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Rachid Yahi, Dahmane Achour & Elhadj Dahia

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LIPSCHITZ CLOSED INJECTIVE HULL IDEALS AND LIPSCHITZ INTERPOLATIVE IDEALS

Rachid Yahi

Laboratory of Functional Analysis and Geometry of Spaces, University of M'sila, University Pole, Road Bourdj Bou Arreiridj, M'sila 28000, Algeria.
E-Mail rachid.yahi@univ-msila.dz

Dahmane Achour

Laboratory of Functional Analysis and Geometry of Spaces, University of M'sila, University Pole, Road Bourdj Bou Arreiridj, M'sila 28000, Algeria.
E-Mail dahmane.achour@univ-msila.dz

Elhadj Dahia*

Laboratory of Functional Analysis and Geometry of Spaces, University of M'sila, University Pole, Road Bourdj Bou Arreiridj, M'sila 28000, Algeria, and Laboratoire de Mathématiques et Physique Appliquées, École Normale Supérieure de Bousaada, 28001 Bousaada, Algeria.
E-Mail dahia.elhadj@ens-bousaada.dz

Abstract. In this paper we present two constructions: the Lipschitz closed injective hull ideals and the Lipschitz interpolative ideals of a Lipschitz operator ideal. These procedures aim to construct new Lipschitz operator ideals between pointed metric spaces and Banach spaces. The new ideals are characterized by specific criteria that determine whether a Lipschitz operator belongs to them, using summability properties and interpolation formulas.

Mathematics Subject Classification (2020): Primary: 47L20, 47H99; Secondary: 47B10, 26A16.

Key words: Lipschitz operator ideals, closed injective hull, interpolative ideals.

1. Introduction and notation. Inspired by Farmer and Johnson's work [13], which extends the concept of p-summing linear operators to the Lipschitz framework, several researchers have proposed various classes of Lipschitz operators between pointed metric spaces and Banach spaces. In specific contexts, these concepts broaden different categories of ideals of linear operators between Banach spaces. (see [1, 2, 3, 6, 10, 11, 12, 14, 21, 23, 25], and the references therein). It is worth mentioning that Achour et al. provide a detailed and systematic study of Lipschitz operator ideals in [5] (see also [2, 6]). Jarchow and Pełczyński outlined a method for generating new Banach operator ideals from a given one, known as the *closed*

*Corresponding author.

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injective hull (see [15]). Subsequently, Matter [18] (see also [16]), used this method to introduce another procedure that assigns new Banach operator ideals from a given one, known as the *interpolative ideal procedure.*

Our study aims to introduce and characterize the Lipschitz closed injective hull ideals and the Lipschitz interpolative ideals. In addition, we show some properties and describe the closed injective hull of a Banach Lipschitz operator ideals of the composition type.

This paper is organized as follows. In Section 2, we introduce and characterize the closed injective hull of Lipschitz operator ideals and determine the closed injective hull of a Lipschitz operator ideal obtained, through the composition method from a given linear operator ideal. Several examples illustrating this concept are also provided. In Section 3, we define the interpolative method by introducing and characterizing the interpolative Lipschitz operator ideals and establish the relationship between the three notions, the interpolative Lipschitz operator ideals, the Lipschitz injective hull, and the Lipschitz closed injective hull. As an application, starting from the well-known Banach operator ideal of p-summing linear operators, we construct a new Banach interpolative Lipschitz ideal. This is achieved through the composition of the space of all Lipschitz operators and the class of (p,σ)-absolutely continuous operators. We characterize this new class through integral domination inequalities and summability inequalities.

As usual, X, Y and Z will denote pointed metric spaces with the base point denoted by 0, and the metric will be denoted by d. We denote by

$$B_X = \{x \in X : d(x,0) \leq 1\}$$

the closure of the ball centered at 0 with radius 1. Also, E, F and G will stand for Banach spaces over the same field $\mathbb{K}$ (either $\mathbb{R}$ or $\mathbb{C}$). The topological dual of E is denoted by E^*. A Banach space E will be considered as a pointed metric space with the distinguished point 0 (the null vector) and metric $d(x,x') = \|x - x'\|$. The Lipschitz space $Lip_0(X,E)$ is the Banach space of all Lipschitz mappings T from X to E such that $T(0) = 0$, under the Lipschitz norm $Lip(\cdot)$, where $Lip(T)$ is the infimum of all constants $C \geq 0$ such that $\|T(x) - T(x')\| \leq Cd(x,x')$ for all $x, x' \in X$. The space $Lip_0(X,\mathbb{K})$ is called the Lipschitz dual of X and will be denoted by $X^{\#}$. It is worth mentioning that the Banach space of all linear operators $\mathcal{L}(E,F)$ is a subspace of $Lip_0(E,F)$, and thus E^* is a subspace of $E^{\#}$. The space of molecules $\mathcal{M}(X)$ for the pointed metric space X is given by the linear span of all functions $X \longrightarrow \mathbb{R}$ that can be written as differences of characteristic functions, i.e., $\mathfrak{m}_{x,x'} = \chi_{\{x\}} - \chi_{\{x'\}}$ for $x, x' \in X$. The norm for this space is given by the formula

$$\|\mathfrak{m}\|_{\mathcal{M}(X)} = \inf \sum_{j=1}^{n} |a_j|\, d(x_j, x'_j), \quad a_j \in \mathbb{R},$$

where the infimum is computed over all representations of $\mathfrak{m}$ as $\mathfrak{m} = \sum_{j=1}^{n} a_j \mathfrak{m}_{x_j,x'_j}$. The completion of $\left(\mathcal{M}(X), \|\cdot\|_{\mathcal{M}(X)}\right)$ is denoted by Æ(X) and is called Arens–Ells space associated with X (see [7]). Consider the canonical Lipschitz injection map

$\delta_X : X \longrightarrow Æ(X)$ defined by $\delta_X(x) = \mathfrak{m}_{x,0}$, which isometrically embeds X in $Æ(X)$. A Lipschitz map $T \in Lip_0(X, E)$ always satisfies a factorization scheme through the space $Æ(X)$ as $T = T_L \circ \delta_X : X \longrightarrow Æ(X) \longrightarrow E$. The map $T_L : Æ(X) \longrightarrow E$ is the unique continuous linear operator (referred to as the linearization of T) that satisfies $T = T_L \circ \delta_X$ and $\|T_L\| = Lip(T)$ (see [24, Theorem 3.6]).

The notion of linear operator ideals is extended to Lipschitz version by Achour et al. in [5]. See also [2, Definition 2.3] and [6, Definition 2.1].

Definition 1. A Lipschitz operator ideal $\mathcal{I}_{Lip}$ is a subclass of Lip_0 such that for every pointed metric space X and every Banach space E, the components

$$\mathcal{I}_{Lip}(X, E) := Lip_0(X, E) \cap \mathcal{I}_{Lip}$$

satisfy

(i) $\mathcal{I}_{Lip}(X, E)$ is a linear subspace of $Lip_0(X, E)$.

(ii) $vg \in \mathcal{I}_{Lip}(X, E)$ for $v \in E$ and $g \in X^{\#}$, where $(vg)(x) = g(x)v$ for all $x \in X$.

(iii) The ideal property: if $S \in Lip_0(Y, X)$, $T \in \mathcal{I}_{Lip}(X, E)$ and $w \in \mathcal{L}(E, F)$, then the composition $w \circ T \circ S$ is in $\mathcal{I}_{Lip}(Y, F)$.

A Lipschitz operator ideal $\mathcal{I}_{Lip}$ is a normed (Banach) Lipschitz operator ideal if there is $\|\cdot\|_{\mathcal{I}_{Lip}} : \mathcal{I}_{Lip} \longrightarrow [0, +\infty[$ that satisfies

(i') $(\mathcal{I}_{Lip}(X, E), \|\cdot\|_{\mathcal{I}_{Lip}})$ is a normed (Banach) space and $Lip(T) \leq \|T\|_{\mathcal{I}_{Lip}}$ for all $T \in \mathcal{I}_{Lip}(X, E)$.

(ii') $\|id_{\mathbb{K}} : \mathbb{K} \longrightarrow \mathbb{K}, id_{\mathbb{K}}(\lambda) = \lambda\|_{\mathcal{I}_{Lip}} = 1$.

(iii') If $S \in Lip_0(Y, X)$, $T \in \mathcal{I}_{Lip}(X, E)$ and $w \in \mathcal{L}(E, F)$, then

$$\|w \circ T \circ S\|_{\mathcal{I}_{Lip}} \leq Lip(S) \|T\|_{\mathcal{I}_{Lip}} \|w\| .$$

Now, we recall the composition method introduced in [5] to produce the Lipschitz operator ideal $\mathcal{I} \circ Lip_0$, from a given linear operator ideal $\mathcal{I}$. The Lipschitz operator $T : X \longrightarrow E$ belongs to $\mathcal{I} \circ Lip_0(X, E)$ if there are a Banach space G, a Lipschitz operator $S \in Lip_0(X, G)$, and a linear operator $u \in \mathcal{I}(G, E)$ such that $T = u \circ S$. This definition is equivalent to say that $T_L \in \mathcal{I}(Æ(X), E)$. If $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is a normed operator ideal, we write

$$\|T\|_{\mathcal{I} \circ Lip_0} = \inf \|u\|_{\mathcal{I}} Lip(S) = \|T_L\|_{\mathcal{I}} ,$$

where the infimum is taken over all u, G, S as described (see [5, Proposition 3.2]).

2. The closed injective hull of Lipschitz operator ideals. Let $\mathcal{I}_{Lip}$ be a Lipschitz operator ideal. Recall that the closed hull $\overline{\mathcal{I}_{Lip}}$ of a Lipschitz operator ideal $\mathcal{I}_{Lip}$ is the class of Lipschitz operators formed by components $\overline{\mathcal{I}_{Lip}}(X,E)$ that are given by the closure of $\mathcal{I}_{Lip}(X,E)$ in $Lip_0(X,E)$ (see [5]). According to [5] or [4], the Lipschitz injective hull $\mathcal{I}_{Lip}^{inj}$ of $\mathcal{I}_{Lip}$ consists of all Lipschitz operators $T \in \mathrm{Lip}_0(X,E)$ such that $J_E \circ T \in \mathcal{I}_{Lip}(X, \ell_\infty(B_{E^*}))$, where $J_E : E \longrightarrow \ell_\infty(B_{E^*})$ is the natural metric injection given by $J_E(y) = (\langle y, y^* \rangle)_{y^* \in B_{E^*}}$. Also, $\mathcal{I}_{Lip}$ is said to be injective if $\mathcal{I}_{Lip} = \mathcal{I}_{Lip}^{inj}$. In [4] it is proved that if $\mathcal{I}_{Lip}$ is a Banach Lipschitz operator ideal, then $\mathcal{I}_{Lip}^{inj}$ is also a Banach Lipschitz operator with $\|T\|_{\mathcal{I}_{Lip}^{inj}} = \|J_E \circ T\|_{\mathcal{I}_{Lip}}$ for all $T \in \mathcal{I}_{Lip}^{inj}(X,E)$.

Our first result provides a criterion for determining that the class $\mathcal{I}_{Lip} \subset Lip_0$ of Lipschitz operators is a Banach Lipschitz operator ideal.

THEOREM 2. *The class $\mathcal{I}_{Lip}$ endowed with a map $\|\cdot\|_{\mathcal{I}_{Lip}} : \mathcal{I}_{Lip} \longrightarrow [0,+\infty)$ is a Banach Lipschitz operator ideal if and only if the following conditions are satisfied:*

(i) *$Lip(\cdot) \leq \|\cdot\|_{\mathcal{I}_{Lip}}$ on $\mathcal{I}_{Lip}$ and $id_{\mathbb{K}} \in \mathcal{I}_{Lip}$ with $\|id_{\mathbb{K}}\|_{\mathcal{I}_{Lip}} = 1$.*

(ii) *$(\mathcal{I}_{Lip}, \|\cdot\|_{\mathcal{I}_{Lip}})$ enjoys the ideal property and the ideal inequality.*

(iii) *If $(T_n)_n \subset \mathcal{I}_{Lip}(X,E)$ with $\sum_{n\geq 1} \|T_n\|_{\mathcal{I}_{Lip}} < \infty$, then*

$$T := \sum_{n\geq 1} T_n \in \mathcal{I}_{Lip}(X,E) \quad \text{and} \quad \|T\|_{\mathcal{I}_{Lip}} \leq \sum_{n\geq 1} \|T_n\|_{\mathcal{I}_{Lip}}.$$

Proof. Firstly, we will show that conditions (i), (ii) and (iii) implies $\mathcal{I}_{Lip}$ is a Banach Lipschitz operator ideal. By the inequality $Lip(\cdot) \leq \|\cdot\|_{\mathcal{I}_{Lip}}$, if $\|T\|_{\mathcal{I}_{Lip}} = 0$ then $T \equiv 0 \in \mathcal{I}_{Lip}$. Also by (iii) we have $S + T \in \mathcal{I}_{Lip}$ for every $S, T \in \mathcal{I}_{Lip}$. For $\alpha \in \mathbb{K}$,

$$\alpha T = u_\alpha \circ T \circ id_X \in \mathcal{I}_{Lip}(X,E),$$

where $u_\alpha \in \mathcal{L}(E,E)$ given by $u_\alpha(x) = \alpha x$. Thus, we have shown that $\mathcal{I}_{Lip}(X,E)$ is a vector subspace of $Lip_0(X,E)$. The triangular inequality and the homogeneity of $\|\cdot\|_{\mathcal{I}_{Lip}}$ are derived directly from (iii). Now let $0 \neq g \in X^\#$ and $v \in E$. Then

$$g \cdot v = u \circ id_{\mathbb{K}} \circ g \in \mathcal{I}_{Lip}(X,E),$$

where $u \in \mathcal{L}(\mathbb{K}, E)$ given by $u(\alpha) = \alpha v$. In order to prove that $\left(\mathcal{I}_{Lip}(X,E), \|\cdot\|_{\mathcal{I}_{Lip}}\right)$ is a Banach space, we will consider a Cauchy sequence $(T_n)_{n\geq 1} \subset \mathcal{I}_{Lip}(X,E)$. Then, it is possible to obtain a subsequence $(T_{n_k})_{k\geq 1}$ of $(T_n)_{n\geq 1}$ such that

$$\|T_{n_k+1} - T_{n_k}\|_{\mathcal{I}_{Lip}} < 2^{-k},$$

which means that $\sum_{k\geq 1} \|T_{n_k+1} - T_{n_k}\|_{\mathcal{I}_{Lip}} < \infty$. By (iii) we have $S_j = \sum_{k\geq j} (T_{n_k+1} - T_{n_k})$ belongs to $\mathcal{I}_{Lip}(X,E)$ and $\|S_j\|_{\mathcal{I}_{Lip}} \leq 2^{-(j-1)}$ for all $j \geq 1$. Now, note that $S_j = (S_1 + T_{n_1}) - T_{n_j}$, and then

$$\lim_{j\longrightarrow+\infty} \left\|T_{n_j} - (S_1 + T_{n_1})\right\|_{\mathcal{I}_{Lip}} = \lim_{j\longrightarrow+\infty} \|S_j\|_{\mathcal{I}_{Lip}} \leq \lim_{j\longrightarrow+\infty} \frac{1}{2^{j-1}} = 0.$$

This shows that $\left(T_{n_j}\right)_{j\geq 1}$ converges to $S_1 + T_{n_1} \in \mathcal{I}_{Lip}(X,E)$ with respect to the norm $\|\cdot\|_{\mathcal{I}_{Lip}}$. Which means that the Cauchy sequence $(T_n)_{n\geq 1}$ has a convergent subsequence $\left(T_{n_j}\right)_{j\geq 1}$. Then $(T_n)_{n\geq 1}$ is convergent in $\left(\mathcal{I}_{Lip}(X,E), \|\cdot\|_{\mathcal{I}_{Lip}}\right)$.

Conversely, if $\left(\mathcal{I}_{Lip}, \|\cdot\|_{\mathcal{I}_{Lip}}\right)$ is a Banach Lipschitz operator ideal, then the conditions (i), (ii) are valid and (iii) is derived directly from the completeness of $\left(\mathcal{I}_{Lip}(X,E), \|\cdot\|_{\mathcal{I}_{Lip}}\right)$. □

Before studying and characterizing the closed injective hull of a Lipschitz operator ideal, we present the following result, which can be seen as a generalization of [22, Satz 4.1] to the Lipschitz case. Our result can be compared to the the linear and polynomial versions stated respectively in [8, Lemma 3.1] and [9, Theorem 3.4].

THEOREM 3. *Let $\mathcal{I}_{Lip}$ be a Banach Lipschitz operator ideal. A Lipschitz operator $T : X \longrightarrow E$ belongs to $\mathcal{I}_{Lip}^{inj}(X,E)$ if and only if there are a Banach space G and a Lipschitz operator $S \in \mathcal{I}_{Lip}(X,G)$ such that*

$$\left\| \sum_{i=1}^{n} a_i \left(T(x_i) - T(x_i')\right) \right\| \leq \left\| \sum_{i=1}^{n} a_i \left(S(x_i) - S(x_i')\right) \right\|, \tag{1}$$

for all $(x_i)_{i=1}^n, (x_i')_{i=1}^n$ in X and $(a_i)_{i=1}^n$ in $\mathbb{R}$. Moreover, $\|T\|_{\mathcal{I}_{Lip}^{inj}} = \inf \|S\|_{\mathcal{I}_{Lip}}$, where the infimum is taken over all G and S in (1).

Proof. For the "if" part, suppose that inequality (1) holds. Consider the normed space $G_0 \subset G$ spanned by $S(X)$ and the linear operator $R_0 : G_0 \longrightarrow E$ given by

$$R_0\left(\sum_{i=1}^{n} a_i S(x_i)\right) = \sum_{i=1}^{n} a_i T(x_i).$$

By using (1), we can check that R_0 is well-defined, continuous with $\|R_0\| \leq 1$ and then has a unique continuous linear extension R to the completion $\overline{G_0}$ of G_0. The metric extension property of $\ell_\infty(B_{E^*})$, (see [20, p. 33]), guarantees the existence of a continuous linear operator $\widetilde{R} : G \longrightarrow \ell_\infty(B_{E^*})$, satisfying that

$$J_E \circ T = \widetilde{R} \circ S \in \mathcal{I}_{Lip}(X, \ell_\infty(B_{E^*})).$$

This means that $T \in \mathcal{I}_{Lip}^{inj}(X,E)$. In addition, we have

$$\|T\|_{\mathcal{I}_{Lip}^{inj}} \leq \left\|\widetilde{R}\right\| \|S\|_{\mathcal{I}_{Lip}} \leq \|S\|_{\mathcal{I}_{Lip}}.$$

To prove the "only if" part, it is enough to take $G = \ell_\infty(B_{E^*})$ and $S = J_E \circ T$. □

The next result provides a characterization of the closed injective hull $\overline{\mathcal{I}_{Lip}}^{\text{inj}}$ of a given ideal $\mathcal{I}_{Lip}$ of Lipschitz operators by means of a norm inequality.

Recall that if $p \geq 1$ and $(E_i)_{i=1}^{\infty}$ is a family of Banach spaces, then the ℓ_p direct sum $\left(\bigoplus_{i=1}^{\infty} E_i\right)_{\ell_p}$ is the space consisting of all $(x_i)_{i=1}^{\infty} \in \prod_{i=1}^{\infty} E_i$ such that $\sum_{i=1}^{\infty} \|x_i\|^p < \infty$. This is a Banach space under the norm

$$\|(x_i)_{i=1}^{\infty}\| = \left(\sum_{i=1}^{\infty} \|x_i\|^p\right)^{\frac{1}{p}}$$

(see [20, p. 35]).

THEOREM 4. *Let $\mathcal{I}_{Lip}$ be a Lipschitz operator ideal. The Lipschitz operator $T : X \longrightarrow E$ belongs to $\overline{\mathcal{I}}_{Lip}^{inj}(X, E)$ if and only if for each $\varepsilon > 0$, there are a Banach space G_ε and $S_\varepsilon \in \mathcal{I}_{Lip}(X, G_\varepsilon)$ such that*

$$\left\|\sum_{i=1}^{n} a_i(T(x_i) - T(x_i'))\right\| \leq \left\|\sum_{i=1}^{n} a_i(S_\varepsilon(x_i) - S_\varepsilon(x_i'))\right\| + \varepsilon \sum_{i=1}^{n} |a_i| d(x_i, x_i') \qquad (2)$$

for all $(x_i)_{i=1}^{n}, (x_i')_{i=1}^{n}$ in X and $(a_i)_{i=1}^{n}$ in $\mathbb{R}$.

Proof. Take $T \in \overline{\mathcal{I}}_{Lip}^{inj}(X, E)$ and $\varepsilon > 0$. Then $J_E \circ T$ is a limit of a sequence $(S_m)_m \subset \mathcal{I}_{Lip}(X, \ell_\infty(B_{E^*}))$ with respect to the norm $Lip(\cdot)$. We can choose positive integer m_ε such that $Lip(J_E \circ T - S_{m_\varepsilon}) \leq \varepsilon$. On the other hand, for each $(x_i)_{i=1}^{n}, (x_i')_{i=1}^{n}$ in X and $(a_i)_{i=1}^{n}$ in $\mathbb{R}$ we have

$$\begin{aligned}
& \left\|\sum_{i=1}^{n} a_i(T(x_i) - T(x_i'))\right\| \\
= & \left\|\sum_{i=1}^{n} a_i(J_E \circ T(x_i) - J_E \circ T(x_i'))\right\| \\
\leq & \left\|\sum_{i=1}^{n} a_i(S_{m_\varepsilon}(x_i) - S_{m_\varepsilon}(x_i'))\right\| \\
& + \left\|\sum_{i=1}^{n} a_i\left(\left(J_E \circ T - S_{m_\varepsilon}\right)(x_i) - \left(J_E \circ T - S_{m_\varepsilon}\right)(x_i')\right)\right\| \\
\leq & \left\|\sum_{i=1}^{n} a_i(S_{m_\varepsilon}(x_i) - S_{m_\varepsilon}(x_i'))\right\| + \varepsilon \sum_{i=1}^{n} |a_i| d(x_i, x_i').
\end{aligned}$$

Conversely, assume that (2) holds for a given $\varepsilon > 0$ and a Banach space G_ε. By using the linearization of the Lipschitz operators T and S_ε we obtain

$$\|T_L(\mathfrak{m})\| \leq \|(S_\varepsilon)_L(\mathfrak{m})\| + \varepsilon \sum_{i=1}^{n} |a_i| d(x_i, x_i'),$$

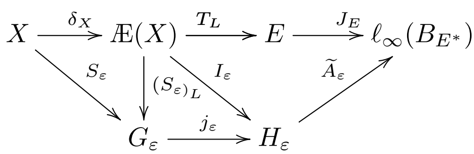
for all molecule $\mathfrak{m} \in \mathcal{M}(X)$ with representation $\mathfrak{m} = \sum_{i=1}^{n} a_i \mathfrak{m}_{x_i,x'_i}$. Therefore,

$$\|T_L(\mathfrak{m})\| \leq \|(S_\varepsilon)_L(\mathfrak{m})\| + \varepsilon \|\mathfrak{m}\|_{\text{Æ}(X)} \, .$$

Consider $H_\varepsilon = (G_\varepsilon \oplus \text{Æ}\,(X))_{\ell_1}$. Define the injective linear operator $I_\varepsilon : \text{Æ}\,(X) \longrightarrow H_\varepsilon$ by $I_\varepsilon(\mathfrak{m}) := ((S_\varepsilon)_L, \varepsilon m)$. According to the injectivity of I_ε, we can define a linear mapping $A_\varepsilon : R(I_\varepsilon) \longrightarrow E$ by $A_\varepsilon\,(I_\varepsilon(\mathfrak{m})) := T_L(\mathfrak{m})$. This mapping is continuous with $\|A_\varepsilon\| \leq 1$, indeed, for all $\mathfrak{m} \in \mathcal{M}(X)$ we have

$$\|A_\varepsilon(I_\varepsilon(\mathfrak{m}))\| \leq \|(S_\varepsilon)_L(\mathfrak{m})\| + \varepsilon \|\mathfrak{m}\|_{\text{Æ}(X)} = \|I_\varepsilon(\mathfrak{m})\|.$$

Using the metric extension property of $\ell_\infty(B_{E^*})$, we can extend $J_E \circ A_\varepsilon$ to the linear operator $\widetilde{A_\varepsilon} \in \mathcal{L}(H_\varepsilon, \ell_\infty(B_{E^*}))$ with $\left\|\widetilde{A_\varepsilon}\right\| \leq 1$.



We put

$$B_\varepsilon := \widetilde{A_\varepsilon} \circ j_\varepsilon \circ S_\varepsilon \in \mathcal{I}_{Lip}(X, \ell_\infty(B_{E^*})),$$

where $j_\varepsilon : G_\varepsilon \to H_\varepsilon$ is the linear operator defined by $j_\varepsilon(t) := (t, 0)$. Define $C_\varepsilon : X \longrightarrow \ell_\infty(B_{E^*})$ by $C_\varepsilon(x) := \widetilde{A_\varepsilon}(0, \varepsilon \mathfrak{m}_{x,0})$. Because of

$$\|C_\varepsilon(x) - C_\varepsilon(x')\| \leq \left\|\widetilde{A_\varepsilon}\right\| \|(0, \varepsilon \mathfrak{m}_{x,x'})\|_{H_\varepsilon} \leq \varepsilon d(x, x'),$$

for all $x, x' \in X$, the mapping C_ε belongs to $Lip_0(X, \ell_\infty(B_{E^*}))$ with $Lip(C_\varepsilon) \leq \varepsilon$. Taking into account that $B_\varepsilon(x) = \widetilde{A_\varepsilon}\,((S_\varepsilon)_L(\mathfrak{m}_{x,0}), 0)$ for every $x \in X$, we obtain $B_\varepsilon + C_\varepsilon = J_E \circ T$ and then $Lip(J_E \circ T - B_\varepsilon) = Lip(C_\varepsilon) \leq \varepsilon$. Since this holds for all $\varepsilon > 0$, it follows that $J_E \circ T \in \overline{\mathcal{I}}_{Lip}(X, E)$. The proof is concluded. □

In the following result, we characterize the closed injective hull of a given Banach Lipschitz operator ideal $(\mathcal{I}_{Lip}, \|\cdot\|_{\mathcal{I}_{Lip}})$, which will be used in the sequel.

COROLLARY 5. *Let $(\mathcal{I}_{Lip}, \|\cdot\|_{\mathcal{I}_{Lip}})$ be a Banach Lipschitz operator ideal. The Lipschitz operator $T : X \longrightarrow E$ belongs to $\overline{\mathcal{I}}^{inj}_{Lip}(X, E)$ if and only if for each $\varepsilon > 0$, there are a Banach space G, a Lipschitz operator $S \in \mathcal{I}_{Lip}(X, G)$, and a function $N : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that*

$$\left\|\sum_{i=1}^{n} a_i (T(x_i) - T(x'_i))\right\| \leq N(\varepsilon) \left\|\sum_{i=1}^{n} a_i (S(x_i) - S(x'_i))\right\| + \varepsilon \sum_{i=1}^{n} |a_i| d(x_i, x'_i), \quad (3)$$

for all $(x_i)_{i=1}^n, (x'_i)_{i=1}^n$ in X and $(a_i)_{i=1}^n$ in $\mathbb{R}$.

Proof. It is enough to show that (3) is holds for $T \in \overline{\mathcal{I}}^{inj}_{Lip}(X,E)$. According to (2), for each $m \in \mathbb{N}$ there are a Banach space G_m and $S_m \in \mathcal{I}_{Lip}(X,G_m)$ such that

$$\left\|\sum_{i=1}^{n} a_i(T(x_i)-T(x_i'))\right\| \leq \left\|\sum_{i=1}^{n} a_i(S_m(x_i)-S_m(x_i'))\right\| + 2^{-m}\sum_{i=1}^{n}|a_i|d(x_i,x_i') \quad (4)$$

for all $(x_i)_{i=1}^n, (x_i')_{i=1}^n$ in X and $(a_i)_{i=1}^n$ in $\mathbb{R}$. Putting $G = \left(\bigoplus_{n\in\mathbb{N}} G_n\right)_{\ell_1}$ and for each $k \in \mathbb{N}$ consider the canonical inclusion $I_k : G_k \longrightarrow G$, then $I_k \circ S_k \in \mathcal{I}_{Lip}(X,G)$ and

$$\sum_{k=1}^{\infty} \frac{1}{2^k \|S_k\|_{\mathcal{I}_{Lip}}} \|I_k \circ S_k\|_{\mathcal{I}_{Lip}} \leq \sum_{k=1}^{\infty} \frac{1}{2^k \|S_k\|_{\mathcal{I}_{Lip}}} \|I_k\| \|S_k\|_{\mathcal{I}_{Lip}} \leq \sum_{k=1}^{\infty} \frac{1}{2^k} = 1,$$

which means that the series $\sum_{k=1}^{\infty} 2^{-k} \|S_k\|_{\mathcal{I}_{Lip}}^{-1} I_k \circ S_k$ converges in the Banach space $\left(\mathcal{I}_{Lip}(X,G), \|\cdot\|_{\mathcal{I}_{Lip}}\right)$ to the Lipschitz operator

$$S = \sum_{k=1}^{\infty} \frac{1}{2^k \|S_k\|_{\mathcal{I}_{Lip}}} I_k \circ S_k \in \mathcal{I}_{Lip}(X,G).$$

It follows from $\|\cdot\| \leq \|\cdot\|_{\mathcal{I}_{Lip}}$ that such series is pointwise convergent. The inequality (4) implies

$$\begin{aligned}
&\left\|\sum_{i=1}^{n} a_i(T(x_i)-T(x_i'))\right\| \\
&\leq 2^m \|S_m\|_{\mathcal{I}_{Lip}} \left\|\sum_{i=1}^{n} 2^{-m} \|S_m\|_{\mathcal{I}_{Lip}}^{-1} a_i \left(S_m(x_i)-S_m(x_i')\right)\right\| + 2^{-m}\sum_{i=1}^{n}|a_i|\, d(x_i,x_i') \\
&\leq 2^m \|S_m\|_{\mathcal{I}_{Lip}} \sum_{k=1}^{\infty} \left\|\sum_{i=1}^{n} 2^{-k} \|S_k\|_{\mathcal{I}_{Lip}}^{-1} a_i \left(S_k(x_i)-S_k(x_i')\right)\right\| + 2^{-m}\sum_{i=1}^{n}|a_i|\, d(x_i,x_i') \\
&= 2^m \|S_m\|_{\mathcal{I}_{Lip}} \left\|\sum_{i=1}^{n} a_i \left(S(x_i)-S(x_i')\right)\right\| + 2^{-m}\sum_{i=1}^{n}|a_i|\, d(x_i,x_i').
\end{aligned}$$

Finally, for all $m \in \mathbb{N}$, if we consider the function $N : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ defined by $N(\varepsilon) = 2\|S_1\|_{\mathcal{I}_{Lip}}$ for $\varepsilon > 1$ and $N(\varepsilon) = 2^m \|S_m\|_{\mathcal{I}_{Lip}}$ for $2^{-m} < \varepsilon \leq 2^{-m+1}$, we get the result. □

The next result describes the closed injective hull of a Banach Lipschitz operator ideal of composition type.

PROPOSITION 6. *For a Banach linear operator ideal $\mathcal{I}$, we have*

$$\overline{\mathcal{I} \circ Lip_0}^{inj} = \overline{\mathcal{I}}^{inj} \circ Lip_0.$$

Proof. Since $\overline{\mathcal{I}}^{inj}$ is closed then $\overline{\mathcal{I}}^{inj} \circ Lip_0$ is closed. On the other hand, for a given T in $Lip_0(X,E)$, suppose that $J_E \circ T$ belongs to $\overline{\mathcal{I}}^{inj} \circ Lip_0(X, \ell_\infty(B_{E^*}))$. The uniqueness of the linearization maps gives

$$(J_E \circ T)_L = J_E \circ T_L \in \overline{\mathcal{I}}^{inj}(Æ(X), \ell_\infty(B_{E^*})).$$

Then, the injectivity of $\overline{\mathcal{I}}^{inj}$ gives $T_L \in \overline{\mathcal{I}}^{inj}(Æ(X), E)$, so

$$T = T_L \circ \delta_X \in \overline{\mathcal{I}}^{inj} \circ Lip_0(X, E),$$

and this shows that $\overline{\mathcal{I}}^{inj} \circ Lip_0$ is an injective Lipschitz operator ideal. Now, since $\mathcal{I} \subset \overline{\mathcal{I}}^{inj}$, we obtain

$$\overline{\mathcal{I} \circ Lip_0}^{inj} \subset \overline{\overline{\mathcal{I}}^{inj} \circ Lip_0}^{inj} = \overline{\mathcal{I}}^{inj} \circ Lip_0.$$

For the converse inclusion, given T in $\overline{\mathcal{I}}^{inj} \circ Lip_0(X,E)$ then there are a Banach space G, a Lipschitz operator $S \in Lip_0(X,G)$ and a linear operator $u \in \overline{\mathcal{I}}^{inj}(G,E)$ such that $T = u \circ S$. For $\varepsilon > 0$, choose $v \in \mathcal{I}(G, \ell_\infty(B_{E^*}))$ such that $\|v - J_E \circ u\| \leq \dfrac{\varepsilon}{1 + Lip(S)}$ and $v \circ S$ belongs to $\mathcal{I} \circ Lip_0(X, \ell_\infty(B_{E^*}))$. We have

$$Lip(v \circ S - J_E \circ T) \leq (1 + Lip(S))\|v - J_E \circ u\| \leq \varepsilon.$$

Therefore $J_E \circ T \in \overline{\mathcal{I} \circ Lip_0}(X, \ell_\infty(B_{E^*}))$, which means that $T \in \overline{\mathcal{I} \circ Lip_0}^{inj}(X,E)$. □

The subsequent result provides several illustrations of the notion of a closed injective hull. Indeed, Jiménez-Vargas et al. in [17] introduced $Lip_{0\mathcal{F}}$, $Lip_{0\mathcal{K}}$, and $Lip_{0\mathcal{W}}$, the ideals of Lipschitz finite-dimensional rank, Lipschitz compact operators, and Lipschitz weakly compact operators, respectively, as an extension of the linear case.

Proposition 7. *We have $(Lip_{0\mathcal{K}})^{inj} = Lip_{0\mathcal{K}}$, $(Lip_{0\mathcal{W}})^{inj} = Lip_{0\mathcal{W}}$. Consequently, $\overline{Lip_{0\mathcal{F}}}^{inj} = Lip_{0\mathcal{K}}$.*

Proof. Firstly, note that the ideals $\mathcal{K}$ and $\mathcal{W}$ of linear compact and linear weakly compact operators are injective (see [20, p. 72]). From [5, Proposition 3.6] and [4, Proposition 2.4], we get the injectivity of $Lip_{0\mathcal{K}}$ and $Lip_{0\mathcal{W}}$, indeed

$$\begin{aligned} (Lip_{0\mathcal{K}})^{inj} &= (\mathcal{K} \circ Lip_0)^{inj} = \mathcal{K}^{inj} \circ Lip_0 = Lip_{0\mathcal{K}}, \\ (Lip_{0\mathcal{W}})^{inj} &= (\mathcal{W} \circ Lip_0)^{inj} = \mathcal{W}^{inj} \circ Lip_0 = Lip_{0\mathcal{W}}. \end{aligned}$$

Now, since $\ell_\infty(B_{E^*})$ has the approximation property, then it follows from [5, Proposition 2.8] that if $T \in Lip_0(X,E)$, then $J_E \circ T \in \overline{Lip_{0\mathcal{F}}}(X, \ell_\infty(B_{E^*}))$ if and only if $J_E \circ T \in Lip_{0\mathcal{K}}(X, \ell_\infty(B_{E^*}))$. This is equivalent to the fact that the operator T

belongs to $Lip_{0\mathcal{K}}(X,E)$. □

Taking into account that $Lip_{0\mathcal{K}}, Lip_{0\mathcal{W}}$ are closed ([5, Corollary 3.3 and Proposition 3.6]), we obtain the following.

PROPOSITION 8. *$\overline{Lip_{0\mathcal{K}}}^{inj} = Lip_{0\mathcal{K}}$ and $\overline{Lip_{0\mathcal{W}}}^{inj} = Lip_{0\mathcal{W}}$.*

3. The interpolative Lipschitz operator ideal. In this section, we present another method for generating new Lipschitz operators ideals from a given Lipschitz operator ideal, which is called the interpolative method. From (3), by taking $N(\varepsilon) = \varepsilon^{-r}$ for a constant $1 \leq r < +\infty$ and computing the minimum of the right-hand side with respect to ε, we infer that condition (3) is equivalent to the following inequality

$$\left\|\sum_{i=1}^{n} a_i(T(x_i) - T(x_i'))\right\| \leq$$

$$\leq \left(r^{-\frac{r}{r+1}} + r^{\frac{1}{r+1}}\right)\left\|\sum_{i=1}^{n} a_i(S(x_i) - S(x_i'))\right\|^{\frac{1}{r+1}}\left(\sum_{i=1}^{n}|a_i|d(x_i,x_i')\right)^{\frac{r}{r+1}}.$$

DEFINITION 9. Let $\mathcal{I}_{Lip}$ be a Lipschitz operators ideal and $0 \leq \sigma < 1$. A Lipschitz operator $T \in Lip_0(X,E)$ belongs to $(\mathcal{I}_{Lip})_\sigma(X,E)$ if there are a Banach space G and a Lipschitz operator $S \in \mathcal{I}_{Lip}(X,G)$ such that

$$\left\|\sum_{i=1}^{n} a_i(T(x_i) - T(x_i'))\right\| \leq \left\|\sum_{i=1}^{n} a_i(S(x_i) - S(x_i'))\right\|^{1-\sigma}\left(\sum_{i=1}^{n}|a_i|d(x_i,x_i')\right)^{\sigma}, \quad (5)$$

for all $(x_i)_{i=1}^n, (x_i')_{i=1}^n$ in X and $(a_i)_{i=1}^n$ in $\mathbb{R}$. For each $T \in (\mathcal{I}_{Lip})_\sigma(X,E)$ denote

$$\|T\|_{(\mathcal{I}_{Lip})_\sigma} := \inf \|S\|_{\mathcal{I}_{Lip}}^{1-\sigma},$$

where the infimum is taken over all Banach spaces G and operators $S \in \mathcal{I}_{Lip}(X,G)$ working in the above inequality.

Note that if we take $n = 1$ in (5) we get $Lip(\cdot) \leq \|\cdot\|_{(\mathcal{I}_{Lip})_\sigma}$.

PROPOSITION 10. *If $(\mathcal{I}_{Lip}, \|\cdot\|_{\mathcal{I}_{Lip}})$ is an injective Banach Lipschitz operator ideal then $((\mathcal{I}_{Lip})_\sigma, \|\cdot\|_{(\mathcal{I}_{Lip})_\sigma})$ is injective Banach Lipschitz operator ideal.*

Proof. We check the conditions (ii) and (iii) in Theorem 2.
(ii) Let $R \in Lip_0(Y,X)$, $T \in (\mathcal{I}_{Lip})_\sigma(X,E)$ and $w \in \mathcal{L}(E,F)$. For all $\varepsilon > 0$ there are a Banach space G and $S \in \mathcal{I}_{Lip}(X,G)$ satisfying (5) and $\|S\|_{\mathcal{I}_{Lip}}^{1-\sigma} \leq (1+\varepsilon)\|T\|_{(\mathcal{I}_{Lip})_\sigma}$. For every $(y_i)_{i=1}^n, (y_i')_{i=1}^n$ in Y and $(a_i)_{i=1}^n$ in $\mathbb{R}$,

$$\left\|\sum_{i=1}^{n} a_i(w \circ T \circ R(y_i) - w \circ T \circ R(y_i'))\right\| \leq$$

$$\leq \|w\|\, Lip(R)^{\sigma} \left\| \sum_{i=1}^{n} a_i \left(S \circ R(y_i) - S \circ R(y_i')\right) \right\|^{1-\sigma} \left(\sum_{i=1}^{n} |a_i| d(y_i, y_i') \right)^{\sigma}.$$

This means that $w \circ T \circ R$ is in $(\mathcal{I}_{Lip})_{\sigma}(Y, F)$ and

$$\begin{aligned} \|w \circ T \circ R\|_{(\mathcal{I}_{Lip})_{\sigma}} &\leq \|w\|\, Lip(R)^{\sigma} \|S \circ R\|_{\mathcal{I}_{Lip}}^{1-\sigma} \\ &\leq \|w\|\, Lip(R)(1+\varepsilon)\|T\|_{(\mathcal{I}_{Lip})_{\sigma}}, \end{aligned}$$

as ε is arbitrary we obtain (ii).

(iii) Let $(T_n)_n$ be a sequence in $(\mathcal{I}_{Lip})_{\sigma}(X, E)$ such that $\sum_{n=1}^{\infty} \|T_n\|_{(\mathcal{I}_{Lip})_{\sigma}} < \infty$. Since $Lip(\cdot) \leq \|\cdot\|_{(\mathcal{I}_{Lip})_{\sigma}}$ on $(\mathcal{I}_{Lip})_{\sigma}$ and $Lip_0(X, E)$ is a Banach space, there exists $T = \sum_{n=1}^{\infty} T_n \in Lip_0(X, E)$. For each $k \in \mathbb{N}$ and all $\varepsilon > 0$ choose a Banach space G_k and $S_k \in \mathcal{I}_{Lip}(X, G_k)$ satisfying (5) and $\|S_k\|_{\mathcal{I}_{Lip}}^{1-\sigma} \leq \|T_k\|_{(\mathcal{I}_{Lip})_{\sigma}} + \varepsilon/2^k$. Hence

$$\sum_{k=1}^{\infty} \|S_k\|_{\mathcal{I}_{Lip}}^{1-\sigma} \leq \sum_{k=1}^{\infty} \|T_k\|_{(\mathcal{I}_{Lip})_{\sigma}} + \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} < \infty.$$

Introducing $G = \left(\bigoplus_{n \in \mathbb{N}} G_n \right)_{\ell_1}$ and the k-th canonical injection $I_k : G_k \longrightarrow G$, then the series $\sum_{k=1}^{\infty} \|S_k\|_{\mathcal{I}_{Lip}}^{-\sigma} I_k S_k$ converge to $S \in \mathcal{I}_{Lip}(X, G)$. Now select $(x_i)_{i=1}^n, (x_i')_{i=1}^n$ in X and $(a_i)_{i=1}^n$ in $\mathbb{R}$. An application of Hölder inequality reveals that

$$\begin{aligned} \left\| \sum_{i=1}^{n} a_i (T(x_i) - T(x_i')) \right\| &\leq \sum_{k=1}^{\infty} \left\| \sum_{i=1}^{n} a_i (T_k(x_i) - T_k(x_i')) \right\| \\ &\leq \sum_{k=1}^{\infty} \left\| \sum_{i=1}^{n} a_i \|S_k\|_{\mathcal{I}_{Lip}}^{-\sigma} (S_k(x_i) - S_k(x_i')) \right\|^{1-\sigma} \\ &\quad \times \left(\sum_{i=1}^{n} |a_i| \|S_k\|_{\mathcal{I}_{Lip}}^{1-\sigma} d(x_i, x_i') \right)^{\sigma} \\ &\leq \left(\sum_{k=1}^{\infty} \left\| \sum_{i=1}^{n} a_i \|S_k\|_{\mathcal{I}_{Lip}}^{-\sigma} (S_k(x_i) - S_k(x_i')) \right\| \right)^{1-\sigma} \\ &\quad \times \left(\sum_{k=1}^{\infty} \left(\sum_{i=1}^{n} |a_i| \|S_k\|_{\mathcal{I}_{Lip}}^{1-\sigma} d(x_i, x_i') \right) \right)^{\sigma} \qquad (6) \end{aligned}$$

We have to estimate the latter expression of the right-hand side of (6), we get

$$\begin{aligned}
&\left(\sum_{k=1}^{\infty}\left\|\sum_{i=1}^{n} a_i \|S_k\|_{\mathcal{I}_{Lip}}^{-\sigma}(S_k(x_i)-S_k(x_i'))\right\|\right)^{1-\sigma}\\
&\qquad\qquad\times\left(\sum_{k=1}^{\infty}\left(\sum_{i=1}^{n}|a_i|\|S_k\|_{\mathcal{I}_{Lip}}^{1-\sigma} d(x_i,x_i')\right)\right)^{\sigma}\\
&\qquad\qquad\leq\left\|\sum_{i=1}^{n} a_i(S(x_i)-S(x_i'))\right\|^{1-\sigma}\\
&\qquad\qquad\times\left(\sum_{k=1}^{\infty}\left(\sum_{i=1}^{n}|a_i|\|S_k\|_{\mathcal{I}_{Lip}}^{1-\sigma} d(x_i,x_i')\right)\right)^{\sigma}\\
&\qquad\qquad\leq\left(\sum_{k=1}^{\infty}\|S_k\|_{\mathcal{I}_{Lip}}^{1-\sigma}\right)^{\sigma}\left\|\sum_{i=1}^{n} a_i(S(x_i)-S(x_i'))\right\|^{1-\sigma}\\
&\qquad\qquad\times\left(\sum_{i=1}^{n}|a_i| d(x_i,x_i')\right)^{\sigma}.
\end{aligned}$$

Then

$$\begin{aligned}
&\left\|\sum_{i=1}^{n} a_i(T(x_i)-T(x_i'))\right\|\\
&\quad\leq\left(\sum_{k=1}^{\infty}\|S_k\|_{\mathcal{I}_{Lip}}^{1-\sigma}\right)^{\sigma}\left\|\sum_{i=1}^{n} a_i(S(x_i)-S(x_i'))\right\|^{1-\sigma}\times\left(\sum_{i=1}^{n}|a_i| d(x_i,x_i')\right)^{\sigma}.
\end{aligned}$$

So $\sum\limits_{n=1}^{\infty} T_n$ converges to $T=\sum\limits_{n=1}^{\infty} T_n \in (\mathcal{I}_{Lip})_{\sigma}(X,E)$, moreover

$$\begin{aligned}
\|T\|_{(\mathcal{I}_{Lip})_{\sigma}} &\leq \left(\sum_{k=1}^{\infty}\|S_k\|_{\mathcal{I}_{Lip}}^{1-\sigma}\right)^{\sigma}\|S\|_{\mathcal{I}_{Lip}}^{1-\sigma}\leq\sum_{k=1}^{\infty}\|S_k\|_{\mathcal{I}_{Lip}}^{1-\sigma}\\
&\leq \sum_{k=1}^{\infty}\|T_k\|_{(\mathcal{I}_{Lip})_{\sigma}}+\sum_{k=1}^{\infty}\frac{\varepsilon}{2^k}=\sum_{k=1}^{\infty}\|T_k\|_{(\mathcal{I}_{Lip})_{\sigma}}+\varepsilon,
\end{aligned}$$

and the proof follows. □

Next, we show that the inclusion between a pair of Lipschitz operator ideals of the class with different parameters behaves as expected.

PROPOSITION 11. *Let $0 \leq \sigma_1, \sigma_2 < 1$ such that $\sigma_1 \leq \sigma_2$ and let $\mathcal{I}_{Lip}$ be a normed Lipschitz operator ideal. Then*

$$(\mathcal{I}_{Lip})_{\sigma_1}(X,E) \subset (\mathcal{I}_{Lip})_{\sigma_2}(X,E).$$

Moreover, we have $\|T\|_{(\mathcal{I}_{Lip})_{\sigma_2}} \leq \|T\|_{(\mathcal{I}_{Lip})_{\sigma_1}}$ for every $T \in (\mathcal{I}_{Lip})_{\sigma_1}(X,E)$.

Proof. Let $T \in (\mathcal{I}_{Lip})_{\sigma_1}(X,E)$ and $\varepsilon > 0$. Choose a Banach space G and $S \in \mathcal{I}_{Lip}(X,G)$ such that $\|S\|_{\mathcal{I}_{Lip}}^{1-\sigma_1} \leq (1+\varepsilon)\|T\|_{(\mathcal{I}_{Lip})_{\sigma_1}}$. For all $(x_i)_{i=1}^n, (x_i')_{i=1}^n$ in X and $(a_i)_{i=1}^n$ in $\mathbb{R}$, we have

$$\begin{aligned} &\left\|\sum_{i=1}^n a_i(T(x_i)-T(x_i'))\right\| \\ \leq\ &\left\|\sum_{i=1}^n a_i(S(x_i)-S(x_i'))\right\|^{1-\sigma_1}\left(\sum_{i=1}^n |a_i| d(x_i,x_i')\right)^{\sigma_1} \\ \leq\ & Lip(S)^{\sigma_2-\sigma_1}\left\|\sum_{i=1}^n a_i(S(x_i)-S(x_i'))\right\|^{1-\sigma_2}\left(\sum_{i=1}^n |a_i| d(x_i,x_i')\right)^{\sigma_2}. \end{aligned}$$

Hence $T \in (\mathcal{I}_{Lip})_{\sigma_2}(X,E)$ and

$$\|T\|_{(\mathcal{I}_{Lip})_{\sigma_2}} \leq Lip(S)^{\sigma_2-\sigma_1}\|S\|_{\mathcal{I}_{Lip}}^{1-\sigma_2} \leq \|S\|_{\mathcal{I}_{Lip}}^{\sigma_2-\sigma_1}\|S\|_{\mathcal{I}_{Lip}}^{1-\sigma_2} \leq (1+\varepsilon)\|T\|_{(\mathcal{I}_{Lip})_{\sigma_1}}.$$

The result follows. □

Let us establish the relationship between the Lipschitz injective hull, the Lipschitz closed injective hull, and the interpolative Lipschitz operator ideals.

PROPOSITION 12. *If $\mathcal{I}_{Lip}$ is a normed Lipschitz operator ideal, then*

$$\mathcal{I}_{Lip}^{inj}(X,E) \subset (\mathcal{I}_{Lip})_{\sigma}(X,E) \subset \overline{\mathcal{I}_{Lip}}^{inj}(X,E).$$

In addition, $\|T\|_{\overline{\mathcal{I}_{Lip}}^{inj}} \leq \|T\|_{(\mathcal{I}_{Lip})_{\sigma}} \leq \|T\|_{\mathcal{I}_{Lip}^{inj}}$ for every $T \in \mathcal{I}_{Lip}^{inj}(X,E)$.

Proof. Take $T \in \mathcal{I}_{Lip}^{inj}(X,E)$ and fix $\varepsilon > 0$. According to Theorem 3, choose Banach space G and $S \in \mathcal{I}_{Lip}(X,G)$ such that $\|S\|_{\mathcal{I}_{Lip}} \leq (1+\varepsilon)\|T\|_{\mathcal{I}_{Lip}^{inj}}$ and for every $(x_i)_{i=1}^n, (x_i')_{i=1}^n$ in X and $(a_i)_{i=1}^n$ in $\mathbb{R}$, we have

$$\begin{aligned} &\left\|\sum_{i=1}^n a_i(T(x_i)-T(x_i'))\right\| \\ \leq\ & Lip(S)^{\sigma}\left\|\sum_{i=1}^n a_i(S(x_i)-S(x_i'))\right\|^{1-\sigma}\left(\sum_{i=1}^n |a_i| d(x_i,x_i')\right)^{\sigma} \end{aligned}$$

from which we deduce that $T \in (\mathcal{I}_{Lip})_{\sigma}(X,E)$ with

$$\|T\|_{(\mathcal{I}_{Lip})_{\sigma}} \leq Lip(S)^{\sigma}\|S\|_{\mathcal{I}_{Lip}}^{1-\sigma} \leq (1+\varepsilon)\|T\|_{\mathcal{I}_{Lip}^{inj}}.$$

The second inclusion is a straightforward consequence derived from the introductory statements in this section. □

The following outcome provides the interpolative Lipschitz operator ideal of a normed Lipschitz operator ideal of composition type.

PROPOSITION 13. *Let $\mathcal{I}$ be a normed linear operator ideal and $0 \leq \sigma < 1$. Then*

$$(\mathcal{I} \circ Lip_0)_\sigma = (\mathcal{I})_\sigma \circ Lip_0. \tag{7}$$

Moreover, we have

$$\|\cdot\|_{(\mathcal{I})_\sigma \circ Lip_0} = \|\cdot\|_{(\mathcal{I} \circ Lip_0)_\sigma}.$$

Proof. Take T in $(\mathcal{I} \circ Lip_0)_\sigma(X, E)$, $\mathfrak{m} \in \mathcal{M}(X)$ and fix $\varepsilon > 0$. Choose a representation $\sum_{i=1}^n a_i \mathfrak{m}_{x_i x_i'}$ of m, where $x_i, x_i' \in X$, and $a_i \in \mathbb{R}$, such that

$$\left(\sum_{i=1}^n |a_i| d(x_i, x_i')\right) \leq (1+\varepsilon) \|\mathfrak{m}\|.$$

Then there are a Banach space G and $S \in \mathcal{I} \circ Lip_0(X, G)$ with

$$\|S\|_{\mathcal{I} \circ Lip_0}^{1-\sigma} \leq (1+\varepsilon) \|T\|_{(\mathcal{I} \circ Lip_0)_\sigma}$$

and

$$\begin{aligned}
\|T_L(\mathfrak{m})\| &= \left\|\sum_{i=1}^n a_i (T(x_i) - T(x_i'))\right\| \\
&\leq \left\|\sum_{i=1}^n a_i (S(x_i) - S(x_i'))\right\|^{1-\sigma} \left(\sum_{i=1}^n |a_i| d(x_i, x_i')\right)^\sigma \\
&\leq \|S_L(\mathfrak{m})\|^{1-\sigma} (1+\varepsilon) \|\mathfrak{m}\|^\sigma.
\end{aligned}$$

Since $S_L \in \mathcal{I}(\text{Æ}(X), G)$ and the above inequality is holds for every $\varepsilon > 0$, we get $T_L \in (\mathcal{I})_\sigma(\text{Æ}(X), E)$ with

$$\|T_L\|_{(\mathcal{I})_\sigma} \leq \|S_L\|_{\mathcal{I}}^{1-\sigma} = \|S\|_{\mathcal{I} \circ Lip_0}^{1-\sigma} \leq (1+\varepsilon) \|T\|_{(\mathcal{I} \circ Lip_0)_\sigma}.$$

Which means that T belongs to $(\mathcal{I})_\sigma \circ Lip_0(X, E)$ and $\|T\|_{(\mathcal{I})_\sigma \circ Lip_0} \leq \|T\|_{(\mathcal{I} \circ Lip_0)_\sigma}$. Conversely, if $T \in (\mathcal{I})_\sigma \circ Lip_0(X, E)$ then for a fix $\varepsilon > 0$ there are a Banach space F, a Lipschitz operator $w \in Lip_0(X, F)$ and $u \in (\mathcal{I})_\sigma(F, E)$ such that $T = u \circ w$ with $Lip(w)\|u\|_{(\mathcal{I})_\sigma} \leq (1+\varepsilon)\|T\|_{(\mathcal{I})_\sigma \circ Lip_0}$. The definition of $(\mathcal{I})_\sigma$ assures the existence of a Banach space G and $v \in \mathcal{I}(F, G)$ such that $\|v\|_{\mathcal{I}}^{1-\sigma} \leq (1+\varepsilon)\|u\|_{(\mathcal{I})_\sigma}$. We have

$$\begin{aligned}
&\left\|\sum_{i=1}^n a_i (T(x_i) - T(x_i'))\right\| \\
= &\left\|u\left(\sum_{i=1}^n a_i (w(x_i) - w(x_i'))\right)\right\| \\
\leq &\ Lip(w)^\sigma \left\|\sum_{i=1}^n a_i (v \circ w(x_i) - v \circ w(x_i'))\right\|^{1-\sigma} \left(\sum_{i=1}^n |a_i| d(x_i, x_i')\right)^\sigma,
\end{aligned}$$

for all $x_i, x_i' \in X$, and $a_i \in \mathbb{R}, (1 \leq i \leq n)$. Since $v \circ w$ is in $\mathcal{I} \circ Lip_0(X, G)$, the Lipschitz operator T belongs to $(\mathcal{I} \circ Lip_0)_\sigma(X, E)$ and

$$\begin{aligned}\|T\|_{(\mathcal{I}\circ Lip_0)_\sigma} &\leq Lip(w)^\sigma \|v \circ w\|_{\mathcal{I}\circ Lip_0}^{1-\sigma} \\ &\leq Lip(w)\,\|v\|_{\mathcal{I}}^{1-\sigma} \leq Lip(w)(1+\varepsilon)\|u\|_{(\mathcal{I})_\sigma} \\ &\leq (1+\varepsilon)^2 \|T\|_{(\mathcal{I})_\sigma \circ Lip_0}.\end{aligned}$$

Then it follows that $\|T\|_{(\mathcal{I}\circ Lip_0)_\sigma} \leq \|T\|_{(\mathcal{I})_\sigma \circ Lip_0}$. □

By the following example, let us complete the panorama of summability for Lipschitz operators defined between pointed metric spaces and Banach spaces.

EXAMPLE 14. Let $(\Pi_p, \|\cdot\|_{\Pi_p})$ be the famous Banach operator ideal of p-summing linear operators ($p \geq 1$) which was introduced by Pietsch in [19]. When we utilize the composition method for Π_p, and subsequently apply the interpolative Lipschitz ideal procedure for the Banach Lipschitz operator ideal $\Pi_p \circ Lip_0$, we acquire the Banach Lipschitz operator ideal $(\Pi_p)_\sigma \circ Lip_0$. Note that $(\Pi_p)_\sigma$ is exactly the Banach ideal $\Pi_{p,\sigma}$ of (p, σ)-absolutely continuous operators building from Π_p by the interpolative ideal procedure (see [18, Section 4]).

Taking into consideration that $T \in (\Pi_p \circ Lip_0)_\sigma(X, E)$ if and only if T_L belongs to $\Pi_{p,\sigma}(\text{Æ}(X), E)$ and by using [18, Theorem 4.1], we characterize the mappings that belongs to $(\Pi_p \circ Lip_0)_\sigma$ by means of integral domination inequality and summability inequality.

Note that the unit ball $B_{X^\#}$ of $X^\#$ is a compact Hausdorff space in the topology of pointwise convergence on X.

THEOREM 15. *Let $1 \leq p < \infty$, $0 \leq \sigma < 1$. The following are equivalent for a Lipschitz operator $T \in Lip_0(X, E)$.*

(a) $T \in (\Pi_p \circ Lip_0)_\sigma(X, E)$.

(b) *There exist a constant $C > 0$ and a regular Borel probability measure μ on $B_{X^\#}$ such that*

$$\left\|\sum_{i=1}^n a_i(T(x_i) - T(x_i'))\right\| \leq$$

$$C\left(\int_{B_{X^\#}} \left(\left|\sum_{i=1}^n a_i\left(f(x_i) - f(x_i')\right)\right|^{1-\sigma} \left(\sum_{i=1}^n |a_i| d(x_i, x_i')\right)^\sigma\right)^{\frac{p}{1-\sigma}} d\mu(f)\right)^{\frac{1-\sigma}{p}},$$

for any $x_i, x_i' \in X$, $a_i \in \mathbb{R}$ $(1 \leq i \leq n)$.

(c) *There is a constant* $C > 0$ *such that for any* $x_i^j, x_i'^j$ *in* X *and all* a_i^j *in* $\mathbb{R}$ $(1 \le i \le n,\ 1 \le j \le s)$*, we have*

$$\left(\sum_{j=1}^{s}\left\|\sum_{i=1}^{n} a_i^j(T(x_i^j) - T(x_i^{j\prime}))\right\|^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}}$$

$$\le C \sup_{f\in B_{X^\#}}\left(\sum_{j=1}^{s}\left(\left|\sum_{i=1}^{n} a_i^j(f(x_i^j) - f(x_i'^j))\right|^{1-\sigma}\left(\sum_{i=1}^{n}|a_i^j| d(x_i^j, x_i'^j)\right)^{\sigma}\right)^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}}.$$

Furthermore, the infimum of the constants $C > 0$ *in (b) and (c) is equal to* $\|T\|_{(\Pi_p\circ Lip_0)_\sigma}$.

REMARK 16. Note that if T belongs to $(\Pi_p \circ Lip_0)_\sigma(X, E)$, then setting $n = 1$ in (c) we have $T \in \Pi^L_{p,\sigma}(X, E)$, the space of (p, σ)-absolutely Lipschitz operators was introduced by Achour et al. in [6]. It is currently unknown whether this latter class satisfies the linearization theorem. However, based on the above, the class $(\Pi_p \circ Lip_0)_\sigma$ indeed fulfills it. This means that this class preserves nearly all the fundamental properties of the linear theory of (p, σ)-absolutely continuous operators.

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