

## A class of solutions of the equation

$$d(n^2) = d(\varphi(n))$$

**Zahra Amroune<sup>1</sup>, Djamel Bellaouar<sup>2</sup>  
and Abdelmadjid Boudaoud<sup>3</sup>**

<sup>1</sup> Laboratory of Pure and Applied Mathematics (LMPA),  
University of M'sila, B.P. 166, Ichbilia, 28000 M'sila, Algeria  
e-mail: zahra.amroune@univ-msila.dz

<sup>2</sup> Department of Mathematics, University 08 Mai 1945 Guelma,  
B.P. 401 Guelma 24000, Algeria  
e-mail: bellaouar.djamel@univ-guelma.dz

<sup>3</sup> Laboratory of Pure and Applied Mathematics (LMPA),  
University of M'sila, B.P. 166, Ichbilia, 28000 M'sila, Algeria  
e-mail: abdelmadjid.boudaoud@univ-msila.dz

**Received:** 27 July 2022

**Accepted:** 26 April 2023

**Revised:** 29 March 2023

**Online First:** 2 May 2023

**Abstract:** For any positive integer  $n$  let  $d(n)$  and  $\varphi(n)$  be the number of divisors of  $n$  and the Euler's phi function of  $n$ , respectively. In this paper we present some notes on the equation  $d(n^2) = d(\varphi(n))$ . In fact, we characterize a class of solutions that have at most three distinct prime factors. Moreover, we show that Dickson's conjecture implies that  $d(n^2) = d(\varphi(n))$  infinitely often.

**Keywords:** Diophantine equations, Euler's phi function, Divisor function.

**2010 Mathematics Subject Classification:** 11A25, 11A41, 11D99.



# 1 Introduction

Let  $d(n)$  be the divisor function, which counts the number of positive divisors of  $n$ , i.e., if  $n$  has the prime factorization  $n = q_1^{a_1} q_2^{a_2} \cdots q_k^{a_k}$  with distinct primes  $q_1, q_2, \dots, q_k$  and positive integers  $a_1, a_2, \dots, a_k$ , then

$$d(n) = (a_1 + 1)(a_2 + 1) \cdots (a_k + 1).$$

Let  $\varphi(n)$  be the Euler function, which counts the number of positive integers  $m \leq n$  with  $(m, n) = 1$ . It is well-known that

$$\varphi(n) = q_1^{a_1-1}(q_1 - 1)q_2^{a_2-1}(q_2 - 1) \cdots q_k^{a_k-1}(q_k - 1).$$

Various diophantine equations involving the divisor function and Euler's phi function were investigated by many authors (see [1, 2, 6–8, 10, 11]). In [4, Problem 705, page 78], it is shown that  $\varphi(d(n)) = d(\varphi(n))$  has infinitely many solutions; while in [12, pages 110–111], it is shown that  $d(n) = \varphi(n)$  has the only solutions 1, 3, 8, 10, 24 and 30, where  $d(n) < \varphi(n)$  for  $n \geq 31$ . Using these multiplicative functions, we are interested here in problems involving the number of positive divisors of  $\varphi(n)$ . In fact, in the present work, we compare the value of the divisor function to its value at Euler's functions. More precisely, we aim to prove that the diophantine equation

$$d(n^2) = d(\varphi(n)) \tag{1}$$

has infinitely many integer solutions as well as we identify large families of solutions. The first few terms are:

$$1, 5, 57, 74, 202, 292, 394, 514, 652, 1354, 2114, 2125, \dots$$

For this purpose, define

$$\mathbb{S} := \{n \in \mathbb{N} : d(n^2) = d(\varphi(n))\}.$$

In this paper, we characterize the elements of  $\mathbb{S}$  that have at most three distinct prime factors. The problem is interesting because it can force us to solve some diophantine equations involving prime numbers. Note also that the proofs are all on the elementary side and depend on long case by case analysis type arguments.

Recall that the Fermat numbers are the sequence  $(F_n)$  of positive integers defined by

$$F_n = 2^{2^n} + 1, n = 0, 1, \dots$$

If a particular  $F_m$  is prime it is called a Fermat prime. The only known Fermat primes are  $F_0, F_1, F_2, F_3$  and  $F_4$  and it has been conjectured that there are only finitely many. On the other hand, if  $p = 2^k + 1$  is a prime then  $k = 2^n$  for some  $n$  and  $p$  is a Fermat prime.

It is well-known that  $d(n) = 2$  if and only if  $n$  is prime and that  $d(n)$  is prime if and only if  $n = p^{a-1}$ , where  $p$  and  $q$  are both prime. Note also that if  $n$  is a prime power, namely  $n = p^a$  with  $p \geq 2$  and  $a \geq 1$ , then  $n \in \mathbb{S}$  implies  $(2a + 1) = d(p - 1)a$ . But the last equation is only true for  $a = 1$  and  $p = 5$ . Hence,  $n = 5$ . Observe first of all that there is a connection between Fermat primes and the solutions of the equation (1), where  $F_1$  is the unique prime solution. Moreover,

if  $n = q_1^{a_1} q_2^{a_2} \cdots q_k^{a_k} \in \mathbb{S}$ , where  $q_1 < q_2 < \cdots < q_k$  are primes and  $a_1, a_2, \dots, a_k$  are positive integers, then  $a_k$  must be odd. In fact, since  $(q_1 \cdots q_{k-1} (q_1 - 1) \cdots (q_k - 1), q_k) = 1$  we conclude that

$$d(\varphi(n)) = d(q_1^{a_1-1} q_2^{a_2-1} \cdots q_k^{a_k-1} (q_1 - 1)(q_2 - 1) \cdots (q_k - 1)) a_k,$$

and so  $a_k$  must be odd since  $d(n^2) = \prod_{i=1}^k (2a_i + 1)$ .

Now, let  $n$  as above and put

$$\prod_{i=1}^k (q_i - 1) = 2^{x_1+x_2+\cdots+x_k} \cdot q_1^{\alpha_1^{(2)}+\alpha_1^{(3)}+\cdots+\alpha_1^{(k)}} \cdots q_{k-2}^{\alpha_{k-2}^{(k-1)}+\alpha_{k-2}^{(k)}} \cdot q_{k-1}^{\alpha_{k-1}^{(k)}} \cdot m,$$

where  $x_1, x_2, \dots, x_k, m \geq 1$  and  $\alpha_1^{(2)}, \dots, \alpha_1^{(k)}, \alpha_2^{(3)}, \dots, \alpha_2^{(k)}, \dots, \alpha_{k-2}^{(k-1)}, \alpha_{k-2}^{(k)}, \alpha_{k-1}^{(k)}$  are non-negative integers with  $(2q_1 q_2 \cdots q_{k-1}, m) = 1$ . Thus in order to prove that  $n$  satisfies (1), it suffices to confirm that the exponents of the prime factors of  $n$  and the above variables satisfy the following diophantine equation:

$$\prod_{i=1}^k (2a_i + 1) = \left(1 + \sum_{i=1}^k x_i\right) \left(a_1 + \sum_{i=2}^k \alpha_1^{(i)}\right) \left(a_2 + \sum_{i=3}^k \alpha_2^{(i)}\right) \cdots \left(a_{k-1} + \alpha_{k-1}^{(k)}\right) a_k d(m). \quad (2)$$

In particular, if  $k = 3$  and  $n$  is odd then we need to solve the diophantine equation

$$(2a + 1)(2b + 1)(2c + 1) = (x_1 + x_2 + x_3 + 1)(a_1 + \alpha_1 + \alpha_2)(b + \alpha_3)c \cdot d(m),$$

where  $m, x_1, x_2, x_3 \geq 1$  and  $\alpha_1, \alpha_2, \alpha_3 \geq 0$  with  $(2q_1 q_2, m) = 1$ .

Now we are in a position to state the main results of the paper.

## 2 Solutions having two distinct prime factors

Assume that  $n = q_1^a q_2^b \in \mathbb{S}$ , where  $q_1, q_2$  are distinct primes with  $2 \leq q_1 < q_2$  and  $a, b \geq 1$ . Since  $((q_1 - 1)(q_2 - 1), q_2) = 1$ , we obtain

$$(2a + 1)(2b + 1) = d((q_1 - 1)(q_2 - 1)q_1^{a-1})b. \quad (3)$$

### 2.1 $n$ is square-free

We have the following result:

**Proposition 2.1.** *The only square-free solutions of the form  $q_1 q_2$  are:*

- i)  $n = 3 \cdot 19$ .
- ii)  $n = 2F_3$ , where  $F_3 = 257$ .
- iii)  $n = 2(4p^2 + 1)$ , where  $p$  and  $4p^2 + 1$  are simultaneously primes.

We need the following lemma.

**Lemma 2.1.** *Let  $p$  be a prime number with  $p \geq 5$ . Then the number  $2p^{2a} + 1$  is composite for every  $a \geq 1$ . In particular, if  $p \equiv 1 \pmod{3}$ , then the number  $2p^a + 1$  is composite for every  $a \geq 2$ .*

*Proof.* First, it is clear that if  $p = 3k + 1$  with  $k \geq 2$ , then  $2p^{2a} + 1 = 2(3k + 1)^{2a} \equiv 0 \pmod{3}$ . That is,  $2p^{2a} + 1$  is a multiple of 3. But, if  $p = 3k + 2$  for some  $k \geq 1$ , then  $p = 3k' - 1$  with  $k' = k + 1$  and so

$$2p^{2a} + 1 = 2(3k' - 1)^{2a} + 1 = 2 \left[ \sum_{i=1}^{2n} (-1)^{2n-i} \binom{i}{2n} (3k')^i \right] + 1 \equiv 0 \pmod{3},$$

which is also a multiple of 3. By the same way, if  $a \geq 2$  and  $p \equiv 1 \pmod{3}$ , then  $2p^a + 1$  is a multiple of 3. This completes the proof.  $\square$

**Remark 2.2.** Let  $p$  be a prime number with  $p \geq 5$  and let  $a \geq 1$ . Similar to what we have shown in Lemma 2.1, if  $r$  is odd then the number  $2^r p^{2a} + 1$  is composite. However, if  $r$  is odd and  $p \equiv 1 \pmod{3}$ , then the number  $2^r p^a + 1$  is also composite.

*Proof of Proposition 2.1.* Suppose that  $n = pq$ , where  $p$  and  $q$  are odd primes with  $p < q$ . By (3), we have

$$9 = d((p-1)(q-1)). \quad (4)$$

We put  $p-1 = 2^s m_1$  and  $q-1 = 2^{s'} m_2$ , where  $m_1, m_2$  are odd and  $s, s' \geq 1$ . From (4), we obtain  $9 = (s + s' + 1) d(m_1 m_2)$ . We distinguish the following cases:

Case 1.  $s + s' = 2$  and  $d(m_1 m_2) = 3$ . That is,  $s = s' = 1$  and  $m_1 m_2 = r^2$  for some prime  $r \geq 3$ . On the other hand, since  $p < q$  we conclude that  $m_1 = 1$  and  $m_2 = r^2$ . Hence,  $p = 3$  and  $q = 2r^2 + 1$ . But, by Lemma 2.1, the number  $2r^2 + 1$  is a multiple of 3 for  $r \geq 5$ , in which case  $n = 3 \cdot 19$  is the only solution of this form.

Case 2.  $s + s' = 8$  and  $d(m_1 m_2) = 1$ . That is,  $m_1 = m_2 = 1$ . Therefore,  $p$  and  $q$  are Fermat primes and hence  $s, s'$  are powers of 2. This case is not valid since  $p < q$ .

Now, assume that  $n = 2q$  with  $q \geq 3$  is prime. By (3),  $9 = d(q-1)$  from which it follows that  $q-1$  is either  $2^8$  or  $2^2 p^2$  for some prime  $p \geq 3$ . Hence,  $n = 2 \cdot 257 = 2F_3$  or  $n = 2(4p^2 + 1)$  with  $p$  and  $4p^2 + 1$  are simultaneously primes.

This completes the proof.  $\square$

## 2.2 $n$ is not square-free with $n$ odd

Assume that  $n$  is **odd**. We have the following results:

**Proposition 2.3.** Let  $n = q_1^a q_2$ , where  $3 \leq q_1 < q_2$  and  $a \geq 2$ . If  $n \in \mathbb{S}$ , then  $n$  is one of the numbers:

- $n = F_1^3 F_2$ .
- $n = 3^{5t-3} (2^3 \cdot 3^t + 1)$ , where  $t \geq 2$  and  $2^3 \cdot 3^t + 1$  is prime.
- $n = 5^{5t-3} (2^2 \cdot 5^t + 1)$ , where  $t \geq 2$  and  $2^2 \cdot 5^t + 1$  is prime.
- $n = 3^{t-1} (2 \cdot 3^t + 1)$ , where  $t \geq 4$  and  $2 \cdot 3^t + 1$  is prime.

*Proof.* By (3), we have

$$3(2a + 1) = d((q_1 - 1)(q_2 - 1)q_1^{a-1}). \quad (5)$$

There are two cases:

Case 1.  $(q_2 - 1, q_1) = 1$ . We put  $q_1 - 1 = 2^s m_1$  and  $q_2 - 1 = 2^{s'} m_2$ , where  $(2, m_1 m_2) = 1$  and  $s, s' \geq 1$ . From (5), we have  $3 = a(d(m_1 m_2)(s + s' + 1) - 6)$ . Since  $a \geq 2$ , it follows that  $a = 3$  and  $s + s' = 6$ . Hence,  $m_1 = m_2 = 1$  and so we must have  $s = 2$  and  $s' = 4$ . That is,  $q_1 = 5$  and  $q_2 = 17$ , in which case  $n = 5^3 \cdot 17$ .

Case 2.  $(q_2 - 1, q_1) = q_1$ . As above, we put  $q_1 - 1 = 2^s m_1$  and  $q_2 - 1 = 2^{s'} q_1^t m_2$ , where  $(2, m_1 m_2) = 1$  and  $s, s', t \geq 1$ . By (5) we have

$$3(2a + 1) = d(m_1 m_2)(s + s' + 1)(a + t). \quad (6)$$

It is clear that  $d(m_1 m_2)$  cannot be  $\geq 3$ , otherwise

$$3(2a + 1) \geq 3(a + t)(s + s' + 1) \geq 9(a + 1) > 6a + 3,$$

a contradiction. Moreover, if  $d(m_1 m_2) = 2$ , then the right-hand side of (6) is even, while its left-hand side is odd, also a contradiction. Therefore,  $d(m_1 m_2) = 1$  and so by (6) once again,  $3(2a + 1) = (2^i + s' + 1)(a + t)$  for some  $i \geq 0$ . Note also that  $2^i + s' + 1$  cannot be  $\geq 6$ , otherwise  $3(2a + 1) \geq 6(a + t) > 6a + 3$ , a contradiction. Consequently, we have either  $2^i + s' = 4$  or  $2^i + s' = 2$ .

(i)  $2^i + s' = 4$ . There are two possibilities:

- $i = 0$  and  $s' = 3$ . It follows that  $a = 5t - 3$ ,  $q_1 = 3$  and  $q_2 = 2^3 \cdot 3^t + 1$ , thus  $n = 3^{5t-3}(2^3 \cdot 3^t + 1)$ , where  $2^3 \cdot 3^t + 1$  is prime. For example, for  $t = 2$ , we get  $n = 3^7 \cdot 73$ .
- $i = 1$  and  $s' = 2$ . That is,  $a = 5t - 3$ ,  $q_1 = 5$  and  $q_2 = 2^2 \cdot 5^t + 1$ , thus  $n = 5^{5t-3}(2^2 \cdot 5^t + 1)$ , where  $2^2 \cdot 5^t + 1$  is prime. For example, for  $t = 2$ , we have  $n = 5^7 \cdot 101$ .

(ii)  $2^i + s' = 2$ . That is,  $i = 0$ ,  $s' = 1$  and so  $a = t - 1$ . Hence,  $q_1 = 3$  and  $q_2 = 2 \cdot 3^t + 1$ . Consequently,  $n = 3^{t-1}(2 \cdot 3^t + 1)$ , where  $(2 \cdot 3^t + 1)$  is prime. For example, for  $t = 4$ , we have  $n = 3^3 \cdot 163$ .

This completes the proof.  $\square$

**Proposition 2.4.** *The number  $n = F_1 F_2^3$  is the only solution of the form  $q_1 q_2^b$ , where  $3 \leq q_1 < q_2$  and  $b \geq 2$ .*

*Proof.* Assume that  $n = q_1 q_2^b \in \mathbb{S}$ , where  $3 \leq q_1 < q_2$  and  $b \geq 2$ . Applying (3), we obtain  $3(2b + 1) = d((q_1 - 1)(q_2 - 1))b$ , from which it follows that  $b(d((q_1 - 1)(q_2 - 1)) - 6) = 3$ , and so  $d((q_1 - 1)(q_2 - 1)) = 7$  and  $b = 3$ . Or, equivalently,  $(q_1 - 1)(q_2 - 1) = 2^6$ . Thus,  $q_1 = 5$  and  $q_2 = 17$ , in which case  $n = 5 \cdot 17^3$ .  $\square$

**Theorem 2.5.** *Let  $n = q_1^a q_2^b$ , where  $3 \leq q_1 < q_2$  and  $a, b \geq 2$ . If  $n \in \mathbb{S}$ , then  $n = 3^a (2 \cdot 3^t + 1)^b$ , where  $2 \cdot 3^t + 1$  is prime and  $ab + 2a + 2b + 1 = 3bt$ .*

*Proof.* By (3), we have

$$(2a + 1)(2b + 1) = d((q_1 - 1)(q_2 - 1)q_1^{a-1})b. \quad (7)$$

There are two cases:

**Case 1.**  $(q_2 - 1, q_1) = 1$ . We put  $q_1 - 1 = 2^x m_1$  and  $q_2 - 1 = 2^y m_2$ , where  $x, y \geq 1$  and  $(2, m_1 m_2) = 1$ . It then follows from (7) that

$$(2a + 1)(2b + 1) = (x + y + 1) d(m_1 m_2) ab. \quad (8)$$

It is clear that  $d(m_1 m_2)$  cannot be  $\geq 3$ . Otherwise,  $(x + y + 1)a \geq 3a > 2a + 1$  and  $d(m_1 m_2)b \geq 3b > 2b + 1$ , a contradiction. Moreover, if  $d(m_1 m_2) = 2$ , then the right-hand side of (8) is even, while its left-hand side is odd, also a contradiction. Therefore,  $m_1 = m_2 = 1$  and so by (8) once again,  $(2a + 1)(2b + 1) = (2^j + 2^i + 1)ab$  for some  $j > i \geq 0$ . Note also that  $2^j + 2^i + 1$  cannot be  $\geq 6$ , and hence  $i = 0$  and  $j = 1$ . That is,  $(2a + 1)(2b + 1) = 4ab$ , which is impossible.

**Case 2.**  $(q_2 - 1, q_1) = q_1$ . We put  $q_1 - 1 = 2^x m_1$  and  $q_2 - 1 = 2^y q_1^t m_2$ , where  $x, y, t \geq 1$  and  $(2, m_1 m_2) = 1$ . By (7),  $(2a + 1)(2b + 1) = (x + y + 1) d(m_1 m_2) (a + t)b$  from which it follows that  $m_1 = m_2 = 1$  and  $b \geq 3$ . Moreover, we see that  $x + y$  is even and  $x + y \leq 4$ . Therefore, if  $x + y = 4$ , then

$$a(b - 2) = b(2 - 5t) + 1. \quad (9)$$

This is impossible since the left-hand side of (9) is positive, while its right-hand side is negative. If  $x + y = 2$ , then  $x = y = 1$ . It follows that  $n = 3^a (2 \cdot 3^t + 1)^b$ , where  $2 \cdot 3^t + 1$  is prime and  $ab + 2a + 2b + 1 = 3bt$ . For example, for  $a = 10$ ,  $b = 7$  and  $t = 5$ , we obtain  $n = 3^{10} (2 \cdot 3^5 + 1)^7 = 3^{10} \cdot 487^7$ .  $\square$

### 2.3 $n$ is not square-free with $n$ even

Now, assume that  $n$  is even. We have the following notes:

**Lemma 2.2.**  $2^x - 3$  is divisible by 5 if and only if  $x \equiv 3 \pmod{4}$ .

*Proof.* Clearly,  $2^{4k} \equiv 1 \pmod{5}$  for every  $k \geq 0$ . Hence,  $2^x \equiv 3 \pmod{5}$  if and only if  $x \equiv 3 \pmod{4}$ .  $\square$

**Proposition 2.6.** Let  $n = 2^a q_2$ , where  $q_2 \geq 3$  and  $a \geq 2$ . If  $n \in \mathbb{S}$ , then  $n$  is one of the numbers:

- $n = 2^{5t-3} (2^t \cdot p^4 + 1)$ , where  $p$  is an odd prime with  $2^t \cdot p^4 + 1$  is also prime.
- $n = 2^{t-1} (2^t \cdot p^2 + 1)$ , where  $p$  is an odd prime with  $2^t \cdot p^2 + 1$  is also prime.
- $n = 2^{(2^i-3)/5} F_i$ , where  $i \equiv 3 \pmod{4}$  and  $F_i$  is a Fermat prime.

*Proof.* By (3), we have  $3(2a + 1) = d((q - 1)2^{a-1})$ . Put  $q - 1 = 2^t m$ , where  $(2, m) = 1$  and  $t \geq 1$ . It follows that  $3(2a + 1) = d(m)(a + t)$ , and hence

$$(6 - d(m))a = d(m)t - 3. \quad (10)$$

It is clear from (10) that  $d(m)$  is odd and cannot be  $\geq 6$ . Now, we consider separately the following possibilities:

- $d(m) = 5$ . It follows that  $m = p^4$ , where  $p \geq 3$  is prime and  $a = 5t + 3$ . Thus,  $n = 2^{5t-3} (2^t \cdot p^4 + 1)$ , where  $2^t \cdot p^4 + 1$  is also prime. For example, for  $t = 1$  and  $p = 3$ , we get  $n = 2^2 (2 \cdot 3^4 + 1) = 2^2 \cdot 163$ .

- $d(m) = 3$ . It follows that  $m = p^2$ , where  $p \geq 3$  is prime and  $a = t - 1$ . Thus,  $n = 2^{t-1} (2^t \cdot p^2 + 1)$ , where  $2^t \cdot p^2 + 1$  is also prime. For example, for  $t = p = 3$ , we have  $n = 2^2 (2^3 \cdot 3^2 + 1) = 2^2 \cdot 73$ .
- $d(m) = 1$ . Then  $t = 2^i$  for some  $i \geq 0$ , and by (10) we have  $a = (2^i - 3)/5$ . By Lemma 2.2,  $i \equiv 3 \pmod{4}$ . Hence,  $n = 2^{(2^i-3)/5} F_i$ , where  $F_i$  is a Fermat prime.

This completes the proof.  $\square$

**Proposition 2.7.** *Let  $n = 2q_2^b$ , where  $q_2$  is odd prime and  $b \geq 2$ . Then  $n \notin \mathbb{S}$ .*

*Proof.* By (3), we have  $3(2b+1) = d(q_2-1)b$ . We put  $q_2-1 = 2^s m$ , where  $(2, m) = 1$  and  $s \geq 1$ . It follows that  $3(2b+1) = d(m)(s+1)b$ , and so

$$3 = (d(m)(s+1) - 6)b. \quad (11)$$

By (11), we must have  $b = 3$ ,  $d(m) = 7$  and  $s = 0$  or  $d(m) = 1$  and so  $s = 2^i = 6$  for some  $i$ . Thus there is no solution in both cases.  $\square$

**Proposition 2.8.** *Let  $n = 2^a q_2^b$ , where  $q_2 \geq 3$  and  $a, b \geq 2$ . If  $n \in \mathbb{S}$ , then  $n$  is one of the numbers:*

- $n = 2^a (2^s \cdot p^2 + 1)^b$ , where  $p$  and  $2^s \cdot p^2 + 1$  are simultaneously primes with  $ab + 2a + 2b + 1 = 3bs$ ,
- $n = 2^a (2^{(3ab+2a+2b+1)/b} + 1)^b$ , where  $b$  divides  $2a + 1$  and  $2^{(3ab+2a+2b+1)/b} + 1$  is prime.

*Proof.* By (3), we have  $(2a+1)(2b+1) = d(2^{a-1}(q_2-1))b$ . If we put  $q_2-1 = 2^s m$ , where  $(2, m) = 1$  and  $s \geq 1$ , it follows that

$$(2b+1)(2b+1) = d(m)(s+a)b. \quad (12)$$

By (12),  $d(m)$  cannot be  $\geq 6$ ; otherwise,  $d(m)(s+a)b \geq (3b)(2(a+s)) > (2a+1)(2b+1)$ , a contradiction. Moreover,  $d(m)$  cannot be even. So the rest cases are:

- $d(m) = 5$ . By (12), we have  $b(2-5s) = a(b-2) - 1$ , which has no sense.
- $d(m) = 3$ . Then  $m = p^2$  for some prime  $p \geq 3$ , and by (12) we have  $ab + 2a + 2b + 1 = 3bs$ . Thus,  $n = 2^a (2^s \cdot p^2 + 1)^b$  where  $p$  and  $2^s \cdot p^2 + 1$  are simultaneously primes with  $ab + 2a + 2b + 1 = 3bs$ . For example, for  $a = 13$ ,  $b = 9$  and  $s = 6$ , we get  $n = 2^{13} \cdot 577^9$ .
- $d(m) = 1$ . It follows that  $q = 2^{2^i} + 1$  for some  $i \geq 0$ , and so  $n = 2^a F_i^b$ , where  $3ab + 2a + 2b + 1 = 2^i b$  by (12). Or equivalently,  $n = 2^a (2^{(3ab+2a+2b+1)/b} + 1)^b$ , where  $b$  divides  $2a + 1$  and  $2^{(3ab+2a+2b+1)/b} + 1$  is prime.

The proof is finished.  $\square$

### 3 Solutions having three distinct prime factors

Let  $n = q_1^a q_2^b q_3^c \in \mathbb{S}$ , where  $q_1, q_2, q_3$  are distinct primes with  $2 \leq q_1 < q_2 < q_3$  and  $a, b, c \geq 1$ . Be definition, we see that

$$(2a+1)(2b+1)(2c+1) = d((q_1-1)(q_2-1)(q_3-1)q_1^{a-1}q_2^{b-1})c. \quad (13)$$

We consider separately the cases  $n$  is square-free and  $n$  is not square-free (odd and even).

### 3.1 $n$ is square-free

First, assume that  $n$  is **square-free odd**.

**Proposition 3.1.** *The only possible solutions of the form  $q_1 q_2 q_3$  are:*

- $F_1 F_3 F_4, F_0 F_2 \cdot 73, 11 \cdot F_2 \cdot 41.$
- $F_1 F_2 (4p^2 + 1)$ , where  $p$  is an odd prime with  $4p^2 + 1$  is prime.
- $F_1 (2p + 1) (2p^5 + 1)$ , where  $p$  is an odd prime with  $2p + 1$  and  $2p^5 + 1$  are primes.
- $F_1 (2^2 p + 1) (2^4 p + 1)$ , where  $p$  is an odd prime with  $2^2 p + 1$  and  $2^4 p + 1$  are primes.
- $F_2 (2p + 1) (2^3 p + 1)$ , where  $p$  is an odd prime with  $2p + 1$  and  $2^3 p + 1$  are primes.

For the proof we need the following lemma:

**Lemma 3.1.** *If  $p$  is prime greater than 3, then the numbers  $2^{2\alpha+1}p + 1$  and  $2^{2\beta}p + 1$  with  $\alpha \geq 0$  and  $\beta \geq 1$  cannot be simultaneously primes.*

*Proof.* First we note that  $2^{2\alpha+1} \equiv 2 \pmod{3}$  and  $2^{2\beta} \equiv 1 \pmod{3}$ . Thus if  $p$  is of the form  $3k + 1$ , then  $2^{2\alpha+1}p + 1 = 2^{2\alpha+1}(3k + 1) + 1 \equiv 0 \pmod{3}$  and if  $p$  is of the form  $3k + 2$ , then  $2^{2\beta}p + 1 = 2^{2\beta}(3k + 2) + 1 \equiv 0 \pmod{3}$ .  $\square$

*Proof of Proposition 3.1.* We put  $q_1 - 1 = 2^x m_1$ ,  $q_2 - 1 = 2^y m_2$ , and  $q_3 - 1 = 2^z m_3$ , where  $m_i$  is odd and  $x, y, z \geq 1$ . Let  $m = m_1 m_2 m_3$ . Applying (13), we have  $27 = d(m)(x + y + z + 1)$ . We distinguish two cases:

Case 1.  $d(m) = 1$ . Then  $2^i + 2^j + 2^k = 26$  for some  $0 \leq i < j < k$ , which is only true for  $i = 1, j = 3$  and  $k = 4$ . Hence,  $n = F_1 F_3 F_4$ .

Case 2.  $d(m) = 3$  and  $x + y + z = 8$ . It follows that  $m = p^2$ , where  $p \geq 3$  is prime. Also we consider the following subcases:

Case 2.1.  $m_1 = m_2 = 1$  and  $m_3 = p^2$ . Since  $x < y$ , by Lemma 2.1, there are only two possibilities:

- $n = 3 \cdot 17 \cdot 73.$
- $n = 5 \cdot 17 \cdot (4p^2 + 1)$ , where  $p$  and  $4p^2 + 1$  are both primes. For example,  $p = 3$ .

Case 2.2.  $m_1 = 1$  and  $m_2 = m_3 = p$ . It follows that  $x = 2^i$ , where  $i \geq 0$  and  $y < z$ . By Lemma 3.1, we have three possibilities:

- $n = 5(2p + 1)(2^5 p + 1)$ , where  $p, 2p + 1$  and  $2^5 p + 1$  are simultaneously primes. For example, for  $p = 11$ , we obtain  $n = 5 \cdot 23 \cdot 353$ .
- $n = 5(4p + 1)(16p + 1)$ , where  $p, 4p + 1$  and  $16p + 1$  are simultaneously primes. For example, for  $p = 7$ , we have  $n = 5 \cdot 29 \cdot 113$ .
- $n = 17(2p + 1)(2^3 p + 1)$ , where  $p, 2p + 1$  and  $2^3 p + 1$  are simultaneously primes. For example, for  $p = 11$ , we have  $n = 17 \cdot 23 \cdot 89$ .

Case 2.3.  $m_1 = m_2 = p$  and  $m_3 = 1$ . This case is not valid. We have the same for  $m_1 = m_3 = 1$  and  $m_2 = p^2$  or  $m_2 = m_3 = 1$  and  $m_1 = p^2$ .

Case 2.4.  $m_1 = m_3 = p$  and  $m_2 = 1$ . It follows that  $y = 2^i$ , where  $i \geq 0$  and  $x < z$ . Thus we must have  $p = 5$ . Hence,  $n = 11 \cdot 17 \cdot 41$ .  $\square$



Second, assume that  $n$  is **square-free even**.

**Proposition 3.2.** *The only possible solutions of the form  $2q_2q_3$  are:*

- $2 \cdot 3 \cdot 1153, 2 \cdot 19 \cdot 1459.$
- $2 \cdot 5 (2^6p^2 + 1)$ , where  $p$  is an odd prime with  $2^6p^2 + 1$  is prime.
- $2 \cdot 17 (2^4p^2 + 1)$ , where  $p$  is an odd prime with  $2^4p^2 + 1$  is prime.
- $2 (2p + 1) (2^7p + 1)$ , where  $p$  is an odd prime with  $2p + 1$  and  $2^7p + 1$  are primes.
- $2 (2^2p + 1) (2^6p + 1)$ , where  $p$  is an odd prime with  $2^2p + 1$  and  $2^6p + 1$  are primes.
- $2 (2^3p + 1) (2^5p + 1)$ , where  $p$  is an odd prime with  $2^3p + 1$  and  $2^5p + 1$  are primes.
- $2 (2p + 1) (2p^7 + 1)$ , where  $p$  is an odd prime with  $2p + 1$  and  $2p^7 + 1$  are primes.
- $2 (2p^3 + 1) (2p^5 + 1)$ , where  $p$  is an odd prime with  $2p^3 + 1$  and  $2p^5 + 1$  are primes.
- $2 (2p_1 + 1) (2p_1p_2^2 + 1)$ , where  $p_1, p_2, 2p_1 + 1$  and  $2p_1p_2^2 + 1$  are simultaneously primes.

For the proof, we need the following lemma:

**Lemma 3.2.** *If  $p_1$  and  $p_2$  are primes greater than 3, then  $2p_1^2p_2^2 + 1$  is composite.*

*Proof.* This follows immediately from the proof of Lemma 2.1. □

*Proof of Proposition 3.2.* We put  $q_1 - 1 = 2^x m_1$  and  $q_2 - 1 = 2^y m_2$ , where  $(2, m_1 m_2) = 1$  and  $x, y \geq 1$ . By (13),  $27 = d(m_1 m_2)(x + y + 1)$ . There are three cases to consider.

Case 1.  $d(m_1 m_2) = 1$ . That is,  $2^i + 2^j = 26$  for some  $i, j \geq 0$ . This is impossible.

Case 2.  $d(m_1 m_2) = 3$  and  $x + y = 8$ . It follows that  $m_1 m_2 = p^2$ , where  $p \geq 3$  is prime. We have three subcases:

Case 2.1.  $m_1 = 1$  and  $m_2 = p^2$ . Then the solutions are given by:

- $n = 2 \cdot 3(2^7p^2 + 1)$  with  $p$  and  $2^7p^2 + 1$  are both prime, and by Lemma 2.1,  $p = 3$  is the only prime with this property, in which case  $n = 2 \cdot 3 \cdot 1153$ .
- $n = 2 \cdot 5(2^6p^2 + 1)$  with  $p$  and  $2^6p^2 + 1$  are both prime. For example, for  $p = 3$ , we have  $n = 2 \cdot 5 \cdot 577$ .
- $n = 2 \cdot 17(16p^2 + 1)$  with  $p$  and  $16p^2 + 1$  are both prime. For example, for  $p = 5$ , we have  $n = 2 \cdot 17 \cdot 401$ .

Case 2.2.  $m_1 = m_2 = p$ . Then  $x < y$ , and the solutions are:

- $2 \cdot (2p + 1)(2^7p + 1)$ , where  $p, 2p + 1$  and  $2^7p + 1$  are primes. For example, for  $p = 5$ , we get  $n = 2 \cdot 11 \cdot 641$ .
- $n = 2(4p + 1)(2^6p + 1)$ , where  $p, 4p + 1$  and  $2^6p + 1$  are primes. For example, for  $p = 7$ , we get  $n = 2 \cdot 29 \cdot 449$ .
- $n = 2(8p + 1)(2^5p + 1)$ , where  $p, 8p + 1$  and  $2^5p + 1$  are primes. For example, for  $p = 11$ , we have  $n = 2 \cdot 89 \cdot 353$ .

Case 2.3.  $m_1 = p^2$  and  $m_2 = 1$ . This case is not valid.

Case 3.  $d(m_1m_2) = 9$  and  $x + y = 2$ . We have two subcases:

Case 3.1.  $m_1m_2 = p^8$ , where  $p \geq 3$  is prime. Then the solutions are:

- $n = 2(2p + 1)(2p^7 + 1)$ , where  $p$ ,  $2p + 1$  and  $2p^7 + 1$  are primes.
- $n = 2(2p^2 + 1)(2p^6 + 1)$ , where  $p$ ,  $2p^2 + 1$  and  $2p^6 + 1$  are primes, which is only true for  $p = 3$ , and so  $n = 2 \cdot 19 \cdot 1459$ .
- $n = 2(2p^3 + 1)(2p^5 + 1)$ , where  $p$ ,  $2p^3 + 1$  and  $2p^5 + 1$  are primes. For example, for  $p = 29$ , we have  $n = 2 \cdot 48779 \cdot 41022299$ .

Case 3.2.  $m_1m_2 = p_1^2p_2^2$ , where  $p_1, p_2$  are odd primes with  $p_1 < p_2$ . By Lemma 3.2, the number  $2 \cdot 3 \cdot (2p_1^2p_2^2 + 1)$  is composite and by Lemma 2.1, we obtain

- $n = 2(2p_1 + 1)(2p_1p_2^2 + 1)$  with  $p_1, p_2, 2p_1 + 1$  and  $2p_1p_2^2 + 1$  are primes. For example, for  $p_1 = 3$  and  $p_2 = 5$ ,  $n = 2 \cdot 7 \cdot 151$ .
- $n = 2(2p_2 + 1)(2p_1^2p_2 + 1)$  with  $p_1, p_2, 2p_2 + 1$  and  $2p_1^2p_2 + 1$  are primes. For example, for  $p_1 = 3$  and  $p_2 = 11$  we obtain  $n = 2 \cdot 23 \cdot 199$ .

This completes the proof. □

## 3.2 $n$ is not square-free with $n$ odd

Assume that  $n$  is **not square-free** with  $n$  is **odd**. Here, we characterize all odd solutions having only one power prime.

**Proposition 3.3.** *The only possible solutions of the form  $q_1^a q_2 q_3$ , where  $a \geq 2$  and  $3 \leq q_1 < q_2 < q_3$  are:*

- $5^3(2p + 1)(8p + 1)$ , where  $p$ ,  $2p + 1$  and  $8p + 1$  are prime numbers.
- $q_1^{17(t+t')-9}(2^y q_1^t + 1)(2^z q_1^{t'} + 1)$ , where  $q_1$  is a Fermat prime and  $y, z$  are positive integers with  $(q_1, y + z) \in \{(3, 15), (5, 14), (17, 12), (257, 8)\}$  and  $2^y q_1^t + 1, 2^z q_1^{t'} + 1$  are primes.
- $q_1^{5(t+t')-3}(2^y q_1^t + 1)(2^z q_1^{t'} + 1)$ , where  $q_1$  is a Fermat prime and  $y, z, t, t'$  are positive integers with  $(q_1, y + z) \in \{(3, 13), (5, 12), (17, 10), (257, 6)\}$  and  $2^y q_1^t + 1, 2^z q_1^{t'} + 1$  are primes.
- $q_1^{\frac{13(t+t')-9}{5}}(2^y q_1^t + 1)(2^z q_1^{t'} + 1)$ , where  $q_1$  is a Fermat prime and  $y, z, t, t'$  are positive integers, with  $(q_1, y + z) \in \{(3, 11), (5, 10), (17, 8), (257, 4)\}$  and  $2^y q_1^t + 1, 2^z q_1^{t'} + 1$  are primes.
- $q_1^{\frac{11(t+t')-9}{7}}(2^y q_1^t + 1)(2^z q_1^{t'} + 1)$ , where  $q_1$  is a Fermat prime and  $y, z, t, t'$  are positive integers with  $(q_1, y + z) \in \{(3, 9), (5, 8), (17, 6)\}$  and  $2^y q_1^t + 1, 2^z q_1^{t'} + 1$  are primes.
- $q_1^{t+t'-1}(2^y q_1^t + 1)(2^z q_1^{t'} + 1)$ , where  $q_1$  is a Fermat prime and  $y, z, t, t'$  are positive integers with  $(q_1, y + z) \in \{(3, 7), (5, 6), (17, 4)\}$  and  $2^y q_1^t + 1, 2^z q_1^{t'} + 1$  are primes.
- $q_1^{\frac{7(t+t')-9}{11}}(2^y q_1^t + 1)(2^z q_1^{t'} + 1)$ , where  $q_1$  is a Fermat prime and  $y, z, t, t'$  are positive integers with  $(q_1, y + z) \in \{(3, 5), (5, 4)\}$  and  $2^y q_1^t + 1, 2^z q_1^{t'} + 1$  are primes.
- $q_1^{\frac{5(t+t')-9}{13}}(2^y q_1^t + 1)(2^z q_1^{t'} + 1)$ , where  $q_1$  is a Fermat prime and  $y, z, t, t'$  are positive integers with  $(q_1, y + z) \in \{(3, 3), (5, 2)\}$  and  $2^y q_1^t + 1, 2^z q_1^{t'} + 1$  are primes.

- $n = (2^x q + 1)^{5(t+t')-3} (2^y (2^x q + 1)^t q + 1) (2^z (2^x q + 1)^{t'} + 1)$ , where  $q, 2^x q + 1, 2^y (2^x q + 1)^t q + 1, 2^z (2^x q + 1)^{t'} + 1$  are primes and  $x, y, z, t, t'$  are positive integers with  $x + y + z = 4$ .
- $n = (2^x q + 1)^{5(t+t')-3} (2^y (2^x q + 1)^t + 1) (2^z (2^x q + 1)^{t'} q + 1)$ , where  $q, 2^x q + 1, 2^y (2^x q + 1)^t + 1, 2^z (2^x q + 1)^{t'} q + 1$  are primes and  $x, y, z, t, t'$  are positive integers with  $x + y + z = 4$ .
- $n = (2^x + 1)^{5(t+t')-3} (2^y (2^x + 1)^t q + 1) (2^z (2^x + 1)^{t'} q + 1)$ , where  $q, 2^x + 1, 2^y (2^x q + 1)^t q + 1, 2^z (2^x q + 1)^{t'} q + 1$  are primes and  $x, y, z, t, t'$  are positive integers with  $x + y + z = 4$ .
- $n = (2^x + 1)^{5(t+t')-3} (2^y (2^x + 1)^t q^2 + 1) (2^z (2^x + 1)^{t'} + 1)$ , where  $q, 2^x + 1, 2^y (2^x q^2 + 1)^t q + 1, 2^z (2^x q + 1)^{t'} + 1$  are primes and  $x, y, z, t, t'$  are positive integers with  $x + y + z = 4$ .
- $n = (2^x + 1)^{5(t+t')-3} (2^y (2^x + 1)^t + 1) (2^z (2^x + 1)^{t'} q^2 + 1)$ , where  $q, 2^x + 1, 2^y (2^x q + 1)^t + 1, 2^z (2^x q + 1)^{t'} q^2 + 1$  are primes and  $x, y, z, t, t'$  are positive integers with  $x + y + z = 4$ .

*Proof.* Let  $n = q_1^a q_2 q_3 \in \mathbb{S}$ . Therefore, by (13) we get

$$9(2a + 1) = d((q_1 - 1)(q_2 - 1)(q_3 - 1)q_1^{a-1}). \quad (14)$$

There are two cases:

Case 1. Suppose that  $((q_2 - 1)(q_3 - 1), q_1) = 1$ . It follows from (14) that

$$9 = (d((q_1 - 1)(q_2 - 1)(q_3 - 1)) - 18)a. \quad (15)$$

We distinguish the following subcases:

Case 1.1.  $a = 9$ . From (15),  $d((q_1 - 1)(q_2 - 1)(q_3 - 1)) = 19$ . We must have

$$(q_1 - 1)(q_2 - 1)(q_3 - 1) = 2^{18},$$

and hence  $q_1, q_2$  and  $q_3$  are Fermat primes. This is impossible.

Case 1.2.  $a = 3$ . From (15),  $d((q_1 - 1)(q_2 - 1)(q_3 - 1)) = 21$ . If

$$(q_1 - 1)(q_2 - 1)(q_3 - 1) = 2^{20},$$

then  $q_1, q_2$  and  $q_3$  are Fermat primes, which is impossible. Thus, we must have

$$(q_1 - 1)(q_2 - 1)(q_3 - 1) = 2^6 q^2,$$

where  $q$  is odd prime, with  $(q_1, q) = 1$ . By Lemma 2.1,  $8q^2 + 1$  and  $2q^2 + 1$  are composite, and by Lemma 3.1,  $2q + 1$  and  $2^4 q + 1$  are not simultaneously primes and the same for  $2^2 q + 1$  and  $2^3 q + 1$ , so we conclude that  $n$  is of the form:  $n = 5^3(2q + 1)(8q + 1)$ , where  $q, 2q + 1$  and  $8q + 1$  are prime numbers with  $q > 5$ . For example, if  $q = 11$  then  $n = 5^3 \cdot 23 \cdot 89$ .

Case 2. Suppose that  $((q_2 - 1)(q_3 - 1), q_1) = q_1$ . We put  $q_1 - 1 = 2^x m_1$ ,  $q_2 - 1 = 2^y q_1^t m_2$  and  $q_3 - 1 = 2^z q_1^{t'} m_3$ , where  $x, y, z \geq 1$ ,  $\max(t, t') \geq 1$  and  $(2p_1, m_1 m_2 m_3) = 1$ . Let us take  $m = m_1 m_2 m_3$ . It follows from (14) that

$$(18 - d(m)(x + y + z + 1))a = d(m)(x + y + z + 1)(t + t') - 9. \quad (16)$$

As above,  $d(m)(x + y + z + 1)$  cannot be  $\geq 18$ . Thus, we distinguish the following subcases:

Case 2.1.  $d(m)(x + y + z + 1) = 17$ . Thus,  $d(m) = 1$  and so  $x + y + z = 16$ . Or equivalently,  $m_1 = m_2 = m_3 = 1$ ,  $q_1$  is a Fermat prime and by (16),  $a = 17(t + t') - 9$ . Therefore, we get

$$n = q_1^{17(t+t')-9} (2^y \cdot q_1^t + 1) (2^z \cdot q_1^{t'} + 1),$$

where  $2^y \cdot q_1^t + 1$  and  $2^z \cdot q_1^{t'} + 1$  are primes with  $2^y \cdot q_1^t < 2^z \cdot q_1^{t'}$ , and we have

$$(q_1, y + z) \in \{(3, 15), (5, 14), (17, 12), (257, 8)\}.$$

For example, for  $q_1 = 3$ ,  $y = 2$ ,  $z = 13$ ,  $t = 2$  and  $t' = 5$  we obtain  $n = 3^{110} \cdot 37 \cdot 1990657$ .

Case 2.2.  $d(m)(x + y + z + 1) = 15$ . We will consider separately the two cases  $d(m) = 1$  and  $d(m) = 3$ .

- $d(m) = 1$ . Then  $x + y + z = 14$ ,  $q_1$  is a Fermat prime and by (16),  $a = 5(t + t') - 3$ . Thus, we have

$$n = q_1^{5(t+t')-3} (2^y \cdot q_1^t + 1) (2^z \cdot q_1^{t'} + 1),$$

where  $2^y \cdot q_1^t + 1$  and  $2^z \cdot q_1^{t'} + 1$  are primes with  $2^y \cdot q_1^t < 2^z \cdot q_1^{t'}$ , and we have  $(q_1, y + z) \in \{(3, 13), (5, 12), (17, 10), (257, 6)\}$ . For example, for  $x = 1$ ,  $y = 2$ ,  $z = 11$ ,  $t = 1$  and  $t' = 2$  we obtain  $n = 3^{12} \cdot 13 \cdot 18433$ .

- $d(m) = 3$ . Then  $x + y + z = 4$  and  $m = q^2$ , where  $q$  is an odd prime with  $(q_1, q) = 1$ . Thus, by (16),  $a = 5(t + t') - 3$ . In this case,  $n$  is one of the numbers:
  - $n = (2^x q + 1)^{5(t+t')-3} (2^y q_1^t q + 1) (2^z q_1^{t'} + 1)$ , where  $2^x q + 1$ ,  $2^y q_1^t q + 1$  and  $2^z q_1^{t'} + 1$  are primes. For example, for  $x = y = 1$ ,  $z = 2$ ,  $q = 3$ ,  $t = 1$  and  $t' = 2$ , we get  $n = 7^{12} \cdot 43 \cdot 197$ .
  - $n = (2^x q + 1)^{5(t+t')-3} (2^y q_1^t + 1) (2^z q_1^{t'} q + 1)$ , where  $2^x q + 1$ ,  $2^y q_1^t + 1$  and  $2^z q_1^{t'} q + 1$  are primes. For example, for  $x = z = 1$ ,  $y = 2$ ,  $q = 3$ ,  $t = 1$  and  $t' = 1$ , we get  $n = 7^7 \cdot 29 \cdot 43$ .
  - $n = (2^x + 1)^{5(t+t')-3} (2^y q_1^t q + 1) (2^z q_1^{t'} q + 1)$ , where  $2^x + 1$ ,  $2^y q_1^t q + 1$  and  $2^z q_1^{t'} q + 1$  are primes. For example, for  $x = z = 1$ ,  $y = 2$ ,  $q = 13$ ,  $t = 0$  and  $t' = 1$ , we get  $n = 3^2 \cdot 53 \cdot 79$ .
  - $n = (2^x + 1)^{5(t+t')-3} (2^y q_1^t q^2 + 1) (2^z q_1^{t'} + 1)$ , where  $2^x + 1$ ,  $2^y q_1^t q^2 + 1$  and  $2^z q_1^{t'} + 1$  are primes. For example, for  $x = y = 1$ ,  $z = 2$ ,  $q = 5$ ,  $t = 1$  and  $t' = 6$ , we get  $n = 3^{32} \cdot 151 \cdot 2917$ .
  - $n = (2^x + 1)^{5(t+t')-3} (2^y q_1^t + 1) (2^z q_1^{t'} q^2 + 1)$ , where  $2^x + 1$ ,  $2^y q_1^t + 1$  and  $2^z q_1^{t'} q^2 + 1$  are primes. For example, for  $y = z = t = 1$ ,  $x = 2$ ,  $q = 3$  and  $t' = 0$ , we get  $n = 5^2 \cdot 11 \cdot 19$ .

Case 2.3.  $d(m)(x+y+z+1) = 13$ . Then  $d(m) = 1$  and  $x+y+z = 12$ . Thus,  $m_1 = m_2 = m_3 = 1$  and  $q_1$  is a Fermat prime. By (16),  $5a = 13(t+t') - 9$ . That is,

$$n = (q_1)^{\frac{13(t+t')-9}{5}} (2^y q_1^t + 1) (2^z q_1^{t'} + 1),$$

where  $2^x + 1$ ,  $2^y q_1^t + 1$  and  $2^z q_1^{t'} + 1$  are primes with  $2^y q_1^t < 2^z q_1^{t'}$  and  $5 \mid 13(t+t') - 9$ , and we have  $(q_1, y+z) \in \{(3, 11), (5, 10), (17, 8), (257, 4)\}$ . For example, for  $x = t = 1$ ,  $y = 5$ ,  $z = 6$  and  $t' = 2$  we get  $n = 3^6 \cdot 97 \cdot 577$ .

Case 2.4.  $d(m)(x+y+z+1) = 11$ . We also have  $m = 1$  and  $x+y+z = 10$ . So,  $q_1$  is a Fermat prime and by (16),  $7a = 11(t+t') - 9$ . Thus,

$$n = q_1^{\frac{11(t+t')-9}{7}} (2^y q_1^t + 1) (2^z q_1^{t'} + 1),$$

where  $2^y q_1^t + 1$  and  $2^z q_1^{t'} + 1$  are primes with  $2^y q_1^t < 2^z q_1^{t'}$  and  $7 \mid 11(t+t') - 9$ , and we have  $(q_1, x+y) \in \{(3, 9), (5, 8), (17, 6)\}$ . For example, for  $x = t = 1$ ,  $y = 2$ ,  $z = 7$  and  $t' = 3$  we have  $n = 3^5 \cdot 13 \cdot 3457$ .

Case 2.5.  $d(m)(x+y+z+1) = 9$ . Here, we must have  $d(m) = 1$  and  $x+y+z = 8$ , from which it follows that  $m = 1$  and  $q_1$  is a Fermat prime. Thus, by (16),  $a = t+t' - 1$ . Hence,

$$n = q_1^{t+t'-1} (2^y q_1^t + 1) (2^z q_1^{t'} + 1),$$

where  $2^y q_1^t + 1$  and  $2^z q_1^{t'} + 1$  are primes with  $2^y q_1^t < 2^z q_1^{t'}$ , and we have

$$(q_1, x+y) \in \{(3, 7), (5, 6), (17, 4)\}.$$

For example, for  $x = t = 1$ ,  $y = 2$ ,  $z = 5$  and  $t' = 4$  we have  $n = 3^4 \cdot 13 \cdot 2593$ .

Case 2.6.  $d(m)(x+y+z+1) = 7$ . As above  $m = 1$  and  $x+y+z = 6$ . By (16),  $11a = 7(t+t') - 9$ . That is,

$$n = q_1^{(7(t+t')-9)/11} (2^y q_1^t + 1) (2^z q_1^{t'} + 1),$$

where  $2^y q_1^t + 1$  and  $2^z q_1^{t'} + 1$  are primes with  $2^y q_1^t < 2^z q_1^{t'}$  and  $11 \mid 7(t+t') - 9$ , and we have  $(q_1, x+y) \in \{(3, 5), (5, 4)\}$ . For example, for  $x = y = 1$ ,  $z = t' = 4$  and  $t = 2$  we get  $n = 3^3 \cdot 19 \cdot 1297$ .

Case 2.7.  $d(m)(x+y+z+1) = 5$ . We must have  $m = 1$  and  $x+y+z = 4$ . By (16), we deduce that  $13a = 5(t+t') - 9$ , and so

$$n = q_1^{\frac{5(t+t')-9}{13}} (2^y q_1^t + 1) (2^z q_1^{t'} + 1),$$

where  $2^y q_1^t + 1$  and  $2^z q_1^{t'} + 1$  are primes with  $2^y q_1^t < 2^z q_1^{t'}$  and  $13 \mid 5(t+t') - 9$ , and we have  $(q_1, x+y) \in \{(3, 3), (5, 2)\}$ . For example, for  $x = t = 1$ ,  $y = 1$ ,  $z = 2$  and  $t' = 6$  we get  $n = 3^2 \cdot 7 \cdot 2917$ .  $\square$

**Proposition 3.4.** *The only possible solutions of the form  $q_1 q_2^b q_3$ , where  $b \geq 2$  and  $3 \leq q_1 < q_2 < q_3$  are:*

- $5(2q+1)^3(8q+1)$ , where  $q, 2q+1, 8q+1$  are primes with  $(q, 2q+1) = 1$ .
- $q_1 q_2^{17t-9}(2^z q_2^t + 1)$ , where  $q_1, q_2$  are Fermat primes and  $z, t$  are positive integers with  $(q_1, q_2, z) \in \{(3, 5, 13), (3, 17, 11), (3, 257, 7), (5, 17, 10), (5, 257, 6), (17, 257, 4)\}$  and  $2^z q_2^t + 1$  is prime.
- $n = q_1 q_2^{5t-3}(2^z \cdot q_2^t + 1)$ , where  $q_1, q_2$  are Fermat primes and  $z, t$  are positive integers with  $(q_1, q_2, z) \in \{(3, 5, 11), (3, 17, 9), (3, 257, 5), (5, 17, 8), (5, 257, 4), (17, 257, 2)\}$  and  $2^z q_2^t + 1$  is prime.
- $n = q_1 q_2^{\frac{13t-9}{5}}(2^z \cdot q_2^t + 1)$ , where  $q_1, q_2$  are Fermat primes and  $z, t$  are positive integers with  $(q_1, q_2, z) \in \{(3, 5, 9), (3, 17, 7), (3, 257, 3), (5, 17, 6), (5, 257, 2)\}$  and  $2^z q_2^t + 1$  is prime.
- $n = q_1 q_2^{\frac{11t-9}{7}}(2^z \cdot q_2^t + 1)$ , where  $q_1, q_2$  are Fermat primes and  $z, t$  are positive integers with  $(q_1, q_2, z) \in \{(3, 5, 7), (3, 17, 5), (3, 257, 1), (5, 17, 4)\}$  and  $2^z q_2^t + 1$  is prime.
- $n = q_1 q_2^{t-1}(2^z \cdot q_2^t + 1)$ , where  $q_1, q_2$  are Fermat primes and  $z, t$  are positive integers with  $(q_1, q_2, z) \in \{(3, 5, 5), (3, 17, 3), (5, 17, 2)\}$  and  $2^z q_2^t + 1$  is prime.
- $n = q_1 q_2^{\frac{7t-9}{11}}(2^z \cdot q_2^t + 1)$ , where  $q_1, q_2$  are Fermat primes and  $z, t$  are positive integers with  $(q_1, q_2, z) \in \{(3, 5, 3), (3, 17, 1)\}$  and  $2^z q_2^t + 1$  is prime.
- $n = (2^x + 1)(2^y q + 1)^{5t-3}(2^z(2^y q + 1)^t q + 1)$ , where  $q, 2^x + 1, 2^y q + 1, 2^z(2^y q + 1)^t q + 1$  are primes.
- $n = (2^x + 1)(2^y q^2 + 1)^{5t-3}(2^z(2^y q^2 + 1)^t + 1)$ , where  $q, 2^x + 1, 2^y q^2 + 1$  and  $2^z(2^y q^2 + 1)^t + 1$  are primes.
- $n = 3 \cdot 5^{5t-3}(2 \cdot 5^t q^2 + 1)$ , where  $q$  and  $2 \cdot 5^t q^2$  are primes.
- $n = 3 \cdot 5^{\frac{5t-9}{13}}(2 \cdot 5^t + 1)$ , where  $2 \cdot 5^t + 1$  is prime.

*Proof.* Let  $n = q_1 q_2^b q_3$ , where  $b \geq 2$  and  $3 \leq q_1 < q_2 < q_3$ . Assume further that  $n \in \mathbb{S}$ . It follows that

$$9(2b+1) = d((q_1-1)(q_2-1)(q_3-1)q_2^{b-1}). \quad (17)$$

There are two cases:

Case 1. Assume that  $((q_3-1), q_2) = 1$ . By (17), we have

$$9 = (d((q_1-1)(q_2-1)(q_3-1)) - 18)b. \quad (18)$$

We distinguish the following subcases:

Case 1.1.  $b = 9$ . From (18),  $d((q_1 - 1)(q_2 - 1)(q_3 - 1)) = 19$ . We must have

$$(q_1 - 1)(q_2 - 1)(q_3 - 1) = 2^{18},$$

and hence  $q_1, q_2$  and  $q_3$  are Fermat numbers. This is impossible.

Case 1.2.  $b = 3$ . By (18),  $d((q_1 - 1)(q_2 - 1)(q_3 - 1)) = 21$ . If  $(q_1 - 1)(q_2 - 1)(q_3 - 1) = 2^{20}$ , then  $q_1, q_2$  and  $q_3$  are Fermat numbers, which is impossible. Thus, we must have

$$(q_1 - 1)(q_2 - 1)(q_3 - 1) = 2^6 q^2,$$

where  $q$  is odd prime with  $(q_2, q) = 1$ . By Lemma 2.1,  $8q^2 + 1$  and  $2q^2 + 1$  are composite and by Lemma 3.1,  $2q + 1$  and  $2^4 q + 1$  are not simultaneously primes. A similar argument holds for  $2^2 q + 1$  and  $2^3 q + 1$ , so we conclude that  $n$  is of the form:

$$n = 5(2q + 1)^3(8q + 1),$$

where  $2q + 1$  and  $8q + 1$  are simultaneously primes. For example, if  $q = 5$  then  $n = 5 \cdot 11^3 \cdot 41$ .

Case 2. Assume that  $((q_3 - 1), q_2) = q_2$ . We put  $q_1 - 1 = 2^x m_1$ ,  $q_2 - 1 = 2^y m_2$  and  $q_3 - 1 = 2^z q_2^t m_3$ , where  $x, y, z \geq 1$ ,  $t \geq 1$  and  $(2q_2, m_1 m_2 m_3) = 1$ . Put  $m = m_1 m_2 m_3$ . It follows from (17) that

$$(18 - d(m)(x + y + z + 1))b = (d(m)(x + y + z + 1)t - 9). \quad (19)$$

Note that  $d(m)(x + y + z + 1)$  cannot  $\geq 18$ . Thus, we distinguish the following subcases:

Case 2.1.  $d(m)(x + y + z + 1) = 17$ . Then  $d(m) = 1$ , and so  $x + y + z = 16$ . Or equivalently,  $m_1 = m_2 = m_3 = 1$ ,  $q_1$  and  $q_2$  are Fermat numbers and by (19),  $b = 17t - 9$ . Therefore,  $n = q_1 q_2^{17t-9} (2^z q_2^t + 1)$ , where  $2^z q_2^t + 1$  is prime with

$$(q_1, q_2, z) \in \{(3, 5, 13), (3, 17, 11), (3, 257, 7), (5, 17, 10), (5, 257, 6), (17, 257, 4)\}.$$

For example, for  $x = 1, y = 2, z = 13, t = 1$  we have  $n = 3 \cdot 5^8 \cdot 40961$ .

Case 2.2.  $d(m)(x + y + z + 1) = 15$ . We will consider separately the two cases  $d(m) = 1$  and  $d(m) = 3$ .

- When  $d(m) = 1$ . Then  $x + y + z = 14$ ,  $q_1$  and  $q_2$  are Fermat numbers and by (19),  $b = 5t - 3$ . Thus,  $n = q_1 q_2^{5t-3} (2^z q_2^t + 1)$ , where  $2^z q_2^t + 1$  is prime with

$$(q_1, q_2, z) \in \{(3, 5, 11), (3, 17, 9), (3, 257, 5), (5, 17, 8), (5, 257, 4), (17, 257, 2)\}.$$

For example, for  $x = 1, y = 2, z = 11$ , we have  $t = 15$  which is the first value with this property. That is,  $n = 3 \cdot 5^{72} \cdot 6250000000001$ .

- When  $d(m) = 3$ . Then  $x + y + z = 4$  and  $m = q^2$ , where  $q$  is an odd prime with  $(q_2, q) = 1$ . Thus, by (19),  $b = 5t - 3$ . Hence,
  - $n = (2^x + 1)(2^y q + 1)^{5t-3} (2^z q_2^t + 1)$ . For example, for  $x = z = 1, y = 2$  and  $t = 1$  we have  $n = 3 \cdot 13^2 \cdot 79$ .
  - $n = (2^x + 1)(2^y q^2 + 1)^{5t-3} (2^z q_2^t + 1)$ . For example, for  $x = y = 1, z = 2$  and  $t = 3$  we have  $n = 3 \cdot 19^{12} \cdot 27437$ .

- $n = (2^x + 1)(2^y + 1)^{5t-3}(2^z q_2^t q^2 + 1)$ , we must have  $x = z = 1$ ,  $y = 2$ , hence  $n = (3)(5)^{5t-3}(2 \cdot 5^t q^2 + 1)$ . For example for  $q = 7$  and  $t = 1$  we get  $n = 3 \cdot 5^2 \cdot 491$ .

**Case 2.3.**  $d(m)(x + y + z + 1) = 13$ . Then  $d(m) = 1$  and  $x + y + z = 12$ . Thus,  $m_1 = m_2 = m_3 = 1$  and  $q_1, q_2$  are Fermat primes. By (19),  $5b = 13t - 9$ . That is,

$$n = q_1 q_2^{(13t-9)/5} (2^z \cdot q_2^t + 1),$$

where  $5 \mid 13t - 9$  and  $2^z \cdot q_2^t + 1$  is prime and we have

$$(q_1, q_2, z) \in \{(3, 5, 9), (3, 17, 7), (3, 257, 5), (5, 17, 6), (5, 257, 2)\}.$$

**Case 2.4.**  $d(m)(x + y + z + 1) = 11$ . We also have  $m = 1$  and  $x + y + z = 10$ . So,  $q_1$  and  $q_2$  are Fermat numbers and by (19),  $7b = 11t - 9$ . Thus,  $n = q_1 q_2^{(11t-9)/7} (2^z \cdot q_2^t + 1)$ , where  $7 \mid 11t - 9$  and  $2^z \cdot q_2^t + 1$  is prime and

$$(q_1, q_2, z) \in \{(3, 5, 7), (3, 17, 5), (3, 257, 1), (5, 17, 4)\}.$$

For example, if  $x = 1, y = 2, z = 7$  and  $t = 333$  then  $n = 3 \cdot 5^{522} \cdot (2^7 \cdot 5^{333} + 1)$ .

**Case 2.5.**  $d(m)(x + y + z + 1) = 9$ . Here, we must have  $m = 1$  and  $x + y + z = 8$ , from which it follows that  $q_1$  and  $q_2$  are Fermat primes. Thus, by (19),  $b = t - 1$  and so

$$n = q_1 q_2^{t-1} (2^z \cdot q_2^t + 1),$$

where  $2^z \cdot q_2^t + 1$  is prime and  $(q_1, q_2, z) \in \{(3, 5, 5), (3, 17, 3), (5, 17, 2)\}$ . For example, for  $x = 1, y = 2, z = 5$ . For  $t = 3$ , we get  $n = 3 \cdot 5^2 \cdot 4001$ .

**Case 2.6.**  $d(m)(x + y + z + 1) = 7$ . Obviously  $m = 1$  and  $x + y + z = 6$ . By (16),  $11b = 7t - 9$ . That is,  $n = q_1 q_2^{(7t-9)/11} (2^z \cdot q_2^t + 1)$ , where  $11 \mid 7t - 9$  and  $2^z \cdot q_2^t + 1$  is prime and we have

$$(q_1, q_2, z) \in \{(3, 5, 3), (3, 17, 1)\}.$$

**Case 2.7.**  $d(m)(x + y + z + 1) = 5$ . We must have  $m = 1$  and  $x + y + z = 4$ . By (19), we deduce that  $13b = 5t - 9$ , and so  $n = 3 \cdot 5^{(5t-9)/13} (2 \cdot 5^t + 1)$ , where  $13 \mid 5t - 9$  and  $2 \cdot 5^t + 1$  is prime. After computation,  $t = 3699$  is the first value with this property. That is,  $n = 3 \cdot 5^{1422} \cdot (2 \cdot 5^{3699} + 1)$ .  $\square$

**Proposition 3.5.** *The only solutions of the form  $q_1 q_2 q_3^c$ , where  $c \geq 2$  and  $3 \leq q_1 < q_2 < q_3$  are:  $3 \cdot 5 \cdot 73^3, 3 \cdot 17 \cdot 19^3$  and  $5(2q + 1)(8q + 1)^3$ , where  $q, 2q + 1$  and  $8q + 1$  are primes.*

*Proof.* Let  $n = q_1 q_2 q_3^c$ , where  $c \geq 2$  and  $3 \leq q_1 < q_2 < q_3$ . Since  $n \in \mathbb{S}$ , then

$$9 = (d((q_1 - 1)(q_2 - 1)(q_3 - 1)) - 18)c. \quad (20)$$

We distinguish the following two cases:



Case 1.  $c = 9$  and  $d((q_1 - 1)(q_2 - 1)(q_3 - 1)) = 19$ . It follows that

$$(q_1 - 1)(q_2 - 1)(q_3 - 1) = 2^{18}$$

and so  $q_1, q_2, q_3$  are Fermat numbers, which is impossible.

Case 2.  $c = 3$  and  $d((q_1 - 1)(q_2 - 1)(q_3 - 1)) = 21$ . As above, if  $(q_1 - 1)(q_2 - 1)(q_3 - 1) = 2^{20}$ , then  $q_1, q_2, q_3$  are also Fermat number, which is impossible. But, if  $(q_1 - 1)(q_2 - 1)(q_3 - 1) = 2^6 q^2$ , with  $q$  is odd prime, then by applying Lemma 3.1,  $2q + 1$  and  $16q + 1$  are not simultaneously primes (the same for  $4q + 1$  and  $8q + 1$ ). Then  $n$  is one of the numbers:

- $n = 3 \cdot 5 \cdot (8q^2 + 1)^3$ , where  $8q^2 + 1$  is prime. By Lemma 2.1,  $q = 3$  is the only prime with this property, hence  $n = 3 \cdot 5 \cdot 73^3$ .
- $n = 3 \cdot 17 \cdot (2q^2 + 1)^3$ , where  $2q^2 + 1$  is prime. From Lemma 2.1,  $q = 3$  is the only prime with this property, hence  $n = 3 \cdot 17 \cdot 19^3$ .
- $n = 5 \cdot (2q + 1)(8q + 1)^3$ , where  $2q + 1$  and  $8q + 1$  are prime numbers. The first primes with these properties are  $q = 5, 11, 29, 131, 179, 239, 431, 491, \dots$ .  $\square$

### 3.3 $n$ is not square-free with $n$ even

Now, assume that  $n$  is **not square-free** with  $n$  is even. Here, we characterize all even solutions having only one power prime.

**Proposition 3.6.** *The only possible solutions of the form  $2^a pq$ , where  $a \geq 2$  and  $3 \leq p < q$  are:*

- $2^{17(x+y)-9}(2^x m_1 + 1)(2^y m_2 + 1)$ , where  $2^x m_1 + 1, 2^y m_2 + 1$  are primes with  $2^x m_1 < 2^y m_2$  and  $m_1 m_2 = r^{16}$  such that  $r$  is an odd prime.
- $2^{5(x+y)-3}(2^x m_1 + 1)(2^y m_2 + 1)$ , where  $2^x m_1 + 1, 2^y m_2 + 1$  are primes with  $2^x m_1 < 2^y m_2$  and  $m_1 m_2 = r^{14}$  or  $m_1 m_2 = r_1^4 r_2^2$  such that  $r, r_1, r_2$  are odd primes.
- $2^{\frac{13(x+y)-9}{5}}(2^x m_1 + 1)(2^y m_2 + 1)$ , where  $2^x m_1 + 1, 2^y m_2 + 1$  are primes with  $2^x m_1 < 2^y m_2$  and  $m_1 m_2 = r^{12}$  such that  $r$  is an odd prime.
- $2^{\frac{11(x+y)-9}{7}}(2^x m_1 + 1)(2^y m_2 + 1)$ , where  $2^x m_1 + 1, 2^y m_2 + 1$  are primes with  $2^x m_1 < 2^y m_2$  and  $m_1 m_2 = r^{10}$  such that  $r$  is an odd prime.
- $2^{(x+y)-1}(2^x m_1 + 1)(2^y m_2 + 1)$ , where  $2^x m_1 + 1, 2^y m_2 + 1$  are primes with  $2^x m_1 < 2^y m_2$  and  $m_1 m_2 = r^8$  or  $m_1 m_2 = r_1^2 r_2^2$  such that  $r, r_1, r_2$  are odd primes.
- $2^{\frac{7(x+y)-9}{11}}(2^x m_1 + 1)(2^y m_2 + 1)$ , where  $2^x m_1 + 1, 2^y m_2 + 1$  are primes with  $2^x m_1 < 2^y m_2$  and where  $m_1 m_2 = r^6$  such that  $r$  is an odd prime.
- $2^{\frac{5(x+y)-9}{13}}(2^x m_1 + 1)(2^y m_2 + 1)$ , where  $2^x m_1 + 1, 2^y m_2 + 1$  are primes with  $2^x m_1 < 2^y m_2$  and  $m_1 m_2 = r^4$ , such that  $r$  is an odd prime.
- $2^{\frac{(x+y)-3}{5}}(2^x m_1 + 1)(2^y m_2 + 1)$ , where  $2^x m_1 + 1, 2^y m_2 + 1$  are primes with  $2^x m_1 < 2^y m_2$  and  $m_1 m_2 = r^2$  such that  $r$  is an odd prime.
- $2^{\frac{(x+y)-1}{9}}(2^x + 1)(2^y + 1)$ , where  $2^x + 1$  and  $2^y + 1$  are primes with  $x < y$ .

*Proof.* Let  $n = 2^a pq$ , where  $a \geq 2$  and  $3 \leq p < q$ . Since  $n \in \mathbb{S}$ , then

$$9(2a + 1) = d((p - 1)(q - 1)2^{a-1}). \quad (21)$$

We put  $p - 1 = 2^x m_1$ ,  $q - 1 = 2^y m_2$ , where  $x, y \geq 1$  and  $(2, m_1 m_2) = 1$ , it follow from (21) that

$$(18 - d(m_1 m_2))a = d(m_1 m_2)(x + y) - 9. \quad (22)$$

We observe that  $d(m_1 m_2)$  is odd and cannot be  $\geq 18$ , so we distinguish the following cases:

Case 1.  $d(m_1 m_2) = 17$ . It follows that  $m_1 m_2 = r^{16}$ , where  $r \geq 3$  is prime and so, by (22),  $a = 17(x + y) - 9$ . Then  $n = 2^{17(x+y)-9}(2^x m_1 + 1)(2^y m_2 + 1)$ , where  $2^x m_1 + 1$  and  $2^y m_2 + 1$  are primes with  $2^x m_1 < 2^y m_2$ . For example, for  $m_1 = r$ ,  $m_2 = r^{15}$ ,  $y = 2$ ,  $x = 2$  and  $r = 3$ , we get  $n = 2^{59} \cdot 13 \cdot 57395629$ .

Case 2.  $d(m_1 m_2) = 15$ . It follows that  $m_1 m_2 = r^{14}$  or  $m_1 m_2 = r_1^4 r_2^2$ , where  $r_1$  and  $r_2$  are distinct odd primes and by (22),  $a = 5(x + y) - 3$ . Therefore,

$$n = 2^{5(x+y)-3}(2^x m_1 + 1)(2^y m_2 + 1),$$

where  $2^x m_1 + 1$  and  $2^y m_2 + 1$  are primes with  $2^x m_1 < 2^y m_2$ . For example, for  $m_1 = 5^2$ ,  $m_2 = 3^4$ ,  $x = 2$ ,  $y = 1$ ,  $r_1 = 5$  and  $r_2 = 3$  we have  $n = 2^{12} \cdot 101 \cdot 163$ .

Case 3.  $d(m_1 m_2) = 13$ . It follows that  $m_1 m_2 = r^{12}$ , where  $r \geq 3$  is prime and by (22),  $5a = 13(x + y) - 9$ . Thus we obtain  $n = 2^{(13(x+y)-9)/5}(2^x m_1 + 1)(2^y m_2 + 1)$ , where  $5|13(x + y) - 9$ ,  $2^x m_1 + 1$  and  $2^y m_2 + 1$  are primes with  $2^x m_1 < 2^y m_2$ . For example, for  $x = 1$ ,  $y = 2$  and  $m_1 = m_2 = 3^6$  we obtain  $n = 2^6 \cdot 1459 \cdot 2917$ .

Case 4.  $d(m_1 m_2) = 11$ . Therefore,  $m_1 m_2 = r^{10}$ , where  $r \geq 3$  is prime. From (22),  $7a = 11(x + y) - 9$ . Hence,  $n = 2^{(11(x+y)-9)/7}(2^x m_1 + 1)(2^y m_2 + 1)$ , where  $7|11(x + y) - 9$ ,  $2^x m_1 + 1$  and  $2^y m_2 + 1$  are primes with  $2^x m_1 < 2^y m_2$ . For example, for  $x = 2$ ,  $y = 16$ ,  $m_1 = 3^6$  and  $m_2 = 3^4$  we obtain  $n = 2^{27} \cdot 2917 \cdot 5308417$ .

Case 5.  $d(m_1 m_2) = 9$ . It follows that  $m_1 m_2 = r^8$  or  $m_1 m_2 = r_1^2 r_2^2$ , where  $r_1$  and  $r_2$  are distinct odd primes. By (22),  $a = (x + y) - 1$ . Hence,  $n = 2^{(x+y)-1}(2^x m_1 + 1)(2^y m_2 + 1)$ , where  $2^x m_1 + 1$  and  $2^y m_2 + 1$  are primes with  $2^x m_1 < 2^y m_2$ . For example, for  $x = 1$ ,  $y = 4$  and  $m_1 = m_2 = 3^4$  we have  $n = 2^4 \cdot 163 \cdot 1297$ . Also, for  $x = 1$ ,  $y = 2$ ,  $m_1 = 3^2$  and  $m_2 = 5^2$  we get  $n = 2^2 \cdot 19 \cdot 101$ .

Case 6.  $d(m_1 m_2) = 7$ . That is,  $m_1 m_2 = r^6$ , where  $r \geq 3$  is prime, and by (22),  $11a = 7(x + y) - 9$ . Hence,  $n = 2^{(7(x+y)-9)/11}(2^x m_1 + 1)(2^y m_2 + 1)$ , where  $11|7(x + y) - 9$ ,  $(2^x m_1 + 1)$  and  $(2^y m_2 + 1)$  are primes with  $2^x m_1 < 2^y m_2$ . For example, for  $x = 2$ ,  $y = 4$ ,  $m_1 = 3^2$  and  $m_2 = 3^4$  we get  $n = 2^3 \cdot 37 \cdot 1297$ .

Case 7.  $d(m_1 m_2) = 5$ . Then  $m_1 m_2 = r^4$ , where  $r \geq 3$  is prime and by (22),  $13a = 5(x + y) - 9$ . Hence,  $n = 2^{(5(x+y)-9)/13}(2^x m_1 + 1)(2^y m_2 + 1)$ , where  $13|5(x + y) - 9$ ,  $(2^x m_1 + 1)$  and  $(2^y m_2 + 1)$  are primes with  $2^x m_1 < 2^y m_2$ . For example, for  $x = 1$ ,  $y = 6$  and  $m_1 = m_2 = 3^2$  we have  $n = 2^2 \cdot 19 \cdot 577$ .

Case 8.  $d(m_1 m_2) = 3$ . Then  $m_1 m_2 = r^2$ , where  $r \geq 3$  is prime. By (22),  $5a = (x + y) - 3$ . Hence,  $n = 2^{((x+y)-3)/5}(2^x m_1 + 1)(2^y m_2 + 1)$ , where  $5|(x + y) - 3$ ,  $2^x m_1 + 1$  and  $2^y m_2 + 1$  are primes with  $2^x m_1 < 2^y m_2$ . For example, for  $x = 1$ ,  $y = 12$  and  $m_1 = 1$ ,  $m_2 = 11^2$  we have  $n = 2^2 \cdot 3 \cdot 495617$ .

Case 9.  $d(m_1 m_2) = 1$ . That  $m_1 = m_2 = 1$  and by (22),  $9a = (x + y) - 1$ . Hence,

$$n = 2^{(x+y-1)/9}(2^x + 1)(2^y + 1),$$

where  $9|(x + y) - 1$ ,  $(2^x + 1)$  and  $(2^y + 1)$  are Fermat primes with  $2^x + 1 < 2^y + 1$ .  $\square$

**Proposition 3.7.** *The only possible solutions of the form  $2p^b q$ , where  $b \geq 2$  and  $3 \leq p < q$  are:*

- $2F_1^9 F_4$ ,  $2F_2^3 F_4$ ,  $2 \cdot 19^3 \cdot 163$ .
- $2 \cdot 5^3(2^4 r^2 + 1)$ , where  $r$  and  $2^4 r^2 + 1$  are primes.
- $2 \cdot 17^3(2^2 r^2 + 1)$ , where  $r$  and  $2^2 r^2 + 1$  are primes.
- $2(2r + 1)^3(2^5 r + 1)$ , where  $r$ ,  $2r + 1$  and  $2^5 r + 1$  are primes.
- $2(2r + 1)^3(2r^5 + 1)$ , where  $r$ ,  $2r + 1$  and  $2r^5 + 1$  are primes.
- $2(2^x + 1)^{17t-9}(2^y(2^x + 1)^t + 1)$ , where  $2^x + 1$ ,  $2^y(2^x + 1)^t + 1$  are primes and  $x, y$  are positive integers with  $x + y = 16$ .
- $2(2^x + 1)^{5t-3}(2^y(2^x + 1)^t + 1)$ , where  $2^x + 1$ ,  $2^y(2^x + 1)^t + 1$  are primes and  $x, y$  are positive integers with  $x + y = 14$ .
- $2 \cdot 3^{5t-3}(2^3 \cdot 3^t \cdot r^2 + 1)$ , where  $r$  and  $2^3 \cdot 3^t \cdot r^2 + 1$  are primes with  $(3, r) = 1$ .
- $2 \cdot 5^{5t-3}(2^2 \cdot 5^t \cdot r^2 + 1)$ , where  $r$  and  $2^3 \cdot 5^t \cdot r^2 + 1$  are primes with  $(5, r) = 1$ .
- $2(2r + 1)^{5t-3}(2^3 r(2r + 1)^t + 1)$ , where  $r$ ,  $2r + 1$  and  $2^3 r(2r + 1)^t + 1$  are primes.
- $2(4r + 1)^{5t-3}(2^2 r(4r + 1)^t + 1)$ , where  $r$ ,  $4r + 1$  and  $2^2 r(4r + 1)^t + 1$  are primes.
- $2(8r + 1)^{5t-3}(2r(8r + 1)^t + 1)$ , where  $r$ ,  $8r + 1$  and  $2r(8r + 1)^t + 1$  are primes.
- $2(4r^2 + 1)^{5t-3}(4(4r^2 + 1)^t + 1)$ , where  $r$ ,  $4r^2 + 1$  and  $4(4r^2 + 1)^t + 1$  are primes.
- $2 \cdot 3^{5t-3}(2 \cdot 3^t \cdot r^4 + 1)$ , where  $r$  and  $2 \cdot 3^t \cdot r^4 + 1$  are primes with  $(3, r) = 1$ .
- $2(2r + 1)^{5t-3}(2r^3(2r + 1)^t + 1)$ , where  $r$ ,  $2r + 1$  and  $2r^3(2r + 1)^t + 1$  are primes.
- $2 \cdot 19^{5t-3}(2 \cdot 19^t r^2 + 1)$ , where  $r$  and  $2 \cdot 19^t r^2 + 1$  are primes with  $(19, r) = 1$ .
- $2(2r^3 + 1)^{5t-3}(2r(2r^3 + 1)^t + 1)$ , where  $r$ ,  $2r^3 + 1$  and  $2r(2r^3 + 1)^t + 1$  are primes.
- $2(2r^4 + 1)^{5t-3}(2(2r^4 + 1)^t + 1)$ , where  $r$ ,  $2r^4 + 1$  and  $2(2r^4 + 1)^t + 1$  are primes.
- $2(2^x + 1)^{\frac{13t-9}{5}}(2^y(2^x + 1)^t + 1)$ , where  $2^x + 1$ ,  $2^y(2^x + 1)^t + 1$  are primes and  $x, y, t$  are positive integers with  $x + y = 12$ .
- $2(2^x + 1)^{\frac{11t-9}{7}}(2^y(2^x + 1)^t + 1)$ , where  $2^x + 1$ ,  $2^y(2^x + 1)^t + 1$  are primes and  $x, y, t$  are positive integers with  $x + y = 10$ .
- $2(2^x + 1)^{t-1}(2^y(2^x + 1)^t + 1)$ , where  $2^x + 1$ ,  $2^y(2^x + 1)^t + 1$  are primes and  $x, y, t$  are positive integers with  $x + y = 8$ .
- $2 \cdot 3^{t-1}(2 \cdot 3^t r^2 + 1)$  where  $r$  and  $2 \cdot 3^t r^2 + 1$  are primes with  $(3, r) = 1$ .
- $2(2r + 1)^{t-1}(2r(2r + 1)^t + 1)$ , where  $r$ ,  $2r + 1$  and  $2r(2r + 1)^t + 1$  are primes.
- $2(2^x + 1)^{\frac{7t-9}{11}}(2^y(2^x + 1)^t + 1)$  where  $2^x + 1$ ,  $2^y(2^x + 1)^t + 1$  are primes and  $x, y, t$  are positive integers with  $x + y = 6$ .
- $2(2^x + 1)^{\frac{5t-9}{13}}(2^y(2^x + 1)^t + 1)$ , where  $2^x + 1$ ,  $2^y(2^x + 1)^t + 1$  are primes and  $x, y, t$  are positive integers with  $x + y = 4$ .

*Proof.* Let  $n = 2p^bq$ , where  $b \geq 2$  and  $3 \leq p < q$ . Since  $n \in \mathbb{S}$ , we have

$$9(2b+1) = d((p-1)(q-1)p^{b-1}) \quad (23)$$

Case 1. Assume that  $(q-1, p) = 1$ . It follows from (23) that  $9 = (d((p-1)(q-1)) - 18)b$ . We distinguish the following subcases:

Case 1.1.  $b = 9$  and  $d((p-1)(q-1)) = 19$ . Thus we must have  $(p-1)(q-1) = 2^{18}$  and  $p, q$  are Fermat primes. Hence,  $n = 2F_1^9F_4$ .

Case 1.2.  $b = 3$  and  $d((p-1)(q-1)) = 21$ . Here we have the following possibilities:

- $(p-1)(q-1) = 2^{20}$  and  $p, q$  are Fermat number. As above,  $n = 2F_2^3F_4$ .
- $(p-1)(q-1) = 2^6r^2$ , where  $r \geq 3$  is prime. By Lemma 2.1,  $2^2 \cdot r + 1$  and  $2^4 \cdot r^2 + 1$  cannot be simultaneously primes. Thus,  $n$  is one of the numbers:
  - $n = 2 \cdot 5^3(2^4r^2 + 1)$ , where  $2^4 \cdot r^2 + 1$  is prime. For example, for  $r = 29$  we have  $n = 2 \cdot 5^3 \cdot 13457$ .
  - $n = 2 \cdot 17^3(2^2r^2 + 1)$ , where  $2^2 \cdot r^2 + 1$  is prime. For example, for  $r = 3$  we have  $n = 2 \cdot 17^3 \cdot 37$ .
  - $n = 2(2r+1)^3(2^5r+1)$ , where  $2 \cdot r + 1$  and  $2^5 \cdot r + 1$  are both prime. For example, for  $r = 3$  we have  $n = 2 \cdot 7^3 \cdot 97$ .
- $(p-1)(q-1) = 2^2r^6$ , where  $r \geq 3$  is prime with  $(r, q) = 1$ . Thus,  $n$  is one of the numbers:
  - $n = 2(2 \cdot r + 1)^3(2 \cdot r^5 + 1)$ , where  $2 \cdot r + 1$  and  $2r^5 + 1$  are both prime. For example, for  $r = 3$  we get  $n = 2 \cdot 7^3 \cdot 487$ .
  - $n = 2(2 \cdot r^2 + 1)^3(2 \cdot r^4 + 1)$ , where  $2 \cdot r^2 + 1$  and  $2 \cdot r^4 + 1$  are both prime. By Lemma 2.1,  $r = 3$  is the only solution for this case, hence we get  $n = 2 \cdot 19^3 \cdot 163$ .

Case 2. Assume that  $(q-1, p) = p$ . We put  $p-1 = 2^xm_1$  and  $q-1 = 2^yp^tm_2$ , where  $x, y \geq 1$ ,  $t \geq 1$  and  $(2p, m_1m_2) = 1$ . Let  $m = m_1m_2$ , it follows from (23) that

$$(18 - d(m)(x+y+1))b = (d(m)(x+y+1)t - 9). \quad (24)$$

We observe that  $d(m)(x+y+1)$  is odd and cannot be  $\geq 18$ , so we have the following possibilities:

Case 2.1.  $d(m)(x+y+1) = 17$ . That is,  $m = 1$  and  $x+y = 16$ . So,  $p$  is a Fermat prime.

By (24),  $b = 17t - 9$  and therefore  $n = 2(2^x + 1)^{17t-9}(2^y(2^x + 1)^t + 1)$ , where  $(2^x + 1)$  and  $(2^y(2^x + 1)^t + 1)$  are primes. For example, for  $x = 1$ ,  $t = 4$  and  $y = 15$  we have  $n = 2 \cdot 3^{59} \cdot 2654209$ .

Case 2.2.  $d(m)(x+y+1) = 15$ . There are three possibilities:

- $d(m) = 1$  and  $x+y = 14$ . So,  $m_1 = m_2 = 1$  and  $p$  is a Fermat prime. By (24),  $b = 5t - 3$ , in which case  $n = 2(2^x + 1)^{5t-3}(2^y(2^x + 1)^t + 1)$ , where  $2^x + 1$  and  $2^y(2^x + 1)^t + 1$  are primes. For example, for  $x = 1$ ,  $y = 13$ ,  $t = 5$  we have  $n = 2 \cdot 3^{22} \cdot 1990657$ .
- $d(m) = 3$  and  $x+y = 4$ . Therefore,  $m = r^2$ , where  $r \geq 3$  is prime. From (24),  $b = 5t - 3$ . By Lemma 2.1,  $2 \cdot 73^t + 1$  is composite. Also, by Remark 2.2, the number  $2^3 \cdot 19^t + 1$  is composite. Thus,  $n$  is one of the numbers:

- $n = 2 \cdot 3^{5t-3}(2^3 \cdot 3^t \cdot r^2 + 1)$ , where  $2^3 \cdot 3^t \cdot r^2 + 1$  is prime. For example, for  $r = 7$  and  $t = 2$  we obtain  $n = 2 \cdot 3^7 \cdot 3529$ .
- $n = 2 \cdot 5^{5t-3}(2^2 \cdot 5^t \cdot r^2 + 1)$ , where  $2^2 \cdot 5^t \cdot r^2 + 1$  is prime. For example, for  $r = 7$  and  $t = 4$  we obtain  $n = 2 \cdot 5^{17} \cdot 122501$ .
- $n = 2(2r + 1)^{5t-3}(2^3 \cdot r \cdot (2r + 1)^t + 1)$ , where  $2r + 1$  and  $2^3 \cdot r \cdot (2r + 1)^t + 1$  are primes. For example, for  $r = t = 3$  we have  $n = 2 \cdot 7^{12} \cdot 8233$ .
- $n = 2(4r + 1)^{5t-3}(2^2 \cdot r \cdot (4r + 1)^t + 1)$ , where  $4r + 1$  and  $2^2 \cdot r \cdot (4r + 1)^t + 1$  are primes. For example, for  $r = 3$  and  $t = 1$  we have  $n = 2 \cdot 13^2 \cdot 157$ .
- $n = 2(8r + 1)^{5t-3}(2 \cdot r \cdot (8r + 1)^t + 1)$ , where  $8r + 1$  and  $2 \cdot r \cdot (8r + 1)^t + 1$  are primes. For example, for  $r = 5$  and  $t = 2$  we get  $n = 2 \cdot 41^7 \cdot 16811$ .
- $n = 2(4r^2 + 1)^{5t-3}(4 \cdot (4r^2 + 1)^t + 1)$ , where  $4r^2 + 1$  and  $4 \cdot r \cdot (4r^2 + 1)^t + 1$  are primes. For example, for  $r = 7$  and  $t = 6$ , we get  $n = 2 \cdot 197^{27} \cdot 233806913236517$ .
- $d(m) = 5$  and  $x = y = 1$ . Thus  $m = r^4$ , where  $r \geq 3$  is prime. It follows from (24) that  $b = 5t - 3$ . Therefore, by Lemma 2.1,  $n$  is one of the numbers:
  - $n = 2 \cdot 3^{5t-3}(2 \cdot 3^t \cdot r^4 + 1)$ , where  $2 \cdot 3^t \cdot r^4 + 1$  is prime. For example, if  $r = 5$  and  $t = 2$  then  $n = 2 \cdot 3^7 \cdot 11251$ .
  - $n = 2(2r + 1)^{5t-3}(2 \cdot r^3 \cdot (2r + 1)^t + 1)$ , where  $2r + 1$  and  $2 \cdot r^3 \cdot (2r + 1)^t + 1$  are primes. For example, for  $r = 3$  and  $t = 1$  we have  $n = 2 \cdot 7^2 \cdot 379$ .
  - $n = 2 \cdot 19^{5t-3}(2 \cdot 19^t \cdot r^2 + 1)$ , where  $2 \cdot 19^t \cdot r^2 + 1$  is prime. For example, if  $r = 3$  and  $t = 29$ , then  $n = 2 \cdot 19^{142} \cdot 218336795902605993201009018384568383223$ .
  - $n = 2(2r^3 + 1)^{5t-3}(2 \cdot r \cdot (2r^3 + 1)^t + 1)$ , where  $2r^3 + 1$  and  $2 \cdot r \cdot (2r^3 + 1)^t + 1$  are primes. For example, if  $r = 5$  and  $t = 12$ , then
 
$$n = 2 \cdot 251^{57} \cdot 625294570645574159995353780011.$$
  - $n = 2(2r^4 + 1)^{5t-3}(2 \cdot (2r^4 + 1)^t + 1)$ , where  $2r^4 + 1$  and  $2 \cdot (2r^4 + 1)^t + 1$  are primes.

Case 2.3.  $d(m)(x + y + 1) = 13$ . It follows that  $m = 1$  and  $x + y = 12$ , which gives that  $p$  is a Fermat prime. By (24),  $5b = 13t - 9$  and so  $n = 2(2^x + 1)^{(13t-9)/5}(2^y(2^x + 1)^t + 1)$ , where  $2^x + 1$  and  $2^y(2^x + 1)^t + 1$  are primes, with  $5 \mid 13t - 9$ .

Case 2.4.  $d(m)(x + y + 1) = 11$ . Obviously,  $m = 1$  and  $x + y = 12$ . So,  $p$  is a Fermat prime. By (24),  $7b = 11t - 9$  and therefore  $n = 2(2^x + 1)^{(11t-9)/7}(2^y(2^x + 1)^t + 1)$ , where  $2^x + 1$  and  $2^y(2^x + 1)^t + 1$  are both prime, with  $7 \mid 11t - 9$ .

Case 2.5.  $d(m)(x + y + 1) = 9$ . There are two possibilities to consider:

- $d(m) = 1$  and so  $x + y = 8$ . Thus,  $m_1 = m_2 = 1$  and  $q_2$  is a Fermat prime. By (24),  $b = t - 1$ , from which it follows that  $n = 2(2^x + 1)^{t-1}(2^y(2^x + 1)^t + 1)$ , where  $2^x + 1$  and  $2^y(2^x + 1)^t + 1$  are prime. For example, for  $x = 2$ ,  $y = 6$  and  $t = 14$  we have  $n = 2 \cdot 5^{13} \cdot 390625000001$ .
- $d(m) = 3$  and  $x + y = 2$ . So,  $m_1 m_2 = r^2$ , where  $r$  is odd prime and  $(p, r) = 1$ ,  $x = y = 1$  and  $b = t - 1$ . Therefore, by Lemma 2.1, it follows that  $n$  is one of the numbers:

- $n = 2 \cdot 3^{t-1}(2 \cdot 3^t \cdot r^2 + 1)$ , where  $2 \cdot 3^t \cdot r^2 + 1$  is prime. For example, for  $r = 5$  and  $t = 4$  we get  $n = 2 \cdot 3^3 \cdot 4051$ .
- $n = 2(2r + 1)^{t-1}(2 \cdot r \cdot (2r + 1)^t + 1)$ , where  $2r + 1$  and  $2 \cdot r \cdot (2r + 1)^t + 1$  are primes. For example for  $r = 3$  and  $t = 4$ , we have  $n = 2 \cdot 7^3 \cdot 14407$ .
- $n = 2 \cdot 19^{t-1}(2 \cdot 19^t + 1)$ . By Lemma 2.1, the number  $2 \cdot 19^t + 1$  is divisible by 3.

**Case 2.6.**  $d(m)(x + y + 1) = 7$ , it follows that  $d(m) = 1$ , and  $(x + y + 1) = 7$ . So,  $m_1 = m_2 = 1$ ,  $p$  is Fermat numbers,  $x + y = 6$ . From (24),  $11b = 7t - 9$  and hence

$$n = 2(2^x + 1)^{(7t-9)/11}(2^y(2^x + 1)^t + 1),$$

where  $2^x + 1$  and  $2^y(2^x + 1)^t + 1$  are primes, with  $11|7t - 9$ . For example, for  $x = 4$ ,  $y = 2$  and  $t = 6$  we have  $n = 2 \cdot 17^3 \cdot 96550277$ .

**Case 2.7.**  $d(m)(x + y + 1) = 5$ . It follows that  $m_1 = m_2 = 1$  and  $x + y = 4$ . So,  $p$  is a Fermat prime and by (24),  $13b = 5t - 9$ . Therefore,  $n = 2(2^x + 1)^{(5t-9)/13}(2^y(2^x + 1)^t + 1)$ , where  $(2^x + 1)$  and  $(2^y(2^x + 1)^t + 1)$  are both prime with  $13|5t - 9$ . For example, for  $x = 1$ ,  $y = 3$  and  $t = 7$  we have  $n = 2 \cdot 3^2 \cdot 17497$ .  $\square$

**Proposition 3.8.** *The only possible solutions of the form  $2pq^c$ , with  $c \geq 2$  and  $3 \leq p < q$  are:*

- $2F_1F_4^9, 2F_2F_4^3, 2 \cdot 3 \cdot 1459^3, 2 \cdot 19 \cdot 163^3$ .
- $2 \cdot 5 \cdot (2^4r^2 + 1)^3$ , where  $r$  and  $2^4r^2 + 1$  are odd primes.
- $2 \cdot 17 \cdot (2^2r^2 + 1)^3$ , where  $r$  and  $2^2r^2 + 1$  are odd primes.
- $2(4r + 1)(2^4r + 1)^3$ , where  $r$ ,  $4r + 1$  and  $2^4r + 1$  are odd primes.
- $2(2r + 1)(2r^5 + 1)^3$ , where  $r$ ,  $2r + 1$  and  $2r^5 + 1$  are odd primes.

*Proof.* Let  $n = 2pq^c$  where  $c \geq 2$  and  $3 \leq p < q$ . Since  $n \in \mathbb{S}$ , we have

$$9 = (d((p - 1)(q - 1)) - 18)c.$$

We distinguish the following cases:

**Case 1.**  $d((p - 1)(q - 1)) = 19$  and  $c = 9$ . It follows that  $(p - 1)(q - 1) = 2^{18}$  and so  $p, q$  are Fermat primes. Then  $n = 2F_1F_4^9$  is the only solution.

**Case 2.**  $d((p - 1)(q - 1)) = 21$  and  $c = 3$ . Here  $(p - 1)(q - 1)$  is either  $2^{20}$ ,  $2^6 \cdot r^2$  or  $2^2 \cdot r^6$  where  $r \geq 3$  is prime. We study these subcases separately.

**Case 2.1.**  $(p - 1)(q - 1) = 2^{20}$ . Then  $p, q$  are Fermat primes, in which case  $n = 2F_2F_4^3$ .

**Case 2.2.**  $(p - 1)(q - 1) = 2^6 \cdot r^2$ . Then  $n$  is one of the numbers:

- $n = 2 \cdot 5(2^4 \cdot r^2 + 1)^3$ , where  $2^4 \cdot r^2 + 1$  is also prime. For example, for  $r = 5$  we have  $n = 2 \cdot 5 \cdot 401^3$ .
- $n = 2 \cdot 17(2^2 \cdot r^2 + 1)^3$ , where  $2^2 \cdot r^2 + 1$  is prime. For example, for  $r = 3$  we get  $n = 2 \cdot 17 \cdot 37^3$ .
- $n = 2(2^2 \cdot r + 1)(2^4 \cdot r + 1)^3$ , where  $2^2 \cdot r + 1$  and  $2^4 \cdot r + 1$  are prime. For example, for  $r = 7$  we get  $n = 2 \cdot 29 \cdot 113^3$ .

**Case 2.3.**  $(p-1)(q-1) = 2^2 \cdot r^6$ . Then  $n$  is one of the numbers:

- $n = 2 \cdot 3(2 \cdot r^6 + 1)^3$ , where  $2 \cdot r^6 + 1$  is prime. By Lemma 2.1,  $r = 3$  is the only possible value, i.e.,  $n = 2 \cdot 3 \cdot 1459^3$ .
- $n = 2(2 \cdot r^2 + 1)(2 \cdot r^4 + 1)^3$ , where  $2 \cdot r^2 + 1$  and  $2 \cdot r^4 + 1$  are primes. By Lemma 2.1, we get  $r = 3$  and so  $n = 2 \cdot 19 \cdot 163^3$ .
- $n = 2(2 \cdot r + 1)(2 \cdot r^5 + 1)^3$ , where  $r$ ,  $2 \cdot r + 1$  and  $2 \cdot r^5 + 1$  are primes. The first primes  $r$  with these properties are  $r = 3, 23, 29, 53, 251, 443, 953, \dots$ .  $\square$

## 4 Are there infinitely many $n$ such that $d(n^2) = d(\varphi(n))$ ?

The crucial question that remains: Is the set  $\mathbb{S}$  infinite? The answer to this question seems difficult because we have, in the previous section, a system of polynomials in which, for a given prime  $p$ , each polynomial must takes in  $p$  a value which is also a prime number.

Recall that Dickson's Conjecture was formulated by Leonard Dickson in [5]: *Let  $s \geq 1$  and let  $f_i(x) = a_i \cdot x + b_i$  with  $a_i, b_i$  integers,  $b_i \geq 1$  for  $i = 1, 2, \dots, s$ . If there does not exist any integer  $n > 1$  dividing all the products  $f_1(k) f_2(k) \cdots f_s(k)$ , for every integer  $k$ , then there exist infinitely many natural numbers  $m$  such that all numbers  $f_1(m), f_2(m), \dots, f_s(m)$  are prime.*

As in [3], let us take the system of integer valued polynomials whose leading coefficients are positive:

$$\begin{cases} f_1(x) = x, \\ f_2(x) = 4x + 1, \\ f_3(x) = 16x + 1. \end{cases}$$

Assume further that there exists an integer  $n > 1$  which is a common divisor for the integers

$$f_1(k) f_2(k) f_3(k), \text{ for } k \in \mathbb{Z}.$$

That is,  $n$  is a common divisor for the integers  $k(4k+1)(16k+1)$ ,  $k \in \mathbb{Z}$ . Then  $n$  is a common divisor for the integers  $(n+1)(4n+5)(16n+17)$ . Since  $n \nmid (n+1)$ ,  $n$  divide  $4n+5$  or  $n$  divide  $16n+17$ . This means that  $n = 5$  or  $n = 17$ . But either  $n = 5$  or  $n = 17$  does not divide  $f_1(2) f_2(2) f_3(2) = 2 \cdot 3^3 \cdot 11$ . So there is no integer  $n > 1$  which is a common divisor for the integers  $f_1(k) f_2(k) f_3(k)$ ,  $k \in \mathbb{Z}$ . Consequently, we have the following result:

**Theorem 4.1.** *Assuming Dickson's conjecture, there exist infinitely many primes  $p$  such that  $4p+1$  and  $16p+1$  are primes.*

**Corollary 4.1.** *There exist infinitely many positive integers  $n$  such that  $n \in \mathbb{S}$ .*

*Proof.* Recall that the integer  $n = p(4p+1)(16p+1)$  with  $p$ ,  $4p+1$  and  $16p+1$  primes are in  $\mathbb{S}$ . Since, by the above theorem there exist infinitely many primes  $p$  such that  $4p+1$  and  $16p+1$  are primes. Thus, we have infinitely many integers  $n = p(4p+1)(16p+1) \in \mathbb{S}$ .  $\square$

We also use Dickson's conjecture to create families of prime numbers

$$\begin{cases} f_1(x) = a_1x + 1 \\ f_2(x) = a_2x + 1 \\ \vdots \\ f_s(x) = a_sx + 1 \end{cases}$$

where  $a_1, \dots, a_s$  are positive integers. We can easily check that the above polynomials verify Dickson's hypothesis. Indeed, suppose that there exists an integer  $n > 1$  such that  $n$  is a common divisor of all the integers  $f_1(k) f_2(k) \cdots f_s(k)$ ,  $k \in \mathbb{Z}$ . Then  $n \mid f_1(0) f_2(0) \cdots f_s(0)$ , i.e.,  $n \mid 1$  which implies that  $n = 1$ . Then there exist infinitely many positive integers  $n$  such that  $f_1(n), f_2(n), \dots, f_s(n)$  are simultaneously primes.

## 5 Miscellaneous examples

In the following, we present some examples of solutions that cannot be deduced from the previous theorems and propositions.

**Example 5.1.** *The set  $\mathbb{S}$  contains the following numbers:*

1.  $n = F_1^a \cdot F_2^b \cdot F_3$ ,  $n = F_1^a \cdot F_2 \cdot F_3^b$  and  $n = F_1 \cdot F_2^a \cdot F_3^b$ , where  $F_n$  is the  $n$ -th Fermat prime and  $(a, b) = (3, 7)$  or  $(7, 3)$ .
2.  $n = F_1^a \cdot F_2^b \cdot F_3 \cdot F_4$ ,  $n = F_1^a \cdot F_2 \cdot F_3^b \cdot F_4$ ,  $F_1^a \cdot F_2 \cdot F_3 \cdot F_4^b$ ,  $F_1 \cdot F_2^a \cdot F_3^b \cdot F_4$ ,  $F_1 \cdot F_2^a \cdot F_3 \cdot F_4^b$  and  $F_1 \cdot F_2 \cdot F_3^a \cdot F_4^b$ , where  $(a, b) = (3, 7)$ .

**Example 5.2.** *We have:*

1. Let  $p, q, r$  be distinct primes such that  $2p + 1$ ,  $4q + 1$  and  $2pqr^2 + 1$  are prime. Then  $n = 2 \cdot 17 \cdot (2p + 1)(4q + 1)(2pqr^2 + 1) \in \mathbb{S}$ .
2. Let  $q_1, q_1, \dots, q_k$  be distinct primes such that  $4q_1 + 1, \dots, 4q_k + 1$  and  $4q_1 \cdots q_k + 1$  are prime for some  $k \geq 1$ . If  $n = 2(4q_1 + 1) \cdots (4q_k + 1)(4q_1 \cdots q_k + 1) \in \mathbb{S}$ , then  $k = 3$ . For example, for  $(q_1, q_2, q_3) = (7, 13, 57)$  we get

$$n = 2(4q_1 + 1)(4q_2 + 1)(4q_3 + 1)(4q_1q_2q_3 + 1) = 2 \cdot 29 \cdot 53 \cdot 229 \cdot 20749 \in \mathbb{S}.$$

3. If  $n \geq 7$  is the product of safe primes\*, then  $n \notin \mathbb{S}$ .

Let  $q_1, q_1, \dots, q_k$  be Sophie Germain primes for some  $k \geq 1$  such that  $2q_1 \cdots q_k + 1$  is also prime.

\*A prime  $p$  is said to be a *Sophie Germain prime* if  $2p + 1$  is also a prime, in which case, this last prime is called a *safe prime*. It has been conjectured that there are infinitely many Sophie Germain primes, but this remains unproved, see [9].



**Example 5.3.** *We have:*

- If  $n = 2 \cdot (2q_1 + 1) \cdots (2q_k + 1) (2q_1 \cdots q_k + 1) \in \mathbb{S}$ , then  $k = 7$ . For example, if

$$(q_1, q_2, q_3, q_4, q_5, q_6, q_7) = (3, 5, 11, 23, 29, 41, 131),$$

then  $n = 2 \cdot (2q_1 + 1) \cdots (2q_7 + 1) (2q_1 \cdots q_7 + 1) = 253470367109666245154 \in \mathbb{S}$ .

- If  $n = 2 \cdot F_0 \cdot (2q_1 + 1) \cdots (2q_k + 1) (2q_1 \cdots q_k + 1) \in \mathbb{S}$ , then  $k = 24$ . For example, let  $q_i$  ( $1 \leq i \leq 24$ ) be the following Sophie Germain primes: 3, 5, 11, 23, 29, 41, 53, 83, 89, 113, 131, 173, 179, 191, 233, 239, 251, 281, 293, 359, 719, 1439, 1481, 3413. After computation, the number:

$$\begin{aligned} n &= 2 \cdot F_0 \cdot (2q_1 + 1) \cdots (2q_k + 1) (2q_1 \cdots q_k + 1) \\ &= 2 \cdot 3 \cdot 7 \cdot 11 \cdot 23 \cdot 47 \cdot 59 \cdot 83 \cdot 107 \cdot 167 \cdot 179 \cdot 227 \cdot 263 \\ &\quad \cdot 347 \cdot 359 \cdot 383 \cdot 467 \cdot 479 \cdot 503 \cdot 563 \cdot 587 \cdot 719 \cdot 1439 \cdot 2879 \cdot 2963 \\ &\quad \cdot 6827 \cdot 668385166547574839150402388419262454473804930401971. \end{aligned}$$

is an element of  $\mathbb{S}$ .

- If  $n = 2 \cdot F_0 \cdot F_2 \cdot (2q_1 + 1) \cdots (2q_k + 1) (2q_1 \cdots q_k + 1) \in \mathbb{S}$ , then  $k = 74$ . Let us take  $q_1, q_2, \dots, q_{73}$  be the first odd Sophie Germain primes. That is,  $(q_1, q_2, \dots, q_{73}) = (3, 5, \dots, 2945)$ . Then the result holds for  $q_{74} = 3863$ .
- If  $n = 2 \cdot F_0 \cdot F_1 \cdot F_2 \cdot (2q_1 + 1) \cdots (2q_k + 1) (2q_1 \cdots q_k + 1) \in \mathbb{S}$ , then  $k = 234$ .
- If  $n = 2 \cdot F_0 \cdot F_1 \cdot F_2 \cdot F_3 \cdot (2q_1 + 1) \cdots (2q_k + 1) (2q_1 \cdots q_k + 1) \in \mathbb{S}$ , then  $k = 712$ .
- If  $n = 2 \cdot F_0 \cdot F_1 \cdot F_2 \cdot F_3 \cdot F_4 \cdot (2q_1 + 1) \cdots (2q_k + 1) (2q_1 \cdots q_k + 1) \in \mathbb{S}$ , then  $k = 2153$ .

## 6 Conclusion

As a conclusion, the different results that we have proved give rise to diophantine equations that deserve to be studied. Here, we give some examples.

1. In Proposition 2.1, we need to find primes  $p$  such that  $4p^2 + 1$  is also prime.
2. In Theorem 2.5, we need to solve the system:

$$\begin{cases} 2 \cdot 3^t + 1 & \text{is prime} \\ ab + 2a + 2b + 1 = 3bt \end{cases},$$

where  $a, b, t$  are non-negative integers.

3. In Proposition 2.6, we need to solve the system:

$$\begin{cases} p & \text{is prime} \\ 2^t p^4 + 1 & \text{is prime} \\ t & \text{positive integer} \end{cases}.$$

4. In Proposition 2.8, we need to solve the system:

$$\begin{cases} p & \text{is prime} \\ 2^s p^2 + 1 & \text{is prime} \\ ab + 2a + 2b + 1 = 3bs \end{cases},$$

where  $a, b, s$  are positive integers.

## Acknowledgements

The authors thank the referees for comments and very helpful suggestions. This research work is supported by the General Direction of Scientific Research and Technological Development (DGRSDT)-Algeria.

## References

- [1] Bellaouar, D. (2016). Notes on certain arithmetic inequalities involving two consecutive primes. *Malaysian Journal of Mathematical Sciences*, 10(3), 253–268.
- [2] Bellaouar, D., Boudaoud, A., & Özer, Ö. (2019). On a sequence formed by iterating a divisor operator. *Czechoslovak Mathematical Journal*, 69(144), 1177–1196.
- [3] Boudaoud, A. (2006). La conjecture de Dickson et classes particulière d'entiers. *Annales mathématiques Blaise Pascal*, 13, 103–109.
- [4] De Koninck, J. M., & Mercier, A. (2007). *1001 Problems in Classical Number Theory*. Providence, RI: American Mathematical Society.
- [5] Dickson, L. E. (1904). A new extension of Dirichlet's theorem on prime numbers. *Messenger of Mathematics*, 33, 155–161.
- [6] Guy, R. K. (1994). *Unsolved Problems in Number Theory* (2nd ed.). New York: Springer-Verlag.
- [7] Heath-Brown, D. R. (1984). The divisor function at consecutive integers. *Mathematika*, 31, 141–149.
- [8] Iannucci, D. E. (2017). On the equation  $\sigma(n) = n + \varphi(n)$ . *Journal of Integer Sequences*, 20, Article 17.6.2.
- [9] Liu, F. (2011). On the Sophie Germain prime conjecture. *WSEAS Transactions on Mathematics*, 10, 421–430.
- [10] Luca, F. (2000). Equations involving arithmetic functions of factorials. *Divulgaciones Matemáticas*, 8, 15–23.
- [11] Nicol, C. A. (1966). Some diophantine equations involving arithmetic functions. *Journal of Mathematical Analysis and Applications*, 15, 154–161.
- [12] Sándor, J. (2002). *Geometric Theorems, Diophantine Equations, and Arithmetic Functions*. Rehoboth: American Research Press.