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Solutions of the equation $d(kn) = \varphi(\varphi(n))$

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Abstract: Let $d(n)$ and $\varphi(n)$ denote the number of positive integers dividing the positive integer n and the Euler's phi function representing the numbers less than and prime to n, respectively. In this paper, we determine all solutions of the equation $d(n) = \varphi(\varphi(n))$ and we prove that the equation $d(kn) = \varphi(\varphi(n))$ has a finite number of solutions for any $k \geq 1$. Further, we characterize all solutions of the last equation when k is prime.

1 Introduction

Let $\mathbb{N}$ be the set of all positive integers and let $n \in \mathbb{N}$. Let $d(n)$ be the divisor function, which counts the number of positive divisors of n, i.e., if n has the prime factorization $n = q_1^{a_1} q_2^{a_2}...q_k^{a_k}$ with distinct primes $q_1, q_2, ..., q_k$ and positive integers $a_1, a_2, ..., a_k$, then

$$d(n) = (a_1 + 1)(a_2 + 1)...(a_k + 1).$$

Let $\varphi(n)$ be the Euler function, which counts the number of positive integers $m \leq n$ with $\gcd(m, n) = 1$. From now on $\gcd(m, n)$ will be denoted by (m, n). It is well-known that

$$\varphi(n) = q_1^{a_1-1}(q_1 - 1)q_2^{a_2-1}(q_2 - 1)...q_k^{a_k-1}(q_k - 1).$$

In our main results, we will use the following inequalities (for details, we refer the reader to [5, Problem 522] and [6, pages 110, 116, 117, 183]). At first, for all positive integers m and n we have

$$d(n) \leq 2\sqrt{n} \tag{1}$$

and

$$d(mn) \leq d(m)d(n). \tag{2}$$

Moreover, if $(m, n) > 1$, then

$$d(mn) < d(m)d(n). \tag{3}$$

If m divides n, then

$$d(n) \geq d(m). \tag{4}$$

Next, for $n \neq 2$ and $n \neq 6$, we have

$$\varphi(n) \geq \sqrt{n}. \tag{5}$$

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For all positive integers m and n, we also have

$$\varphi(mn) = \varphi(m)\varphi(n)\frac{d}{\varphi(d)}, \tag{6}$$

where $d = (m,n)$. From which it follows that

$$\varphi(m)\varphi(n) \leq \varphi(mn) \tag{7}$$

In particular, if $d > 1$ then (6) and (7) give us the inequality

$$\varphi(m)\varphi(n) < \varphi(mn). \tag{8}$$

Various diophantine equations involving the divisor function and Euler's phi function were investigated by many authors. For example, see [2], [3] and [4]. In [5, Problem 705, page 78], it is shown that $\varphi(d(n)) = d(\varphi(n))$ has infinitely many solutions; while in [6, pages 110-111], it is shown that $d(n) = \varphi(n)$ has the only solutions $1,3,8,10,24$ and 30, where $d(n) < \varphi(n)$ for $n \geq 31$. In the same context, in [7] it is shown that the equation $\varphi(n) + d(n) = n$ has the only solutions $n = 8$ and $n = 9$. Other similar problems have been discussed in publications such as Sándor [8] and [9].

The present work is a continuation of the authors' articles [1] and [3]. We first define for any positive integer k the following sets:

$$\mathbb{E}_k = \{n \in \mathbb{Z}^+ : d(kn) = \varphi(\varphi(n))\}, \tag{9}$$

$$\mathbb{L}_k = \{n \in \mathbb{Z}^+ : d(kn) < \varphi(\varphi(n))\}, \tag{10}$$

$$\mathbb{G}_k = \{n \in \mathbb{Z}^+ : d(kn) > \varphi(\varphi(n))\}. \tag{11}$$

So, the main focus of this paper is to examine the set $\mathbb{E}_k$ of solutions n of the equation $d(kn) = \varphi(\varphi(n))$ and related inequalities. In fact, we characterize all the elements of $\mathbb{E}_1$, $\mathbb{L}_1$ and $\mathbb{G}_1$, respectively. Then we deduce that the sets $\mathbb{G}_k$ and $\mathbb{E}_k$ are finite, while $\mathbb{L}_k$ is infinite. Moreover, we prove that if p is a prime number with $p \geq 23$ and $p \neq 31$, then $\mathbb{E}_p = \{11, 13, 33, 34, 35, 39, 62, 63, 76, 88, 98, 102, 104, 105, 110, 130, 154, 186, 228, 234, 264, 280, 294, 312, 330, 390, 462, 504, 540, 630, 840\}$.

2 The equation $d(n) = \varphi(\varphi(n))$

This section is devoted to investigate the elements of (9), (10) and (11) for $k = 1$.

Theorem 1. *The numbers* $1, 5, 7, 15, 22, 26, 40, 56, 66, 70, 78, 108, 120, 126, 168, 210$ *are the only solutions of the equation* $d(n) = \varphi(\varphi(n))$. *Moreover, the numbers* $2, 3, 4, 6, 8, 9, 10, 12, 14, 16, 18, 20, 24, 28, 30, 36, 42, 48, 54, 60, 72, 84, 90, 180$ *are the only solutions of the inequality* $d(n) > \varphi(\varphi(n))$.

For the proof, we start by proving the following results in which, if the solution $n = q_1^{\alpha_1} q_2^{\alpha_2} ... q_s^{\alpha_s}$ with $q_1 < q_2 < ... < q_s$ and α_i are positive integers, then $s \leq 5$ and $\alpha_i \leq 3$ for $1 \leq i \leq s$. This means that the number of solutions is finite. Let us start with square-free solutions.

2.1 Square-free solutions

Proposition 1. *The only prime numbers that satisfy the equation* $d(n) = \varphi(\varphi(n))$ *are* 5 *and* 7.

Proof. Let p be a prime number. If $p \in \mathbb{E}_1$, then $d(p) = 2 = \varphi(p-1)$. If $p - 1 \neq 2,6$, then by (5), $p \leq 5$. In this case, $p = 5$ is the only solution. If $p - 1 = 2$ or 6, it follows that $p = 3$ or 7, where $7 \in \mathbb{E}_1$ while $3 \notin \mathbb{E}_1$. Finally, we conclude that if $p \in \mathbb{E}_1$, then p is either 5 or 7.

We deduce the following corollary.

Corollary 1. *Let p be a prime number. We have:*

- *If p is either* 5 *or* 7*, then $p \in \mathbb{E}_1$.*
- *If p is either* 2 *or* 3*, then $p \in \mathbb{G}_1$.*
- *If $p \geq 11$, then $p \in \mathbb{L}_1$.*

Proof. By the same way of the proof of Proposition 1, we conclude that if $n = p$ is prime with $p \geq 11$, then the inequality $d(n) \geq \varphi(\varphi(n))$ cannot be true, and so $p \in \mathbb{L}_1$.

Proposition 2. *The only square-free solutions of the form $q_1 q_2$, where q_1 and q_2 are distinct primes, are $2 \cdot 11$, $2 \cdot 13$, $3 \cdot 5$ and $3 \cdot 7$.*

Proof. Suppose that $q_1 q_2 \in \mathbb{E}_1$, where q_1, q_2 are distinct primes with $2 \leq q_1 < q_2$. We obtain from (9) that

$$\varphi((q_1 - 1)(q_2 - 1)) = 4.$$

If $(q_1 - 1)(q_2 - 1) \neq 2, 6$, then by (5) we have $(q_1 - 1)(q_2 - 1) \leq 16$. Here, q_1 cannot be ≥ 5. There are two possibilities:
  1. If $q_1 = 2$, then q_2 must be in $\{5, 7, 11, 13, 17\}$.
  2. If $q_1 = 3$, then q_2 must be in $\{5, 7\}$.
  Thus, since $2 \cdot 5$, $2 \cdot 7$ and $2 \cdot 17$ are in $\mathbb{G}_1$, we get $(q_1, q_2) \in \{(2, 11), (2, 13), (3, 5), (3, 7)\}$.

By the same way we can prove the following proposition:

Proposition 3. *If a solution n is square-free and has 3 or 4 distinct primes, then n is one the numbers $2 \cdot 3 \cdot 11$, $2 \cdot 5 \cdot 7$, $2 \cdot 3 \cdot 13$ and $2 \cdot 3 \cdot 5 \cdot 7$.*

Proposition 4. *If a solution n is square-free and has more than 4 distinct primes, then $n \in \mathbb{L}_1$.*

For the proof we need the following lemma:

Lemma 1. *Let $k \geq 6$ and let $q_1, q_2, ..., q_k$ be distinct primes. Then*

$$(q_1 - 1)(q_2 - 1)...(q_k - 1) > 2^{2k}.$$

Proof. The proof holds by induction on k, since $(q_1 - 1)(q_2 - 1)...(q_6 - 1) \geq 2^{2 \cdot 6}$ for every 6-tuple $(q_1, q_2, ..., q_6)$ of distinct primes.

Proof(Proposition 4). Let $n = q_1 q_2 ... q_k$, where $q_1, q_2, ..., q_k$ are distinct primes. We assume further that $n \in \mathbb{E}_1 \cup \mathbb{G}_1$. In the case when $k = 5$. We see that

$$32 = d(q_1 q_2 ... q_5) \geq \varphi((q_1 - 1)(q_2 - 1)...(q_5 - 1)).$$

By (5), we obtain

$$\varphi((q_1 - 1)(q_2 - 1)...(q_5 - 1)) \geq \sqrt{(q_1 - 1)(q_2 - 1)...(q_5 - 1)},$$

and hence $(q_1 - 1)(q_2 - 1)...(q_5 - 1) \leq 1024$. Since $q_1 < q_2 < ... < q_5$, we distinguish the following cases:

• $q_1 = 2$ and (q_2, q_3, q_4, q_5) must belong to the set

$$\{(3, 5, 7, 11), (3, 5, 7, 13), (3, 5, 7, 17), (3, 5, 7, 19), (3, 5, 11, 13)\}.$$

But, the corresponding numbers $n = q_1 q_2 ... q_5$ belong to $\mathbb{L}_1$, which is impossible since $n \in \mathbb{E}_1 \cup \mathbb{G}_1$.
• $q_1 \geq 3$. We get $(q_1 - 1)(q_2 - 1)...(q_5 - 1) \geq 2 \cdot 4 \cdot 6 \cdot 10 \cdot 12 > 1024$, which is impossible as well. Thus, q_1 cannot be greater than 3.

In the case when $k \geq 6$, we also see that

$$2^k = d(q_1 q_2 ... q_k) \geq \varphi((q_1 - 1)(q_2 - 1)...(q_k - 1)),$$

and by (5) we have

$$\varphi((q_1 - 1)(q_2 - 1)...(q_k - 1)) \geq \sqrt{(q_1 - 1)(q_2 - 1)...(q_k - 1)}.$$

It follows that

$$2^{2k} \geq (q_1 - 1)(q_2 - 1)...(q_k - 1).$$

This contradicts Lemma 1. Hence, $d(n) < \varphi(\varphi(n))$.

2.2 Nonsquare-free odd solutions

Proposition 5. *Let $s \geq 2$ and let $n_1, n_2, ..., n_s$ be relatively prime positive integers with $n_i \geq 3$ for $i = 1, 2, ..., s$. If $n_1, n_2, ..., n_s \in \mathbb{E}_1 \cup \mathbb{L}_1$, then $n_1 n_2 ... n_s \in \mathbb{L}_1$.*

Proof. Let $n_1, n_2, ..., n_s$ be as above and assume that $n_1, n_2, ..., n_s \in \mathbb{E}_1 \cup \mathbb{L}_1$. Since $(\varphi(n_i), \varphi(n_j)) > 1$ for $1 \leq i, j \leq s$, it follows from (8) and the multiplicativity of d and φ that

$$d(n_1 n_2 ... n_s) = d(n_1) d(n_2) ... d(n_s) \leq \varphi(\varphi(n_1)) \varphi(\varphi(n_2)) ... \varphi(\varphi(n_s))$$
$$< \varphi(\varphi(n_1) \varphi(n_2) ... \varphi(n_s)) = \varphi(\varphi(n_1 n_2 ... n_s)).$$

Hence, $n_1 n_2 ... n_s \in \mathbb{L}_1$.

Proposition 6. *Let $\alpha \geq 2$ and let $n = q^\alpha$ be a prime power with q is odd. Then $n \in \mathbb{L}_1$ except for $n = 3^2 \in \mathbb{G}_1$.*

For the proof we need the following lemma.

Lemma 2. *Let n be a positive integer with $n \geq 3$ and let q be an odd prime number. Then*

$$(n+1)^2 < q^{n-1}(q-1).$$

Proof. This follows immediately using mathematical induction.

Proof(Proposition 6). Assume by way of contradiction that $d(n) \geq \varphi(\varphi(n))$. Therefore, $\alpha + 1 \geq \varphi(q^{\alpha-1}(q-1))$. Also, by (5), we get

$$\varphi(q^{\alpha-1}(q-1)) \geq \sqrt{q^{\alpha-1}(q-1)},$$

and so $(\alpha+1)^2 \geq q^{\alpha-1}(q-1)$, which is impossible by Lemma 2 except for $(q, \alpha) = (3, 2)$.

Proposition 7. *Let $s \geq 2$ and let $n = q_1^{\alpha_1} q_2^{\alpha_2} ... q_s^{\alpha_s}$ be an odd number with $\max(\alpha_1, \alpha_2, ..., \alpha_s) \geq 2$. Then $n \in \mathbb{L}_1$.*

Proof. We distinguish three cases:

    Case 1. $s = 2$. There are two subcases:
    Subcase 1.1. $q_1 = 3$. This means that $n = 3^{\alpha_1} q^{\alpha_2}$, and we have three possibilities:

- $\alpha_1 = 1$, so α_2 must be > 1 because $q_2 \geq 5$. Thus, it is clear that

$$2^2 (\alpha_2 + 1)^2 < 2 q_2^{\alpha_2 - 1}(q_2 - 1).$$

If we assume that $n \in \mathbb{E}_1 \cup \mathbb{G}_1$, this means that $2(\alpha_2 + 1) \geq \varphi(2 q_2^{\alpha_2 - 1}(q_2 - 1))$, and by (5) we obtain

$$\varphi(2 q_2^{\alpha_2 - 1}(q_2 - 1)) \geq \sqrt{2 q_2^{\alpha_2 - 1}(q_2 - 1)}.$$

Thus, $2^2(\alpha_2 + 1)^2 \geq 2 q_2^{\alpha_2 - 1}(q_2 - 1)$. This is a contradiction. Hence, $n \in \mathbb{L}_1$.

- $\alpha_1 = 2$. Since $q_2 \geq 5$, we conclude that $3^2(\alpha_2 + 1)^2 < 2 \cdot 3 \cdot q_2^{\alpha_2 - 1}(q_2 - 1)$. Hence, $n \in \mathbb{L}_1$.
- $\alpha_1 > 2$. By Corollary 1 and Proposition 6, $q_i^{\alpha_i} \in \mathbb{E}_1 \cup \mathbb{L}_1$ for $1 \leq i \leq s$, so by Proposition 5, $n \in \mathbb{L}_1$.

    Subcase 1.2. $q_1 > 3$. By Corollary 1 and Proposition 6, $q_i^{\alpha_i} \in \mathbb{E}_1 \cup \mathbb{L}_1$ for $1 \leq i \leq s$, so it is clear by Proposition 5 that $n \in \mathbb{L}_1$.

    Case 2. $s = 3$. Here, we also have two subcases:
    Subcase 2.1. $q_1 = 3$. We also distinguish three possibilities:

- $\alpha_1 = 1$. That is, $\alpha_2 > 1$ or $\alpha_3 > 1$. Then we can easily check that

$$2^2(\alpha_2 + 1)^2(\alpha_3 + 1)^2 < 2(q_2 - 1) q_2^{\alpha_2 - 1}(q_3 - 1) q_3^{\alpha_3 - 1},$$

and hence $n \in \mathbb{L}_1$.

- $\alpha_1 = 2$. By the same way we find $n \in \mathbb{L}_1$.
- $\alpha_1 > 2$. By Corollary 1 and Proposition 6, $q_i^{\alpha_i} \in \mathbb{E}_1 \cup \mathbb{L}_1$ for $1 \leq i \leq s$. So by Proposition 5, $n \in \mathbb{L}_1$.

    Subcase 2.2. $q_1 > 3$. By Corollary 1 and Proposition 6, $q_i^{\alpha_i} \in \mathbb{E}_1 \cup \mathbb{L}_1$ for $1 \leq i \leq s$. So by Proposition 5, $n \in \mathbb{L}_1$.
    Case 3. $s \geq 4$. Here, we distinguish two subcases:
    Subcase 3.1. $q_1 = 3$. By induction on s we can easily prove the following inequality

$$(\alpha_1 + 1)^2(\alpha_2 + 1)^2 ... (\alpha_s + 1)^2 < 2 \cdot 3^{\alpha_1 - 1}(q_2 - 1) q_2^{\alpha_2 - 1} ... (q_s - 1) q_s^{\alpha_s - 1},$$

which gives $n \in \mathbb{L}_1$.
    Subcase 3.2. $q_1 > 3$. By Corollary 1 and Proposition 6, $q_i^{\alpha_i} \in \mathbb{E}_1 \cup \mathbb{L}_1$ for $1 \leq i \leq s$. Similarly, by Proposition 5, $n \in \mathbb{L}_1$.

2.3 Nonsquare-free even solutions

In this subsection, we can prove the following proposition as those appearing with the case when *n* is nonsquare-free odd.

Proposition 8. *Let $n = 2^{\alpha_1} q_2^{\alpha_2} ... q_s^{\alpha_s}$ be an even number such that $\max(\alpha_1, \alpha_2, ..., \alpha_s) \geq 2$. There are four possibilities:*

- $s = 1$. *Here n is a prime power, where $n = 2^{\alpha_1}$ with $\alpha_1 \geq 2$. We have:*
  - *If n is either 2^2 or 2^3 or 2^4, then $n \in \mathbb{G}_1$.*
  - *If $\alpha_1 \geq 5$, then $n \in \mathbb{L}_1$.*
- $s = 2$. *We have:*
  - *If $n = 2^2 \cdot 3, 2 \cdot 3^2, 2^2 \cdot 5, 2^3 \cdot 3, 2^2 \cdot 7, 2^2 \cdot 3^2, 2^4 \cdot 3, 2 \cdot 3^3, 2^3 \cdot 3^2$, then $n \in \mathbb{G}_1$.*
  - *If $n = 2^3 \cdot 5, 2^3 \cdot 7, 2^2 \cdot 3^3$, then $n \in \mathbb{E}_1$.*
  - *If n is different from the previous mentioned numbers, then $n \in \mathbb{L}_1$.*
- $s = 3$. *We have:*
  - *If $n = 2^2 \cdot 3 \cdot 5, 2^2 \cdot 3 \cdot 7, 2 \cdot 3^2 \cdot 5, 2^2 \cdot 3^2 \cdot 5$, then $n \in \mathbb{G}_1$.*
  - *If $n = 2^3 \cdot 3 \cdot 5, 2 \cdot 3^2 \cdot 7, 2^3 \cdot 3 \cdot 7$, then $n \in \mathbb{E}_1$.*
  - *If n is different from the previous mentioned numbers, then $n \in \mathbb{L}_1$.*
- $s \geq 4$. *Here, we have $n \in \mathbb{L}_1$.*

Now, we are ready to prove Theorem 1.

Proof(Theorem 1). Clearly, $1 \in E_1$. Assume that $n = q_1^{\alpha_1} q_2^{\alpha_2} ... q_s^{\alpha_s}$ with $n \neq$ 1, 2, 3, 4, 5, 6, 7 , 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 26 , 28, 30, 36, 40, 42, 48, 54, 56, 60, 66, 70, 72, 78, 84, 90, 108, 120, 126, 168, 180 and 210, where $q_1 < q_2 < ... < q_s$ are primes and $\alpha_1, \alpha_2, ..., \alpha_s$ are positive integers. There are five cases to consider:
  Case 1. $s = 1$. There are two possibilities:

- $\alpha_1 = 1$. Since $n \neq 2, 3, 5, 7$, it follows from Corollary 1 that $n \in \mathbb{L}_1$.
- $\alpha_1 \geq 2$. Since $n \neq 2^2, 2^3, 3^2, 2^4$, it follows from Propositions 6 and 8 that $n \in \mathbb{L}_1$.

  Case 2. $s = 2$. There are two possibilities:

- $\alpha_1 = \alpha_2 = 1$. Since $n \neq 10, 14, 15, 21, 22, 26, 34$, by Proposition 2 we conclude that $n \in \mathbb{L}_1$.
- α_1 or $\alpha_2 \geq 2$. Since $n \neq 12, 18, 20, 24, 28, 36, 40, 48, 54, 56, 72, 108$, it follows from Proposition 7 and Proposition 8 that $n \in \mathbb{L}_1$.

  Case 3. $s = 3$.

- *n* is square-free. Since $n \neq 30, 42, 66, 70, 78$, by Propositions 3 and 4 we have $n \in \mathbb{L}_1$.
- *n* is not square-free. Since $n \neq 60, 84, 90, 120, 126, 168, 180$, it follows from Propositions 7 and 8 that $n \in \mathbb{L}_1$.

  Case 4. $s = 4$.

- *n* is square-free. Since $n \neq 210$, it follows from Propositions 3 and 4 that $n \in \mathbb{L}_1$.
- *n* is not square-free. It follows from Propositions 7 and 8 that $n \in \mathbb{L}_1$.

  Case 5. $s \geq 5$. By Propositions 4, 7 and 8 we have $n \in \mathbb{L}_1$.
  The proof is finished.

3 On the equation $d(k \cdot n) = \varphi(\varphi(n))$ with $k \geq 2$

Proposition 9. *Let $s, k \geq 2$ and let $n_1, n_2, ..., n_s$ be relatively prime positive integers with $n_i \geq 3$ for $i = 1, 2, ..., s$. If $n_1, n_2, ..., n_s \in \mathbb{E}_k \cup \mathbb{L}_k$, then $n_1 n_2 ... n_s \in \mathbb{L}_k$.*

Proof. This is similar to the proof of Proposition 5.

Theorem 2. *Let $k \geq 2$. The sets $\mathbb{G}_k$ and $\mathbb{E}_k$ are finite, while $\mathbb{L}_k$ is infinite.*

Proof. At first, we prove that for any prime p there exists an exponent α_0 such that $p^x \in \mathbb{L}_k$ for every $x \geq \alpha_0$. Let $n = p^\alpha$ be a prime power such that $p^\alpha \in \mathbb{E}_k \cup \mathbb{G}_k$. Then

$$d(k \cdot p^\alpha) \geq \varphi(p^{\alpha-1}(p-1)).$$

Put $k = p^a m$, where $a \geq 0$, $m \geq 1$ and $(p,m) = 1$. Then

$$d(m)(a + \alpha + 1) \geq p^{\alpha-2}(p-1)\varphi(p-1)$$

and by (1) and (5) we have

$$2\sqrt{m}(a + \alpha + 1) \geq d(m)(a + \alpha + 1) \geq p^{\alpha-2}(p-1)\varphi(p-1) \geq p^{\alpha-2}(p-1)\sqrt{p-1}. \tag{12}$$

Therefore,

$$4m(a + \alpha + 1)^2 \geq p^{2\alpha-4}(p-1)^3.$$

Since a and m are fixed and α is the exponent of p, the last inequality has only finite number of solutions. Thus, the set $\mathbb{E}_k \cup \mathbb{G}_k$ contains only finitely many prime powers, namely $l_1^{\gamma_1}, l_2^{\gamma_2}, ..., l_s^{\gamma_s}$. Consequently, it suffices to choose $\alpha_0 = \max(\gamma_1 + 1, \gamma_2 + 1, ..., \gamma_s + 1)$.

Now, let $n = q_1 q_2 ... q_m$, where $q_1 < q_2 < ... < q_m$ are primes ($m \geq 2$). Assume that $n \in \mathbb{E}_k \cup \mathbb{G}_k$. Therefore,

$$d(k \cdot q_1 q_2 ... q_m) \geq \varphi(\varphi(q_1 q_2 ... q_m)).$$

By (1), (2) and (5) we obtain

$$(2\sqrt{k})(2^{m+1}) \geq d(k)d(q_1...q_m) \geq d(k \cdot q_1...q_m) \geq \varphi(\varphi(q_1...q_m)) \geq \sqrt{(q_1-1)...(q_m-1)},$$

from which it follows that

$$2^{2m+2}4k \geq (q_1-1)(q_2-1)...(q_m-1). \tag{13}$$

Since k is fixed and $(q_1-1)(q_2-1)...(q_m-1)$ can be sufficiently large, we deduce that there exists a positive integer m_0 for which the inequality (13) is not true for every $m \geq m_0$. Thus the inequality $d(k \cdot n) \geq \varphi(\varphi(n))$ holds for finitely many square-free integers.

Finally, let $n = q_1^{\alpha_1} q_2^{\alpha_2} ... q_m^{\alpha_m}$, where $q_1 < q_2 < ... < q_m$ are prime numbers and $\alpha_1, \alpha_2, ..., \alpha_m$ are positive integers. Note that the number m cannot be sufficiently large as we wish such that $d(k \cdot s) \geq \varphi(\varphi(s))$ for $s = q_1 q_2 ... q_m$. Moreover, from above there exist positive integers $\alpha_1^{(0)}, \alpha_2^{(0)}, ..., \alpha_m^{(0)}$ such that the numbers $q_1^{\alpha_1'}, q_2^{\alpha_2'}, ..., q_m^{\alpha_m'}$ satisfy the inequality

$$d\left(k \cdot q_i^{\alpha_i'}\right) < \varphi\left(\varphi\left(q_i^{\alpha_i'}\right)\right)$$

for every $\alpha_i' \geq \alpha_i^{(0)}$ ($1 \leq i \leq m$). Applying Proposition 9, the numbers $n' = q_1^{\alpha_1'} q_2^{\alpha_2'} ... q_m^{\alpha_m'}$ with $\alpha_i' \geq \alpha_i^{(0)}$ ($1 \leq i \leq m$) satisfy the inequality $d(k \cdot n') < \varphi(\varphi(n'))$. Thus, the inequality $d(k \cdot n) \geq \varphi(\varphi(n))$ has only a finite number of solutions, while the inequality $d(n) < \varphi(\varphi(n))$ has infinitely many solutions.

Proposition 10. *We have* $\underset{k \geq 1}{\cap} \mathbb{G}_k = \mathbb{G}_1$.

Proof. It suffices to prove that $\mathbb{G}_1 \subset \underset{k \geq 1}{\cap} \mathbb{G}_k$. As we have already seen

$$\mathbb{G}_1 = \{2, 3, 4, 6, 8, 9, 10, 12, 14, 16, 18, 20, 24, 28, 30, 36, 42, 48, 54, 60, 72, 84, 90, 180\}.$$

Then for every $k \geq 1$ and x in $\mathbb{G}_1$ we see that $d(kx) > \varphi(\varphi(x))$. This proves $\mathbb{G}_1 \subset \mathbb{G}_k$. This completes the proof.

Proposition 11. *Let* $r, s \geq 2$. *If* r *divides* s, *then* $\mathbb{G}_r \subset \mathbb{G}_s$.

Proof. Let $n \in \mathbb{G}_r$. Since r divides s, we see by (4) that

$$d(s \cdot n) \geq d(r \cdot n) > \varphi(\varphi(n)),$$

and so $n \in \mathbb{G}_s$, as required.

Theorem 3. *Let p be a prime number with $p \geq 23$ and $p \neq 31$. Then $\mathbb{E}_p = \{11, 13, 33, 34, 35, 39, 62, 63, 76, 88, 98, 102, 104, 105, 110, 130, 154, 186, 228, 234, 264, 280, 294, 312, 330, 390, 462, 504, 540, 630, 840\}$.*

Proof. Let $p \geq 23$ be prime with $p \neq 31$. Clearly, if n is one of the above numbers, then we can easily check that $d(p \cdot n) = 2d(n) = \varphi(\varphi(n))$. Then the numbers mentioned in this theorem are part of the solution set.

Now, let $n \geq 1$ such that $n \in \mathbb{E}_p \cup \mathbb{G}_p$. It follows that

$$2d(n) \geq d(p \cdot n) \geq \varphi(\varphi(n)).$$

As in the proof of Theorem 1, we can show that the only solutions of $2d(n) \geq \varphi(\varphi(n))$ are $n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 24, 26, 27, 28, 30, 32, 33, 34, 35, 36, 38, 39, 40, 42, 44, 45, 48, 50, 52, 54, 56, 60, 62, 63, 66, 70, 72, 76, 78, 80, 84, 88, 90, 96, 98, 100, 102, 104, 105, 108, 110, 112, 114, 120, 126, 130, 132, 140, 144, 150, 154, 156, 162, 168, 180, 186, 198, 210, 216, 228, 234, 240, 252, 264, 270, 280, 288, 294, 300, 312, 330, 336, 360, 390, 396, 420, 450, 462, 504, 540, 630$ and 840. In addition, it is easy to check that the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 17, 18, 19, 20, 21, 22, 24, 26, 27, 28, 30, 32, 36, 38, 40, 42, 44, 45, 48, 50, 52, 54, 56, 60, 66, 70, 72, 78, 80, 84, 90, 96, 100, 108, 112, 114, 120, 126, 132, 140, 144, 150, 156, 162, 168, 180, 198, 210, 216, 240, 252, 270, 288, 300, 336, 360, 396, 420$ and 450 satisfy the inequality $d(p \cdot n) > \varphi(\varphi(n))$. Thus, the numbers quoted in the text of the present theorem are the only solutions of the equation $d(p \cdot n) = \varphi(\varphi(n))$, where $p \geq 23$ is prime with $p \neq 31$. This completes the proof.

Remark. By the same argument as above and by a brute force search with Maple in the range $1 \leq n \leq 10^{10}$, one can show that

- $\mathbb{E}_2 = \{11, 13, 33, 35, 38, 39, 44, 52, 63, 105, 114, 132, 140, 156, 252, 270, 420\}$.
- $\mathbb{E}_3 = \{11, 13, 34, 35, 45, 62, 76, 88, 98, 104, 110, 114, 130, 154, 198, 252, 280, 360\}$.
- $\mathbb{E}_5 = \{11, 13, 33, 34, 39, 50, 62, 63, 76, 88, 98, 102, 104, 150, 154, 186, 228, 234, 264, 270, 294, 312, 462, 504\}$.
- $\mathbb{E}_7 = \{11, 13, 33, 34, 39, 62, 76, 88, 102, 104, 110, 130, 186, 228, 234, 264, 312, 330, 390, 540\}$.
- $\mathbb{E}_{11} = \{13, 34, 35, 39, 62, 63, 76, 98, 102, 104, 105, 130, 186, 228, 234, 280, 294, 312, 390, 504, 540, 630, 840\}$.
- $\mathbb{E}_{13} = \{11, 33, 34, 35, 62, 63, 76, 88, 98, 102, 105, 110, 154, 186, 228, 264, 280, 294, 330, 462, 504, 540, 630, 840\}$.
- $\mathbb{E}_{17} = \{11, 13, 33, 35, 39, 62, 63, 76, 88, 98, 104, 105, 110, 130, 154, 186, 228, 234, 264, 280, 294, 312, 330, 390, 462, 504, 540, 630, 840\}$.
- $\mathbb{E}_{19} = \{11, 13, 33, 34, 35, 38, 39, 62, 63, 88, 98, 102, 104, 105, 110, 114, 130, 154, 186, 234, 264, 280, 294, 312, 330, 390, 462, 504, 540, 630, 840\}$.
- $\mathbb{E}_{31} = \{11, 13, 33, 34, 35, 39, 63, 76, 88, 98, 102, 104, 105, 110, 130, 154, 228, 234, 264, 280, 294, 312, 330, 390, 462, 504, 540, 630, 840\}$.

Moreover, from the proof of Theorem 3, for any prime $p \geq 23$ with $p \neq 31$ we deduce that $\mathbb{G}_p = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 17, 18, 19, 20, 21, 22, 24, 26, 27, 28, 30, 32, 36, 38, 40, 42, 44, 45, 48, 50, 52, 54, 56, 60, 66, 70, 72, 78, 80, 84, 90, 96, 100, 108, 112, 114, 120, 126, 132, 140, 144, 150, 156, 162, 168, 180, 198, 210, 216, 240, 252, 270, 288, 300, 336, 360, 396, 420, 450\}$. Thus, if I is a finite subset of positive integers, say $I = \{1, 2, ..., N\}$ with $N \geq 100$, then $\mathbb{L}_p \cap I = I - \mathbb{E}_p - \mathbb{G}_p = \{17, 19, 23, 25, 29, 31, 37, 41, 43, 46, 47, 49, 51, 53, 55, 57, 58, 59, 61, 64, 65, 67, 68, 69, 71, 73, 74, 75, 77, 79, 81, 82, 83, 85, 86, 87, 89, 91, 92, 93, 94, 95, 97, 99, ...\}$.

4 Conclusion

In the previous sections, we investigated the solutions n of the equation (*i*) $d(kn) = \varphi(\varphi(n))$ and also the respective solutions of the corresponding inequalities (*ii*) $d(kn) < \varphi(\varphi(n))$ and (*iii*) $d(kn) > \varphi(\varphi(n))$. Since the positive integers are naturally partitioned into 3 subsets according to (*i*), (*ii*) and (*iii*), we have also characterized the elements of the sets $\mathbb{E}_1, \mathbb{L}_1, \mathbb{G}_1, \mathbb{E}_p, \mathbb{L}_p$ and $\mathbb{G}_p$ whenever p is prime. As a conclusion, we gave the relation between the sizes of these sets, where $\cap \mathbb{E}_p = \{\emptyset\}$ as p runs through the sequence of all primes. The same technique is applied to characterize the elements $\mathbb{E}_k, \mathbb{L}_k$ and $\mathbb{G}_k$ whenever k is composite.

Declarations

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