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Some representations of unlimited natural numbers

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Abstract. Based on the authors' article [5] and the work of Hrbáček [11], we prove that every unlimited natural number ω is of the form $\omega = \omega_1 \cdot \omega_2 + \omega_3 \cdot \omega_4$ in at least k different ways ($k \geq 1$ is limited), where $\omega_i \in \mathbb{N}$ is unlimited and ω_i/ω_j is appreciable for $1 \leq i, j \leq 4$. Other similar representations of unlimited natural numbers are also presented.

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1. Introduction

The study of which integers are represented by a given quadratic form is one of the most celebrated in the theory of numbers. In Guy [10, D4, p. 229], Waring's problem is that of representation of positive integers as a sum of a fixed number s of nonnegative k-th powers, i.e., whether for a given k there is any fixed $s = s(k)$ such that

$$n = x_1^k + x_2^k + \cdots + x_s^k$$

is solvable for any n. In 1640, Fermat stated his conjecture that every prime number $p \equiv 1 (\mathrm{mod}\ 4)$ can be written in the form $p = x^2 + y^2$. A century later, Euler proved Fermat's conjecture and worked seriously on related problems and generalizations. In 1770, Lagrange and Euler (see, e.g., Adler [1, Theorem 8.22, p. 234]) proved that every positive integer is a sum of four squares. In 1798, Legendre and Gauss ([1, Theorem 8.25, p. 236]) classified the integers that could be represented as a sum of three squares. More precisely, they proved that a positive integer can be represented as a sum of three squares if and only if it is not of the form $4^m(8k + 7)$. This result is deeper and more difficult than either of the two-square or four-square theorems. Motivated by Lagrange's result, it is natural to ask about the collection of quadratic forms that represent all positive integers, or more generally, to fix in advance a collection S of integers, and ask about quadratic forms that represent all numbers in S. In this context, Iwaniec [12] considered a more general problem of the number of representations of an integer n by a positive definite quadratic form $Q(x_1, \ldots, x_s)$.

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For example, in [1, p. 259], it is shown that each nonnegative integer is either of the form $x^2+y^2+z^2$ or of the form $x^2+y^2+2z^2$, where x, y and z are positive integers.

In the context of nonstandard analysis [6], we shall need the following definition and principle which are used throughout this paper.

Definition 1. *Two positive real numbers x and y are of the same order, written $x \sim y$, if x/y is appreciable. Or, equivalently, there exist standard real numbers $r_1, r_2 \in \mathbb{R}_+$ such that $r_1 < x/y < r_2$.*

Principle 1. [Cauchy's principle [6, p. 19]] *No external set is internal.*

For details about internal and external sets, one can see [3, definitions 2.2, 2.3] and [6, pp. 5,6]. Furthermore, we explain here how to apply this principle. Let ω be unlimited. The set $\{n \in \mathbb{N} : \ \omega > n\}$ is internal and contains all limited positive integers. By Cauchy's principle, $\omega > n_0$ for some unlimited positive integer n_0.

As a continuation of our previous works [3, 4, 5] and Hrbáček's work [11], we prove in the present paper that every unlimited positive integer n can be written in the form:

$$\begin{cases} n = \omega_1 \cdot \omega_2 + \omega_3 \cdot \omega_4 \\ \omega_i \sim \omega_j \text{ for } 1 \leq i, j \leq 4, \end{cases} \tag{A_2}$$

where $\omega_i \in \mathbb{N}$ for $1 \leq i \leq 4$. Note that the second condition of (A_2) implies that each ω_i is unlimited. As a consequence, if $k \geq 2$ is a limited positive integer, then we can generalize the above form as follows:

$$\begin{cases} n = \omega_1 \cdot \omega_2 + \omega_3 \cdot \omega_4 + \cdots + \omega_{2k-1} \cdot \omega_{2k}, \\ \\ \omega_i \sim \omega_j \text{ for } 1 \leq i, j \leq 2k. \end{cases} \tag{A_k}$$

Moreover, we present some families of unlimited positive integers which can be represented as in (A_2) by giving the values of ω_i $(1 \leq i \leq 4)$ in terms of n. Other similar types of representation of unlimited natural numbers are also discussed.

To start with our main results, we need the following lemmas:

Lemma 1. *Let $a, b, c, d \in \mathbb{R}_+$.*

(1) $a \sim a$. If $a \sim b$, then $b \sim a$. If $a \sim b$ and $b \sim c$, then $a \sim c$.

(2) If $a \sim b$ and $r, s \in \mathbb{R}^+$ are appreciable, then $r \cdot a \sim s \cdot b$.

(3) If $a \sim c$ and $b \sim d$, then $a + b \sim c + d$.

(4) If $a \sim c$ and $b \sim d$, then $a \cdot b \sim c \cdot d$.

(5) If $a \sim b$ and $n \in \mathbb{N}^+$ is standard, then $a^n \sim b^n$ and $\sqrt[n]{a} \sim \sqrt[n]{b}$.

Proof. Proof of *(3)*. We have $r_1 \cdot c < a < r_2 \cdot c$ and $s_1 \cdot d < b < s_2 \cdot d$ for some standard $r_1, r_2, s_1, s_2 \in \mathbb{R}^+$. Hence $u_1 \cdot (c+d) \leq r_1 \cdot c + s_1 \cdot d < a + b < r_2 \cdot c + s_2 \cdot d \leq u_2 \cdot (c+d)$ for $u_1 = \min\{r_1, s_1\}$ and $u_2 = \max\{r_2, s_2\}$. □

To state the second lemma, we need the result known as Bertrand's postulate: For every $n \in \mathbb{N}$, $n \geq 2$, there is a prime p such that $n < p < 2n$.

Lemma 2. *For every $x \in \mathbb{R}$, $x \geq 2$, there is a prime p such that $x < p < 2x$.*

Proof. Recall that $[x]$ denotes the integer part of the real number x. There is a prime p such that $[x] < p < 2\,[x]$. Then $x < [x] + 1 \leq p < 2\,[x] \leq 2x$. □

2. Unlimited integers of the form $\omega_1 \cdot \omega_2 + \omega_3 \cdot \omega_4$

One of the main results is the following:

Theorem 1. *Every unlimited $\omega \in \mathbb{N}$ can be written in the form $\omega_1 \cdot \omega_2 + \omega_3 \cdot \omega_4$, where $\omega_i \sim \sqrt{\omega}$ and $\omega_i > 0$ for $1 \leq i \leq 4$.*

Proof. By Bertrand's postulate, there is a prime number p_1 such that $\frac{\sqrt{\omega}}{2} < p_1 < \sqrt{\omega}$ and a prime number p_2 such that $\frac{\sqrt{\omega}}{4} < p_2 < \frac{\sqrt{\omega}}{2}$.

The Diophantine equation $p_1 \cdot x + p_2 \cdot y = \omega$ has a particular solution x_0, y_0 in integers (Euclid's algorithm) since $\gcd(p_1, p_2) = 1$. Moreover, all solutions are given by $x_t = x_0 + t \cdot p_2$ and $y_t = y_0 - t \cdot p_1$, where t is an arbitrary integer. Now, we can choose t so that

$$\frac{\sqrt{\omega}}{4} < x_t < \frac{3\sqrt{\omega}}{4}. \tag{1}$$

In fact, let t^* be the largest integer for which $x_{t^*} \leq \sqrt{\omega}/4$. Then clearly $x_{t^*+1} > \sqrt{\omega}/4$ and since $x_{t^*+1} = x_{t^*} + p_2$, it follows that

$$x_{t^*+1} - \frac{\sqrt{\omega}}{4} \leq x_{t^*+1} - x_{t^*} = p_2 < \frac{\sqrt{\omega}}{2},$$

and so $x_{t^*+1} < \frac{\sqrt{\omega}}{4} + \frac{\sqrt{\omega}}{2} = \frac{3\sqrt{\omega}}{4}$. Thus, we let $t = t^* + 1$. This proves (1). For this t we get $\frac{\omega}{8} < p_1 \cdot x_t < \frac{3\omega}{4}$ and hence $\omega/4 < p_2 \cdot y_t = \omega - p_1 \cdot x_t < 7\omega/8$. It follows that $\frac{\sqrt{\omega}}{2} < y_t < \frac{7\sqrt{\omega}}{2}$. We let $\omega_1 = p_1$, $\omega_2 = x_t$, $\omega_3 = p_2$ and $\omega_4 = y_t$. This completes the proof. □

We now consider the basic question: Can every unlimited natural number n be represented in the form $n = \omega_1 \cdot \omega_2 + \omega_3 \cdot \omega_4$, where $\omega_i \sim \omega_j$ holds for all $1 \leq i, j \leq 4$ in at least k different ways ($k \geq 1$ limited)? For the answer, fix a standard k. By Bertrand's postulate, there is a prime number p_1 such that $\frac{\sqrt{\omega}}{2k} < p_1 < \frac{\sqrt{\omega}}{k}$ and a prime number p_2 such that $\frac{\sqrt{\omega}}{4k} < p_2 < \frac{\sqrt{\omega}}{2k}$, so $p_1 \sim \sqrt{\omega}$ and $p_2 \sim \sqrt{\omega}$. The Diophantine equation $p_1 \cdot x + p_2 \cdot y = \omega$ has a solution x_0, y_0 in integers. Moreover, every solution is of the form $x_t = x_0 + t \cdot p_2$, $y_t = y_0 - t \cdot p_1$ for some $t \in \mathbb{Z}$. We can now choose t so that $\frac{\sqrt{\omega}}{4k} < x_t < \frac{3\sqrt{\omega}}{4k}$, so $x_t \sim \sqrt{\omega}$. For this t we get $\frac{\omega}{8k^2} < p_1 \cdot x_t < \frac{3\omega}{4k^2}$ and hence

$$\frac{\left(4k^2 - 3\right)\omega}{4k^2} < p_2 \cdot y_t = \omega - p_1 \cdot x_t < \frac{\left(8k^2 - 1\right)\omega}{8k^2}.$$

It follows that $\frac{\left(4k^2-3\right)\sqrt{\omega}}{2k} < y_t < \frac{\left(8k^2-1\right)\sqrt{\omega}}{2k}$. Different values of k give different values of the quadruple p_1, p_2, x_t, y_t.

Proposition 1. *Let $k \geq 1$ be limited. Every unlimited positive integer ω can be represented as $\omega = \omega_1 \cdot \omega_2 + \omega_3 \cdot \omega_4$ in at least k different ways with the same values of ω_1, ω_3 for all k, where $\omega_i \in \mathbb{N}$ is unlimited for $1 \leq i \leq 4$.*

Proof. Let p_1, p_2, p_3 be distinct unlimited primes such that $\omega \geq p_1 p_2 p_3$ (such prime numbers exist by Cauchy's principle and the fact that there are infinitely many primes, since ω is greater than any product of three standard prime numbers). Since $\gcd(p_1, p_2) = 1$, we conclude that there exist integers x_0 and y_0 such that $p_1 \cdot x_0 + p_2 \cdot y_0 = 1$. Therefore, the integer solutions of $p_1 \cdot x + p_2 \cdot y = \omega$ are given by $x_t = \omega x_0 - p_2 t$ and $y_t = \omega y_0 + p_1 t$, where $p_1 \cdot x_0 + p_2 \cdot y_0 = 1$ and $t \in \mathbb{Z}$. Thus, this equation has positive solutions if $\omega x_0 > p_2 t$ and $\omega y_0 > -p_1 t$, from which it follows that

$$\frac{-\omega y_0}{p_1} < t < \frac{\omega x_0}{p_2}. \tag{2}$$

Now let $k \geq 1$ be limited. Since $\omega > p_1 p_2 k$, or equivalently $\omega(p_1 x_0 + p_2 y_0) > p_1 p_2 k$, we conclude that

$$\frac{-\omega y_0}{p_1} < \left[\frac{-\omega y_0}{p_1}\right] + k < \frac{\omega x_0}{p_2}. \tag{3}$$

Therefore, inequalities (2) hold for at least k different values of t with $t = [-\omega y_0/p_1] + i$ for $1 \leq i \leq k$.

Next, note that x_t and y_t are not both limited; otherwise $p_3 \leq \frac{x_t}{p_2} + \frac{y_t}{p_1} \cong 0$, which is a contradiction. In fact, without loss of generality, assume that x_t is unlimited with $x_0 > 0$, i.e., $y_0 < 0$ and we show that y_t is also unlimited.

Let $a \geq 1$ be limited. Since $\omega(p_1 x_0 + p_2 y_0) > a p_2$, we deduce that $p_2(a - \omega y_0) < p_1 \omega x_0$. Moreover, as in the proof of (3), we can prove that $\frac{a - \omega y_0}{p_1} + t' < \frac{\omega x_0}{p_2}$ for every limited $t' \geq 1$. Indeed, the last inequality holds since $\omega(p_1 x_0 + p_2 y_0) > p_2(a + t' p_1)$, and so the following inequalities:

$$\frac{a - \omega y_0}{p_1} < t < \frac{\omega x_0}{p_2} \tag{4}$$

hold at least for k different values of t. It follows from the left-hand side of (4) that $p_1 t > a - \omega y_0$. Thus, $y_t = \omega y_0 + p_1 t > a$, which shows that y_t is unlimited. We let $\omega_1 = p_1$, $\omega_2 = x_t$, $\omega_3 = p_2$ and $\omega_4 = y_t$, which are unlimited positive integers. This completes the proof. □

Remark 1. *One can give a proof of Proposition 1 as follows: By Bertrand's postulate there exist prime numbers p_1 and p_2 such that $\frac{\sqrt[3]{\omega}}{2} < p_1 < \sqrt[3]{\omega}$ and $\frac{\sqrt[3]{\omega}}{4} < p_2 < \frac{\sqrt[3]{\omega}}{2}$. The solutions of the equation $p_1 x + p_2 y = \omega$ are of the form $x_t = x_0 - t p_2$ and $y_t = y_0 + t p_1$, where t is an integer. Fix t so that $(\sqrt[3]{\omega})^2 - \sqrt[3]{\omega} < y_t < (\sqrt[3]{\omega})^2$. If $k \geq 0$ is standard, then $y_{t+k} = y_t + k p_1$, so y_{t+k} is unlimited and $y_{t+k} < (\sqrt[3]{\omega})^2 + k\sqrt[3]{\omega}$, so that $p_1 x_{t+k} = \omega - p_2 y_{t+k} > \omega - \sqrt[3]{\omega}\left((\sqrt[3]{\omega})^2 + k\sqrt[3]{\omega}\right)/2 > \omega/4$ and $x_{t+k} > (\sqrt[3]{\omega})^2/4$ is also unlimited. We can let $\omega_1 = p_1$, $\omega_2 = x_{t+k}$, $\omega_3 = p_2$, $\omega_4 = y_{t+k}$ and $k \geq 0$.*

Corollary 1. *Let $k \geq 2$ be a standard natural number. Every unlimited $\omega \in \mathbb{N}$ can be written in the form* (A_k).

Proof. By induction. Note that $\omega_{2k-1}\cdot\omega_{2k}\sim\omega$, so Theorem 1 enables the inductive step by writing $\omega_{2k-1}\cdot\omega_{2k}=\omega'_{2k-1}\cdot\omega'_{2k}+\omega_{2k+1}\cdot\omega_{2k+2}$ with $\omega'_{2k-1},\omega'_{2k},\omega_{2k+1},\omega_{2k+2}\sim\sqrt{\omega}$. □

Lemma 3. *Every unlimited $\omega\in\mathbb{N}$ can be written in the form $\omega=\omega_1^2\cdot\omega_3+\omega_4\cdot\eta$, where $\omega_1,\omega_3,\omega_4\sim\sqrt[3]{\omega}$ and $\eta\sim\sqrt[3]{\omega^2}$.*

Proof. We closely follow the proof of Theorem 1. We fix prime numbers p_1 such that $\frac{\sqrt[3]{\omega}}{2}<p_1<\sqrt[3]{\omega}$ and p_2 such that $\frac{\sqrt[3]{\omega}}{4}<p_2<\frac{\sqrt[3]{\omega}}{2}$. The general solution of the Diophantine equation $p_1^2\cdot x+p_2\cdot y=\omega$ has the form $x_t=x_0+t\cdot p_2$, $y_t=y_0-t\cdot p_1^2$, $t\in\mathbb{Z}$. We can now choose t so that $\frac{\sqrt[3]{\omega}}{4}<x_t<\frac{3\sqrt[3]{\omega}}{4}$. For this t we get $\frac{\omega}{16}<p_1^2\cdot x_t<\frac{3\omega}{4}$ and hence $\frac{\omega}{4}<p_2\cdot y_t=\omega-p_1^2\cdot x_t<\frac{15\omega}{16}$. It follows that $\frac{\sqrt[3]{\omega^2}}{2}<y_t<\frac{15\sqrt[3]{\omega^2}}{4}$. We let $\omega_1=p_1$, $\omega_3=x_t$, $\omega_4=p_2$, $\eta=y_t$. □

Theorem 2. *Every unlimited $\omega\in\mathbb{N}$ can be written in the form*

$$\omega=\omega_1\cdot\omega_2\cdot\omega_3+\omega_4\cdot\omega_5\cdot\omega_6+\omega_7\cdot\omega_8\cdot\omega_9,$$

where $\omega_i>0$ and $\omega_i\sim\sqrt[3]{\omega}$ for $1\le i\le 9$.

Proof. Use Theorem 1 to write $\eta=\omega_5\cdot\omega_6+\omega_8\cdot\omega_9$ where $\omega_5,\omega_6,\omega_8,\omega_9\sim\sqrt{\eta}\sim\sqrt[3]{\omega}$, then substitute into the expression from Lemma 3 and let $\omega_2=\omega_1$, $\omega_7=\omega_4$. □

Corollary 2. *Let $k\ge 3$ be a standard natural number. Every unlimited $\omega\in\mathbb{N}$ can be written in the form*

$$\omega=\sum_{i=1}^{k}\omega_{i,1}\cdot\omega_{i,2}\cdot\omega_{i,3},\tag{5}$$

where $\omega_{i,j}>0$ and $\omega_{i,j}\sim\sqrt[3]{\omega}$ for $1\le i\le k$, $1\le j\le 3$.

Proof. By induction, starting with $k=3$ and using the observation that $\eta=\omega_1\cdot\omega_2\cdot\omega_3+\omega_4\cdot\omega_5\cdot\omega_6\sim\omega$ and hence, by Theorem 2, it can be expressed as $\eta=\omega'_1\cdot\omega'_2\cdot\omega'_3+\omega'_4\cdot\omega'_5\cdot\omega'_6+\omega'_7\cdot\omega'_8\cdot\omega'_9$, where $\omega'_i>0$ and $\omega'_i\sim\sqrt[3]{\omega}$ for $1\le i\le 9$. □

Lemma 3 generalizes as follows. Note that $r=2$ gives Theorem 1.

Lemma 4. *Let $r\ge 2$ be a standard natural number. Every unlimited $\omega\in\mathbb{N}$ can be written in the form $\omega=\omega_1^{r-1}\cdot\omega_3+\omega_4\cdot\eta$ where $\omega_1,\omega_3,\omega_4\sim\sqrt[r]{\omega}$ and $\eta\sim\sqrt[r]{\omega^{r-1}}$.*

Proof. We fix prime numbers p_1 such that $\frac{\sqrt[r]{\omega}}{2}<p_1<\sqrt[r]{\omega}$ and p_2 such that $\frac{\sqrt[r]{\omega}}{4}<p_2<\frac{\sqrt[r]{\omega}}{2}$. The general solution of the Diophantine equation $p_1^{r-1}\cdot x+p_2\cdot y=\omega$ has the form $x_t=x_0+t\cdot p_2$, $y_t=y_0-t\cdot p_1^{r-1}$, $t\in\mathbb{Z}$. We can now choose t so that $\frac{\sqrt[r]{\omega}}{4}<x_t<\frac{3\sqrt[r]{\omega}}{4}$. For this t we get $\frac{\omega}{2^{r+1}}<p_1^{r-1}\cdot x_t<\frac{3\omega}{4}$ and hence $\frac{\omega}{4}<p_2\cdot y_t=\omega-p_1^{r-1}\cdot x_t<\frac{(2^{r+1}-1)\omega}{2^{r+1}}$. It follows that $\frac{1}{2}\cdot\sqrt[r]{\omega^{r-1}}<y_t<\frac{2^{r+1}-1}{2^{r-1}}\cdot\sqrt[r]{\omega^{r-1}}$. We let $\omega_1=p_1$, $\omega_3=x_t$, $\omega_4=p_2$, $\eta=y_t$. □

Theorem 3. *Let $r\ge 2$ and $k\ge r$ be standard natural numbers. Every unlimited $\omega\in\mathbb{N}$ can be written in the form $\omega=\sum_{i=1}^{k}\prod_{j=1}^{r}\omega_{i,j}$, where $\omega_{i,j}>0$ and $\omega_{i,j}\sim\sqrt[r]{\omega}$ for $1\le i\le k$, $1\le j\le 3$.*

Proof. By induction on r. For $r = 2$, this is Corollary 1. Assume the theorem is true for $r-1$. Then $k-1 \geq r-1$ and we can write $\eta = \sum_{i=1}^{k-1} \prod_{j=1}^{r-1} \omega_{i,j}$ with all $\omega_{i,j} \sim \sqrt[r-1]{\eta} = \sqrt[r]{\omega}$ and substitute the result into the formula from Lemma 4. □

Next, we present an explicit method to prove that all numbers that are similar in structure to $n!$ can be written in the form (A_2).

Theorem 4. *Let $(a_i)_{1\leq i\leq k}$ be a sequence of positive integers such that a_1 is limited, k is unlimited and $a_{i+1} - a_i$ is limited positive for $i = 1, 2, \ldots, k-1$, and let $n = a_1 a_2 \cdots a_k$. There exist two unlimited positive integers R_1 and R_2 such that $n = R_1 \cdot R_2$ with $R_1 \sim R_2$.*

Proof. Let λ be a limited positive integer such that $0 < a_{i+1} - a_i \leq \lambda$ for $1 \leq i \leq k-1$. Indeed, such number exists since the set $\{a_{i+1} - a_i : i < k\}$ is internal, so it has a maximal element $a_{i^*+1} - a_{i^*}$ which is limited.

Now, we show that there exists a unique unlimited positive integer t such that

$$\begin{cases} a_1 a_2 \cdots a_{t-1} a_t < a_{t+1} a_{t+2} \cdots a_{k-1} a_k, \\ a_1 a_2 \cdots a_t a_{t+1} \geq a_{t+2} \cdots a_{k-1} a_k. \end{cases} \tag{6}$$

Otherwise,

$$\begin{cases} a_1 < a_2 a_3 \cdots a_{k-1} a_k \\ a_1 a_2 < a_3 a_4 \cdots a_{k-1} a_k \\ \qquad\qquad \vdots \\ a_1 a_2 \cdots a_{k-3} a_{k-2} < a_{k-1} a_k \\ a_1 a_2 \cdots a_{k-2} a_{k-1} < a_k. \end{cases} \tag{7}$$

But the last inequality of (7) leads to a contradiction because $a_{k-2} a_{k-1} > a_k$. Indeed, the numbers a_{k-2}, a_{k-1} and a_k are unlimited with $0 < a_k - a_{k-1} < \lambda$ and $0 < a_k - a_{k-2} < 2\lambda$, which implies that $a_{k-1} = a_k - \lambda_1$ and $a_{k-2} = a_k - \lambda_2$ for some limited integers λ_1 and λ_2, since λ is limited. Therefore,

$$a_{k-1} a_{k-2} = a_k^2 \left(1 - \frac{\lambda_1}{a_k}\right)\left(1 - \frac{\lambda_2}{a_k}\right) = a_k^2 (1 - \phi) > a_k,$$

where $\phi \cong 0$. A contradiction. This proves (6).

Next, from (6) we also have

$$\frac{1}{a_{t+1}} \leq \frac{a_1 a_2 \cdots a_{t-1} a_t}{a_{t+2} \cdots a_{k-1} a_k} < a_{t+1}. \tag{8}$$

There are three cases to consider:

Case 1. $a_1 a_2 \cdots a_{t-1} a_t / a_{t+2} \cdots a_{k-1} a_k$ is appreciable. Since $a_{i+1} - a_i \leq \lambda$ with λ limited, i.e., the elements $(a_i)_{1\leq i\leq k}$ are increasing by a limited quantity, there exists a positive integer i_0 with $i_0 \leq t$ such that a_{i_0} and $\sqrt{a_{t+1}}$ have the same order, that is, $a_{i_0}/\sqrt{a_{t+1}}$ is appreciable. We put $R_1 = a_1 a_2 \cdots a_{t-1} a_t a_{t+1} / a_{i_0}$ and $R_2 = a_{t+2} \cdots a_{k-1} a_k a_{i_0}$. It is clear that $n = R_1 \cdot R_2$, where

$$\frac{R_1}{R_2} = \frac{a_1 a_2 \cdots a_{t-1} a_t a_{t+1}}{a_{i_0}^2 a_{t+2} \cdots a_{k-1} a_k} = \frac{a_1 a_2 \cdots a_{t-1} a_t}{a_{t+2} \cdots a_{k-1} a_k} \cdot \frac{a_{t+1}}{a_{i_0}^2}$$

is appreciable since $a_{t+1} \sim a_{i_0}^2$.

Case 2. $a_1a_2\cdots a_{t-1}a_t/a_{t+2}\cdots a_{k-1}a_k \cong 0$. Here by (8), there exists an unlimited positive integer $l \leq a_{t+1}$ such that $\frac{a_1a_2\cdots a_{t-1}a_t}{a_{t+2}\cdots a_{k-1}a_k} \cdot l$ is appreciable. We have the following subcases:

Case 2.1. $a_{t+1}/l = A$ with A appreciable. Here, we put $R_1 = a_1a_2\cdots a_{t-1}a_ta_{t+1}$ and $R_2 = a_{t+2}\cdots a_{k-1}a_k$, in which case $n = R_1 \cdot R_2$, where

$$\frac{R_1}{R_2} = \frac{a_1a_2\cdots a_{t-1}a_ta_{t+1}}{a_{t+2}\cdots a_{k-1}a_k} = \frac{a_1a_2\cdots a_{t-1}a_t}{a_{t+2}\cdots a_{k-1}a_k} \cdot lA,$$

which is appreciable.

Case 2.2. a_{t+1}/l is unlimited. As above, let i_0 be a positive integer with $i_0 \leq t$ such that a_{i_0} and $\sqrt{a_{t+1}/l}$ have the same order. We put $R_1 = a_1a_2\cdots a_{t-1}a_ta_{t+1}/a_{i_0}$ and $R_2 = a_{t+2}\cdots a_{k-1}a_ka_{i_0}$. It follows that $R_1/R_2 = \frac{a_1a_2\cdots a_{t-1}a_t}{a_{t+2}\cdots a_{k-1}a_k} \cdot \frac{a_{t+1}}{a_{i_0}^2 l}$ is appreciable since $a_{t+1}/l \sim a_{i_0}^2$.

Case 3. $a_1a_2\cdots a_{t-1}a_t/a_{t+2}\cdots a_{k-1}a_k \cong +\infty$. In this case, by (8), there exists an unlimited positive integer $m \leq a_{t+1}$ such that $\frac{a_1a_2\cdots a_{t-1}a_t}{a_{t+2}\cdots a_{k-1}a_k} \cdot \frac{1}{m}$ is appreciable. We also have the following subcases:

Case 3.1. $a_{t+1}/m = A$ with A appreciable. Here we put $R_1 = a_1a_2\cdots a_{t-1}a_t$ and $R_2 = a_{t+2}\cdots a_{k-1}a_ka_{t+1}$, where $n = R_1 \cdot R_2$ and $R_1/R_2 = \frac{a_1a_2\cdots a_{t-1}a_t}{a_{t+2}\cdots a_{k-1}a_ka_{t+1}} = \left(\frac{a_1a_2\cdots a_{t-1}a_t}{a_{t+2}\cdots a_{k-1}a_k} \cdot \frac{1}{m}\right) \cdot \frac{1}{A}$ which is appreciable.

Case 3.2. $a_{t+1}/m = \omega$ with ω unlimited. Let i_0, j_0 be two positive integers not exceeding t with $i_0 \neq j_0$ such that $a_{i_0} \sim m$ and $a_{j_0} \sim \sqrt{\omega}$. Then we put $R_1 = a_1a_2\cdots a_{t-1}a_ta_{t+1}/a_{i_0}a_{j_0}$ and $R_2 = a_{t+2}\cdots a_{k-1}a_ka_{i_0}a_{j_0}$. We also observe that $n = R_1 \cdot R_2$, where

$$\frac{R_1}{R_2} = \frac{a_1a_2\cdots a_{t-1}a_ta_{t+1}}{a_{i_0}^2a_{j_0}^2a_{t+2}\cdots a_{k-1}a_k} = \left(\frac{a_1a_2\cdots a_{t-1}a_t}{a_{t+2}\cdots a_{k-1}a_k} \cdot \frac{1}{m}\right) \cdot \frac{ma_{t+1}}{a_{i_0}^2a_{j_0}^2}$$

is appreciable since $ma_{t+1} = m^2\omega \sim a_{i_0}^2a_{j_0}^2$.

This completes the proof. □

Applying Theorem 4, we obtain the following corollaries.

Corollary 3. *Let n be as in Theorem 4. Then n is of the form $\omega_1 \cdot \omega_2 + \omega_3 \cdot \omega_4$, where $\omega_i \in \mathbb{N}$ is unlimited and $\omega_i \sim \omega_j$ for $1 \leq i, j \leq 4$.*

Proof. Since $n = R_1 \cdot R_2$ with $R_1 \sim R_2$, we conclude that if one of these numbers is even, say R_1, then $n = (R_1/2) \cdot R_2 + (R_1/2) \cdot R_2$. If R_1 and R_2 are both odd, then $n = \left(\frac{R_1-1}{2}\right) R_2 + \left(\frac{R_1-1}{2} + 1\right) \cdot R_2$, as required. □

Corollary 4. *Let n be unlimited. Then $n!$ is of the form $n! = \omega_1 \cdot \omega_2 + \omega_3 \cdot \omega_4$, where $\omega_i \in \mathbb{N}$ is unlimited and $\omega_i \sim \omega_j$ for $1 \leq i, j \leq 4$.*

Proof. By definition $n! = a_1 a_2 \cdots a_n$, where $a_i = i$ $(1 \leq i \leq n)$, that is, $(a_i)_{1 \leq i \leq n}$ satisfy conditions of Theorem 4. Then the result follows by applying Corollary 3. □

The proof of Theorem 4 can be adapted straightforwardly to obtain the following corollary.

Corollary 5. *Let k be unlimited and let $(a_i)_{1 \leq i \leq k}$ be a sequence of positive integers such that a_1 is limited and $a_{i+1} = s_i \cdot a_i$, where $s_i > 1$ is limited for $i = 1, 2, \ldots, k-1$, and let $n = a_1 a_2 \cdots a_k$. Then there exist two unlimited positive integers R_1 and R_2 such that $n = R_1 \cdot R_2$, where $R_1 \sim R_2$.*

3. Other similar representations

In this subsection, we provide some other representations of unlimited natural numbers. First, we need the following lemma:

Lemma 5 (see [9])**.** *Let $n! = \prod_{p \leq n} p^{v_p(n!)}$ be the prime factorization of $n!$. If $v_p(n!) > v_q(n!)$, then $p^{v_p(n!)} > q^{v_q(n!)}$.*

Remark 2. *By Nathanson [16, Theorem 1.12, p. 29], for every positive integer n and prime p, $v_p(n!) = \sum_{\alpha=1}^{+\infty} \left[\frac{n}{p^\alpha}\right] = \sum_{\alpha=1}^{\left[\frac{\log n}{\log p}\right]} \left[\frac{n}{p^\alpha}\right]$. It follows that for primes p and q with $p < q$ we have $v_p(n!) \geq v_q(n!)$. In particular, if $n \geq 4$, $p = 2$ and $q \geq 3$, then clearly $v_p(n!) = v_2(n!) > v_q(n!)$. Hence by Lemma 5, $2^{v_2(n!)} > q^{v_q(n!)}$.*

Theorem 5. *Let n be unlimited. Then $n!$ can be written as $R_1 \cdot R_2$ where, R_1, R_2 are two unlimited positive integers with $R_1 \sim \sqrt[3]{n!} \sim \left[(n!)^{\frac{1}{3}}\right]$.*

Proof. By Stirling's formula we have $n! = n^n e^{-n} \sqrt{2\pi n}\,(1 + \phi_1)$, $\phi_1 \cong 0$ (see [7, p. 49]). On the other hand, in 1808, Legendre determined the exact power t of the prime p that divides $n!$ (so p^{t+1} does not divide $n!$) [18, p. 18], namely,

$$t = \sum_{\alpha=1}^{\infty} \left[\frac{n}{p^\alpha}\right] = \frac{n - (a_0 + a_1 + .. + a_r)}{p - 1},$$

where the integers $a_0, a_1, \ldots, a_r$ are the digits of n in base p, that is, $n = a_r p^r + a_{r-1} p^{r-1} + \cdots + a_1 p + a_0$ such that $0 \leq a_i \leq p - 1$ for $i = 0, 1, \ldots, r$.

Now, assume that $n! = \prod_{i=1}^{m} p_i^{\alpha_i}$, where $2 = p_1 < p_2 < \cdots < p_m$ are primes and $\alpha_i \geq 1$ for all i. We have $\left[(n!)^{\frac{1}{3}}\right] = (n!)^{\frac{1}{3}}\,(1 + \phi_2)$, $\phi_2 \cong 0$. By the formula above, the exponent α_2 of 3 satisfies $\alpha_2 \leq n/2$. Since $\left[(n!)^{\frac{1}{3}}\right] / p_2^{\alpha_2} = \left[(n!)^{\frac{1}{3}}\right] / 3^{\alpha_2} \geq \left[(n!)^{\frac{1}{3}}\right] / 3^{n/2}$, it is easily seen that $\left[(n!)^{\frac{1}{3}}\right] / p_2^{\alpha_2} \cong +\infty$. Then there exists a positive integer k such that

$$p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_k^{\alpha_k} \leq \left[(n!)^{\frac{1}{3}}\right] < p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_k^{\alpha_k} \cdot p_{k+1}^{\alpha_{k+1}}.$$

Since in the prime factorization of $n!$ we have $\alpha_1 > \alpha_{k+1}$, it follows from Lemma 5 that $p_1^{\alpha_1} > p_{k+1}^{\alpha_{k+1}}$. Hence there exists an integer s with $0 \leq s < \alpha_1$ such that

$$p_1^s \cdot p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_k^{\alpha_k} \leq \left[(n!)^{\frac{1}{3}}\right] < p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_k^{\alpha_k} \cdot p_1^{s+1}.$$

Therefore, $1 \leq \left[(n!)^{\frac{1}{3}}\right] / p_1^s p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_k^{\alpha_k} < 2$, that is, $\left[(n!)^{\frac{1}{3}}\right] \sim p_1^s p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_k^{\alpha_k}$. Hence, $n! = p_1^s p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_k^{\alpha_k} \cdot p_1^{\alpha_1 - s} p_{k+1}^{\alpha_{k+1}} \cdots p_m^{\alpha_m}$, which is of the form $R_1 \cdot R_2$, where $R_1 = p_1^s p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_k^{\alpha_k}$ and $R_2 = p_1^{\alpha_1 - s} p_{k+1}^{\alpha_{k+1}} \cdots p_m^{\alpha_m}$. This completes the proof. □

Corollary 6. *$n!$ is of the form $\omega_1 \cdot \omega_2 \cdot \omega_3 + \omega_4 \cdot \omega_5 \cdot \omega_6$, where $\omega_i \in \mathbb{N}$ is unlimited with $\omega_i \sim \sqrt[3]{n!}$ for $1 \leq i, j \leq 6$.*

Proof. Since $n! = R_1 \cdot R_2$, *where* $R_1 \sim \sqrt[3]{n!}$, *we have* $R_2 \sim \sqrt[3]{(n!)^2}$. Use Theorem 1 to write $R_2 = \omega_2 \cdot \omega_3 + \omega_4 \cdot \omega_5$ where $\omega_2, \omega_3, \omega_4, \omega_5 \sim \sqrt{R_2} = \sqrt[3]{n!}$. □

Consider the sequence of Fibonacci numbers (F_n), where $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$, $n \geq 2$. It is well-known that the generalized Fibonacci sequence is defined by $G_n = G_{n-1} + G_{n-2}$, where $G_1 = a$ and $G_2 = b$ ($a, b \in \mathbb{N}$ and $n \geq 3$), see Koshy [14, page 109].

Theorem 6. *Let n be unlimited. If a and b are limited, then $G_{3n}^2 - G_n^2$ is of the form $\omega_1 \cdot \omega_2 \cdot \omega_3 + \omega_4 \cdot \omega_5 \cdot \omega_6$, where $\omega_i \in \mathbb{N}$ is unlimited with $\omega_i \sim \omega_j$ for $1 \leq i, j \leq 6$.*

Proof. By [14, Theorem 7.1, p. 109], we have

$$G_n = aF_{n-2} + bF_{n-1}. \tag{9}$$

Moreover, the terms of this sequence verify the following equality: $G_{m+n}^2 - G_{m-n}^2 = G_{m+1} G_m F_{2n} + G_{m-1} G_m F_{2n}$ (see [14, Identity 3, p. 214]. In particular, for $m = 2n$ we get $G_{3n}^2 - G_n^2 = G_{2n+1} G_{2n} F_{2n} + G_{2n-1} G_{2n} F_{2n}$, which is of the form $\omega_1 \cdot \omega_2 \cdot \omega_3 + \omega_4 \cdot \omega_5 \cdot \omega_6$, where $\omega_i \in \mathbb{N}$ are unlimited ($1 \leq i \leq 6$). Applying (9) we have $\omega_i \sim \omega_j$ for $1 \leq i, j \leq 6$. □

Note that Corollary 5 and Theorem 6 are interesting because it is not known whether every unlimited ω is of the form $\omega_1 \cdot \omega_2 \cdot \omega_3 + \omega_4 \cdot \omega_5 \cdot \omega_6$ with $\omega_i \sim \omega_j$ for $1 \leq i, j \leq 6$.

Proposition 2. *There are infinitely many unlimited positive integers n such that $F_n = \omega_1 \cdot \omega_2 + \omega_3 \cdot \omega_4$, where $\omega_1, \omega_2, \omega_3, \omega_4 \in \mathbb{N}$ are unlimited, pairwise relatively prime with $\omega_i \sim \omega_j$ for $1 \leq i, j \leq 4$.*

Proof. Let k be a positive integer with $3 \nmid (k+1)$ and let $n = 2k$. Applying Andrica [2, Equation (2), p. 194] ($F_{m+n} = F_{m+1} \cdot F_n + F_m \cdot F_{n-1}$), if $m = n+1$, then $F_{2n+1} = F_{n+2} \cdot F_n + F_{n+1} \cdot F_{n-1}$. Let $x, y \in \{n-1, n, n+1, n+2\}$. We can verify easily that $\gcd(x, y) = 1$ or 2, and by Koshy [14, Theorem 16.3, p. 198] we have $\gcd(F_x, F_y) = F_{\gcd(x,y)} = 1$ since $F_1 = F_2 = 1$. On the other hand, we see that F_x / F_y is appreciable since $|x - y| \leq 3$. □

Theorem 7. *Every unlimited positive integer n can be written in the form* (A_2)*, where $\omega_i \in \mathbb{Z}$ is unlimited and $|\omega_i/\omega_j| \in \{1/2, 1, 2\}$ for $1 \leq i, j \leq 4$.*

The proof is based on the fact that a positive integer n can be represented as the difference of two squares if and only if n is not of the form $4k+2$ (see, e.g. Dujella [8]).

***Proof of Theorem** 7.* Let n be an unlimited positive integer. If n is not of the form $4k+2$, then $n = x^2 - y^2$ for some positive integers x, y with x unlimited, and if n is of the form $4k+2$, then $n = 2m$ with m odd, i.e., m is not of the form $4k+2$. Thus, n is of the form $2x^2 - 2y^2$. In both cases, n is of the form $\lambda\left(x^2 - y^2\right)$, where $\lambda \in \{1, 2\}$. There are two cases to consider:

Case 1. x and y are of the same order. In this case we have nothing to prove and we can put $\omega_1 = \lambda x$, $\omega_2 = x$, $\omega_3 = -\lambda y$ and $\omega_4 = y$.

Case 2. $y/x \cong 0$. We distinguish two cases:

Case 2.1. Assume that $x + y$ is even. Then

$$n = \lambda\,(x-y)\,(x+y) = \lambda\,(x-y)\left(\frac{x+y}{2}\right) + \lambda\,(x-y)\left(\frac{x+y}{2}\right),$$

which is of the form $\omega_1 \cdot \omega_2 + \omega_3 \cdot \omega_4$, where $\omega_i \in \mathbb{Z}$ is unlimited and $|\omega_i/\omega_j| \in \{1/2, 1, 2\}$ for $1 \leq i, j \leq 4$.

Case 2.2. Assume that $x + y$ is odd. Then

$$\begin{aligned} n &= \lambda\,(x-y)\,(x+y-1) + \lambda\,(x-y) \\ &= \lambda\,(x-y)\left(\frac{x+y-1}{2}\right) + \lambda\,(x-y)\left(\frac{x+y-1}{2}\right) + \lambda\,(x-y) \\ &= \lambda\,(x-y)\left(\frac{x+y-1}{2}\right) + \lambda\,(x-y)\left(\frac{x+y+1}{2}\right), \end{aligned}$$

which is also of the form $\omega_1 \cdot \omega_2 + \omega_3 \cdot \omega_4$ with $\omega_i \in \mathbb{Z}$ unlimited and $|\omega_i/\omega_j| \in \{1/2, 1, 2\}$ for $1 \leq i, j \leq 4$. This completes the proof. □

Theorem 8. *Every unlimited positive integer is either of the form $\omega_1^2 - \omega_2^2$, where $\omega_1, \omega_2 \in \mathbb{N}$ are unlimited with $\omega_1/\omega_2 \cong 1$, or of the form $\omega_1^2/2 - \omega_2^2/2$, where $\omega_1, \omega_2 \in \mathbb{N}$ are even and unlimited with $\omega_1/\omega_2 \cong 1$.*

Proof. We distinguish two cases:

Case 1. Assume that n is not of the form $4k+2$. Then $n = a^2 - b^2$ for some positive integers a, b. This means that either n is odd or it is of the form $4k$. If it is odd, then $n-1$ and $n+1$ are both even, in which case

$$n = \left(\frac{n+1}{2}\right)^2 - \left(\frac{n-1}{2}\right)^2. \tag{10}$$

On the other hand, if n is divisible by 4, then $n = \left(\frac{n}{4}+1\right)^2 - \left(\frac{n}{4}-1\right)^2$. In both cases, n is of the form $\omega_1^2 - \omega_2^2$, where $\omega_1, \omega_2 \in \mathbb{N}$ are unlimited and $\omega_1/\omega_2 \cong 1$.

Case 2. Assume that $n = 4k+2$, then $n = 2m$ with m odd. Since m satisfies (10), we conclude that $n = (m+1)\,(m+1)\,/2 - (m-1)\,(m-1)\,/2$, which is of the

form $\omega_1^2/2 - \omega_2^2/2$, where $\omega_1, \omega_2 \in \mathbb{N}$ are unlimited and $\omega_1/\omega_2 \cong 1$. This completes the proof. □

Proposition 3. *Let p be a limited prime number such that $p \equiv 1\,(mod\ 4)$. There exist infinitely many positive integers n such that n is of the form (A_2) with $\omega_1/\omega_2 = \omega_3/\omega_4 = p$.*

Proof. Let a and b be two limited positive integers such that $p = a^2 + b^2$ and $\gcd(a, b) = 1$. Consider the Diophantine equation $a \cdot x + b \cdot y = 1$. Then there are limited integers x_0 and y_0 for which $a \cdot x_0 + b \cdot y_0 = 1$ and all solutions are given by $x_t = x_0 + bt$ and $y_t = y_0 - at$, where $t \in \mathbb{Z}$. For $t \cong \infty$ we see that $|x_t| \sim |y_t|$. For each such values of t it follows from Lagrange's identity (Jarvis [13, Lemma 1.18, p. 9]) that $p\left(x_t^2 + y_t^2\right) = (ax_t + by_t)^2 + (ay_t - bx_t)^2 = 1 + k^2$, where $k = ay_t - bx_t$. Thus, $1 + k^2 = px_t^2 + py_t^2$. The proof is finished if we put $n = 1 + k^2$, $\omega_1 = p\,|x_t|$, $\omega_2 = |x_t|$, $\omega_3 = p\,|y_t|$ and $\omega_4 = |y_t|$. □

Proposition 4. *Every unlimited positive integer n can be written as one of the following four forms:*

(*1*) $n = \lambda\omega_1^2 + \omega_2^2 + \omega_2^2$, *where* $\lambda \in \{1, 2\}$ *and* $\omega_i \sim \omega_j$ *for* $1 \leq i, j \leq 3$.

(*2*) $n = (\lambda + 1)\,\omega_1^2 + \omega_2^2 - \omega_3 \cdot \omega_4$, *where* $\lambda \in \{1, 2\}$ *and* $\omega_i \sim \omega_j$ *for* $1 \leq i, j \leq 4$.

(*3*) $n = (\lambda + 2)\,\omega_1^2 - \omega_2 \cdot \omega_3 - \omega_4 \cdot \omega_5$, *where* $\lambda \in \{1, 2\}$ *and* $\omega_i \sim \omega_j$ *for* $1 \leq i, j \leq 5$.

(*4*) $n = 2\omega_1^2 + 2\omega_2^2 - \omega_3 \cdot \omega_4$, *where* $\omega_i \sim \omega_j$ *for* $1 \leq i, j \leq 4$.

Proof. Let n be an unlimited positive integer. From [1, Theorem 8.25, p. 236], n can be written in the form $x^2 + y^2 + \lambda z^2$, where $\lambda = 1$ or $\lambda = 2$.

First, assume that $z = \max\{x, y, z\}$. We distinguish the following cases:

Case 1. x and y are of the same order as z. In this case, we have nothing to prove and we can put $\omega_1 = z$, $\omega_2 = y$ and $\omega_3 = x$. Then n is in form (*1*).

Case 2. $x/z \cong 0$ and y/z is appreciable. Here, $n = (x + z)(x - z) + y^2 + (\lambda + 1)\,z^2$. Hence, $\omega_1 = z$, $\omega_2 = y$, $\omega_3 = x + z$ and $\omega_4 = z - x$. Thus, n is in form (*2*).

Case 3. $y/z \cong 0$ and x/z is appreciable. This case is very similar to that of Case 2 with x, y exchanged. Thus, n is in form (*2*).

Case 4. $x/z \cong 0$ and $y/z \cong 0$. Then, $n = (x + z)(x - z) + (y + z)(y - z) + (\lambda + 2)\,z^2$. Hence we can put $\omega_1 = z$, $\omega_2 = z + x$, $\omega_3 = z - x$, $\omega_4 = z + y$ and $\omega_5 = z - y$. Then n is in form (*3*).

Now, assume that $\lambda = 2$ and $\max\{x, y, z\}$ is either x or y, say x. We also have the following cases:

Case 1. y and z are of the same order as x. Here n is in form (*1*).

Case 2. $y/x \cong 0$ and z/x is appreciable. In this case, $n = 2x^2 + 2z^2 - (x + y)(x - y)$. Hence, $\omega_1 = x$, $\omega_2 = z$, $\omega_3 = x + y$ and $\omega_4 = x - y$. Then n is in form (*4*).

Case 3. $z/x \cong 0$ and y/x is appreciable. We can do the same reasoning as above, that is, n is in form (*4*).

Case 4. $y/x \cong 0$ and $z/x \cong 0$. Then, $n = 4x^2 - 2(x+z)(x-z) - (x+y)(x-y)$. Hence, $\omega_1 = 2x$, $\omega_2 = 2(x+z)$, $\omega_3 = x - z$, $\omega_4 = x + y$ and $\omega_5 = x - y$. Then n is in form *(3)*.

This completes the proof. □

4. Unlimited integers of the form $a \cdot \omega_1^2 + b \cdot \omega_2^2$, where $\omega_1 \sim \omega_2$

Let n be an arbitrary unlimited number and let a, b be limited. We want to represent n in the form: $a \cdot \omega_1^2 + b \cdot \omega_2^2$, where $\omega_1 \sim \omega_2$.

Let ω be unlimited and let F_ω be the ω-th Fibonacci number. Then $F_{2\omega+1}$ is of the form $\omega_1^2 + \omega_2^2$, where $\omega_1 \sim \omega_2$ and $\gcd(\omega_1, \omega_2) = 1$. In fact, from Koshy [14, Identity 30, p. 97] we have $F_{2\omega+1} = F_\omega^2 + F_{\omega+1}^2$, where $\gcd(F_\omega, F_{\omega+1}) = 1$ by [14, Theorem 16.3, p. 198].

Let us start with the following result:

Proposition 5. *There exist unlimited prime numbers p such that $p = \omega_1^2 + \omega_2^2$, where $\omega_1, \omega_2 \in \mathbb{N}$ are unlimited.*

Proof. From Dirichlet's theorem about primes in arithmetic progressions there exists an unlimited prime q of the form $4k - 1$. Let n be an unlimited positive integer with $n < q$. It is not difficult to see that the numbers q and

$$4\left(q+1^2\right)^2\left(q+2^2\right)^2\cdots\left(q+n^2\right)^2$$

are coprime. By Dirichlet's theorem once again, there exists a positive integer k' such that the number $p = 4\left(q+1^2\right)^2\left(q+2^2\right)^2\cdots\left(q+n^2\right)^2 \cdot k' - q$ is prime. Clearly, it is of the form $4t + 1$. By Nathanson [16, Theorem 13.3, p. 407], there exist two positive integers ω_1, ω_2 with $\omega_1 < \omega_2$ such that $p = \omega_1^2 + \omega_2^2$. Now, assume by way of contradiction that ω_1 is limited, i.e., $\omega_1 < n$. It follows that

$$\begin{aligned}\omega_2^2 = p - \omega_1^2 &= 4\left(q+1^2\right)^2\left(q+2^2\right)^2\cdots\left(q+n^2\right)^2 \cdot k' - \left(q+\omega_1^2\right)\\ &= \left(q+\omega_1^2\right)\left[\begin{array}{c} 4\left(q+1^2\right)^2\cdots\left(q+(\omega_1-1)^2\right)^2\left(q+\omega_1^2\right)\left(q+(\omega_1+1)^2\right)^2\cdots \\ \left(q+n^2\right)^2\cdot k' - 1\end{array}\right].\end{aligned}$$

Note also that the above factors are relatively prime, i.e.,

$$\gcd\left(q+\omega_1^2, 4\left(q+1^2\right)^2\cdots\left(q+(\omega_1-1)^2\right)^2\left(q+\omega_1^2\right)\cdots\left(q+n^2\right)^2\cdot k' - 1\right) = 1,$$

and so $4\left(q+1^2\right)^2\cdots\left(q+(\omega_1-1)^2\right)^2\left(q+\omega_1^2\right)\left(q+(\omega_1+1)^2\right)^2\cdots\left(q+n^2\right)^2\cdot k' - 1$ must be square. This is impossible because it is of the form $4t - 1$. Thus, $\omega_2 > \omega_1 \geq n \cong \infty$. This completes the proof. □

Proposition 6. *Let $n \in \mathbb{N}$ be unlimited such that n is representable as the sum of two squares. Then either $n = a^2 + b^2$ with $a \sim b$ or $2n = a^2 + b^2$ with $a \sim b$.*

Proof. Suppose that $n = a^2 + b^2$ with $b \leq a$. If $a \sim b$, the desired assertion holds in this case; otherwise, $b/a \cong 0$ and so $2n = (a-b)^2 + (a+b)^2$, where in this case $a - b \sim a + b$. This completes the proof. □

5. Representation of unlimited integers using quadratic forms

In this section, we aim to represent unlimited positive integers as in (A_2), where some of the factors ω_i $(1 \leq i \leq 4)$ are in $\mathbb{Z}$. In addition, we give the values of ω_i $(1 \leq i \leq 4)$.

Recall that a quadratic form is a homogeneous polynomial of degree two. The quadratic form $Q(x, y, \ldots, z)$ represents the integer n if there exist integers $a, b, \ldots, c$ such that $n = Q(a, b, \ldots, c)$. A binary quadratic form is a quadratic form in two variables. We consider the following definition:

Definition 2. *Let $f(x, y) = ax^2 + bxy + cy^2$. We say that f represents an integer n if $f(u, v) = n$ for some integers u and v, and that f properly represents n if $f(u, v) = n$ with $\gcd(u, v) = 1$.*

In what follows, we give two results, where in the first we show that every unlimited integer n, which can be represented by a quadratic form $f(x, y) = ax^2+bxy+cy^2$ such that a, b and c are all nonzero limited integers with $b^2 - ac \neq 0$, can be written in the form (A_2), where $\omega_i \in \mathbb{Z}$ is unlimited for $1 \leq i \leq 4$. More precisely, we give the value of ω_i in terms of n for $1 \leq i, j \leq 4$. In the second theorem, we present some types of quadratic forms for which any unlimited positive integer n that can be represented by one of these forms is of the form:

$$\begin{cases} n = \omega_1 \cdot \omega_2 + \omega_3 \cdot \omega_4 \\ \omega_i \sim \omega_j \ (1 \leq i, j \leq 4), \\ \gcd(\omega_1 \cdot \omega_2, \omega_3 \cdot \omega_4) \text{ is limited} \end{cases} \tag{A_2'}$$

where $\omega_i \in \mathbb{Z}$ is unlimited for $1 \leq i \leq 4$. Here we also give the value of ω_i in terms of n for $1 \leq i, j \leq 4$.

Theorem 9. *Let n be an unlimited positive integer. Assume that n is represented by the quadratic form $f(x, y) = ax^2+bxy+cy^2$, where a, b and c are all nonzero limited integers with $b^2 - ac \neq 0$. Then by rewriting this quadratic form n can always be represented explicitly in the form (A_2), where some of the ω_i may be negative integers.*

Proof. We suppose that n is represented by f, i.e., $n = ax^2 + bxy + cy^2$. We have the following cases:

I. ($x = 0$ and $y \neq 0$) or ($x \neq 0$ and $y = 0$). In this case, $n = cy^2$ with $c \neq 0$ or $n = ax^2$ with $a \neq 0$. Let us take, for instance, $n = cy^2$. Then $n = c\,(y - t + t)^2$. Hence, $n = c\left((y - t)^2 + t^2 + 2t\,(y - t)\right) = c\,(y - t)^2 + ct\,(2y - t)$. We end this case if we take $t = [y/2]$ and put $\omega_1 = y - t$, $\omega_2 = c\,(y - t)$, $\omega_3 = ct$ and $\omega_4 = 2y - t$.

II. $x, y \neq 0$. We distinguish two subcases:

II-1. $a, b, c \neq 0$. Consider the following possibilities:

II-1-1. y/x is appreciable. Clearly, we have $n = x\,(ax + by) + cy^2$. Since $ax + by$ is of the same order as x and y, we put $\omega_1 = x$, $\omega_2 = ax + by$, $\omega_3 = cy$ and $\omega_4 = y$. Then n can be represented in the form (A_2).

II-1-2. y/x is unlimited. Here we see that

$$\begin{aligned} n &= ax^2+bxy+cy^2 = ax^2+y\left(bx+cy\right) = a\left(x-y+y\right)^2+y\left(bx+cy\right) \\ &= a\left(x-y\right)\left(x+y\right)+y\left(ay+bx+cy\right) = a\left(x-y\right)\left(x+y\right)+y\left(y\left(a+c\right)+bx\right). \end{aligned}$$

We end this case if $a+c\neq 0$ because we can put $\omega_1 = a\left(x-y\right)$, $\omega_2 = x+y$, $\omega_3 = y$ and $\omega_4 = y\left(a+c\right)+bx$. Otherwise, $c=-a$ and so $n = ax^2+bxy-ay^2$. Since $x = x-y+y$, we conclude that $n = \left(x-y\right)\left(ax+\left(a+b\right)y\right)+by^2$. Similarly, when $a+b\neq 0$, we put $\omega_1 = x-y$, $\omega_2 = ax+\left(a+b\right)y$, $\omega_3 = by$ and $\omega_4 = y$. Otherwise, $b=-a$ and then $n = ax^2-axy-ay^2$. Here, we can easily see that $n = a\left(x+2y\right)^2-5ya\left(x+y\right)$. To finish the proof for this case, we only need to put $\omega_1 = a\left(x+2y\right)$, $\omega_2 = x+2y$, $\omega_3 = -5ya$ and $\omega_4 = x+y$. In addition, the proof of our claim for the case that x/y is unlimited is similar to our previous discussion.

II-2. At least one of the coefficients a, b and c is zero.

II-2-1. Only one coefficient among the numbers a, b and c is zero. We have the following cases:

• $b=0$. Then $n = ax^2+cy^2$. Here we can assume that x and y are positive with $y\geq x$. If y/x is appreciable, then the proof in this case is obviously met by taking appropriate values for ω_i $(1\leq i\leq 4)$. Otherwise, y/x is unlimited from which we get $n = ax^2+cy^2 = a\left(x-y+y\right)^2+cy^2 = a\left(x-y\right)\left(x+y\right)+\left(a+c\right)y^2$. Hence,

$$n = \begin{cases} a\left(x-y\right)\left(x+y\right)+\left(a+c\right)y^2, \text{ if } a+c\neq 0 \\ a\left(x-y\right)^2+2ay\left(x-y\right), \text{ otherwise.} \end{cases}$$

The proof in this case is met by taking appropriate values for ω_i $(1\leq i\leq 4)$. The case $x>y$ is treated in the same way.

• $a=0$. Then $n = bxy+cy^2$. Suppose that $|y|\geq|x|$. If y/x is appreciable, then the proof in this case is obviously met by taking appropriate values for ω_i $(1\leq i\leq 4)$. Otherwise, y/x is unlimited and then $n = b\left(x-y\right)y+\left(c+b\right)y^2$. If $c+b\neq 0$, then the proof is finished for this case by taking appropriate values for ω_i $(1\leq i\leq 4)$. Otherwise, $c+b=0$ and then

$$n = b\left(x-y\right)y = b\left(x-y\right)\left(y-t+t\right) = b\left(x-y\right)\left(y-t\right)+b\left(x-y\right)t$$

where $t=[y/2]$. Also the proof is finished for this case by taking appropriate values for ω_i $(1\leq i\leq 4)$. The case $|x|>|y|$ is treated in the same way.

• $c=0$. Then $n = ax^2+bxy$. This case is treated in the same way as the case $(a=0)$.

II-2-2. Exactly two coefficients among a, b and c are zero. We distinguish the following possibilities:

• $a=b=0$. Then $n = cy^2$. This case is treated in the same way as the case (**I**).

• $a=c=0$. Then $n = bxy$. Suppose that $|y|\geq|x|$. If y/x is appreciable, then

$$n = bx\left(y-t+t\right) = bx\left(y-t\right)+bxt,$$

where $t = [y/2]$. This complete the proof for this case by taking $\omega_1 = bx$, $\omega_2 = y - t$, $\omega_3 = bx$ and $\omega_4 = t$. If y/x is unlimited, then $n = by\,(x - y + y) = by\,(x - y) + by^2$. This completes the proof by taking $\omega_1 = by$, $\omega_2 = x - y$, $\omega_3 = by$ and $\omega_4 = y$. The case $|x| > |y|$ is treated in the same way.

• $b = c = 0$. Then $n = ax^2$. This case is treated in the same way as the case $(a = b = 0)$ of the previous case.

This completes the proof of Theorem 9. □

By a similar proof we obtain the following result:

Theorem 10. *Let n be an unlimited positive integer represented by a quadratic form $f(x, y) = ax^2 + bxy + cy^2$, where a, b and c are limited integers with $\gcd(x, y) = 1$. Then n is represented as in (A'_2) whenever f corresponds to one of the following cases:*

(1) $f(x, y) = ax^2$.

(2) $f(x, y) = ax^2 + cy^2$ *with* $a \neq -c$.

(3) $f(x, y) = ax^2 + bxy + cy^2$ *such that* $a, b, c \neq 0$ *and* y/x *is appreciable.*

(4) $f(x, y) = ax^2 + bxy + cy^2$ *such that* $a, b \neq 0, c = -a$ *and* y/x *is not appreciable.*

(5) $f(x, y) = ax^2 + bxy + cy^2$ *such that* $b = c = -a$.

Proof. (1) $n = ax^2$. Then $a, x \neq 0$. Put $n = a\,(x - t + t)^2$, where $t \geq 3x$ is prime with $t \sim 3x$. Therefore, $n = a\left((x - t)^2 + t^2 + 2t\,(x - t)\right) = a\,(x - t)^2 + at\,(2x - t)$. Let $\omega_1 = a\,(x - t)$, $\omega_2 = x - t$, $\omega_3 = at$ and $\omega_4 = 2x - t$. Clearly, ω_i is unlimited for $1 \leq i \leq 4$ and $\omega_i \sim \omega_j$ for $1 \leq i, j \leq 4$. Moreover, we can prove that $\gcd(\omega_1 \cdot \omega_2\ , \omega_3 \cdot \omega_4)$ is limited. Indeed, first we see that $\gcd(t, 2x - t) = 1$ since t is prime and $t \geq 3x$. Suppose further that $\gcd(\omega_1\omega_2, \omega_3\omega_4) = ad_1$, where $d_1 \geq 2$. Then $d_1 \mid (x - t)^2$ and $d_1 \mid t\,(2x - t)$. Hence, $d_1 \mid (x - t)^2 + t\,(2x - t) = x^2$. There are two possibilities:

- $d_1 \mid x$. Then $d_1 \mid t$ since $d_1 \mid t\,(2x - t)$, which is impossible since $\gcd(x, t) = 1$.
- $d_1 \nmid x$. We put $x^2 = q_1^{2\alpha_1} q_2^{2\alpha_2} \cdots q_r^{2\alpha_r}$, where $q_1, q_2, \ldots, q_r$ are distinct primes and $\alpha_1, \alpha_2, \ldots, \alpha_r$ are positive integers, and let $d_1 = q_1^{a_1} q_2^{a_2} \cdots q_r^{a_r}$ with $0 \leq a_i \leq 2\alpha_i$ for $1 \leq i \leq r$. We prove that every prime factor of d_1 is limited; otherwise, if p is an unlimited prime number with $p \mid d_1$, then $p \mid t$ and so $p = t$. A contradiction. Now, let $q_{i_0}^{a_0}$ be an unlimited prime power such that $q_{i_0}^{a_{i_0}} \mid d_1$, i.e., q_{i_0} is limited and a_{i_0} is unlimited. Since $q_{i_0}^{a_{i_0}} \mid x^2$, we conclude that $q_{i_0}^{\omega} \mid x$, where $\omega = a_{i_0}/2$ if a_{i_0} is even or $\omega = (a_{i_0} - 1)/2$; otherwise. Since $q_{i_0}^{\omega} \mid 2x - t$, we deduce that $q_{i_0}^{\omega} = t$. This is a contradiction since t is prime. Therefore, all the prime powers $q_1^{a_1}, q_2^{a_2}, \ldots, q_r^{a_r}$ are limited and so d_1 is also limited.

(2) Here we can assume that x and y are positive and a, c are both non-zero; otherwise, if a or c is zero, then we are in case (1). Suppose that $y > x$. If y/x is appreciable, then the proof is easy. In the case when y/x is unlimited, we see that

$$n = ax^2 + cy^2 = a\left(x - y + y\right)^2 + cy^2 = a\left(x - y\right)\left(x + y\right) + \left(a + c\right)y^2.$$

Let $\omega_1 = a\left(x - y\right)$, $\omega_2 = \left(x + y\right)$, $\omega_3 = \left(a + c\right)y$ and $\omega_4 = y$. Clearly, ω_i is unlimited for $1 \leq i \leq 4$ and $\omega_i \sim \omega_j$ for $1 \leq i, j \leq 4$. Moreover, $\gcd\left(\omega_1 \cdot \omega_2\ ,\ \omega_3 \cdot \omega_4\right)$ is limited. Indeed, if $d = \left(a\left(x - y\right)\left(x + y\right),\ \left(a + c\right)y^2\right) \cong +\infty$, then $d \mid a\left(x - y\right)\left(x + y\right)$ and $d \mid \left(a + c\right)y^2$. As in case (1), let p^a be an unlimited prime power such that p^a divides both d and y, from which it follows that $p^a \mid a\left(x - y\right)\left(x + y\right)$. This contradicts the fact that x and y are relatively prime, i.e., d is limited.

(3) Assume that $n = ax^2 + bxy + cy^2$, where $a, b, c \neq 0$ and y/x is appreciable. In this case, $n = x\left(ax + by\right) + cy^2$. Now, if $ax + by = 0$, then $n = cy^2$ and this case can be treated as in case (1); otherwise, if $\left(ax + by\right)/x$ is appreciable, then we put $\omega_1 = x$, $\omega_2 = ax + by$, $\omega_3 = cy$ and $\omega_4 = y$. Then we can easily prove that $\gcd\left(\omega_1 \cdot \omega_2\ , \omega_3 \cdot \omega_4\right)$ is limited since $\gcd\left(x, y\right) = 1$. But, if $\left(ax + by\right)/x \cong 0$, then we can write n as $n = ax^2 + y\left(bx + cy\right)$, where $\left(bx + cy\right)/y$ must be appreciable and we end the proof as before. It remains to prove that $\left(ax + by\right)/x$ and $\left(bx + cy\right)/y$ cannot be simultaneously infinitesimal. Indeed, suppose we have $ax + by = \phi_1 x = w_1$ and $bx + cy = \phi_2 y = w_2$, where ϕ_1 and ϕ_2 are two infinitesimal numbers, that is, we have the following system:

$$\begin{cases} a \cdot x + b \cdot y = w_1 \\ b \cdot x + c \cdot y = w_2. \end{cases}$$

The solution of this system is $y = \dfrac{b \cdot w_1 - a \cdot w_2}{b^2 - ac}$ and $x = \dfrac{b \cdot w_2 - c \cdot w_1}{b^2 - ac}$. But this is a contradiction because this means that $y = \phi y$ and $x = \widetilde{\phi} x$, where ϕ and $\widetilde{\phi}$ are also infinitesimal.

(4) Consider the case when $n = ax^2 + bxy + cy^2$, where $a, b \neq 0, c = -a$ and y/x is unlimited. Then $n = ax^2 + bxy - ay^2$. Put $x = x - y + y$ we get

$$n = \left(x - y\right)\left(ax + \left(a + b\right)y\right) + by^2.$$

If $a + b \neq 0$, then the proof is completed for this case by choosing $\omega_1 = x - y$, $\omega_2 = ax + \left(a + b\right)y$, $\omega_3 = by$ and $\omega_4 = y$. Otherwise, $b = -a$, and so $n = ax^2 - axy - ay^2$, in which case we get $n = a\left(x + 2y\right)^2 - 5ya\left(x + y\right)$. This ends the proof for this case by setting $\omega_1 = a\left(x + 2y\right)$, $\omega_2 = x + 2y$, $\omega_3 = -5ya$ and $\omega_4 = x + y$. As before, we can prove that $\gcd\left(\omega_1 \cdot \omega_2\ , \omega_3 \cdot \omega_4\right)$ is limited. Using the same way as above we can consider the case when $n = ax^2 + bxy + cy^2$, where $a, b \neq 0, c = -a$ and x/y is unlimited.

(5) Here we can follow the same argument as in the proof of (4).

The proof of Theorem 10 is finished. □

5.1. Examples

Applying the above theorems we find the following examples:

1) Let p be an unlimited prime number with $p \equiv 1(\text{mod } 4)$. By Niven [17, Lemma 2.13, p. 54], there exist positive integers s, t for which $p = s^2 + t^2$. Hence by Theorem 9, p can be written as in (A'_2).

2) Let p be an unlimited prime number such that $(p/13) = (p/17) = 1$. By [17, Proposition 11.3.3, p. 324], either $p = x^2 + xy - 55y^2$ or $p = -x^2 + xy + 55y^2$, but not both represent p. Hence by Theorem 9, p can be written as in (A'_2).

3) Let p be an unlimited prime number such that $(-2/p) = (p/13) = 1$. Then at least one of the following statements is true: (a) both p and $2p$ can be written as in (A'_2). (b) both $3p$ and $5p$ can be written as in (A'_2). Indeed, by Lehman [15, Proposition 7.3.2, p. 216], one and only one of the following is true: (a) The equations $x^2 + 26y^2 = p$ and $2x^2 + 13y^2 = 2p$ both have solutions in integers. (b) The equations $x^2 + 26y^2 = 3p$ and $2x^2 + 13y^2 = 5p$ both have solutions in integers. Hence, Theorem 9 gives us the response. Here, we remark that if we can write p and $2p$ as in (A'_2), then we can do the same for $3p$ and $5p$, while the converse is not true.

4) Let p be an unlimited prime number which is not congruent to 13, 17, 19, or 23 modulo 24. Since p is not divisible by 4 and 9, we conclude from Lehman [15, Proposition 7.2.3, p. 207] that p is either properly represented by $x^2 + 6y^2$ or by $2x^2 + 3y^2$. Hence, by Theorem 10, p can be written as in (A'_2).

5) Let p be an unlimited prime number which is not divisible by any prime congruent to $3, 5, 6 \text{ (mod } 7)$. Then p is represented as in (A'_2). Indeed, in this case, p is not divisible by 49. Then, by [15, Corollary 2.5.4, p. 84], p is properly represented $x^2 + 7y^2$. Applying Theorem 10, p can be written as in (A'_2).

6. Some equivalent internal statements

All variables range over positive integers. First, let us consider (F_3): Every unlimited v can be written in the form $v = a \cdot x^2 + b \cdot y^2$, where a, b are limited. The external statement (F_3) is equivalent to the following internal statement (S_3): *There is a finite set* $\{\langle a_1, b_1\rangle, \ldots, \langle a_k, b_k\rangle\}$ *and a number* s *such that for every* $n \geq s$ *there exist* $i \leq k$ *and* x, y *such that* $n = a_i \cdot x^2 + b_i \cdot y^2$.

Proposition 7. $(F_3) \Leftrightarrow (S_3)$.

Proof. First, assume that (S_3) holds. By transfer, the set $\{\langle a_1, b_1\rangle, \ldots, \langle a_k, b_k\rangle\}$ and the number s can be taken to be standard. If v is unlimited, then $v > s$, so $a_i \cdot x^2 + b_i \cdot y^2$ for some standard i, a_i and b_i. This proves (F_3). Conversely, assume that (S_3) holds. Then for every standard finite set $\{\langle a_1, b_1\rangle, \ldots, \langle a_k, b_k\rangle\}$ and every standard number s there exists n such that for every $i \leq k$ we have $n \geq s \wedge \forall x, y$ $(n \neq a_i \cdot x^2 + b_i \cdot y^2$. By idealization[‡], there is v such that for every standard $\langle a, b\rangle$

[‡]Idealization (see F. Diener [6, pp.9, 21]): $\forall^{\text{stfin}} z \exists y \forall x \in z \;\; B(x, y, t) \Leftrightarrow \exists y \forall^{st} x \; B(x, y, t)$. The only nonlogical symbol of B must be $\in$ (that is, B must be internal). The parameter t may take

and every standard s we have $v \geq s \wedge \forall x, y\ (v \neq a \cdot x^2 + b \cdot y^2)$. So v is unlimited and it cannot be written in the desired form. □

Next, let us consider (F_3^*), which is obtained from (F_3) by adding the requirement that x/y be appreciable. Note that (F_3^*) is equivalent to the following internal statement (S_3^*): *There is a finite set* $\{\langle a_1, b_1\rangle, \ldots, \langle a_k, b_k\rangle\}$ *and numbers* m, s *such that for every* $n \geq s$ *there exist* $i \leq k$ *and* $x, y \geq \sqrt{n}/m$ *such that* $n = a_i \cdot x^2 + b_i \cdot y^2$.

Proposition 8. $(F_3^*) \Leftrightarrow (S_3^*)$.

Proof. Assume (S_3^*) holds. By transfer, the set $\{\langle a_1, b_1\rangle, \ldots, \langle a_k, b_k\rangle\}$ and the numbers m, s can be taken to be standard. If v is unlimited, then $v > s$, so $v = a_i \cdot x^2 + b_i \cdot y^2$ for some standard a, b and $x, y \geq \frac{\sqrt{v}}{m}$. Of course, also $x, y \leq \sqrt{v}$, hence $1/m \leq x/y \leq m$.

Assume the negation of (S_3^*) holds. As in the proof of "(F_3) implies (S_3)", we obtain v such that for every standard $\langle a, b\rangle$ and every standard m, s we have $v \geq s \wedge \forall x, y \geq \frac{\sqrt{v}}{m}\ (v \neq a \cdot x^2 + b \cdot y^2)$.

Suppose that for some standard a, b we have $v = a \cdot x^2 + b \cdot y^2$, where x/y is appreciable. Then $1/\ell \leq x/y \leq 1/\ell$ holds for some standard ℓ. It follows that $y \leq x \cdot \ell$ and $x \leq y \cdot \ell$, hence $v \leq (a + b \cdot \ell^2) \cdot x^2$ and $v \leq (a \cdot \ell^2 + b) \cdot y^2$. Fix a standard $m \geq \max(\sqrt{a + b \cdot \ell^2}, \sqrt{a \cdot \ell^2 + b})$. Then $x, y \geq \frac{\sqrt{v}}{m}$, a contradiction. □

If (F_3^*) is true, then (F_2): *Every unlimited* v *can be written in the form* $v = x_1 \cdot x_2 + x_3 \cdot x_4$, *where all* x_i *are unlimited and* x_i/x_j *is always appreciable is true.* Statement (F_2) is equivalent to the internal statement

(S_2): *There are numbers* m, s *such that for every* $n \geq s$ *there exist* x_1, x_2, x_3, x_4 *such that* $n = x_1 \cdot x_2 + x_3 \cdot x_4$ *and* $\sqrt{v}/m \leq x_i \leq m \cdot \sqrt{v}$ *holds for* $1 \leq i \leq 4$.

Proposition 9. $(F_2) \Leftrightarrow (S_2)$.

Proof. Similar to the preceding proof. On the one hand, note that the condition $\sqrt{v}/m \leq x_i \leq m \cdot \sqrt{v}$ implies that $1/m^2 \leq x_i/x_j \leq m^2$, so all the ratios x_i/x_j are appreciable. On the other hand, if $1/k \leq x_i/x_j \leq k$ holds for all i, j (where k is standard), we have $(1/k)x_j \leq x_i \leq k \cdot x_j$ for all i, j. From this one gets $(1/k + 1/k^2) \cdot x_i^2 \leq x_1 \cdot x_2 + x_3 \cdot x_4 = v \leq (k + k^2) \cdot x_i^2$. Let $m \geq \max\left(\sqrt{k + k^2}, k/\sqrt{1 + k}\right)$ be standard. The above inequality gives $(1/m^2) \cdot x_i^2 \leq v \leq m^2 \cdot x_i^2$ and $\sqrt{v}/m \leq x_i \leq m \cdot \sqrt{v}$ for all i. □

In addition, Theorem 1 is equivalent to the following internal statement:

Theorem 11. *There exists* $(i, j) \in \mathbb{N}^2$ *such that every* $\omega \geq i$ *can be written as* $\omega = \omega_1 \cdot \omega_2 + \omega_3 \cdot \omega_4$, *where* ω_l *is a positive integer with* $\omega_l/\sqrt{\omega} \in [1/j, j]$ *for* $1 \leq l \leq 4$.

Proof. We write Theorem 1 as follows:

$$\forall\ \omega\ [\ \forall^{st}\ i\quad (\ \omega > i) \Rightarrow \exists\ (\omega_1, \omega_2, \omega_3, \omega_4)$$
$$\exists^{st}\ j\ \forall\ l \in \{1, \ldots, 4\}\quad \left(\omega_l/\sqrt{\omega} \in [1/j, j]\right)\ \&\ \omega = \omega_1 \cdot \omega_2 + \omega_3 \cdot \omega_4\ ],$$

any value.

where all variables range over positive integers. This is equivalent to

$$\begin{aligned}&\forall\ \omega\ \exists^{st}\ j\ \exists^{st}\ i\quad [\ (\ \omega > i)\\ &\Rightarrow \exists\ (\omega_1,\omega_2,\omega_3,\omega_4)\ \forall\ l\in\{1,\dots,4\}\,,\big(\omega_l/\sqrt{\omega}\in[1/j,j]\big)\ \&\ \omega=\omega_1\cdot\omega_2+\omega_3\cdot\omega_4\ ]\,.\end{aligned}$$

By idealization, we obtain

$$\begin{aligned}&\exists^{st}\ i\ \exists^{st}\ j\ \forall\ \omega\ [\ (\ \omega > i)\\ &\Rightarrow \exists\ (\omega_1,\omega_2,\omega_3,\omega_4)\ \forall\ l\in\{1,\dots,4\}\,,\big(\ \omega_l/\sqrt{\omega}\in[1/j,j]\big)\ \&\ \omega=\omega_1\cdot\omega_2+\omega_3\cdot\omega_4\ ]\,.\end{aligned}$$

Now, by transfer, the last formula is equivalent to

$$\begin{aligned}&\exists\ i\ \exists\ j\ \forall\ \omega\ [\ (\ \omega > i)\\ &\Rightarrow \exists\ (\omega_1,\omega_2,\omega_3,\omega_4)\ \forall\ l\in\{1,\dots,4\}\,,\big(\omega_l/\sqrt{\omega}\in[1/j,j]\big)\ \&\ \omega=\omega_1\cdot\omega_2+\omega_3\cdot\omega_4\ ]\,.\end{aligned}$$

This completes the proof. □

Finally, we obtain a generalization of the above theorem as follows:

Corollary 7. *Let $k\geq 2$ be a fixed standard integer. Then there exists $(i,j)\in\mathbb{N}^2$ such that every $\omega\geq i$ can be written as $\omega=\omega_1\cdot\omega_2+\cdots+\omega_{2k-1}\cdot\omega_{2k}$, where ω_l is a positive integer with $\omega_l/\sqrt{\omega}\in[1/j,j]$ for $l=1,2,\dots,2k$.*

Proof. Corollary 1 is equivalent to the following internal statement:

$$\begin{aligned}\forall\ \omega\ \big[\ \forall^{st}\ i\quad (\ \omega > i)\ \Rightarrow\ &\exists\,\{\omega_1,\dots,\omega_{2k}\}\,\exists^{st}j\ \forall l\in\{1,\dots,2k\}\,,\\ &\big(\omega_l/\sqrt{\omega}\in[1/j,j]\big)\ \&\ \omega=\omega_1\omega_2+\cdots+\omega_{2k-1}\omega_{2k}\ ]\,,\end{aligned}$$

where k is a standard positive integer. The unique free variable is k and it is standard, so we can apply the same method as before to show that the last formula is equivalent to

$$\begin{aligned}\exists\ i\ \exists\ j\ \forall\ \omega\ [\ (\ \omega > i)\ \Rightarrow\ &\exists\ (\omega_1,\dots,\omega_{2k})\ \forall\ l\in\{1,\dots,2k\}\,,\\ &\big(\omega_l/\sqrt{\omega}\in[1/j,j]\big)\ \&\ \omega=\omega_1\omega_2+\cdots+\omega_{2k-1}\omega_{2k}\ ]\,,\end{aligned}$$

as required. □

7. Open questions

For further research, we propose the following questions on the representation of unlimited integers as in (A_2).

1. We ask if every unlimited positive integer n is of the form $n=\omega_1\cdot\omega_2+\omega_3\cdot\omega_4$, where $\omega_i\in\mathbb{N}$ is unlimited and $\omega_i\sim\omega_j$ for $1\leq i,j\leq 4$ with $\gcd(\omega_i,\omega_j)=1$ for $i\neq j$.

2. Let ω be unlimited. Consider the numbers $n=a_1a_2\cdots a_\omega$, where a_i is standard for every i standard and $a_{i+1}/a_i\cong\infty$ for $i\cong\infty$. For example, n is the product of Fermat numbers, i.e., $n=f_0f_1\cdots f_\omega$ with $\omega\cong\infty$, where $f_n=2^{2^n}+1$ ($n\geq 0$). As in the proof of Theorem 4, we ask if we can determine effective values $\omega_1,\omega_2,\omega_3,\omega_4$ such that $n=\omega_1\cdot\omega_2+\omega_3\cdot\omega_4$, where $\omega_i\sim\omega_j$ for $1\leq i,j\leq 4$.

3. Does result (5) in Corollary 2 in Section 2 hold for $k = 2$? In other words, we ask whether every unlimited positive integer n is of the form $n = \omega_1 \cdot \omega_2 \cdot \omega_3 + \omega_4 \cdot \omega_5 \cdot \omega_6$, where $\omega_i \in \mathbb{N}$ is unlimited with $\omega_i \sim \omega_j$ for $1 \leq i, j \leq 6$.

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