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NUMERICAL SOLUTIONS OF NONLINEAR QUADRATIC VOLTERRA INTEGRAL EQUATIONS USING VIETA-LUCAS WAVELETS

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Abstract. In this article, we present a numerical approach for solving Nonlinear Quadratic Volterra Integral Equations (NQVIEs) with the collocation method using Vieta-Lucas Wavelets (VLWs) and the Legendre-Gauss Quadrature Rule (LGQR). First, we prove the existence and uniqueness of the main problem under specific conditions. Then, we apply the proposed method; the NQVIEs well be reduced to a system of nonlinear algebraic equations that can be solved by Newton's method. We also estimate the error bound and the convergence of the presented method. Several numerical examples are mentioned in order to demonstrate its effectiveness and accuracy in solving NQVIEs.

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1. Introduction

Integral equations are very important tools for many scientific fields. Volterra integral equations, linear or nonlinear, are an important class of integral equations. Recently, many studies and much research aim to develop numerical methods to find the approximate solutions of these equations. For example, a combination of the quasilinearization method and linear barycentric rational interpolation is used to solve nonlinear multi-dimensional Volterra integral equations [1]. An iterative scheme based on the quasilinearization method and the Jacobi-Galerkin method is used to solve a family of functional Volterra integral equations [2]. Spectral methods are widely used to solve these kinds of equations using orthogonal polynomials. For example, the Legendre-collocation method for solving systems of Volterra integral equations of the second kind [3]. Using Vieta-Lucas Polynomials (VLPs) has several benefits because of their many interesting and useful characteristics, i.e., they produce precise approximations with less computational work. Furthermore, the related errors are very low [4].

Special kinds of equations called Quadratic Integral Equations (QIEs) are used to characterize various phenomena of the real world. The NQVIEs of the second kind have the following general form:

$$u(t) = g(t) + f(t,u(t)) \int_0^t k(t,s,u(s))\, ds\,, \quad 0 \leq t < 1, \tag{1}$$

where $u(t)$ is the unidentified function, $g(t)$, $f(t,u(t))$, and $k(t,s,u(s))$ three functions are given.
We consider two cases for the function $f(t,u(t))$:

- **Case** 1**:** If $f(t,u(t))$ is given directly.
- **Case** 2**:** If $f(t,u(t)) = \int_0^t v(t,s,u(s))\, ds$, where $v(t,s,u(s))$ is given.

This class of equations is used for modeling a lot of problems in engineering and science, such as gas kinetic theory, traffic theory, and neutron transport theory, etc. [5–7]. In the last few years, scientific focus has been directed toward the existence, uniqueness, positive solutions, and monotonic solutions of this class of equations [8–11], and the numerical methods for approximating them. In general, these types of equations have a certain level of difficulty and are sometimes unsolvable. Therefore, numerous methods have been proposed to approximate the QIEs solutions, such as the Fixed Point Method (FPM) [12], Homotopy Analysis Method (HAM) [13], Homotopy Perturbation Methods (HPM) [14], Müntz-Legendre Wavelets Method (MLWM) [15], and several other methods [9, 16]. However, achieving both high accuracy and computational efficiency remains a challenge, especially when dealing with complex integral kernels and nonlinearities.

In this study, we look for numerical approximations of solutions for NQVIEs using VLWs. The wavelets method serves as a powerful tool for function approximation and for solving NQVIEs. VLWs, derived from VLPs, provide orthogonality and fast convergence, making them well-suited for solving integral equations. To further enhance the accuracy of the proposed approach, we also estimate the integral using LGQR, which enables efficient numerical evaluation of integrals.

The remainder of this paper is structured as follows: we begin by discussing the **existence and uniqueness** of solutions to NQVIEs. In the **Basic Definitions** section, we introduce fundamental concepts relevant to LGQR, VLPs, and VLWs. The **Proposed Method** section describes the approach for solving NQVIEs using VLWs and LGQR, followed by the **Convergence Analysis** section. The **Numerical Experiments** section presents illustrative examples. Finally, we **conclude** with a **discussion** of the obtained numerical results.

2. Existence and uniqueness

The problem of the existence and uniqueness of solutions for NQVIEs was discussed in [8, 11, 17]. In our work, we present a new case in which NQVIEs admit a solution.

Let $I=[0,1]$. Denote by $E=C(I)$ the space of continuous functions defined on I with norm $\|u\|=\sup\limits_{t\in I}|u(t)|$. Consider the NQVIE in (1),

$$u(t)=g(t)+f(t,u(t))\int_0^t k(t,s,u(s))ds.$$

Given the following assumptions:
(i) $g: I\to\mathbb{R}$ is continuous.
(ii) $f: I\times E\subset\mathbb{R}\to\mathbb{R}$ is continuous.
(iii) $k,v: I\times I\times E\subset\mathbb{R}\to\mathbb{R}$ are continuous.
(iv) f,v,k satisfy the Lipschitz condition; there exist two positive constants, L_1 and L_2, so that $|k(t,s,x)-k(t,s,y)|\leq L_1|x-y|$ and $|f(t,x)-f(t,y)|\leq L_2|x-y|$ (respectively, $|v(t,s,x)-v(t,s,y)|\leq L_2|x-y|$) for **case 1** (respectively, for **case 2**) for all $t,s\in I$ and $x,y\in E$.
(v) $\lambda_1=\sup\{|k(t,s,x)|: t,s\in I, x\in E\}$.
(vi) $\lambda_2=\sup\{|f(t,x)|: t\in I, x\in E\}$ (respectively, $\lambda_2=\sup\{|v(t,s,x)|: t,s\in I$, $x\in E\}$) for **case 1** (respectively, for **case 2**).

Theorem 1 *Let the conditions (i)-(vi) and $(L_2\lambda_1+L_1\lambda_2)<1$ be satisfied. Then, the NQVIE* (1) *has a unique solution $u\in C(I)$.*

PROOF Let F be a mapping defined as $F: E\to E$, $Fu=g(t)+f(t,u(t))\int_0^t k(t,s,u(s))ds$,
case 1: Let $x,y\in E$, then

$$\begin{aligned}(Fx)(t)-(Fy)(t)&=f(t,x(t))\int_0^t k(t,s,x(s))ds-f(t,y(t))\int_0^t k(t,s,y(s))ds,\\ &=f(t,x(t))\int_0^t k(t,s,x(s))ds-f(t,y(t))\int_0^t k(t,s,x(s))ds\\ &+f(t,y(t))\int_0^t k(t,s,x(s))ds-f(t,y(t))\int_0^t k(t,s,y(s))ds, \qquad (2)\\ &=(f(t,x(t))-f(t,y(t)))\int_0^t k(t,s,x(s))ds\\ &+f(t,y(t))\left(\int_0^t k(t,s,x(s))-k(t,s,y(s))ds\right),\end{aligned}$$

so, we have

$$\begin{aligned}\|(Fx)(t)-(Fy)(t)\|&\leq\max_{t\in I}|f(t,x(t))-f(t,y(t))|\int_0^t|k(t,s,x(s))|ds\\ &+\max_{t\in I}|f(t,y(t))|\int_0^t|k(t,s,x(s))-k(t,s,y(s))|ds, \qquad (3)\\ &\leq L_2\lambda_1\max_{t\in I}|x(t)-y(t)|+L_1\lambda_2\max_{t\in I}|x(t)-y(t)|,\\ &\leq(L_2\lambda_1+L_1\lambda_2)\|x(t)-y(t)\|,\end{aligned}$$

under the condition $L_2\lambda_1 + L_1\lambda_2 < 1$, the mapping F is a contraction on E. Then, there exists a unique solution of Eq. (1). For **case 2**, with similar steps and techniques, we reach the same constant $L_2\lambda_1 + L_1\lambda_2 < 1$. ■

3. Basic definitions

This section provides definitions that will be used to solve the NQVIEs in the following section.

3.1. Legendre-Gauss Quadrature Rule (LGQR)

Let σ_i be the roots of the Legendre polynomial $L_G(x)$ of degree G, where $i = 1,\dots,G$. The G-point Legendre-Gauss Quadrature Rule is used to approximate the integral of a function f from a to b as follows [18]:

$$\int_a^b f(x)\,dx \simeq \sum_{i=1}^{G} w_i f(\eta_i), \tag{4}$$

where the Legendre-Gauss quadrature nodes η_i and weights w_i are given by:

$$w_i = \frac{b-a}{(1-\sigma_i^2)(L_G'(\sigma_i))^2}, \quad \eta_i = \frac{b-a}{2}\sigma_i + \frac{b+a}{2}, \quad i = 1,\dots,G. \tag{5}$$

The error of the Legendre-Gauss quadrature estimation is given by [18]

$$\int_a^b f(x)\,dx - \sum_{i=1}^{G} w_i f(\eta_i) = \frac{2^{2G+3}((G+1)!)^4 f^{(2G+2)}(c)}{(2G+3)((2G+2)!)^3}, c \in [a,b], \tag{6}$$

so that

$$\left| \int_a^b f(x)\,dx - \sum_{i=1}^{G} w_i f(\eta_i) \right| \leq \Gamma_G, \tag{7}$$

where $\Gamma_G = \dfrac{2^{2G+3}((G+1)!)^4 \max_{c\in[a,b]} \left|f^{(2G+2)}(c)\right|}{(2G+3)((2G+2)!)^3}$.

3.2. Vieta-Lucas Polynomials (VLPs)

The Vieta-Lucas Polynomials (VLPs) $V_m(t)$ of degree m ($m \in \mathbb{N}$) are defined as follows [19, 20]:

$$V_m(t) = 2\cos\left(m \arccos\left(\tfrac{t}{2}\right)\right), \quad t \in [-2,2]. \tag{8}$$

The recurrence relation for $V_m(t)$ is given by:

$$\begin{cases} V_0(t) = 2, \\ V_1(t) = t, \\ V_m(t) = tV_{m-1}(t) - V_{m-2}(t), \quad m = 2,3,4,\dots. \end{cases} \tag{9}$$

The VLPs can also be expressed as a power series:

$$V_m(t) = \begin{cases} 2, & m = 0, \\ \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^i \frac{m(m-i-1)!}{i!(m-2i)!} t^{m-2i}, & m = 1,2,3,4,\dots. \end{cases} \tag{10}$$

Here, $\lfloor \frac{m}{2} \rfloor$ denotes the floor function of $\frac{m}{2}$.

Additionally, the VLPs $V_n(t)$ and $V_m(t)$ are orthogonal over the area $[-2,2]$ with regard to the weight function $w(t) = \frac{1}{\sqrt{4-t^2}}$. Therefore, they fulfill this resulting orthogonality condition:

$$\langle V_n(t), V_m(t) \rangle_{w(t)} = \int_{-2}^{2} V_n(t) V_m(t) w(t)\, dt = \begin{cases} 4\pi, & n = m = 0, \\ 2\pi, & n = m \neq 0, \\ 0, & n \neq m \neq 0. \end{cases} \tag{11}$$

3.3. Vieta-Lucas Wavelets (VLWs)

The Vieta-Lucas Wavelets $\Upsilon_{s,m}(t) = \Upsilon(k,s,m,t)$ are defined on the interval $[0,1)$ as follows [19]:

$$\Upsilon_{s,m}(t) = \begin{cases} \frac{2^{\frac{k+2}{2}}}{\sqrt{\alpha_m}} V_m\left(2^{k+2}t - 4s + 2\right), & \frac{s-1}{2^k} \leq t < \frac{s}{2^k}, \\ 0, & \text{otherwise}. \end{cases} \tag{12}$$

Here, k is a positive integer, $s = 1,2,\dots,2^k$, and $m = 0,1,2,\dots,M+1$, where $M+1$ is the highest degree of the VLPs. The normalization factor α_m is defined as

$$\alpha_m = \begin{cases} 4\pi, & m = 0, \\ 2\pi, & m \geq 1. \end{cases}$$

The VLWs functions form an orthogonal system in $L^2_{w_s}[0,1]$ with regard to the weight function $w_s(t) = w\left(2^{k+2}t - 4s + 2\right)$. This means that the following orthogonality condition holds:

$$\left\langle \Upsilon_{s,m}(t), \Upsilon_{s',m'}(t) \right\rangle_{w_s(t)} = \int_0^1 \Upsilon_{s,m}(t) \Upsilon_{s',m'}(t) w_s(t)\, dt = \begin{cases} 1, & (s,m) = (s',m'), \\ 0, & (s,m) \neq (s',m'). \end{cases} \tag{13}$$

Using the VLWs series, a function $u(t)$ defined over $L^2_{w_s}[0,1]$ can be approximated as

$$u(t) = \sum_{s=1}^{\infty} \sum_{m=0}^{\infty} u_{s,m} \Upsilon_{s,m}(t), \tag{14}$$

where the coefficients $u_{s,m}$ are given by

$$u_{s,m} = \int_0^1 u(t)\Upsilon_{s,m}(t)w_s(t)\,dt. \tag{15}$$

The truncated form of the VLWs series can be written as

$$u(t) \simeq u^*(t) = \sum_{s=1}^{2^k}\sum_{m=0}^{M+1} u_{s,m}\Upsilon_{s,m}(t) = U^T\Upsilon(t), \tag{16}$$

where U and $\Upsilon(t)$ are $\mathfrak{M}\times 1$ matrices, where $\mathfrak{M} = 2^k(M+2)$. These matrices are defined as

$$U = \left[u_{1,0}, u_{1,1}, \ldots, u_{1,M+1} \mid u_{2,0}, \ldots, u_{2,M+1} \mid \ldots \mid u_{2^k,0}, \ldots, u_{2^k,M+1}\right]^T,$$
$$\Upsilon(t) = \left[\Upsilon_{1,0}(t), \ldots, \Upsilon_{1,M+1}(t) \mid \Upsilon_{2,0}(t), \ldots, \Upsilon_{2,M+1}(t) \mid \ldots \mid \Upsilon_{2^k,0}(t), \ldots, \Upsilon_{2^k,M+1}(t)\right]^T. \tag{17}$$

4. The proposed method

In this section, we apply the VLWs method, supported by LGQR, to solve the NQVIEs of the form:

$$u(t) = g(t) + f(t, u(t)) \int_0^t k(t, s, u(s))\,ds, \quad 0 \le t < 1. \tag{18}$$

First, we approximate the function $u(t)$ in Eq. (18) using the VLWs method in the following way:

$$u(t) \simeq u^*(t) = U^T\Upsilon(t), \tag{19}$$

where U is an $\mathfrak{M}$-dimensional column vector containing unknown coefficients, and $\Upsilon(t)$ is as defined in (16) and (17). Substituting Eq. (19) into Eq. (18), we obtain

$$U^T\Upsilon(t) = g(t) + f\left(t, U^T\Upsilon(t)\right) \int_0^t k\left(t, s, U^T\Upsilon(s)\right) ds. \tag{20}$$

Next, we set $s = t\varepsilon$, where $\varepsilon \in [0,1]$, and apply LGQR to approximate the integral:

$$\begin{aligned} \int_0^t k\left(t, s, U^T\Upsilon(s)\right) ds &= \int_0^1 tk\left(t, t\varepsilon, U^T\Upsilon(t\varepsilon)\right) d\varepsilon \\ &= \sum_{i=1}^{G} tw_i k\left(t, t\eta_i, U^T\Upsilon(t\eta_i)\right), \end{aligned} \tag{21}$$

where w_i and η_i are the weights and nodes of the LGQR, respectively.

Substituting Eq. (21) into Eq. (20), we derive

$$U^T\Upsilon(t) = g(t) + f\left(t, U^T\Upsilon(t)\right)\sum_{i=1}^{G} tw_i k\left(t, t\eta_i, U^T\Upsilon(t\eta_i)\right). \quad (22)$$

Now, we deal with two cases for the function $f(t,u(t))$:

Case 1: If $f(t,u(t))$ is explicitly given, we collocate the Newton-Cotes nodes $t_j = \frac{2j-1}{2\mathfrak{M}}$, for $j = 1, \ldots, \mathfrak{M}$ into Eq. (22), resulting in a nonlinear algebraic system for the unknown coefficient vector U, which can be solved using the Newton method.

Case 2: If $f(t,u(t)) = \int_0^t v(t,s,u(s))\,ds$, as in Eq. (18), we approximate the integral using the same LGQR method from Eq. (21), yielding

$$\int_0^t v\left(t,s,U^T\Upsilon(s)\right)ds = \sum_{l=1}^{G'} tw_l v\left(t, t\eta_l, U^T\Upsilon(t\eta_l)\right). \quad (23)$$

Substituting Eqs. (23) and (21) into Eq. (20), we obtain

$$U^T\Upsilon(t) = g(t) + \sum_{l=1}^{G'} tw_l v\left(t, t\eta_l, U^T\Upsilon(t\eta_l)\right)\sum_{i=1}^{G} tw_i k\left(t, t\eta_i, U^T\Upsilon(t\eta_i)\right). \quad (24)$$

We consistently choose $G = G'$. Substituting the Newton-Cotes nodes $t_j = \frac{2j-1}{2\mathfrak{M}}$, for $j = 1, \ldots, \mathfrak{M}$ into Eq. (24), resulting in a nonlinear algebraic system for the unknown coefficient vector U, which is also solved using the Newton method.

5. Convergence analysis

Theorem 2 *Assume that $u(t) \in L^2_{w_s}[0,1]$ with bounded second derivative $\left|u''(t)\right| \leq H$, $H > 0$. Then, we can expand function $u(t)$ as an infinite sum of VLWs, and the series is uniformly convergent to $u(t)$ as $k, M \to \infty$. Additionally, the following inequalities are satisfied by the coefficients $u_{s,m}$ defined in* (15)*:*

$$|u_{s,m}| \leq \frac{H\sqrt{\pi}}{2^{\frac{5}{2}} s^{\frac{5}{2}} (m^2-1)}, m > 1, s \geq 1 \text{ and } |u_{s,1}| \leq \frac{\sqrt{\pi}}{2^{\frac{5}{2}} s^{\frac{3}{2}}} \max_{0 \leq t \leq 1} \left|u'(t)\right|, m = 1, s \geq 1.$$

PROOF see [19]. ∎

Lemma 1 *Let $f(t)$ be a continuous, positive, decreasing function for $t \geq s$ if $f(k) = \Lambda_k$, provided that $\sum \Lambda_s$ is convergent, then $\sum_{k=s+1}^{\infty} \Lambda_k \leq \int_s^{\infty} f(t)dt$.*

Theorem 3 *Let us assume that $u(t)$ satisfies Theorem 2. And let $u^*(t)$ be the approximation of $u(t)$ using VLWs. Then, the estimate of the error bound between $u(t)$ and $u^*(t)$ is*

$$|u(t) - u^*(t)| \leq \Theta_M^k, \text{ where } \Theta_M^k = \frac{H(2M+3)}{3 \times 2^{k+1}(M+1)(M+2)}. \tag{25}$$

PROOF Considering the VLWs expansion as

$$u^*(t) = \sum_{s=1}^{2^k} \sum_{m=0}^{M+1} u_{s,m} \Upsilon_{s,m}(t),$$

from (14), we obtain

$$|u(t) - u^*(t)| \leq \sum_{s=2^k+1}^{\infty} \sum_{m=M+2}^{\infty} |u_{s,m}| \, |\Upsilon_{s,m}(t)| \, .$$

By using the definition of $\Upsilon_{s,m}(t)$ in (12) and Theorem 2 for $m > 1$, it can be expressed as

$$\begin{aligned} |u(t) - u^*(t)| &\leq \frac{2^{\frac{k+3}{2}}}{\sqrt{\pi}} \sum_{s=2^k+1}^{\infty} \sum_{m=M+2}^{\infty} |u_{s,m}| \, , \\ &\leq \frac{2^{\frac{k+3}{2}}}{\sqrt{\pi}} \sum_{s=2^k+1}^{\infty} \sum_{m=M+2}^{\infty} \frac{H\sqrt{\pi}}{2^{\frac{5}{2}} s^{\frac{5}{2}} |m^2 - 1|}, \\ &= H 2^{\frac{k-2}{2}} \left(\sum_{s=2^k+1}^{\infty} \frac{1}{s^{\frac{5}{2}}} \right) \left(\sum_{m=M+2}^{\infty} \frac{1}{|m^2 - 1|} \right), \\ &= H 2^{\frac{k-2}{2}} \left(\sum_{s=2^k+1}^{\infty} \frac{1}{s^{\frac{5}{2}}} \right) \left(\frac{2M+3}{2(M+1)(M+2)} \right), \end{aligned} \tag{26}$$

from Lemma 1, we obtain

$$\begin{aligned} |u(t) - u^*(t)| &\leq H 2^{\frac{k-2}{2}} \left(\int_{\xi=2^k}^{\infty} \frac{d\xi}{\xi^{\frac{5}{2}}} \right) \left(\frac{2M+3}{2(M+1)(M+2)} \right), \\ &= \Theta_M^k \xrightarrow[k,m \to \infty]{} 0. \end{aligned} \tag{27}$$

The inequality (27) implies that the error between $u(t)$ and $u^*(t)$ using VLWs is inversely proportional to k and M, and it is reduced while increasing M and k. ∎

Theorem 4 *Let $u(t)$ and $u^*(t)$ be the exact and approximate solutions of equation (1), respectively. We assume that the conditions (i)-(vi), $(L_1\lambda_2 + L_2\lambda_1) < 1$, and Theorem 2 are satisfied. Then, respectively in **case 1** and **case 2**, we have*

$\|u(t)-u^*(t)\| \leq \max\left\{\Theta_M^k, (L_2\lambda_1+L_1\lambda_2)\Theta_M^k+\lambda_2\Gamma_G\right\}$ and

$\|u(t)-u^*(t)\| \leq \max\left\{\Theta_M^k, L_1(\lambda_2+\Gamma'_G)\Theta_M^k+\Gamma_G(\lambda_2+\Gamma'_G)+\Gamma'_G\lambda_1\right\}.$

PROOF Consider equation (22) in **case 1**:

$$u^*(t)=g(t)+f(t,u^*(t))\sum_{i=1}^{G} tw_i k(t,t\eta_i,u^*(t\eta_i)), \tag{28}$$

where $u^*(t)=U^T\Upsilon(t)$, by subtracting (1) and (28), we get

$$\begin{aligned}
u(t)-u^*(t) &= f(t,u(t))\int_0^t k(t,s,u(s))ds - f(t,u^*(t))\sum_{i=1}^{G} tw_i k(t,t\eta_i,u^*(t\eta_i)),\\
&= f(t,u(t))\int_0^t k(t,s,u(s))ds - f(t,u^*(t))\int_0^t k(t,s,u(s))ds\\
&+ f(t,u^*(t))\int_0^t k(t,s,u(s))ds - f(t,u^*(t))\sum_{i=1}^{G} tw_i k(t,t\eta_i,u^*(t\eta_i)),\\
&= \left(f(t,u(t))-f(t,u^*(t))\right)\int_0^t k(t,s,u(s))ds\\
&+ f(t,u^*(t))\left(\int_0^t k(t,s,u(s))ds - \sum_{i=1}^{G} tw_i k(t,t\eta_i,u^*(t\eta_i))\right),\\
&= \left(f(t,u(t))-f(t,u^*(t))\right)\int_0^t k(t,s,u(s))ds + f(t,u^*(t))\left(\int_0^t k(t,s,u(s))\right.\\
&\left. -k(t,s,u^*(s))ds + \int_0^t k(t,s,u^*(s))ds - \sum_{i=1}^{G} tw_i k(t,t\eta_i,u^*(t\eta_i))\right),
\end{aligned} \tag{29}$$

Now, suppose that $u(t)$ satisfies the conditions (i)-(vi), and from Eq. (7) we get

$$\begin{aligned}
\|u(t)-u^*(t)\| &\leq |f(t,u(t))-f(t,u^*(t))|\int_0^t |k(t,s,u(s))|ds + |f(t,u^*(t))|\times\\
&\left(\int_0^t |k(t,s,u(s))-k(t,s,u^*(s))|ds + \left|\int_0^t k(t,s,u^*(s))ds - \sum_{i=1}^{G} tw_i k(t,t\eta_i,u^*(t\eta_i))\right|\right),\\
&\leq L_2\|u(t)-u^*(t)\|\lambda_1 + \lambda_2\left(L_1\|u(t)-u^*(t)\|+\Gamma_G\right).
\end{aligned} \tag{30}$$

If $u(t)$ satisfies Theorem 2, then we get

$$\|u(t)-u^*(t)\| \leq \max\left\{\Theta_M^k, (L_2\lambda_1+L_1\lambda_2)\Theta_M^k+\lambda_2\Gamma_G\right\}. \tag{31}$$

For **case 2**, from (24) we have

$$u^*(t)=g(t)+\sum_{l=1}^{G'} tw_l v(t,t\eta_l,u^*(t\eta_l))\sum_{i=1}^{G} tw_i k(t,t\eta_i,u^*(t\eta_i)). \tag{32}$$

By similar steps and techniques, we can establish that

$$\begin{aligned}\|u(t)-u^*(t)\| \leq & \left|\int_0^t v(t,s,u(s))ds - \sum_{l=1}^{G'} tw_l v(t,t\eta_l,u^*(t\eta_l))\right| \int_0^t |k(t,s,u(s))|ds \\ & + \left|\sum_{l=1}^{G'} tw_l v(t,t\eta_l,u^*(t\eta_l))\right| \left|\int_0^t k(t,s,u(s))ds - \sum_{i=1}^{G} tw_i k(t,t\eta_i,u^*(t\eta_i))\right| \\ & \leq \Gamma'_G\lambda_1 + (\lambda_2+\Gamma'_G)\left(L_1\|u(t)-u^*(t)\| + \Gamma_G\right), \\ & = L_1(\lambda_2+\Gamma'_G)\|u(t)-u^*(t)\| + \Gamma_G(\lambda_2+\Gamma'_G) + \Gamma'_G\lambda_1. \end{aligned} \tag{33}$$

If $u(t)$ satisfies Theorem 2, we get

$$\|u(t)-u^*(t)\| \leq \max\left\{\Theta_M^k, L_1(\lambda_2+\Gamma'_G)\Theta_M^k + \Gamma_G(\lambda_2+\Gamma'_G) + \Gamma'_G\lambda_1\right\}. \tag{34}$$

6. Numerical experiments

In this section, we define the maximum absolute error E as follows:

$$E = \|u(t)-u^*(t)\| = \max_{0\leq t<1} |u(t)-u^*(t)|, \tag{35}$$

where $u(t)$ is the exact solution and $u^*(t)$ is the approximate solution from our method. We implement the numerical scheme on some examples to clarify the accuracy of the presented method. In these examples, $0 \leq t < 1$ and $k = 1$. The computations for the examples were performed using Mathematica 10.3 software.

Example 1. Consider the NQVIE as follows:

$$u(t) = \sin(t) - \frac{t^3\sin(t)}{30(1+\sin(t))} + f(t,u(t))\int_0^t k(t,s,u(s))ds, \tag{36}$$

where $f(t,u(t)) = \frac{|u(t)|}{3(1+|u(t)|)}$, $|f(t,u(t))| \leq \lambda_2 = \frac{1}{3}$, $\left|\frac{\partial f(t,u)}{\partial u}\right| \leq L_2 = \frac{1}{3}, k(t,s,u(s)) =$ $= \frac{t}{5}\arcsin(|u(s)|)$, $|k(t,s,u(s))| \leq \lambda_1 = \frac{\pi}{10}$, $\left|\frac{\partial k(t,s,u)}{\partial u}\right| \leq L_1 = \frac{1}{5}$, according to Theorem 1, the Eq. (36) has a unique solution $u(t) = \sin(t)$. By applying the proposed method for $G = 2,4,6$ and $M = 2,6$, we obtain the maximum absolute error E as shown in Table 1. Figure 1 represents the absolute errors for $G = 2,6$ at $M = 6$.

Table 1. Maximum absolute error E for Example 1

	$G=2$		$G=4$		$G=6$	
M	2	6	2	6	2	6
E	$4.81551e^{-5}$	$3.1488e^{-11}$	$4.73769e^{-5}$	$3.16498e^{-11}$	$4.72393e^{-5}$	$3.12172e^{-11}$
CPU Time	0.0311923	0.137649	0.0473989	0.190662	0.0519349	0.249198

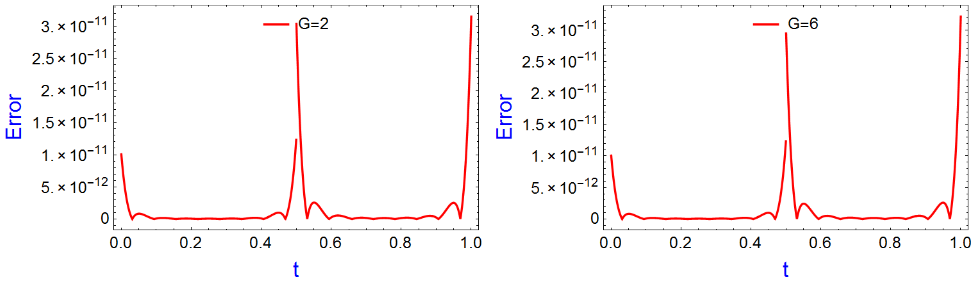


Fig. 1. Absolute errors at $M = 6$ for Example 1

Example 2. Consider the following NQVIE [12]:

$$u(t) = e^t + \frac{t\sqrt{e^t}}{10}\left(\ln(e^{-t}+1) - \ln 2\right) + \sqrt{u(t)}\int_0^t \frac{0.1t}{1+u(s)}\,ds. \tag{37}$$

Eq. (37) has the unique solution $u(t) = e^t$ according to [12]. By applying the proposed method for $G = 4, 6$ and $M = 4, 6, 8$, we obtain the maximum absolute error E as shown in Table 2. In Table 3, we present a comparison between our method and the best errors obtained by the FPM [12]. Figure 2 indicates the absolute errors at different values of M for $G = 6$. Figure 3 represents the graph of the solution at $M = 8$ and $G = 6$.

Table 2. Maximum absolute error E for Example 2

	$G = 4$			$G = 6$		
M	4	6	8	4	6	8
E	$1.655201e^{-7}$	$1.546555e^{-10}$	$2.120720e^{-10}$	$1.647666e^{-7}$	$9.978618e^{-11}$	$7.286847e^{-14}$
CPU Time	0.129105	0.25691	0.46221	0.169522	0.342608	0.594012

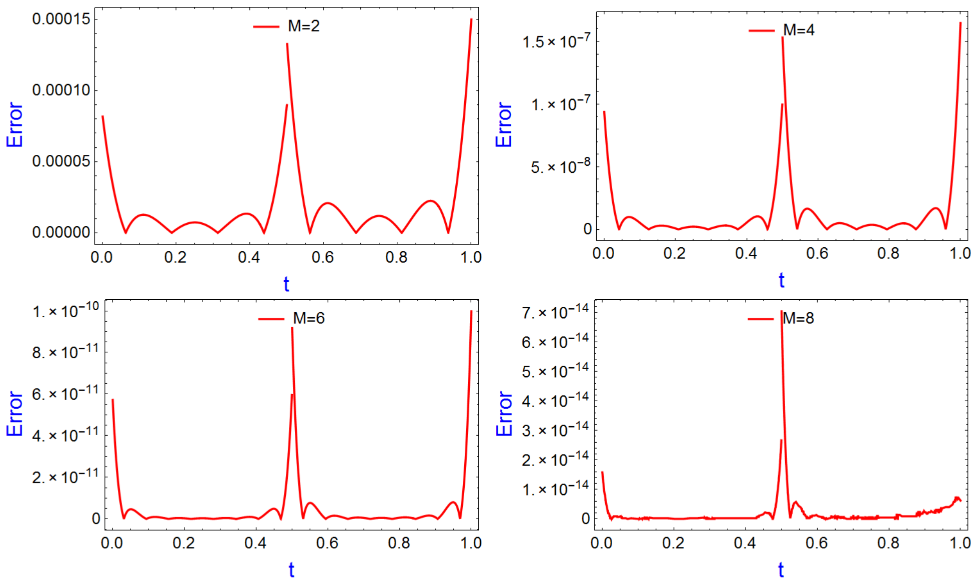


Fig. 2. Absolute errors for different values of M at $G = 6$ for Example 2

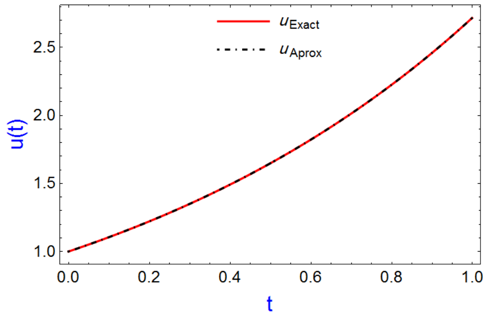


Fig. 3. Exact and approximate solutions at $M = 8$ and $G = 6$ for Example 2

Table 3. Best errors for Example 2

FPM [12] $N = 100$	Our Method at $M = 8, G = 6$	Our CPU time
$1.166e^{-07}$	$7.286847e^{-14}$	0.620423

Example 3. Consider the following NQVIE [15]:

$$u(t) = t^2 - \frac{t^{15}}{1350} + \left(\int_0^t su^2(s)\,ds\right)\left(\int_0^t \frac{s^2}{25}u^3(s)\,ds\right). \tag{38}$$

The unique solution of Eq. (38) is $u(t) = t^2$; see [17]. We provide a comparison between MLWM [15] and the proposed method for $G = 5$, $M = 2$ in Table 4.

Table 4. Best errors for Example 3

MLWM [15] $\hat{m} = 16$	Our Method at $M = 2, G = 5$	Our CPU time
$7.2494e^{-06}$	$2.688612e^{-16}$	0.049396

Example 4. Consider the following problem [16]:

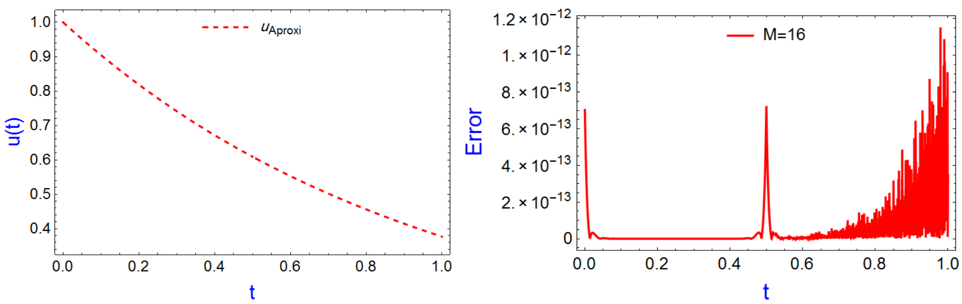
$$u(t) = e^{-t} + u(t)\int_0^t \frac{t^2 \ln(1 + s|u(s)|)}{2e^{t+s}}\,ds. \tag{39}$$

Eq. (39) has at least one undetermined solution; see [8]. We construct the approximate solution $u^*(t)$ in $[0,1)$. By applying the proposed method for $G = 6$ and $M = 6, 10, 14$, we obtain the maximum absolute residual error (RE_M), and a comparison between this result and the result in the Avazzadeh method [16] is all provided in Table 5. Figure 4 represents the absolute residual errors and the graph of the approximate solution at $M = 16$ and $G = 6$. Where

$$RE_M = \max_{0 \le t < 1}\left|u^*(t) - g(t) - f(t, u^*(t))\int_0^t k(t, s, u^*(s))\,ds\right|.$$

Table 5. Maximum absolute residual error RE_M for Example 4

	Method of Avazzadeh [16]			Our method at $G = 6$		
M	5	10	15	6	10	14
RE_M	$5.426e^{-06}$	$1.161e^{-08}$	$5.172e^{-12}$	$7.928569e^{-07}$	$6.689707e^{-10}$	$1.944288e^{-12}$
CPU Time				0.0391214	0.0487566	0.0958534



(a) (b)

Fig. 4. Graph of approximate solution (a), absolute residual error for Example 4 (b)

Discussion: We observe on some figures (Figs. 1 and 2) a jump in the absolute error at $t = 0.5$. This is because of how the wavelets method works; the family of Vieta-Lucas wavelets for $k = 1$ is 2^k subfamily $\{\Upsilon_{1,m}(t)\}_m$, $\{\Upsilon_{2,m}(t)\}_m$ defined in $[0, 0.5[$, $[0.5, 1[$ respectively. There are different polynomials in each interval, and $t = 0.5$ is the time to switch between these two subfamilies of polynomials. It can be said that this is one of the issues of the wavelets method, although all the functions are continuous. However, these jumps in absolute error are considered very small compared to the convergence speed of this method, which provides good approximations to the solution. We notice from the figures that these jumps decrease as we increase the value of M. We also notice from the tables (Tables 1 and 2) for the case $G = 4$, as M increases from 6 to 8, the error increases slightly instead of decreasing. The slight increased error is relevant to the jump in absolute error at $t = 0.5$, for $M = 8$ and the interplay between the number of collocation points and the degree of the polynomial approximation M.

7. Conclusion

In this paper, we use the Vieta-Lucas wavelets method and the Legendre-Gauss quadrature rule to find the approximate solutions of NQVIEs. Using this method, the NQVIEs are converted to an algebraic system of nonlinear equations, which Newton's technique can solve. Numerical experiments indicate that the suggested approach is precise, accurate, and effective for a small number of Vieta-Lucas Wavelets. Moreover, the obtained results show that the maximum absolute error decreases with increasing the degree or the number of polynomials M and the G-point Legendre-Gauss Quadrature Rule. Therefore, it was verified that the method employed in the present work is valid and applicable.

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