorat troisième cycle

Spécialité:
Mathématiques
Option:
EDPs et Optimisation
Par:

Bilal Basti

Thème

Existence et unicité de solutions auto-similaires générales pour certaines équations fractionnaires non-linéaires

"Existence and uniqueness of general self-similar solutions for some nonlinear fractional equations"

Soutenu publiquement le : 04/05/2019, devant le jury :

Benabderahmane Benyettou	Prof.	Université Mohamed Boudiaf - M'sila	Président
Benhamidouche Noureddine	Prof.	Université Mohamed Boudiaf - M'sila	Rapporteur
Arioua Yacine	M.C.A.	Université Mohamed Boudiaf - M'sila	Co-Rapporteur
Hachama Mohamed	Prof.	Université de Khemis Miliana	Examinateur
Benbachir Maamar	Prof.	Université de Khemis Miliana	Examinateur
Nouiri Brahim	M.C.A.	Université Mohamed Boudiaf - M'sila	Examinateur.

Promotion 2018/2019.

Acknowledgments

Above all, my praise and gratitude is due to Allah the almighty who granted me strength and patience to complete this thesis. I would like to express my sincere gratitude to everyone who has been part of this journey. First and foremost, I am indebted to my supervisor *Prof.* Benhamidouche Noureddine for his patience, valuable advice and guidance and consistent encouragement that he provided throughout this research. I am also deeply grateful to *Dr.* Arioua Yacine who introduced me to this field. Furthermore, I would like to extend my thanks and appreciation to the panel of examiners, *Prof.* Benbachir Maamer, *Prof.* Hachama Mohamed, *Dr.* Nouiri Brahim, and president of the committee *Prof.* Benabderahmane Benyettou. My journey has been made more endurable with the help and thoughtful encouragement I have received from my family and friends, and so my greatest thanks go to them.

Notation

$\mathbb{N}$	Natural numbers $\{0, 1, 2, 3, \ldots\}$.
$\mathbb{N}^*$	Nonzero natural numbers $\{1, 2, 3, \ldots\}$.
$\mathbb{Z}$	Integers $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.
$\mathbb{Z}_0^-$	Negative integers $\{\ldots, -3, -2, -1, 0\}$.
$\mathbb{R}$	Real numbers $(-\infty, \infty)$.
$\mathbb{R}_+$	Positive real numbers $(0, \infty)$.
$\mathbb{R}^*$	Nonzero real numbers $(-\infty, 0) \cup (0, \infty)$.
$\mathbb{C}$	Complex numbers, $z \in \mathbb{C}$, then $z = x + iy$, where $x, y \in \mathbb{R}$, and $i^2 = -1$.
$\mathrm{Re}\,(\alpha)$	Real part of complex α.
Ω	Finite closed interval of the real axis $\mathbb{R}$.
$L^1(\Omega)$	Space of LEBESGUE complex-valued measurable functions u on Ω, for which $\|u\|_{L^1} = \int_\Omega |u(s)|\, ds < \infty$.
$L^p(\Omega)$	Space of all measurable functions u, for which $|u|^p \in L^1(\Omega)$, for any $1 < p < \infty$.
$L^\infty(\Omega)$	Space of all measurable functions u on Ω, for which $\exists C > 0$, $|u(t)| \le C$, p.p. $t \in \Omega$.
$X_c^p(\Omega)$	All measurable functions u on Ω for which $\|u\|_{X_c^p} = \underset{c\in\mathbb{R},\ 1\le p<\infty}{\left(\int_\Omega |s^c u(s)|^p \frac{ds}{s}\right)^{\frac{1}{p}}} < \infty$.
$X_c^\infty(\Omega)$	All measurable functions u on Ω for which $\|u\|_{X_c^\infty} = ess \sup\limits_{a\le t\le b} \underset{c\in\mathbb{R}}{[t^c |u(t)|]} < \infty$.
$C(\Omega)$	The BANACH space of all continuous functions from Ω into $\mathbb{R}$.

$\mathcal{M}u$	MELLIN transform of a function u.
$\mathcal{M}^{-1}v$	Inverse of MELLIN transform of a function v.
$\Gamma(\cdot)$	EULER gamma function.
$(\alpha)_n$	POCHHAMMER symbol, where $(\alpha)_0 = 1,\ (\alpha)_n = \prod_{i=0}^{n-1}(\alpha+i),\ \alpha \in \mathbb{C},\ n \in \mathbb{N}^*$.
$B(\cdot,\cdot)$	Beta function.
$E_\alpha(\cdot)$	Standard MITTAG-LEFFLER function.
$E_{\alpha,\beta}(\cdot)$	MITTAG-LEFFLER function in two arguments, α and β.
ODE	Ordinary Differential Equation.
FDE	Fractional Differential Equation.
PDE	Partial Differential Equation.
FPDE	Fractional-order's Partial Differential Equation.
BVP	Boundary Value Problem.
IVP	Initial Value Problem.
ICP	Problem With an Integral Condition.
$\mathcal{I}^1u$	Primitive of LEBESGUE summable function u.
$\mathcal{I}^n u$	CAUCHY formula for the n^{th} integrals, $n \in \mathbb{N}$.
$\mathcal{I}^\alpha u$	RIEMANN-LIOUVILLE's fractional integral of order α.
${}^{RL}\mathcal{D}^\alpha u$	RIEMANN-LIOUVILLE's fractional derivative of order α.
${}^{C}\mathcal{D}^\alpha u$	CAPUTO's fractional derivative of order α.
${}^{H}\mathcal{I}^\alpha u$	HADAMARD's fractional integral of order α.
${}^{H}\mathcal{D}^\alpha u$	HADAMARD's fractional derivative of order α.
${}^{\rho}\mathcal{I}^\alpha u$	KATUGAMPOLA's fractional integral of order α.
${}^{\rho}\mathcal{D}^\alpha u$	KATUGAMPOLA's fractional derivative of order α.

Contents

Introduction

Our understanding of some real-world phenomena and technology today is largely based on partial differential equations, which will be abbreviated as PDE in what follows. It is indeed thanks to the modeling of these phenomena through PDE that we have been able to understand the role of certain parameters, and especially to obtain sometimes extremely precise forecasts.

One major consideration when dealing with PDEs is that there is usually no way of getting their solutions out explicitly. What mathematics can do, however, is to determine whether one or more solutions exist, and sometimes very precisely describe some properties of these solutions.

Fractional-order's partial differential equations (FPDEs) are generalizations of classical partial differential equations. They have been of considerable interest in the recent literature. They have received a great deal of attention in many fields.

The origin of the fractional calculus dates back to the end of 17^{th} century, the era when NEWTON and LEIBNIZ developed the foundations of differential and integral calculus. But during the last three centuries, the fractional calculus has been a subject of interest to scholars, and the application of fractional derivatives have developed and become more diversified.

In this thesis we recall a class of solutions to the radially symmetric FPDE that are invariant under a scaling group of the variables, and, therefore, we take the so-called self-similar form.

A time-dependent phenomenon is called self-similar if the spatial distributions of its variables at different times can be obtained from one another by a similarity transformation [10], [11], [17], [18], [27], [32], [34]; a transformation that maintains certain features of a function or curve. A particular similarity transformation is a scale-invariant transformation where the variables are scaled by powers of a common factor λ.

Self-similarity occurs when the solution of the problem (as opposed to the FPDE) is invariant under the scaling transformation and is a property of the FPDE which moving mesh method preserves [27]. We obtain a self-similar solution of the FPDE by assuming that there is a functional relationship $w(x,t) = t^{C_1} u\left(xt^{C_2}\right)$ between the similarity variables C_1 and C_2, based on our rescaling. The self-similar solution is then dependent only on the solution to a fractional differential equation (FDE).

The theory of fractional differential equations, the second key notion on which this thesis is based, have emerged as an interesting area to explore in recent years. Note that this theory has many applications in the description of various phenomena in the real world, (Samko et al. 1993 [35], Podlubny 1999 [33], Kilbas et al. 2006 [23], Diethelm 2010 [13]).

The fixed-point methods play a particularly major role in solving the problems of fractional differential equations. They are essential mathematical tools that show the existence of solutions in various kinds of equations. A fixed point theory is at the heart of nonlinear analysis since it provides the apparatus necessary to prove theories of the existence of solutions in various nonlinear problems.

Recently, other results dealing with the existence, uniqueness and multiplicity of real or positive solutions of nonlinear fractional problems have appeared. They use nonlinear analysis techniques such as the fixed point theorems.

It can be noted here that most of the literature on fractional calculus was devoted to solving boundary or initial value problems, or those with an integral conditions, generated by nonlinear fractional differential equations at the base of special functions [2], [3], [6]-[9], [20], [23], [26], [29], [37].

This thesis will use the documentations [7], [8] to give a progress report on all these studies by giving several existence and uniqueness results of generalized self-similar solutions for certain classes of FPDEs, thus, realizing the fractional derivative of Katugampola in Banach spaces. These studies will be done mainly using Banach's contraction principle, Schauder's and Guo-Krasnosel'skii's fixed point theorems, and the technique of the nonlinear alternative of Leray-Schauder's type (see [15], [25]).

The first chapter, entitled "Basic Concepts and Elements of Fractional Calculus", will be devoted to the elements of fractional calculus. We introduce the history of integration and derivation of non-integer orders, and then some definitions and results that are basic for the study.

The second chapter will be devoted to the different basic definitions and results (lemmas, theorems) crucial to self-similar form in relation to the theory of partial differential equations with fractional operators, and we present the main results of this work.

We will discuss in the third chapter, the existence and uniqueness of generalized self-similar solutions for the following nonlinear FPDE:

$$ {}_{x}^{\rho}\mathcal{D}_{0+}^{\alpha}w+\beta f\left(x,t,w\right)=0,\ \left(x,t\right)\in\left(0,X\right)\times\left[0,T\right], $$

supplemented with the boundary conditions:

$$ w\left(0,t\right)=0,\ w\left(X,t\right)=0, $$

where $\beta\in\mathbb{R}$, and ${}_{x}^{\rho}\mathcal{D}_{0+}^{\alpha}$ for $\rho>0$, presents the Katugampola's space-fractional derivative of order $1<\alpha\leq 2$, and $f:[0,X]\times[0,T]\times\mathbb{R}\to\mathbb{R}$ for some $X,T\in\mathbb{R}_{+}$ is a given function.

In the fourth chapter, we are interested in the existence and uniqueness of generalized self-similar solutions for the following implicit problem of nonlinear FPDE:

$$ {}_{x}^{\rho}\mathcal{D}_{0+}^{\alpha}w=f\left(x,t,w,{}_{x}^{\rho}\mathcal{D}_{0+}^{\alpha}w\right),\ 0<\alpha\leq 1,\ \left(x,t\right)\in\left(0,X\right]\times\left[0,T\right], $$

with the initial condition:

$$ w\left(0,t\right)=0, $$

where $f:[0,X]\times[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, with $X,T\in\mathbb{R}_{+}$ is a given function.

The fifth chapter, entitled "Nonlinear Fractional Equations With an Integral Condition", will be devoted to the existence and uniqueness in a general manner of generalized self-similar solutions for the following nonlinear FPDE:

$$ {}_{x}^{\rho}\mathcal{D}_{0+}^{\alpha}w=f\left(x,t,w,{}_{x}^{\rho}\mathcal{D}_{0+}^{\beta}w\right),\ 0<\beta<\alpha\leq 1,\ \left(x,t\right)\in\left[0,X\right]\times\left[0,T\right], $$

with the integral condition:

$$ \left({}_{x}^{\rho}\mathcal{I}_{0+}^{1-\alpha}w\right)\left(0^{+},t\right)=0. $$

Here $f:[0,X]\times[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ with $X,T\in\mathbb{R}_{+}$, is a given function.

Chapter 1

Basic Concepts and Elements of Fractional Calculus

This chapter will be devoted to the primary definitions and basic concepts related to fractional calculus such as the EULER gamma, Beta and MITTAG-LEFFLER functions, and the MELLIN transform. In addition to that, it will also present other elements of functional analysis, such as the fractional derivation, fractional integration, relative definitions of operators of fractional order, among others, which will all be at the core of this work.

1.1 Historical Overview

On the topic of fractional integrals and derivatives, we cite a particular date as that of the first appearance of the so called "Fractional Calculus". In a letter dated September 30^{th}, 1695, L'HÔSPITAL wrote to LEIBNIZ asking him about a particular notation he had used in his writings for the n^{th}-derivative of the linear function $u(t) = t$, $\frac{d^n t}{dt^n}$. L'HÔSPITAL wondered what the result would be if $n = 1/2$. LEIBNIZ's response was: "An apparent paradox, from which one day useful consequences will be drawn." In these words fractional calculus was born.

Following L'HÔSPITAL's and LEIBNIZ's first inquisition, fractional calculus was primarily a study reserved for the best minds in mathematics. FOURIER, EULER and LAPLACE are among the many who tackled the fractional calculus and the mathematical consequences [31].

Several mathematicians used their own notation and methodology to introduce definitions that fit the concept of an integral or derivative non-integral order. The most famous of these definitions in the world of fractional calculus are RIEMANN-LIOUVILLE and CAPUTO definitions. The sheer numbers of actual definitions are no doubt as numerous as the scholars in this field, and they are addressed in detail in this thesis.

Most of the mathematical theory applicable to the study of fractional calculus was developed prior to the turn of the twentieth century. However, it is only during the last century that the most intriguing advances in engineering and scientific application have been achieved. Mathematics had in some cases to change in order to meet the requirements of physical reality.

CAPUTO reformulated the more classic definition of the RIEMANN-LIOUVILLE's fractional derivative in order to use integer order initial conditions to solve his fractional order differential equations [33]. In 1996, KOLOWANKAR reformulated again RIEMANN-LIOUVILLE's fractional derivative [24].

1.2 Background Materials of Functional Analysis

Let $\Omega = [a, b]$ $(-\infty < a < b < \infty)$ be a finite closed interval of the real axis $\mathbb{R} = (-\infty, \infty)$. We denote by $L^1(\Omega)$ the space of those LEBESGUE complex-valued measurable functions u on Ω for which $\|u\|_{L^1} < \infty$, where

$$\|u\|_{L^1} = \int_\Omega |u(s)|\, ds.$$

Let $1 \leq p < \infty$, we denote by $L^p(\Omega)$ the space of those LEBESGUE complex-valued measurable functions u on Ω for which $|u|^p \in L^1(\Omega)$.

For the case $p = \infty$, we denote by

$$L^\infty(\Omega) = \{u : \Omega \to \mathbb{R};\ \text{that } u \text{ is measurable, and } \exists C > 0 \text{ such that } |u(t)| \leq C \text{ p.p. on } \Omega\},$$

the space of those bounded LEBESGUE complex-valued measurable functions u on Ω. We note

$$\|u\|_{L^p} = \left(\int_\Omega |u(s)|^p\, ds\right)^{\frac{1}{p}}, \text{ and } \|u\|_{L^\infty} = ess \sup_{a\leq t\leq b} |u(t)|.$$

Here $ess \sup_{a\leq t\leq b} |u(t)|$ is the essential maximum of the function $|u(t)|$ [see, for example, NIKOL'SKII [30], p. 12-13].

As in [23], consider the space $X_c^p(\Omega)$, of those complex-valued LEBESGUE measurable functions u on Ω for which $\|u\|_{X_c^p} < \infty$, where the norm is defined by:

$$\|u\|_{X_c^p} = \left(\int_\Omega |s^c u(s)|^p \frac{ds}{s}\right)^{\frac{1}{p}} < \infty,$$

for $1 \leq p < \infty$, $c \in \mathbb{R}$. For the case $p = \infty$;

$$\|u\|_{X_c^\infty} = ess \sup_{a \leq t \leq b} \left[t^c |u(t)|\right], \; c \in \mathbb{R}.$$

By $C(\Omega)$ we denote the BANACH space of all continuous functions from Ω into $\mathbb{R}$ with the norm:

$$\|u\|_\infty = \sup_{a \leq t \leq b} |u(t)|.$$

1.3 Special Functions of Fractional Calculus

In this section, we present the functions EULER gamma, Beta and MITTAG-LEFFLER. These functions play an important role in the theory of fractional calculus.

Euler Gamma Function

As we will explain later, the EULER gamma function is intrinsically tied to fractional calculus by definition (see the sections 1.4, 1.5). The simplest interpretation of the EULER gamma function is simply the generalization of the factorial for all real numbers.

Firstly, we give the MELLIN transform definition, which has an important role in the definition of EULER gamma function.

Definition 1.1 (Mellin transform [23]) *The* MELLIN *transform of a function $u(t)$ of a real variable $t \in \mathbb{R}^+ = (0, \infty)$ is defined by:*

$$(\mathcal{M}u)(s) = \mathcal{M}[u(t)](s) = u^*(s) = \int_0^\infty t^{s-1} u(t)\, dt, \; \mathrm{Re}(s) > 0,$$

and the inverse of MELLIN *transform is given for $t \in \mathbb{R}^+$ by the formula:*

$$\left(\mathcal{M}^{-1}u^*\right)(t) = \mathcal{M}^{-1}[u^*(s)](t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} t^{-s} u^*(s)\, ds, \; \gamma = \mathrm{Re}(s).$$

The direct and inverse of MELLIN *transforms are inverse to each other for "sufficiently good" functions u and v,*

$$\mathcal{M}^{-1}\mathcal{M}u = u, \; \text{and } \mathcal{M}\mathcal{M}^{-1}v = v.$$

The definition of the EULER gamma function is given as follows:

Definition 1.2 ([23]) *The* EULER *gamma function is defined by the so-called* EULER *integral of the second kind and is given with a direct* MELLIN *transform formula, as follows:*

$$\Gamma(\alpha) = \mathcal{M}\left[e^{-s}\right](\alpha) = \int_0^\infty s^{\alpha-1}e^{-s}ds, \tag{1.1}$$

where $s^{\alpha-1} = e^{(\alpha-1)\ln(s)}$. *This integral is convergent for all complex* $\mathrm{Re}(\alpha) > 0$, *with* $\Gamma(1) = 1$, $\Gamma(0^+) = +\infty$, $\Gamma(\alpha)$ *is a monotonous and strictly decreasing function for* $0 < \alpha \leq 1$.

The "beauty" of the EULER gamma function can be found in its properties. First, as seen in property 1.1, this function is unique in that the value for any quantity is, by consequence of the form of the integral, equivalent to that quantity α minus one times the EULER gamma of the quantity minus one.

Property 1.1 ([23]) *An important property of the* EULER *gamma function* $\Gamma(\alpha)$ *is the following recurrence relation:*

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha), \quad \mathrm{Re}(\alpha) > 0, \tag{1.2}$$

when we can demonstrate by an integration by parts, as follows:

$$\Gamma(\alpha+1) = \int_0^\infty s^{\alpha}e^{-s}ds = \left[-s^{\alpha}e^{-s}\right]_0^\infty + \alpha\int_0^\infty s^{\alpha-1}e^{-s}ds = \alpha\Gamma(\alpha).$$

Definition 1.3 ([23]) *The* POCHHAMMER *symbol* $(\alpha)_n$ *is defined for complex* $\alpha \in \mathbb{C}$ *and non-negative integer* $n \in \mathbb{N}$ *by*

$$(\alpha)_0 = 1, \ \textit{and} \ (\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1), \ n \in \mathbb{N}^*. \tag{1.3}$$

Note that $(1)_n = n!$.

Property 1.2 ([23]) *Using the last definition and relation* (1.2), *the* EULER *gamma function is extended to the half-plane* $\mathrm{Re}(\alpha) \leq 0$ *by*

$$\Gamma(\alpha) = \frac{\Gamma(\alpha+n)}{(\alpha)_n}, \ \mathrm{Re}(\alpha) > -n, \ \textit{for}\ n \in \mathbb{N}^*, \ \mathrm{Re}(\alpha) \notin \mathbb{Z}_0^- = \{\ldots,-3,-2,-1,0\}. \tag{1.4}$$

Here $(\alpha)_n$ *is the* POCHHAMMER *symbol.*

For a better understanding, the graph of the EULER gamma function $y = \Gamma(t)$ for real values of t is given in figure (1).

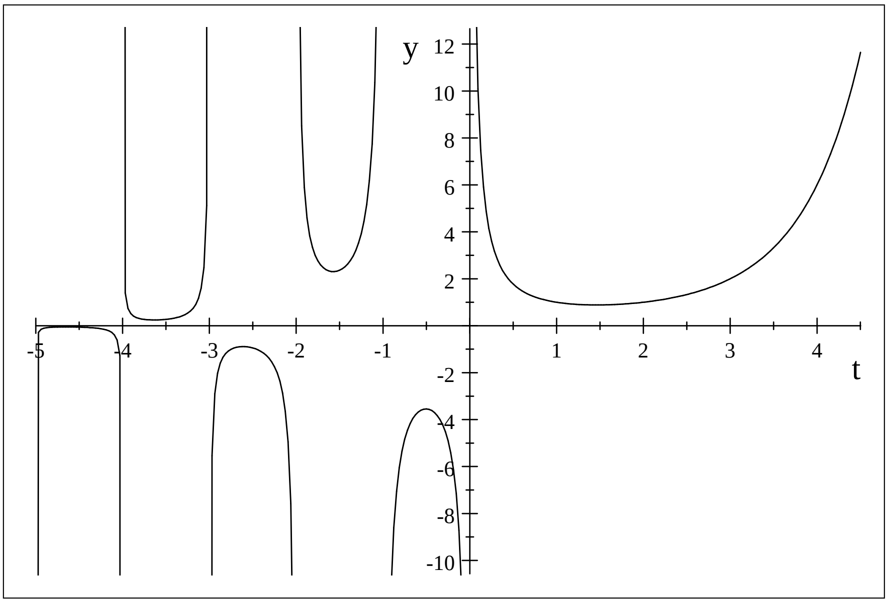

Figure 1 Approximation of Euler gamma function

Figure (1) demonstrates the EULER gamma function at and around zero. Note that at negative integer values, the EULER gamma function goes to infinity.

Property 1.3 ([23]) *The* EULER *gamma function generalizes the factorial because*

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\cdots\Gamma(1) = n!, \ \forall n \in \mathbb{N},$$

also, from (1.3) *and* (1.4) *, we get*

$$\Gamma(1) = \frac{\Gamma(n+1)}{(1)_n}, \text{ which implies that } \Gamma(n+1) = n!,$$

for any $n \in \mathbb{N}$.

Beta Function

Beta function, also known as the EULER integral of the first kind, is an important relationship in fractional calculus.

Definition 1.4 ([23]) *The Beta function is a type of* EULER *integral defined by:*

$$B(p,q) = \int_0^1 s^{p-1}(1-s)^{q-1}\,ds, \ p,q \in \mathbb{C}\backslash\mathbb{Z}_0^-, \tag{1.5}$$

Property 1.4 ([23]) *For all* $p,q \in \mathbb{C}\backslash\mathbb{Z}_0^-$*, we have:*

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}. \tag{1.6}$$

Equation (1.6) provides the analytical continuation of the Beta function to the entire complex plane via the analytical continuation of the EULER gamma function. It should also be mentioned that Beta function is symmetric, i.e.,

$$B(p,q) = B(q,p),\ \forall p,q \in \mathbb{C}\backslash\mathbb{Z}_0^-.$$

Mittag-Leffler Function

The MITTAG-LEFFLER function is an important function that is widely used in the field of fractional calculus. Just as the exponential naturally arises out of the solution to integer order differential equations, the MITTAG-LEFFLER function plays an analogous role in the solution of non-integer order differential equations. The generalization of the single-parameter exponential function has been introduced by G. M. MITTAG-LEFFLER [28] and is designated by the following definition:

Definition 1.5 ([23]) *The standard definition of the* MITTAG-LEFFLER *function is given by*

$$E_\alpha(t) = \sum_{k=0}^{+\infty} \frac{t^k}{\Gamma(\alpha k + 1)},\ \ \alpha > 0. \tag{1.7}$$

It is also common to represent the MITTAG-LEFFLER *function in two arguments, α and β. Such that*

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{+\infty} \frac{t^k}{\Gamma(\alpha k + \beta)},\ \ \alpha,\beta > 0. \tag{1.8}$$

The last relation is the more generalized form of the function. For $\beta = 1$, we find the relationship (1.7) *.*

Example 1.1 *From the relation* (1.8) *, we find that*

$$\begin{aligned} E_{1,1}(t) &= \sum_{k=0}^{+\infty} \frac{t^k}{\Gamma(k+1)} = \sum_{k=0}^{+\infty} \frac{t^k}{k!} = e^t, \\ E_{1,2}(t) &= \sum_{k=0}^{+\infty} \frac{t^k}{\Gamma(k+2)} = \sum_{k=0}^{+\infty} \frac{t^k}{(k+1)!} = \frac{1}{t}\sum_{k=0}^{+\infty} \frac{t^{k+1}}{(k+1)!} = \frac{1}{t}\left(e^t - 1\right), \end{aligned}$$

1.4 Basic Fractional Integrals and Derivatives

Our goal in this section is to introduce the methods and results used in our work. We begin by giving the definitions of the fractional integrals and then fractional derivatives. Following that, we observe that - only - certain properties of classical derivatives can be generalized to the fractional case. The majority of the definitions in this chapter are taken from [23] and [35], which we refer to for a thorough analysis of the subject.

1.4.1 Riemann-Liouville Fractional Integrals and Derivatives

As is the case with the majority of introductory works on fractional calculus, we will follow RIEMANN's approach to propose a first definition of fractional integral. We will see that all the definitions we have given are left definitions, and there are symmetrical versions on the right. These are rarely used because they are anti-causal (they depend on the nature of the functions).

I. The Riemann-Liouville Fractional Integral Operators

I.1 Functions defined on a bounded interval

Let $[a, b]$ be a finite closed interval of the real axis $\mathbb{R} = (-\infty, \infty)$, and let u be a measurable continuous function on $[a, b]$ in $\mathbb{R}$. Let's start by noting $\mathcal{I}^1_{a^+}$ the primitive of u, and we give

$$\mathcal{I}^1_{a^+} u(t) = \int_a^t u(s)\, ds. \tag{1.9}$$

The iteration of $\mathcal{I}^1_{a^+}$ allows to obtain the primitive second of u. Moreover, according to the theorem of FUBINI,

$$\begin{aligned} \mathcal{I}^2_{a^+} u(t) &= \mathcal{I}^1_{a^+} \circ \mathcal{I}^1_{a^+} u(t) = \int_a^t \left(\int_a^s u(\tau)\, d\tau \right) ds = \int_a^t u(\tau) \left(\int_\tau^t ds \right) d\tau \\ &= \int_a^t (t - \tau)\, u(\tau)\, d\tau. \end{aligned}$$

The RIEMANN-LIOUVILLE's approach is based on the CAUCHY formula (1.10) for the n^{th} integral which uses only a simple integration so as to provide a good basis for generalization.

$$\mathcal{I}^n_{a^+} u(t) = \int_a^t \int_a^{t_1} \cdots \int_a^{t_{n-1}} u(t_n)\, dt_n dt_{n-1} \ldots dt_1 = \frac{1}{(n-1)!} \int_a^t (t - s)^{n-1} u(s)\, ds. \tag{1.10}$$

Now it is clear how to get an integral of arbitrary order. We simply generalize the CAUCHY formula (1.10), the integer n is substituted by a positive real number α and the EULER gamma function is used instead of the factorial:

$$\mathcal{I}^\alpha_{a^+} u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha-1} u(s)\, ds.$$

This formula represents the integral of arbitrary order $\alpha > 0$, but it does not permit the order $\alpha = 0$ which formally corresponds to the identity operator. This expectation is fulfilled under certain reasonable assumptions at least if we consider the limit for $\alpha \to 0$ (see [33]).

Hence, we extend the above definition by setting:

$$\mathcal{I}_{a^+}^{0} u(t) = u(t).$$

Definition 1.6 (Left-sided Riemann-Liouville fractional integral [23]) *The left-sided* RIEMANN-LIOUVILLE*'s fractional integral of order $\alpha > 0$ of a continuous function $u : [a, b] \to \mathbb{R}$ is given by:*

$$\mathcal{I}_{a^+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} u(s)\, ds, \;\; t \in [a, b]. \tag{1.11}$$

I.2 Functions defined on $\mathbb{R}$ and $\mathbb{R}^+$

It is natural to extend the definition (1.11) to the axes $\mathbb{R}$ and $\mathbb{R}^+$. Let us note these operators $\mathcal{I}_{+}^{\alpha}$ and $\mathcal{I}_{0^+}^{\alpha}$ resp.,

$$\mathcal{I}_{+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^t (t-s)^{\alpha-1} u(s)\, ds, \;\; t \in \mathbb{R},$$

we give $u : \mathbb{R} \to \mathbb{R}$ is a continuous function, and

$$\mathcal{I}_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds, \;\; t > 0,$$

where $u : \mathbb{R}^+ \to \mathbb{R}$ is a continuous function.

I.3 Right-sided fractional integral

If we go back to the starting relationship (1.9) for a function $u : [a, b] \to \mathbb{R}$, we can notice that the integral

$$\mathcal{I}_{b^-}^{1} u(t) = \int_b^t u(s)\, ds = -\int_t^b u(s)\, ds,$$

is also a primitive of u, which this time involves the values to the right of u.

From the relationship:

$$\int_b^t (t-s)^{n-1} u(s)\, ds = (-1)^n \int_t^b (s-t)^{n-1} u(s)\, ds,$$

we could define in the same way the right-sided integral of order n of u by:

$$\forall t \in [a, b], \;\; \mathcal{I}_{b^-}^{n} u(t) = \frac{(-1)^n}{(n-1)!} \int_t^b (s-t)^{n-1} u(s)\, ds.$$

Definition 1.7 (Right-sided Riemann-Liouville fractional integral [23]) *The right-sided* RIEMANN-LIOUVILLE*'s fractional integral of order* $\alpha > 0$ *of a continuous function* $u : [a, b] \to \mathbb{R}$ *is given by:*

$$\mathcal{I}_{b^-}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} u(s)\, ds, \ t \in [a, b]. \tag{1.12}$$

The extension on $[a, +\infty)$ *and* $\mathbb{R}$ *is noted* $\mathcal{I}_{-}^{\alpha}$ *:*

$$\mathcal{I}_{-}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_t^{+\infty} (s-t)^{\alpha-1} u(s)\, ds.$$

II. The Riemann-Liouville Fractional Differential Operators

The definition of fractional integrals is very straightforward and there are no complications. A more difficult question to deal with is how to define a fractional derivative. There is no formula for the n^{th} derivative analogous to (1.10) so we have to generalize the derivatives through a fractional integral.

If $\alpha > 0$, we denote $[\alpha]$ the integer part of α, $[\alpha]$ is the unique integer satisfying

$$[\alpha] \le \alpha < [\alpha] + 1.$$

Let $u : [a, b] \to \mathbb{R}$. From the classic relationship $\frac{d}{dt} = \frac{d^2}{dt^2} \circ \mathcal{I}_{a^+}^{1}$ we can define a fractional derivative of order $0 \le \alpha < 1$ by:

$$\frac{d^{\alpha}}{dt^{\alpha}} = \frac{d}{dt} \circ \mathcal{I}_{a^+}^{1-\alpha}.$$

More generally, if $\alpha > 0$ and if $n = [\alpha] + 1$, we can put:

$$\frac{d^{\alpha}}{dt^{\alpha}} = \left(\frac{d}{dt}\right)^{n} \circ \mathcal{I}_{a^+}^{n-\alpha}. \tag{1.13}$$

We obtain exactly the left-sided RIEMANN-LIOUVILLE's fractional derivative.

Definition 1.8 (Left-sided Riemann-Liouville fractional derivative [23]) *The left-sided* RIEMANN-LIOUVILLE*'s fractional derivative of order* $\alpha > 0$ *of a continuous function* $u : [a, b] \to \mathbb{R}$ *is given by:*

$${}^{RL}\mathcal{D}_{a^+}^{\alpha} u(t) = \left(\frac{d}{dt}\right)^{n} \circ \mathcal{I}_{a^+}^{n-\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^{n} \int_a^t (t-s)^{n-\alpha-1} u(s)\, ds, \ n = [\alpha] + 1.$$

As $n = [\alpha] + 1$, this formula includes even the integer order derivatives. If $\alpha = k \in \mathbb{N}$, then $n = k + 1$, and we obtain:

$${}^{RL}\mathcal{D}_{a^+}^{k} u(t) = \frac{1}{\Gamma(1)} \left(\frac{d}{dt}\right)^{k+1} \int_a^t u(s)\, ds = \frac{d^k u(t)}{dt^k}, \ \forall t \in [a, b]. \tag{1.14}$$

If we note that ${}^{RL}\mathcal{D}_{a^+}^{-\alpha} = \mathcal{I}_{a^+}^{\alpha} u(t)$, we get ${}^{RL}\mathcal{D}_{a^+}^{0} u(t) = u(t)$. We can write both fractional integral and derivative using one expression and formulate the definition of the Riemann-Liouville's differ-integral.

Moreover, we saw that the definition (1.7) of integrals on the right was associated with $-\frac{d}{dt}$. The preceding reasoning thus leads to the following definition:

Definition 1.9 (Right-sided Riemann Liouville-fractional derivative [23]) *The right-sided* Riemann Liouville-*fractional derivative of order* $\alpha > 0$ *of a continuous function* $u : [a,b] \to \mathbb{R}$ *is given by:*

$${}^{RL}\mathcal{D}_{b^-}^{\alpha} u(t) = \left(-\frac{d}{dt}\right)^n \circ \mathcal{I}_{a^+}^{n-\alpha} u(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_t^b (s-t)^{n-\alpha-1} u(s)\, ds, \ \ n = [\alpha]+1.$$

Now, if $u : \mathbb{R} \to \mathbb{R}$, the preceding definitions are generalized directly and are called Liouville derivatives.

Definition 1.10 (Left-sided Liouville fractional derivative [23]) *Let* $\alpha > 0$, *and* $n = [\alpha]+1$. *The left-sided* Liouville *fractional derivative of order* α *of a continuous function* $u : [a,b] \to \mathbb{R}$ *is given by:*

$$\forall t \in \mathbb{R}, \ {}^{RL}\mathcal{D}_{+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_{-\infty}^t (t-s)^{n-\alpha-1} u(s)\, ds.$$

Definition 1.11 (Right-sided Liouville fractional derivative [23]) *Let* $\alpha > 0$, *and* $n = [\alpha]+1$. *The right-sided* Liouville *fractional derivative of order* α *of a continuous function* $u : [a,b] \to \mathbb{R}$ *is given by:*

$$\forall t \in \mathbb{R}, \ {}^{RL}\mathcal{D}_{-}^{\alpha} u(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_t^{+\infty} (s-t)^{n-\alpha-1} u(s)\, ds.$$

According to (1.13) and (1.14), all these derivatives coincide with the usual derivatives for integer orders:

$$\forall n \in \mathbb{N}, \begin{cases} {}^{RL}\mathcal{D}_{a^+}^{n} u = {}^{RL}\mathcal{D}_{+}^{n} u = u^{(n)}, \\ {}^{RL}\mathcal{D}_{b^-}^{n} u = {}^{RL}\mathcal{D}_{-}^{n} u = (-1)^n u^{(n)}. \end{cases}$$

Property 1.5 ([23]) *If* $\alpha, \beta > 0$, *then*

$$\mathcal{I}_{a^+}^{\alpha} (t-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (t-a)^{\beta+\alpha-1}, \tag{1.15}$$

$${}^{RL}\mathcal{D}_{a^+}^{\alpha} (t-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (t-a)^{\beta-\alpha-1}. \tag{1.16}$$

In particular, if $\beta = 1$ *and* $\alpha > 1$, *then* Riemann-Liouville*'s fractional derivatives of a constant are, in general, not equal to zero:*

$${}^{RL}\mathcal{D}_{a^+}^{\alpha} 1 = \frac{1}{\Gamma(1-\alpha)} (t-a)^{-\alpha}.$$

1.4.2 Caputo-Type Fractional Derivatives

Moreover, if we go back to $[a,b]$, the inversion of the compositions in the right side of (1.13) seems also reasonable to define a fractional derivative:

$$\frac{d^\alpha}{dt^\alpha} = \mathcal{I}_{a^+}^{n-\alpha} \circ \left(\frac{d}{dt}\right)^n. \tag{1.17}$$

It should be noted, however, that this definition is less natural than the previous one, since $\frac{d}{dt} \circ \mathcal{I}_{a^+}^1 u(t) = u(t)$, while $\mathcal{I}_{a^+}^1 \circ \frac{d}{dt} u(t) = u(t) - u(a)$,

This problem of terms of border (here $u(a)$) is in fact very often found in the fractional calculation. The definition given by (1.17) is called the CAPUTO's derivative.

Definition 1.12 (Left-sided Caputo fractional derivative [23]) *The left-sided* CAPUTO*'s fractional derivative of order $\alpha > 0$ of a function $u \in C^n([a,b], \mathbb{R})$, where $n = [\alpha] + 1$ is given by:*

$$^C\mathcal{D}_{a^+}^{\alpha} u(t) = \mathcal{I}_{a^+}^{n-\alpha} \circ \left(\frac{d}{dt}\right)^n u(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} \left(\frac{d}{ds}\right)^n u(s)\, ds,$$

also

$$\forall t \in \mathbb{R},\ ^C\mathcal{D}_{+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \int_{-\infty}^t (t-s)^{n-\alpha-1} \left(\frac{d}{ds}\right)^n u(s)\, ds.$$

Let's also define its analog on the right.

Definition 1.13 (Right-sided Caputo fractional derivative [23]) *The right-sided* CAPUTO*'s fractional derivative of order $\alpha > 0$ of a function $u \in C^n([a,b], \mathbb{R})$, where $n = [\alpha] + 1$ is given by:*

$$^C\mathcal{D}_{b^-}^{\alpha} u(t) = \mathcal{I}_{b^-}^{n-\alpha} \circ \left(-\frac{d}{dt}\right)^n u(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b (s-t)^{n-\alpha-1} \left(\frac{d}{ds}\right)^n u(s)\, ds,$$

also

$$\forall t \in \mathbb{R},\ ^C\mathcal{D}_{-}^{\alpha} u(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^{+\infty} (s-t)^{n-\alpha-1} \left(\frac{d}{ds}\right)^n u(s)\, ds.$$

Fortunately, the following result shows that they approach classical derivatives by lower bound.

Lemma 1.1 ([35]) *Let $\alpha \in \mathbb{R}^+ \backslash \mathbb{N}$, be such that $n = [\alpha] + 1$. If $u^{(n)}$ is a continuous function, so almost everywhere*

$$\lim_{\alpha \to n^-} {}^C\mathcal{D}_{a^+}^{\alpha} u(t) = u^{(n)}(t),$$
$$\lim_{\alpha \to n^-} {}^C\mathcal{D}_{b^-}^{\alpha} u(t) = (-1)^n u^{(n)}(t).$$

Proof. As $u^{(n)}$ is a continuous function, according to [35], by posing $\beta = n - \alpha$, we have:

$$\lim_{\beta \to 0^+} \mathcal{I}_{a^+}^{\beta} u^{(n)} = u^{(n)},$$

almost everywhere. The same reasoning applies for ${}^{C}\mathcal{D}_{b^-}^{\alpha} u$. The proof is complete. ∎

If $\alpha \notin \mathbb{N}$ and $u(t)$ is a function for which the CAPUTO's fractional derivatives ${}^{C}\mathcal{D}_{a^+}^{\alpha} u(t)$ and ${}^{C}\mathcal{D}_{b^-}^{\alpha} u(t)$ of order $\alpha > 0$ exist together with the RIEMANN-LIOUVILLE's fractional derivatives ${}^{RL}\mathcal{D}_{a^+}^{\alpha} u(t)$ and ${}^{RL}\mathcal{D}_{b^-}^{\alpha} u(t)$, then, in accordance with (1.11) and (1.12), they are connected with each other through the following relations:

$${}^{C}\mathcal{D}_{a^+}^{\alpha} u(t) = {}^{RL}\mathcal{D}_{a^+}^{\alpha} u(t) - \sum_{k=0}^{n-1} \frac{u^{(k)}(a)}{\Gamma(k-\alpha+1)} (t-a)^{k-\alpha}, \text{ where } n = [\alpha] + 1,$$

then

$${}^{C}\mathcal{D}_{a^+}^{\alpha} u(t) = {}^{RL}\mathcal{D}_{a^+}^{\alpha} u(t), \text{ if } u(a) = u'(a) = \cdots = u^{(n-1)}(a) = 0.$$

Also

$${}^{C}\mathcal{D}_{b^-}^{\alpha} u(t) = {}^{RL}\mathcal{D}_{b^-}^{\alpha} u(t) - \sum_{k=0}^{n-1} \frac{u^{(k)}(b)}{\Gamma(k-\alpha+1)} (b-t)^{k-\alpha}, \text{ where } n = [\alpha] + 1,$$

then

$${}^{C}\mathcal{D}_{b^-}^{\alpha} u(t) = {}^{RL}\mathcal{D}_{b^-}^{\alpha} u(t), \text{ if } u(b) = u'(b) = \cdots = u^{(n-1)}(b) = 0.$$

Property 1.6 ([23]) *If* $\alpha, \beta > 0$, *then*

$${}^{C}\mathcal{D}_{a^+}^{\alpha} (t-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (t-a)^{\beta-\alpha-1}.$$

In particular, if $\beta = 1$, *we get*

$${}^{C}\mathcal{D}_{a^+}^{\alpha} 1 = 0.$$

1.4.3 Hadamard-Type Fractional Integrals and Derivatives

In this part we present the definitions and some properties of the HADAMARD-type fractional integrals and fractional derivatives. Some of these definitions and results were presented in SAMKO et al. [35].

Let $[a, b]$ $(0 < a < b < \infty)$ be a finite or infinite interval of the half-axis $\mathbb{R}^+$. We consider the left-sided and right-sided integrals of fractional order $\alpha > 0$ defined by:

Definition 1.14 (Left-sided Hadamard fractional integral [23]) *The left-sided* HADAMARD*'s fractional integral of order $\alpha > 0$ of a continuous function $u : [a, b] \to \mathbb{R}$ is given by:*

$$^{H}\mathcal{I}_{a^+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} u(s) \frac{ds}{s},\ t \in [a, b]. \tag{1.18}$$

Definition 1.15 (Right-sided Hadamard fractional integral [23]) *The right-sided* HADAMARD*'s fractional integral of order $\alpha > 0$ of a continuous function $u : [a, b] \to \mathbb{R}$ is given by:*

$$^{H}\mathcal{I}_{b^-}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \left(\log \frac{s}{t}\right)^{\alpha-1} u(s) \frac{ds}{s},\ t \in [a, b].$$

Respectively. When $a = 0$ and $b = \infty$, these relations are given by:

$$^{H}\mathcal{I}_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \left(\log \frac{t}{s}\right)^{\alpha-1} u(s) \frac{ds}{s},\ t > 0,$$

and

$$^{H}\mathcal{I}_{-}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_t^\infty \left(\log \frac{s}{t}\right)^{\alpha-1} u(s) \frac{ds}{s},\ t > 0.$$

Definition 1.16 (Left-sided Hadamard fractional derivative [23]) *The left-sided* HADAMARD*'s fractional derivative of order $\alpha > 0$ of a continuous function $u : [a, b] \to \mathbb{R}$ is given by*

$$^{H}\mathcal{D}_{a^+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(t\frac{d}{dt}\right)^n \int_a^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} u(s) \frac{ds}{s},\ t \in [a, b],\ n = [\alpha] + 1. \tag{1.19}$$

Definition 1.17 (Right-sided Hadamard fractional derivative [23]) *The right-sided* HADAMARD*'s fractional derivative of order $\alpha > 0$ of a continuous function $u : [a, b] \to \mathbb{R}$ is given by*

$$^{H}\mathcal{D}_{b^-}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(-t\frac{d}{dt}\right)^n \int_t^b \left(\log \frac{s}{t}\right)^{n-\alpha-1} u(s) \frac{ds}{s},\ t \in [a, b],\ n = [\alpha] + 1.$$

Respectively. For $n = [\alpha] + 1$, when $a = 0$ and $b = \infty$, these relations are given by:

$$^{H}\mathcal{D}_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(t\frac{d}{dt}\right)^n \int_0^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} u(s) \frac{ds}{s},\ t > 0,$$

and

$$^{H}\mathcal{D}_{-}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(-t\frac{d}{dt}\right)^n \int_t^\infty \left(\log \frac{s}{t}\right)^{n-\alpha-1} u(s) \frac{ds}{s},\ t > 0.$$

1.5 Generalized Fractional Integrals and Derivatives of Katugampola

A recent generalization in 2011, introduced by UDITA KATUGAMPOLA [22], combines the RIEMANN-LIOUVILLE's fractional integral and the HADAMARD's fractional integral into a single form (see [23]). The generalized fractional integral ${}^{\rho}\mathcal{I}_{a^+}^{\alpha}u$ of order $\alpha > 0$ of a function $u \in X_c^p[a,b]$ for $-\infty < a < b < +\infty$, is now known as KATUGAMPOLA's fractional integral, it is given in the following definition.

Definition 1.18 (Katugampola fractional integral [22]) *The left-sided (resp. right-sided) generalized fractional integral of* KATUGAMPOLA *of order $\alpha > 0$ of a function $u \in X_c^p[a,b]$ is defined by:*

$$\left({}^{\rho}\mathcal{I}_{a^+}^{\alpha}u\right)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t \frac{s^{\rho-1}u(s)}{(t^{\rho}-s^{\rho})^{1-\alpha}}ds, \ \ \rho > 0, \ \ t \in [a,b], \tag{1.20}$$

respectively

$$\left({}^{\rho}\mathcal{I}_{b^-}^{\alpha}u\right)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_t^b \frac{s^{\rho-1}u(s)}{(s^{\rho}-t^{\rho})^{1-\alpha}}ds, \ \ \rho > 0, \ \ t \in [a,b]. \tag{1.21}$$

These are the fractional generalizations of the n-fold left-sided and right-sided integrals of the form

$$\int_a^t s_1^{\rho-1}\int_a^{s_1} s_2^{\rho-1}\dots\int_a^{s_{n-1}} u(s_n)\,ds_n ds_{n-1}\dots ds_1,$$

and

$$\int_t^b s_1^{\rho-1}\int_{s_1}^{b} s_2^{\rho-1}\dots\int_{s_{n-1}}^{b} u(s_n)\,ds_n ds_{n-1}\dots ds_1,$$

for $n \in \mathbb{N}$, respectively.

When $b = \infty$, the generalized fractional integral of KATUGAMPOLA is called a LIOUVILLE-type integral and the case $a = \infty$ is referred to as the WEYL-derivative [36].

Now, consider the generalized fractional derivatives of KATUGAMPOLA defined below.

Definition 1.19 (Katugampola fractional derivatives [19]) *Let $\alpha, \rho \in \mathbb{R}^+$, and $n = [\alpha]+1$. The* KATUGAMPOLA*'s fractional derivative corresponding to the* KATUGAMPOLA*'s fractional integral (1.20) (resp. (1.21)) are defined by:*

$${}^{\rho}\mathcal{D}_{a^+}^{\alpha}u(t) = \left(t^{1-\rho}\frac{d}{dt}\right)^n \left({}^{\rho}\mathcal{I}_{a^+}^{n-\alpha}u\right)(t) = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\left(t^{1-\rho}\frac{d}{dt}\right)^n \int_a^t \frac{s^{\rho-1}u(s)}{(t^{\rho}-s^{\rho})^{\alpha-n+1}}ds, \tag{1.22}$$

respectively

$${}^{\rho}\mathcal{D}_{b^-}^{\alpha}u(t) = \left(-t^{1-\rho}\frac{d}{dt}\right)^n \left({}^{\rho}\mathcal{I}_{b^-}^{n-\alpha}u\right)(t) = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\left(-t^{1-\rho}\frac{d}{dt}\right)^n \int_t^b \frac{s^{\rho-1}u(s)}{(s^{\rho}-t^{\rho})^{\alpha-n+1}}ds.$$

When $\rho = 1$ we arrive at the standard RIEMANN-LIOUVILLE's fractional integral (resp. derivative) definition 1.6 (resp. 1.8).

Using L'HÔSPITAL's rule to the KATUGAMPOLA's fractional integral (1.20) , when $\rho \to 0^+$ we have:

$$\begin{aligned}\lim_{\rho\to 0^+} \left({}^{\rho}\mathcal{I}^{\alpha}_{a^+}u\right)(t) &= \lim_{\rho\to 0^+} \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{s^{\rho-1}u(s)}{(t^{\rho}-s^{\rho})^{1-\alpha}} ds \\ &= \frac{1}{\Gamma(\alpha)} \int_a^t \lim_{\rho\to 0^+} \left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1} s^{\rho-1}u(s)\,ds \\ &= \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1} \frac{u(s)}{s} ds.\end{aligned}$$

This is the famous HADAMARD's fractional integral (1.18) . Similarly, when $\rho \to 0^+$, we find the HADAMARD's fractional derivative (1.19) . (For more explanations see [19]-[22]).

This could all be summed up in the following theorem:

Theorem 1.1 ([19]-[22]) *Let $\alpha, \rho \in \mathbb{R}^+$, then*

$$\begin{aligned}\lim_{\rho\to 1} \left({}^{\rho}\mathcal{I}^{\alpha}_{a^+}u\right)(t) &= \mathcal{I}^{\alpha}_{a^+}u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} u(s)\,ds, \\ \lim_{\rho\to 0^+} \left({}^{\rho}\mathcal{I}^{\alpha}_{a^+}u\right)(t) &= {}^{H}\mathcal{I}^{\alpha}_{a^+}u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1} \frac{u(s)}{s} ds, \\ \lim_{\rho\to 1} \left({}^{\rho}\mathcal{D}^{\alpha}_{a^+}u\right)(t) &= {}^{RL}\mathcal{D}^{\alpha}_{a^+}u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} u(s)\,ds, \\ \lim_{\rho\to 0^+} \left({}^{\rho}\mathcal{D}^{\alpha}_{a^+}u\right)(t) &= {}^{H}\mathcal{D}^{\alpha}_{a^+}u(t) = \frac{1}{\Gamma(n-\alpha)} \left(t\frac{d}{dt}\right)^n \int_a^t \left(\log\frac{t}{s}\right)^{n-\alpha-1} \frac{u(s)}{s} ds.\end{aligned}$$

Remark 1.1 *we have the same result for the right-sided case.*

1.5.1 Properties and Preliminary Results on Half-Axis $\mathbb{R}^+$

Let $[a, b] \subset \mathbb{R}^+$, be a finite closed interval on the half-axis $\mathbb{R}^+$.

Remark 1.2 ([7], [8]) *As an example, for $\alpha, \rho > 0$, and $\mu > -\rho$, we have:*

$${}^{\rho}\mathcal{D}^{\alpha}_{0^+}t^{\mu} = \frac{\rho^{\alpha-1}\Gamma\left(1+\frac{\mu}{\rho}\right)}{\Gamma\left(1-\alpha+\frac{\mu}{\rho}\right)} t^{\mu-\alpha\rho}. \tag{1.23}$$

In particular

$${}^{\rho}\mathcal{D}^{\alpha}_{0^+}t^{\rho(\alpha-m)} = 0, \ \textit{for each } m = 1, 2, \ldots, n.$$

Similarly, we have:

$$ {}^{\rho}\mathcal{I}_{0^+}^{\alpha} t^{\mu} = \frac{\rho^{-\alpha}\Gamma\left(1+\frac{\mu}{\rho}\right)}{\Gamma\left(1+\alpha+\frac{\mu}{\rho}\right)} t^{\mu+\alpha\rho}, \ \forall \mu > -\rho. \tag{1.24} $$

Remark 1.3 *When $\rho = 1$ and $\mu > -1$, we obtain the* RIEMANN-LIOUVILLE*'s fractional integral (resp. derivative) of the power function* 1.15, 1.16*, given by* [35], [33], [23], [13],

$$ {}^{1}\mathcal{I}_{0^+}^{\alpha} t^{\mu} = \frac{\Gamma(1+\mu)}{\Gamma(1+\alpha+\mu)} t^{\mu+\alpha}, $$

respectively

$$ {}^{1}\mathcal{D}_{0^+}^{\alpha} t^{\mu} = \frac{\Gamma(1+\mu)}{\Gamma(1-\alpha+\mu)} t^{\mu-\alpha}. $$

Lemma 1.2 ([7], [8]) *Let $p \geq 1$, $c > 0$. Then*

$$ C[a,b] \hookrightarrow X_c^p[a,b], $$

and far all $u \in C[a,b]$, we get

$$ \|u\|_{X_c^p} \leq \|u\|_{\infty}, \ \forall b \leq (pc)^{\frac{1}{pc}}. $$

In the sequel, a, b, α, n, p, and c are positive real constants such that

$$ p \geq 1, \ n = [\alpha] + 1, \text{ and } a < b \leq (pc)^{\frac{1}{pc}}. $$

We express some properties of KATUGAMPOLA's fractional integral and derivative in the following results.

Theorem 1.2 (Index property [19]) *Let $\alpha, \beta, \rho > 0$. Then for any $u \in C[a,b]$, we have:*

$$ {}^{\rho}\mathcal{I}_{a^+}^{\alpha}\, {}^{\rho}\mathcal{I}_{a^+}^{\beta} u(t) = {}^{\rho}\mathcal{I}_{a^+}^{\alpha+\beta} u(t), $$

and

$$ {}^{\rho}\mathcal{D}_{a^+}^{\alpha}\, {}^{\rho}\mathcal{D}_{a^+}^{\beta} u(t) = {}^{\rho}\mathcal{D}_{a^+}^{\alpha+\beta} u(t), \ \text{for all } 0 < \alpha, \beta < 1. $$

Theorem 1.3 (Inverse property [19]) *Let $\alpha, \rho > 0$. Then for any $u \in C[a,b]$, we have:*

$$ {}^{\rho}\mathcal{D}_{a^+}^{\alpha}\, {}^{\rho}\mathcal{I}_{a^+}^{\alpha} u(t) = u(t), \ \text{for all } \alpha \in (0,1). \tag{1.25} $$

Theorem 1.4 (Linearity property [19]) *Let $\alpha \in (0,1)$, and $\rho > 0$. Then for any $u, v \in C[a,b]$, we have:*

$$ \begin{cases} {}^{\rho}\mathcal{D}_{a^+}^{\alpha}(u+v)(t) = {}^{\rho}\mathcal{D}_{a^+}^{\alpha} u(t) + {}^{\rho}\mathcal{D}_{a^+}^{\alpha} v(t), \\ {}^{\rho}\mathcal{I}_{a^+}^{\alpha}(u+v)(t) = {}^{\rho}\mathcal{I}_{a^+}^{\alpha} u(t) + {}^{\rho}\mathcal{I}_{a^+}^{\alpha} v(t). \end{cases} \tag{1.26} $$

Remark 1.4 ([7], [8]) *Let $\alpha, \rho > 0$. Then for any $u \in C[a,b]$, we have:*

$$ {}^{\rho}\mathcal{I}_{a^+}^{1}\left(t^{1-\rho}\frac{d}{dt}\right)u(t) = u(t) - u(a), \tag{1.27}$$

and

$$ {}^{\rho}\mathcal{I}_{a^+}^{\alpha}u(t) = \left(t^{1-\rho}\frac{d}{dt}\right){}^{\rho}\mathcal{I}_{a^+}^{\alpha+1}u(t). \tag{1.28}$$

Theorem 1.5 (Mellin transforms [21]) *Let $\alpha, \rho > 0$. Then for any $u \in C[a,b]$, we have:*

$$\mathcal{M}\left[{}^{\rho}\mathcal{D}_{a^+}^{\alpha}u(t)\right](s) = \frac{\rho^{\alpha}\Gamma\left(1-\frac{s}{\rho}+\alpha\right)}{\Gamma\left(1-\frac{s}{\rho}\right)}(\mathcal{M}u)(s-\rho\alpha), \ s<\rho. \tag{1.29}$$

Remark 1.5 ([23]) *When $\rho = 1$ and $\alpha = m \in \mathbb{N}$, we obtain the arbitrary* Mellin *transform of the usual derivative of a function u,*

$$\mathcal{M}\left[\frac{d^m}{dt^m}u(t)\right](s) = \frac{\Gamma(1+m-s)}{\Gamma(1-s)}(\mathcal{M}u)(s-m). \tag{1.30}$$

This equation is valid for $u \in C^m(\mathbb{R}^+)$, such that $u^{(m)}(t) \in L^1(\mathbb{R}^+)$.

1.6 Studies and Results of FDE's Solutions

Recently, some results dealing with the existence, uniqueness and multiplicity of real or positive solutions of nonlinear fractional problems have appeared. Quoting on this subject the work [6], Bai and Lü used some fixed point theorems on a cone to show the existence and multiplicity of positive solutions for a Dirichlet-type problem of the nonlinear FDE:

$$\begin{cases} \mathcal{D}_{0^+}^{\alpha}u(t) + f(t,u(t)) = 0, \ 0<t<1, \\ u(0) = u(1) = 0, \end{cases}$$

where $\mathcal{D}_{0^+}^{\alpha}u$ is the standard Riemann-Liouville's fractional derivative of order $1 < \alpha \leq 2$ and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is a continuous function.

In [9], Benchohra and Lazreg applied the Banach's contraction principle, Schauder's fixed-point theorem and Leray-Schauder type to show the existence and uniqueness of solutions for an initial value problem of the nonlinear implicit FDE:

$$\begin{cases} {}^{C}\mathcal{D}_{0^+}^{\alpha}u(t) = f\left(t, u(t), {}^{C}\mathcal{D}_{0^+}^{\alpha}u(t)\right), \ t \in [0,\lambda], \ \lambda > 0, \ 0 < \alpha \leq 1, \\ u(0) = u_0, \end{cases}$$

where ${}^{C}\mathcal{D}_{0+}^{\alpha}u$ is the CAPUTO's fractional derivative, $f:[0,\lambda]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a given function, and $u_0\in\mathbb{R}$.

In a recent work [20], KATUGAMPOLA studied the existence and uniqueness of solutions for the following initial value problem:

$$\begin{cases} {}_{C}^{\rho}\mathcal{D}_{0+}^{\alpha}u(t)=f(t,u(t)),\ \alpha>0, \\ D^k u(0)=u_0^{(k)},\ k=1,2,...,m-1, \end{cases}$$

where $m=[\alpha]$, ${}_{C}^{\rho}\mathcal{D}_{0+}^{\alpha}$ is the CAPUTO-type generalized fractional derivative, of order α, and $f:G\to\mathbb{R}$ is a given continuous function with:

$$G=\left\{(t,u):t\in[0,h^*],\ \left|u-\sum_{k=0}^{m-1}\frac{t^k u_0^{(k)}}{k!}\right|\leq K,\ K,h^*>0\right\}.$$

In [29], MURAD and HADID, by means of SCHAUDER's fixed-point theorem and the BANACH's contraction principle, considered the boundary value problem of the FDE:

$$\begin{cases} \mathcal{D}_{0+}^{\alpha}u(t)=f\left(t,u(t),\mathcal{D}_{0+}^{\beta}u(t)\right),t\in(0,1),\ 1<\alpha\leq 2,\ 0<\beta<1 \\ u(0)=0,\ u(1)=\ \mathcal{I}_{0+}^{\gamma}u(s),\ 0<\gamma\leq 1, \end{cases}$$

where $\mathcal{D}_{0+}^{\alpha}u$ $\left[\text{resp. } \mathcal{I}_{0+}^{\alpha}u\right]$ is the RIEMANN-LIOUVILLE's fractional derivative (resp. fractional integral), and $f:[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous function.

1.7 Fixed Point Theorems

In the remainder of this section, we introduce the notations, definitions and theorems necessary for this study.

Definition 1.20 ([1]) *Let E be a real space. Then the set of all convex combinations of $u,v\in E$ is the set of points*

$$\{w_\sigma\in E: w_\sigma=(1-\sigma)u+\sigma v,\ 0\leq\sigma\leq 1\}. \tag{1.31}$$

Also, $P\subset E$ is said to be convex provided that the given two points $u,v\in P$; the set (1.31) is a subset of P.

Definition 1.21 ([14]) *Let E be a real BANACH space. A nonempty closed convex set $P\subset E$ is called a cone of E if it satisfies the following conditions:*

(i) $u\in P$, $\sigma\geq 0$, implies $\sigma u\in P$.

(ii) $u\in P$, $-u\in P$, implies $u=0$.

Definition 1.22 (Equicontinuous [6]) *Let E be a* Banach *space. A part P in $C(E)$ is called equicontinuous if*

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall u, v \in E,\ \forall \mathcal{A} \in P,\ \ \|u - v\| < \delta \Rightarrow \|\mathcal{A}(u) - \mathcal{A}(v)\| < \varepsilon.$$

Theorem 1.6 (Ascoli-Arzelà [1]) *Let E be a compact space. If $\mathcal{A}$ is an equicontinuous, bounded subset of $C(E)$, then $\mathcal{A}$ is relatively compact.*

Definition 1.23 (Completely continuous [14]) *We say $\mathcal{A} : E \to E$ is completely continuous if for any bounded subset $P \subset E$, the set $\mathcal{A}(P)$ is relatively compact.*

Definition 1.24 ([15]) *Let E be any space and $\mathcal{A}$ a map of E, or of a subset of E, into E.*
- The map $\mathcal{A}$ is called a contraction mapping if there exists $k \in (0,1)$ such that

$$\forall u, v \in E,\ \ \|\mathcal{A}u - \mathcal{A}v\| \leq k\|u - v\|.$$

- A point $u \in E$ is called a fixed point for $\mathcal{A}$ if $\mathcal{A}u = u$.

Lemma 1.3 (Gronwall [16]) *Let $u(t)$ and $v(t)$ be nonnegative, continuous functions on $0 \leq t \leq \lambda$, for which the inequality:*

$$u(t) \leq \mu + \int_0^t v(s)\, u(s)\, ds,\ \ 0 \leq t \leq \lambda,$$

holds, where μ is a nonnegative constant. Then:

$$u(t) \leq \mu \exp\left(\int_0^t v(s)\, ds\right),\ \ 0 \leq t \leq \lambda.$$

For subsequent applications, the following fixed-point theorems are fundamental in the proofs of our main results.

Theorem 1.7 (Guo-Krasnosel'skii fixed point [25]) *Let E be a* Banach *space, $P \subseteq E$ a cone, and Ω_1, Ω_2 two bounded open balls of E centered at the origin with $\bar{\Omega}_1 \subset \Omega_2$. Suppose that $\mathcal{A} : P \cap (\bar{\Omega}_2 \backslash \Omega_1) \to P$ is a completely continuous operator such that either*
(i) $\|\mathcal{A}u\| \leq \|u\|$, $u \in P \cap \partial\Omega_1$ *and* $\|\mathcal{A}u\| \geq \|u\|$, $u \in P \cap \partial\Omega_2$, *or*
(ii) $\|\mathcal{A}u\| \geq \|u\|$, $u \in P \cap \partial\Omega_1$ *and* $\|\mathcal{A}u\| \leq \|u\|$, $u \in P \cap \partial\Omega_2$,
holds. Then $\mathcal{A}$ has a fixed point in $P \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

Theorem 1.8 (Banach's fixed point [15]) *Let P be a non-empty closed subset of a* Banach *space E, then any contraction mapping $\mathcal{A}$ of P into itself has a unique fixed point.*

Theorem 1.9 (Schauder's fixed point [15]) *Let E be a* BANACH *space, and P be a closed, convex and nonempty subset of E. Let $\mathcal{A}: P \to P$ be a continuous mapping such that $\mathcal{A}(P)$ is a relatively compact subset of E. Then $\mathcal{A}$ has at least one fixed point in P.*

Theorem 1.10 (Nonlinear Alternative of Leray-Schauder type [15]) *Let E be a* BANACH *space with $P \subset E$ be closed and convex. Assume U is a relatively open subset of P with $0 \in U$ and $\mathcal{A}: \bar{U} \to P$ is a compact map. Then either,*

(i) *$\mathcal{A}$ has a fixed point in $\bar{U}$; or*

(ii) *there is a point $u \in \partial U$ and $\mu \in (0,1)$ with $u = \mu\mathcal{A}(u)$.*

Chapter 2

Partial Differential Equations With Fractional Operators

In this chapter we will introduce some notions of fractional differential equations and fractional-order's partial differential equations. We will as well present the notions of the self-similar solution and apply them to the fractional-order's partial differential equations, then we present some principles to calculate generalized self-similar solutions, and we give the main theorems of our work.

2.1 Introduction to Fractional-order's PDEs

Fractional-order's partial differential equations (FPDEs) are generalizations of classical partial differential equations. They have been of considerable interest to the recent literature. A considerable attention has been especially devoted to these topics in the fields of viscoelasticity materials, electrochemical processes, dielectric polarization, among others. Increasingly, these models are used in applications such as fluid flow and finance.

The solutions of FPDEs play an important role in the proper understanding of qualitative features of many phenomena and processes in various areas of natural sciences. Furthermore, simple solutions are often used in teaching many courses as specific examples which illustrate basic tenets of a theory that admits mathematical formulation.

2.2 Definitions and Properties

What is a fractional-order's partial differential equation?

To answer this question, we first recall some notions related to fractional differential equations.

Definition 2.1 (FDEs [23]) *A fractional differential equation is a relationship of the type*

$$F\left(\eta, u\left(\eta\right), \mathcal{D}^{\alpha_1}u\left(\eta\right), \mathcal{D}^{\alpha_2}u\left(\eta\right), \ldots\right) = 0,\ \ \alpha_1, \alpha_2, \ldots > 0,$$

between the variable $\eta \in \mathbb{R}$, and the fractional derivatives of order $\alpha_1, \alpha_2, \ldots$ of the unknown function u at the point η. Here $\mathcal{D}^{\alpha}u$ presents a fractional differential operator of order $\alpha > 0$.

Definition 2.2 (FPDEs) *A fractional-order's partial differential equation for the function w is a relationship between w, the independent variables $(\eta_1, \eta_2, \ldots, \eta_n) \in \mathbb{R}^n$ and one or more fractional derivatives $\mathcal{D}^{\alpha_1}_{\eta_1}w, \mathcal{D}^{\alpha_2}_{\eta_2}w, \ldots, \mathcal{D}^{\alpha_3}_{\eta_3}\mathcal{D}^{\alpha_4}_{\eta_4}w \ldots$, that we can write in the form:*

$$F\left(w, \eta_1, \ldots, \mathcal{D}^{\alpha_1}_{\eta_1}w, \mathcal{D}^{\alpha_2}_{\eta_2}w, \ldots, \mathcal{D}^{\alpha_3}_{\eta_3}\mathcal{D}^{\alpha_4}_{\eta_4}w \ldots\right) = 0,\ \ \alpha_1, \alpha_2, \ldots > 0.$$

The symbol $\mathcal{D}^{\alpha}_{\eta_i}w$ presents a fractional differential operator of order α at η_i, $i = 1, 2, \ldots, n$.

2.2.1 Self-Similar Solutions of Fractional Equations

In general, for some FPDEs which have the characterization of symmetries, (see for example [18], [32], [34]), we can determine the exact solutions with certain (finite or infinite) transformations. Here, a FPDE becomes a FDE, in this case the solutions are called "self-similar solutions" ([10], [17]) , which often play a central role in the study of a FPDE, since it is equivalent to these solutions to solve locally or globally.

Let

$$\mathcal{D}^{\alpha}_{x}w = f\left(x, t, w, \mathcal{D}^{\alpha_1}_{t}w, \mathcal{D}^{\alpha_2}_{x}w, \ldots\right),\ \ \alpha > \alpha_1 > \alpha_2 > \ldots > 0, \tag{2.1}$$

be a FPDE, and $w = w\left(x, t\right)$ is a scalar function of space and time variables $(x, t) \in \mathbb{R}^2$. The symbol $\mathcal{D}^{\alpha}_{x}w$ presents a space-fractional differential operator of order α.

Self-similar solutions are very important in physics because they model phenomena that are independent of the scale of measurement. We suggest finding the solution of the equation (2.1) in the following "self-similar" form:

$$w\left(x, t\right) = \lambda^{\mu}w\left(\lambda^{\gamma}x, \lambda t\right),\ \ (x, t) \in \mathbb{R} \times \mathbb{R}_{+},\ \ \lambda > 0. \tag{2.2}$$

On the basis of this consideration, we search the values of $\mu, \gamma \in \mathbb{R}$, for which $\lambda^{\mu} w(\lambda^{\gamma} x, \lambda t)$ is a solution of the equation (2.1) , for all $\lambda > 0$ knowing that w is a solution of the same equation.

Definition 2.3 (Self-similar [10], [17], [27]) *A function which is invariant by a change of scale in time is called "self-similar". The principle of the search for self-similar solutions consists in replacing the form (2.2) in the equation (2.1) , which makes it possible to transform the FPDE (2.1) in a FDE.*

Remark 2.1 ([10], [11], [17], [27]) *In general, several forms of self-similar solutions exist. To admit such solutions of the equation (2.1) must check the so-called "similarity" conditions.*

If we take for example $\lambda = \frac{1}{t} > 0$, the self-similar solution (2.2) equivalent to

$$w(x,t) = \frac{1}{t^{\mu}} w\left(\frac{x}{t^{\gamma}}, 1\right) = t^{-\mu} u(\eta), \text{ where } \eta = \frac{x}{t^{\gamma}}, \ (x,t) \in \mathbb{R} \times \mathbb{R}_{+}, \ \mu, \gamma \in \mathbb{R}. \tag{2.3}$$

In this case, the function u, called the "basic profile," is not known in advance and is to be identified.

To discuss the self-similar solutions, we should first deduce the equation satisfied by the function $u(\eta)$ in (2.3) used for the definition of self-similar solutions.

Theorem 2.1 *Let $p, q, r, \mu, \gamma \in \mathbb{R}$, be such that*

$$f\left(x, t, w, \mathcal{D}_{t}^{\alpha_1} w, \mathcal{D}_{x}^{\alpha_2} w, \ldots\right) = t^{q\mu + r\gamma} f\left(\eta, u, \mathcal{D}^{\alpha_1} u, \mathcal{D}^{\alpha_2} u, \ldots\right). \tag{2.4}$$

If the similarity condition

$$(q+1)\mu + (r-p)\gamma = 0, \tag{2.5}$$

is satisfied. Then, the self-similar form (2.3) is a solution of the fractional equation (2.1) , if the basic profile u is a solution of following FDE:

$$\mathcal{D}^{\alpha} u = f\left(\eta, u, \mathcal{D}^{\alpha_1} u, \mathcal{D}^{\alpha_2} u, \ldots\right), \ \textit{where } \alpha > \alpha_1 > \alpha_2 > \ldots > 0.$$

Proof. Let $p, q, r, \mu, \gamma \in \mathbb{R}$, be such that the self-similar form (2.3) satisfies (2.4) . Then

$$\mathcal{D}_{x}^{\alpha} w = \mathcal{D}_{x}^{\alpha} t^{-\mu} u\left(\frac{x}{t^{\gamma}}\right) = t^{p\gamma - \mu} \mathcal{D}^{\alpha} u(\eta), \tag{2.6}$$

where $\mathcal{D}^{\alpha} u$ presents a fractional differential operator of order α.
If we replace (2.6) , and (2.4) in the FPDE (2.1) , we have:

$$t^{p\gamma - \mu} \mathcal{D}^{\alpha} u(\eta) = t^{q\mu + r\gamma} f\left(\eta, u, \mathcal{D}^{\alpha_1} u, \mathcal{D}^{\alpha_2} u, \ldots\right),$$

then

$$\mathcal{D}^{\alpha}u(\eta) = t^{(q+1)\mu+(r-p)\gamma} f(\eta, u, \mathcal{D}^{\alpha_1}u, \mathcal{D}^{\alpha_2}u, \ldots).$$

If the similarity condition (2.5) is satisfied, then the transformation:

$$w(x,t) = t^{-\mu}u(\eta), \text{ with } \eta = \frac{x}{t^{\gamma}},$$

reduces the fractional-order's partial differential equation (2.1) to the ordinary differential equation of fractional order of the form

$$\mathcal{D}^{\alpha}u(\eta) = f(\eta, u, \mathcal{D}^{\alpha_1}u, \mathcal{D}^{\alpha_2}u, \ldots).$$

The proof is complete. ∎

2.2.2 Generalized Self-Similar Solutions of Fractional Equations

In the general case, we consider the FPDE (2.1) of space and time variables $(x,t) \in \mathbb{R} \times \mathbb{R}_+$, which is

$$\mathcal{D}_x^{\alpha}w = f(x, t, w, \mathcal{D}_t^{\alpha_1}w, \mathcal{D}_x^{\alpha_2}w, \ldots), \ \alpha > \alpha_1 > \alpha_2 > \ldots > 0. \tag{2.7}$$

On the basis of this consideration, our main goal in this part is to determine the main properties of the solution of the FPDE (2.7), under the generalized self-similar form which is:

$$w(x,t) = \psi(t)\,u(\eta), \text{ with } \eta = \frac{x}{\varphi(t)}, \ (x,t) \in \mathbb{R} \times \mathbb{R}_+. \tag{2.8}$$

The functions $\varphi(t)$ and $\psi(t)$, depend on time t and the "basic profile" u, are not known in advance and are to be identified.

To discuss the generalized self-similar solutions, we should first deduce the equation satisfied by the function $u(\eta)$ in (2.8) as it is used for the definition of self-similar solutions.

Theorem 2.2 *Let $\varphi(t), \psi(t) \in \mathbb{R}_+$ are continuous functions, and $p, q, r \in \mathbb{R}$, be such that the generalized self-similar form (2.8), satisfies*

$$\mathcal{D}_x^{\alpha}w = \frac{\psi(t)}{\varphi^{p}(t)}\mathcal{D}^{\alpha}u(\eta), \tag{2.9}$$

and

$$f(x, t, w, \mathcal{D}_t^{\alpha_1}w, \mathcal{D}_x^{\alpha_2}w, \ldots) = \frac{\psi^{-q}(t)}{\varphi^{r}(t)} f(\eta, u, \mathcal{D}^{\alpha_1}u, \mathcal{D}^{\alpha_2}u, \ldots). \tag{2.10}$$

Then the transformation

$$w(x,t) = \psi(t)\,u(\eta), \ \textit{with } \eta = \frac{x}{\varphi(t)}, \ \textit{for any } (x,t) \in \mathbb{R} \times \mathbb{R}_+,$$

reduces the fractional-order's partial differential equation (2.7) *to the ordinary differential equation of fractional order of the form:*

$$\mathcal{D}^{\alpha}u(\eta) = f(\eta, u, \mathcal{D}^{\alpha_1}u, \mathcal{D}^{\alpha_2}u, \ldots),$$

if and only if the generalized similarity conditions

$$q = -1, \text{ and } p = r, \text{ or } \psi(t) = \varphi^{\frac{p-r}{q+1}}(t) \text{ for } q \neq -1, \tag{2.11}$$

are satisfied.

Proof. Let $p, q, r \in \mathbb{R}$, be such that the self-similar form (2.8), satisfies (2.9) and (2.10). If we replace (2.9), and (2.10) in the FPDE (2.7), we have:

$$\frac{\psi(t)}{\varphi^{p}(t)}\mathcal{D}^{\alpha}u(\eta) = \frac{\psi^{-q}(t)}{\varphi^{r}(t)}f(\eta, u, \mathcal{D}^{\alpha_1}u, \mathcal{D}^{\alpha_2}u, \ldots).$$

Consequently

$$\mathcal{D}^{\alpha}u(\eta) = \frac{\varphi^{p-r}(t)}{\psi^{q+1}(t)}f(\eta, u, \mathcal{D}^{\alpha_1}u, \mathcal{D}^{\alpha_2}u, \ldots).$$

From the condition (2.11), we get easily that the transformation (2.8) reduces the fractional-order's partial differential equation (2.7) to the ordinary differential equation of fractional order of the form

$$\mathcal{D}^{\alpha}u(\eta) = f(\eta, u, \mathcal{D}^{\alpha_1}u, \mathcal{D}^{\alpha_2}u, \ldots).$$

The proof is complete. ■

For a special case, if we choose

$$\psi(t) = t^{-\mu}, \text{ and } \varphi(t) = t^{\gamma}, \text{ for some } \mu, \gamma \in \mathbb{R},$$

we have the classical self-similar solution (2.3), which is given by the following form:

$$w(x,t) = t^{-\mu}u(\eta), \text{ where } \eta = \frac{x}{t^{\gamma}}, \ (x,t) \in \mathbb{R}\times\mathbb{R}_{+}, \ \mu, \gamma \in \mathbb{R}.$$

The condition (2.11) is equivalent to the condition (2.5), which is:

$$(q+1)\mu + (r-p)\gamma = 0.$$

2.3 Generalized Self-Similar Solutions for Katugampola FPDEs

Many problems and models in physics, chemistry, biology and economics are modeled by partial differential equations of fractional order. We shall give in this part a basic example of a class of nonlinear fractional equations (see [11], [27], [32]), (which we are going to tackle in this thesis), and they are written as follows:

$$ {}_{x}^{\rho}\mathcal{D}_{0^+}^{\alpha} w = f\left(x, t, w, {}_{t}^{\rho}\mathcal{D}_{0^+}^{\alpha} w\ , {}_{x}^{\rho}\mathcal{D}_{0^+}^{\beta} w\ , \ldots\right),\ \alpha > \beta > \ldots > 0, \tag{2.12}$$

where $w = w(x,t)$ is a scalar function of space variables $x > 0$, and time $t \in \mathbb{R}$. The symbol ${}_{x}^{\rho}\mathcal{D}_{0^+}^{\alpha} w$ is the linear generalization of KATUGAMPOLA's space-fractional derivative, that is given in the following definition.

Definition 2.4 (Katugampola's space-fractional derivatives [19]) *Let $\alpha, \rho \in \mathbb{R}^+$, and $n = [\alpha] + 1$. The space-fractional derivative of* KATUGAMPOLA *for a scalar function $w(x,t)$ of space variable $x > 0$, and time $t \in \mathbb{R}$, is defined by:*

$$ {}_{x}^{\rho}\mathcal{D}_{0^+}^{\alpha} w = \begin{cases} \frac{\partial^n w(x,t)}{\partial x^n}, & \alpha = n \in \mathbb{N}^*, \\ \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \left(x^{1-\rho}\frac{d}{dx}\right)^n \int_0^x s^{\rho-1}\left(x^\rho - s^\rho\right)^{n-\alpha-1} w(s,t)\, ds, & n-1 < \alpha < n. \end{cases} \tag{2.13}$$

The used differential operator of KATUGAMPOLA for a FPDE, generalizes the RIEMANN-LIOUVILLE's and the HADAMARD's fractional derivatives into a single form.

When $\rho = 1$, we arrive at the standard RIEMANN-LIOUVILLE's fractional derivative (see [11]), which is:

$$ \frac{\partial^\alpha w}{\partial x^\alpha} = \begin{cases} \frac{\partial^n w(x,t)}{\partial x^n}, & \alpha = n \in \mathbb{N}^*, \\ \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_0^x (x-s)^{n-\alpha-1} w(s,t)\, ds, & n-1 < \alpha < n. \end{cases} $$

When $\rho \to 0^+$ we have the famous HADAMARD's fractional derivative for a scalar function w of several independent variables $(\eta_1, \eta_2, \ldots, \eta_n) \in \mathbb{R}^n$, (see [23]).

Our main goal in this work is to determine the existence, uniqueness and main properties of the solution of the FPDE (2.12), under the generalized self-similar form (2.8).

Theorem 2.3 *Let $\varphi(t), \psi(t) \in \mathbb{R}_+$ are continuous functions, and $p, q \in \mathbb{R}$, be such that*

$$ f\left(x, t, w, {}_{t}^{\rho}\mathcal{D}_{0^+}^{\alpha} w\ , {}_{x}^{\rho}\mathcal{D}_{0^+}^{\beta} w\ , \ldots\right) = \frac{\psi^p(t)}{\varphi^q(t)} f\left(\eta, u, {}^{\rho}\mathcal{D}_{0^+}^{\alpha} u\ , {}^{\rho}\mathcal{D}_{0^+}^{\beta} u\ , \ldots\right), \tag{2.14}$$

for any $(x,t) \in \mathbb{R}_+^2$. *Then the transformation:*

$$w(x,t) = \psi(t)\, u(\eta), \ \textit{with } \eta = \frac{x}{\varphi(t)},$$

reduces the fractional-order's partial differential equation (2.12) *to the ordinary differential equation of fractional order of the form:*

$$^{\rho}\mathcal{D}_{0+}^{\alpha} u = f\left(\eta, u, {}^{\rho}\mathcal{D}_{0+}^{\alpha} u\,, {}^{\rho}\mathcal{D}_{0+}^{\beta} u\,, \ldots\right), \ \eta > 0. \tag{2.15}$$

If and only if the generalized similarity conditions

$$p = 1, \ \textit{and } q = \rho\alpha, \ \textit{or } \psi(t) = \varphi^{\frac{\rho\alpha - q}{1-p}}(t) \ \textit{for } p \neq 1, \tag{2.16}$$

are satisfied.

Proof. The fractional equation resulting from the substitution of expression (2.8) in the original PDE (2.12), should be reduced to the standard bilinear functional equation (see [34]). First, for $s = \frac{\tau}{\varphi(t)}$, we get:

$$\begin{aligned}
{}_{x}^{\rho}\mathcal{D}_{0+}^{\alpha} w &= \frac{\psi(t)\,\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \left(x^{1-\rho}\frac{d}{dx}\right)^{n} \int_0^x \tau^{\rho-1}\left(x^{\rho}-\tau^{\rho}\right)^{n-\alpha-1} u\left(\frac{\tau}{\varphi(t)}\right) d\tau \\
&= \frac{\varphi(t)\,\psi(t)\,\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \left(\left(\eta\varphi(t)\right)^{1-\rho}\frac{d}{\varphi(t)\, d\eta}\right)^{n} \\
&\quad \int_0^{\eta} \left(s\varphi(t)\right)^{\rho-1}\left(x^{\rho}-\left(s\varphi(t)\right)^{\rho}\right)^{n-\alpha-1} u(s)\, ds \\
&= \frac{\psi(t)\,\rho^{\alpha-n+1}}{\varphi^{\rho\alpha}(t)\,\Gamma(n-\alpha)} \left(\eta^{1-\rho}\frac{d}{d\eta}\right)^{n} \int_0^{\eta} s^{\rho-1}\left(\eta^{\rho}-s^{\rho}\right)^{n-\alpha-1} u(s)\, ds \\
&= \frac{\psi(t)}{\varphi^{\rho\alpha}(t)}\, {}^{\rho}\mathcal{D}_{0+}^{\alpha} u(\eta)\,.
\end{aligned} \tag{2.17}$$

If we replace (2.14) and (2.17) in (2.12), we get easily

$$\frac{\psi(t)}{\varphi^{\rho\alpha}(t)}\, {}^{\rho}\mathcal{D}_{0+}^{\alpha} u(\eta) = \frac{\psi^{p}(t)}{\varphi^{q}(t)} f\left(\eta, u(\eta), {}^{\rho}\mathcal{D}_{0+}^{\alpha} u(\eta)\,, {}^{\rho}\mathcal{D}_{0+}^{\beta} u(\eta)\,, \ldots\right).$$

If the condition (2.16) is satisfied, we get easily (2.15). The proof is complete. ■

2.3.1 Generalized Self-Similar Solutions for Nonlinear BVP

In this section, we study the existence and uniqueness of solutions of the following problem of the nonlinear partial differential equations of space-fractional order [7], [12]:

$${}_{x}^{\rho}\mathcal{D}_{0+}^{\alpha} w + \beta f(x,t,w) = 0, \ \beta \in \mathbb{R}, \ 1 < \alpha \leq 2, \ (x,t) \in (0,X) \times [0,T]\,, \tag{2.18}$$

with the boundary conditions:

$$w(0,t)=0,\ w(X,t)=0, \tag{2.19}$$

under the generalized self-similar form which is:

$$w(x,t)=\psi(t)\,u(\eta),\ \text{with } \eta=\frac{x}{\varphi(t)},\ \text{and } \varphi,\psi\in C[0,T]\to\mathbb{R}_+, \tag{2.20}$$

where $f:[0,X]\times[0,T]\times\mathbb{R}\to\mathbb{R}$ is a given function which satisfies the hypotheses of theorem 2.3, with finite positive constants X,T.

Existence of positive solutions

Let $\alpha,\beta,\rho,h,\bar{\eta},T,L,X,\lambda\in\mathbb{R}_+$, be such that $1<\alpha\leq 2$, $0<\rho\leq 1$. We define the function:

$$b(\eta)=\begin{cases} b_1(\eta), & \text{for } \eta\in[0,\bar{\eta}],\\ b_2(\eta), & \text{for } \eta\in[\bar{\eta},\lambda],\end{cases}$$

where

$$b_1(\eta)=\left(\frac{\eta}{\lambda}\right)^{\rho(\alpha-1)},\ b_2(\eta)=\frac{(\alpha-1)(\lambda-\eta)}{8\lambda},$$

and $\bar{\eta}\in(0,\lambda)$ is the unique solution of the equation $b_1(\eta)=b_2(\eta)$.

Also, we define Green's function associated with the boundary value problem (2.18)-(2.19), (see lemma 3.2) which is:

$$G(\eta,s)=\begin{cases} \frac{\rho^{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)}\left[\left[\frac{\eta^\rho}{\lambda^\rho}(\lambda^\rho-s^\rho)\right]^{\alpha-1}-(\eta^\rho-s^\rho)^{\alpha-1}\right], & 0\leq s\leq\eta\leq\lambda,\\ \frac{\rho^{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)}\left[\frac{\eta^\rho}{\lambda^\rho}(\lambda^\rho-s^\rho)\right]^{\alpha-1}, & 0\leq\eta\leq s\leq\lambda,\end{cases}$$

and the constants:

$$\begin{aligned}\sigma &= 1+\frac{8^{\rho\alpha}L(\alpha+1)\left[8^{\rho\alpha}-(8^\rho-1)^\alpha\right]}{h(8^\rho-1)^\alpha\left[8^\rho(\alpha+1)+8^{\rho(\alpha-1)}(\alpha-1)(8^\rho-1)\right]},\\ \omega_1 &= \int_0^\lambda G(s,s)\,ds,\\ \omega_2 &= \frac{\bar{b}}{\sigma^2}\int_0^\lambda G(s,s)\,b(s)\,ds,\ \text{where } \bar{b}=b(\bar{\eta}).\end{aligned}$$

We suggest some conditions on f, which allow us to obtain some results on the existence of positive solutions to the boundary value problem (2.18)-(2.19), under the generalized self-similar form (2.20).

Let $f:[0,X]\times[0,T]\times[0,\infty)\to[h,\infty)$ be a continuous function, with finite positive constant h. We define

$$f_0^*(t)=\varphi^{\rho\alpha}(t)\lim_{w\to 0^+}\min_{0\leq x\leq X}\frac{f(x,t,w)}{w},\quad f_\infty^*(t)=\varphi^{\rho\alpha}(t)\lim_{w\to+\infty}\min_{0\leq x\leq X}\frac{f(x,t,w)}{w},$$
$$F_0^*(t)=\varphi^{\rho\alpha}(t)\lim_{w\to 0^+}\max_{0\leq x\leq X}\frac{f(x,t,w)}{w},\quad F_\infty^*(t)=\varphi^{\rho\alpha}(t)\lim_{w\to+\infty}\max_{0\leq x\leq X}\frac{f(x,t,w)}{w}.$$

Theorem 2.4 (Existence of positive solutions) *Let $X, T \in \mathbb{R}_+$, and let $1 < \alpha \leq 2$.*
1) If $\omega_2 f_\infty^* (t) > \omega_1 F_0^* (t)$ holds for any $t \in [0, T]$, then for each:

$$\beta \in \left((\omega_2 f_\infty^* (t))^{-1}, (\omega_1 F_0^* (t))^{-1} \right),$$

the boundary value problem (2.18)-(2.19) has at least one positive solution under the generalized self-similar form:

$$w(x,t) = \psi(t)\, u\left(\frac{x}{\varphi(t)}\right), \text{ for } x \in [0, X], \text{ and } \varphi, \psi \in C[0,T] \to \mathbb{R}_+. \tag{2.21}$$

2) If $\omega_2 f_0^* (t) > \omega_1 F_\infty^* (t)$ holds for any $t \in [0, T]$, then for each:

$$\beta \in \left((\omega_2 f_0^* (t))^{-1}, (\omega_1 F_\infty^* (t))^{-1} \right),$$

the boundary value problem (2.18)-(2.19) has at least one positive solution under the generalized self-similar form (2.21).
3) Suppose there exists $r_2 > r_1 > 0$, such that

$$\sup_{0 \leq w \leq \psi(t) r_2} \max_{0 \leq x \leq X} f(x,t,w) \leq \frac{r_2 \psi(t)}{\beta \omega_1 \varphi^{\rho\alpha}(t)}, \ \forall t \in [0, T],$$

and

$$\inf_{0 \leq w \leq \psi(t) r_1} f(x,t,w) \geq \frac{r_1 \psi(t)}{\beta \sigma \omega_2 \varphi^{\rho\alpha}(t)} b\left(\frac{x}{\varphi(t)}\right), \ \forall (x,t) \in [0, X] \times [0, T].$$

Then, the boundary value problem (2.18)-(2.19) has at least one positive solution under the generalized self-similar form (2.21), with $r_1 \leq \frac{\|w(\cdot,t)\|_\infty}{\psi(t)} \leq r_2$.

The proof of theorem is studied in detail in the third chapter.

Existence and uniqueness of real solutions

In this part, we assume that $\beta \in \mathbb{R}$ and $\rho > 0$, and $f : [0, X] \times [0, T] \times \mathbb{R} \to \mathbb{R}$.

Theorem 2.5 (Existence and uniqueness of real solutions) *Let $\alpha, X, T \in \mathbb{R}_+$, be such that $1 < \alpha \leq 2$. Suppose that $f(x,t,w)$ is a continuous function with respect to w on $\mathbb{R}$, and a* Lebesgue *measurable function with respect to x on $[0, X]$, $\forall t \in [0, T]$. If there exists a constant $\xi > 0$, such that*

$$|f(x,t,w_1) - f(x,t,w_2)| \leq \frac{\xi}{\varphi^{\rho\alpha}(t)} |w_1 - w_2|, \text{ for any } (x,t) \in [0, X] \times [0, T],$$

and there exists a constant $\bar{\varphi} = \min_{0 \le t \le T} \varphi(t) > 0$, *such that*

$$|\beta| < \frac{(\rho\bar{\varphi}^{\rho})^{\alpha}\,\Gamma(\alpha+1)}{\xi X^{\alpha\rho}}.$$

Then, there exists a unique solution of the boundary value problem (2.18)-(2.19) *under the generalized self-similar form:*

$$w(x,t) = \psi(t)\,u\left(\frac{x}{\varphi(t)}\right), \ \textit{for } x \in [0,X], \ \textit{and } \varphi, \psi \in C[0,T] \to \mathbb{R}_{+}.$$

The proof of theorem is studied in detail in the third chapter.

2.3.2 Generalized Self-Similar Solutions for Nonlinear IVP

In this section, we study the existence and uniqueness of solutions of the following implicit problem of the nonlinear partial differential equations of space-fractional order [8], [34]:

$$ {}_{x}^{\rho}\mathcal{D}_{0+}^{\alpha} w = f\left(x,t,w,{}_{x}^{\rho}\mathcal{D}_{0+}^{\alpha} w\right), \ 0 < \alpha \le 1, \ (x,t) \in (0,X] \times [0,T], \tag{2.22}$$

with the initial condition:

$$w(0,t) = 0, \tag{2.23}$$

under the generalized self-similar form (2.20) , where $f : [0,X] \times [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function which satisfies the hypotheses of theorem 2.3, with finite positive constants X, T.

Now, we give our existence and uniqueness results of the initial value problem (2.22)-(2.23) . We suggest the following hypotheses:

$\overline{(\text{H1})}$ $f : [0,X] \times [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

$\overline{(\text{H2})}$ There exist two constants $\sigma, \beta > 0$, where $\beta < 1$ such that:

$$|f(x,t,w,z) - f(x,t,\tilde{w},\tilde{z})| \le \sigma\,|w-\tilde{w}| + \beta\,|z-\tilde{z}|,$$

for any $w, z, \tilde{w}, \tilde{z} \in \mathbb{R}$ and $(x,t) \in [0,X] \times [0,T]$.

$\overline{(\text{H3})}$ There exist three positive continuous functions a, b, c such that:

$$|f(x,t,w,z)| \le a\left(\frac{x}{\varphi(t)}\right) + b\left(\frac{x}{\varphi(t)}\right)|w| + c\left(\frac{x}{\varphi(t)}\right)|z|,$$

for any $(x,t) \in [0,X] \times [0,T]$ and $w, z \in \mathbb{R}$, where

$$\varphi(t) \ge \bar{\varphi} > 0, \ \text{for any } t \in [0,T].$$

We denote:

$$M_0 = \frac{a^*}{1-c^*}, \text{ and } M_1 = \frac{b^*}{1-c^*},$$

where

$$\begin{aligned} a^* &= \sup_{x\in[0,X]} \max_{t\in[0,T]} a\left(\frac{x}{\varphi(t)}\right), \; b^* = \sup_{x\in[0,X]} \max_{t\in[0,T]} b\left(\frac{x}{\varphi(t)}\right), \\ c^* &= \sup_{x\in[0,X]} \max_{t\in[0,T]} c\left(\frac{x}{\varphi(t)}\right), \text{ with } c^* < 1. \end{aligned}$$

Theorem 2.6 (Existence and uniqueness of solutions) *Let* $\alpha, \rho, X, T \in \mathbb{R}_+$, *be such that* $0 < \alpha \leq 1$.

1) *Assume that hypotheses* $\overline{(\text{H1})}$-$\overline{(\text{H3})}$ *hold. Then, the initial value problem* (2.22)-(2.23) *has at least one positive solution under the generalized self-similar form:*

$$w(x,t) = \psi(t)\, u\left(\frac{x}{\varphi(t)}\right), \text{ for } x \in [0,X], \text{ and } \varphi, \psi \in C[0,T] \to \mathbb{R}_+. \tag{2.24}$$

2) *Assume that hypotheses* $\overline{(\text{H1})}$-$\overline{(\text{H3})}$ *hold. If we put*

$$\frac{M_1 X^{\rho\alpha}}{(\rho\bar{\varphi}^{\rho})^{\alpha}\,\Gamma(\alpha+1)} < 1.$$

Then, the initial value problem (2.22)-(2.23) *has at least one positive solution under the generalized self-similar form* (2.24).

3) *Assume the hypotheses* $\overline{(\text{H1})}$, $\overline{(\text{H2})}$ *hold. If*

$$\frac{\sigma X^{\rho\alpha}}{(1-\beta)(\rho\bar{\varphi}^{\rho})^{\alpha}\,\Gamma(\alpha+1)} < 1.$$

Then, there exists a unique solution of the initial value problem (2.22)-(2.23) *under the generalized self-similar form* (2.24).

The proof of theorem is studied in detail in the fourth chapter.

2.3.3 Generalized Self-Similar Solutions for Nonlinear ICP

In this section, we study the existence and uniqueness of solutions of the following problem of the nonlinear partial differential equations of space-fractional order [29], [34]:

$${}^{\rho}_{x}\mathcal{D}^{\alpha}_{0+} w = f\left(x, t, w, {}^{\rho}_{x}\mathcal{D}^{\beta}_{0+} w\right), \; (x,t) \in [0,X] \times [0,T], \tag{2.25}$$

with the integral condition:

$$\left({}^{\rho}_{x}\mathcal{I}^{1-\alpha}_{0+} w\right)(0^+, t) = 0, \tag{2.26}$$

under the generalized self-similar form (2.20) . Here $0<\beta<\alpha\leq 1$, $f:[0,X]\times[0,T]\times\mathbb{R}\times\mathbb{R}\rightarrow\mathbb{R}$ is a given function which satisfies the hypotheses of theorem 2.3, with finite positive constants X,T.

Now, we give our existence and uniqueness results of the problem (2.25) with the integral condition (2.26) . We suggest the following hypotheses:

$\overline{(\mathrm{H1})}$ $f:[0,X]\times[0,T]\times\mathbb{R}\times\mathbb{R}\rightarrow\mathbb{R}$ is a continuous function.

$\overline{(\mathrm{H2})}$ For all $0<\beta<\alpha\leq 1$, there exist two constants $\sigma,\gamma>0$, where $\gamma<\frac{\rho^{\alpha-\beta}\Gamma(1+\alpha-\beta)}{\lambda^{\rho(\alpha-\beta)}}$ such that:

$$\left|f\left(x,t,w,z\right)-f\left(x,t,\tilde{w},\tilde{z}\right)\right|\leq\sigma\left|w-\tilde{w}\right|+\gamma\left|z-\tilde{z}\right|,$$

for any $w,z,\tilde{w},\tilde{z}\in\mathbb{R}$ and $(x,t)\in[0,X]\times[0,T]$.

$\overline{(\mathrm{H3})}$ There exist three positive continuous functions a,b,c such that:

$$\left|f\left(x,t,w,z\right)\right|\leq a\left(\frac{x}{\varphi\left(t\right)}\right)+b\left(\frac{x}{\varphi\left(t\right)}\right)\left|w\right|+c\left(\frac{x}{\varphi\left(t\right)}\right)\left|z\right|,$$

for any $(x,t)\in[0,X]\times[0,T]$ and $w,z\in\mathbb{R}$, where

$$\varphi\left(t\right)\geq\bar{\varphi}>0,\ \text{for any } t\in[0,T].$$

We denote:

$$M_0=\frac{\left(\rho\bar{\varphi}^{\rho}\right)^{\alpha-\beta}\Gamma\left(1+\alpha-\beta\right)a^{*}}{\left(\rho\bar{\varphi}^{\rho}\right)^{\alpha-\beta}\bar{\varphi}^{\rho(\alpha-\beta)}\Gamma\left(1+\alpha-\beta\right)-c^{*}X^{\rho(\alpha-\beta)}},$$

and

$$M_1=\frac{\left(\rho\bar{\varphi}^{\rho}\right)^{\alpha-\beta}\Gamma\left(1+\alpha-\beta\right)b^{*}}{\left(\rho\bar{\varphi}^{\rho}\right)^{\alpha-\beta}\Gamma\left(1+\alpha-\beta\right)-c^{*}X^{\rho(\alpha-\beta)}},$$

where $0<\beta<\alpha\leq 1$, and

$$\begin{aligned} a^{*} &= \sup_{x\in[0,X]}\max_{t\in[0,T]} a\left(\frac{x}{\varphi\left(t\right)}\right),\ b^{*}=\sup_{x\in[0,X]}\max_{t\in[0,T]} b\left(\frac{x}{\varphi\left(t\right)}\right),\\ c^{*} &= \sup_{x\in[0,X]}\max_{t\in[0,T]} c\left(\frac{x}{\varphi\left(t\right)}\right),\ \text{with } c^{*}<\frac{\left(\rho\bar{\varphi}^{\rho}\right)^{\alpha-\beta}\Gamma\left(1+\alpha-\beta\right)}{X^{\rho(\alpha-\beta)}}.\end{aligned}$$

Theorem 2.7 (Existence and uniqueness of solutions) *Let $\alpha,\beta,\rho,X,T\in\mathbb{R}_{+}$, be such that $0<\beta<\alpha\leq 1$.*

1) *Assume that hypotheses $\overline{(\mathrm{H1})}$-$\overline{(\mathrm{H3})}$ hold. Then, the problem (2.25) with the integral condition (2.26) has at least one positive solution under the generalized self-similar form:*

$$w\left(x,t\right)=\psi\left(t\right)u\left(\frac{x}{\varphi\left(t\right)}\right),\ \text{for } x\in[0,X],\ \text{and } \varphi,\psi\in C\left[0,T\right]\rightarrow\mathbb{R}_{+}.\tag{2.27}$$

2) *Assume that hypotheses* $\overline{(\mathrm{H1})}$-$\overline{(\mathrm{H3})}$ *hold. If we put*

$$\frac{M_1 X^{\rho\alpha}}{(\rho\bar{\varphi}^{\rho})^{\alpha}\,\Gamma(\alpha+1)} < 1.$$

Then, the problem (2.25) *with the integral condition* (2.26) *has at least one positive solution under the generalized self-similar form* (2.27) *.*

3) *Assume the hypotheses* $\overline{(\mathrm{H1})}$, $\overline{(\mathrm{H2})}$ *hold. If*

$$\frac{\sigma X^{\rho\alpha}\Gamma(1+\alpha-\beta)}{\Gamma(\alpha+1)\left[(\rho\bar{\varphi}^{\rho})^{\alpha}\,\Gamma(1+\alpha-\beta)-\gamma\,(\rho\bar{\varphi}^{\rho})^{\beta}\,X^{\rho(\alpha-\beta)}\right]} < 1.$$

Then, there exists a unique solution of the problem (2.25) *with the integral condition* (2.26) *under the generalized self-similar form* (2.27) *.*

The proof of theorem is studied in detail in the fifth chapter.

Chapter 3

Existence of Solutions to a Nonlinear Boundary Value Problem

This chapter has been published in: Journal of Mathematics and Applications **42** (2019), (see [7]).

3.1 Introduction

This chapter deals with the existence and uniqueness of generalized self-similar solutions for a boundary value problem of nonlinear FPDE with generalized fractional derivative. The main results are proved by means of Guo-Krasnosel'skii's (theorem 1.7) and Banach's (theorem 1.8) fixed point theorems. The used differential operator is developed by Katugampola. For application purposes, some examples are provided to demonstrate the applicability of our main results.

This chapter will be dedicated to the existence and uniqueness of solutions of the following problem of the nonlinear FPDE [7], [12]:

$$ {}_{x}^{\rho}\mathcal{D}_{0+}^{\alpha} w + \beta f\left(x,t,w\right) = 0,\ 1 < \alpha \leq 2,\ \ (x,t) \in (0,X) \times [0,T]\,, \tag{3.1} $$

with the boundary conditions:

$$ w\left(0,t\right) = 0,\ w\left(X,t\right) = 0, $$

under the generalized self-similar form which is:

$$w(x,t) = \psi(t)\, u(\eta), \text{ with } \eta = \frac{x}{\varphi(t)}, \text{ and } \varphi, \psi \in C[0,T] \to \mathbb{R}_+, \tag{3.2}$$

where $\beta \in \mathbb{R}$, and $f : [0,X] \times [0,T] \times \mathbb{R} \to \mathbb{R}$ for some $X, T \in \mathbb{R}_+$, is a given function which satisfies the hypotheses of theorem 2.3.

Then the transformation (3.2) reduces the fractional-order's partial differential equation (3.1) to the ordinary differential equation of fractional order of the form:

$${}^{\rho}\mathcal{D}_{0+}^{\alpha} u(\eta) + \beta f(\eta, u(\eta)) = 0, \; 0 < \eta < \lambda, \tag{3.3}$$

supplemented with the boundary conditions:

$$u(0) = 0, \; u(\lambda) = 0, \tag{3.4}$$

where $f : [0, \lambda] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, with finite positive constant $\lambda = X\bar{\varphi}^{-1}$, with $\bar{\varphi} = \min\limits_{0 \le t \le T} \varphi(t)$.

According to chapter 2 (subsection 2.3.1), we discuss the existence and uniqueness of solutions of nonlinear FDEs (3.3), with the boundary conditions (3.4).

We obtain several existence and uniqueness results for the problem (3.3)-(3.4).

3.2 Definition of Integral Solution

In the sequel, λ, p, n and c are real constants such that

$$p \ge 1, \; c > 0, \; n = [\alpha] + 1, \text{ and } 0 < \lambda \le (pc)^{\frac{1}{pc}}.$$

Now, we present some important lemmas which play a key role in the proofs of the main results.

Lemma 3.1 ([7]) *Let $\alpha, \rho \in \mathbb{R}^+$. If $u \in C[0,\lambda]$, then:*

(i) The fractional equation ${}^{\rho}\mathcal{D}_{0+}^{\alpha} u(\eta) = 0$, has a solution as follows:

$$u(\eta) = C_1 \eta^{\rho(\alpha-1)} + C_2 \eta^{\rho(\alpha-2)} + \cdots + C_n \eta^{\rho(\alpha-n)}, \text{ where } C_m \in \mathbb{R}, \text{ with } m = 1, 2, \ldots, n.$$

(ii) If ${}^{\rho}\mathcal{D}_{0+}^{\alpha} u \in C[0,\lambda]$ and $1 < \alpha \le 2$, then:

$${}^{\rho}\mathcal{I}_{0+}^{\alpha}\, {}^{\rho}\mathcal{D}_{0+}^{\alpha}\, u(\eta) = u(\eta) + C_1 \eta^{\rho(\alpha-1)} + C_2 \eta^{\rho(\alpha-2)}, \text{ for some } C_1, C_2 \in \mathbb{R}. \tag{3.5}$$

Proof. (i) Let $\alpha, \rho \in \mathbb{R}^+$. From remark 1.2, we have:

$$^{\rho}\mathcal{D}^{\alpha}_{0+}\eta^{\rho(\alpha-m)} = 0, \text{ for each } m = 1, 2, \ldots, n.$$

Then, the fractional equation $^{\rho}\mathcal{D}^{\alpha}_{0+}u(\eta) = 0$, admits a solution as follows:

$$u(\eta) = C_1\eta^{\rho(\alpha-1)} + C_2\eta^{\rho(\alpha-2)} + \cdots + C_n\eta^{\rho(\alpha-n)}, \; C_m \in \mathbb{R}, \; m = 1, 2, \ldots, n.$$

(ii) Let $^{\rho}\mathcal{D}^{\alpha}_{0+}u \in C[0,\lambda]$ be the fractional derivative (1.22) of order $1 < \alpha \leq 2$. If we apply the operator $^{\rho}\mathcal{I}^{\alpha}_{0+}$ to $^{\rho}\mathcal{D}^{\alpha}_{0+}u(\eta)$ and use definitions 1.18, 1.19, theorems 1.2, 1.3, 1.4 and property (1.28), we get

$$\begin{aligned}
^{\rho}\mathcal{I}^{\alpha}_{0+}\, {}^{\rho}\mathcal{D}^{\alpha}_{0+}u(\eta) &= \left(\eta^{1-\rho}\frac{d}{d\eta}\right) {}^{\rho}\mathcal{I}^{\alpha+1}_{0+}\, {}^{\rho}\mathcal{D}^{\alpha}_{0+}u(\eta) \\
&= \left(\eta^{1-\rho}\frac{d}{d\eta}\right)\left[\frac{\rho^{-\alpha}}{\Gamma(\alpha+1)}\int_0^{\eta}(\eta^{\rho}-s^{\rho})^{\alpha}s^{\rho-1}\, {}^{\rho}\mathcal{D}^{\alpha}_{0+}u(s)\,ds\right] \\
&= \left(\eta^{1-\rho}\frac{d}{d\eta}\right)\left[\frac{\rho^{-\alpha}}{\Gamma(\alpha+1)}\int_0^{\eta}(\eta^{\rho}-s^{\rho})^{\alpha}s^{\rho-1}\left[\left(s^{1-\rho}\frac{d}{ds}\right)^2 {}^{\rho}\mathcal{I}^{2-\alpha}_{0+}u(s)\right]ds\right] \\
&= \left(\eta^{1-\rho}\frac{d}{d\eta}\right)\left[\frac{\rho^{-\alpha}}{\Gamma(\alpha+1)}\int_0^{\eta}(\eta^{\rho}-s^{\rho})^{\alpha}\frac{d}{ds}\left[\left(s^{1-\rho}\frac{d}{ds}\right) {}^{\rho}\mathcal{I}^{2-\alpha}_{0+}u(s)\right]ds\right] \\
&= \left(\eta^{1-\rho}\frac{d}{d\eta}\right)\left[\frac{\rho^{-\alpha}}{\Gamma(\alpha+1)}\left(\left[(\eta^{\rho}-s^{\rho})^{\alpha}\left(s^{1-\rho}\frac{d}{ds}\right) {}^{\rho}\mathcal{I}^{2-\alpha}_{0+}u(s)\right]_0^{\eta}\right.\right. \\
&\quad \left.\left. + \,\alpha\rho\int_0^{\eta}s^{\rho-1}(\eta^{\rho}-s^{\rho})^{\alpha-1}\left(s^{1-\rho}\frac{d}{ds}\right) {}^{\rho}\mathcal{I}^{2-\alpha}_{0+}u(s)\,ds\right)\right].
\end{aligned}$$

From (1.28) , we have

$$\left(s^{1-\rho}\frac{d}{ds}\right) {}^{\rho}\mathcal{I}^{2-\alpha}_{0+}u(s) = {}^{\rho}\mathcal{I}^{1-\alpha}_{0+}u(s)\,. \tag{3.6}$$

On the other hand, from (1.22) , we have

$$\left(s^{1-\rho}\frac{d}{ds}\right) {}^{\rho}\mathcal{I}^{2-\alpha}_{0+}u(s) = \left(s^{1-\rho}\frac{d}{ds}\right)^1 {}^{\rho}\mathcal{I}^{1-(\alpha-1)}_{0+}u(s) = {}^{\rho}\mathcal{D}^{\alpha-1}_{0+}u(s)\,. \tag{3.7}$$

Then

$$^{\rho}\mathcal{I}^{\alpha}_{0+}\, {}^{\rho}\mathcal{D}^{\alpha}_{0+}u(\eta) = \underbrace{\left(\eta^{1-\rho}\frac{d}{d\eta}\right)\left(\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\eta}(\eta^{\rho}-s^{\rho})^{\alpha-1}\frac{d}{ds}\, {}^{\rho}\mathcal{I}^{2-\alpha}_{0+}u(s)\,ds\right)}_{\mathcal{Q}} - \frac{\rho^{1-\alpha}\, {}^{\rho}\mathcal{I}^{1-\alpha}_{0+}u(0^+)}{\Gamma(\alpha)}\eta^{\rho(\alpha-1)},$$

where

$$\begin{aligned}
\mathcal{Q} &= \left(\eta^{1-\rho}\frac{d}{d\eta}\right)\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\left(\left[(\eta^{\rho}-s^{\rho})^{\alpha-1}\,{}^{\rho}\mathcal{I}_{0^+}^{2-\alpha}u(s)\right]_0^{\eta}+\rho(\alpha-1)\int_0^{\eta}s^{\rho-1}(\eta^{\rho}-s^{\rho})^{\alpha-2}\,{}^{\rho}\mathcal{I}_{0^+}^{2-\alpha}u(s)\,ds\right)\\
&= \left(\eta^{1-\rho}\frac{d}{d\eta}\right)\left(\frac{\rho^{2-\alpha}}{\Gamma(\alpha-1)}\int_0^{\eta}s^{\rho-1}(\eta^{\rho}-s^{\rho})^{\alpha-2}\,{}^{\rho}\mathcal{I}_{0^+}^{2-\alpha}u(s)\,ds-\frac{\rho^{1-\alpha}\,{}^{\rho}\mathcal{I}_{0^+}^{2-\alpha}u(0^+)}{\Gamma(\alpha)}\eta^{\rho(\alpha-1)}\right)\\
&= \left(\eta^{1-\rho}\frac{d}{d\eta}\right)\left({}^{\rho}\mathcal{I}_{0^+}^{\alpha-1}\,{}^{\rho}\mathcal{I}_{0^+}^{2-\alpha}u(\eta)-\frac{\rho^{1-\alpha}\,{}^{\rho}\mathcal{I}_{0^+}^{2-\alpha}u(0^+)}{\Gamma(\alpha)}\eta^{\rho(\alpha-1)}\right)\\
&= \left(\eta^{1-\rho}\frac{d}{d\eta}\right)\left({}^{\rho}\mathcal{I}_{0^+}^{1}u(\eta)-\frac{\rho^{1-\alpha}\,{}^{\rho}\mathcal{I}_{0^+}^{2-\alpha}u(0^+)}{\Gamma(\alpha)}\eta^{\rho(\alpha-1)}\right)\\
&= u(\eta)-\frac{\rho^{2-\alpha}\,{}^{\rho}\mathcal{I}_{0^+}^{2-\alpha}u(0^+)}{\Gamma(\alpha-1)}\eta^{\rho(\alpha-2)}.
\end{aligned}$$

Finally, for $1<\alpha\leq 2$, we have:

$$ {}^{\rho}\mathcal{I}_{0^+}^{\alpha}\,{}^{\rho}\mathcal{D}_{0^+}^{\alpha}u(\eta)=u(\eta)-\frac{\rho^{1-\alpha}\,{}^{\rho}\mathcal{I}_{0^+}^{1-\alpha}u(0^+)}{\Gamma(\alpha)}\eta^{\rho(\alpha-1)}-\frac{\rho^{2-\alpha}\,{}^{\rho}\mathcal{I}_{0^+}^{2-\alpha}u(0^+)}{\Gamma(\alpha-1)}\eta^{\rho(\alpha-2)}. \tag{3.8}$$

As

$$ {}^{\rho}\mathcal{I}_{0^+}^{\alpha}\eta^{\mu}=\frac{\rho^{-\alpha}\Gamma\left(1+\frac{\mu}{\rho}\right)}{\Gamma\left(1+\alpha+\frac{\mu}{\rho}\right)}\eta^{\mu+\alpha\rho},\ \forall\mu>-\rho,$$

we use (3.6) , (3.7) , to prove that

$$\begin{aligned}
{}^{\rho}\mathcal{I}_{0^+}^{1-\alpha}\left[C_1\eta^{\rho(\alpha-1)}\right] &= C_1\frac{\rho^{-(1-\alpha)}\Gamma\left(1+\frac{\rho(\alpha-1)}{\rho}\right)}{\Gamma\left(1+(1-\alpha)+\frac{\rho(\alpha-1)}{\rho}\right)}\eta^{\rho(\alpha-1)+(1-\alpha)\rho}=C_1\rho^{\alpha-1}\Gamma(\alpha), && (3.9)\\
{}^{\rho}\mathcal{I}_{0^+}^{1-\alpha}\left[C_2\eta^{\rho(\alpha-2)}\right] &= C_2\,{}^{\rho}\mathcal{D}_{0^+}^{\alpha-1}\eta^{\rho(\alpha-2)}=C_2\,{}^{\rho}\mathcal{D}_{0^+}^{\alpha-1}\eta^{\rho((\alpha-1)-1)}=0, && (3.10)
\end{aligned}$$

for some $C_1,C_2\in\mathbb{R}$, and

$$\begin{aligned}
{}^{\rho}\mathcal{I}_{0^+}^{2-\alpha}\left[C_1\eta^{\rho(\alpha-1)}\right] &= C_1\frac{\rho^{-(2-\alpha)}\Gamma\left(1+\frac{\rho(\alpha-1)}{\rho}\right)}{\Gamma\left(1+(2-\alpha)+\frac{\rho(\alpha-1)}{\rho}\right)}\eta^{\rho(\alpha-1)+(2-\alpha)\rho}=\frac{C_1\Gamma(\alpha)\,\eta^{\rho}}{\rho^{2-\alpha}} && (3.11)\\
{}^{\rho}\mathcal{I}_{0^+}^{2-\alpha}\left[C_2\eta^{\rho(\alpha-2)}\right] &= C_2\frac{\rho^{-(2-\alpha)}\Gamma\left(1+\frac{\rho(\alpha-2)}{\rho}\right)}{\Gamma\left(1+(2-\alpha)+\frac{\rho(\alpha-2)}{\rho}\right)}\eta^{\rho(\alpha-2)+(2-\alpha)\rho}=\frac{C_2\Gamma(\alpha-1)}{\rho^{2-\alpha}}. && (3.12)
\end{aligned}$$

Then, for $u(\eta)=C_1\eta^{\rho(\alpha-1)}+C_2\eta^{\rho(\alpha-2)}$, we have respectively:

$$\begin{aligned}
{}^{\rho}\mathcal{I}_{0^+}^{1-\alpha}u(0^+) &= {}^{\rho}\mathcal{I}_{0^+}^{1-\alpha}\left[C_1\eta^{\rho(\alpha-1)}\right](0^+)+{}^{\rho}\mathcal{I}_{0^+}^{1-\alpha}\left[C_2\eta^{\rho(\alpha-2)}\right](0^+)=\frac{C_1\Gamma(\alpha)}{\rho^{1-\alpha}}, && (3.13)\\
{}^{\rho}\mathcal{I}_{0^+}^{2-\alpha}u(0^+) &= {}^{\rho}\mathcal{I}_{0^+}^{2-\alpha}\left[C_1\eta^{\rho(\alpha-1)}\right](0^+)+{}^{\rho}\mathcal{I}_{0^+}^{2-\alpha}\left[C_2\eta^{\rho(\alpha-2)}\right](0^+)=\frac{C_2\Gamma(\alpha-1)}{\rho^{2-\alpha}} && (3.14)
\end{aligned}$$

From (3.8) , (3.9) , (3.10) , (3.11) , (3.12) , (3.13) and (3.14) we get (3.5) . The proof is complete.

■

In the following lemma, we define the integral solution of the boundary value problem (3.3)-(3.4) .

Lemma 3.2 ([7]) *Let $\alpha, \rho \in \mathbb{R}^+$, be such that $1 < \alpha \leq 2$. We give ${}^{\rho}\mathcal{D}_{0+}^{\alpha}u \in C[0, \lambda]$, and $f(\eta, u)$ is a continuous function. Then the boundary value problem (3.3)-(3.4), is equivalent to the fractional integral equation*

$$u(\eta) = \beta \int_0^{\lambda} G(\eta, s) f(s, u(s))\, ds, \ \eta \in [0, \lambda],$$

where

$$G(\eta, s) = \begin{cases} \frac{\rho^{1-\alpha} s^{\rho-1}}{\Gamma(\alpha)} \left[\left[\frac{\eta^{\rho}}{\lambda^{\rho}} (\lambda^{\rho} - s^{\rho}) \right]^{\alpha-1} - (\eta^{\rho} - s^{\rho})^{\alpha-1} \right], & 0 \leq s \leq \eta \leq \lambda, \\ \frac{\rho^{1-\alpha} s^{\rho-1}}{\Gamma(\alpha)} \left[\frac{\eta^{\rho}}{\lambda^{\rho}} (\lambda^{\rho} - s^{\rho}) \right]^{\alpha-1}, & 0 \leq \eta \leq s \leq \lambda, \end{cases} \tag{3.15}$$

is the Green's function associated with the boundary value problem (3.3)-(3.4).

Proof. Let $\alpha, \rho \in \mathbb{R}^+$, be such that $1 < \alpha \leq 2$. We apply lemma 3.1 to reduce the fractional equation (3.3) to an equivalent fractional integral equation. It is easy to prove the operator ${}^{\rho}\mathcal{I}_{0+}^{\alpha}$ has the linearity property for all $\alpha > 0$ after direct integration. Then by applying ${}^{\rho}\mathcal{I}_{0+}^{\alpha}$ to equation (3.3), we get

$${}^{\rho}\mathcal{I}_{0+}^{\alpha}\, {}^{\rho}\mathcal{D}_{0+}^{\alpha} u(\eta) + \beta\, {}^{\rho}\mathcal{I}_{0+}^{\alpha} f(\eta, u(\eta)) = 0.$$

From lemma 3.1, we find for $1 < \alpha \leq 2$,

$${}^{\rho}\mathcal{I}_{0+}^{\alpha}\, {}^{\rho}\mathcal{D}_{0+}^{\alpha} u(\eta) = u(\eta) + C_1 \eta^{\rho(\alpha-1)} + C_2 \eta^{\rho(\alpha-2)},$$

for some $C_1, C_2 \in \mathbb{R}$. Then, the integral solution of the equation (3.3) is:

$$u(\eta) = -\frac{\beta \rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{\eta} \frac{s^{\rho-1} f(s, u(s))}{(\eta^{\rho} - s^{\rho})^{1-\alpha}} ds - C_1 \eta^{\rho(\alpha-1)} - C_2 \eta^{\rho(\alpha-2)}. \tag{3.16}$$

The conditions (3.4) imply that:

$$u(0) = -\lim_{\eta \to 0} C_2 \eta^{\rho(\alpha-2)} = 0.$$

Then $C_2 = 0$, also

$$u(\lambda) = -\frac{\beta \rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{\lambda} \frac{s^{\rho-1} f(s, u(s))}{(\lambda^{\rho} - s^{\rho})^{1-\alpha}} ds - C_1 \lambda^{\rho(\alpha-1)} = 0,$$

then

$$C_1 = -\frac{\beta \rho^{1-\alpha}}{\lambda^{\rho(\alpha-1)} \Gamma(\alpha)} \int_0^{\lambda} \frac{s^{\rho-1} f(s, u(s))}{(\lambda^{\rho} - s^{\rho})^{1-\alpha}} ds.$$

The integral equation (3.16) is equivalent to:

$$u(\eta) = -\frac{\beta\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^\eta \frac{s^{\rho-1}f(s,u(s))}{(\eta^\rho - s^\rho)^{1-\alpha}}ds + \frac{\beta\eta^{\rho(\alpha-1)}\rho^{1-\alpha}}{\lambda^{\rho(\alpha-1)}\Gamma(\alpha)}\int_0^\lambda \frac{s^{\rho-1}f(s,u(s))}{(\lambda^\rho - s^\rho)^{1-\alpha}}ds.$$

Therefore, the unique solution of problem (3.3)-(3.4) is:

$$\begin{aligned} u(\eta) &= \beta\int_0^\eta \frac{\rho^{1-\alpha}s^{\rho-1}\left[\left[\frac{\eta^\rho}{\lambda^\rho}(\lambda^\rho - s^\rho)\right]^{\alpha-1} - (\eta^\rho - s^\rho)^{\alpha-1}\right]}{\Gamma(\alpha)}f(s,u(s))\,ds \\ &\quad +\beta\int_\eta^\lambda \frac{\rho^{1-\alpha}s^{\rho-1}\left[\frac{\eta^\rho}{\lambda^\rho}(\lambda^\rho - s^\rho)\right]^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s))\,ds \\ &= \beta\int_0^\lambda G(\eta,s)\,f(s,u(s))\,ds. \end{aligned}$$

The proof is complete. ■

3.3 Existence and Uniqueness Results

3.3.1 Existence Results of Positive Solutions

In this section, we assume that $\beta > 0$ and $0 < \rho \leq 1$. We suggest some conditions on f, which allow us to obtain some results on existence of positive solutions for the boundary value problem (3.3)-(3.4). Let $f : [0,\lambda] \times [0,\infty) \to [h,\infty)$ be a continuous function, with finite positive constant h.

We note that $u(\eta)$ is a solution of (3.3)-(3.4) if and only if:

$$u(\eta) = \beta\int_0^\lambda G(\eta,s)\,f(s,u(s))\,ds,\ \eta \in [0,\lambda].$$

Now we prove some properties of the Green's function $G(\eta,s)$ given in (3.15).

Lemma 3.3 ([7]) *Let $1 < \alpha \leq 2$ and $0 < \rho \leq 1$, then the Green's function $G(\eta,s)$ given by (3.15) satisfies:*

(1) *$G(\eta,s) > 0$ for $\eta, s \in (0,\lambda)$.*

(2) *$\max_{0\leq\eta\leq\lambda} G(\eta,s) = G(s,s)$, for each $s \in [0,\lambda]$.*

(3) *For any $\eta \in [0,\lambda]$,*

$$G(\eta,s) \geq b(\eta)\,G(s,s),\ \textit{for any } \frac{\lambda}{8} \leq s \leq \lambda \textit{ and some } b \in C[0,\lambda]. \tag{3.17}$$

Proof. (1) Let $1 < \alpha \leq 2$ and $0 < \rho \leq 1$. In the case $0 < \eta \leq s < \lambda$, we have:

$$\frac{\rho^{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)}\left[\frac{\eta^{\rho}}{\lambda^{\rho}}(\lambda^{\rho}-s^{\rho})\right]^{\alpha-1} > 0.$$

Moreover, for $0 < s \leq \eta < \lambda$, we have $\frac{\eta^{\rho}}{\lambda^{\rho}} < 1$, then $\frac{\eta^{\rho}}{\lambda^{\rho}}s^{\rho} < s^{\rho}$ and $\eta^{\rho} - \frac{\eta^{\rho}}{\lambda^{\rho}}s^{\rho} > \eta^{\rho} - s^{\rho}$, thus

$$\eta^{\rho} - \frac{\eta^{\rho}}{\lambda^{\rho}}s^{\rho} = \frac{\eta^{\rho}}{\lambda^{\rho}}(\lambda^{\rho}-s^{\rho}) > \eta^{\rho} - s^{\rho} \Rightarrow \left[\frac{\eta^{\rho}}{\lambda^{\rho}}(\lambda^{\rho}-s^{\rho})\right]^{\alpha-1} - (\eta^{\rho}-s^{\rho})^{\alpha-1} > 0,$$

which implies that $G(\eta, s) > 0$ for any $\eta, s \in (0, \lambda)$.

(2) To prove that

$$\max_{0\leq\eta\leq\lambda} G(\eta, s) = G(s, s) = \frac{\rho^{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)}\left[\frac{s^{\rho}}{\lambda^{\rho}}(\lambda^{\rho}-s^{\rho})\right]^{\alpha-1}, \ \forall s \in [0, \lambda], \tag{3.18}$$

we choose

$$g_1(\eta, s) = \tfrac{\rho^{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)}\left[\left[\tfrac{\eta^{\rho}}{\lambda^{\rho}}(\lambda^{\rho}-s^{\rho})\right]^{\alpha-1} - (\eta^{\rho}-s^{\rho})^{\alpha-1}\right], \quad g_2(\eta, s) = \tfrac{\rho^{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)}\left[\tfrac{\eta^{\rho}}{\lambda^{\rho}}(\lambda^{\rho}-s^{\rho})\right]^{\alpha-1}.$$

Indeed, we put $\max\limits_{0\leq\eta\leq\lambda} G(\eta, s) = G(\eta^{*}, s)$, where $0 \leq \eta^{*} \leq \lambda$. Then, we get for some $0 < \eta_1 < \eta_2 < \lambda$, that

$$\begin{aligned}\max_{0\leq\eta\leq\lambda} G(\eta, s) &= \begin{cases} g_1(\eta^{*}, s), & s \in [0, \eta_1], \\ \max\{g_1(\eta^{*}, s), g_2(\eta^{*}, s)\}, & s \in [\eta_1, \eta_2], \\ g_2(\eta^{*}, s), & s \in [\eta_2, \lambda], \end{cases} \\ &= \begin{cases} g_1(\eta^{*}, s), & s \in [0, r], \\ g_2(\eta^{*}, s), & s \in [r, \lambda], \end{cases}\end{aligned}$$

where $r \in [\eta_1, \eta_2]$, is the unique solution of equation

$$g_1(\eta^{*}, s) = g_2(\eta^{*}, s) \Leftrightarrow \eta^{*} = s,$$

which shows the equality (3.18).

(3) In the following, we divide the proof into two parts, to show the existence of $b \in C[0, \lambda]$, such that

$$G(\eta, s) \geq b(\eta) G(s, s), \ \text{for any } \frac{\lambda}{8} \leq s \leq \lambda.$$

(i) Firstly, if $0 \leq \eta \leq s \leq \lambda$, we see that $\frac{G(\eta,s)}{G(s,s)}$ is decreasing with respect to s.

Consequently

$$\frac{G(\eta, s)}{G(s, s)} = \frac{\left[\frac{\eta^{\rho}}{\lambda^{\rho}}(\lambda^{\rho}-s^{\rho})\right]^{\alpha-1}}{\left[\frac{s^{\rho}}{\lambda^{\rho}}(\lambda^{\rho}-s^{\rho})\right]^{\alpha-1}} = \left(\frac{\eta}{s}\right)^{\rho(\alpha-1)} \geq \left(\frac{\eta}{\lambda}\right)^{\rho(\alpha-1)} = b_1(\eta), \ \forall \eta \in [0, s].$$

(ii) In the same way, if $0 \leq s \leq \eta \leq \lambda$, we have $\frac{s^\rho}{\lambda^\rho} < \frac{\eta^\rho}{\lambda^\rho} \leq 1$, $\left(\frac{\eta^\rho}{\lambda^\rho}\right)^{\alpha-2} \geq 1$, $\forall \alpha \in (1,2]$, and

$$\begin{aligned}
G(\eta, s) &= \frac{\rho^{1-\alpha} s^{\rho-1}}{\Gamma(\alpha)} \left[\left[\frac{\eta^\rho}{\lambda^\rho} (\lambda^\rho - s^\rho) \right]^{\alpha-1} - (\eta^\rho - s^\rho)^{\alpha-1} \right] \\
&= \frac{(\alpha-1)\rho^{1-\alpha} s^{\rho-1}}{\Gamma(\alpha)} \int_{\eta^\rho - s^\rho}^{\frac{\eta^\rho}{\lambda^\rho}(\lambda^\rho - s^\rho)} t^{\alpha-2} dt \\
&\geq \frac{(\alpha-1)\rho^{1-\alpha} s^{\rho-1}}{\Gamma(\alpha)} \left(\frac{\eta^\rho}{\lambda^\rho}\right)^{\alpha-2} (\lambda^\rho - s^\rho)^{\alpha-2} \left(\frac{\eta^\rho}{\lambda^\rho}(\lambda^\rho - s^\rho) - (\eta^\rho - s^\rho) \right) \\
&\geq \frac{(\alpha-1)\rho^{1-\alpha} s^{\rho-1}}{\Gamma(\alpha)} (\lambda^\rho - s^\rho)^{\alpha-1} \frac{s^\rho(\lambda^\rho - \eta^\rho)}{\lambda^\rho(\lambda^\rho - s^\rho)}.
\end{aligned}$$

As $0 < \rho \leq 1$, we get

$$\lambda^\rho - \eta^\rho = \rho \int_\eta^\lambda t^{\rho-1} dt \geq \rho \lambda^{\rho-1}(\lambda - \eta), \text{ and } \lambda^\rho - s^\rho = \rho \int_s^\lambda t^{\rho-1} dt \leq \rho s^{\rho-1}(\lambda - s).$$

Therefore

$$\begin{aligned}
\frac{G(\eta, s)}{G(s, s)} &\geq \frac{\frac{(\alpha-1)\rho^{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)} (\lambda^\rho - s^\rho)^{\alpha-1} \frac{s^\rho(\lambda^\rho-\eta^\rho)}{\lambda^\rho(\lambda^\rho - s^\rho)}}{\frac{\rho^{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)} \left[\frac{s^\rho}{\lambda^\rho}(\lambda^\rho - s^\rho)\right]^{\alpha-1}} = (\alpha-1) \frac{s^\rho(\lambda^\rho - \eta^\rho)}{\lambda^\rho(\lambda^\rho - s^\rho)} \left(\frac{\lambda^\rho}{s^\rho}\right)^{\alpha-1} \\
&\geq (\alpha-1) \frac{s(\lambda - \eta)}{\lambda(\lambda - s)} \\
&\geq (\alpha-1) \frac{s(\lambda-\eta)}{\lambda^2}.
\end{aligned}$$

Finally, for $s \in \left[\frac{\lambda}{8}, \eta\right]$, we have:

$$\frac{G(\eta, s)}{G(s, s)} \geq \frac{(\alpha-1)(\lambda-\eta)}{8\lambda} = b_2(\eta).$$

It is clear that $b_1(\eta)$ and $b_2(\eta)$ are positive functions, and this is enough to choose:

$$b(\eta) = \begin{cases} \left(\frac{\eta}{\lambda}\right)^{\rho(\alpha-1)}, & \text{for } \eta \in [0, \bar{\eta}], \\ \frac{(\alpha-1)(\lambda-\eta)}{8\lambda}, & \text{for } \eta \in [\bar{\eta}, \lambda], \end{cases}$$

where $\bar{\eta} \in (0, \lambda)$ is the unique solution of the equation $b_1(\eta) = b_2(\eta)$. We see that

$$b(\eta) \leq \bar{b} = b(\bar{\eta}) = \left(\frac{\bar{\eta}}{\lambda}\right)^{\rho(\alpha-1)} = \frac{(\alpha-1)(\lambda-\bar{\eta})}{8\lambda} < 1 \text{ for all } \eta \in [0, \lambda].$$

Finally, we have $\forall s \in \left[\frac{\lambda}{8}, \lambda\right]$,

$$G(\eta, s) \geq b(\eta) G(s, s), \ \forall \eta \in [0, \lambda].$$

The proof is complete. ■

Lemma 3.4 ([7]) *Let $1<\alpha\leq 2$ and $0<\rho\leq 1$, then there exists a positive constant*

$$\sigma=1+\frac{8^{\rho\alpha}L\left(\alpha+1\right)\left[8^{\rho\alpha}-\left(8^{\rho}-1\right)^{\alpha}\right]}{h\left(8^{\rho}-1\right)^{\alpha}\left[8^{\rho}\left(\alpha+1\right)+8^{\rho(\alpha-1)}\left(\alpha-1\right)\left(8^{\rho}-1\right)\right]},\ \textit{for some } h,L>0,$$

such that

$$\int_{0}^{\lambda}G\left(s,s\right)f\left(s,u\left(s\right)\right)ds\leq\sigma\int_{\frac{\lambda}{8}}^{\lambda}G\left(s,s\right)f\left(s,u\left(s\right)\right)ds. \tag{3.19}$$

Proof. As $f\left(\eta,u\left(\eta\right)\right)\geq h$, for any $\eta\in\left[0,\lambda\right]$, we get

$$\begin{aligned}\int_{\frac{\lambda}{8}}^{\lambda}G\left(s,s\right)f\left(s,u\left(s\right)\right)ds &\geq h\int_{\frac{\lambda}{8}}^{\lambda}\frac{\rho^{1-\alpha}s^{\rho-1}}{\Gamma\left(\alpha\right)}\left[\frac{s^{\rho}}{\lambda^{\rho}}\left(\lambda^{\rho}-s^{\rho}\right)\right]^{\alpha-1}ds\\ &\geq -\frac{h}{\alpha\rho^{\alpha}\lambda^{\rho(\alpha-1)}\Gamma\left(\alpha\right)}\int_{\frac{\lambda}{8}}^{\lambda}s^{\rho(\alpha-1)}\left[-\rho\alpha s^{\rho-1}\left(\lambda^{\rho}-s^{\rho}\right)^{\alpha-1}\right]ds.\end{aligned}$$

The integral by part gives:

$$\begin{aligned}\int_{\frac{\lambda}{8}}^{\lambda}G\left(s,s\right)f\left(s,u\left(s\right)\right)ds &\geq \frac{h\left[\frac{\lambda^{\rho(\alpha-1)}}{8^{\rho(\alpha-1)}}\left(\lambda^{\rho}-\frac{\lambda^{\rho}}{8^{\rho}}\right)^{\alpha}+\rho\left(\alpha-1\right)\int_{\frac{\lambda}{8}}^{\lambda}s^{\rho(\alpha-1)-1}\left(\lambda^{\rho}-s^{\rho}\right)^{\alpha}ds\right]}{\rho^{\alpha}\lambda^{\rho(\alpha-1)}\Gamma\left(\alpha+1\right)}\\ &\geq \frac{h\left[\frac{\lambda^{\rho}}{8^{\rho(\alpha-1)}}\left(\lambda^{\rho}-\frac{\lambda^{\rho}}{8^{\rho}}\right)^{\alpha}+\rho\left(\alpha-1\right)\int_{\frac{\lambda}{8}}^{\lambda}\frac{s^{\rho(\alpha-2)}}{\lambda^{\rho(\alpha-2)}}s^{\rho-1}\left(\lambda^{\rho}-s^{\rho}\right)^{\alpha}ds\right]}{\rho^{\alpha}\lambda^{\rho}\Gamma\left(\alpha+1\right)}\\ &\geq \frac{h\left[\frac{\lambda^{\rho}}{8^{\rho(\alpha-1)}}\left(\lambda^{\rho}-\frac{\lambda^{\rho}}{8^{\rho}}\right)^{\alpha}-\frac{\alpha-1}{\alpha+1}\int_{\frac{\lambda}{8}}^{\lambda}\left[-\rho\left(\alpha+1\right)s^{\rho-1}\left(\lambda^{\rho}-s^{\rho}\right)^{\alpha}\right]ds\right]}{\rho^{\alpha}\lambda^{\rho}\Gamma\left(\alpha+1\right)}\\ &\geq \frac{h\lambda^{\rho\alpha}\left(8^{\rho}-1\right)^{\alpha}}{\rho^{\alpha}8^{\rho\alpha}\Gamma\left(\alpha+1\right)}\left[\frac{8^{\rho}\left(\alpha+1\right)+8^{\rho(\alpha-1)}\left(\alpha-1\right)\left(8^{\rho}-1\right)}{8^{\rho\alpha}\left(\alpha+1\right)}\right].\end{aligned}$$

Then

$$\frac{\rho^{\alpha}8^{\rho\alpha}\Gamma\left(\alpha+1\right)}{h\lambda^{\rho\alpha}\left(8^{\rho}-1\right)^{\alpha}}\left[\frac{8^{\rho\alpha}\left(\alpha+1\right)}{8^{\rho}\left(\alpha+1\right)+8^{\rho(\alpha-1)}\left(\alpha-1\right)\left(8^{\rho}-1\right)}\right]\int_{\frac{\lambda}{8}}^{\lambda}G\left(s,s\right)f\left(s,u\left(s\right)\right)ds\geq 1. \tag{3.20}$$

On the other hand, if $\max\limits_{0\leq\eta\leq\lambda}f\left(\eta,u\right)$ is bounded for $u\in\left[0,\infty\right)$, then there exists $L_0>0$, such that

$$\left|f\left(\eta,u\left(\eta\right)\right)\right|\leq L_0,\ \forall\eta\in\left[0,\lambda\right].$$

In a similar way, if $\max\limits_{0\leq\eta\leq\lambda}f\left(\eta,u\right)$ is unbounded for $u\in\left[0,\infty\right)$, then there exists $M_0>0$, such that

$$\sup_{0\leq u\leq M_0}\max_{0\leq\eta\leq\lambda}\left|f\left(\eta,u\left(\eta\right)\right)\right|\leq L_1,\ \text{for some } L_1>0.$$

In all cases, for $L=\max\left\{L_0,L_1\right\}$, we have:

$$\int_{0}^{\frac{\lambda}{8}}G\left(s,s\right)f\left(s,u\left(s\right)\right)ds\leq L\int_{0}^{\frac{\lambda}{8}}G\left(s,s\right)ds\leq\frac{L\lambda^{\rho\alpha}\left[8^{\rho\alpha}-\left(8^{\rho}-1\right)^{\alpha}\right]}{8^{\rho\alpha}\rho^{\alpha}\Gamma\left(\alpha+1\right)}.$$

From (3.20) , we get

$$
\begin{aligned}
\int_0^\lambda G(s,s) f(s,u(s))\, ds &= \int_{\frac{\lambda}{8}}^\lambda G(s,s) f(s,u(s))\, ds + \int_0^{\frac{\lambda}{8}} G(s,s) f(s,u(s))\, ds \\
&\leq \int_{\frac{\lambda}{8}}^\lambda G(s,s) f(s,u(s))\, ds + \frac{L\lambda^{\rho\alpha}\left[8^{\rho\alpha} - (8^\rho - 1)^\alpha\right]}{\rho^\alpha 8^{\rho\alpha}\Gamma(\alpha+1)} \\
&\leq \int_{\frac{\lambda}{8}}^\lambda G(s,s) f(s,u(s))\, ds + \frac{L\lambda^{\rho\alpha}\left[8^{\rho\alpha} - (8^\rho - 1)^\alpha\right]}{\rho^\alpha 8^{\rho\alpha}\Gamma(\alpha+1)} \times \\
&\quad \frac{\rho^\alpha 8^{\rho\alpha}\Gamma(\alpha+1)}{h\lambda^{\rho\alpha}(8^\rho - 1)^\alpha}\left[\frac{8^{\rho\alpha}(\alpha+1)}{8^\rho(\alpha+1) + 8^{\rho(\alpha-1)}(\alpha-1)(8^\rho-1)}\right] \times \\
&\quad \int_{\frac{\lambda}{8}}^\lambda G(s,s) f(s,u(s))\, ds.
\end{aligned}
$$

Thus

$$
\int_0^\lambda G(s,s) f(s,u(s))\, ds \leq \sigma \int_{\frac{\lambda}{8}}^\lambda G(s,s) f(s,u(s))\, ds.
$$

The proof is complete. ■

Let us define the cone P by:

$$
P = \left\{ u \in C[0,\lambda] \ \mid u(\eta) \geq \frac{b(\eta)}{\sigma}\|u\|_\infty, \ \forall \eta \in [0,\lambda] \right\}. \tag{3.21}
$$

Lemma 3.5 ([7]) *Let $\mathcal{A} : P \to C[0,\lambda]$ be an integral operator defined by:*

$$
\mathcal{A}u(\eta) = \beta \int_0^\lambda G(\eta,s) f(s,u(s))\, ds, \tag{3.22}
$$

equipped with the standard norm

$$
\|\mathcal{A}u\|_\infty = \max_{0\leq\eta\leq\lambda} |\mathcal{A}u(\eta)|.
$$

Then $\mathcal{A}(P) \subset P$.

Proof. For any $u \in P$, we have from (3.17) , (3.19) and (3.21) , that

$$
\begin{aligned}
\mathcal{A}u(\eta) &= \beta \int_0^\lambda G(\eta,s) f(s,u(s))\, ds \\
&\geq \beta b(\eta) \int_{\frac{\lambda}{8}}^\lambda G(s,s) f(s,u(s))\, ds \\
&\geq \frac{\beta b(\eta)}{\sigma} \int_0^\lambda G(s,s) f(s,u(s))\, ds \\
&\geq \frac{b(\eta)}{\sigma} \max_{0\leq\eta\leq\lambda}\left(\beta \int_0^\lambda G(\eta,s) f(s,u(s))\, ds\right) \\
&\geq \frac{b(\eta)}{\sigma}\|\mathcal{A}u\|_\infty, \ \forall \eta \in [0,\lambda].
\end{aligned}
$$

Thus $\mathcal{A}(P) \subset P$. The proof is complete. ■

Lemma 3.6 ([7]) *$\mathcal{A}: P \to P$ is a completely continuous operator.*

Proof. In view of continuity of $G(\eta, s)$ and $f(\eta, u)$, the operator $\mathcal{A}: P \to P$ is continuous.

Let $\Omega \subset P$ be bounded. Then there exists a positive constant $M > 0$, such that:

$$\|u\|_{\infty} \leq M, \ \forall u \in \Omega.$$

By choice

$$L = \sup_{0 \leq u \leq M} \max_{0 \leq \eta \leq \lambda} |f(\eta, u)| + 1.$$

In this case, we get $\forall u \in \Omega$,

$$\begin{aligned}
|\mathcal{A}u(\eta)| &= \left|\beta \int_0^{\lambda} G(\eta, s) f(s, u(s))\, ds\right| \\
&\leq \beta \int_0^{\lambda} |G(\eta, s) f(s, u(s))|\, ds \\
&\leq \beta L \int_0^{\lambda} G(s, s)\, ds \\
&\leq \frac{\beta L}{\rho^{\alpha-1}\Gamma(\alpha)} \int_0^{\lambda} s^{\rho-1} (\lambda^{\rho} - s^{\rho})^{\alpha-1}\, ds \\
&\leq \frac{\beta L \lambda^{\alpha\rho}}{\rho^{\alpha}\Gamma(\alpha+1)}.
\end{aligned}$$

Consequently, $|\mathcal{A}u(\eta)| \leq \frac{\beta L \lambda^{\alpha\rho}}{\rho^{\alpha}\Gamma(\alpha+1)}$, $\forall u \in \Omega$. Hence, $\mathcal{A}(\Omega)$ is bounded.

Now, for $1 < \alpha \leq 2$ and $0 < \rho \leq 1$, we give:

$$\delta(\varepsilon) = \left(\frac{\rho^{\alpha}\Gamma(\alpha)}{\lambda^{\rho}\beta L}\varepsilon\right)^{\frac{1}{\rho(\alpha-1)}}, \text{ for some } \varepsilon > 0.$$

Then $\forall u \in \Omega$, and $\eta_1, \eta_2 \in [0, \lambda]$, where $\eta_1 < \eta_2$, and $\eta_2 - \eta_1 < \delta$, we find $|\mathcal{A}u(\eta_2) - \mathcal{A}u(\eta_1)| < \varepsilon$.

Consequently, for $0 \leq s \leq \eta_1 < \eta_2 \leq \lambda$, we have:

$$\begin{aligned}
G(\eta_2, s) - G(\eta_1, s) &= \frac{\rho^{1-\alpha} s^{\rho-1}}{\Gamma(\alpha)} \left[\left[\eta_2^{\rho(\alpha-1)} - \eta_1^{\rho(\alpha-1)}\right] \left(\frac{\lambda^{\rho} - s^{\rho}}{\lambda^{\rho}}\right)^{\alpha-1} \right. \\
&\quad \left. - \left[(\eta_2^{\rho} - s^{\rho})^{\alpha-1} - (\eta_1^{\rho} - s^{\rho})^{\alpha-1}\right]\right] \\
&< \frac{\rho^{1-\alpha} s^{\rho-1}}{\Gamma(\alpha)} \left[\eta_2^{\rho(\alpha-1)} - \eta_1^{\rho(\alpha-1)}\right] \left(\frac{\lambda^{\rho} - s^{\rho}}{\lambda^{\rho}}\right)^{\alpha-1} \\
&< \frac{\rho^{1-\alpha} s^{\rho-1}}{\Gamma(\alpha)} \left[\eta_2^{\rho(\alpha-1)} - \eta_1^{\rho(\alpha-1)}\right].
\end{aligned}$$

In the same way, for $0 \leq \eta_1 \leq s < \eta_2 \leq \lambda$ or $0 \leq \eta_1 < \eta_2 \leq s \leq \lambda$, we have:

$$G(\eta_2, s) - G(\eta_1, s) < \frac{\rho^{1-\alpha} s^{\rho-1}}{\Gamma(\alpha)} \left[\eta_2^{\rho(\alpha-1)} - \eta_1^{\rho(\alpha-1)}\right].$$

Then

$$\begin{aligned}
|\mathcal{A}u(\eta_2) - \mathcal{A}u(\eta_1)| &= \left|\beta \int_0^\lambda [G(\eta_2, s) - G(\eta_1, s)] f(s, u(s))\, ds\right| \\
&\leq \beta L \int_0^\lambda |G(\eta_2, s) - G(\eta_1, s)|\, ds \\
&< \beta L \int_0^\lambda \frac{\rho^{1-\alpha} s^{\rho-1}}{\Gamma(\alpha)} \left[\eta_2^{\rho(\alpha-1)} - \eta_1^{\rho(\alpha-1)}\right] ds \\
&< \frac{\beta L \rho^{1-\alpha}}{\Gamma(\alpha)} \left[\eta_2^{\rho(\alpha-1)} - \eta_1^{\rho(\alpha-1)}\right] \left[\frac{1}{\rho} s^\rho\right]_0^\lambda.
\end{aligned}$$

Finally

$$|\mathcal{A}u(\eta_2) - \mathcal{A}u(\eta_1)| < \frac{\beta L \lambda^\rho}{\rho^\alpha \Gamma(\alpha)} \left[\eta_2^{\rho(\alpha-1)} - \eta_1^{\rho(\alpha-1)}\right]. \tag{3.23}$$

In the following, we divide the proof into three cases.

(a) If $\delta \leq \eta_1 < \eta_2 \leq \lambda$, we have:

$$\delta \leq \eta_1 < \eta_2 \Leftrightarrow \eta_2^{\rho(\alpha-2)} < \eta_1^{\rho(\alpha-2)} \leq \delta^{\rho(\alpha-2)}, \text{ and } \eta_2^{\rho-1} < \eta_1^{\rho-1} \leq \delta^{\rho-1}.$$

Thus

$$\eta_2^\rho - \eta_1^\rho = \eta_2 \eta_2^{\rho-1} - \eta_1 \eta_1^{\rho-1} < \eta_2 \eta_2^{\rho-1} - \eta_1 \eta_2^{\rho-1} = \eta_2^{\rho-1}(\eta_2 - \eta_1) < \delta^{\rho-1}(\eta_2 - \eta_1) < \delta^\rho.$$

Similarly

$$\begin{aligned}
\eta_2^{\rho(\alpha-1)} - \eta_1^{\rho(\alpha-1)} &= \eta_2^\rho \eta_2^{\rho(\alpha-2)} - \eta_1^\rho \eta_1^{\rho(\alpha-2)} < \eta_2^\rho \eta_2^{\rho(\alpha-2)} - \eta_1^\rho \eta_2^{\rho(\alpha-2)} = \eta_2^{\rho(\alpha-2)}(\eta_2^\rho - \eta_1^\rho) \\
&< \delta^{\rho(\alpha-2)}(\eta_2^\rho - \eta_1^\rho) \\
&< \delta^{\rho(\alpha-1)}.
\end{aligned}$$

Then, the inequality (3.23) gives:

$$\begin{aligned}
|\mathcal{A}u(\eta_2) - \mathcal{A}u(\eta_1)| &< \frac{\beta L \lambda^\rho}{\rho^\alpha \Gamma(\alpha)} \left[\eta_2^{\rho(\alpha-1)} - \eta_1^{\rho(\alpha-1)}\right] < \frac{\beta L \lambda^\rho}{\rho^\alpha \Gamma(\alpha)} \delta^{\rho(\alpha-1)} \\
&< \frac{\beta L \lambda^\rho}{\rho^\alpha \Gamma(\alpha)} \left[\left(\frac{\rho^\alpha \Gamma(\alpha)}{\lambda^\rho \beta L} \varepsilon\right)^{\frac{1}{\rho(\alpha-1)}}\right]^{\rho(\alpha-1)} \\
&< \varepsilon.
\end{aligned} \tag{3.24}$$

(b) If $\eta_1 \leq \delta < \eta_2 < 2\delta$, we have:

$$\eta_1 \leq \delta < \eta_2 \Leftrightarrow \eta_2^{\rho(\alpha-2)} < \delta^{\rho(\alpha-2)} \leq \eta_1^{\rho(\alpha-2)},$$

and

$$\eta_2^{\rho(\alpha-1)} - \eta_1^{\rho(\alpha-1)} = \eta_2^{\rho}\eta_2^{\rho(\alpha-2)} - \eta_1^{\rho}\eta_1^{\rho(\alpha-2)} < \eta_2^{\rho}\delta^{\rho(\alpha-2)} - \eta_1^{\rho}\delta^{\rho(\alpha-2)} < \delta^{\rho(\alpha-2)}\left(\eta_2^{\rho} - \eta_1^{\rho}\right) < \delta^{\rho(\alpha-1)}.$$

Also, we find the same result (3.24) .

(c) If $\eta_1 < \eta_2 \leq \delta$, we have:

$$\begin{aligned} \left|\mathcal{A}u\left(\eta_2\right) - \mathcal{A}u\left(\eta_1\right)\right| &< \frac{\beta L\lambda^{\rho}}{\rho^{\alpha}\Gamma\left(\alpha\right)}\left[\eta_2^{\rho(\alpha-1)} - \eta_1^{\rho(\alpha-1)}\right] \\ &< \frac{\beta L\lambda^{\rho}}{\rho^{\alpha}\Gamma\left(\alpha\right)}\eta_2^{\rho(\alpha-1)} \\ &< \frac{\beta L\lambda^{\rho}}{\rho^{\alpha}\Gamma\left(\alpha\right)}\delta^{\rho(\alpha-1)} \\ &< \varepsilon. \end{aligned}$$

By means of the Ascoli-Arzelà theorem 1.6, we have $\mathcal{A} : P \to P$ is completely continuous. The proof is complete. ■

We define some important constants

$$\begin{aligned} &F_0 = \lim_{u\to 0^+}\max_{\eta\in[0,\lambda]}\tfrac{f(\eta,u)}{u}, \quad F_\infty = \lim_{u\to +\infty}\max_{\eta\in[0,\lambda]}\tfrac{f(\eta,u)}{u}, \\ &f_0 = \lim_{u\to 0^+}\min_{\eta\in[0,\lambda]}\tfrac{f(\eta,u)}{u}, \quad f_\infty = \lim_{u\to +\infty}\min_{\eta\in[0,\lambda]}\tfrac{f(\eta,u)}{u}, \\ &\omega_1 = \textstyle\int_0^{\lambda} G\left(s,s\right)ds, \qquad \omega_2 = \tfrac{\bar{b}}{\sigma^2}\textstyle\int_0^{\lambda} G\left(s,s\right)b\left(s\right)ds. \end{aligned} \tag{3.25}$$

Assume that $\frac{1}{\omega_2 f_\infty} = 0$ if $f_\infty \to \infty$, $\frac{1}{\omega_1 F_0} = \infty$ if $F_0 \to 0$, $\frac{1}{\omega_2 f_0} = 0$ if $f_0 \to \infty$, and $\frac{1}{\omega_1 F_\infty} = \infty$ if $F_\infty \to 0$.

Theorem 3.1 **([7])** *If $\omega_2 f_\infty > \omega_1 F_0$ holds, then for each:*

$$\beta \in \left(\left(\omega_2 f_\infty\right)^{-1}, \left(\omega_1 F_0\right)^{-1}\right), \tag{3.26}$$

the boundary value problem (3.3)-(3.4) has at least one positive solution.

Proof. Let β satisfy (3.26) and $\varepsilon > 0$, be such that

$$\left(\left(f_\infty - \varepsilon\right)\omega_2\right)^{-1} \leq \beta \leq \left(\left(F_0 + \varepsilon\right)\omega_1\right)^{-1}. \tag{3.27}$$

From the definition of F_0, we see that there exists $r_1 > 0$, such that

$$f\left(\eta,u\right) \leq \left(F_0 + \varepsilon\right)u, \ \forall \eta \in \left[0,\lambda\right], \ 0 < u \leq r_1. \tag{3.28}$$

Consequently, for $u \in P$ with $\|u\|_\infty = r_1$, we have from (3.27), (3.28), that

$$\begin{aligned}
\|\mathcal{A}u\|_\infty &= \max_{0<\eta<\lambda} \left| \beta \int_0^\lambda G(\eta, s) f(s, u(s))\, ds \right| \\
&\leq \beta \int_0^\lambda G(s,s)(F_0 + \varepsilon) u(s)\, ds \\
&\leq \beta (F_0 + \varepsilon) \|u\|_\infty \int_0^\lambda G(s,s)\, ds \\
&\leq \beta (F_0 + \varepsilon) \|u\|_\infty \omega_1 \\
&\leq \|u\|_\infty .
\end{aligned}$$

Hence, if we choose

$$\Omega_1 = \{u \in C[0,\lambda] : \ \|u\|_\infty < r_1\},$$

then

$$\|\mathcal{A}u\|_\infty \leq \|u\|_\infty, \text{ for } u \in P \cap \partial\Omega_1. \tag{3.29}$$

By definition of f_∞, there exists $r_3 > 0$, such that

$$f(\eta, u) \geq (f_\infty - \varepsilon) u, \ \forall \eta \in [0,\lambda], \ u \geq r_3. \tag{3.30}$$

Therefore, for $u \in P$ with $\|u\|_\infty = r_2 = \max\{2r_1, r_3\}$, we have from (3.27), (3.30), that

$$\begin{aligned}
\|\mathcal{A}u\|_\infty &\geq \mathcal{A}u(\bar{\eta}) = \beta \int_0^\lambda G(\bar{\eta}, s) f(s, u(s))\, ds \\
&\geq \beta \int_{\frac{\lambda}{8}}^\lambda b(\bar{\eta}) G(s,s) f(s, u(s))\, ds \\
&\geq \frac{\beta \bar{b}}{\sigma} \int_0^\lambda G(s,s) f(s, u(s))\, ds \\
&\geq \frac{\beta \bar{b}}{\sigma} \int_0^\lambda G(s,s) [(f_\infty - \varepsilon) u(s)]\, ds.
\end{aligned}$$

By definition of P in (3.21), we have:

$$\begin{aligned}
\|\mathcal{A}u\|_\infty &\geq \frac{\beta \bar{b}(f_\infty - \varepsilon)}{\sigma^2} \|u\|_\infty \int_0^\lambda G(s,s) b(s)\, ds \\
&\geq \beta (f_\infty - \varepsilon) \|u\|_\infty \omega_2 \\
&\geq \|u\|_\infty .
\end{aligned}$$

If we set

$$\Omega_2 = \{u \in C[0,\lambda] : \|u\|_\infty < r_2\},$$

then

$$\|\mathcal{A}u\|_\infty \geq \|u\|_\infty, \text{ for } u \in P \cap \partial\Omega_2. \tag{3.31}$$

Now, from (3.29) , (3.31) , and lemma 1.7, we guarantee that $\mathcal{A}$ has a fixed point $u \in P \cap \left(\bar{\Omega}_2 \backslash \Omega_1\right)$ with $r_1 \leq \|u\|_\infty \leq r_2$. It is clear that u is a positive solution of (3.3)-(3.4) . The proof is complete. ■

Theorem 3.2 ([7]) *If $\omega_2 f_0 > \omega_1 F_\infty$ holds, then for each:*

$$\beta \in \left((\omega_2 f_0)^{-1}, (\omega_1 F_\infty)^{-1}\right), \tag{3.32}$$

the boundary value problem (3.3)-(3.4) has at least one positive solution.

Proof. Let β satisfy (3.32) and $\varepsilon > 0$, be such that

$$\left((f_0 - \varepsilon)\,\omega_2\right)^{-1} \leq \beta \leq \left((F_\infty + \varepsilon)\,\omega_1\right)^{-1}. \tag{3.33}$$

From definition of f_0, we see that there exists $r_1 > 0$, such that

$$f(\eta, u) \geq (f_0 - \varepsilon)\,u, \ \forall \eta \in [0, \lambda], \ 0 < u \leq r_1.$$

Further, if $u \in P$ with $\|u\|_\infty = r_1$, then similar to the second part of the proof of theorem 3.1, we can find that $\|\mathcal{A}u\|_\infty \geq \|u\|_\infty$. Then, if we choose

$$\Omega_1 = \{u \in C[0, \lambda] : \ \|u\|_\infty < r_1\},$$

thus

$$\|\mathcal{A}u\|_\infty \geq \|u\|_\infty, \text{ for } u \in P \cap \partial\Omega_1. \tag{3.34}$$

Next, and by definition of F_∞, we may choose $R_1 > 0$, such that

$$f(\eta, u) \leq (F_\infty + \varepsilon)\,u, \text{ for } u \geq R_1. \tag{3.35}$$

We consider two cases:

1) If $\max\limits_{0 \leq \eta \leq \lambda} f(\eta, u)$ is bounded for $u \in [0, \infty)$. Then, there exists some $L > 0$, such that

$$f(\eta, u) \leq L, \text{ for all } \eta \in [0, \lambda], \ u \in P.$$

Let us denote by $r_3 = \max\{2r_1, \beta L \omega_1\}$, if $u \in P$ with $\|u\|_\infty = r_3$, then

$$\begin{aligned} \|\mathcal{A}u\|_\infty &= \max_{0 \leq \eta \leq \lambda} \left|\beta \int_0^\lambda G(\eta, s)\, f(s, u(s))\, ds\right| \\ &\leq \beta L \int_0^\lambda G(s, s)\, ds = \beta L \omega_1 \\ &\leq r_3 = \|u\|_\infty. \end{aligned}$$

Hence,

$$\|\mathcal{A}u\|_\infty \leq \|u\|_\infty, \text{ for } u \in \partial P_{r_3} = \{u \in P : \|u\|_\infty \leq r_3\}. \tag{3.36}$$

2) If $\max\limits_{0\leq\eta\leq\lambda} f(\eta, u)$ is unbounded for $u \in [0, \infty)$, then there exists some $r_4 = \max\{2r_1, R_1\}$, such that

$$f(\eta, u) \leq \max_{0\leq\eta\leq\lambda} f(\eta, r_4), \text{ for all } 0 < u \leq r_4, \ \eta \in [0, \lambda].$$

Let $u \in P$ with $\|u\|_\infty = r_4$. Then, from (3.33), (3.35), we have:

$$\begin{aligned}
\|\mathcal{A}u\|_\infty &= \max_{0<\eta<\lambda} \left|\beta \int_0^\lambda G(\eta, s) f(s, u(s)) ds\right| \\
&\leq \beta \int_0^\lambda G(s, s)(F_\infty + \varepsilon) u(s) ds \\
&\leq \beta(F_\infty + \varepsilon)\|u\|_\infty \int_0^\lambda G(s, s) ds = \beta(F_\infty + \varepsilon)\|u\|_\infty \omega_1 \\
&\leq \|u\|_\infty.
\end{aligned}$$

Thus, (3.36) is also true for $u \in \partial P_{r_4}$.

In both cases 1 and 2, if we set

$$\Omega_2 = \{u \in C[0, \lambda] : \ \|u\|_\infty < r_2 = \max\{r_3, r_4\}\},$$

then

$$\|\mathcal{A}u\|_\infty \leq \|u\|_\infty, \text{ for } u \in P \cap \partial\Omega_2. \tag{3.37}$$

Now, from (3.34), (3.37), and lemma 1.7, we guarantee that $\mathcal{A}$ has a fixed point $u \in P \cap (\bar{\Omega}_2 \backslash \Omega_1)$ with $r_1 \leq \|u\|_\infty \leq r_2$. It is clear that u is a positive solution of (3.3)-(3.4). The proof is complete. ■

Theorem 3.3 ([7]) *Suppose there exists $r_2 > r_1 > 0$, such that*

$$\sup_{0\leq u\leq r_2} \max_{0\leq\eta\leq\lambda} f(\eta, u) \leq \frac{r_2}{\beta\omega_1}, \ \textit{and} \ \inf_{0\leq u\leq r_1} f(\eta, u) \geq \frac{r_1}{\beta\sigma\omega_2} b(\eta), \ \forall\eta \in [0, \lambda]. \tag{3.38}$$

Then, the boundary value problem (3.3)-(3.4) has a positive solution $u \in P$, with $r_1 \leq \|u\|_\infty \leq r_2$.

Proof. Choose

$$\Omega_1 = \{u \in C[0, \lambda] : \ \|u\|_\infty < r_1\}.$$

Then, for $u \in P \cap \partial\Omega_1$, we get

$$\begin{aligned} \|\mathcal{A}u\|_\infty &\geq \mathcal{A}u(\bar{\eta}) = \beta \int_0^\lambda G(\bar{\eta}, s) f(s, u(s))\, ds \\ &\geq \beta \int_{\frac{\lambda}{8}}^\lambda b(\bar{\eta}) G(s,s) f(s, u(s))\, ds \\ &\geq \frac{\beta \bar{b}}{\sigma} \int_0^\lambda G(s,s) \inf_{0 \leq u \leq r_1} f(s, u(s))\, ds \\ &\geq \frac{\beta \bar{b}}{\sigma} \int_0^\lambda G(s,s) \frac{r_1}{\beta \sigma \omega_2} b(s)\, ds \\ &\geq r_1 = \|u\|_\infty. \end{aligned}$$

On the other hand, we choose

$$\Omega_2 = \{u \in C[0,\lambda] : \ \|u\|_\infty < r_2\}.$$

Then, for $u \in P \cap \partial\Omega_2$, we get

$$\begin{aligned} \|\mathcal{A}u\|_\infty &= \max_{0<\eta<\lambda} \left| \beta \int_0^\lambda G(\eta, s) f(s, u(s))\, ds \right| \\ &\leq \beta \int_0^\lambda G(s,s) \sup_{0 \leq u \leq r_2} \max_{0 \leq \eta \leq \lambda} f(s, u(s))\, ds \\ &\leq \beta \int_0^\lambda G(s,s) \frac{r_2}{\beta \omega_1} ds = r_2 = \|u\|_\infty. \end{aligned}$$

Now, from lemma 1.7, we guarantee that $\mathcal{A}$ has a fixed point $u \in P \cap (\bar{\Omega}_2 \backslash \Omega_1)$ with $r_1 \leq \|u\|_\infty \leq r_2$. It is clear that u is a positive solution of (3.3)-(3.4). The proof is complete. ■

3.3.2 Existence and Uniqueness Results of Real Solution

In this section, we assume that $\beta \in \mathbb{R}$ and $\rho > 0$, and $f : [0,\lambda] \times \mathbb{R} \to \mathbb{R}$ satisfies the conditions:

(H1) $f(\eta, u)$ is Lebesgue measurable function with respect to η on $[0,\lambda]$,

(H2) $f(\eta, u)$ is a continuous function with respect to u on $\mathbb{R}$.

Theorem 3.4 ([7]) *Assume* (H1), (H2) *hold, and there exists a constant* $\xi > 0$, *such that*

$$|f(\eta, u) - f(\eta, v)| \leq \xi |u - v|, \text{ for almost every } \eta \in [0,\lambda], \text{ and all } u, v \in C[0,\lambda]. \quad (3.39)$$

If

$$|\beta| < \frac{\rho^\alpha \Gamma(\alpha+1)}{\xi \lambda^{\alpha\rho}}. \quad (3.40)$$

Then, there exists a unique solution of the boundary value problem (3.3)-(3.4) *on* $[0,\lambda]$.

Proof. Assume that $|\beta| < \frac{\rho^{\alpha}\Gamma(\alpha+1)}{\xi\lambda^{\alpha\rho}}$, and consider the operator $\mathcal{A} : C[0,\lambda] \to C[0,\lambda]$ defined by (3.22) as follows

$$\mathcal{A}u(\eta) = \beta \int_0^{\lambda} G(\eta, s) f(s, u(s))\, ds.$$

We shall show that $\mathcal{A}$ is a contraction mapping. In fact, for any $u, v \in C[0,\lambda]$, we have

$$\begin{aligned} |\mathcal{A}u(\eta) - \mathcal{A}v(\eta)| &= \left|\beta \int_0^{\lambda} G(\eta, s)[f(s, u(s)) - f(s, v(s))]\, ds\right| \\ &\le |\beta| \int_0^{\lambda} G(\eta, s) |f(s, u(s)) - f(s, v(s))|\, ds \\ &\le |\beta| \xi \int_0^{\lambda} G(s, s) |u(s) - v(s)|\, ds, \end{aligned}$$

then

$$\begin{aligned} \|\mathcal{A}u - \mathcal{A}v\|_{\infty} &\le |\beta| \xi \|u - v\|_{\infty} \int_0^{\lambda} G(s, s)\, ds \\ &\le \frac{|\beta| \xi \lambda^{\alpha\rho}}{\rho^{\alpha}\Gamma(\alpha+1)} \|u - v\|_{\infty}. \end{aligned} \tag{3.41}$$

This implies from (3.41) that $\mathcal{A}$ is a contraction operator. As a consequence of theorem 1.8, using BANACH's contraction principle [15], we deduce that $\mathcal{A}$ has a unique fixed point which is the unique solution of the problem (3.3)-(3.4) on $[0,\lambda]$. The proof is complete. ■

3.4 Examples

In this section, we present some examples to illustrate the applicability of our main results.

Example 1. Consider the following boundary value problem

$$\begin{cases} {}_{x}^{1}\mathcal{D}_{0+}^{\frac{3}{2}} w(x,t) + \beta \left(\varphi^{-\frac{3}{2}}(t) + x\varphi^{-\frac{5}{2}}(t)\right) w(x,t) \ln\left(1 + \frac{w(x,t)}{\psi(t)}\right) = 0, \ x \in (0, \varphi(t)). \\ w(0,t) = w(\varphi(t), t) = 0, \ t \in [0,T], \text{ for any } T > 0. \end{cases}$$

Set the function

$$f(x,t,w) = \left(\varphi^{-\frac{3}{2}}(t) + x\varphi^{-\frac{5}{2}}(t)\right) w(x,t) \ln\left(1 + \frac{w(x,t)}{\psi(t)}\right),$$

which satisfies the hypotheses of theorem 2.3. As

$${}_{x}^{1}\mathcal{D}_{0+}^{\frac{3}{2}} w(x,t) = \frac{\psi(t)}{\varphi^{\frac{3}{2}}(t)}\, {}^{1}\mathcal{D}_{0+}^{\frac{3}{2}} u(\eta),$$

and

$$\begin{aligned} f(x,t,w) &= \left(\varphi^{-\frac{3}{2}}(t) + x\varphi^{-\frac{5}{2}}(t)\right) w(x,t) \ln\left(1 + \frac{w(x,t)}{\psi(t)}\right) \\ &= \frac{\psi(t)}{\varphi^{\frac{3}{2}}(t)} (1+\eta) u(\eta) \ln(1 + u(\eta)). \end{aligned}$$

Then, the transformation (3.2) reduces the previous boundary value problem of fractional-order's partial differential equation to a boundary value problem of fractional differential equation of the form (see [7]):

$$\begin{cases} {}^{1}\mathcal{D}_{0^+}^{\frac{3}{2}} u(\eta) + \beta(1+\eta) u(\eta) \ln(1+u(\eta)) = 0, \ \eta \in [0,1]. \\ u(0) = u(1) = 0. \end{cases} \tag{3.42}$$

Set $\beta > 0$ any finite positive real number, and

$$f(\eta, u) = (1+\eta) u \ln(1+u).$$

In this case, the function f is jointly continuous for any $\eta \in [0,1]$, and any $u > 0$.

We get

$$F_0 = \lim_{u \to 0^+} \max_{\eta \in [0,1]} \tfrac{f(\eta,u)}{u} = 0^+, \quad f_\infty = \lim_{u \to +\infty} \min_{\eta \in [0,1]} \tfrac{f(\eta,u)}{u} = \infty.$$

On the other hand, we get

$$\omega_1 = \int_0^1 G(s,s)\, ds = \frac{1}{\Gamma\left(\frac{3}{2}\right)} \int_0^1 \sqrt{s(1-s)} ds = \frac{1}{\frac{1}{2}\sqrt{\pi}} \frac{\pi}{8} = \frac{\sqrt{\pi}}{4}, \tag{3.43}$$

and

$$b(\eta) = \begin{cases} \sqrt{\eta} & \text{for } \eta \in [0, \bar{\eta}], \\ \frac{1-\eta}{16} & \text{for } \eta \in [\bar{\eta}, 1]. \end{cases} \tag{3.44}$$

Then

$$\omega_2 = \frac{\bar{b}}{\sigma^2 \Gamma\left(\frac{3}{2}\right)} \left[\int_0^{\bar{\eta}} s\sqrt{(1-s)} ds + \frac{1}{16} \int_{\bar{\eta}}^1 \sqrt{s}\,(1-s)^{\frac{3}{2}}\, ds\right] \simeq \frac{\bar{b}\sqrt{\pi}}{128\sigma^2}. \tag{3.45}$$

Where $\bar{\eta} \simeq 0,003876\ldots$ and $\bar{b} \simeq 0,062258\ldots$ and the choice of σ depends directly on the choice of r_1, r_2 in (3.29), (3.31).

Because $\omega_1, \omega_2 > 0$, two finite constants for any choice of $0 < r_1 < r_2 < \infty$. We have always:

$$\frac{1}{\omega_2 f_\infty} = 0, \text{ and } \frac{1}{\omega_1 F_0} = \infty.$$

Then, the condition (3.26) is satisfied for any $0 < \beta < \infty$.

It follows from theorem 3.1 that the problem (3.42) has at least one solution.

Example 2. Consider

$$\begin{cases} {}_x^1\mathcal{D}_{0^+}^{\frac{3}{2}} w(x,t) + \beta\left(\varphi^{-\frac{3}{2}}(t) + x\varphi^{-\frac{5}{2}}(t)\right) w(x,t)\exp\left(\frac{\psi^3(t)-w^3(x,t)}{\psi^2(t)w(x,t)}\right) = 0, \ x\in(0,\varphi(t)). \\ w(0,t) = w(\varphi(t),t) = 0, \ t\in[0,T], \text{ for any } T>0. \end{cases}$$

Set the function

$$f(x,t,w) = \left(\varphi^{-\frac{3}{2}}(t) + x\varphi^{-\frac{5}{2}}(t)\right) w(x,t)\exp\left(\frac{\psi^3(t)-w^3(x,t)}{\psi^2(t)\,w(x,t)}\right),$$

which satisfies the hypotheses of theorem 2.3. As

$$\begin{aligned} f(x,t,w) &= \left(\varphi^{-\frac{3}{2}}(t) + x\varphi^{-\frac{5}{2}}(t)\right) w(x,t)\exp\left(\frac{\psi^3(t)-w^3(x,t)}{\psi^2(t)\,w(x,t)}\right) \\ &= \frac{\psi(t)}{\varphi^{\frac{3}{2}}(t)}(1+\eta)\,u(\eta)\exp\left(\frac{1}{u(\eta)} - [u(\eta)]^2\right). \end{aligned}$$

Then, the transformation (3.2) reduces the previous boundary value problem of fractional-order's partial differential equation to a boundary value problem of fractional differential equation of the form (see [7]):

$$\begin{cases} {}^1\mathcal{D}_{0^+}^{\frac{3}{2}} u(\eta) + \beta(1+\eta)\,u(\eta)\exp\left(\frac{1}{u(\eta)} - [u(\eta)]^2\right) = 0, \ \eta\in[0,1]. \\ u(0) = u(1) = 0. \end{cases} \tag{3.46}$$

Set $\beta > 0$ any finite positive real number, and

$$f(\eta,u) = (1+\eta)\,u\exp\left(\frac{1}{u} - u^2\right).$$

Clearly, for any $\eta\in[0,1]$ and any $u>0$, the function f is jointly continuous.

Here, we have:

$$f_0 = \lim_{u\to 0^+}\min_{\eta\in[0,1]}\frac{f(\eta,u)}{u} = \infty, \quad F_\infty = \lim_{u\to+\infty}\max_{\eta\in[0,1]}\frac{f(\eta,u)}{u} = 0^+.$$

Also, we find the same function $b(\eta)$ in (3.44), and the same constant ω_1, ω_2 respectively in (3.43), (3.45).

The choice of $\sigma > 1$ depends directly on the choice of r_1, r_2 in (3.34), (3.37).

Because $\omega_1, \omega_2 > 0$ are two finite constants for any choice of $0 < r_1 < r_2 < \infty$. We have always:

$$\frac{1}{\omega_2 f_0} = 0, \text{ and } \frac{1}{\omega_1 F_\infty} = \infty.$$

Then, the condition (3.32) is satisfied for any $0 < \beta < \infty$.

It follows from theorem 3.2 that the problem (3.46) has at least one solution.

Example 3. Consider the following boundary value problem

$$\begin{cases} {}_x^1\mathcal{D}_{0^+}^{\frac{3}{2}} w(x,t) + \frac{1}{\sqrt{\pi}} \varphi^{-\frac{5}{2}}(t)\left(\varphi(t)+x\right)\left(\psi(t)+w(x,t)\right) = 0, \ x \in (0,\varphi(t)). \\ w(0,t) = w(\varphi(t),t) = 0, \ t \in [0,T], \text{ for any } T > 0. \end{cases}$$

Set the function

$$f(x,t,w) = \varphi^{-\frac{5}{2}}(t)\left(\varphi(t)+x\right)\left(\psi(t)+w(x,t)\right),$$

which satisfies the hypotheses of theorem 2.3. As

$$\begin{aligned} f(x,t,w) &= \varphi^{-\frac{5}{2}}(t)\left(\varphi(t)+x\right)\left(\psi(t)+w(x,t)\right) \\ &= \frac{\psi(t)}{\varphi^{\frac{3}{2}}(t)}(1+\eta)(1+u(\eta)). \end{aligned}$$

Then, the transformation (3.2) reduces the previous boundary value problem of fractional-order's partial differential equation to a boundary value problem of fractional differential equation of the form (see [7]):

$$\begin{cases} {}^1\mathcal{D}_{0^+}^{\frac{3}{2}} u(\eta) + \frac{1}{\sqrt{\pi}}(1+\eta)(1+u(\eta)) = 0, \ \eta \in [0,1]. \\ u(0) = u(1) = 0. \end{cases} \tag{3.47}$$

Set $\beta = \frac{1}{\sqrt{\pi}}$, and

$$f(\eta,u) = (1+\eta)(1+u).$$

The function f is jointly continuous for any $\eta \in [0,1]$ and any $u > 0$.

We find the same function $b(\eta)$ in (3.44), such that $0 \leq b(\eta) < 1$, and

$$\omega_1 = \int_0^1 G(s,s)\,ds = \frac{\sqrt{\pi}}{4}.$$

Choosing $r_1 = \frac{1}{10^4} < r_2 = 2$. Then, for all $\eta \in [0,1]$, we have:

$$h = 1 \leq f(\eta,u) \leq 6 = L.$$

In this case

$$\begin{aligned} \sigma &= 1 + \frac{8^{\rho\alpha} L(\alpha+1)\left[8^{\rho\alpha} - (8^{\rho}-1)^{\alpha}\right]}{h(8^{\rho}-1)^{\alpha}\left[8^{\rho}(\alpha+1) + 8^{\rho(\alpha-1)}(\alpha-1)(8^{\rho}-1)\right]} \\ &= 1 + \frac{8^{\frac{3}{2}} \times 6 \times \frac{5}{2} \times \left(8^{\frac{3}{2}} - 7^{\frac{3}{2}}\right)}{7^{\frac{3}{2}} \times \left(8 \times \frac{5}{2} + \sqrt{8} \times \frac{7}{2}\right)} \\ &\simeq 3,517426\ldots \end{aligned}$$

Then

$$\omega_2 \simeq \frac{\bar{b}\sqrt{\pi}}{128\sigma^2} \simeq \frac{0,062258 \times \sqrt{\pi}}{128 \times 3,517426^2} \simeq \frac{3,9313\sqrt{\pi}}{10^5}.$$

It remains to show that the conditions in (3.38) , which is

$$\sup_{0\leq u\leq r_2}\max_{0\leq\eta\leq 1} f(\eta,u)=6\leq\frac{r_2}{\beta\omega_1}\simeq 8,$$

and

$$\inf_{0\leq u\leq r_1} f_3(\eta,u)=1+\eta\geq\frac{r_1}{\beta\sigma\omega_2}b(\eta)\simeq 0,72317\times b(\eta)\,,\ \forall\eta\in[0,1]\,.$$

Are satisfied. It follows from theorem 3.3 that the problem (3.47) has at least one solution.

Example 4. Let

$$\begin{cases} {}_x^{\frac{2}{3}}\mathcal{D}_{0+}^{\frac{3}{2}}w(x,t)+\frac{\psi(t)\cos\left(\frac{x}{\varphi(t)}\right)[2\psi(t)+|w(x,t)|]}{\pi\varphi(t)\left(\sqrt{2}\cos\left(\frac{x}{\varphi(t)}\right)+\sin\left(\frac{x}{\varphi(t)}\right)\right)[\psi(t)+|w(x,t)|]}=0,\ x\in\left(0,\frac{\pi}{4}\varphi(t)\right). \\ w(0,t)=w\left(\frac{\pi}{4}\varphi(t),t\right)=0,\ t\in[0,T]\,,\text{ for any } T>0. \end{cases}$$

Set the function

$$f(x,t,w)=\frac{\psi(t)\cos\left(\frac{x}{\varphi(t)}\right)[2\psi(t)+|w(x,t)|]}{\varphi(t)\left(\sqrt{2}\cos\left(\frac{x}{\varphi(t)}\right)+\sin\left(\frac{x}{\varphi(t)}\right)\right)[\psi(t)+|w(x,t)|]},$$

which satisfies the hypotheses of theorem 2.3. As

$${}_x^{\frac{2}{3}}\mathcal{D}_{0+}^{\frac{3}{2}}w(x,t)=\frac{\psi(t)}{\varphi(t)}\,{}^{\frac{2}{3}}\mathcal{D}_{0+}^{\frac{3}{2}}u(\eta)\,,$$

and

$$\begin{aligned} f(x,t,w) &= \frac{\psi(t)\cos\left(\frac{x}{\varphi(t)}\right)[2\psi(t)+|w(x,t)|]}{\varphi(t)\left(\sqrt{2}\cos\left(\frac{x}{\varphi(t)}\right)+\sin\left(\frac{x}{\varphi(t)}\right)\right)[\psi(t)+|w(x,t)|]} \\ &= \frac{\psi(t)}{\varphi(t)}\frac{\cos(\eta)[2+|u(\eta)|]}{\left(\sqrt{2}\cos(\eta)+\sin(\eta)\right)[1+|u(\eta)|]}. \end{aligned}$$

Then, the transformation (3.2) reduces the previous boundary value problem of fractional-order's partial differential equation to a boundary value problem of fractional differential equation of the form (see [7]):

$$\begin{cases} {}^{\frac{2}{3}}\mathcal{D}_{0+}^{\frac{3}{2}}u(\eta)+\frac{\cos(\eta)[2+|u(\eta)|]}{\pi\left(\sqrt{2}\cos(\eta)+\sin(\eta)\right)[1+|u(\eta)|]}=0,\ \eta\in\left[0,\frac{\pi}{4}\right], \\ u(0)=u\left(\frac{\pi}{4}\right)=0. \end{cases}\tag{3.48}$$

Set $\beta=\frac{1}{\pi}$ and

$$f(\eta,u)=\frac{\cos(\eta)[2+|u|]}{\left(\sqrt{2}\cos(\eta)+\sin(\eta)\right)[1+|u|]},\ \eta\in\left[0,\frac{\pi}{4}\right],\ u,v\in\mathbb{R}.$$

As $\sin(\eta)$, $\cos(\eta)$ are continuous positive functions $\forall \eta \in \left[0, \frac{\pi}{4}\right]$, the function f is jointly continuous. For any $u, v \in \mathbb{R}$ and $\eta \in \left[0, \frac{\pi}{4}\right]$, we have:

$$\frac{\sqrt{2}}{2} \leq \cos(\eta) \leq 1, \text{ and } 0 \leq \sin(\eta) \leq \frac{\sqrt{2}}{2},$$

then

$$\begin{aligned} |f(\eta, u) - f(\eta, v)| &= \left| \frac{\cos(\eta)\,[2 + |u|]}{\left(\sqrt{2}\cos(\eta) + \sin(\eta)\right)[1 + |u|]} - \frac{\cos(\eta)\,[2 + |v|]}{\left(\sqrt{2}\cos(\eta) + \sin(\eta)\right)[1 + |v|]} \right| \\ &= \left| \frac{\cos(\eta)}{\sqrt{2}\cos(\eta) + \sin(\eta)} \right| \left| \frac{2 + |u|}{1 + |u|} - \frac{2 + |v|}{1 + |v|} \right| \\ &\leq ||u| - |v|| \leq |u - v|. \end{aligned}$$

Hence, the condition (3.39) is satisfied with $\xi = 1$. It remains to show that the condition (3.40)

$$0 < \beta = \frac{1}{\pi} \simeq 0,318309\ldots < \frac{\rho^{\alpha}\Gamma(\alpha + 1)}{\xi \lambda^{\alpha\rho}} = \frac{\frac{2}{3}^{\frac{3}{2}} \times \Gamma\left(\frac{5}{2}\right)}{\frac{\pi}{4}} \simeq 0,921317\ldots$$

is satisfied. It follows from theorem 3.4 that the problem (3.48), has a unique solution.

3.5 Proof of Main Theorems

In this section, we prove the existence and uniqueness of solutions of the following problem of the nonlinear partial differential equations of space-fractional order [7], [12]:

$$ {}_{x}^{\rho}\mathcal{D}_{0+}^{\alpha} w + \beta f(x, t, w) = 0, \;\; \beta \in \mathbb{R}, \;\; 1 < \alpha \leq 2, \;\; (x, t) \in (0, X) \times [0, T], \tag{3.49}$$

with the boundary conditions:

$$w(0, t) = 0, \;\; w(X, t) = 0, \tag{3.50}$$

under the generalized self-similar form which is:

$$w(x, t) = \psi(t)\, u(\eta), \text{ with } \eta = \frac{x}{\varphi(t)}, \text{ and } \varphi, \psi \in C[0, T] \rightarrow \mathbb{R}_{+}. \tag{3.51}$$

The transformation (3.51) reduces the fractional-order's partial differential equation (3.49) to the ordinary differential equation of fractional order of the form:

$$ {}^{\rho}\mathcal{D}_{0+}^{\alpha} u(\eta) + \beta f(\eta, u(\eta)) = 0, \;\; 0 < \eta < \lambda, \tag{3.52}$$

supplemented with the boundary conditions:

$$u(0) = 0, \;\; u(\lambda) = 0, \tag{3.53}$$

where $\lambda = X\bar{\varphi}^{-1}$, with $\bar{\varphi} = \min_{0 \leq t \leq T} \varphi(t)$.

Now we proceed to demonstrate the theorems 2.4 and 2.5.

Proof of theorem 2.4:

Let $f:[0,X]\times[0,T]\times\mathbb{R}_{+}\to[h,\infty)$ be a continuous function, with $h>0$. By using (3.51), the functions

$$f_0^*(t)=\varphi^{\rho\alpha}(t)\lim_{w\to0^+}\min_{0\le x\le X}\frac{f(x,t,w)}{w},\quad f_\infty^*(t)=\varphi^{\rho\alpha}(t)\lim_{w\to+\infty}\min_{0\le x\le X}\frac{f(x,t,w)}{w},$$
$$F_0^*(t)=\varphi^{\rho\alpha}(t)\lim_{w\to0^+}\max_{0\le x\le X}\frac{f(x,t,w)}{w},\quad F_\infty^*(t)=\varphi^{\rho\alpha}(t)\lim_{w\to+\infty}\max_{0\le x\le X}\frac{f(x,t,w)}{w},$$

are equivalent to

$$f_0=\lim_{u\to0^+}\min_{0\le\eta\le\lambda}\frac{f(\eta,u)}{u},\quad f_\infty=\lim_{u\to+\infty}\min_{0\le\eta\le\lambda}\frac{f(\eta,u)}{u},$$
$$F_0=\lim_{u\to0^+}\max_{0\le\eta\le\lambda}\frac{f(\eta,u)}{u},\quad F_\infty=\lim_{u\to+\infty}\max_{0\le\eta\le\lambda}\frac{f(\eta,u)}{u},$$

which are given in (3.25).

1) We already proved in theorem 3.1 the existence of positive solutions of the boundary value problem (3.52)-(3.53) provided that $\omega_2 f_\infty>\omega_1F_0$ holds, and $\beta\in\left((\omega_2f_\infty)^{-1},(\omega_1F_0)^{-1}\right)$. Consequently, if $\omega_2f_\infty^*(t)>\omega_1F_0^*(t)$ holds for any $t\in[0,T]$, then for each:

$$\beta\in\left((\omega_2f_\infty^*(t))^{-1},(\omega_1F_0^*(t))^{-1}\right),$$

the boundary value problem (3.49)-(3.50) has at least one positive solution under the generalized self-similar form (3.51).

2) In theorem 3.2, we have proved the existence of positive solutions of the boundary value problem (3.52)-(3.53) provided that $\omega_2f_0>\omega_1F_\infty$ holds, and $\beta\in\left((\omega_2f_0)^{-1},(\omega_1F_\infty)^{-1}\right)$. Consequently, if $\omega_2f_0^*(t)>\omega_1F_\infty^*(t)$ holds for any $t\in[0,T]$, then for each:

$$\beta\in\left((\omega_2f_0^*(t))^{-1},(\omega_1F_\infty^*(t))^{-1}\right),$$

the boundary value problem (3.49)-(3.50) has at least one positive solution under the generalized self-similar form (3.51).

3) Suppose there exists $r_2>r_1>0$. By using (3.51), the conditions

$$\sup_{0\le w\le\psi(t)r_2}\max_{0\le x\le X}f(x,t,w)\le\frac{r_2\psi(t)}{\beta\omega_1\varphi^{\rho\alpha}(t)},\ \forall t\in[0,T],$$

and

$$\inf_{0\le w\le\psi(t)r_1}f(x,t,w)\ge\frac{r_1\psi(t)}{\beta\sigma\omega_2\varphi^{\rho\alpha}(t)}b\left(\frac{x}{\varphi(t)}\right),\ \forall(x,t)\in[0,X]\times[0,T],$$

are equivalent to

$$\sup_{0\le u\le r_2}\max_{0\le\eta\le\lambda}f(\eta,u)\le\frac{r_2}{\beta\omega_1},\ \text{and}\ \inf_{0\le u\le r_1}f(\eta,u)\ge\frac{r_1}{\beta\sigma\omega_2}b(\eta),\ \forall\eta\in[0,\lambda],\tag{3.54}$$

respectively.

We already proved in theorem 3.3 the existence of positive solutions of the boundary value problem (3.52)-(3.53) provided that (3.54) holds. Then, the boundary value problem (3.49)-(3.50) has at least one positive solution under the generalized self-similar form (3.51), with $r_1 \leq \frac{\|w(\cdot,t)\|_\infty}{\psi(t)} \leq r_2$.

The proof of theorem 2.4 is complete.

Proof of theorem 2.5:

In this part, we assume that $\beta \in \mathbb{R}$ and $\rho > 0$, and $f : [0, X] \times [0, T] \times \mathbb{R} \to \mathbb{R}$.

Let $f(x, t, w)$ be a continuous function with respect to w on $\mathbb{R}$, and LEBESGUE measurable function with respect to x on $[0, X]$, $\forall t \in [0, T]$. By using (3.51), the conditions

$$|f(x, t, w_1) - f(x, t, w_2)| \leq \frac{\xi}{\varphi^{\rho\alpha}(t)} |w_1 - w_2|, \text{ for any } (x, t) \in [0, X] \times [0, T],$$

and

$$|\beta| < \frac{(\rho\bar{\varphi}^\rho)^\alpha \Gamma(\alpha + 1)}{\xi X^{\alpha\rho}},$$

are equivalent to (3.39) and (3.40), which is

$$|f(\eta, u) - f(\eta, v)| \leq \xi |u - v|, \text{ for almost every } \eta \in [0, \lambda],$$

and

$$|\beta| < \frac{\rho^\alpha \Gamma(\alpha + 1)}{\xi \lambda^{\alpha\rho}},$$

respectively.

We already proved in theorem 3.4, the existence and uniqueness of a solution of the boundary value problem (3.52)-(3.53) provided that (3.39), (3.40) holds true. Then, there exists a unique solution of the boundary value problem (3.49)-(3.50) under the generalized self-similar form (3.51).

The proof of theorem 2.5 is complete.

Chapter 4

Existence of Solutions to a Nonlinear Implicit Initial Value Problem

This chapter has been published in: Jornal of Applied Mathematics E-Notes **19** (2019), (see [8]).

4.1 Introduction

This chapter studies the existence and uniqueness of generalized self-similar solutions for a class of nonlinear implicit FPDE with an initial condition. The arguments for the study are based upon BANACH's contraction principle (theorem 1.8), SCHAUDER's fixed point theorem (theorem 1.9), and the nonlinear alternative of LERAY-SCHAUDER type (theorem 1.10). The used differential operator is developed by KATUGAMPOLA. For application purposes, some examples are provided to demonstrate the applicability of our main results.

We study the existence and uniqueness of solutions of the following implicit problem of the nonlinear FPDE using KATUGAMPOLA's fractional derivative [8], [34]:

$$ {}_{x}^{\rho}\mathcal{D}_{0^+}^{\alpha} w = f\left(x,t,w,{}_{x}^{\rho}\mathcal{D}_{0^+}^{\alpha} w\right),\ 0<\alpha\leq 1,\ \ (x,t)\in(0,X]\times[0,T], \tag{4.1}$$

with the initial condition:

$$w\left(0,t\right)=0,$$

under the generalized self-similar form which is:

$$w\left(x,t\right)=\psi\left(t\right)u\left(\eta\right),\ \text{with } \eta=\frac{x}{\varphi\left(t\right)},\ \text{and } \varphi,\psi\in C\left[0,T\right]\rightarrow\mathbb{R}_{+}, \tag{4.2}$$

where $f : [0, X] \times [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, with $X, T \in \mathbb{R}_+$, is a given function which satisfies the hypotheses of theorem 2.3.

Then the transformation (4.2) reduces the fractional-order's partial differential equation (4.1) to the ordinary differential equation of fractional order of the form:

$$^{\rho}\mathcal{D}^{\alpha}_{0+}u(\eta) = f\left(\eta, u(\eta), {}^{\rho}\mathcal{D}^{\alpha}_{0+}u(\eta)\right), \ \eta \in (0, \lambda], \tag{4.3}$$

with the initial condition:

$$u(0) = 0. \tag{4.4}$$

Where $\lambda = X\bar{\varphi}^{-1}$, with $\bar{\varphi} = \min_{0 \leq t \leq T} \varphi(t)$ is a finite positive constant, and $f : [0, \lambda] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

According to chapter 2 (subsection 2.3.2), we discuss in a general manner the existence and uniqueness of solutions of nonlinear implicit FDEs (4.3) , with the initial condition (4.4) .

We obtain several existence and uniqueness results for the problem (4.3)-(4.4) .

4.2 Definition of Integral Solution

In the sequel, λ, p, n and c are real constants such that

$$p \geq 1, \ c > 0, \ n = [\alpha] + 1, \ \text{and } 0 < \lambda \leq (pc)^{\frac{1}{pc}}.$$

In what follows, we present some significant lemmas to show the principal theorems, we have:

Lemma 4.1 ([8]) *Let $\alpha, \rho > 0$. If $u \in C[0, \lambda]$, then:*

(i) *The fractional deferential equation $^{\rho}\mathcal{D}^{\alpha}_{0+}u(\eta) = 0$, has a unique solutions:*

$$u(\eta) = C_1\eta^{\rho(\alpha-1)} + C_2\eta^{\rho(\alpha-2)} + \cdots + C_n\eta^{\rho(\alpha-n)}, \ \textit{where } C_m \in \mathbb{R}, \ \textit{with } m = 1, 2, \ldots, n.$$

(ii) *If $^{\rho}\mathcal{D}^{\alpha}_{0+}u \in C[0, \lambda]$ and $0 < \alpha \leq 1$, then:*

$$^{\rho}\mathcal{I}^{\alpha}_{0+} \, {}^{\rho}\mathcal{D}^{\alpha}_{0+} \, u(\eta) = u(\eta) + C\eta^{\rho(\alpha-1)}, \ \textit{for some constant } C \in \mathbb{R}. \tag{4.5}$$

Proof. **(i)** Let $\alpha, \rho > 0$, from remark 1.2, we have:

$$^{\rho}\mathcal{D}^{\alpha}_{0+}\eta^{\rho(\alpha-m)} = 0, \ \text{for each } m = 1, 2, \ldots, n.$$

Then the fractional equation $^{\rho}\mathcal{D}^{\alpha}_{0+}u(\eta) = 0$, has a particular solution, as follows:

$$u(\eta) = C_m\eta^{\rho(\alpha-m)}, \ C_m \in \mathbb{R}, \ \text{for each } m = 1, 2, \ldots, n. \tag{4.6}$$

Thus, the given general solution of ${}^{\rho}\mathcal{D}^{\alpha}_{0+}u(\eta) = 0$ is a sum of particular solutions (4.6) , i.e.

$$u(\eta) = C_1\eta^{\rho(\alpha-1)} + C_2\eta^{\rho(\alpha-2)} + \cdots + C_n\eta^{\rho(\alpha-n)},\ C_m \in \mathbb{R}\ \ (m = 1, 2, \ldots, n).$$

(ii) Let ${}^{\rho}\mathcal{D}^{\alpha}_{0+}u \in C[0,\lambda]$ be the fractional derivatives (1.22) of order $0 < \alpha \leq 1$.
If we apply the operator ${}^{\rho}\mathcal{D}^{\alpha}_{0+}$ to ${}^{\rho}\mathcal{I}^{\alpha}_{0+}\ {}^{\rho}\mathcal{D}^{\alpha}_{0+}u(\eta) - u(\eta)$, and use the properties (1.25) , (1.26) we have:

$$\begin{aligned} {}^{\rho}\mathcal{D}^{\alpha}_{0+}\left[{}^{\rho}\mathcal{I}^{\alpha}_{0+}\ {}^{\rho}\mathcal{D}^{\alpha}_{0+}u(\eta) - u(\eta)\right] &= {}^{\rho}\mathcal{D}^{\alpha}_{0+}\ {}^{\rho}\mathcal{I}^{\alpha}_{0+}\ {}^{\rho}\mathcal{D}^{\alpha}_{0+}u(\eta) - {}^{\rho}\mathcal{D}^{\alpha}_{0+}u(\eta) \\ &= {}^{\rho}\mathcal{D}^{\alpha}_{0+}u(\eta) - {}^{\rho}\mathcal{D}^{\alpha}_{0+}u(\eta) \\ &= 0. \end{aligned}$$

After the step **(i)** we deduce there exists $C \in \mathbb{R}$, such that:

$${}^{\rho}\mathcal{I}^{\alpha}_{0+}\ {}^{\rho}\mathcal{D}^{\alpha}_{0+}u(\eta) - u(\eta) = C\eta^{\rho(\alpha-1)},$$

which implies the law of composition (4.5) . The proof is complete. ■

Based on the previous lemma, we will define the integral solution of the problem (4.3)-(4.4) .

Lemma 4.2 ([8]) *Let $\alpha, \rho \in \mathbb{R}$, be such that $0 < \alpha \leq 1$, and $\rho > 0$. We give $u, {}^{\rho}\mathcal{D}^{\alpha}_{0+}u \in C[0,\lambda]$, and $f(\eta, u, v)$ is a continuous function. Then the problem (4.3)-(4.4) is equivalent to the fractional integral equation:*

$$u(\eta) = \int_0^{\eta} G(\eta, s)\, f\left(s, u(s), {}^{\rho}\mathcal{D}^{\alpha}_{0+}u(s)\right) ds,$$

where

$$G(\eta, s) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} s^{\rho-1}\left(\eta^{\rho} - s^{\rho}\right)^{\alpha-1}. \tag{4.7}$$

Proof. Let $0 < \alpha \leq 1$ and $\rho > 0$, we may apply lemma 4.1 to reduce the fractional equation (4.3) to an equivalent fractional integral equation.
By applying ${}^{\rho}\mathcal{I}^{\alpha}_{0+}$ to equation (4.3) we obtain:

$${}^{\rho}\mathcal{I}^{\alpha}_{0+}\ {}^{\rho}\mathcal{D}^{\alpha}_{0+}u(\eta) = {}^{\rho}\mathcal{I}^{\alpha}_{0+} f\left(\eta, u(\eta), {}^{\rho}\mathcal{D}^{\alpha}_{0+}u(\eta)\right). \tag{4.8}$$

From lemma 4.1, we find easily:

$${}^{\rho}\mathcal{I}^{\alpha}_{0+}\ {}^{\rho}\mathcal{D}^{\alpha}_{0+}u(\eta) = u(\eta) + C\eta^{\rho(\alpha-1)},$$

for some $C \in \mathbb{R}$. Then, the fractional integral equation (4.8), gives:

$$u(\eta) = {}^{\rho}\mathcal{I}_{0+}^{\alpha} f\left(\eta, u(\eta), {}^{\rho}\mathcal{D}_{0+}^{\alpha} u(\eta)\right) - C\eta^{\rho(\alpha-1)}. \tag{4.9}$$

If we use the condition (4.4) in equation (4.9) we find:

$$u(0) = 0 = -C \lim_{\eta \to 0^+} \eta^{\rho(\alpha-1)} \Rightarrow C = 0.$$

Therefore, the problem (4.3)-(4.4) is equivalent to:

$$u(\eta) = \int_0^{\eta} G(\eta, s) f\left(s, u(s), {}^{\rho}\mathcal{D}_{0+}^{\alpha} u(s)\right) ds,$$

where $G(\eta, s)$, which is given by the equality (4.7). The proof is complete. ■

4.3 Some Explicit Solutions

Example 1: (Simple explicit solution, see [8])

Let $\alpha, \rho > 0$, and $\mu > 0$. Then, the problem (4.3)-(4.4) for

$$f\left(\eta, u(\eta), {}^{\rho}\mathcal{D}_{0+}^{\alpha} u(\eta)\right) = \frac{2\rho^{\alpha-1}\Gamma\left(1+\frac{\mu}{\rho}\right)}{\Gamma\left(1-\alpha+\frac{\mu}{\rho}\right)}\eta^{-\alpha\rho}u(\eta) - {}^{\rho}\mathcal{D}_{0+}^{\alpha} u(\eta),$$

has an explicit solution on $[0, \lambda]$, which is given by:

$$u(\eta) = C\eta^{\mu}, \text{ for some } C \in \mathbb{R}. \tag{4.10}$$

In fact, the solution (4.10) satisfies the condition (4.4), which is

$$u(0) = 0.$$

Also, from remark 1.2 and equation (1.23), we have:

$$\begin{aligned}
f\left(\eta, u(\eta), {}^{\rho}\mathcal{D}_{0+}^{\alpha} u(\eta)\right) &= \frac{2\rho^{\alpha-1}\Gamma\left(1+\frac{\mu}{\rho}\right)}{\Gamma\left(1-\alpha+\frac{\mu}{\rho}\right)}\eta^{-\alpha\rho}u(\eta) - {}^{\rho}\mathcal{D}_{0+}^{\alpha} u(\eta) \\
&= \frac{2\rho^{\alpha-1}\Gamma\left(1+\frac{\mu}{\rho}\right)}{\Gamma\left(1-\alpha+\frac{\mu}{\rho}\right)}\eta^{-\alpha\rho}\left[C\eta^{\mu}\right] - \frac{C\rho^{\alpha-1}\Gamma\left(1+\frac{\mu}{\rho}\right)}{\Gamma\left(1-\alpha+\frac{\mu}{\rho}\right)}\eta^{\mu-\alpha\rho} \\
&= \frac{C\rho^{\alpha-1}\Gamma\left(1+\frac{\mu}{\rho}\right)}{\Gamma\left(1-\alpha+\frac{\mu}{\rho}\right)}\eta^{\mu-\alpha\rho} \\
&= {}^{\rho}\mathcal{D}_{0+}^{\alpha}\left[C\eta^{\mu}\right] \\
&= {}^{\rho}\mathcal{D}_{0+}^{\alpha} u(\eta).
\end{aligned}$$

Example 2: (Explicit solution for $\alpha = \rho = 1,$ see [8])

Let $\alpha, \rho > 0$. Then, the problem (4.3)-(4.4) for

$$f\left(\eta, u\left(\eta\right), {}^{\rho}\mathcal{D}_{0+}^{\alpha}u\left(\eta\right)\right) = \frac{3}{2\eta}\left(e^{\eta} - u\left(\eta\right) - 1\right) - \frac{1}{2}\, {}^{\rho}\mathcal{D}_{0+}^{\alpha}u\left(\eta\right),$$

has an explicit solution on $[0, \lambda]$, which is given by:

$$u\left(\eta\right) = E_{1,2}\left(\eta\right) - 1, \tag{4.11}$$

where $E_{\sigma,\beta}\left(\eta\right)$ presented in (1.8), is the MITTAG-LEFFLER function for $\sigma = 1,\ \beta = 2.$

In fact, after using L'HÔSPITAL's rule, the solution (4.11) satisfies the condition (4.4), which is

$$u\left(0\right) = \lim_{\eta\to 0}\left[E_{1,2}\left(\eta\right) - 1\right] = \lim_{\eta\to 0}\frac{e^{\eta} - \eta - 1}{\eta} = \lim_{\eta\to 0}\left[e^{\eta} - 1\right] = 0.$$

Also, from theorem 1.5, equation (1.29), we have:

$$\begin{aligned}\mathcal{M}\left[f\left(\eta, u\left(\eta\right), {}^{\rho}\mathcal{D}_{0+}^{\alpha}u\left(\eta\right)\right)\right](s) &= \mathcal{M}\left[\frac{3}{2\eta}\left(e^{\eta} - u\left(\eta\right) - 1\right)\right](s) - \frac{1}{2}\mathcal{M}\left[{}^{\rho}\mathcal{D}_{0+}^{\alpha}u\left(\eta\right)\right](s)\\ &= \frac{3}{2}\mathcal{M}\left[\frac{e^{\eta} - u\left(\eta\right) - 1}{\eta}\right](s) - \frac{\rho^{\alpha}\Gamma\left(1 - \frac{s}{\rho} + \alpha\right)}{2\Gamma\left(1 - \frac{s}{\rho}\right)}\left(\mathcal{M}u\right)\left(s - \rho\alpha\right).\end{aligned}$$

If we put $u\left(\eta\right) = E_{1,2}\left(\eta\right) - 1$, we have

$$\begin{aligned}\frac{e^{\eta} - u\left(\eta\right) - 1}{\eta} &= \frac{e^{\eta} - E_{1,2}\left(\eta\right)}{\eta}\\ &= \frac{e^{\eta} - \frac{e^{\eta}-1}{\eta}}{\eta}\\ &= \frac{\eta e^{\eta} - e^{\eta} + 1}{\eta^2}\\ &= \frac{d}{d\eta}\left(E_{1,2}\left(\eta\right) - 1\right)\\ &= \frac{d}{d\eta}u\left(\eta\right).\end{aligned}$$

Then

$$\mathcal{M}\left[f\left(\eta, u\left(\eta\right), {}^{\rho}\mathcal{D}_{0+}^{\alpha}u\left(\eta\right)\right)\right](s) = \frac{3}{2}\mathcal{M}\left[\frac{d}{d\eta}u\left(\eta\right)\right](s) - \frac{\rho^{\alpha}\Gamma\left(1 - \frac{s}{\rho} + \alpha\right)}{2\Gamma\left(1 - \frac{s}{\rho}\right)}\left(\mathcal{M}u\right)\left(s - \rho\alpha\right)$$

After using remark 1.5 and equation (1.30), we have:

$$\mathcal{M}\left[f\left(\eta, u\left(\eta\right), {}^{\rho}\mathcal{D}_{0+}^{\alpha}u\left(\eta\right)\right)\right](s) = \frac{3\Gamma\left(2 - s\right)}{2\Gamma\left(1 - s\right)}\left(\mathcal{M}u\right)\left(s - 1\right) - \frac{\rho^{\alpha}\Gamma\left(1 - \frac{s}{\rho} + \alpha\right)}{2\Gamma\left(1 - \frac{s}{\rho}\right)}\left(\mathcal{M}u\right)\left(s - \rho\alpha\right).$$

If $\alpha = \rho = 1$, we have:

$$\mathcal{M}\left[f\left(\eta, u(\eta), {}^{1}\mathcal{D}_{0^+}^{1} u(\eta)\right)\right](s) = \frac{\Gamma(2-s)}{\Gamma(1-s)}(\mathcal{M}u)(s-1) = \mathcal{M}\left[{}^{1}\mathcal{D}_{0^+}^{1} u(\eta)\right](s).$$

If we apply the inverse of MELLIN transform, we get easily that the solution u presented in (4.11) is a solution for the problem (4.3)-(4.4).

4.4 Existence and Uniqueness Results

Now, we will prove our first existence result for the problem (4.3)-(4.4) which is based on BANACH's fixed point theorem.

We suggest the following hypotheses:

(H1) $f : [0, \eta] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

(H2) There exist two constants $\sigma, \beta > 0$, where $\beta < 1$ such that:

$$|f(\eta, u, v) - f(\eta, \tilde{u}, \tilde{v})| \leq \sigma |u - \tilde{u}| + \beta |v - \tilde{v}|,$$

for any $u, v, \tilde{u}, \tilde{v} \in \mathbb{R}$ and $\eta \in [0, \lambda]$.

(H3) There exists three positive functions $a, b, c \in C[0, \lambda]$ such that:

$$|f(\eta, u, v)| \leq a(\eta) + b(\eta)|u| + c(\eta)|v| \text{ for all } \eta \in [0, \lambda] \text{ and } u, v \in \mathbb{R}.$$

We denote:

$$M_0 = \frac{a^*}{1 - c^*}, \text{ and } M_1 = \frac{b^*}{1 - c^*},$$

where

$$a^* = \sup_{\eta \in [0,\lambda]} a(\eta), \; b^* = \sup_{\eta \in [0,\lambda]} b(\eta), \; c^* = \sup_{\eta \in [0,\lambda]} c(\eta), \text{ with } c^* < 1.$$

In what follows, we present the principal theorems:

Theorem 4.1 ([8]) *Assume the hypotheses* (H1), (H2) *hold. We give* $0 < \alpha \leq 1$, *and* $\rho > 0$. *If*

$$\frac{\sigma \lambda^{\rho\alpha}}{(1-\beta)\rho^{\alpha}\Gamma(\alpha+1)} < 1. \tag{4.12}$$

Then the problem (4.3)-(4.4) *admits a unique solution on* $[0, \lambda]$.

Proof. To begin the proof, we will transform the problem (4.3)-(4.4) into a fixed point problem. Define the operator $\mathcal{A} : C[0, \lambda] \to C[0, \lambda]$ by:

$$\mathcal{A}u(\eta) = \int_0^{\eta} G(\eta, s) f\left(s, u(s), {}^{\rho}\mathcal{D}_{0^+}^{\alpha} u(s)\right) ds. \tag{4.13}$$

Because the problem (4.3)-(4.4) is equivalent to the fractional integral equation (4.13) , the fixed points of $\mathcal{A}$ are solutions of the problem (4.3)-(4.4) .
Let $u, v \in C[0,\lambda]$, be such that:

$$^{\rho}\mathcal{D}^{\alpha}_{0+}u(\eta) = f(\eta, u(\eta), {}^{\rho}\mathcal{D}^{\alpha}_{0+}u(\eta)), \quad {}^{\rho}\mathcal{D}^{\alpha}_{0+}v(\eta) = f(\eta, v(\eta), {}^{\rho}\mathcal{D}^{\alpha}_{0+}v(\eta)).$$

Which implies that:

$$\mathcal{A}u(\eta) - \mathcal{A}v(\eta) = \int_0^{\eta} G(\eta,s)\left[f(s,u(s), {}^{\rho}\mathcal{D}^{\alpha}_{0+}u(s)) - f(s,v(s), {}^{\rho}\mathcal{D}^{\alpha}_{0+}v(s))\right] ds.$$

Then, for all $\eta \in [0,\lambda]$

$$|\mathcal{A}u(\eta) - \mathcal{A}v(\eta)| \leq \int_0^{\eta} G(\eta,s)\left|{}^{\rho}\mathcal{D}^{\alpha}_{0+}u(s) - {}^{\rho}\mathcal{D}^{\alpha}_{0+}v(s)\right| ds. \tag{4.14}$$

By (H2) we have:

$$\begin{aligned} \left|{}^{\rho}\mathcal{D}^{\alpha}_{0+}u(\eta) - {}^{\rho}\mathcal{D}^{\alpha}_{0+}v(\eta)\right| &= \left|f(\eta,u(\eta), {}^{\rho}\mathcal{D}^{\alpha}_{0+}u(\eta)) - f(\eta,v(\eta), {}^{\rho}\mathcal{D}^{\alpha}_{0+}v(\eta))\right| \\ &\leq \sigma|u(\eta)-v(\eta)| + \beta\left|{}^{\rho}\mathcal{D}^{\alpha}_{0+}u(\eta) - {}^{\rho}\mathcal{D}^{\alpha}_{0+}v(\eta)\right|. \end{aligned}$$

Thus

$$\left|{}^{\rho}\mathcal{D}^{\alpha}_{0+}u(\eta) - {}^{\rho}\mathcal{D}^{\alpha}_{0+}v(\eta)\right| \leq \frac{\sigma}{1-\beta}|u(\eta)-v(\eta)|.$$

From (4.14) we have:

$$|\mathcal{A}u(\eta) - \mathcal{A}v(\eta)| \leq \frac{\sigma}{1-\beta}\int_0^{\eta} G(\eta,s)|u(s)-v(s)| ds.$$

Then:

$$\|\mathcal{A}u - \mathcal{A}v\|_{\infty} \leq \frac{\sigma\lambda^{\rho\alpha}}{(1-\beta)\rho^{\alpha}\Gamma(\alpha+1)}\|u-v\|_{\infty}.$$

This implies that by (4.12) , $\mathcal{A}$ is a contraction operator.
As a consequence of theorem 1.8, using BANACH's contraction principle [15], we deduce that $\mathcal{A}$ has a unique fixed point which is the unique solution of the problem (4.3)-(4.4) on $[0,\lambda]$.
The proof is complete. ■

Theorem 4.2 ([8]) *Assume that hypotheses* (H1)-(H3) *hold. We give* $0 < \alpha \leq 1$, *and* $\rho > 0$. *If we put*

$$\frac{M_1\lambda^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} < 1,$$

then the problem (4.3)-(4.4) *has at least one solution on* $[0,\lambda]$.

Proof. In the previous theorem, we already transformed the problem (4.3)-(4.4) into a fixed point problem

$$\mathcal{A}u(\eta) = \int_0^\eta G(\eta, s) f\left(s, u(s), {}^\rho\mathcal{D}_{0+}^\alpha u(s)\right) ds.$$

We demonstrate that $\mathcal{A}$ satisfies the assumption of SCHAUDER's fixed point theorem 1.9. This could be proved through three steps:

Step 1. $\mathcal{A}$ is a continuous operator.

Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence such that $\lim_{n\to\infty} u_n = u$ in $C[0,\lambda]$. Then for each $\eta \in [0,\lambda]$,

$$|\mathcal{A}u_n(\eta) - \mathcal{A}u(\eta)| \leq \int_0^\eta G(\eta, s) \left| f\left(s, u_n(s), {}^\rho\mathcal{D}_{0+}^\alpha u_n(s)\right) - f\left(s, u(s), {}^\rho\mathcal{D}_{0+}^\alpha u(s)\right)\right| ds, \tag{4.15}$$

where

$${}^\rho\mathcal{D}_{0+}^\alpha u_n(\eta) = f\left(\eta, u_n(\eta), {}^\rho\mathcal{D}_{0+}^\alpha u_n(\eta)\right), \text{ and } {}^\rho\mathcal{D}_{0+}^\alpha u(\eta) = f\left(\eta, u(\eta), {}^\rho\mathcal{D}_{0+}^\alpha u(\eta)\right).$$

As a consequence of (H2), we find easily ${}^\rho\mathcal{D}_{0+}^\alpha u_n \to {}^\rho\mathcal{D}_{0+}^\alpha u$ in $C[0,\lambda]$. In fact we have:

$$\begin{aligned} \left|{}^\rho\mathcal{D}_{0+}^\alpha u_n(\eta) - {}^\rho\mathcal{D}_{0+}^\alpha u(\eta)\right| &= \left| f\left(\eta, u_n(\eta), {}^\rho\mathcal{D}_{0+}^\alpha u_n(\eta)\right) - f\left(\eta, u(\eta), {}^\rho\mathcal{D}_{0+}^\alpha u(\eta)\right)\right| \\ &\leq \sigma |u_n(\eta) - u(\eta)| + \beta \left|{}^\rho\mathcal{D}_{0+}^\alpha u_n(\eta) - {}^\rho\mathcal{D}_{0+}^\alpha u(\eta)\right|. \end{aligned}$$

Thus:

$$\left|{}^\rho\mathcal{D}_{0+}^\alpha u_n(\eta) - {}^\rho\mathcal{D}_{0+}^\alpha u(\eta)\right| \leq \frac{\sigma}{1-\beta} |u_n(\eta) - u(\eta)|$$

Since $u_n \to u$, then we get ${}^\rho\mathcal{D}_{0+}^\alpha u_n(\eta) \to {}^\rho\mathcal{D}_{0+}^\alpha u(\eta)$ as $n \to \infty$ for each $\eta \in [0,\lambda]$.

Now let $K_0 > 0$, be such that for each $\eta \in [0,\lambda]$, we have:

$$\left|{}^\rho\mathcal{D}_{0+}^\alpha u_n(\eta)\right| \leq K_0, \left|{}^\rho\mathcal{D}_{0+}^\alpha u(\eta)\right| \leq K_0.$$

Then, we have:

$$\begin{aligned} |\mathcal{A}u_n(\eta) - \mathcal{A}u(\eta)| &\leq \int_0^\eta G(\eta, s) \left| f\left(s, u_n(s), {}^\rho\mathcal{D}_{0+}^\alpha u_n(s)\right) - f\left(s, u(s), {}^\rho\mathcal{D}_{0+}^\alpha u(s)\right)\right| ds \\ &\leq \int_0^\eta G(\eta, s) \left|{}^\rho\mathcal{D}_{0+}^\alpha u_n(s) - {}^\rho\mathcal{D}_{0+}^\alpha u(s)\right| ds \\ &\leq \int_0^\eta G(\eta, s) \left[\left|{}^\rho\mathcal{D}_{0+}^\alpha u_n(s)\right| + \left|{}^\rho\mathcal{D}_{0+}^\alpha u(s)\right|\right] ds \\ &\leq \int_0^\eta 2K_0 G(\eta, s)\, ds. \end{aligned}$$

For each $\eta \in [0,\lambda]$, the function $s \rightarrow 2K_0 G(\eta,s)$ is integrable on $[0,\eta]$, then the Lebesgue dominated convergence theorem and (4.15) imply that:

$$|\mathcal{A}u_n(\eta) - \mathcal{A}u(\eta)| \rightarrow 0 \text{ as } n \rightarrow \infty,$$

and hence:

$$\lim_{n\rightarrow\infty} \|\mathcal{A}u_n - \mathcal{A}u\|_\infty = 0.$$

Consequently, $\mathcal{A}$ is continuous.

Step 2. Let

$$r \geq \frac{M_0 \lambda^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1) - M_1 \lambda^{\rho\alpha}}.$$

We define:

$$P_r = \{u \in C[0,\lambda] : \|u\|_\infty \leq r\}.$$

It is clear that P_r is a bounded, closed and convex subset of $C[0,\lambda]$.

Let $u \in P_r$, and $\mathcal{A} : P_r \rightarrow C[0,\lambda]$ be the integral operator defined in (4.13), then $\mathcal{A}(P_r) \subset P_r$.

In fact, for each $\eta \in [0,\lambda]$, we have from (H3):

$$\begin{aligned} |{}^\rho\mathcal{D}^\alpha_{0+} u(\eta)| &= |f(\eta, u(\eta), {}^\rho\mathcal{D}^\alpha_{0+} u(\eta))| \\ &\leq a(\eta) + b(\eta)|u(\eta)| + c(\eta)|{}^\rho\mathcal{D}^\alpha_{0+} u(\eta)|. \end{aligned}$$

Then

$$|{}^\rho\mathcal{D}^\alpha_{0+} u(\eta)| \leq \frac{a^*}{1-c^*} + \frac{b^*}{1-c^*} r = M_0 + M_1 r. \tag{4.16}$$

Thus

$$\begin{aligned} |\mathcal{A}u(\eta)| &\leq \int_0^\eta G(\eta,s)|f(s,u(s),{}^\rho\mathcal{D}^\alpha_{0+} u(s))| ds \\ &\leq \int_0^\eta G(\eta,s)[M_0 + M_1 r] ds \\ &\leq \frac{M_0\lambda^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)} + \frac{M_1\lambda^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)} r \\ &\leq \frac{[\rho^\alpha\Gamma(\alpha+1) - M_1\lambda^{\rho\alpha}]\frac{M_0\lambda^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)-M_1\lambda^{\rho\alpha}} + M_1\lambda^{\rho\alpha} r}{\rho^\alpha\Gamma(\alpha+1)} \\ &\leq \frac{[\rho^\alpha\Gamma(\alpha+1) - M_1\lambda^{\rho\alpha}] r + M_1\lambda^{\rho\alpha} r}{\rho^\alpha\Gamma(\alpha+1)} \\ &\leq r. \end{aligned}$$

Then $\mathcal{A}(P_r) \subset P_r$.

Step 3. $\mathcal{A}(P_r)$ is relatively compact.

Let $\eta_1, \eta_2 \in [0, \lambda]$, $\eta_1 < \eta_2$, and $u \in P_r$. Then

$$\begin{aligned} |\mathcal{A}u(\eta_2) - \mathcal{A}u(\eta_1)| &= \left| \int_0^{\eta_2} G(\eta_2, s) f\left(s, u(s), {}^\rho\mathcal{D}_{0+}^\alpha u(s)\right) ds \right. \\ &\quad \left. - \int_0^{\eta_1} G(\eta_1, s) f\left(s, u(s), {}^\rho\mathcal{D}_{0+}^\alpha u(s)\right) ds \right| \\ &\leq \int_0^{\eta_1} \left| \left[G(\eta_2, s) - G(\eta_1, s) \right] f\left(s, u(s), {}^\rho\mathcal{D}_{0+}^\alpha u(s)\right) \right| ds \\ &\quad + \int_{\eta_1}^{\eta_2} G(\eta_2, s) \left| f\left(s, u(s), {}^\rho\mathcal{D}_{0+}^\alpha u(s)\right) \right| ds \\ &\leq (M_0 + M_1 r) \times \\ &\quad \left[\int_0^{\eta_1} \left| (G(\eta_2, s) - G(\eta_1, s)) \right| ds + \int_{\eta_1}^{\eta_2} G(\eta_2, s)\, ds \right] \end{aligned} \tag{4.17}$$

We have:

$$\begin{aligned} G(\eta_2, s) - G(\eta_1, s) &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} s^{\rho-1} \left[(\eta_2^\rho - s^\rho)^{\alpha-1} - (\eta_1^\rho - s^\rho)^{\alpha-1} \right] \\ &= \frac{-1}{\alpha \rho^\alpha \Gamma(\alpha)} \frac{d}{ds} \left[(\eta_2^\rho - s^\rho)^\alpha - (\eta_1^\rho - s^\rho)^\alpha \right] \end{aligned}$$

then

$$\int_0^{\eta_1} \left| (G(\eta_2, s) - G(\eta_1, s)) \right| ds \leq \frac{1}{\rho^\alpha \Gamma(\alpha+1)} \left[(\eta_2^\rho - \eta_1^\rho)^\alpha + (\eta_2^{\rho\alpha} - \eta_1^{\rho\alpha}) \right]$$

we have also

$$\begin{aligned} \int_{\eta_1}^{\eta_2} G(\eta_2, s)\, ds &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{\eta_1}^{\eta_2} s^{\rho-1} (\eta_2^\rho - s^\rho)^{\alpha-1} ds \\ &= \frac{-1}{\alpha \rho^\alpha \Gamma(\alpha)} \left[(\eta_2^\rho - s^\rho)^\alpha \right]_{\eta_1}^{\eta_2} \\ &\leq \frac{1}{\rho^\alpha \Gamma(\alpha+1)} (\eta_2^\rho - \eta_1^\rho)^\alpha . \end{aligned}$$

Then (4.17) gives

$$|\mathcal{A}u(\eta_2) - \mathcal{A}u(\eta_1)| \leq \frac{M_0 + M_1 r}{\rho^\alpha \Gamma(\alpha+1)} \left[2 (\eta_2^\rho - \eta_1^\rho)^\alpha + (\eta_2^{\rho\alpha} - \eta_1^{\rho\alpha}) \right].$$

As $\eta_1 \to \eta_2$, the right side of the above inequality tends to zero.

As a consequence of steps 1 to 3 together, and by means of the Ascoli-Arzelà theorem 1.6, we deduce that $\mathcal{A} : P_r \to P_r$ is continuous, compact and satisfies the assumption of Schauder's fixed point theorem 1.9. Then $\mathcal{A}$ has a fixed point which is a solution of the problem (4.3)-(4.4) on $[0, \lambda]$. The proof is complete.

■

Our next existence result is based on the nonlinear alternative of LERAY-SCHAUDER type.

Theorem 4.3 **([8])** *Assume* (H1)-(H3) *hold. Then the problem* (4.3)-(4.4) *has at least one solution on* $[0,\lambda]$.

Proof. Let $\alpha,\rho>0$, be such that $0<\alpha\leq 1$.
We shall show that the operator $\mathcal{A}$ defined in (4.13), satisfies the assumption of LERAY-SCHAUDER's fixed point theorem 1.10. The proof will be given in several steps.

Step 1. Clearly $\mathcal{A}$ is continuous.

Step 2. $\mathcal{A}$ maps bounded sets into bounded sets in $C[0,\lambda]$.

Indeed, it is enough to show that for any $\omega>0$ there exist a positive constant ℓ such that for each $u\in B_\omega=\{u\in C[0,\lambda]:\|u\|_\infty\leq\omega\}$, we have $\|\mathcal{A}u\|_\infty\leq\ell$.
For $u\in B_\omega$, we have, for each $\eta\in[0,\lambda]$,

$$|\mathcal{A}u(\eta)|\leq\int_0^\eta G(\eta,s)\left|f\left(s,u(s),{}^\rho\mathcal{D}_{0+}^\alpha u(s)\right)\right|ds. \tag{4.18}$$

By (H3), similarly of (4.16), for each $\eta\in[0,\lambda]$, we have:

$$\left|f\left(\eta,u(\eta),{}^\rho\mathcal{D}_{0+}^\alpha u(\eta)\right)\right|\leq M_0+M_1\omega.$$

Thus (4.18) implies that:

$$\|\mathcal{A}u\|_\infty\leq\frac{M_0\lambda^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)}+\frac{M_1\lambda^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)}\omega=\ell.$$

Step 3. Clearly, $\mathcal{A}$ maps bounded sets into equicontinuous sets of $C[0,\lambda]$.

We conclude that $\mathcal{A}:C[0,\lambda]\to C[0,\lambda]$ is continuous and completely continuous.

Step 4. A priori bounds.

We now show there exists an open set $U\subset C[0,\lambda]$ with $u\neq\mu\mathcal{A}(u)$ for $\mu\in(0,1)$ and $u\in\partial U$.
Let $u\in C[0,\lambda]$ and $u=\mu\mathcal{A}(u)$ for some $0<\mu<1$. Thus for each $\eta\in[0,\lambda]$, we have:

$$u(\eta)\leq\mu\int_0^\eta G(\eta,s)\left|f\left(s,u(s),{}^\rho\mathcal{D}_{0+}^\alpha u(s)\right)\right|ds.$$

By (H3), for all solution $u\in C[0,\lambda]$, of the problem (4.3)-(4.4), we have:

$$\begin{aligned}|u(\eta)| &= \left|\int_0^\eta G(\eta,s)f\left(s,u(s),{}^\rho\mathcal{D}_{0+}^\alpha u(s)\right)ds\right|\\ &\leq \int_0^\eta G(\eta,s)\left|{}^\rho\mathcal{D}_{0+}^\alpha u(s)\right|ds.\end{aligned}$$

Then for each $\eta \in [0, \lambda]$, we have:

$$\begin{aligned} \left|{}^{\rho}\mathcal{D}_{0^+}^{\alpha} u(\eta)\right| &= \left| f\left(\eta, u(\eta), {}^{\rho}\mathcal{D}_{0^+}^{\alpha} u(\eta)\right)\right| \\ &\leq a(\eta) + b(\eta)\left|u(\eta)\right| + c(\eta)\left|{}^{\rho}\mathcal{D}_{0^+}^{\alpha} u(\eta)\right|. \end{aligned}$$

Then

$$\begin{aligned} \left|{}^{\rho}\mathcal{D}_{0^+}^{\alpha} u(\eta)\right| &\leq \frac{1}{1-c^*}\left(a^* + b^*\left|u(\eta)\right|\right) \\ &\leq M_0 + M_1\left|u(\eta)\right|. \end{aligned}$$

Hence

$$\left|u(\eta)\right| \leq \frac{M_0\lambda^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} + \int_0^{\eta} M_1 G(\eta, s)\left|u(s)\right| ds.$$

After the GRONWALL lemma [16], we have:

$$\left|u(\eta)\right| \leq \frac{M_0\lambda^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} \exp\left(\frac{M_1\lambda^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\right).$$

Thus

$$\|u\|_{\infty} \leq \frac{M_0\lambda^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} \exp\left(\frac{M_1\lambda^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\right) = M_2.$$

Let

$$U = \left\{u \in C[0, \lambda] : \|u\|_{\infty} < M_2 + 1\right\}.$$

By choosing U, there is no $u \in \partial U$, such that $u = \mu\mathcal{A}(u)$, for $\mu \in (0,1)$. As a consequence of LERAY-SCHAUDER's theorem 1.10, $\mathcal{A}$ has a fixed point u in U which is a solution to (4.3)-(4.4). The proof is complete.

■

4.5 Examples

Example 1. Consider the following CAUCHY problem:

$$\begin{cases} {}_{x}^{1}\mathcal{D}_{0^+}^{\frac{1}{2}} w(x,t) = \dfrac{\psi^2(t)\cos\left(\frac{x}{\varphi(t)}\right)}{\pi\varphi^{\frac{1}{2}}(t)\left(\sqrt{2}\cos\left(\frac{x}{\varphi(t)}\right)+\sin\left(\frac{x}{\varphi(t)}\right)\right)\left[\psi(t)+|w(x,t)|+\varphi^{\frac{1}{2}}(t)\left|{}_{x}^{1}\mathcal{D}_{0^+}^{\frac{1}{2}} w(x,t)\right|\right]}, \ x \in \left(0, \frac{\pi}{4}\varphi(t)\right]. \\ w(0,t) = 0, \ \ t \in [0, T], \ \text{ for any } T > 0. \end{cases}$$

Set the function

$$f(x,t,w) = \frac{\psi^2(t)\cos\left(\frac{x}{\varphi(t)}\right)}{\pi\varphi^{\frac{1}{2}}(t)\left(\sqrt{2}\cos\left(\frac{x}{\varphi(t)}\right)+\sin\left(\frac{x}{\varphi(t)}\right)\right)\left[\psi(t)+|w(x,t)|+\varphi^{\frac{1}{2}}(t)\left|{}_{x}^{1}\mathcal{D}_{0^+}^{\frac{1}{2}} w(x,t)\right|\right]},$$

which satisfies the hypotheses of theorem 2.3. As

$$ {}_{x}^{1}\mathcal{D}_{0+}^{\frac{1}{2}} w(x,t) = \frac{\psi(t)}{\varphi^{\frac{1}{2}}(t)} \, {}^{1}\mathcal{D}_{0+}^{\frac{1}{2}} u(\eta), $$

and

$$ \begin{aligned} f(x,t,w) &= \frac{\psi^2(t)\cos\left(\frac{x}{\varphi(t)}\right)}{\pi\varphi^{\frac{1}{2}}(t)\left(\sqrt{2}\cos\left(\frac{x}{\varphi(t)}\right)+\sin\left(\frac{x}{\varphi(t)}\right)\right)\left[\psi(t)+|w(x,t)|+\varphi^{\frac{1}{2}}(t)\left|{}_{x}^{1}\mathcal{D}_{0+}^{\frac{1}{2}} w(x,t)\right|\right]} \\ &= \frac{\psi(t)}{\varphi^{\frac{1}{2}}(t)} \frac{\cos(\eta)}{\pi\left(\sqrt{2}\cos(\eta)+\sin(\eta)\right)\left[1+|u(\eta)|+\left|{}^{1}\mathcal{D}_{0+}^{\frac{1}{2}} u(\eta)\right|\right]}. \end{aligned} $$

Then, the transformation (4.2) reduces the previous CAUCHY problem of fractional-order's partial differential equation to CAUCHY problem of fractional differential equation of the form (see [8]):

$$ \begin{cases} {}^{1}\mathcal{D}_{0+}^{\frac{1}{2}} u(\eta) = \frac{\cos(\eta)}{\pi\left(\sqrt{2}\cos(\eta)+\sin(\eta)\right)\left[1+|u(\eta)|+\left|{}^{1}\mathcal{D}_{0+}^{\frac{1}{2}} u(\eta)\right|\right]}, & \eta \in \left(0, \frac{\pi}{4}\right], \\ u(0) = 0. \end{cases} \tag{4.19} $$

Set:

$$ f(\eta,u,v) = \frac{\cos(\eta)}{\pi\left(\sqrt{2}\cos(\eta)+\sin(\eta)\right)\left[1+|u|+|v|\right]}, \quad \eta \in \left[0, \frac{\pi}{4}\right], \quad u,v \in \mathbb{R}. $$

Because $\sin(\eta), \cos(\eta)$ are continuous positive functions $\forall \eta \in \left[0, \frac{\pi}{4}\right]$, the function f is jointly continuous. For any $u, v, \tilde{u}, \tilde{v} \in \mathbb{R}$ and $\eta \in \left[0, \frac{\pi}{4}\right]$, we have:

$$ \frac{\sqrt{2}}{2} \leq \cos(\eta) \leq 1, \text{ and } 0 \leq \sin(\eta) \leq \frac{\sqrt{2}}{2}, $$

then:

$$ |f(\eta,u,v) - f(\eta,\tilde{u},\tilde{v})| \leq \frac{1}{\pi}\left(|u-\tilde{u}| + |v-\tilde{v}|\right). $$

Hence, the condition (H2) is satisfied with:

$$ \sigma = \beta = \frac{1}{\pi} \simeq 0.3183 < 1. $$

It remains to show that the condition (4.12):

$$ \frac{\sigma\lambda^{\rho\alpha}}{(1-\beta)\rho^{\alpha}\Gamma(\alpha+1)} = \frac{\left(\frac{1}{\pi}\right)\left(\frac{\pi}{4}\right)^{\frac{1}{2}}}{\left(1-\frac{1}{\pi}\right)\Gamma\left(\frac{1}{2}+1\right)} = \frac{\sqrt{\pi}}{2(\pi-1)\Gamma\left(\frac{3}{2}\right)} \simeq 0.4669 < 1, $$

is satisfied. It follows from theorem 4.1 that the problem (4.19) has a unique solution.

Example 2. Consider the following CAUCHY problem

$$ \begin{cases} {}_{x}^{1}\mathcal{D}_{0+}^{\frac{1}{2}} w(x,t) = \frac{\psi(t)\cos\left(\frac{x}{\varphi(t)}\right)\left[2\psi(t)+|w(x,t)|+\varphi^{\frac{1}{2}}(t)\left|{}_{x}^{1}\mathcal{D}_{0+}^{\frac{1}{2}} w(x,t)\right|\right]}{\pi\varphi^{\frac{1}{2}}(t)\left(\sqrt{2}\cos\left(\frac{x}{\varphi(t)}\right)+\sin\left(\frac{x}{\varphi(t)}\right)\right)\left[\psi(t)+|w(x,t)|+\varphi^{\frac{1}{2}}(t)\left|{}_{x}^{1}\mathcal{D}_{0+}^{\frac{1}{2}} w(x,t)\right|\right]}, & x \in \left(0, \frac{\pi}{4}\varphi(t)\right]. \\ w(0,t) = 0, \ t \in [0,T], \text{ for any } T > 0. \end{cases} $$

Set the function

$$f(x,t,w)=\frac{\psi(t)\cos\left(\frac{x}{\varphi(t)}\right)\left[2\psi(t)+|w(x,t)|+\varphi^{\frac{1}{2}}(t)\left|{}^{1}_{x}\mathcal{D}^{\frac{1}{2}}_{0^+}w(x,t)\right|\right]}{\pi\varphi^{\frac{1}{2}}(t)\left(\sqrt{2}\cos\left(\frac{x}{\varphi(t)}\right)+\sin\left(\frac{x}{\varphi(t)}\right)\right)\left[\psi(t)+|w(x,t)|+\varphi^{\frac{1}{2}}(t)\left|{}^{1}_{x}\mathcal{D}^{\frac{1}{2}}_{0^+}w(x,t)\right|\right]},$$

which satisfies the hypotheses of theorem 2.3. As

$$\begin{aligned}f(x,t,w)&=\frac{\psi(t)\cos\left(\frac{x}{\varphi(t)}\right)\left[2\psi(t)+|w(x,t)|+\varphi^{\frac{1}{2}}(t)\left|{}^{1}_{x}\mathcal{D}^{\frac{1}{2}}_{0^+}w(x,t)\right|\right]}{\pi\varphi^{\frac{1}{2}}(t)\left(\sqrt{2}\cos\left(\frac{x}{\varphi(t)}\right)+\sin\left(\frac{x}{\varphi(t)}\right)\right)\left[\psi(t)+|w(x,t)|+\varphi^{\frac{1}{2}}(t)\left|{}^{1}_{x}\mathcal{D}^{\frac{1}{2}}_{0^+}w(x,t)\right|\right]}\\&=\frac{\psi(t)}{\varphi^{\frac{1}{2}}(t)}\frac{\cos(\eta)\left[2+|u(\eta)|+\left|{}^{1}\mathcal{D}^{\frac{1}{2}}_{0^+}u(\eta)\right|\right]}{\pi\left(\sqrt{2}\cos(\eta)+\sin(\eta)\right)\left[1+|u(\eta)|+\left|{}^{1}\mathcal{D}^{\frac{1}{2}}_{0^+}u(\eta)\right|\right]}.\end{aligned}$$

Then, the transformation (4.2) reduces the previous Cauchy problem of fractional-order's partial differential equation to Cauchy problem of fractional differential equation of the form (see [8]):

$$\begin{cases}{}^{1}\mathcal{D}^{\frac{1}{2}}_{0^+}u(\eta)=\dfrac{\cos(\eta)\left[2+|u(\eta)|+\left|{}^{1}\mathcal{D}^{\frac{1}{2}}_{0^+}u(\eta)\right|\right]}{\pi\left(\sqrt{2}\cos(\eta)+\sin(\eta)\right)\left[1+|u(\eta)|+\left|{}^{1}\mathcal{D}^{\frac{1}{2}}_{0^+}u(\eta)\right|\right]},\ \eta\in\left[0,\frac{\pi}{4}\right],\\ u(0)=0\end{cases}\tag{4.20}$$

Set

$$f(\eta,u,v)=\frac{\cos(\eta)\left[2+|u|+|v|\right]}{\pi\left(\sqrt{2}\cos(\eta)+\sin(\eta)\right)\left[1+|u|+|v|\right]},\ \eta\in\left[0,\frac{\pi}{4}\right],\ u,v\in\mathbb{R}.$$

Clearly, the function f is jointly continuous. For any $u,v,\tilde{u},\tilde{v}\in\mathbb{R}$ and $\eta\in\left[0,\frac{\pi}{4}\right]$, we have

$$|f(\eta,u,v)-f(\eta,\tilde{u},\tilde{v})|\leq\frac{1}{\pi}\left(|u-\tilde{u}|+|v-\tilde{v}|\right).$$

Hence, the hypothese (H2) is satisfied with $\sigma=\beta=\frac{1}{\pi}<1$. Also, we have:

$$|f(\eta,u,v)|\leq\frac{\cos(\eta)}{\pi\left(\sqrt{2}\cos(\eta)+\sin(\eta)\right)}\left(2+|u|+|v|\right).$$

Thus, the hypothese (H3) is satisfied with

$$a(\eta)=\frac{2\cos(\eta)}{\pi\left(\sqrt{2}\cos(\eta)+\sin(\eta)\right)},\ \text{and } b(\eta)=c(\eta)=\frac{\cos(\eta)}{\pi\left(\sqrt{2}\cos(\eta)+\sin(\eta)\right)}.$$

We have also

$$a^*=\frac{2}{\pi},\ \text{and } b^*=c^*=\frac{1}{\pi}<1,\ \text{and } M_0=\frac{2}{\pi-1},\ \text{and } M_1=\frac{1}{\pi-1}.$$

And the condition

$$\frac{M_1\lambda^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)}=\frac{\left(\frac{1}{\pi-1}\right)\left(\frac{\pi}{4}\right)^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}+1\right)}=\frac{\sqrt{\pi}}{2(\pi-1)\Gamma\left(\frac{3}{2}\right)}\simeq 0.4669<1.$$

It follows from theorem 4.2 and theorem 4.3, that the problem (4.20) has at least one solution.

4.6 Proof of Main Theorems

In this section, we prove the existence and uniqueness of solutions of the following implicit problem of the nonlinear partial differential equations of space-fractional order [8], [34]:

$$ {}_{x}^{\rho}\mathcal{D}_{0+}^{\alpha} w = f\left(x, t, w, {}_{x}^{\rho}\mathcal{D}_{0+}^{\alpha} w\right), \; 0 < \alpha \leq 1, \; (x,t) \in (0, X] \times [0, T], \tag{4.21} $$

with the initial condition:

$$ w(0,t) = 0, \tag{4.22} $$

under the generalized self-similar form which is:

$$ w(x,t) = \psi(t)\, u(\eta), \text{ with } \eta = \frac{x}{\varphi(t)}, \text{ and } \varphi, \psi \in C[0,T] \to \mathbb{R}_{+}. \tag{4.23} $$

Then the transformation (4.23) reduces the fractional-order's partial differential equation (4.21) to the ordinary differential equation of fractional order of the form:

$$ {}^{\rho}\mathcal{D}_{0+}^{\alpha} u(\eta) = f\left(\eta, u(\eta), {}^{\rho}\mathcal{D}_{0+}^{\alpha} u(\eta)\right), \; \eta \in (0, \lambda], \tag{4.24} $$

with the initial condition:

$$ u(0) = 0. \tag{4.25} $$

Where $\lambda = X\bar{\varphi}^{-1}$, with $\bar{\varphi} = \min\limits_{0 \leq t \leq T} \varphi(t)$ is a finite positive constant.

Now we proceed to demonstrate the theorem 2.6.

Proof of theorem 2.6:

Now, we prove our existence and uniqueness results for the initial value problem (4.21)-(4.22). By using (4.23), the hypotheses $\overline{(\mathrm{H1})}$, $\overline{(\mathrm{H2})}$ and $\overline{(\mathrm{H3})}$, presented in theorem 2.6, are equivalent to the hypotheses (H1), (H2) and (H3), presented in the begin of section 4.4, respectively. Also the conditions

$$ \frac{M_1 X^{\rho\alpha}}{(\rho\bar{\varphi}^{\rho})^{\alpha}\,\Gamma(\alpha+1)} < 1, \text{ and } \frac{\sigma X^{\rho\alpha}}{(1-\beta)\,(\rho\bar{\varphi}^{\rho})^{\alpha}\,\Gamma(\alpha+1)} < 1, $$

are equivalent to

$$ \frac{M_1 \lambda^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} < 1, \text{ and } \frac{\sigma \lambda^{\rho\alpha}}{(1-\beta)\,\rho^{\alpha}\Gamma(\alpha+1)} < 1, $$

respectively.

1) We already proved in theorem 4.3, the existence of solution of the initial value problem (4.24)-(4.25) provided that (H1), (H2) and (H3) hold. Consequently, if $\overline{(\mathrm{H1})}$, $\overline{(\mathrm{H2})}$ and $\overline{(\mathrm{H3})}$

hold, the initial value problem (4.21)-(4.22) has at least one solution under the generalized self-similar form (4.23) .

2) We already proved in theorem 4.2, the existence of solution of the initial value problem (4.24)-(4.25) provided that (H1) , (H2) and (H3) hold, and

$$\frac{M_1\lambda^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} < 1.$$

Consequently, if $\overline{(\mathrm{H1})}$, $\overline{(\mathrm{H2})}$ and $\overline{(\mathrm{H3})}$ hold, and

$$\frac{M_1 X^{\rho\alpha}}{(\rho\bar{\varphi}^{\rho})^{\alpha}\Gamma(\alpha+1)} < 1,$$

the initial value problem (4.21)-(4.22) has at least one solution under the generalized self-similar form (4.23) .

3) We already proved in theorem 4.1, the existence and uniqueness of solution of the initial value problem (4.24)-(4.25) provided that (H1) , (H2) hold, and

$$\frac{\sigma\lambda^{\rho\alpha}}{(1-\beta)\rho^{\alpha}\Gamma(\alpha+1)} < 1.$$

Consequently, if $\overline{(\mathrm{H1})}$, $\overline{(\mathrm{H2})}$ hold, and

$$\frac{\sigma X^{\rho\alpha}}{(1-\beta)(\rho\bar{\varphi}^{\rho})^{\alpha}\Gamma(\alpha+1)} < 1.$$

Then, there exists a unique solution of the initial value problem (4.21)-(4.22) under the generalized self-similar form (4.23) .

The proof of theorem 2.6 is complete.

Chapter 5

Nonlinear Fractional Equations With an Integral Condition

5.1 Introduction

This chapter studies the existence and uniqueness of generalized self-similar solutions for a class of nonlinear FPDE with an integral condition. The arguments for the study are based upon BANACH's contraction principle (theorem 1.8), SCHAUDER's fixed point theorem (theorem 1.9), and the nonlinear alternative of LERAY-SCHAUDER type (theorem 1.10). The used differential operator is developed by KATUGAMPOLA. For application purposes, some examples are provided to demonstrate the applicability of our main results.

We study in a general manner, the existence and uniqueness of solutions of the following problem of the nonlinear FPDE using KATUGAMPOLA's fractional derivative [29], [34]:

$$ {}_{x}^{\rho}\mathcal{D}_{0^+}^{\alpha} w = f\left(x, t, w, {}_{x}^{\rho}\mathcal{D}_{0^+}^{\beta} w\right), \; (x,t) \in [0, X] \times [0, T], \tag{5.1} $$

with the integral condition:

$$ \left({}_{x}^{\rho}\mathcal{I}_{0^+}^{1-\alpha} w\right)\left(0^+, t\right) = 0, $$

under the generalized self-similar form which is:

$$ w(x,t) = \psi(t)\, u(\eta), \; \text{with } \eta = \frac{x}{\varphi(t)}, \; \text{and } \varphi, \psi \in C[0,T] \to \mathbb{R}_+. \tag{5.2} $$

Here $0 < \beta < \alpha \leq 1$, $f : [0, X] \times [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with $X, T \in \mathbb{R}_+$, is a given function which satisfies the hypotheses of theorem 2.3.

Then the transformation (5.2) reduces the fractional-order's partial differential equation (5.1) to the ordinary differential equation of fractional order of the form:

$$ {}^{\rho}\mathcal{D}_{0+}^{\alpha} u(\eta) = f\left(\eta, u(\eta), {}^{\rho}\mathcal{D}_{0+}^{\beta} u(\eta)\right), \ \eta \in [0, \lambda], \tag{5.3} $$

with the integral condition:

$$ \left({}^{\rho}\mathcal{I}_{0+}^{1-\alpha} u\right)\left(0^+\right) = 0. \tag{5.4} $$

Where $\lambda = X\bar{\varphi}^{-1}$, with $\bar{\varphi} = \min\limits_{0 \leq t \leq T} \varphi(t)$ is a finite positive constant, and $f : [0, \lambda] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

According to chapter 2 (subsection 2.3.3), we discuss in a general manner the existence and uniqueness of solutions of nonlinear FDEs (5.3), with the integral condition (5.4).

We obtain several existence and uniqueness results for the problem (5.3)-(5.4).

5.2 Definition of Integral Solution

In the sequel, λ, p, n and c are real constants such that

$$ p \geq 1, \ c > 0, \ n = [\alpha] + 1, \ \text{and } 0 < \lambda \leq (pc)^{\frac{1}{pc}}. $$

In what follows, we present some significant lemmas to show the principal theorems, we have:

Lemma 5.1 *Let $\alpha, \rho \in \mathbb{R}$, be such that $0 < \alpha \leq 1$, and $\rho > 0$. We define:*

$$ P = \left\{u \in C[0, \lambda] \mid \ \left({}^{\rho}\mathcal{I}_{0+}^{1-\alpha} u\right)\left(0^+\right) = 0\right\}. $$

Then $(P, \|\cdot\|_{\infty})$ is a Banach *space.*

Proof. Let $0 < \alpha \leq 1$ and $\rho > 0$.
It is clear that the space P with the norm $\|\cdot\|_{\infty}$ is a subspace of $C[0, \lambda]$ which is a Banach space.
It remains to prove that P is a closed subspace in $C[0, \lambda]$.
Let $(u_n)_{n\in\mathbb{N}} \in P$ be a real sequence such that $\lim\limits_{n\to\infty} u_n = u$ in $C[0, \lambda]$. Then for each $\eta \in [0, \lambda]$, we have:

$$ |u_n(\eta)| \leq K_0, \ |u(\eta)| \leq K_0, \ \text{for some } K_0 > 0. $$

Since $u_n \to u$, then we get ${}^{\rho}\mathcal{I}_{0^+}^{1-\alpha}u_n(\eta) \to {}^{\rho}\mathcal{I}_{0^+}^{1-\alpha}u(\eta)$ as $n \to \infty$ for each $\eta \in [0, \lambda]$. In fact

$$\begin{aligned} \left|{}^{\rho}\mathcal{I}_{0^+}^{1-\alpha}u_n(\eta) - {}^{\rho}\mathcal{I}_{0^+}^{1-\alpha}u(\eta)\right| &= \left|\frac{\rho^{\alpha}}{\Gamma(1-\alpha)}\int_0^{\eta}\frac{s^{\rho-1}u_n(s)}{(\eta^{\rho}-s^{\rho})^{\alpha}}ds - \frac{\rho^{\alpha}}{\Gamma(1-\alpha)}\int_0^{\eta}\frac{s^{\rho-1}u(s)}{(\eta^{\rho}-s^{\rho})^{\alpha}}ds\right| \\ &\leq \frac{\rho^{\alpha}}{\Gamma(1-\alpha)}\int_0^{\eta}\frac{s^{\rho-1}}{(\eta^{\rho}-s^{\rho})^{\alpha}}\left|u_n(s)-u(s)\right|ds. \end{aligned} \tag{5.5}$$

Then

$$\begin{aligned} \left|{}^{\rho}\mathcal{I}_{0^+}^{1-\alpha}u_n(\eta) - {}^{\rho}\mathcal{I}_{0^+}^{1-\alpha}u(\eta)\right| &\leq \frac{\rho^{\alpha}}{\Gamma(1-\alpha)}\int_0^{\eta} s^{\rho-1}(\eta^{\rho}-s^{\rho})^{-\alpha}\left(|u_n(s)|+|u(s)|\right)ds \\ &\leq \frac{2\rho^{\alpha}K_0}{\Gamma(1-\alpha)}\left[-\frac{(\eta^{\rho}-s^{\rho})^{1-\alpha}}{1-\alpha}\right]_0^{\eta} \\ &\leq \frac{2\rho^{\alpha}\lambda^{\rho(1-\alpha)}K_0}{\Gamma(2-\alpha)}. \end{aligned}$$

Thus, for each $\eta \in [0, \lambda]$, the LEBESGUE dominated convergence theorem and (5.5) imply that

$$\left|{}^{\rho}\mathcal{I}_{0^+}^{1-\alpha}u_n(\eta) - {}^{\rho}\mathcal{I}_{0^+}^{1-\alpha}u(\eta)\right| \to 0 \text{ as } n \to \infty.$$

Hence

$$\lim_{n\to\infty}\left\|{}^{\rho}\mathcal{I}_{0^+}^{1-\alpha}u_n(\eta) - {}^{\rho}\mathcal{I}_{0^+}^{1-\alpha}u(\eta)\right\|_{\infty} = 0,$$

and for $\eta \to 0^+$, we have also:

$$\lim_{n\to\infty}\left({}^{\rho}\mathcal{I}_{0^+}^{1-\alpha}u_n\right)(0^+) = \left({}^{\rho}\mathcal{I}_{0^+}^{1-\alpha}u\right)(0^+) = 0, \text{ then } u \in P.$$

Consequently, P is closed in $C[0, \lambda]$, and hence $(P, \|\cdot\|_{\infty})$ is a BANACH space. The proof is complete. ■

Lemma 5.2 *Let* $0 < \beta < \alpha \leq 1$, *be such that* $u, {}^{\rho}\mathcal{D}_{0^+}^{\alpha}u \in C[0, \lambda]$, *then:*

$${}^{\rho}\mathcal{I}_{0^+}^{\alpha-\beta}\, {}^{\rho}\mathcal{D}_{0^+}^{\alpha}u(\eta) = {}^{\rho}\mathcal{D}_{0^+}^{\beta}u(\eta) - \frac{\rho^{1-\alpha+\beta}\left({}^{\rho}\mathcal{I}_{0^+}^{1-\alpha}u\right)(0^+)}{\Gamma(\alpha-\beta)}\eta^{\rho(\alpha-\beta-1)}. \tag{5.6}$$

Moreover; $\forall u \in P$, *we have for every* $\eta \in [0, \lambda]$ *that:*

$$\left|{}^{\rho}\mathcal{D}_{0^+}^{\beta}u(\eta)\right| \leq \frac{\lambda^{\rho(\alpha-\beta)}}{\rho^{\alpha-\beta}\Gamma(1+\alpha-\beta)}\left\|{}^{\rho}\mathcal{D}_{0^+}^{\alpha}u\right\|_{\infty}. \tag{5.7}$$

Proof. Let $u, {}^{\rho}\mathcal{D}_{0^+}^{\alpha}u \in C[0, \lambda]$. If we apply the operator ${}^{\rho}\mathcal{I}_{0^+}^{\alpha-\beta}$ to ${}^{\rho}\mathcal{D}_{0^+}^{\alpha}u(\eta)$, and using theorems 1.2, 1.3, 1.4 and property (1.28), which is:

$$\left(\eta^{1-\rho}\frac{d}{d\eta}\right){}^{\rho}\mathcal{I}_{0^+}^{\delta+1}u(\eta) = {}^{\rho}\mathcal{I}_{0^+}^{\delta}u(\eta), \ \forall\delta > 0,$$

we have:

$$\begin{aligned}
{}^{\rho}\mathcal{I}_{0^+}^{\alpha-\beta}\ {}^{\rho}\mathcal{D}_{0^+}^{\alpha}u(\eta) &= \left(\eta^{1-\rho}\frac{d}{d\eta}\right){}^{\rho}\mathcal{I}_{0^+}^{\alpha-\beta+1}\ {}^{\rho}\mathcal{D}_{0^+}^{\alpha}u(\eta)\\
&= \left(\eta^{1-\rho}\frac{d}{d\eta}\right)\left[\frac{\rho^{\beta-\alpha}}{\Gamma(1+\alpha-\beta)}\int_0^{\eta}(\eta^{\rho}-s^{\rho})^{\alpha-\beta}\frac{d}{ds}\,{}^{\rho}\mathcal{I}_{0^+}^{1-\alpha}u(s)\,ds\right]\\
&= \left(\eta^{1-\rho}\frac{d}{d\eta}\right)\frac{\rho^{\beta-\alpha}}{\Gamma(1+\alpha-\beta)}\left[\left[(\eta^{\rho}-s^{\rho})^{\alpha-\beta}\,{}^{\rho}\mathcal{I}_{0^+}^{1-\alpha}u(s)\right]_0^{\eta}\right.\\
&\quad\left.+\rho(\alpha-\beta)\int_0^{\eta}s^{\rho-1}(\eta^{\rho}-s^{\rho})^{\alpha-\beta-1}\,{}^{\rho}\mathcal{I}_{0^+}^{1-\alpha}u(s)\,ds\right].
\end{aligned}$$

Then

$$\begin{aligned}
{}^{\rho}\mathcal{I}_{0^+}^{\alpha-\beta}\ {}^{\rho}\mathcal{D}_{0^+}^{\alpha}u(\eta) &= \left(\eta^{1-\rho}\frac{d}{d\eta}\right)\frac{\rho^{1-\alpha+\beta}}{\Gamma(\alpha-\beta)}\int_0^{\eta}s^{\rho-1}(\eta^{\rho}-s^{\rho})^{\alpha-\beta-1}\,{}^{\rho}\mathcal{I}_{0^+}^{1-\alpha}u(s)\,ds\\
&\quad-\frac{\rho^{\beta-\alpha}\left({}^{\rho}\mathcal{I}_{0^+}^{1-\alpha}u\right)(0^+)}{\Gamma(1+\alpha-\beta)}\left(\eta^{1-\rho}\frac{d}{d\eta}\right)\eta^{\rho(\alpha-\beta)}\\
&= \left(\eta^{1-\rho}\frac{d}{d\eta}\right){}^{\rho}\mathcal{I}_{0^+}^{\alpha-\beta}\ {}^{\rho}\mathcal{I}_{0^+}^{1-\alpha}u(s)-\frac{\rho^{\beta-\alpha}\left({}^{\rho}\mathcal{I}_{0^+}^{1-\alpha}u\right)(0^+)}{\Gamma(1+\alpha-\beta)}\rho(\alpha-\beta)\,\eta^{1-\rho}\eta^{\rho(\alpha-\beta)-1}\\
&= {}^{\rho}\mathcal{D}_{0^+}^{\beta}u(\eta)-\frac{\rho^{1-\alpha+\beta}\left({}^{\rho}\mathcal{I}_{0^+}^{1-\alpha}u\right)(0^+)}{\Gamma(\alpha-\beta)}\eta^{\rho(\alpha-\beta-1)}.
\end{aligned}$$

Moreover, $\forall u\in P$, we have $\left({}^{\rho}\mathcal{I}_{0^+}^{1-\alpha}u\right)(0^+)=0$, then for every $\eta\in[0,\lambda]$:

$$ {}^{\rho}\mathcal{I}_{0^+}^{\alpha-\beta}\ {}^{\rho}\mathcal{D}_{0^+}^{\alpha}u(\eta) = {}^{\rho}\mathcal{D}_{0^+}^{\beta}u(\eta)$$

and

$$\begin{aligned}
\left|{}^{\rho}\mathcal{D}_{0^+}^{\beta}u(\eta)\right| &= \left|{}^{\rho}\mathcal{I}_{0^+}^{\alpha-\beta}\ {}^{\rho}\mathcal{D}_{0^+}^{\alpha}u(\eta)\right|\le\frac{\rho^{1-\alpha+\beta}}{\Gamma(\alpha-\beta)}\int_0^{\eta}s^{\rho-1}(\eta^{\rho}-s^{\rho})^{\alpha-\beta-1}\left|{}^{\rho}\mathcal{D}_{0^+}^{\alpha}u(s)\right|ds\\
&\le \left[-\frac{\rho^{\beta-\alpha}}{(\alpha-\beta)\Gamma(\alpha-\beta)}(\eta^{\rho}-s^{\rho})^{\alpha-\beta}\right]_0^{\eta}\left\{\sup_{0\le\eta\le\lambda}\left|{}^{\rho}\mathcal{D}_{0^+}^{\alpha}u(\eta)\right|\right\}\\
&\le \frac{\lambda^{\rho(\alpha-\beta)}}{\rho^{\alpha-\beta}\Gamma(1+\alpha-\beta)}\left\|{}^{\rho}\mathcal{D}_{0^+}^{\alpha}u\right\|_{\infty}.
\end{aligned}$$

The proof is complete. ■

Based on the previous lemma, we will define the integral solution of the problem (5.3)-(5.4).

Lemma 5.3 *Let $\alpha,\beta,\rho\in\mathbb{R}$, be such that $0<\beta<\alpha\le1$, and $\rho>0$. We give u, ${}^{\rho}\mathcal{D}_{0^+}^{\alpha}u\in C[0,\lambda]$, and $f(\eta,u,v)$ is a continuous function. Then the problem (5.3)-(5.4) is equivalent to the integral equation:*

$$u(\eta)=\int_0^{\eta}G(\eta,s)\,f\left(s,u(s),{}^{\rho}\mathcal{D}_{0^+}^{\beta}u(s)\right)ds,$$

where

$$G(\eta, s) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} s^{\rho-1} (\eta^\rho - s^\rho)^{\alpha-1}. \tag{5.8}$$

Proof. Let $0 < \beta < \alpha \leq 1$ and $\rho > 0$, we may apply lemma 4.1 to reduce the fractional equation (5.3) to an equivalent fractional integral equation.
By applying ${}^\rho\mathcal{I}^\alpha_{0+}$ to equation (5.3) we obtain:

$${}^\rho\mathcal{I}^\alpha_{0+}\ {}^\rho\mathcal{D}^\alpha_{0+} u(\eta) = {}^\rho\mathcal{I}^\alpha_{0+} f\left(\eta, u(\eta), {}^\rho\mathcal{D}^\beta_{0+} u(\eta)\right). \tag{5.9}$$

From lemma 4.1, we find easily:

$${}^\rho\mathcal{I}^\alpha_{0+}\ {}^\rho\mathcal{D}^\alpha_{0+} u(\eta) = u(\eta) + C\eta^{\rho(\alpha-1)},$$

for some $C \in \mathbb{R}$. Then, the fractional integral equation (5.9), gives:

$$u(\eta) = {}^\rho\mathcal{I}^\alpha_{0+} f\left(\eta, u(\eta), {}^\rho\mathcal{D}^\beta_{0+} u(\eta)\right) - C\eta^{\rho(\alpha-1)}. \tag{5.10}$$

From (1.24) we have:

$${}^\rho\mathcal{I}^{1-\alpha}_{0+} \eta^{\rho(\alpha-1)} = \rho^{\alpha-1}\Gamma(\alpha).$$

If we use the condition (5.4) in equation (5.10) we find:

$$\left({}^\rho\mathcal{I}^{1-\alpha}_{0+} u\right)\left(0^+\right) = 0 = -C\rho^{\alpha-1}\Gamma(\alpha) \Rightarrow C = 0.$$

Therefore, the problem (5.3)-(5.4) is equivalent to:

$$u(\eta) = \int_0^\eta G(\eta, s) f\left(s, u(s), {}^\rho\mathcal{D}^\beta_{0+} u(s)\right) ds,$$

where $G(\eta, s)$, which is given by the equality (5.8). The proof is complete. ■

5.3 Existence and Uniqueness Results

Lemma 5.4 *Let $\mathcal{A} : P \to C[0, \lambda]$ be an integral operator, which is defined by:*

$$\mathcal{A}u(\eta) = \int_0^\eta G(\eta, s) f\left(s, u(s), {}^\rho\mathcal{D}^\beta_{0+} u(s)\right) ds, \tag{5.11}$$

equipped with the standard norm:

$$\|\mathcal{A}u\|_\infty = \sup_{0 \leq \eta \leq \lambda} |\mathcal{A}u(\eta)|.$$

Then $\mathcal{A}(P) \subset P$.

Proof. Let $u \in P$, be such that

$$ {}^{\rho}\mathcal{D}_{0^+}^{\alpha} u(\eta) = f\left(\eta, u(\eta), {}^{\rho}\mathcal{D}_{0^+}^{\beta} u(\eta)\right). $$

From (5.11) , we have:

$$ \begin{aligned} \left({}^{\rho}\mathcal{I}_{0^+}^{1-\alpha}\mathcal{A}u\right)(\eta) &= {}^{\rho}\mathcal{I}_{0^+}^{1-\alpha}\,{}^{\rho}\mathcal{I}_{0^+}^{\alpha} f\left(\eta, u(\eta), {}^{\rho}\mathcal{D}_{0^+}^{\beta} u(\eta)\right) \\ &= {}^{\rho}\mathcal{I}_{0^+}^{1} f\left(\eta, u(\eta), {}^{\rho}\mathcal{D}_{0^+}^{\beta} u(\eta)\right) \\ &= {}^{\rho}\mathcal{I}_{0^+}^{1}\,{}^{\rho}\mathcal{D}_{0^+}^{\alpha} u(\eta)\ . \end{aligned} $$

If we use (1.22) and (1.27) we have:

$$ \begin{aligned} \left({}^{\rho}\mathcal{I}_{0^+}^{1-\alpha}\mathcal{A}u\right)(\eta) &= {}^{\rho}\mathcal{I}_{0^+}^{1}\,{}^{\rho}\mathcal{D}_{0^+}^{\alpha} u(\eta) \\ &= {}^{\rho}\mathcal{I}_{0^+}^{1}\left(\eta^{1-\rho}\frac{d}{d\eta}\right){}^{\rho}\mathcal{I}_{0^+}^{1-\alpha} u(\eta) \\ &= {}^{\rho}\mathcal{I}_{0^+}^{1-\alpha} u(\eta) - \left({}^{\rho}\mathcal{I}_{0^+}^{1-\alpha} u\right)\left(0^+\right) \\ &= {}^{\rho}\mathcal{I}_{0^+}^{1-\alpha} u(\eta)\ . \end{aligned} $$

Thus $\left({}^{\rho}\mathcal{I}_{0^+}^{1-\alpha}\mathcal{A}u\right)(0^+) = 0$. Consequently $\mathcal{A}(P) \subset P$. The proof is complete. ■

Now, we will prove our first existence result for the problem (5.3)-(5.4) which is based on Banach's fixed point theorem.

We suggest the following hypotheses:

(H1) $f : [0,\lambda] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

(H2) For all $0 < \beta < \alpha \leq 1$, there exist two constants $\sigma, \gamma > 0$, where $\gamma < \frac{\rho^{\alpha-\beta}\Gamma(1+\alpha-\beta)}{\lambda^{\rho(\alpha-\beta)}}$ such that:

$$ |f(\eta, u, v) - f(\eta, \tilde{u}, \tilde{v})| \leq \sigma |u - \tilde{u}| + \gamma |v - \tilde{v}|, $$

for any $u, v, \tilde{u}, \tilde{v} \in \mathbb{R}$ and $\eta \in [0,\lambda]$.

(H3) There exists three positive functions $a, b, c \in C[0,\lambda]$ such that:

$$ |f(\eta, u, v)| \leq a(\eta) + b(\eta)|u| + c(\eta)|v| \text{ for all } \eta \in [0,\lambda] \text{ and } u, v \in \mathbb{R}. $$

We denote

$$ M_0 = \frac{\rho^{\alpha-\beta}\Gamma(1+\alpha-\beta)\,a^*}{\rho^{\alpha-\beta}\Gamma(1+\alpha-\beta) - c^*\lambda^{\rho(\alpha-\beta)}}, $$

and

$$ M_1 = \frac{\rho^{\alpha-\beta}\Gamma(1+\alpha-\beta)\,b^*}{\rho^{\alpha-\beta}\Gamma(1+\alpha-\beta) - c^*\lambda^{\rho(\alpha-\beta)}}, $$

where $0<\beta<\alpha\leq 1$, and

$$a^{*}=\sup_{\eta\in[0,\lambda]}a\left(\eta\right),\ b^{*}=\sup_{\eta\in[0,\lambda]}b\left(\eta\right),\ c^{*}=\sup_{\eta\in[0,\lambda]}c\left(\eta\right),\ \text{with } c^{*}<\frac{\rho^{\alpha-\beta}\Gamma\left(1+\alpha-\beta\right)}{\lambda^{\rho(\alpha-\beta)}}.$$

In what follows, we present the principal theorems:

Theorem 5.1 *Assume the hypotheses* (H1), (H2) *hold. We give* $0<\beta<\alpha\leq 1$, *and* $\rho>0$. *If*

$$\frac{\sigma\lambda^{\rho\alpha}\Gamma\left(1+\alpha-\beta\right)}{\Gamma\left(\alpha+1\right)\left[\rho^{\alpha}\Gamma\left(1+\alpha-\beta\right)-\gamma\rho^{\beta}\lambda^{\rho(\alpha-\beta)}\right]}<1. \tag{5.12}$$

Then the problem (5.3)-(5.4) *admits a unique solution on* $[0,\lambda]$.

Proof. To begin the proof, we will transform the problem (5.3)-(5.4) into a fixed point problem. Define the operator $\mathcal{A}:P\to P$ by:

$$\mathcal{A}u\left(\eta\right)=\int_{0}^{\eta}G\left(\eta,s\right)f\left(s,u\left(s\right),{}^{\rho}\mathcal{D}_{0+}^{\beta}u\left(s\right)\right)ds. \tag{5.13}$$

Because the problem (5.3)-(5.4) is equivalent to the fractional integral equation (5.13), the fixed points of $\mathcal{A}$ are solutions of the problem (5.3)-(5.4).
Let $u,v\in P$, be such that:

$${}^{\rho}\mathcal{D}_{0+}^{\alpha}u\left(\eta\right)=f\left(\eta,u\left(\eta\right),{}^{\rho}\mathcal{D}_{0+}^{\beta}u\left(\eta\right)\right),\ \ {}^{\rho}\mathcal{D}_{0+}^{\alpha}v\left(\eta\right)=f\left(\eta,v\left(\eta\right),{}^{\rho}\mathcal{D}_{0+}^{\beta}v\left(\eta\right)\right).$$

Which implies that:

$$\mathcal{A}u\left(\eta\right)-\mathcal{A}v\left(\eta\right)=\int_{0}^{\eta}G\left(\eta,s\right)\left[f\left(s,u\left(s\right),{}^{\rho}\mathcal{D}_{0+}^{\beta}u\left(s\right)\right)-f\left(s,v\left(s\right),{}^{\rho}\mathcal{D}_{0+}^{\beta}v\left(s\right)\right)\right]ds.$$

Then, for all $\eta\in[0,\lambda]$

$$\left|\mathcal{A}u\left(\eta\right)-\mathcal{A}v\left(\eta\right)\right|\leq\int_{0}^{\eta}G\left(\eta,s\right)\left|{}^{\rho}\mathcal{D}_{0+}^{\alpha}u\left(s\right)-{}^{\rho}\mathcal{D}_{0+}^{\alpha}v\left(s\right)\right|ds. \tag{5.14}$$

By (H2) we have:

$$\begin{aligned}\left|{}^{\rho}\mathcal{D}_{0+}^{\alpha}u\left(\eta\right)-{}^{\rho}\mathcal{D}_{0+}^{\alpha}v\left(\eta\right)\right| &= \left|f\left(\eta,u\left(\eta\right),{}^{\rho}\mathcal{D}_{0+}^{\beta}u\left(\eta\right)\right)-f\left(\eta,v\left(\eta\right),{}^{\rho}\mathcal{D}_{0+}^{\beta}v\left(\eta\right)\right)\right|\\ &\leq \sigma\left|u\left(\eta\right)-v\left(\eta\right)\right|+\gamma\left|{}^{\rho}\mathcal{D}_{0+}^{\beta}u\left(\eta\right)-{}^{\rho}\mathcal{D}_{0+}^{\beta}v\left(\eta\right)\right|.\end{aligned}$$

By using (5.7) from lemma 5.2, we have:

$$\left\|{}^{\rho}\mathcal{D}_{0+}^{\alpha}u-{}^{\rho}\mathcal{D}_{0+}^{\alpha}v\right\|_{\infty}\leq\sigma\left\|u-v\right\|_{\infty}+\frac{\gamma\lambda^{\rho(\alpha-\beta)}}{\rho^{\alpha-\beta}\Gamma\left(1+\alpha-\beta\right)}\left\|{}^{\rho}\mathcal{D}_{0+}^{\alpha}u-{}^{\rho}\mathcal{D}_{0+}^{\alpha}v\right\|_{\infty},$$

thus

$$\left\| {}^{\rho}\mathcal{D}_{0+}^{\alpha}u - {}^{\rho}\mathcal{D}_{0+}^{\alpha}v \right\|_{\infty} \leq \frac{\sigma\rho^{\alpha-\beta}\Gamma\left(1+\alpha-\beta\right)}{\rho^{\alpha-\beta}\Gamma\left(1+\alpha-\beta\right)-\gamma\lambda^{\rho(\alpha-\beta)}}\left\|u-v\right\|_{\infty}.$$

From (5.14) we have:

$$\left\|\mathcal{A}u-\mathcal{A}v\right\|_{\infty} \leq \frac{\sigma\lambda^{\rho\alpha}\Gamma\left(1+\alpha-\beta\right)}{\Gamma\left(\alpha+1\right)\left[\rho^{\alpha}\Gamma\left(1+\alpha-\beta\right)-\gamma\rho^{\beta}\lambda^{\rho(\alpha-\beta)}\right]}\left\|u-v\right\|_{\infty}.$$

This implies that by (5.12), $\mathcal{A}$ is a contraction operator.

As a consequence of theorem 1.8, using BANACH's contraction principle [15], we deduce that $\mathcal{A}$ has a unique fixed point which is the unique solution of the problem (5.3)-(5.4) on $[0,\lambda]$. The proof is complete. ■

Theorem 5.2 *Assume that hypotheses* (H1)-(H3) *hold. We give* $0<\beta<\alpha\leq 1$, *and* $\rho>0$. *If we put*

$$\frac{M_1\lambda^{\rho\alpha}}{\rho^{\alpha}\Gamma\left(\alpha+1\right)}<1,$$

then the problem (5.3)-(5.4) *has at least one solution on* $[0,\lambda]$.

Proof. In the previous theorem, we already transform the problem (5.3)-(5.4) into a fixed point problem

$$\mathcal{A}u\left(\eta\right)=\int_0^{\eta}G\left(\eta,s\right)f\left(s,u\left(s\right),{}^{\rho}\mathcal{D}_{0+}^{\beta}u\left(s\right)\right)ds.$$

We demonstrate that $\mathcal{A}$ satisfies the assumption of SCHAUDER's fixed point theorem 1.9. This could be proved through three steps:

Step 1: $\mathcal{A}$ is a continuous operator.

Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence such that $\lim_{n\to\infty}u_n=u$ in P. Then for each $\eta\in[0,\lambda]$,

$$\left|\mathcal{A}u_n\left(\eta\right)-\mathcal{A}u\left(\eta\right)\right|\leq\int_0^{\eta}G\left(\eta,s\right)\left|f\left(s,u_n\left(s\right),{}^{\rho}\mathcal{D}_{0+}^{\beta}u_n\left(s\right)\right)-f\left(s,u\left(s\right),{}^{\rho}\mathcal{D}_{0+}^{\beta}u\left(s\right)\right)\right|ds, \tag{5.15}$$

where

$${}^{\rho}\mathcal{D}_{0+}^{\alpha}u_n\left(\eta\right)=f\left(\eta,u_n\left(\eta\right),{}^{\rho}\mathcal{D}_{0+}^{\beta}u_n\left(\eta\right)\right),\ \text{and}\ {}^{\rho}\mathcal{D}_{0+}^{\alpha}u\left(\eta\right)=f\left(\eta,u\left(\eta\right),{}^{\rho}\mathcal{D}_{0+}^{\beta}u\left(\eta\right)\right).$$

As a consequence of (H2), we find easily ${}^{\rho}\mathcal{D}_{0+}^{\alpha}u_n\to{}^{\rho}\mathcal{D}_{0+}^{\alpha}u$ in P. In fact we have:

$$\begin{aligned}\left|{}^{\rho}\mathcal{D}_{0+}^{\alpha}u_n\left(\eta\right)-{}^{\rho}\mathcal{D}_{0+}^{\alpha}u\left(\eta\right)\right| &= \left|f\left(\eta,u_n\left(\eta\right),{}^{\rho}\mathcal{D}_{0+}^{\beta}u_n\left(\eta\right)\right)-f\left(\eta,u\left(\eta\right),{}^{\rho}\mathcal{D}_{0+}^{\beta}u\left(\eta\right)\right)\right| \\ &\leq \sigma\left|u_n\left(\eta\right)-u\left(\eta\right)\right|+\gamma\left|{}^{\rho}\mathcal{D}_{0+}^{\beta}u_n\left(\eta\right)-{}^{\rho}\mathcal{D}_{0+}^{\beta}u\left(\eta\right)\right|.\end{aligned}$$

By using (5.7) from lemma 5.2, we have:

$$\left\| {}^{\rho}\mathcal{D}_{0+}^{\alpha}u_n - {}^{\rho}\mathcal{D}_{0+}^{\alpha}u \right\|_{\infty} \leq \sigma \left\| u_n - u \right\|_{\infty} + \frac{\gamma\lambda^{\rho(\alpha-\beta)}}{\rho^{\alpha-\beta}\Gamma\left(1+\alpha-\beta\right)} \left\| {}^{\rho}\mathcal{D}_{0+}^{\alpha}u_n - {}^{\rho}\mathcal{D}_{0+}^{\alpha}u \right\|_{\infty},$$

thus:

$$\left\| {}^{\rho}\mathcal{D}_{0+}^{\alpha}u_n - {}^{\rho}\mathcal{D}_{0+}^{\alpha}u \right\|_{\infty} \leq \frac{\sigma\rho^{\alpha-\beta}\Gamma\left(1+\alpha-\beta\right)}{\rho^{\alpha-\beta}\Gamma\left(1+\alpha-\beta\right) - \gamma\lambda^{\rho(\alpha-\beta)}} \left\| u_n - u \right\|_{\infty}.$$

Since $u_n \to u$, then we get ${}^{\rho}\mathcal{D}_{0+}^{\alpha}u_n\left(\eta\right) \to {}^{\rho}\mathcal{D}_{0+}^{\alpha}u\left(\eta\right)$ as $n \to \infty$ for each $\eta \in [0,\lambda]$.

Now let $K_1 > 0$, be such that for each $\eta \in [0,\lambda]$, we have:

$$\left|{}^{\rho}\mathcal{D}_{0+}^{\alpha}u_n\left(\eta\right)\right| \leq K_1, \left|{}^{\rho}\mathcal{D}_{0+}^{\alpha}u\left(\eta\right)\right| \leq K_1.$$

Then, we have:

$$\begin{aligned} \left|\mathcal{A}u_n\left(\eta\right) - \mathcal{A}u\left(\eta\right)\right| &\leq \int_0^{\eta} G\left(\eta,s\right)\left| f\left(s, u_n\left(s\right), {}^{\rho}\mathcal{D}_{0+}^{\beta}u_n\left(s\right)\right) - f\left(s, u\left(s\right), {}^{\rho}\mathcal{D}_{0+}^{\beta}u\left(s\right)\right)\right| ds \\ &\leq \int_0^{\eta} G\left(\eta,s\right)\left|{}^{\rho}\mathcal{D}_{0+}^{\alpha}u_n\left(s\right) - {}^{\rho}\mathcal{D}_{0+}^{\alpha}u\left(s\right)\right| ds \\ &\leq \int_0^{\eta} G\left(\eta,s\right)\left[\left|{}^{\rho}\mathcal{D}_{0+}^{\alpha}u_n\left(s\right)\right| + \left|{}^{\rho}\mathcal{D}_{0+}^{\alpha}u\left(s\right)\right|\right] ds \\ &\leq \int_0^{\eta} 2K_1 G\left(\eta,s\right) ds. \end{aligned}$$

For each $\eta \in [0,\lambda]$, the function $s \to 2K_1G\left(\eta,s\right)$ is integrable on $[0,\eta]$, then the Lebesgue dominated convergence theorem and (5.15) imply that:

$$\left|\mathcal{A}u_n\left(\eta\right) - \mathcal{A}u\left(\eta\right)\right| \to 0 \text{ as } n \to \infty,$$

and hence:

$$\lim_{n\to\infty} \left\|\mathcal{A}u_n - \mathcal{A}u\right\|_{\infty} = 0.$$

Consequently, $\mathcal{A}$ is continuous.

Step 2: Let

$$r \geq \frac{M_0\lambda^{\rho\alpha}}{\rho^{\alpha}\Gamma\left(\alpha+1\right) - M_1\lambda^{\rho\alpha}},$$

and we define:

$$P_r = \left\{u \in P : \left\|u\right\|_{\infty} \leq r\right\}.$$

It is clear that P_r is a bounded, closed and convex subset of P.

Let $u \in P_r$, and $\mathcal{A} : P_r \to P$ be the integral operator defined in (5.13), then $\mathcal{A}\left(P_r\right) \subset$

P_r.

In fact, by using (5.7) from lemma 5.2, and (H3) we have for each $\eta \in [0, \lambda]$:

$$\begin{aligned} \left|{}^{\rho}\mathcal{D}_{0+}^{\alpha} u(\eta)\right| &= \left| f\left(\eta, u(\eta), {}^{\rho}\mathcal{D}_{0+}^{\beta} u(\eta)\right)\right| \\ &\leq a(\eta) + b(\eta)\left|u(\eta)\right| + c(\eta)\left|{}^{\rho}\mathcal{D}_{0+}^{\beta} u(\eta)\right| \\ &\leq a^* + b^*\left|u(\eta)\right| + \frac{c^*\lambda^{\rho(\alpha-\beta)}}{\rho^{\alpha-\beta}\Gamma(1+\alpha-\beta)}\left\|{}^{\rho}\mathcal{D}_{0+}^{\alpha} u\right\|_{\infty}. \end{aligned}$$

Then

$$\begin{aligned} \left\|{}^{\rho}\mathcal{D}_{0+}^{\alpha} u\right\|_{\infty} &\leq \frac{\rho^{\alpha-\beta}\Gamma(1+\alpha-\beta)\, a^*}{\rho^{\alpha-\beta}\Gamma(1+\alpha-\beta) - c^*\lambda^{\rho(\alpha-\beta)}} + \frac{\rho^{\alpha-\beta}\Gamma(1+\alpha-\beta)\, b^*}{\rho^{\alpha-\beta}\Gamma(1+\alpha-\beta) - c^*\lambda^{\rho(\alpha-\beta)}} r \\ &\leq M_0 + M_1 r. \end{aligned} \tag{5.16}$$

Thus

$$\begin{aligned} \left|\mathcal{A}u(\eta)\right| &\leq \int_0^{\eta} G(\eta, s)\left| f\left(s, u(s), {}^{\rho}\mathcal{D}_{0+}^{\beta} u(s)\right)\right| ds \\ &\leq \frac{M_0\lambda^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} + \frac{M_1\lambda^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} r \\ &\leq \frac{\left[\rho^{\alpha}\Gamma(\alpha+1) - M_1\lambda^{\rho\alpha}\right]\frac{M_0\lambda^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1) - M_1\lambda^{\rho\alpha}} + M_1\lambda^{\rho\alpha} r}{\rho^{\alpha}\Gamma(\alpha+1)} \\ &\leq \frac{\left[\rho^{\alpha}\Gamma(\alpha+1) - M_1\lambda^{\rho\alpha}\right] r + M_1\lambda^{\rho\alpha} r}{\rho^{\alpha}\Gamma(\alpha+1)} \\ &\leq r. \end{aligned}$$

Then $\mathcal{A}(P_r) \subset P_r$.

Step 3: $\mathcal{A}(P_r)$ is relatively compact.

Let $\eta_1, \eta_2 \in [0, \lambda]$, $\eta_1 < \eta_2$, and $u \in P_r$. Then

$$\begin{aligned} \left|\mathcal{A}u(\eta_2) - \mathcal{A}u(\eta_1)\right| &= \left|\int_0^{\eta_2} G(\eta_2, s) f\left(s, u(s), {}^{\rho}\mathcal{D}_{0+}^{\beta} u(s)\right) ds \right. \\ &\quad \left. - \int_0^{\eta_1} G(\eta_1, s) f\left(s, u(s), {}^{\rho}\mathcal{D}_{0+}^{\beta} u(s)\right) ds\right| \\ &\leq \int_0^{\eta_1}\left|\left[G(\eta_2, s) - G(\eta_1, s)\right] f\left(s, u(s), {}^{\rho}\mathcal{D}_{0+}^{\beta} u(s)\right)\right| ds \\ &\quad + \int_{\eta_1}^{\eta_2} G(\eta_2, s)\left| f\left(s, u(s), {}^{\rho}\mathcal{D}_{0+}^{\beta} u(s)\right)\right| ds \\ &\quad (M_0 + M_1 r) \times \\ &\leq \left[\int_0^{\eta_1}\left|(G(\eta_2, s) - G(\eta_1, s))\right| ds + \int_{\eta_1}^{\eta_2} G(\eta_2, s)\, ds\right]. \end{aligned} \tag{5.17}$$

We have:

$$\begin{aligned} G(\eta_2, s) - G(\eta_1, s) &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} s^{\rho-1} \left[(\eta_2^\rho - s^\rho)^{\alpha-1} - (\eta_1^\rho - s^\rho)^{\alpha-1} \right] \\ &= \frac{-1}{\alpha\rho^\alpha \Gamma(\alpha)} \frac{d}{ds} \left[(\eta_2^\rho - s^\rho)^{\alpha} - (\eta_1^\rho - s^\rho)^{\alpha} \right] \end{aligned}$$

then

$$\int_0^{\eta_1} |(G(\eta_2, s) - G(\eta_1, s))| \, ds \leq \frac{1}{\rho^\alpha \Gamma(\alpha+1)} \left[(\eta_2^\rho - \eta_1^\rho)^\alpha + (\eta_2^{\rho\alpha} - \eta_1^{\rho\alpha}) \right]$$

we have also

$$\begin{aligned} \int_{\eta_1}^{\eta_2} G(\eta_2, s) \, ds &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{\eta_1}^{\eta_2} s^{\rho-1} (\eta_2^\rho - s^\rho)^{\alpha-1} \, ds \\ &= \frac{-1}{\alpha\rho^\alpha \Gamma(\alpha)} \left[(\eta_2^\rho - s^\rho)^\alpha \right]_{\eta_1}^{\eta_2} \\ &\leq \frac{1}{\rho^\alpha \Gamma(\alpha+1)} (\eta_2^\rho - \eta_1^\rho)^\alpha . \end{aligned}$$

Then (5.17) gives

$$|\mathcal{A}u(\eta_2) - \mathcal{A}u(\eta_1)| \leq \frac{M_0 + M_1 r}{\rho^\alpha \Gamma(\alpha+1)} \left[2 (\eta_2^\rho - \eta_1^\rho)^\alpha + (\eta_2^{\rho\alpha} - \eta_1^{\rho\alpha}) \right].$$

As $\eta_1 \to \eta_2$, the right side of the above inequality tends to zero.
As a consequence of steps 1 to 3 together, and by means of the Ascoli-Arzelà theorem 1.6, we deduce that $\mathcal{A} : P_r \to P_r$ is continuous, compact and satisfies the assumption of Schauder's fixed point theorem 1.9. Then $\mathcal{A}$ has a fixed point which is a solution of the problem (5.3)-(5.4) on $[0, \lambda]$. The proof is complete.

■

Our next existence result is based on the nonlinear alternative of Leray-Schauder type.

Theorem 5.3 *Assume* (H1)-(H3) *holds. Then the problem* (5.3)-(5.4) *has at least one solution on* $[0, \lambda]$.

Proof. Let $\alpha, \beta, \rho > 0$, be such that $\beta < \alpha \leq 1$.
We shall show that the operator $\mathcal{A}$ defined in (5.13), satisfies the assumption of Leray-Schauder's fixed point theorem 1.10. The proof will be given in several steps.

Step 1: Clearly $\mathcal{A}$ is continuous.

Step 2: $\mathcal{A}$ maps bounded sets into bounded sets in P.

Indeed, it is enough to show that for any $\omega > 0$ there exist a positive constant ℓ such that for each $u \in B_\omega = \{u \in P : \|u\|_\infty \leq \omega\}$, we have $\|\mathcal{A}u\|_\infty \leq \ell$.

For $u \in B_\omega$, we have, for each $\eta \in [0, \lambda]$,

$$|\mathcal{A}u(\eta)| \leq \int_0^\eta G(\eta, s) \left| f\left(s, u(s), {}^\rho\mathcal{D}_{0+}^\beta u(s)\right) \right| ds. \tag{5.18}$$

By $(H3)$, similarly (5.16), for each $\eta \in [0, \lambda]$, we have:

$$\left| f\left(\eta, u(\eta), {}^\rho\mathcal{D}_{0+}^\beta u(\eta)\right) \right| \leq M_0 + M_1\omega.$$

Thus (5.18) implies that:

$$\|\mathcal{A}u\|_\infty \leq \frac{M_0\lambda^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)} + \frac{M_1\lambda^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)}\omega = \ell.$$

Step 3: Clearly, $\mathcal{A}$ maps bounded sets into equicontinuous sets of P.

We conclude that $\mathcal{A} : P \to P$ is continuous and completely continuous.

Step 4: A priori bounds.

We now show there exists an open set $U \subset P$ with $u \neq \mu\mathcal{A}(u)$ for $\mu \in (0, 1)$ and $u \in \partial U$.

Let $u \in P$ and $u = \mu\mathcal{A}(u)$ for some $0 < \mu < 1$. Thus for each $\eta \in [0, \lambda]$, we have:

$$u(\eta) \leq \mu \int_0^\eta G(\eta, s) \left| f\left(s, u(s), {}^\rho\mathcal{D}_{0+}^\beta u(s)\right) \right| ds.$$

By $(H3)$, for all solution $u \in P$, of the problem (5.3)-(5.4), we have:

$$\begin{aligned} |u(\eta)| &= \left| \int_0^\eta G(\eta, s) f\left(s, u(s), {}^\rho\mathcal{D}_{0+}^\beta u(s)\right) ds \right| \\ &\leq \int_0^\eta G(\eta, s) \left|{}^\rho\mathcal{D}_{0+}^\alpha u(s)\right| ds. \end{aligned}$$

Then for each $\eta \in [0, \lambda]$, we have:

$$\begin{aligned} \left|{}^\rho\mathcal{D}_{0+}^\alpha u(\eta)\right| &= \left| f\left(\eta, u(\eta), {}^\rho\mathcal{D}_{0+}^\beta u(\eta)\right) \right| \\ &\leq a(\eta) + b(\eta)|u(\eta)| + c(\eta)\left|{}^\rho\mathcal{D}_{0+}^\beta u(\eta)\right| \\ &\leq a^* + b^*|u(\eta)| + \frac{c^*\lambda^{\rho(\alpha-\beta)}}{\rho^{\alpha-\beta}\Gamma(1+\alpha-\beta)} \sup_{0\leq\eta\leq\lambda} \left|{}^\rho\mathcal{D}_{0+}^\alpha u(\eta)\right|. \end{aligned}$$

Then

$$\begin{aligned} \sup_{0\leq\eta\leq\lambda} \left|{}^\rho\mathcal{D}_{0+}^\alpha u(\eta)\right| &\leq \frac{\rho^{\alpha-\beta}\Gamma(1+\alpha-\beta)}{\rho^{\alpha-\beta}\Gamma(1+\alpha-\beta) - c^*\lambda^{\rho(\alpha-\beta)}} \left(a^* + b^* \sup_{0\leq\eta\leq\lambda}|u(\eta)|\right) \\ &\leq M_0 + M_1 \sup_{0\leq\eta\leq\lambda}|u(\eta)|. \end{aligned}$$

Hence

$$\sup_{0\leq\eta\leq\lambda}|u(\eta)| \leq \frac{M_0\lambda^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)} + \int_0^\eta M_1 G(\eta,s)\left\{\sup_{0\leq s\leq\lambda}|u(s)|\right\}ds.$$

After the GRONWALL lemma [16], we have:

$$\sup_{0\leq\eta\leq\lambda}|u(\eta)| \leq \frac{M_0\lambda^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)}\exp\left(\frac{M_1\lambda^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)}\right).$$

Thus

$$\|u\|_\infty \leq \frac{M_0\lambda^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)}\exp\left(\frac{M_1\lambda^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)}\right) = M_2.$$

Let

$$U = \left\{u \in P : \|u\|_\infty < M_2+1\right\}.$$

By choosing U, there is no $u \in \partial U$, such that $u = \mu\mathcal{A}(u)$, for $\mu \in (0,1)$. As a consequence of LERAY-SCHAUDER's theorem 1.10, $\mathcal{A}$ has a fixed point u in U which is a solution to (5.3)-(5.4). The proof is complete.

■

5.4 Examples

Example 1. Consider the following problem:

$$\begin{cases} {}^1_x\mathcal{D}^{\frac{1}{2}}_{0^+}w(x,t) = \dfrac{\psi^2(t)\cos\left(\frac{x}{\varphi(t)}\right)}{\pi\varphi^{\frac{1}{2}}(t)\left(\sqrt{2}\cos\left(\frac{x}{\varphi(t)}\right)+\sin\left(\frac{x}{\varphi(t)}\right)\right)\left[\psi(t)+|w(x,t)|+\varphi^{\frac{1}{4}}(t)\left|{}^1_x\mathcal{D}^{\frac{1}{4}}_{0^+}w(x,t)\right|\right]}, \; x \in \left[0, \frac{\pi}{4}\varphi(t)\right]. \\ \left({}^1_x\mathcal{I}^{\frac{1}{2}}_{0^+}w\right)(0^+,t) = 0, \; t \in [0,T], \text{ for any } T > 0. \end{cases}$$

Set the function

$$f(x,t,w) = \frac{\psi^2(t)\cos\left(\frac{x}{\varphi(t)}\right)}{\pi\varphi^{\frac{1}{2}}(t)\left(\sqrt{2}\cos\left(\frac{x}{\varphi(t)}\right)+\sin\left(\frac{x}{\varphi(t)}\right)\right)\left[\psi(t)+|w(x,t)|+\varphi^{\frac{1}{4}}(t)\left|{}^1_x\mathcal{D}^{\frac{1}{4}}_{0^+}w(x,t)\right|\right]},$$

which satisfies the hypotheses of theorem 2.3. As

$${}^1_x\mathcal{D}^{\frac{1}{2}}_{0^+}w(x,t) = \frac{\psi(t)}{\varphi^{\frac{1}{2}}(t)}\,{}^1\mathcal{D}^{\frac{1}{2}}_{0^+}u(\eta),$$

and

$$\begin{aligned} f(x,t,w) &= \frac{\psi^2(t)\cos\left(\frac{x}{\varphi(t)}\right)}{\pi\varphi^{\frac{1}{2}}(t)\left(\sqrt{2}\cos\left(\frac{x}{\varphi(t)}\right)+\sin\left(\frac{x}{\varphi(t)}\right)\right)\left[\psi(t)+|w(x,t)|+\varphi^{\frac{1}{4}}(t)\left|{}^1_x\mathcal{D}^{\frac{1}{4}}_{0^+}w(x,t)\right|\right]} \\ &= \frac{\psi(t)}{\varphi^{\frac{1}{2}}(t)}\frac{\cos(\eta)}{\pi\left(\sqrt{2}\cos(\eta)+\sin(\eta)\right)\left[1+|u(\eta)|+\left|{}^1\mathcal{D}^{\frac{1}{4}}_{0^+}u(\eta)\right|\right]}. \end{aligned}$$

Then, the transformation (5.2) reduces the previous problem of fractional-order's partial differential equation to problem of fractional differential equation of the form

$$\begin{cases} {}^{1}\mathcal{D}_{0^+}^{\frac{1}{2}} u(\eta) = \frac{\cos(\eta)}{\pi\left(\sqrt{2}\cos(\eta)+\sin(\eta)\right)\left[1+|u(\eta)|+\left|{}^{1}\mathcal{D}_{0^+}^{\frac{1}{4}} u(\eta)\right|\right]}, \ \eta \in \left[0, \frac{\pi}{4}\right], \\ \left({}^{1}\mathcal{I}_{0^+}^{\frac{1}{2}} u\right)(0^+) = 0. \end{cases} \tag{5.19}$$

Set:

$$f(\eta, u, v) = \frac{\cos(\eta)}{\pi\left(\sqrt{2}\cos(\eta)+\sin(\eta)\right)\left[1+|u|+|v|\right]}, \ \eta \in \left[0, \frac{\pi}{4}\right], \ u, v \in \mathbb{R}.$$

Because $\sin(\eta)$, $\cos(\eta)$ are continuous positive functions $\forall \eta \in \left[0, \frac{\pi}{4}\right]$, the function f is jointly continuous. For any $u, v, \tilde{u}, \tilde{v} \in \mathbb{R}$ and $\eta \in \left[0, \frac{\pi}{4}\right]$, we have $\frac{\sqrt{2}}{2} \leq \cos(\eta) \leq 1$, and $0 \leq \sin(\eta) \leq \frac{\sqrt{2}}{2}$, then:

$$|f(\eta, u, v) - f(\eta, \tilde{u}, \tilde{v})| \leq \frac{1}{\pi}\left(|u-\tilde{u}| + |v-\tilde{v}|\right).$$

Hence, the condition (H2) is satisfied with:

$$\sigma = \gamma = \frac{1}{\pi} \simeq 0.3183 < \frac{\rho^{\alpha-\beta}\Gamma(1+\alpha-\beta)}{\lambda^{\rho(\alpha-\beta)}} = \left(\frac{\pi}{4}\right)^{-\frac{1}{4}} \Gamma\left(\frac{5}{4}\right) \simeq 0.9628.$$

It remains to show that the condition (5.12):

$$\begin{aligned} \frac{\sigma\lambda^{\rho\alpha}\Gamma(1+\alpha-\beta)}{\Gamma(\alpha+1)\left[\rho^{\alpha}\Gamma(1+\alpha-\beta) - \gamma\rho^{\beta}\lambda^{\rho(\alpha-\beta)}\right]} &= \frac{\left(\frac{1}{\pi}\right)\left(\frac{\pi}{4}\right)^{\frac{1}{2}}\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{2}\right)\left[\Gamma\left(\frac{5}{4}\right) - \frac{1}{\pi}\left(\frac{\pi}{4}\right)^{\frac{1}{4}}\right]} \\ &= \frac{\sqrt{\pi}\Gamma\left(\frac{5}{4}\right)}{2\Gamma\left(\frac{3}{2}\right)\left[\pi\Gamma\left(\frac{5}{4}\right) - \left(\frac{\pi}{4}\right)^{\frac{1}{4}}\right]} \\ &\simeq 0.4755 < 1, \end{aligned}$$

is satisfied. It follows from theorem 5.1 that the problem (5.19) has a unique solution.

Example 2. Consider the following problem:

$$\begin{cases} {}_{x}^{1}\mathcal{D}_{0^+}^{\frac{1}{2}} w(x,t) = \frac{\psi(t)\cos\left(\frac{x}{\varphi(t)}\right)\left[2\psi(t)+|w(x,t)|+\varphi^{\frac{1}{4}}(t)\left|{}_{x}^{1}\mathcal{D}_{0^+}^{\frac{1}{4}} w(x,t)\right|\right]}{\pi\varphi^{\frac{1}{2}}(t)\left(\sqrt{2}\cos\left(\frac{x}{\varphi(t)}\right)+\sin\left(\frac{x}{\varphi(t)}\right)\right)\left[\psi(t)+|w(x,t)|+\varphi^{\frac{1}{4}}(t)\left|{}_{x}^{1}\mathcal{D}_{0^+}^{\frac{1}{4}} w(x,t)\right|\right]}, \ x \in \left[0, \frac{\pi}{4}\varphi(t)\right]. \\ \left({}_{x}^{1}\mathcal{I}_{0^+}^{\frac{1}{2}} w\right)(0^+, t) = 0, \ t \in [0, T], \ \text{for any } T > 0. \end{cases}$$

Set the function

$$f(x,t,w) = \frac{\psi(t)\cos\left(\frac{x}{\varphi(t)}\right)\left[2\psi(t)+|w(x,t)|+\varphi^{\frac{1}{4}}(t)\left|{}_{x}^{1}\mathcal{D}_{0^+}^{\frac{1}{4}} w(x,t)\right|\right]}{\pi\varphi^{\frac{1}{2}}(t)\left(\sqrt{2}\cos\left(\frac{x}{\varphi(t)}\right)+\sin\left(\frac{x}{\varphi(t)}\right)\right)\left[\psi(t)+|w(x,t)|+\varphi^{\frac{1}{4}}(t)\left|{}_{x}^{1}\mathcal{D}_{0^+}^{\frac{1}{4}} w(x,t)\right|\right]},$$

which satisfies the hypotheses of theorem 2.3. As

$$
\begin{aligned}
f(x,t,w) &= \frac{\psi(t)\cos\left(\frac{x}{\varphi(t)}\right)\left[2\psi(t)+|w(x,t)|+\varphi^{\frac{1}{4}}(t)\left|{}^{1}_{x}\mathcal{D}^{\frac{1}{4}}_{0^+}w(x,t)\right|\right]}{\pi\varphi^{\frac{1}{2}}(t)\left(\sqrt{2}\cos\left(\frac{x}{\varphi(t)}\right)+\sin\left(\frac{x}{\varphi(t)}\right)\right)\left[\psi(t)+|w(x,t)|+\varphi^{\frac{1}{4}}(t)\left|{}^{1}_{x}\mathcal{D}^{\frac{1}{4}}_{0^+}w(x,t)\right|\right]} \\
&= \frac{\psi(t)}{\varphi^{\frac{1}{2}}(t)}\frac{\cos(\eta)\left[2+|u(\eta)|+\left|{}^{1}\mathcal{D}^{\frac{1}{4}}_{0^+}u(\eta)\right|\right]}{\pi\left(\sqrt{2}\cos(\eta)+\sin(\eta)\right)\left[1+|u(\eta)|+\left|{}^{1}\mathcal{D}^{\frac{1}{4}}_{0^+}u(\eta)\right|\right]}.
\end{aligned}
$$

Then, the transformation (4.2) reduces the previous problem of fractional-order's partial differential equation to problem of fractional differential equation of the form

$$
\begin{cases}
{}^{1}\mathcal{D}^{\frac{1}{2}}_{0^+}u(\eta) = \frac{\cos(\eta)\left[2+|u(\eta)|+\left|{}^{1}\mathcal{D}^{\frac{1}{4}}_{0^+}u(\eta)\right|\right]}{\pi\left(\sqrt{2}\cos(\eta)+\sin(\eta)\right)\left[1+|u(\eta)|+\left|{}^{1}\mathcal{D}^{\frac{1}{4}}_{0^+}u(\eta)\right|\right]}, \ \eta\in\left[0,\frac{\pi}{4}\right], \\
\left({}^{1}\mathcal{I}^{\frac{1}{2}}_{0^+}u\right)(0^+)=0.
\end{cases}
\tag{5.20}
$$

Set:

$$
f(\eta,u,v)=\frac{\cos(\eta)\left[2+|u|+|v|\right]}{\pi\left(\sqrt{2}\cos(\eta)+\sin(\eta)\right)\left[1+|u|+|v|\right]}, \ \eta\in\left[0,\frac{\pi}{4}\right], \ u,v\in\mathbb{R}.
$$

Clearly, the function f is jointly continuous. For any $u,v,\tilde{u},\tilde{v}\in\mathbb{R}$ and $\eta\in\left[0,\frac{\pi}{4}\right]$, we have:

$$
|f(\eta,u,v)-f(\eta,\tilde{u},\tilde{v})|\leq\frac{1}{\pi}\left(|u-\tilde{u}|+|v-\tilde{v}|\right).
$$

Therefore, the condition (H2) is satisfied with:

$$
\sigma=\gamma=\frac{1}{\pi}\simeq 0.3183<\frac{\rho^{\alpha-\beta}\Gamma(1+\alpha-\beta)}{\lambda^{\rho(\alpha-\beta)}}=\left(\frac{\pi}{4}\right)^{-\frac{1}{4}}\Gamma\left(\frac{5}{4}\right)\simeq 0.9628.
$$

Also, we have:

$$
|f(\eta,u,v)|\leq\frac{\cos(\eta)}{\pi\left(\sqrt{2}\cos(\eta)+\sin(\eta)\right)}\left(2+|u|+|v|\right).
$$

Thus, the condition (H3) is satisfied with:

$$
a(\eta)=\frac{2\cos(\eta)}{\pi\left(\sqrt{2}\cos(\eta)+\sin(\eta)\right)}, \ \text{and } b(\eta)=c(\eta)=\frac{\cos(\eta)}{\pi\left(\sqrt{2}\cos(\eta)+\sin(\eta)\right)}.
$$

We have also:

$$
a^*=\frac{2}{\pi},
$$

and

$$
b^*=c^*=\frac{1}{\pi}\simeq 0.3183<\frac{\rho^{\alpha-\beta}\Gamma(1+\alpha-\beta)}{\lambda^{\rho(\alpha-\beta)}}=\left(\frac{\pi}{4}\right)^{-\frac{1}{4}}\Gamma\left(\frac{5}{4}\right)\simeq 0.9628,
$$

and

$$
M_0=\frac{\rho^{\alpha-\beta}\Gamma(1+\alpha-\beta)a^*}{\rho^{\alpha-\beta}\Gamma(1+\alpha-\beta)-c^*\lambda^{\rho(\alpha-\beta)}}=\frac{2\Gamma\left(\frac{5}{4}\right)}{\pi\Gamma\left(\frac{5}{4}\right)-\left(\frac{\pi}{4}\right)^{\frac{1}{4}}},
$$

also:

$$M_1 = \frac{\rho^{\alpha-\beta}\Gamma\left(1+\alpha-\beta\right)b^*}{\rho^{\alpha-\beta}\Gamma\left(1+\alpha-\beta\right)-c^*\lambda^{\rho(\alpha-\beta)}} = \frac{\Gamma\left(\frac{5}{4}\right)}{\pi\Gamma\left(\frac{5}{4}\right)-\left(\frac{\pi}{4}\right)^{\frac{1}{4}}}.$$

And the condition:

$$\begin{aligned}\frac{M_1\lambda^{\rho\alpha}}{\rho^{\alpha}\Gamma\left(\alpha+1\right)} &= \frac{\left(\frac{\Gamma\left(\frac{5}{4}\right)}{\pi\Gamma\left(\frac{5}{4}\right)-\left(\frac{\pi}{4}\right)^{\frac{1}{4}}}\right)\left(\frac{\pi}{4}\right)^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}+1\right)} \\ &= \frac{\sqrt{\pi}\Gamma\left(\frac{5}{4}\right)}{2\Gamma\left(\frac{3}{2}\right)\left[\pi\Gamma\left(\frac{5}{4}\right)-\left(\frac{\pi}{4}\right)^{\frac{1}{4}}\right]} \\ &\simeq 0.4755 < 1.\end{aligned}$$

It follows from theorem 5.2 and theorem 5.3, that the problem (5.20) has at least one solution.

5.5 Proof of Main Theorems

In this section, we prove the existence and uniqueness of solutions of the following implicit problem of the nonlinear partial differential equations of space-fractional order [29], [34]:

$$ {}_x^{\rho}\mathcal{D}_{0^+}^{\alpha} w = f\left(x,t,w,{}_x^{\rho}\mathcal{D}_{0^+}^{\beta} w\right),\ (x,t)\in[0,X]\times[0,T], \tag{5.21}$$

with the integral condition:

$$\left({}_x^{\rho}\mathcal{I}_{0^+}^{1-\alpha} w\right)\left(0^+,t\right) = 0, \tag{5.22}$$

under the generalized self-similar form which is:

$$w\left(x,t\right)=\psi\left(t\right)u\left(\eta\right),\ \text{with } \eta=\frac{x}{\varphi\left(t\right)},\ \text{and } \varphi,\psi\in C\left[0,T\right]\to\mathbb{R}_+. \tag{5.23}$$

Here $0<\beta<\alpha\leq 1$. Then the transformation (5.23) reduces the fractional-order's partial differential equation (5.21) to the ordinary differential equation of fractional order of the form:

$$ {}^{\rho}\mathcal{D}_{0^+}^{\alpha} u\left(\eta\right) = f\left(\eta,u\left(\eta\right),{}^{\rho}\mathcal{D}_{0^+}^{\beta} u\left(\eta\right)\right),\ \eta\in[0,\lambda], \tag{5.24}$$

with the integral condition:

$$\left({}^{\rho}\mathcal{I}_{0^+}^{1-\alpha} u\right)\left(0^+\right) = 0. \tag{5.25}$$

Where $\lambda = X\bar{\varphi}^{-1}$, with $\bar{\varphi}=\min\limits_{0\leq t\leq T}\varphi\left(t\right)$ is a finite positive constant.

Now we proceed to demonstrate the theorem 2.7.

Proof of theorem 2.7:

Now, we prove our existence and uniqueness results of the problem (5.21) with the integral condition (5.22) . By using (5.23) , the hypotheses $\overline{(\mathrm{H1})}$, $\overline{(\mathrm{H2})}$ and $\overline{(\mathrm{H3})}$, presented in theorem 2.7, are equivalent to the hypotheses (H1) , (H2) and (H3) , presented in the section 5.3, respectively. Also the conditions

$$\frac{M_1 X^{\rho\alpha}}{(\rho\bar{\varphi}^{\rho})^{\alpha}\,\Gamma(\alpha+1)} < 1, \text{ and } \frac{\sigma X^{\rho\alpha}\Gamma(1+\alpha-\beta)}{\Gamma(\alpha+1)\left[(\rho\bar{\varphi}^{\rho})^{\alpha}\,\Gamma(1+\alpha-\beta)-\gamma(\rho\bar{\varphi}^{\rho})^{\beta}X^{\rho(\alpha-\beta)}\right]} < 1,$$

are equivalent to

$$\frac{M_1\lambda^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} < 1, \text{ and } \frac{\sigma\lambda^{\rho\alpha}\Gamma(1+\alpha-\beta)}{\Gamma(\alpha+1)\left[\rho^{\alpha}\Gamma(1+\alpha-\beta)-\gamma\rho^{\beta}\lambda^{\rho(\alpha-\beta)}\right]} < 1,$$

respectively.

1) We already proved in theorem 5.3, the existence of the solution of the problem (5.24) with the integral condition (5.25) provided that (H1) , (H2) and (H3) hold. Consequently, if $\overline{(\mathrm{H1})}$, $\overline{(\mathrm{H2})}$ and $\overline{(\mathrm{H3})}$ hold, the problem (5.21) with the integral condition (5.22) has at least one solution under the generalized self-similar form (5.23) .

2) We already proved in theorem 5.2, the existence of the solution of the problem (5.24) with the integral condition (5.25) provided that (H1) , (H2) and (H3) hold, and

$$\frac{M_1\lambda^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} < 1.$$

Consequently, if $\overline{(\mathrm{H1})}$, $\overline{(\mathrm{H2})}$ and $\overline{(\mathrm{H3})}$ hold, and

$$\frac{M_1 X^{\rho\alpha}}{(\rho\bar{\varphi}^{\rho})^{\alpha}\,\Gamma(\alpha+1)} < 1,$$

the problem (5.21) with the integral condition (5.22) has at least one solution under the generalized self-similar form (5.23) .

3) We already proved in theorem 5.1, the existence and uniqueness of the solution of the problem (5.24) with the integral condition (5.25) provided that (H1) , (H2) hold, and

$$\frac{\sigma\lambda^{\rho\alpha}\Gamma(1+\alpha-\beta)}{\Gamma(\alpha+1)\left[\rho^{\alpha}\Gamma(1+\alpha-\beta)-\gamma\rho^{\beta}\lambda^{\rho(\alpha-\beta)}\right]} < 1.$$

Consequently, if $\overline{(\mathrm{H1})}$, $\overline{(\mathrm{H2})}$ hold, and

$$\frac{\sigma X^{\rho\alpha}\Gamma(1+\alpha-\beta)}{\Gamma(\alpha+1)\left[(\rho\bar{\varphi}^{\rho})^{\alpha}\,\Gamma(1+\alpha-\beta)-\gamma(\rho\bar{\varphi}^{\rho})^{\beta}X^{\rho(\alpha-\beta)}\right]} < 1.$$

Then, there exists a unique solution of the problem (5.24) with the integral condition (5.25) under the generalized self-similar form (5.23) .

The proof of theorem 2.7 is complete.

Conclusion

This study is part of the process of applying the methods of analysis of existence and uniqueness results of solutions for certain classes of partial differential equations of fractional order.

In this thesis, we used [7], [8] to give several existence and uniqueness results of generalized self-similar solutions for certain classes of FPDEs, thus, realizing the fractional derivative of Katugampola in Banach spaces. These studies were done mainly using Banach's contraction principle, Schauder's and Guo-Krasnosel'skii's fixed point theorems, and the technique of the nonlinear alternative of Leray-Schauder type (see [15], [25]).

The first chapter allowed us to familiarize some notions related to the fractional calculus and provided some elementary, but useful results for our study. The second chapter was devote to introduce the different basic definitions and results (lemmas, theorems) crucial to self-similar form in relation to the theory of partial differential equations with fractional operators, and we presented our main results of this work.

We have discussed in the third, fourth and fifth chapters the existence and uniqueness of generalized self-similar solutions for some nonlinear FPDEs using Katugampola's fractional derivative, in the form:

$$ {}_{x}^{\rho}\mathcal{D}_{0^+}^{\alpha} w = f\left(x, t, w, {}_{t}^{\rho}\mathcal{D}_{0^+}^{\alpha} w\,, {}_{x}^{\rho}\mathcal{D}_{0^+}^{\beta} w\,, \ldots\right),\ \alpha > \beta > \ldots > 0. $$

The existence results of solutions of previous FPDE, are given with boundary value, with initial value or with integral conditions, and we gave in each chapter some examples to illustrate the applicability of our results.

This work opens the way for other developments on partial differential equations of fractional order. In particular, we can offer the following perspectives:

- The qualitative study of these problems, in particular, will examine the asymptotic behavior of the solutions.

- The search for numerical methods of resolution of partial differential equations or of differential equations of fractional order.
- The application of methods other than those proposed in this thesis, for the study of the existence of self-similar solutions of other problems of FPDEs, in new spaces.
- The application of other direct methods that can be more precise than those proposed in this thesis, for the study of the existence of the solutions of FPDE.

These perspectives are possible directions for future works.

Bibliography

[1] R. P. Agarwal, M. Meehan, D. O'Regan, *Fixed Point Theory and Applications,* Cambridge University Press, Cambridge, 2001.

[2] R. Almeida, A. B. Malinowska, Odzijewicz, *Fractional differential equations with dependence on the Caputo-Katugampola derivative*, J. Comput Nonlinear Dynam. **11** (6) (2016).

[3] Y. Arioua, N. Benhamidouche, *Boundary value problem for Caputo-Hadamard fractional differential equations*, Surveys in Mathematics and its Applications **12** (2017), 103-115.

[4] R. L. Bagley and P. J. Torvik, *A theoritical basis for the application of fractional calculus in viscoelasticity*, Journal of Rheology **27** (1983), 201-210.

[5] R. L. Bagley and P. J. Torvik, *On the fractional calculus model of viscoelasticity behavior*, Journal of Rheology **30** (1986), 133-155.

[6] Z. Bai, H. Lü, *Positive solutions for boundary value problem of nonlinear fractional differential equation*, J. Math. Anal. Appl. **311** (2005), 495-505.

[7] B. Basti, Y. Arioua, N. Benhamidouche, *Existence and uniqueness of solutions for nonlinear Katugampola fractional differential equations*, Journal of Mathematics and Applications, **42** (2019).

[8] Y. Arioua, B. Basti, N. Benhamidouche, *Initial value problem for nonlinear implicit fractional differential equations with Katugampola derivative*, Applied Mathematics E-Notes **19** (2019).

[9] M. Benchohra, J. E. Lazreg, *Existence results for nonlinear implicit fractional differential equations*, Surveys in Mathematics and its Applications. **9** (2014), 79-92.

[10] N. Benhamidouche, *Exact solutions to some nonlinear PDEs, travelling profiles method*, Electronic Journal of Qualitative Theory of differential Equation **15** (2008), 1-7.

[11] E. Buckwar, Y. Luchko, *Invariance of a partial differential equation of fractional order under a certain Lie group*, Preprint No A-01-98, Free University of Berlin (1998) .

[12] A. A. Dezin, *Partial Differential Equations, An Introduction to a General Theory of Linear Boundary Value Problems,* Springer-Verlag, Berlin-New York, (1987).

[13] K. Diethelm, *The analysis of fractional differential equations*, Springer Berlin (2010).

[14] M. El-Shahed, *Positive solutions for boundary value problem of nonlinear fractional differential equation*, Abstract and Applied Analysis 2007, 1-8 (2007).

[15] A. Granas, J. Dugundji, *Fixed point theory*, Springer-Verlag, New York (2003).

[16] T. H. Gronwall, *Note on the derivatives with respect to a parameter of the solutions of a system of differential equations*, Ann. Math. **20** (1918-1919), 292-296.

[17] J. Hulshof, J. L. Vàzquez, *Self similar solutions of the second kind for the modified porous medium equation*, European J. App. Math, **5** (1994) , 391-403.

[18] N. H. Ibragimov, *CRC Handbook of Lie group Analysis of Differential Equations, Vol 1 Symmetries, Exact solutions and Conservation Laws*, CRS Press , Boca Raton, 1994.

[19] U. N. Katugampola, *A new approach to generalized fractional derivatives*, Mathematical Analysis and Applications **6** (4) (2014), 1-15.

[20] U. N. Katugampola, *Existence and uniqueness results for a class of genaralized fractional differential equations*, Department of Maths, University of Delaware, Newark, USA. (2016).

[21] U. N. Katugampola, *Mellin transforms of generalized fractional integrals and derivatives*, Applied Mathematics and Computation. **257** (2015), 566-580.

[22] U. N. Katugampola, *New approach to a genaralized fractional integral*, Applied Mathematics and Computation. **218** (3) (2011), 860-865.

[23] A. A. Kilbas, H. H. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier Science B.V., Amsterdam, (2006).

[24] K. M. Kolwankar, A. D. Gangal, *Fractional Differentiability of nowhere differentiable functions and dimensions*, American Institute of Phyics, No. **4** (1996) .

[25] M. Krasnosel'skii, *Positive solutions of operator equations*, Noordhoff, Groningen (1964).

[26] R. W. Leggett, L. R. Williams, *Multiple positive fixed points of nonlinear operators on ordered Banach spaces*, Indiana Univ. Math. J. **28** (1979), 673-688.

[27] Y. Luchko, R. Gorenflo, *Scale-invariant solutions of a partial differential equation of fractional order*, Arnimallee 2-6, Free University of Berlin (1998).

[28] G. M. Mittag-Leffler, *Sur la nouvelle fonction* $E_\alpha(t)$, C. R. Académie des Sciences **137** (1903), 554-558.

[29] S. A. Murad, S. Hadid, *Existence and uniqueness theorem for fractional differential equation with integral boundary condition*, J. Frac. Calc. Appl., **3** (6) (2012), 1-9.

[30] S. M. Nikol'skii, *Course of Mathematical Analysis,* vol. 1-2, Nauka, Moscow, (1983).

[31] K. Nishimoto, *An essence of Nishimoto's Fractional Calculus*, Descartes Press Co. (1991).

[32] P. J. Olver, *Application of Lie Groups to Differential Equations*, Springer-Verlag, NewYork, 1986.

[33] I. Podlubny, *Fractional differential equations*, Mathematics in Science and Engineering, Academic Press, New York, (1999).

[34] A. D. Polyanin, V. F. Zaitsev, *Handbook of Nonlinear Partial Equation*, Chapman&Hall/CRC, Boca Raton, 2004.

[35] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integral and derivatives (theory and applications)*, Gordon and Breach, Switzerland, (1993).

[36] H. Weyl, *Bemerkungen zum begriff des differential quotienten gebrochener ordnung*, Vierteljschr. Naturforsch. Gesellsch. Zürich, **62** (1) (1917), 296-302.

[37] X. Xu, D. Jiang, C. Yuan, *Multiple positive solutions for the boundary value problem of a nonlinear fractional differential equation*, Nonlinear Anal. **71** (2009), 4676-4688.

الملخص:

نتطرق في هذه الأطروحة لدراسة عدة نتائج حول وجود ووحدانية حلول ذات التشابه العام لمعادلات ذات اشتقاقات جزئية غير خطية ذات رتبة كسرية بمفهوم كاتوغامبولا، بشروط حدودية، بقيمة ابتدائية، أو أخرى بشروط تكاملية في فضاء بناخي، وذلك باستعمال تقنية نظرية النقطة الثابتة لكروسنوسلسكي، شودار، مبدأ بناخ للتقلص، وتقنية البديل غير الخطية لنمط ليري-شودار.

كلمات مفتاحية: معادلات ذات اشتقاقات جزئية ذات رتبة كسرية، معادلات تفاضلية ذات رتبة كسرية، مشتقات كسرية لكاتوغامبولا، حلول مشابهة، نقطة ثابتة، فضاء بناخي، مسألة بشروط حدودية، مسألة بقيمة ابتدائية، شروط تكاملية، وجود، وحدانية.

Résumé :

Dans cette thèse, nous allons discuter plusieurs résultats d'existence et d'unicité de solutions auto-similaires générales pour certaines équations aux dérivées partielles non linéaires d'ordre fractionnaire de type Katugampola, avec des valeurs aux limites, valeur initiale, ou avec des conditions intégrales dans un espace de Banach, en utilisant le principe de contraction de Banach, les théorèmes de point fixe de Schauder et de Guo-Krasnosel'skii, et la technique alternative non linéaire de type Leray-Schauder.

Mots clés: *Equations aux dérivés partielles fractionnaires, équation différentielle fractionnaire, dérivée fractionnaire de Katugampola, solutions auto-similaires, point fixe, espace de Banach, problème aux limites, problème de valeur initial, conditions intégrales, existence, unicité.*

Abstract:

In this thesis, we discuss several existence and uniqueness results of generalized self-similar solutions for some nonlinear partial differential equations of fractional order of Katugampola type, with boundary value, initial value, or with integral conditions in Banach space, we use the Banach contraction principle, Schauder and Guo-Krasnosel'skii fixed point theorems, and the technique of the nonlinear alternative of Leray-Schauder type.

Key words: *Fractional partial differential equation, fractional differential equations, fractional derivative of Katugampola, self-similar solutions, fixed point, Banach space, boundary value problem, initial value problem, integral conditions, existence, uniqueness.*

A.M.S Classifications: *35R11, 35A01, 34A08, 35C06, 34K37.*