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ON A SEQUENCE FORMED BY ITERATING A DIVISOR OPERATOR

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Abstract. Let $\mathbb{N}$ be the set of positive integers and let $s \in \mathbb{N}$. We denote by d^s the arithmetic function given by $d^s(n) = (d(n))^s$, where $d(n)$ is the number of positive divisors of n. Moreover, for every $l, m \in \mathbb{N}$ we denote by $\delta^{s,l,m}(n)$ the sequence

$$\underbrace{d^s(d^s(\ldots d^s(d^s(n)+l)+l\ldots)+l)}_{m\text{-times}} = \begin{cases} d^s(n) & \text{for } m=1,\\ d^s(d^s(n)+l) & \text{for } m=2,\\ d^s(d^s(d^s(n)+l)+l) & \text{for } m=3,\\ \vdots & \end{cases}$$

We present classical and nonclassical notes on the sequence $(\delta^{s,l,m}(n))_{m\geqslant 1}$, where l, n, s are understood as parameters.

Keywords: divisor function; prime number; iterated sequence; internal set theory

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1. Introduction

In the present article we apply nonstandard analysis in the field of number theory. On this topic, we refer to [2], [3], [4], [9]. Here we study a modified repeated divisor operator, with the aim to derive some bounds on growth based on permanence principles of nonstandard analysis.

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Let as usual $d(n)$ denote the number of positive divisors of n. It is well known that d is multiplicative, we can write down an explicit formula for $d(n)$ in terms of the prime powers that exactly divide n. Let

$$n = q_1^{\alpha_1} q_2^{\alpha_2} \dots q_r^{\alpha_r},$$

where $q_1, q_2, \dots, q_r$ are distinct primes, then

$$d(n) = (\alpha_1 + 1)(\alpha_2 + 1) \dots (\alpha_r + 1). \tag{1.1}$$

For the proof of (1.1), see [18], page 67.

We are interested in problems involving a sequence formed by iterating a divisors operator. That is, problems which are related to the iteration of powers of $d(n)$. There seem to exist only a few results in the literature on this subject. First, we would like to have a brief introduction into the history of the subject.

From the wealth of problems involving the divisor function $d(n)$ we shall concentrate on some problems connected with the work of Erdős. We begin with the iterations of $d(n)$. Thus let for $m \in \mathbb{N}$ fixed,

$$d^{(1)}(n) = d(n), \quad d^{(m)}(n) = d(d^{(m-1)}(n)) \quad \text{for } m \geqslant 2 \tag{1.2}$$

be the mth iteration of $d(n)$. We note that $d^{(m)}(n)$ is sometimes called the m-fold iterated of $d(n)$. Already $d^{(2)}(n)$ is not multiplicative; this fact makes the problems involving $d^{(m)}(n)$ and iterates of other multiplicative functions quite difficult.

In 1915, Ramanujan [13] proved that

$$d^{(2)}(n) > 4^{\sqrt{2 \log n} / \log \log n}$$

for an infinity of values of n. This lower bound of $d^{(2)}(n)$ was obtained by considering integers of the form

$$2^1 \cdot 3^2 \cdot 5^4 \cdot \dots \cdot p_m^{p_m - 1},$$

where p_i is the ith prime number.

Important results on the order of $d^{(m)}(n)$ were obtained in 1967 by Erdős and Kátai [8]. They studied the growth of the sequence $d^{(m)}(n)$ using the mth element of the Fibonacci sequence and gave an upper bound for all sufficiently large n and a lower bound for an infinity of values of n. In fact, let l_m denote the mth Fibonacci number given by the recurrence relation

$$l_{-1} = 0, \quad l_0 = 1, \quad l_m = l_{m-1} + l_{m-2} \quad (m \geqslant 1).$$

Then the result of Erdős and Kátai says that

$$d^{(m)}(n) < \exp(\log n)^{1/l_m+\varepsilon} \tag{1.3}$$

for all positive ε and all sufficiently large values of n. Further for every $\varepsilon > 0$,

$$d^{(m)}(n) > \exp(\log n)^{1/l_m-\varepsilon}$$

for an infinity of values of n.

Let l, s be positive integers. Generalizing (1.2), we define a new sequence, denoted by $\delta^{s,l,m}(n)$, and we give it an upper bound as in (1.3). We show further the existence of the maximal order, that is, the existence of the smallest positive integer m_0 such that $\delta^{s,l,m}(n)$ is fixed for every $m \geqslant m_0$. Before launching to explain exactly the problems that we want to solve, we present the following notation.

Notation 1.1. For every positive integer s, d^s denotes the arithmetic function given by $d^s(n) = (d(n))^s$ for $n \geqslant 1$. Moreover for every $l, m \in \mathbb{N}$ we denote by $\delta^{s,l,m}(n)$ the sequence

$$\underbrace{d^s(d^s(\ldots d^s(d^s(n)+l)+l\ldots)+l)}_{m\text{-times}} = \begin{cases} d^s(n) & \text{for } m=1, \\ d^s(d^s(n)+l) & \text{for } m=2, \\ d^s(d^s(d^s(n)+l)+l) & \text{for } m=3, \\ \vdots & \end{cases} \tag{1.4}$$

We note that $\delta^{s,l,1}(n) = \delta^{s,1,1}(n) = d^s(n)$.

In this paper, we study distribution behaviour and growth problems for $\delta^{s,l,m}(n)$. Moreover, we will continue the research from [1] on the sequence $\delta^{1,l,m}(n)$. In fact, let k be a positive integer and let W_k denote the set of positive integers n for which the number of distinct prime factors of n is larger or equal to k. In [1], the author proved some inequalities for an infinity of values of $n \in W_k$. Similarly, in this paper we present a property of $\delta^{1,l,m}(n)$ for an infinity of values of $n \in W_k$. In the framework of internal set theory, see [12], [15], some nonclassical properties on the upper bound of $\delta^{s,l,m}(n)$ are given.

We end this introduction with an outline of the paper. Section 2 contains preliminaries and auxiliary results that we need in the rest of the paper. Moreover, we present a nonstandard version of the theorem (see [11], pages 225–226) which says that for multiplicative functions to converge to 0 it is only needed that they converge to 0 for prime powers; this nonstandard version is needed for our main theorem. In Section 3, we prove that for every $n \geqslant 1$ there exists an order m such that $\delta^{1,1,m}(n) = 2$ or 3. Next, in the equation $\delta^{1,1,m}(n) = 2$, we show that the order m can be arbitrarily given for an infinity of values of n. This is obtained by using Dirichlet's Theorem about primes in an arithmetic progression [17]. In Subsection 3.1, we give some examples to illustrate the results stated in Proposition 3.1, Remark 3.1 and Theorem 3.1, where we consider only $l = 1$ and s is odd. In the end of this section, we will use the theorem from Section 2 to prove our main theorem which deals with the upper bound of $\delta^{s,l,m}(n)$.

2. Basic tools

First of all, we summarize a few auxiliary results about arithmetic functions and internal set theory that we need. One can refer to [5], [11], [12], [15], [17].

Definition 2.1. A prime power is a positive integer power of a single prime number. The first prime powers are

$$2,\ 3,\ 2^2,\ 5,\ 7,\ 2^3,\ 3^2,\ 11,\ 13,\ 2^4,\ 17,\ 19,\ 23,\ 5^2,\ 3^3,\ \ldots$$

The following classical theorem links the convergence of a sequence with the convergence along prime powers. Below we give a nonstandard proof, using the equivalent nonstandard version given in Theorem 2.3; this nonstandard version is needed in the proof of Theorem 3.2.

Theorem 2.1 ([11], page 225). *Let f be an arbitrary multiplicative function. If $\lim_{q^\alpha \to \infty} f(q^\alpha) = 0$ as q^α runs through the sequence of all prime powers, then $\lim_{n\to\infty} f(n) = 0$.*

Notation 2.1. If $n = q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_r^{\alpha_r}$ is the factorization of n as a product of powers of distinct primes, then

$$\operatorname{set}(n) = \{q_1, q_2, \ldots, q_r\} \quad \text{and} \quad |\operatorname{set}(n)| = r.$$

We recall Dirichlet's Theorem, which will be needed in our proofs.

Theorem 2.2 (Dirichlet's Theorem, [11], page 347). *If a and b are relatively prime integers with $a \geqslant 1$, the polynomial $f(n) = an + b$ represents infinitely many primes.*

Historically, Leibniz, Euler and Cauchy are among the first who began the use of infinitely small quantities. In order to use better this notion, Robinson proposed in 1961 another approach, namely, the nonstandard analysis. In 1977, Nelson provided another presentation of the nonstandard analysis, called IST (Internal Set Theory), based on ZFC to which is added a new unary predicate called "standard". The use of this predicate is governed by the following three axioms: Transfer principle, Idealization principle, and Standardization principle. For details, see [7], pages 7–33, [12], [14].

Any real number that can be characterized in the unique classical way is necessarily standard. Thus, $0, 1, \ldots, 100^{100}, \ldots$ are standard. But not all integers are standard. A real number ω is called unlimited, or infinitely large if its absolute value $|\omega|$ is larger than any standard integer n. So a nonstandard integer ω is also an unlimited real number; $\omega + \sqrt{3}$ is an example of an unlimited real number that is not an integer. A real number ε is called infinitesimal, or infinitely small, if its absolute value $|\varepsilon|$ is smaller than $1/n$ for any standard n. Of course, 0 is infinitesimal but (fortunately) it is not the only one: $\varepsilon = 1/\omega$ is infinitesimal, provided ω is unlimited. A real number r is called limited if it is not unlimited and appreciable if it is neither unlimited nor infinitesimal. Finally, two real numbers x and y are equivalent or infinitely close (written $x \simeq y$) if their difference $x - y$ is infinitesimal. For details, see [6], page 2–4.

In mathematics, we describe as internal a formula which is expressible in the classical language (ZFC) and as external a formula of the nonstandard language (IST) which involves the new predicate "standard" or one of its derivatives such as "infinitesimal" or "limited", see [10]. For example, the formula $[x < \varepsilon^2 + 1 \Rightarrow x - 1 < \varepsilon^2]$ is internal whereas the formula $[x \simeq \infty \Rightarrow x/2 + 1 \simeq \infty]$ is external. On the other hand, since sets are defined by using the formulas, we have two types of sets: internal sets and external sets. For example, $\{x \in \mathbb{R} \colon |x| \leqslant \varepsilon\}$ is internal whereas $\{x \in \mathbb{R} \colon x \simeq 0\}$ is external. Observe that a set defined by means of an external formula is not necessarily external. For example, the set $\{x \in \mathbb{R} \colon \operatorname{st}(x) \text{ and } x \simeq 0\}$ which is equal to the singleton $\{0\}$, is not only internal but standard (see [6], pages 5–7). We recognize that in nonstandard literature we find several points of view to define an external set (see [10], [16]) and this reflects the problematic posed by this notion. In this article, we opt for the following definition.

Definition 2.2 ([10]). We call internal any set defined by means of an internal formula and we call external any subset of an internal set defined by means of an external formula, which is not (reduced to) an internal set.

Based on the above facts, we qualify mathematical objects as internal or external. For example, we say that a function is internal or external if its graph is, respectively, internal or external, and so on.

Definition 2.3 (see [6], page 20). Let X be a standard set, and let $(A_x)_{x \in X}$ be an internal family of sets.

(a) A union of the form $G = \bigcup_{\mathrm{st}x \in X} A_x$ is called a *pregalaxy*; if it is external G is called a *galaxy*.
(b) An intersection of the form $H = \bigcap_{\mathrm{st}x \in X} A_x$ is called a *prehalo*; if it is external H is called a *halo*.

Example 2.1 (for details, see [2]). We have:
(1) The set of limited positive integers $\mathbb{N}^\sigma$ is a galaxy.
(2) The set $\mathrm{hal}(0) = \{x \in \mathbb{R} \colon\ x \simeq 0\}$ is a halo.

The following principles are important for the proof of our nonclassical notes.

Principle 2.1 (Cauchy's Principle, [6], page 19). No external set is internal.

Principle 2.2 (Fehrele's Principle, [6], page 20). No halo is a galaxy.

In addition, we recall the following notation which is related to internal set theory. The symbols Φ and $\pounds$ are used, respectively, for an arbitrary infinitesimal and an arbitrary limited number (see [6], page 3). Let x, y be two real numbers where x is unlimited. We write $y \ll x$ when $x - y$ is unlimited positive.

We end this section with a nonstandard version and the proof of Theorem 2.1.

Theorem 2.3 (Nonstandard version of Theorem 2.1). *Let f be a standard multiplicative function. If $f(q^\alpha) \simeq 0$ as q^α runs through the sequence of all unlimited prime powers, then $f(n) \simeq 0$ for every unlimited n.*

P r o o f. Let n be an unlimited positive integer such that $f(q^\alpha) \simeq 0$ whenever q^α runs through the sequence of all unlimited prime powers. We put $n = q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_s^{\alpha_s}$, where $q_1 < q_2 < \ldots < q_s$ are primes and $\alpha_i \geqslant 1$ for $i = 1, 2, \ldots, s$. It follows that

$$f(n) = \prod_{i=1}^{s} f(q_i^{\alpha_i}),$$

because f is multiplicative. Since $q_1, q_2, \ldots, q_s$ are mutually distinct primes, the numbers $q_1^{\alpha_1}, q_2^{\alpha_2}, \ldots, q_s^{\alpha_s}$ are also mutually distinct, which we can order as follows

$$q_{i_1}^{\alpha_{i_1}} < q_{i_2}^{\alpha_{i_2}} < \ldots < q_{i_s}^{\alpha_{i_s}},$$

where $i_j \in \{1, 2, \ldots, s\}$ for $j = 1, 2, \ldots, s$. We distinguish two cases.

Case 1: For every $j = 1, 2, \ldots, s$, $q_{i_j}^{\alpha_{i_j}}$ is unlimited. In this case, from the hypothesis we get $f(q_{i_j}^{\alpha_{i_j}}) \simeq 0$ for $j = 1, 2, \ldots, s$. Consequently,

$$f(n) = \prod_{j=1}^{s} f\bigl(q_{i_j}^{\alpha_{i_j}}\bigr) \simeq 0.$$

Case 2: Some of the numbers $\big(q_{i_j}^{\alpha_{i_j}}\big)_{1\leqslant j\leqslant s}$ are unlimited and some are not. We also distinguish two cases.

Subcase 2.1: Only the first m numbers $q_{i_1}^{\alpha_{i_1}} < q_{i_2}^{\alpha_{i_2}} < \ldots < q_{i_m}^{\alpha_{i_m}}$ are standard, where m is standard and $1 \leqslant m < s$. In this case, $f(q_{i_1}^{\alpha_{i_1}})f(q_{i_2}^{\alpha_{i_2}})\ldots f(q_{i_m}^{\alpha_{i_m}})$ is standard. Since $q_{i_j}^{\alpha_{i_j}} \simeq \infty$ for $j = m+1, m+2, \ldots, s$,

$$\prod_{j=m+1}^{s} f\big(q_{i_j}^{\alpha_{i_j}}\big) \simeq 0.$$

Consequently,

$$f(n) = \prod_{j=1}^{m} f\big(q_{i_j}^{\alpha_{i_j}}\big) \times \prod_{j=m+1}^{s} f\big(q_{i_j}^{\alpha_{i_j}}\big) \simeq 0.$$

Subcase 2.2: For every standard $j \geqslant 1$, $q_{i_j}^{\alpha_{i_j}}$ is standard. In this case, the set

$$T = \left\{a \in \mathbb{N}\colon\ \forall t,\ 1 \leqslant t \leqslant a \Rightarrow \prod_{j=1}^{t} f\big(q_{i_j}^{\alpha_{i_j}}\big) \times f\big(q_{i_s}^{\alpha_{i_s}}\big) \simeq 0\right\}$$

is a prehalo containing the galaxy $\mathbb{N}^\sigma$. In fact, let $a \in \mathbb{N}^\sigma$. Since $f(q_{i_s}^{\alpha_{i_s}}) \simeq 0$ and $\prod_{j=1}^{t} f\big(q_{i_j}^{\alpha_{i_j}}\big)$ is standard for all $t \leqslant a$, also

$$\prod_{j=1}^{t} f\big(q_{i_j}^{\alpha_{i_j}}\big) \times f\big(q_{i_s}^{\alpha_{i_s}}\big) \simeq 0.$$

That is, $a \in T$. Therefore, T contains $\mathbb{N}^\sigma$ strictly according to the principle of Cauchy if it is internal or the principle of Fehrele if it is external. Let $\omega \in T$ be unlimited, then

$$f(n) = \prod_{j=1}^{\omega} f\big(q_{i_j}^{\alpha_{i_j}}\big) \times \prod_{j=\omega+1}^{s} f\big(q_{i_j}^{\alpha_{i_j}}\big) = \prod_{j=1}^{\omega} f\big(q_{i_j}^{\alpha_{i_j}}\big) \times f\big(q_{i_s}^{\alpha_{i_s}}\big) \times \prod_{j=\omega+1}^{s-1} f\big(q_{i_j}^{\alpha_{i_j}}\big) \simeq 0$$

because $f\big(q_{i_1}^{\alpha_{i_1}}\big)f\big(q_{i_2}^{\alpha_{i_2}}\big)\ldots f\big(q_{i_\omega}^{\alpha_{i_\omega}}\big) \times f\big(q_{i_s}^{\alpha_{i_s}}\big) \simeq 0$, and $f\big(q_{i_j}^{\alpha_{i_j}}\big) \simeq 0$ for $j = \omega+1$, $\omega+2, \ldots, s-1$, since $q_{i_j}^{\alpha_{i_j}}$ are unlimited for these indexes. Hence the proof of Theorem 2.3 is complete. $\square$

Remark 2.1. Note that Theorem 2.3 is the nonstandard version of Theorem 2.1 in the following sense: Let f be a multiplicative function. By Theorem 2.1, $\lim_{q^\alpha\to\infty} f(q^\alpha) = 0$ as q^α runs through the sequence of all prime powers is the sufficient condition for $\lim_{n\to\infty} f(n) = 0$, whereas by Theorem 2.3, for standard f, the condition that $f(q^\alpha) \simeq 0$ as q^α runs through the sequence of all unlimited prime powers is sufficient for $f(n) \simeq 0$ whenever $n \simeq \infty$; the latter is the nonstandard characterization of a sequence having the limit equal to 0.

3. Main results

It is well-known (see, for example [18], page 67) that for every positive integer $n \geqslant 2$ there is an order m such that $d^{(m)}(n) = 2$. Indeed, we see that $d(2) = 2$ and for $n \geqslant 3$ with $n = q_1^{\alpha_1} q_2^{\alpha_2} \dots q_r^{\alpha_r}$ we have

$$d(n) = \prod_{i=1}^{r} (\alpha_i + 1) < q_1^{\alpha_1} q_2^{\alpha_2} \dots q_r^{\alpha_r} = n.$$

Thus, $d^{(2)}(n) < d(n)$ whenever $d(n) \geqslant 3$, and $d^{(3)}(n) < d^{(2)}(n)$ whenever $d^{(2)}(n) \geqslant 3$, and so on. Moreover, we note that for any $m \geqslant 1$ there exists a positive integer n such that $d^{(m)}(n) = 2$. In the first result of the present section we prove that $\delta^{1,1,m}(n)$ has the same properties as those appearing for $d^{(m)}(n)$, where the values that we reach after m-fold iterations are 2 or 3. In Theorem 3.1 and Corollary 3.2, we show for infinitely many n that the order m can be arbitrarily given for which $\delta^{1,1,m}(n) = 2$ and $\delta^{1,1,m}(n) = 3$, respectively. Moreover, we deal with problems involving the growth of $\delta^{s,l,m}(n)$, where s, l are limited. In particular, if $d(n)$ is unlimited, we will prove that $\delta^{s,l,1}(n) - \delta^{s,l,m}(n)$ and $\delta^{s,l,1}(n)/\delta^{s,l,m}(n)$ are also unlimited for every $m \geqslant 2$.

Proposition 3.1. *Let $n \geqslant 1$. There exists a positive integer m_0 such that for every $m \geqslant m_0$, one has*

$$\delta^{1,1,m}(n) = 2 \text{ or } 3. \tag{3.1}$$

Proof. First, we will show that for every $n \geqslant 2$,

$$\delta^{1,1,2}(n) \leqslant \delta^{1,1,1}(n) = d(n). \tag{3.2}$$

Obviously the last inequality holds when n is prime. In the case when n is composite, we distinguish two cases.

Case 1: Assume that $d(n) + 1 = p$, where $p \geqslant 5$ is prime (because if $p = 2$ or $p = 3$, then $d(n) = 1$ or $d(n) = 2$, respectively, meaning that $n = 1$ or n is prime). Then

$$\delta^{1,1,2}(n) = 2 < p - 1 = d(n).$$

Case 2: Assume that $d(n) + 1$ is composite. We put $d(n) + 1 = ab$ with $1 < a \leqslant b$ and consider three cases:

Subcase 2.1: $a \neq 2$ and $b \neq 2$. We have

$$a^2(b - d(b)) + b^2(a - d(a)) > a + b,$$

because both $a - d(a)$ and $b - d(b)$ are positive. It follows that

$$a^2 d(b) + b^2 d(a) < (a+b)(ab-1) = (a+b)d(n). \tag{3.3}$$

Therefore,

$$\delta^{1,1,2}(n) = d(ab) = \frac{ad(ab) + bd(ab)}{a+b} < \frac{a^2 d(b) + b^2 d(a)}{a+b} < d(n), \tag{3.4}$$

where the right-hand side of (3.4) holds by (3.3) and the left hand side because $d(ab) \leqslant d(a)d(b)$, $d(a) < a$ and $d(b) < b$.

Subcase 2.2: $a = 2$ and $(2, b) = 1$. Since $b \geqslant 3$, we obtain

$$d(ab) = d(2b) = 2d(b) < 2b.$$

Thus,

$$\delta^{1,1,2}(n) = d(ab) \leqslant 2b - 1 = d(n).$$

Subcase 2.3: $a = 2$ and $(2, b) \neq 1$ with $b \geqslant 2$. We can put $2b = 2^N b'$, where $N \geqslant 2$, $b' \geqslant 1$ and $(2^N, b') = 1$. It follows that

$$d(ab) = d(2^N b') = (N+1)d(b') < 2^N b', \tag{3.5}$$

because $t + 1 < 2^t$ for every $t \geqslant 2$ and $d(b') \leqslant b'$. Therefore,

$$\delta^{1,1,2}(n) = d(ab) \leqslant 2^N b' - 1 = d(n). \tag{3.6}$$

This proves (3.2).

We are now ready to prove (3.1). For $n = 1$, $d(1) + 1 = 2$. Then for every $m \geqslant m_0 = 2$, we have

$$\delta^{1,1,m}(n) = 2.$$

It is the same when n is prime, where $m_0 = 1$. Assume that n is composite with $n \geqslant 4$, that is, $d(n) \geqslant 3$. Note that if $d(n) = 3$, then $\delta^{1,1,m}(n) = 3$ for every $m \geqslant 1$. If $d(n) \geqslant 4$, then by applying (3.2) repeatedly we obtain for every $m \geqslant 1$

$$2 \leqslant \delta^{1,1,m}(n) \leqslant \ldots \leqslant \delta^{1,1,3}(n) \leqslant \delta^{1,1,2}(n) \leqslant \delta^{1,1,1}(n) = d(n), \tag{3.7}$$

noting that $d(t) \geqslant 2$ whenever $t \geqslant 2$. For every $i \geqslant 2$ we will prove that one of the statements:

$$\begin{aligned} &\delta^{1,1,i+1}(n) = 2, \\ &\delta^{1,1,i+1}(n) = 3, \\ &\delta^{1,1,i+1}(n) < \delta^{1,1,i}(n), \\ &\delta^{1,1,i+2}(n) < \delta^{1,1,i+1}(n) \end{aligned} \tag{3.8}$$

holds. Let $i \geqslant 2$. There are two cases:

Case 1: $\delta^{1,1,i}(n)+1$ is prime. Then

$$\delta^{1,1,i+1}(n)=\delta^{1,1,i+2}(n)=\ldots=2. \tag{3.9}$$

Case 2: $\delta^{1,1,i}(n)+1$ is composite. We also consider three cases as in the proof of (3.2).

Subcase 2.1: $\delta^{1,1,i}(n)+1=xy$ with $x\neq 2$ and $y\neq 2$. Using (3.4), we get

$$d(\delta^{1,1,i}(n)+1)=\delta^{1,1,i+1}(n)<\delta^{1,1,i}(n). \tag{3.10}$$

Subcase 2.2: $\delta^{1,1,i}(n)+1=2y$ with $(2,y)=1$ and $y\geqslant 3$. In this case, assume that $y=q_1^{\alpha_1}q_2^{\alpha_2}\ldots q_r^{\alpha_r}$, where $q_1,q_2,\ldots,q_r$ are distinct prime numbers and $\alpha_1,\alpha_2,\ldots,\alpha_r$ are positive integers. Since $q_j\geqslant 3$ for $j=1,2,\ldots,r$,

$$\alpha_j+1<q_j^{\alpha_j}\quad \text{for } j=1,2,\ldots,r,$$

and so

$$2(\alpha_j+1)<2q_j^{\alpha_j}-1\quad \text{for } j=1,2,\ldots,r.$$

Therefore,

$$d(2y)=2\prod_{j=1}^{r}(\alpha_j+1)<2q_1^{\alpha_1}q_2^{\alpha_2}\ldots q_r^{\alpha_r}-1=2y-1.$$

It follows that

$$d(\delta^{1,1,i}(n)+1)=\delta^{1,1,i+1}(n)<\delta^{1,1,i}(n). \tag{3.11}$$

Subcase 2.3: $\delta^{1,1,i}(n)+1=2^My$ with $M\geqslant 2$, $(2^M,y)=1$ and $y\geqslant 1$. As in (3.5) and (3.6), we have $d(\delta^{1,1,i}(n)+1)=\delta^{1,1,i+1}(n)\leqslant 2^My-1$. If

$$\delta^{1,1,i+1}(n)<2^My-1=\delta^{1,1,i}(n), \tag{3.12}$$

we obtain the desired inequality. Otherwise, $\delta^{1,1,i+1}(n)=2^My-1$. In this case, we see that

$$d(\delta^{1,1,i+1}(n)+1)=\delta^{1,1,i+2}(n)=(M+1)d(y),$$

where by (3.7), $(M+1)d(y)\leqslant 2^My-1$. If $(M+1)d(y)<2^My-1$, we have

$$\delta^{1,1,i+2}(n)=(M+1)d(y)<2^My-1=\delta^{1,1,i+1}(n), \tag{3.13}$$

which is the inequality we need. In the remaining case, $(M+1)d(y)=2^My-1$. Here we distinguish two cases, $y=1$ and $y\geqslant 2$.

Assume that $y = 1$. That is $M + 1 = 2^M - 1$. Obviously the last equality holds if and only if $M = 2$. Hence $\delta^{1,1,i+1}(n) = 3$ and by (3.12),

$$\delta^{1,1,i}(n) = \delta^{1,1,i+1}(n) = \ldots = 3. \tag{3.14}$$

But, when $M \geqslant 3$, we have

$$\delta^{1,1,i+2}(n) < \delta^{1,1,i+1}(n), \tag{3.15}$$

since $M + 1 < 2^M - 1$.

Assume that $y = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r} \geqslant 2$, where $p_1, p_2, \ldots, p_r$ are distinct prime numbers and $\alpha_1, \alpha_2, \ldots, \alpha_r$ are positive integers. Since $M + 1 < 2^M$ and $\alpha_j + 1 \leqslant q_j^{\alpha_j}$ for $j = 1, 2, \ldots, r$, it follows for $M = 2$ that

$$2^M y - 1 - (M+1)d(y) = 4p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r} - 3\prod_{j=1}^{r}(\alpha_j + 1) - 1 \geqslant 1, \tag{3.16}$$

which is impossible. For $M \geqslant 3$, we also see that

$$2^M y - 1 - (M+1)d(y) \geqslant (2^M - (M+1))\prod_{j=1}^{r}(\alpha_j + 1) - 1 \geqslant 7, \tag{3.17}$$

which is impossible as well. Then one has $(M+1)d(y) < 2^M y - 1$ when $y \geqslant 2$. Hence by (3.13), (3.16) and (3.17),

$$\delta^{1,1,i+2}(n) < \delta^{1,1,i+1}(n). \tag{3.18}$$

Combining (3.9)–(3.18), we obtain (3.8). Since there exists no infinite descending chain on the natural numbers, as every chain of natural numbers has a minimal element, we obtain $\delta^{1,1,m}(n) = 2$ or 3 for some $m \geqslant 1$. This completes the proof. □

Let k be a positive integer and let W_k be the subset given by

$$W_k = \{n \in \mathbb{N} \colon \omega(n) \geqslant k\},$$

where $\omega(n)$ denotes the number of distinct prime factors of n. In [1] some inequalities were studied for infinitely many $n \in W_k$. In the next theorem we show with respect to the equation $\delta^{1,1,m}(n) = 2$ that the order m can be arbitrarily given for infinitely many $n \in W_k$. This is obtained by using Dirichlet's Theorem about primes in an arithmetic progression.

Theorem 3.1. *Let M_0 be a positive integer with $M_0 \geqslant 2$. There are infinitely many $n \in W_k$ such that for every $m \geqslant M_0$,*

$$\delta^{1,1,m}(n) = \delta^{1,1,M_0}(n). \tag{3.19}$$

P r o o f. We divide the proof into two parts:

(i) We put $m_0 = M_0 - 1$. It suffices to prove that there are infinitely many primes p such that

$$p = \delta^{1,1,m_0}(n) + 1, \tag{3.20}$$

where $n \in W_k$. Indeed, by Dirichlet's Theorem, the arithmetic progression $2^{k-1}t+1$; $t \geqslant 1$ contains infinitely many primes. We denote $\Theta = 2^{k-1}$. Let p, q_1, q_2 be distinct primes of the form $\Theta t + 1$. Then Θ divides both $p-1$ and $q_1^a q_2^b - 1$ for every $a, b \geqslant 1$. Let $(l_1, l_2, \ldots, l_k)$ be an arbitrary k-tuple of distinct primes.

In the case when $m_0 = 1$, we put

$$n = l_1 l_2 \ldots l_{k-1} \times l_k^{(p-1)/\Theta - 1} \in W_k.$$

It follows that $d(n) + 1 = p$. Thus, (3.19) is true for every $m \geqslant 2$.

In the case when $m_0 \geqslant 2$, we can put

$$\begin{cases} s_1 = \dfrac{q_1^{\Theta-1} q_2^{s_2} - 1}{\Theta} - 1, \\ s_2 = \dfrac{q_1^{\Theta-1} q_2^{s_3} - 1}{\Theta} - 1, \\ \quad \vdots \\ s_{m_0-1} = \dfrac{q_1^{\Theta-1} q_2^{s_{m_0}} - 1}{\Theta} - 1, \\ s_{m_0} = \dfrac{p-1}{\Theta} - 1. \end{cases}$$

From now on we denote $\Psi = q_1^{\Theta-1}$ and

$$\Xi = \frac{\Psi q_2^{\frac{\Psi q_2^{\cdot^{\cdot^{\cdot^{\frac{\Psi q_2^{\frac{\Psi q_2^{\frac{p-1}{\Theta}-1}-1}{\Theta}-1}-1}{\Theta}-1}}}}-1}{\Theta}-1}-1}{\Theta} - 1.$$

Let $n = l_1 l_2 \ldots l_{k-1} \times l_k^{s_1} \in W_k$, or, equivalently,

$$n = l_1 l_2 \ldots l_{k-1} l_k^{\Xi}, \tag{3.21}$$

where the first exponentiation contains $m_0 - 1$ fractions involving q_1 and q_2. This written in the form of towering storeys allows to calculate successively $\delta^{1,1,1}(n)$, $\delta^{1,1,2}(n), \ldots, \delta^{1,1,M_0}(n)$. Indeed, for each such integer n, it follows from the definition of d that

$$\delta^{1,1,1}(n) + 1 = \Xi + 1,$$

where the first exponentiation of q_2 contains $m_0 - 2$ fractions involving q_1 and q_2. By repeating these steps we can reach the top of (3.21) as follows:

$$\delta^{1,1,m_0-1}(n) + 1 = \Psi q_2^{(p-1)/\Theta - 1},$$

and so

$$\delta^{1,1,m_0}(n) + 1 = p. \tag{3.22}$$

This ends the proof of Part (i).

(ii) Now, from (3.22) we obtain $\delta^{1,1,m_0+1}(n) = \delta^{1,1,M_0}(n) = 2$, and therefore (3.19) is true for every $m \geqslant M_0$. This completes the proof of Theorem 3.1. □

Corollary 3.1. *For any positive integer m, there are infinitely many n such that $\delta^{1,1,m}(n) = 2$.*

Proof. Let $m \geqslant 1$. If $m = 1$, then $\delta^{1,1,1}(p) = d(p) = 2$ for any prime p. If $m \geqslant 2$, it follows from the proof of (3.20) that there exist infinitely many $n \in W_k$ such that $\delta^{1,1,m-1}(n) + 1$ is prime. Therefore,

$$\delta^{1,1,m}(n) = d(\delta^{1,1,m-1}(n) + 1) = 2.$$

This completes the proof. □

Corollary 3.2. *For any positive integer m, there are infinitely many n such that $\delta^{1,1,m}(n) = 3$.*

Proof. Let $m \geqslant 1$, and let p be an odd prime. Assume that $m = 1$, and put $n = p^2$. Then $\delta^{1,1,m}(n) = 3$. Assume that $m \geqslant 2$, and define the positive integer

$$n = p^{p^{p^{\cdot^{\cdot^{\cdot^{p^{p^2-2}-2}}}}-2}-2}$$

which contains m exponentiations involving the prime number p. As in the proof of Theorem 3.1, we obtain

$$\delta^{1,1,1}(n) = p^{p^{\cdot^{\cdot^{\cdot^{p^{p^2-2}-2}}}}-2} - 1,$$

which contains $(m-1)$ exponentiations involving the prime number p. In this way, we successively compute $\delta^{1,1,2}(n), \delta^{1,1,3}(n), \ldots$. At the end, we obtain

$$\begin{cases} \delta^{1,1,m-2}(n) = p^{p^2-2} - 1, \\ \delta^{1,1,m-1}(n) = p^2 - 1, \\ \quad \delta^{1,1,m}(n) = 3. \end{cases}$$

This completes the proof. $\square$

Remark 3.1. It seems very likely that Proposition 3.1 can be generalized. Computation suggests that for every $n \geqslant 1$ there exists an order m such that

$$\begin{cases} \delta^{3,1,m}(n) = 64 \text{ or } 512, \\ \delta^{9,1,m}(n) = 68719476736 \text{ or } 18014398509481984, \\ \delta^{10,1,m}(n) = 1048576 \text{ or } 61917364224. \end{cases} \tag{3.23}$$

See also Example 3.2 below.

Following the same idea as in the proof of Theorem 3.1, we want to present a property of the growth of $\delta^{1,l,m}(n)$ for an infinity of values of $n \in W_k$.

Proposition 3.2. *Let a, m_0, k, l be positive integers with a odd and 2^k dividing l. There are infinitely many $n \in W_k$ such that $d(\delta^{1,l,m_0}(n) + a + l) = 2$.*

P r o o f. Since $(\Theta, a) = 1$, by Dirichlet's Theorem there are infinitely many primes of the form $\Theta t + a$. Let b be the odd positive integer given by $b = l/\Theta + 1$, and let p be a prime number of the form $\Theta t + a$, where $(p-a)/\Theta > b$. We study the following two cases.

In the first case, we assume that $m_0 = 1$. For

$$n = l_1 l_2 \ldots l_{k-1} \times l_k^{(p-a)/\Theta - b} \in W_k,$$

where $l_1, l_2, \ldots, l_k$ are distinct primes, it follows that $d(n) = \delta^{1,l,1}(n) = p - a - l$. Thus, $d(\delta^{1,l,1}(n) + a + l) = 2$.

In the second case, we assume that $m_0 \geqslant 2$. Similarly, let q_1, q_2 be two distinct primes of the form $\Theta t + b$, since $(\Theta, b) = 1$. We denote

$$\Upsilon = \frac{\Psi q_2^{\frac{\Psi q_2^{\cdot^{\cdot^{\cdot^{\frac{\Psi q_2^{\frac{\Psi q_2^{\frac{p-a}{\Theta} - b}}{\Theta} - b}}{\Theta} - b}}}}}{\Theta} - b}}{\Theta} - b.$$

Consider the positive integer of the form

$$n = l_1 l_2 \dots l_{k-1} \times l_k^{\Upsilon},$$

where the first exponentiation contains $m_0 - 1$ fractions involving q_1 and q_2 and $l_1, l_2, \dots, l_k$ are also distinct primes. Since $l = \Theta(b-1)$, it follows that

$$\delta^{1,l,1}(n) + l = \Upsilon + b,$$

where the first exponentiation of q_2 contains $m_0 - 2$ fractions involving q_1 and q_2. Repeating the process as in the proof of Theorem 3.1 and Corollary 3.2, we obtain

$$\begin{cases} \delta^{1,l,m_0-1}(n) + l = \Psi q_2^{(p-a)/\Theta - b}, \\ \delta^{1,l,m_0}(n) + l = p - a, \\ d(\delta^{1,l,m_0}(n) + a + l) = 2. \end{cases}$$

This completes the proof. $\square$

Now, we present some examples to illustrate the results stated in Proposition 3.1, Remark 3.1 and Theorem 3.1, respectively.

3.1. Examples.

Example 3.1. With respect to the result of Proposition 3.1, the following table shows the first value of m for which $\delta^{1,1,m}(n) = 2$ or 3, for different values of n.

n	m	$\delta^{1,1,m}(n)$	n	m	$\delta^{1,1,m}(n)$	n	m	$\delta^{1,1,m}(n)$
1	2	2	$2^{2^{10}}$	3	2	3	1	2
2	1	2	$2^{2^{50}}$	4	2	3^3	2	2
2^2	1	3	$2^{2^{100}}$	4	3	3^{25}	3	2
2^{2^2}	3	2	$2^{2^{150}}$	4	2	$3^{847\,288\,609\,441}$	4	2
2^{2^3}	3	2	$2^{2^{200}}$	3	3	$3^{3^{847\,288\,609\,441}-2}$	5	2
2^{2^4}	3	2	$2^{2^{220}}$	4	3	3^2	1	3
2^{2^5}	3	2	$2^{2^{250}}$	4	2	3^7	2	3
2^{2^6}	3	3	$2^{2^{300}}$	4	3	3^{2185}	3	3
2^{2^7}	3	3	$2^{2^{350}}$	3	2	$3^{3^{2185}-2}$	4	3
2^{2^8}	3	3	$2^{2^{380}}$	3	2	$3^{3^{3^{2185}-2}-2}$	5	3

The above values are obtained by hand and also by using a program in Maple as follows: Let N be a positive integer with $N \notin \{3, 4\}$. We can verify that $\delta^{1,1,m}(N) = 2$ or 3 for some $m \geqslant 1$. In fact, we have the following algorithm.

Algorithm 3.1

$> n := N$:

$u := n$:

$k := 0$:

while $u \neq 4$ **and** $u \neq 3$ **do**

$k := k + 1$:

$u := \mathrm{tau}(u) + 1$:

od:

print("m-fold"$= k$, "δ^{1, 1, m}(n)"$= u - 1$);

Note that this program solves the equality $\delta^{1,1,m}(n) = 2$ or 3 for some $m \geqslant 1$, provided only that $n \notin \{3, 4\}$. If $n = 3$ or $n = 4$, then the program gives (m-fold, $\delta^{1,1,m}(n)) = (0, 2)$ or (m-fold, $\delta^{1,1,m}(n)) = (0, 3)$, respectively. However, we can check by hand again that $\delta^{1,1,m}(3) = 2$ and $\delta^{1,1,m}(4) = 3$ for any $m \geqslant 1$.

In the next example we give two positive integers n and n' which have the same distinct prime factors and satisfy the first equation of (3.23), and this after 4-fold iterations. That is, $\mathrm{set}(n) = \mathrm{set}(n')$, $\delta^{3,1,m}(n) = 64$ and $\delta^{3,1,m}(n') = 512$ for every $m \geqslant 4$.

Example 3.2. Let $(q_1, q_2, \ldots, q_{13})$ be an arbitrary 13-tuple of distinct primes. We put

$$n = q_1^{25} \times q_2^{22} \times q_3^{15} \times q_4^{70} \times q_5^{11} \times q_6^{15} \times q_7^{13} \times q_8 \times q_9 \times q_{10} \times q_{11} \times q_{12}^{15} \times q_{13}^{10},$$
$$n' = q_1^{2000} \times q_2^{302} \times q_3^{105} \times q_4^{700} \times q_5^{101} \times q_6^{15} \times q_7^{13} \times q_8 \times q_9 \times q_{10} \times q_{11} \times q_{12}^{1500} \times q_{13}^{999}.$$

It is clear that n and n' have the same distinct prime factors, where $|\mathrm{set}(n)| = |\mathrm{set}(n')| = 13$. By computation, we see that $\delta^{3,1,m}(n) = 64$ and $\delta^{3,1,m}(n') = 512$ for every $m \geqslant 4$:

$$\delta^{3,1,1}(n) = 135\,964\,112\,015\,285\,579\,850\,731\,807\,751\,719\,092\,224$$
$$\delta^{3,1,2}(n) = 56\,623\,104$$
$$\delta^{3,1,3}(n) = 4096$$
$$\delta^{3,1,4}(n) = 64,$$

and

$$\delta^{3,1,1}(n') = 2^{42}3^9 5^9 7^3 17^3 19^3 23^3 29^3 53^3 79^3 101^3 701^3$$
$$\delta^{3,1,2}(n') = 2097\,152$$
$$\delta^{3,1,3}(n') = 1728$$
$$\delta^{3,1,4}(n') = 512.$$

As an application of Theorem 3.1, the following example gives the smallest positive integer $x \in W_3$ in the form (3.21) such that $\delta^{1,1,4}(x) = 2$.

Example 3.3. In view of Theorem 3.1, assume that $k = 3$ and $M_0 = 4$. Then the positive integer

$$x = 5 \times 3 \times 2^{511\,528\,924\,107\,551\,574\,707\,030} \in W_3$$

is the smallest number that satisfies (3.20) and (3.21). Indeed, we see that

$$\delta^{1,1,1}(x) + 1 = 2046\,115\,696\,430\,206\,298\,828\,125 = 5^{30}13^3.$$

Therefore, we have

$$\begin{aligned} \delta^{1,1,2}(x) + 1 &= 125 = 5^3, \\ \delta^{1,1,3}(x) + 1 &= 5, \\ \delta^{1,1,4}(x) &= 2. \end{aligned}$$

On the other hand, we can write

$$x = 5 \times 3 \times 2^{\dfrac{13^{2^{3-1}-1} \times 5^{\dfrac{5^{2^{3-1}-1} \times 13^{\frac{5-1}{2^{3-1}}-1} - 1}{2^{3-1}} - 1} - 1}{2^{3-1}} - 1}.$$

Thus we have shown that x is the smallest one, since the numbers 5 and 13 are chosen to be the smallest distinct primes satisfying the properties of p, q_1 and q_2 which are stated in the proof of Theorem 3.1.

To prove our main theorem, we will make use of the following lemma.

Lemma 3.1. *Let n be an unlimited positive integer and let $\gamma \in \mathbb{R}_+^*$. If γ is standard, then*

$$\frac{d(n)}{n^\gamma} \simeq 0.$$

P r o o f. Let q be a prime number and $\alpha \geqslant 1$ such that q^α is unlimited. We see that

$$\frac{d(q^\alpha)}{q^{\alpha\gamma}} = \frac{\alpha+1}{q^{\alpha\gamma}} = \frac{\alpha+1}{q^{\alpha\gamma/2}}\frac{1}{q^{\alpha\gamma/2}} \leqslant \frac{\alpha+1}{2^{\alpha\gamma/2}}\frac{1}{q^{\alpha\gamma/2}}. \tag{3.24}$$

Observing that $\alpha\gamma$ is either standard or unlimited, we obtain from (3.24) that

$$\frac{d(q^\alpha)}{q^{\alpha\gamma}} \leqslant \begin{cases} \pounds \cdot \Phi \simeq 0, & \text{if } \alpha \text{ is limited,} \\ \Phi \cdot \Phi \simeq 0, & \text{otherwise.} \end{cases}$$

Therefore, $d(q^\alpha)/q^{\alpha\gamma} \simeq 0$. Since $t \mapsto d(t)/t^\gamma$ is multiplicative and standard, it follows from Theorem 2.3 that for every unlimited n, $d(n)/n^\gamma \simeq 0$. This proves Lemma 3.1. □

Our main result below gives an upper bound for $\delta^{s,l,m}(n)$, where $m \geqslant 2$ and l, s are limited.

Theorem 3.2 (Main Theorem)**.** *Let l, s be limited positive integers and let $n \geqslant 1$. There exists a positive constant A_n such that for every limited $m \geqslant 2$, we have*

$$\delta^{s,l,m}(n) \ll A_n.$$

Proof. We distinguish two cases.

(a) Assume that $d(n)$ is unlimited. We will show that for every limited $m \geqslant 2$, we have

$$\delta^{s,l,m}(n) \ll \delta^{s,l,1}(n) = d^s(n). \tag{3.25}$$

This means that, $A_n = d^s(n)$. First, it suffices to prove that if $\delta^{s,l,i}(n) \simeq \infty$ for some $i \geqslant 1$ (for example $i = 1$), then

$$\delta^{s,l,i+1}(n) \ll \delta^{s,l,i}(n). \tag{3.26}$$

In fact, since s is limited, it follows from Lemma 3.1 that

$$\frac{\delta^{s,l,i+1}(n)}{\delta^{s,l,i}(n) + l} = \Big(\frac{d(\delta^{s,l,i}(n) + l)}{(\delta^{s,l,i}(n) + l)^{1/s}}\Big)^s = \Big(\frac{d(t)}{t^{1/s}}\Big)^s \simeq 0,$$

where $t = \delta^{s,l,i}(n) + l \simeq \infty$ and $1/s$ is not infinitesimal. Thus,

$$\delta^{s,l,i+1}(n) \ll \delta^{s,l,i}(n) + l.$$

That is, (3.26) is satisfied because l is limited.

Next, let $m \geqslant 2$ be limited. We see that

$$\delta^{s,l,1}(n) - \delta^{s,l,m}(n) = \sum_{j=1}^{m-1} \{\delta^{s,l,j}(n) - \delta^{s,l,j+1}(n)\}.$$

In view of (3.26), $\delta^{s,l,1}(n) - \delta^{s,l,2}(n) \simeq \infty$ because $d(n) \simeq \infty$. Moreover, for every $j = 2, 3, \ldots, m-1$, the difference

$$\delta^{s,l,j}(n) - \delta^{s,l,j+1}(n)$$

is either unlimited positive, limited positive or limited negative. This proves (3.25).

(b) Assume that $d(n)$ is limited. Since l, s are limited, hence for every limited $m \geqslant 1$, so is $\delta^{s,l,m}(n)$. Let ω be an unlimited positive integer, then the result of our theorem in fact follows immediately with $A_n = \omega$.

This completes the proof of Theorem 3.2. $\square$

We close this paper by the following corollary.

Corollary 3.3. *Let l, s be limited positive integers and let $n \geqslant 1$. If $d(n)$ is unlimited, then*

$$\delta^{s,l,m}(n) \ll d^s(n) \tag{3.27}$$

holds for any $m \geqslant 2$. However, if $d(n)$ is limited, then $\delta^{s,l,m}(n)$ is also limited for any $m \geqslant 1$.

P r o o f. By (3.26), the terms of (1.4) are decreasing when they are unlimited. If we take $d(n)$ unlimited, then (3.27) holds for every $m \geqslant 2$ whenever $\delta^{s,l,m}(n)$ is limited or unlimited.

Assume that $d(n)$ is limited. Due to the second case of the proof of Theorem 3.2, it suffices to show that $\delta^{s,l,m}(n)$ is limited for every unlimited m. Assume, by way of contradiction, that there exists an unlimited positive integer m_0 such that $\delta^{s,l,m_0}(n) \simeq \infty$. Define the set

$$T' = \{t \in \{1, 2, \dots, m_0 - 1\}\colon \ \delta^{s,l,t}(n) \ < \ \delta^{s,l,m_0}(n)\}, \tag{3.28}$$

which is internal and containing $\mathbb{N}^\sigma$. In fact, since $d(n)$, s, l are limited, the number $\delta^{s,l,t}(n)$ is also limited for every $t \in \mathbb{N}^\sigma$. Therefore, $\mathbb{N}^\sigma \subset T'$. By Cauchy's principle, there exists an unlimited positive integer t_0 that satisfies (3.28), where $t_0 < m_0$ and $\delta^{s,l,t_0}(n)$ is unlimited. This is a contradiction because (3.26) gives the inequality

$$\delta^{s,l,m_0}(n) \ll \delta^{s,l,t_0}(n).$$

Thus, $\delta^{s,l,m}(n)$ is limited for any $m \geqslant 1$. This completes the proof. $\square$

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