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Nonclassical Study on certain Diophantine Inequalities involving Multiplicative Arithmetic Functions

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ABSTRACT

This paper, for the most part, is in the framework of Internal Set Theory (IST), where any real number must be infinitesimal, appreciable or unlimited; theses numbers are called standard or nonstandard. In particular, any positive integer must be standard (limited) or nonstandard (unlimited). In the first part, we estimate for an unlimited positive integer n and to an infinitesimal near, the values of some arithmetic functions of the form $\dfrac{f(n)}{g(n)}$, where f and g are constructed using multiplicative functions. Further, in the classical mathematics, several Diophantine inequalities involving certain multiplicative arithmetic functions are studied.

Keywords: Diophantine Inequalities, Multiplicative Functions, Prime Numbers, Internal Set Theory.

1. Introduction

This work is placed in the framework of Internal Set Theory introduced by Robinson (1974) and developed by Nelson (1977), Van den Berg (1992), Diener and Diener (1995)) and many others. Firstly, we summarize a few auxiliary results about Internal Set Theory that we need, see Bellaouar et al. (2019).

Historically, Leibniz, Euler and Cauchy are among the first who began the use of infinitely small quantities. In order to use better this notion, A. Robinson (1961) proposed another approach, namely, the nonstandard analysis. In 1977, E. Nelson provided another presentation of the nonstandard analysis, called IST (Internal Set Theory), based on ZFC to which is added a new unary predicate called "standard". The use of this predicate is governed by the following three axioms: Idealization principle, Standardization principle and Transfer principle. For details, see Nelson (1977),Robinson (1974),Kanovei and Reeken (2013).

Recall that any real number that can be characterized in unique classical way is necessarily standard. Thus, $0, 1, ..., 10^{1000}, ...$ are standard. But not all integers are standard. A real number ω is called unlimited, or infinitely large if its absolute value $|\omega|$ is larger than any standard integer n. So a nonstandard integer ω is also an unlimited real number; $\omega - \sqrt{2}$ is an example of an unlimited real number that is not an integer. A real number ε is called infinitesimal, or infinitely small, if its absolute value $|\epsilon|$ is smaller than $\frac{1}{n}$ for any standard n. Of course, 0 is infinitesimal but (fortunately) it is not the only one: $\epsilon = \frac{1}{\omega}$ is infinitesimal, provided ω is unlimited. A real number r is called limited if it is not unlimited and appreciable if it is neither unlimited nor infinitesimal. Finally, two real numbers x and y are equivalent or infinitely close (written $x \simeq y$) if their difference $x - y$ is infinitesimal. For details, see (Diener and Diener, 1995, p. 2-4).

In mathematics, we describe as internal a formula which is expressible in the classical language (ZFC) and as external, a formula of the nonstandard language (IST) which involves the new predicate "standard" or one of its derivatives such as " infinitesimal " or " limited " Kanovei and Reeken (2013). For example, the formula $[2x < 2\epsilon + 4 \Rightarrow x < \epsilon + 2]$ is internal whereas the formula $[x \simeq +\infty \Rightarrow \sqrt{x} + 1 \simeq +\infty]$ is external.

Definition 1.1 (Kanovei and Reeken (2013))**.** We call internal any set defined by means of an internal formula and we call external any subset of an internal set defined by means of an external formula, which is not (reduced to) an internal set.

Based on the above facts, we qualify mathematical objects as internal or external. For example, we say that a function is internal (resp. external) if its graph is internal (resp. external), and so on.

Definition 1.2 (see, (Diener and Diener, 1995, p. 20))**.** Let X be a standard set, and let $(A_x)_{x\in X}$ be an internal family of sets.

- A union of the form $G = \bigcup_{stx\in X} A_x$ is called a *pregalaxy*; if it is external G is called a *galaxy*.

Example 1.1 (for details, see Bellaouar and Boudaoud (2015))**.** We have

- The set of limited positive integers $\mathbb{N}^{\sigma}$ is a galaxy.

The following principle is important for the proof of Theorem 3.3.

Theorem 1.1 (Cauchy's principle Diener and Diener (1995))**.** *No external set is internal.*

The main purpose of this paper is using the nonstandard analysis and some notions from elementary number theory to study the following near Diophantine equation:

$$F(n) \simeq l, \tag{1}$$

where F is an arithmetic function and $l \in \mathbb{N}$ is a parameter. The equation of the form (1) is called an external equation. This name comes from the fact that $F(n)$ is equivalent to l, i.e., $F(n)$ is very near to l within the meaning of the theory of IST, that is, an external formulation. Therefore, (1) is more general than the internal equation $F(n) = l$. More precisely, we research for integers n making $F(n)$ equivalent to l. On the other hand, we present classical and nonclassical results concerning some inequalities involving certain multiplicative functions.

The paper is structured as follows: There are three subsections. In Subsection 3.1, some near Diophantine equations are studied. That is, we estimate for an unlimited positive integer n the values of some arithmetic functions of the form $\frac{f(n)}{g(n)}$, where f and g are formed by the sum and the product of well-known multiplicative functions. Moreover, we prove that if $\frac{\sigma(n)}{n}$ is unlimited, then the number of distinct prime factors of n cannot be limited, where $\sigma(n)$ is the sum of all of the positive divisors of n. In Subsection 3.2, we study several different types of Diophantine inequalities, some of which deal with the comparison of two arithmetic functions, where the first member is formed by the product and the sum of certain multiplicative arithmetic functions and the second is an integer-valued polynomial whose leading coefficient is positive. Finally, in Subsection 3.3, we give some properties of the quantity $\dfrac{\tau^s(ab)}{\tau(a^s)\tau(b^s)}$ in the case when a, b are unlimited and $s \geq 2$, where $\tau(n)$ is the number of positive divisors of n and $\tau^s(n) = (\tau(n))^s$.

2. Basic tools

In this paper, we continue the research from Bellaouar (2016). Firstly, let

$$n = q_1^{\alpha_1} q_2^{\alpha_2} \dots q_r^{\alpha_r},$$

where $r, \alpha_1, \alpha_2, ..., \alpha_r$ are natural numbers and $q_1, q_2, ..., q_r$ are different primes. We need to use the following facts, see De Koninck and Mercier (2004):

1. Recall that $\varphi(n)$ is, by definition, the number of congruence classes in the set $(\frac{\mathbb{Z}}{n\mathbb{Z}})^*$ of invertible congruence classes modulo n . Let ψ be the Dedekind function, and let $s \geq 1$. We have
$$\varphi_s(n) = n^s \prod_{i=1}^{r} \left(1 - \frac{1}{q_i^s}\right) \text{ and } \psi_s(n) = n^s \prod_{i=1}^{r} \left(1 + \frac{1}{q_i^s}\right),$$
where $\varphi_s(1) = \psi_s(1) = 1$. Note that for $s = 1$, $\varphi_1 = \varphi$ and $\psi_1 = \psi$.

2. Let $\sigma(n)$ denote the arithmetic function as we stated in the introduction. That is,
$$\sigma(n) = \prod_{i=1}^{r} \frac{q_i^{\alpha_i+1} - 1}{q_i - 1}.$$

3. Let $\tau(n)$ denote the number of positive divisors of n. We have
$$\tau(n) = \prod_{i=1}^{r} (\alpha_i + 1).$$

4. Let $\omega(n)$ denote the number of distinct prime factors of n. That is, $\omega(n) = r$. As used in Bellaouar (2016), we let

$$W_k = \{n \in \mathbb{N} \ / \ \omega(n) \geq k\}.$$

Finally, we also need the following theorems.

Theorem 2.1 (Dickson's conjecture, Dickson (1904)). *Let $s \geq 1$, $f_i(x) = b_i x + a_i$ with a_i, b_i integers, $b_i \geq 1$ (for $i = 1, 2, ..., s$). Assume that the following condition is satisfied:*
$(*)$ " There does not exist any integer $n > 1$ dividing all the products

$$f_1(k) . f_2(k) ... f_s(k),$$

for every integer k ". Then, $f_1(m), f_2(m), \ldots, f_s(m)$ are simultaneously prime for infinitely many values of m.

Theorem 2.2 (for details, see Boudaoud (2006)). *Assuming the Dickson Conjecture, for each couple of integers $q > 0$ and $k > 0$, there exists an infinite subset $L_{q,k} \subset \mathbb{N}$ such that for every $n \in L_{q,k}$ and for every s with $0 < |s| \leq q$, we have*

$$n + s = |s| \, t_1 t_2 ... t_k,$$

where $t_1 < t_2 < ... < t_k$ are prime numbers. In addition, for each couple of integers $q > 0$ and $k > 0$, there exists an infinite subset $M_{q,k} \subset \mathbb{N}$ such that for every $n \in M_{q,k}$ and for every $s \in [-q, q]$, we have

$$n + s = l t_1 t_2 ... t_k,$$

where $t_1 < t_2 < ... < t_k$ are also prime numbers and $l \in [1, 2q + 1]$.

Theorem 2.3 (Bertrand's Theorem,(Wells, 2005, p. 20)). *If n is an integer greater than 2, then there is at least one prime between n and $2n - 1$.*

Theorem 2.4 (Bone's inequality, (Wells, 2005, p. 21)). *If p_n is the n-th prime number, then*

$$p_{n+1}^2 < p_1 p_2 ... p_n$$

provided $n \geq 4$.

3. Main Results

Throughout the paper $p, (p_i)_{1 \leq i \leq r}, (q_i)_{1 \leq i \leq r}$, with or without subscripts, always denote primes; $a, \alpha, m, n, r, s, ...$ denote positive integers. Limited numbers and infinitesimal numbers are denoted by £ and ϕ, respectively.

Our main results are divided into three subsections.

3.1 On certain external equations

Let f and g be two functions expressed by the product and the sum of certain multiplicative arithmetic functions. In this section, we prove the existence of an infinity of values of n such that $\frac{f(n)}{g(n)}$ is equivalent to an appreciable rational number. We start by the following theorem.

Theorem 3.1. *Assuming the Dickson Conjecture as in Theorem 2.2, if k, q are limited then for every unlimited $n \in L_{q,k}$ and for every positive integer $s \leq q$, we have*

$$\frac{\sigma(n+s)\varphi(n+s)}{(n+s)^2} \simeq \frac{\sigma(l)\varphi(l)}{l^2} \tag{2}$$

for some limited positive integer l.

Proof. From Theorem 2.2, we assume that $n + s = st_1t_2...t_k$, where $0 < t_1 < t_2 < ... < t_k$ are prime numbers. There are two cases to consider.

Case 1. Suppose that $t_i \simeq +\infty$ for $i = 1, 2, ..., k$. Since $(s, t_1t_2...t_k) = 1$, it follows that

$$\begin{aligned}
\frac{\sigma(n+s)\varphi(n+s)}{(n+s)^2} &= \frac{\sigma(st_1t_2...t_k)\varphi(st_1t_2...t_k)}{(st_1t_2...t_k)^2} \\
&= \frac{\sigma(s)\sigma(t_1t_2...t_k)\varphi(s)\varphi(t_1t_2...t_k)}{s^2(t_1t_2...t_k)^2} \\
&= \frac{\sigma(s)\varphi(s)}{s^2}\left(\frac{\sigma(t_1t_2...t_k)\varphi(t_1t_2...t_k)}{(t_1t_2...t_k)^2}\right) \qquad (3) \\
&= \frac{\sigma(s)\varphi(s)}{s^2}\left(\frac{(t_1+1)(t_2+1)...(t_k+1)(t_1-1)(t_2-1)...(t_k-1)}{(t_1t_2...t_k)^2}\right) \\
&= \frac{\sigma(s)\varphi(s)}{s^2}\left(\frac{(t_1^2-1)(t_2^2-1)...(t_k^2-1)}{(t_1t_2...t_k)^2}\right) \\
&= \frac{\sigma(s)\varphi(s)}{s^2}\left(\frac{t_1^2t_2^2...t_k^2\prod_{i=1}^{k}(1-\frac{1}{t_i^2})}{(t_1t_2...t_k)^2}\right) \\
&= \frac{\sigma(s)\varphi(s)}{s^2}\prod_{i=1}^{k}(1-\frac{1}{t_i^2}). \qquad (4)
\end{aligned}$$

Since k, s are limited, then

$$\begin{aligned}\frac{\sigma(n+s)\varphi(n+s)}{(n+s)^2} &= \frac{\sigma(s)\varphi(s)}{s^2}(1-\phi) \\ &= \frac{\sigma(s)\varphi(s)}{s^2} - \phi \\ &\simeq \frac{\sigma(s)\varphi(s)}{s^2},\end{aligned} \tag{5}$$

which proves (2) with $l = s$.

Case 2. Suppose that there exists a positive integer i_0 such that t_i is limited for $i = 1, 2, ..., i_0$, and t_j is unlimited for $j = i_0 + 1, ..., k$. From Theorem 2.2 once again, we assume that $n + s = st_1t_2...t_{i_0}t_{i_0+1}...t_k$. Since $(st_1t_2...t_{i_0}, t_{i_0+1}...t_k) = 1$, then

$$\begin{aligned}\frac{\sigma(n+s)\varphi(n+s)}{(n+s)^2} &= \frac{\sigma(st_1t_2..t_{i_0}..t_k)\varphi(st_1t_2..t_{i_0}..t_k)}{(st_1t_2...t_{i_0}t_{i_0+1}...t_k)^2} \\ &= \frac{\sigma(st_1t_2...t_{i_0})\varphi(st_1t_2...t_{i_0})\sigma\left(t_{i_{0+1}}...t_k\right)\varphi(t_{i_{0+1}}...t_k)}{\left(st_1t_2...t_{i_0}\right)^2\left(t_{i_0+1}...t_k\right)^2} \\ &= \frac{\sigma\left(s'\right)\varphi(s')}{\left(s'\right)^2}\left(\frac{\sigma(t_{i_{0+1}}...t_k)\varphi(t_{i_{0+1}}...t_k)}{(t_{i_{0+1}}...t_k)^2}\right),\end{aligned}$$

where $s' = st_1t_2...t_{i_0}$ is limited. Furthermore, as in (3), (4) and (5), we have

$$\frac{\sigma(t_{i_{0+1}}...t_k)\varphi((t_{i_{0+1}}...t_k)}{(t_{i_{0+1}}...t_k)^2} = 1 - \phi.$$

Thus,

$$\begin{aligned}\frac{\sigma(n+s)\varphi(n+s)}{(n+s)^2} &= \frac{\sigma\left(s'\right)\varphi(s')}{\left(s'\right)^2}(1-\phi) \\ &= \frac{\sigma\left(s'\right)\varphi(s')}{\left(s'\right)^2} - \phi \\ &\simeq \frac{\sigma\left(s'\right)\varphi(s')}{\left(s'\right)^2}.\end{aligned}$$

That is, $l = s'$. This completes the proof of Theorem 3.1. □

Now, we let W denote the set

$$W = \{n \in \mathbb{N};\ \omega\left(n\right) \text{ is limited and } p \simeq +\infty \text{ for any } p \mid n\}.$$

In the following proposition we present an arithmetic function of the form $\frac{f}{g}$ such that

$$\frac{f(n)}{g(n)} \simeq 1,$$

for any $n \in W$.

Proposition 3.1. *For each natural number $n \in W$, we have*

$$\frac{\sigma(n)\varphi(n)}{n^2} = 1 - \phi,$$

where ϕ denotes an infinitesimal real number.

Proof. Let $n = q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_r^{\alpha_r} \in W$ with $\alpha_i \geq 1$, for $i = 1, 2, \ldots, r$. Since r is limited, it follows that

$$\begin{aligned}
\frac{\sigma(n)\varphi(n)}{n^2} &= \frac{\sigma\left(q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_r^{\alpha_r}\right)\varphi\left(q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_r^{\alpha_r}\right)}{q_1^{2\alpha_1} q_2^{2\alpha_2} \ldots q_r^{2\alpha_r}} \\
&= \frac{\prod_{i=1}^{r} \frac{q_i^{\alpha_i+1} - 1}{q_i - 1} \prod_{i=1}^{r} \left(1 - \frac{1}{q_i}\right)}{q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_r^{\alpha_r}} \\
&= \prod_{i=1}^{r} \left(\frac{1 - \frac{1}{q_i^{\alpha_i+1}}}{1 - \frac{1}{q_i}} \right) \prod_{i=1}^{r} \left(1 - \frac{1}{q_i}\right) \\
&= \prod_{i=1}^{r} (1 - \phi_i) \prod_{i=1}^{r} (1 - \phi_i) \\
&= 1 - \phi.
\end{aligned}$$

This completes the proof. □

Let $s \geq 1$ be a limited positive integer. In the following corollary we present an arithmetic function of the form $\frac{f}{g}$ such that

$$\frac{f(sn)}{g(sn)} \simeq \frac{f(s)}{g(s)},$$

for any $n \in W$.

Corollary 3.1. *Let s be a limited positive integer. For every $n \in W$, we have*

$$\frac{\sigma(sn)\varphi(sn)}{(sn)^2} = \frac{\sigma(s)\varphi(s)}{s^2} - \phi.$$

Proof. Let $n \in W$ and let $s \geq 1$ be a limited positive integer. Since $(s, n) = 1$, it follows from Proposition 3.1 that

$$\begin{aligned} \frac{\sigma(sn)\varphi(sn)}{(sn)^2} &= \frac{\sigma(s)\varphi(s)}{s^2}\left(\frac{\sigma(n)\varphi(n)}{n^2}\right) \\ &= \frac{\sigma(s)\varphi(s)}{s^2}(1-\phi) \\ &= \frac{\sigma(s)\varphi(s)}{s^2} - \phi. \end{aligned}$$

This completes the proof. □

Proposition 3.2. *Let s be a limited positive integer. For every $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_r^{\alpha_r} \in W$ with $\alpha_i \in \{1,2\}$ for all $1 \leq i \leq r$,*

$$\frac{\varphi(sn) + \sigma(sn)}{\gamma^2(sn)} = \pounds.$$

If $\alpha_i > 2$ for all $1 \leq i \leq r$, then

$$\frac{\varphi(sn) + \sigma(sn)}{\gamma^2(sn)} \simeq +\infty.$$

Proof. Since $(s, n) = 1$, then

$$\begin{aligned} \frac{\varphi(sn) + \sigma(sn)}{\gamma^2(sn)} &= \frac{\varphi(s)\varphi(n) + \sigma(s)\sigma(n)}{\gamma^2(s)\gamma^2(n)} \\ &= \frac{\varphi(s)}{\gamma^2(s)}\left(\frac{\varphi(n)}{\gamma^2(n)}\right) + \frac{\sigma(s)}{\gamma^2(s)}\left(\frac{\sigma(n)}{\gamma^2(n)}\right) \\ &= a.\frac{\varphi(n)}{\gamma^2(n)} + b.\frac{\sigma(n)}{\gamma^2(n)}, \end{aligned}$$

where $a = \dfrac{\varphi(s)}{\gamma^2(s)}$ and $b = \dfrac{\sigma(s)}{\gamma^2(s)}$ are limited rational numbers. Moreover, since $n \in W$, then $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_r^{\alpha_r}$ with $p_1, p_2, ..., p_r$ are distinct unlimited primes and $\alpha_i \geq 1$, for $i = 1, 2, ..., r$. Thus,

$$\begin{aligned}
\frac{\varphi(sn)+\sigma(sn)}{\gamma^2(sn)} &= a.\frac{p_1^{\alpha_1}p_2^{\alpha_2}...p_r^{\alpha_r}\prod_{i=1}^{r}\left(1-\frac{1}{p_i}\right)}{p_1^2p_2^2...p_r^2} + b.\frac{p_1^{\alpha_1}p_2^{\alpha_2}...p_r^{\alpha_r}\prod_{i=1}^{r}\left(\frac{1-\frac{1}{p_i^{\alpha_i+1}}}{1-\frac{1}{p_i}}\right)}{p_1^2p_2^2...p_r^2} \\
&= a\prod_{i=1}^{r}p_i^{\alpha_i-2}\prod_{i=1}^{r}\left(1-\frac{1}{p_i}\right) + b\prod_{i=1}^{r}p_i^{\alpha_i-2}\prod_{i=1}^{r}\left(\frac{1-\frac{1}{p_i^{\alpha_i+1}}}{1-\frac{1}{p_i}}\right).
\end{aligned}$$

Now if $\alpha_i \in \{1,2\}$ for all $1 \leq i \leq r$, then

$$\frac{\varphi(sn)+\sigma(sn)}{\gamma^2(sn)} \leq a\prod_{i=1}^{r}\left(1-\frac{1}{p_i}\right) + b\prod_{i=1}^{r}\left(\frac{1-\frac{1}{p_i^{\alpha_i+1}}}{1-\frac{1}{p_i}}\right). \tag{6}$$

Since r is limited and $\frac{1}{p_i} \simeq 0$ for $i = 1, 2, ..., r$, then the right hand side of (6) is limited. Hence,

$$\frac{\varphi(sn)+\sigma(sn)}{\gamma^2(sn)} = \pounds.$$

The case $\alpha_i > 2$, for all $1 \leq i \leq r$, implies

$$\begin{aligned}
\frac{\varphi(sn)+\sigma(sn)}{\gamma^2(sn)} &= a\prod_{i=1}^{r}p_i^{\alpha_i-2}\prod_{i=1}^{r}\left(1-\frac{1}{p_i}\right) + b\prod_{i=1}^{r}p_i^{\alpha_i-2}\prod_{i=1}^{r}\left(\frac{1-\frac{1}{p_i^{\alpha_i+1}}}{1-\frac{1}{p_i}}\right) \\
&\simeq +\infty,
\end{aligned}$$

since

$$\prod_{i=1}^{r}\left(1-\frac{1}{p_i}\right) \simeq \prod_{i=1}^{r}\left(\frac{1-\frac{1}{p_i^{\alpha_i+1}}}{1-\frac{1}{p_i}}\right) \simeq 1. \tag{7}$$

This completes the proof. □

Proposition 3.3. *Let s be a limited positive integer. For every $n \in W$, we have*

$$\frac{\varphi^2(sn)\sigma^2(sn)}{(sn)^2} \simeq +\infty.$$

Proof. Let $n \in W$. Since $(s,n)=1$, it follows that

$$\begin{aligned}\frac{\sigma^2(sn)\varphi^2(sn)}{(sn)^2} &= \frac{\sigma^2(s)\varphi^2(s)}{s^2}\left(\frac{\varphi^2(n)\sigma^2(n)}{n^2}\right)\\ &= \frac{\sigma^2(s)\varphi^2(s)}{s^2}\left(\frac{\varphi^2(p_1^{\alpha_1}p_2^{\alpha_2}...p_r^{\alpha_r})\sigma^2(p_1^{\alpha_1}p_2^{\alpha_2}...p_r^{\alpha_r})}{(p_1^{\alpha_1}p_2^{\alpha_2}...p_r^{\alpha_r})^2}\right)\\ &= \frac{\sigma^2(s)\varphi^2(s)}{s^2}p_1^{2\alpha_1}p_2^{2\alpha_2}...p_r^{2\alpha_r}\prod_{i=1}^{r}(1-\frac{1}{p_i})^2\prod_{i=1}^{r}\left(\frac{1-\frac{1}{p_i^{\alpha_i+1}}}{1-\frac{1}{p_i}}\right)^2\\ &\simeq +\infty,\end{aligned}$$

which is valid by (7). This completes the proof. □

Next, we prove that if the report $\frac{\sigma(n)}{n}$ is unlimited, then $\omega(n)$ cannot be limited.

Proposition 3.4. *Let a be an unlimited real number and let n be an arbitrary solution of the inequality $\sigma(n) \geq an$. Then $\omega(n)$ is unlimited.*

Proof. By contradiction, suppose that n has a limited number of distinct prime factors $q_1 < q_2 < ... < q_s$, where s is limited. It follows from (De Koninck and Mercier, 2004, Problem 516) that

$$+\infty \simeq a \leq \frac{\sigma(n)}{n} < \prod_{i=1}^{s}\frac{q_i}{q_i-1} \simeq £,$$

where $£ \in \mathbb{Q}$ is limited. This is a contradiction. □

Remark 3.1. Let $\mathbb{P}$ be the sequence of primes in ascending order. Let n be an arbitrary solution of the inequality $\sigma(n) \geq an$, where a is unlimited. Then

$$a \leq \frac{\sigma(n)}{n} = \sum_{d|n}\frac{1}{d} = \sum_{p|n}\frac{1}{p} + \sum_{\substack{d|n\\ d\notin\mathbb{P}}}\frac{1}{d} \simeq +\infty.$$

It follows that at least one of the numbers $\sum_{p|n}\frac{1}{p}$ and $\sum_{\substack{d|n\\ d\notin\mathbb{P}}}\frac{1}{d}$ is unlimited. Thus, in each of the cases, n has an unlimited number of distinct prime factors.

Remark 3.2. We also prove Proposition 3.4 as follows: For every positive integer n, as stated in (De Koninck and Mercier, 2004, Problem 618), we have

$$\frac{\sigma(n)}{n} < \begin{cases} \left(\frac{3}{2}\right)^{\omega(n)} & ; \text{ if } n \text{ is odd,} \\ 2.\left(\frac{3}{2}\right)^{\omega(n)-1} & ; \text{ if } n \text{ is even.} \end{cases}$$

Then if $\dfrac{\sigma(n)}{n}$ is unlimited, so does $\omega(n)$.

3.2 On certain multiplicative functions which are bounded by a polynomial with integer coefficients.

In the present section we study several Diophantine inequalities, some of which are formed by the product and the sum of certain multiplicative arithmetic functions on the left side, and by integer-valued polynomials whose leading coefficients are positive on the right side.

Theorem 3.2. *Let s and n be positive integers with $n \geq 2$. Then,*

$$\varphi_s(n)^{\tau(n)}\psi_s(n)\sigma(n) \geq n^{3s+1} + n^{3s} - n^{2s+1} - n^{2s} - n^{s+1} - n^s + n + 1. \quad (8)$$

Proof. Firstly, for $s = 1$, we note that

$$\varphi(n)^{\tau(n)}\psi(n)\sigma(n) - \left(n^4 - 2n^2 + 1\right) = \begin{cases} 0, \text{ for } n = p, \\ 111, \text{ for } n = 4. \end{cases}$$

Next, it suffices to show that if $\varphi(n)^{\tau(n)}\psi(n)\sigma(n) \geq n^4 - 2n^2 + 1$ for some $n \geq 3$, then it is also true for pn with $p \geq 2$ is prime. Indeed, for each such integer n and for any prime $p \geq 2$ we distinguish two cases.

1. When p does not divide n. Since $\varphi(n) \geq 2$ and $\tau(n) \geq 2$, it follows that

$$\begin{aligned}
\varphi(pn)^{\tau(pn)} \psi(pn) \sigma(pn) &= (p-1)^{2\tau(n)} \varphi(n)^{2\tau(n)} (1+p)^2 \psi(n) \sigma(n) \\
&= (p-1)^{2\tau(n)} \varphi(n)^{\tau(n)} (1+p)^2 \left[\varphi(n)^{\tau(n)} \psi(n) \sigma(n)\right] \\
&\geq (p-1)^{2\tau(n)} \varphi(n)^{\tau(n)} (1+p)^2 \left(n^4 - 2n^2 + 1\right) \\
&\geq (p-1)^4 2^2 (1+p)^2 \left(n^4 - 2n^2 + 1\right) \\
&= n^4 \left(4p^6 - 8p^5 - 4p^4 + 16p^3 - 4p^2 - 8p + 4\right) + \\
&\quad n^2 \left(-8p^6 + 16p^5 + 8p^4 - 32p^3 + 8p^2 + 16p - 8\right) + \\
&\quad 4p^6 - 8p^5 - 4p^4 + 16p^3 - 4p^2 - 8p + 4.
\end{aligned}$$

Thus,

$$\begin{aligned}
& \varphi(pn)^{\tau(pn)} \psi(pn) \sigma(pn) - \left((pn)^4 - 2(pn)^2 + 1\right) \qquad (9) \\
\geq\ & n^4 \left(4p^6 - 8p^5 - 5p^4 + 16p^3 - 4p^2 - 8p + 4\right) + \\
& n^2 \left(-8p^6 + 16p^5 + 8p^4 - 32p^3 + 10p^2 + 16p - 8\right) + \\
& 4p^6 - 8p^5 - 4p^4 + 16p^3 - 4p^2 - 8p + 3.
\end{aligned}$$

Using the graph of the function $x \mapsto 4x^6 - 8x^5 - 4x^4 + 16x^3 - 4x^2 - 8x + 3$, we have

$$4p^6 - 8p^5 - 4p^4 + 16p^3 - 4p^2 - 8p + 3 > 0. \qquad (10)$$

In fact, we see that

$$4p^6 - 8p^5 - 4p^4 + 16p^3 - 4p^2 - 8p = 4p^4 \left(p^2 - 2p - 1\right) + 4p\left(4p^2 - p - 2\right),$$

where $p^2 - 2p - 1 > 0$ holds for every $p \geq 3$ and $4p^2 - p - 2 > 0$ holds for every $p \geq 2$. This proves (10) for every $p \geq 2$, since its value at $p = 2$ is 35. Moreover, from the graph of the function:

$$x \mapsto \frac{8x^6 - 16x^5 - 8x^4 + 32x^3 - 10x^2 - 16x + 8}{4x^6 - 8x^5 - 5x^4 + 16x^3 - 4x^2 - 8x + 4},$$

and by using the same manner as those of the proof of (10) we can prove that

$$0 < \frac{8p^6 - 16p^5 - 8p^4 + 32p^3 - 10p^2 - 16p + 8}{4p^6 - 8p^5 - 5p^4 + 16p^3 - 4p^2 - 8p + 4} \leq 3.2.$$

Since $n \geq 2$, then

$$n^2 > \frac{-\left(-8p^6 + 16p^5 + 8p^4 - 32p^3 + 10p^2 + 16p - 8\right)}{4p^6 - 8p^5 - 5p^4 + 16p^3 - 4p^2 - 8p + 4} > 0. \qquad (11)$$

Setting

$$\begin{array}{rcl} A & = & 4p^6 - 8p^5 - 4p^4 + 16p^3 - 4p^2 - 8p + 3, \\ B & = & -8p^6 + 16p^5 + 8p^4 - 32p^3 + 10p^2 + 16p - 8, \\ C & = & 4p^6 - 8p^5 - 5p^4 + 16p^3 - 4p^2 - 8p + 4. \end{array}$$

Since $A > 0$ and $n^2 > \dfrac{-B}{C}$, it follows from the inequality (9) that

$$\varphi(pn)^{\tau(pn)} \psi(pn) \sigma(pn) - \left((pn)^4 - 2(pn)^2 + 1\right) > n^4 C + n^2 B + A > 0.$$

2. When p divides n. Since $\psi(pn) = p\psi(n)$, $\varphi(pn) = p\varphi(n)$, $\sigma(pn) > p\sigma(n)$ and $\tau(pn) \geq \tau(n) + 1$, then

$$\begin{array}{rcl} \varphi(pn)^{\tau(pn)} \psi(np) \sigma(np) & = & (p\varphi(n))^{\tau(pn)} \psi(pn) \sigma(pn) \\ & > & p^{\tau(pn)+2} \varphi(n)^{\tau(pn)} \psi(n) \sigma(n) \\ & \geq & p^{\tau(n)+3} \varphi(n)^{\tau(n)+1} \psi(n) \sigma(n) \\ & = & p^{\tau(n)+3} \varphi(n) \left[\varphi(n)^{\tau(n)} \psi(n) \sigma(n)\right] \\ & \geq & p^{\tau(n)+3} \varphi(n) \left(n^4 - 2n^2 + 1\right) \\ & \geq & 2n^4 p^5 - 4n^2 p^5 + 2p^5. \end{array}$$

Therefore,

$$\begin{array}{rl} & \varphi(pn)^{\tau(pn)} \psi(pn) \sigma(pn) - \left((pn)^4 - 2(pn)^2 + 1\right) \\ \geq & 2n^4 p^5 - n^4 p^4 - 4n^2 p^5 + 2n^2 p^2 + 2p^5 - 1 \\ = & n^4 \left(2p^5 - p^4\right) + n^2 \left(-4p^5 + 2p^2\right) + 2p^5 - 1. \end{array} \tag{12}$$

Since $p \geq 2$, then $2p^5 - 1 > 0$. Using the graph of the function $x \mapsto \dfrac{4x^5 - 2x^2}{2x^5 - x^4}$ and the proof of (10), we can also prove that

$$0 < \frac{4p^5 - 2p^2}{2p^5 - p^4} \leq \frac{5}{2}.$$

Since $n \geq 2$, then

$$n^2 > \frac{-\left(-4p^5 + 2p^2\right)}{2p^5 - p^4} > 0. \tag{13}$$

It follows from (12),(13) that

$$\varphi(pn)^{\tau(pn)} \psi(pn) \sigma(pn) - \left((pn)^4 - 2(pn)^2 + 1\right) > 0.$$

Hence, for $s = 1$, we have proved that the inequality $\varphi(n)^{\tau(n)} \psi(n) \sigma(n) \geq n^4 - 2n^2 + 1$ is true for every $n \geq 2$.

Now, assume for some $s \geq 1$ that the desired inequality holds for any composite positive integer n. We distinguish two cases:

Case 1. Suppose that n is not the square of a prime number. Then

$$\begin{aligned}
& \varphi_{s+1}(n)^{\tau(n)} \psi_{s+1}(n) \sigma(n) \\
= \; & \left(n^{s+1} \prod_{p|n} \left(1 - \frac{1}{p^{s+1}}\right) \right)^{\tau(n)} n^{s+1} \prod_{p|n} \left(1 + \frac{1}{p^{s+1}}\right) \sigma(n) \\
= \; & n^{\tau(n)} \left(n^{s} \prod_{p|n} \left(1 - \frac{1}{p^{s+1}}\right) \right)^{\tau(n)} n^{s+1} \prod_{p|n} \left(1 + \frac{1}{p^{s+1}}\right) \sigma(n) \\
\geq \; & n^{\tau(n)} \left(n^{s} \prod_{p|n} \left(1 - \frac{1}{p^{s}}\right) \right)^{\tau(n)} n^{s} \prod_{p|n} \left(1 + \frac{1}{p^{s}}\right) \sigma(n) \\
= \; & n^{\tau(n)} \left[\varphi_s(n)^{\tau(n)} \psi_s(n) \sigma(n) \right] \\
\geq \; & n^{\tau(n)} \left(n^{3s+1} + n^{3s} - n^{2s+1} - n^{2s} - n^{s+1} - n^{s} + n + 1 \right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\varphi_{s+1}(n)^{\tau(n)} \psi_{s+1}(n) \sigma(n) \; & \geq \; n^{4} \left(n^{3s+1} + n^{3s} - n^{2s+1} - n^{2s} - n^{s+1} - n^{s} + n + 1 \right) \quad (14) \\
& = \; n^{3s+5} + n^{3s+4} - n^{2s+5} - n^{2s+4} - n^{s+5} - n^{s+4} + n^{5} + n^{4},
\end{aligned}$$

where (14) holds because n is not of the form p^2 with p is prime, and therefore $\tau(n) \geq 4$. Since $n \geq 6$, it follows that

$$\begin{aligned}
& \varphi_{s+1}(n)^{\tau(n)} \psi_{s+1}(n) \sigma(n) - \left(n^{3s+4} + n^{3s+3} - n^{2s+3} - n^{2s+2} - n^{s+2} - n^{s+1} + n + 1 \right) \\
\geq \; & n^{3s+5} - n^{3s+3} - n^{2s+4} + n^{2s+3} - n^{2s+5} + n^{2s+2} - n^{s+5} - n^{s+4} + n^{s+2} + \\
& n^{s+1} + n^{5} + n^{4} - n - 1 \\
\geq \; & 6^{3s+5} - 6^{3s+3} - 2^{2s+4} + 6^{2s+3} - 6^{2s+5} + 6^{2s+2} - 6^{s+5} - 6^{s+4} + 6^{s+2} + \\
& 6^{s+1} + 6^{5} + 6^{4} - 6 - 1 \\
= \; & 7560 \times 6^{3s} - 8820 \times 6^{2s} - 9030 \times 6^{s} + 9065 \\
\geq \; & 1270\,325.
\end{aligned}$$

Note that when n is prime, the inequality (8) becomes

$$\begin{aligned}
\varphi_s(n)^{\tau(n)} \psi_s(n) \sigma(n) \; & = \; (n^s - 1)^2 (n^s + 1)(n + 1) \\
& = \; n^{3s+1} + n^{3s} - n^{2s+1} - n^{2s} - n^{s+1} - n^{s} + n + 1.
\end{aligned}$$

Case 2. Suppose that $n = p^2$ for some prime number $p \geq 2$. We also have

$$\begin{aligned} \varphi_s(n)^{\tau(n)} \psi_s(n) \sigma(n) &= \left(p^{2s} - p^s\right)^3 \left(p^{2s} + p^s\right)\left(1 + p + p^2\right) \\ &= p^{8s+2} + p^{8s+1} + p^{8s} - 2p^{7s+2} - 2p^{7s+1} - 2p^{7s} + \\ &\quad 2p^{5s+2} + 2p^{5s+1} + 2p^{5s} - p^{4s+2} - p^{4s+1} - p^{4s}. \end{aligned}$$

Therefore,

$$\begin{aligned} & \varphi_s(n)^{\tau(n)} \psi_s(n) \sigma(n) - \left(n^{3s+1} + n^{3s} - n^{2s+1} - n^{2s} - n^{s+1} - n^s + n + 1\right) \\ \geq\ & p^{8s+2} + p^{8s+1} + p^{8s} - 2p^{7s+2} - 2p^{7s+1} - 2p^{7s} - p^{6s+2} - p^{6s} + 2p^{5s+2} + \\ & 2p^{5s+1} + 2p^{5s} + p^{2s+2} - p^{4s+1} + p^{2s} - p^2 - 1 \\ \geq\ & 7 \times 2^{8s} - 14 \times 2^{7s} - 5 \times 2^{6s} + 14 \times 2^{5s} - 2 \times 2^{4s} + 5 \times 2^{2s} - 5 \\ \geq\ & 111, \end{aligned}$$

since $p \geq 2$. Hence, (8) is true for $n = p^2$ with p is prime.

Thus, our assertion is proved by induction on s. This completes the proof of Theorem 3.2. ☐

Remark 3.3. In the case when $n = p^2$ with p is prime, then (14) becomes

$$\varphi_{s+1}(n)^{\tau(n)} \psi_{s+1}(n) \sigma(n) \geq n^{3s+4} + n^{3s+3} - n^{2s+3} - n^{2s+4} - n^{s+4} - n^{s+3} + n^4 + n^3,$$

since $\tau(n) = 3$. Hence,

$$\begin{aligned} & \varphi_{s+1}(n)^{\tau(n)} \psi_{s+1}(n) \sigma(n) - \left(n^{3s+4} + n^{3s+3} - n^{2s+3} - n^{2s+2} - n^{s+2} - n^{s+1} + n + 1\right) \\ \geq\ & -n^{2s+4} + n^{2s+2} - n^{s+4} - n^{s+3} + n^{s+2} + n^{s+1} + n^4 + n^3 - n - 1, \end{aligned} \tag{15}$$

where the leading coefficient of (15) is negative. Therefore, in this case, the inequality (8) can not be easily deduced for $s + 1$.

The next that we will study is a Diophantine inequality involving the arithmetic function $\dfrac{\psi_s(n)}{n^s}$. Let p_r be the r-th prime number with $r \geq 2$, and let $s \geq 1$. We define

$$B_{r,s} = \left\{ n \in \mathbb{N} \;/ \psi_s(n) < \frac{p_r}{p_{r-1}} n^s \right\}.$$

Note that if n is an even positive integer, then $n \notin B_{r,s}$. In fact, assume that $n = 2^a N$ with $a \geq 1$ and $(2, N) = 1$, then by applying Bertrand's theorem we obtain

$$p_{r-1}\psi_s(n) = p_{r-1}\psi_s(2^a N) = p_{r-1}(2^s + 1) 2^{s(a-1)} \psi_s(N) > p_r N^s.$$

Proposition 3.5. *Suppose that $n \notin B_{r,s}$ for some $r \geq 2$ and $s \geq 1$, then there are infinity many positive integers s' such that $n \in B_{r,s+s'}$.*

Proof. Let $n \notin B_{r,s}$ for some $r \geq 2$ and $s \geq 1$. Since the sequence

$$\left\{ \prod_{p|n} \left(1 + \frac{1}{p^m}\right) \right\}_{m \geq 1}$$

is decreasing and tends to 1 as m tends to infinity, then there exists a unique positive integer s_0 such that

$$\prod_{p|n} \left(1 + \frac{1}{p^{s+s_0-1}}\right) \geq \frac{p_r}{p_{r-1}} > \prod_{p|n} \left(1 + \frac{1}{p^{s+s_0}}\right) > \prod_{p|n} \left(1 + \frac{1}{p^{s+s_0+1}}\right) > ....$$

That is, for every $s' \geq s_0$, we have $n \in B_{r,s+s'}$. This completes the proof. $\square$

In the paper of Sandor (2008), it is proved that for every $n \geq 2$,

$$\sigma(n) > n + (\omega(n) - 1)\sqrt{n}.$$

A new similar result is given by the following proposition.

Proposition 3.6. *Let $k \geq 1$. There are infinitely many $n \in W_k$ such that*

$$\sigma(n) > 1 + n + \omega(n)\sqrt{n}.$$

Proof. Let s be a positive integer such that $s \geq \max\{4, k\}$, and let p_s be the s-th prime number. For $n = p_1 p_2 ... p_s$, we obtain from Bone's inequality stated by (Wells, 2005, p. 21),

$$p_{s+1}^2 < p_1 p_2 ... p_s = n.$$

Since $p_i < \sqrt{n}$ for $i = 1, 2, ..., s$, it follows that

$$\frac{n}{p_1} > \frac{n}{p_2} > ... > \frac{n}{p_s} > \sqrt{n},$$

and therefore

$$\begin{aligned} \sigma(n) &> 1 + n + \sum_{i=1}^{s} \frac{n}{p_i} \\ &> 1 + n + s\sqrt{n} \\ &= 1 + n + \omega(n)\sqrt{n}. \end{aligned}$$

As required. $\square$

The following theorem improves the result stated in Proposition 3.6.

Theorem 3.3. *Let f be an arbitrary integer-valued polynomial with $f(n) \neq 0$ for any $n \in \mathbb{N}$. There exists an infinite subset of positive integers A such that for every $a \in A$, there exists a positive integer $b \leq \omega(a)$ satisfying*

$$\sigma(a) > 1 + a + f(b)\sqrt{a}.$$

Proof. From Transfer Principle (this is the third Axiom of Internal Set Theory), we prove our theorem in the case when f is standard (i.e., f does not include unlimited coefficients). Let p_n be the n-th prime number. We put

$$A = \{a \in \mathbb{N} \mid a = p_1 p_2 ... p_s \text{ with } s \simeq +\infty\}.$$

Let $a = p_1 p_2 ... p_s \in A$. Since $\sqrt{a}$ is unlimited, then for every limited positive integer $i \leq s$,

$$p_i < \frac{i}{f(i)}\sqrt{a}.$$

Define the internal subset $\left\{ i \in \mathbb{N} \mid p_i < \frac{i}{f(i)}\sqrt{a} \right\}$, which contains the galaxy $\mathbb{N}^{\sigma}$. By Cauchy's principle stated in Theorem 1.1, there exists an unlimited positive integer $b \leq s = \omega(a)$ such that

$$p_b < \frac{b}{f(b)}\sqrt{a}.$$

Therefore,

$$\frac{n}{p_1} > \frac{n}{p_2} > ... > \frac{n}{p_b} > \frac{f(b)}{b}\sqrt{a}.$$

It follows that

$$\begin{aligned} \sigma(a) &> 1 + a + \sum_{i=1}^{b} \frac{a}{p_i} \\ &> 1 + a + f(b)\sqrt{a}. \end{aligned}$$

This completes the proof. $\square$

3.3 On the fraction defined by $\tau^s(ab)$ and $\tau(a^s)\tau(b^s)$ with $s \geq 1$

Let $s \geq 2$ be a positive integer. We aim to study the growth of the sequence

$$\frac{\tau^s(ab)}{\tau(a^s)\tau(b^s)}, \text{ where } (a, b) \in \mathbb{N}^2.$$

In the present section we present some cases for which the above quantity is unlimited.

Proposition 3.7. *Let a, b be unlimited positive integers. If one of the following conditions holds:*

1. *There exists a prime number p such that $p^c|a$ for some unlimited c, p^n does not divide b for every unlimited n, and vice versa.*

2. *The cardinality of the set $\left\{p \in \mathbb{P};\ \exists\ \alpha, \beta \geq 1 \text{ such that } \alpha \neq \beta,\ p^{\alpha}||a \text{ and } p^{\beta}||b\right\}$ is unlimited.*

3. *The cardinality of the set $\{p \in \mathbb{P};\ p|a \text{ and } p \text{ does not divide } b\}$ is unlimited, and vice versa.*

Then for every $s \geq 2$, we have

$$\frac{\tau^s(ab)}{\tau(a^s)\tau(b^s)} \simeq +\infty.$$

Proof. Suppose that $a = \prod\limits_{i=1}^{k} p_i^{\alpha_i}$, $b = \prod\limits_{i=1}^{k} p_i^{\beta_i}$, where $\alpha_i, \beta_i \geq 0$ for $i = 1, 2, ..., k$, p_i is the i-th prime number and $k = \max\{i \geq 1 \mid (\alpha_i, \beta_i) \neq (0,0)\}$. It follows that,

$$\begin{cases} \tau(a^s) = \prod\limits_{i=1}^{k} (s\alpha_i + 1), \\ \tau(b^s) = \prod\limits_{i=1}^{k} (s\beta_i + 1). \end{cases}$$

Since $ab = \prod\limits_{i=1}^{k} p_i^{\alpha_i + \beta_i}$, it follows that

$$\tau^s(ab) = \prod_{i=1}^{k} (\alpha_i + \beta_i + 1)^s.$$

Therefore,

$$\frac{\tau^s(ab)}{\tau(a^s)\tau(b^s)} = \prod_{i=1}^{k} \frac{(\alpha_i + \beta_i + 1)^s}{(s\alpha_i + 1)(s\beta_i + 1)}. \tag{16}$$

Suppose that the first condition holds. Since $\dfrac{\alpha_i^s + \beta_i^s + 1}{(s\alpha_i + 1)(s\beta_i + 1)} \geq 1$ for all

$1 \leq i \leq k$, then it follows from (16) that

$$\begin{aligned}
\frac{\tau^s(ab)}{\tau(a^s)\tau(b^s)} &= \prod_{i=1}^{k} \frac{(\alpha_i+\beta_i+1)^s}{(s\alpha_i+1)(s\beta_i+1)} \\
&\geq \prod_{i=1}^{k} \frac{\alpha_i^s+\beta_i^s+1}{(s\alpha_i+1)(s\beta_i+1)} \\
&= \prod_{i=1}^{k} \frac{\frac{\alpha_i^{s-1}}{s^2\beta_i}+\frac{\beta_i^{s-1}}{s^2\alpha_i}+\frac{1}{s^2\alpha_i\beta_i}}{1+\frac{1}{s\beta_i}+\frac{1}{s\alpha_i}+\frac{1}{s^2\alpha_i\beta_i}} \\
&\simeq +\infty,
\end{aligned}$$

because there exists $i_0 \in \{1, 2, ..., k\}$ such that the number $\dfrac{\alpha_{i_o}^{s-1}}{s^2\beta_{i_o}}$ is unlimited.

When the second condition holds. Since k is unlimited and $s \geq 2$, then from (16) we have

$$\begin{aligned}
\frac{\tau^s(ab)}{\tau(a^s)\tau(b^s)} &= \prod_{i=1}^{k} \frac{(\alpha_i+\beta_i+1)^s}{(s\alpha_i+1)(s\beta_i+1)} \\
&= \prod_{i=1}^{k} (1+a_i) \\
&\simeq +\infty,
\end{aligned}$$

because $\dfrac{(\alpha_i+\beta_i+1)^s}{(s\alpha_i+1)(s\beta_i+1)} - 1 = a_i$ is either appreciable or unlimited, for $i = 1, 2, ..., k$.

When the third condition holds. From (16) we get

$$\begin{aligned}
\frac{\tau^s(ab)}{\tau(a^s)\tau(b^s)} &= \prod_{i=1}^{k} \frac{(\alpha_i+\beta_i+1)^s}{(s\alpha_i+1)(s\beta_i+1)} \\
&> \prod_{i\in\{t\ :1\leq t\leq k,\ \alpha_t\neq 0,\ \beta_t=0\}} \frac{(\alpha_i+1)^s}{(s\alpha_i+1)} \\
&= \prod_{i\in\{t\ :1\leq t\leq k,\ \alpha_t\neq 0,\ \beta_t=0\}} (1+a_i) \\
&\simeq +\infty,
\end{aligned}$$

because $\frac{(\alpha_i+1)^s}{(s\alpha_i+1)} - 1 = a_i$ is either appreciable or unlimited, for every i belongs to the set $\{t\ : 1 \leq t \leq k,\ \alpha_t \neq 0,\ \beta_t = 0\}$. $\square$

Finally, we prove the following corollary.

Corollary 3.2. *Let a, b be positive integers. There exist two integers $R, T \geq 0$ such that*

$$\frac{\tau^2(ab)}{\tau(a^2)\,\tau(b^2)} \geq \left(\frac{4}{3}\right)^{R+T}.$$

Proof. Suppose that $a = \prod_{i=1}^{k} p_i^{\alpha_i}$, $b = \prod_{i=1}^{k} p_i^{\beta_i}$, where $\alpha_i, \beta_i \geq 0$ for $i = 1, 2, ..., k$, p_i is the i-th prime number and $k = \max\{i \geq 1 \mid (\alpha_i, \beta_i) \neq (0,0)\}$. Since

$$\frac{\tau^2(ab)}{\tau(a^2)\,\tau(b^2)} = \prod_{\{t\ :1\leq t\leq k,\ \alpha_t\neq 0,\ \beta_t=0\}} \frac{(\alpha_i+\beta_i+1)^2}{(2\alpha_i+1)(2\beta_i+1)} \cdot \prod_{\{t\ :1\leq t\leq k,\ \alpha_t\neq 0,\ \beta_t=0\}} \frac{(\alpha_i+1)^2}{(2\alpha_i+1)} \cdot \prod_{\{t\ :1\leq t\leq k,\ \alpha_t=0,\ \beta_t\neq 0\}} \frac{(\beta_i+1)^2}{(2\beta_i+1)},$$

where

$$\frac{(\alpha_i+\beta_i+1)^2}{(2\alpha_i+1)(2\beta_i+1)} = \left(1 + \frac{(\alpha_i-\beta_i)^2}{(2\alpha_i+1)(2\beta_i+1)}\right) \geq 1, \text{ for } i = 1, 2, ..., k \text{ with } \alpha_i \neq 0, \beta_i \neq 0.$$

It follows from (16) that

$$\frac{\tau^2(ab)}{\tau(a^2)\,\tau(b^2)} \geq \prod_{\{t\ :1\leq t\leq k,\ \alpha_t\neq 0,\ \beta_t=0\}} \left(1 + \frac{\alpha_i^2}{2\alpha_i+1}\right) \prod_{\{t\ :1\leq t\leq k,\ \alpha_t=0,\ \beta_t\neq 0\}} \left(1 + \frac{\beta_i^2}{2\beta_i+1}\right).$$

We put $R = Card\left\{t \in \overline{1,k};\ \alpha_t \neq 0 \text{ and } \beta_t = 0\right\}$ and $T = Card\left\{t \in \overline{1,k};\ \alpha_t = 0 \text{ and } \beta_t \neq 0\right\}$, then

$$\frac{\tau^2(ab)}{\tau(a^2)\,\tau(b^2)} \geq \left(\frac{4}{3}\right)^{R+T},$$

where $\frac{\alpha_i^2}{2\alpha_i+1} \geq \frac{1}{3}$, for $i = 1, 2, ..., r$ and $\frac{\beta_i^2}{2\beta_i+1} \geq \frac{1}{3}$, for $i = 1, 2, ..., t$. This completes the proof. $\square$

4. Conclusion

In this paper we consider reports of the form $\dfrac{f(n)}{g(n)}$, where f, g are two arithmetic multiplicative functions. Note that for this report there exists a family $S_{f/g}$ of unlimited integers such that for every $n \in S_{f/g}$, $\dfrac{f(n)}{g(n)}$ remains equivalent to the same value, say l. Hence, our work consists of all solutions of certain Diophantine equations of the form:

$$\frac{f(n)}{g(n)} \simeq l. \tag{17}$$

Note that solving equations of the form (17) is easier and more flexible than solving $\dfrac{f(n)}{g(n)} = l$, since all its solutions are also solutions to (17). That is, (17) is more general than $\dfrac{f(n)}{g(n)} = l$. Further, our attack includes a new method to compare an arithmetic function $F(n)$ formed by the sum and the product of certain multiplicative arithmetic functions with an integer-valued polynomial whose leading coefficient is positive, as stated in Theorem 3.2. For details, let us assume furthermore that $F(p)$ is a polynomial in integers whenever p is prime, where the leading coefficient is positive. It is clear that there is a comparison of this polynomial with $F(n)$ whenever n is arbitrary. In fact, define

$$F_s(n) = f(\sigma(n), \tau(n), \varphi_s(n), \psi_s(n), ...),$$

where $s \in \mathbb{N}$ is a parameter and f is an arbitrary integer-valued function of several variables. Assume that for any prime p,

$$F_s(p) = p^{as+b} + c_{as+b-1}p^{as+b-1} + ... + c_1 p + c_0, \tag{18}$$

where $a, b \geq 1$ and $c_i \geq 0$, for $i = 0, 1, ..., as+b-1$. By the help of (18) we ask whether the Diophantine inequality

$$F_s(n) \geq n^{as+b} + c_{as+b-1}n^{as+b-1} + ... + c_1 n + c_0$$

holds for each number $n \geq 2$ or not.

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