Lebesgue and Sobolev Spaces with Variable Exponents

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Core Definitions

Variable Exponent: $p : \Omega \to [1, \infty)$ measurable
Bounds: $p^- = \text{ess inf}\, p(x)$, $p^+ = \text{ess sup}\, p(x)$
Modular: $\rho_{p(\cdot)}(f) = \int_\Omega |f(x)|^{p(x)} dx$
Lebesgue Space: $L^{p(\cdot)}(\Omega) = \{f : \exists\lambda > 0, \rho_{p(\cdot)}(f/\lambda) < \infty\}$
Luxemburg Norm: $\|f\|_{p(\cdot)} = \inf\{\lambda > 0 : \rho_{p(\cdot)}(f/\lambda) \leq 1\}$
Sobolev Space: $W^{k,p(\cdot)}(\Omega) = \{f \in L^{p(\cdot)} : D^\alpha f \in L^{p(\cdot)}, |\alpha| \leq k\}$

Key Properties

- **Banach Spaces**: Complete under Luxemburg norm
- **Reflexive**: When $1 < p^- \leq p^+ < \infty$
- **Separable**: When $p^+ < \infty$
- **Hölder Inequality**: For $\frac{1}{s(x)} = \frac{1}{p(x)} + \frac{1}{q(x)}$:

$$\|fg\|_{s(\cdot)} \leq 2\|f\|_{p(\cdot)}\|g\|_{q(\cdot)}$$

Essential Conditions

Log-Hölder Continuity: $|p(x) - p(y)| \leq \frac{C}{\log(e+1/|x-y|)}$
Sobolev Embedding: If p log-Hölder and $1 < p^- \leq p^+ < n$:
$W^{1,p(\cdot)} \hookrightarrow L^{p^*(\cdot)}$, $\frac{1}{p^*(x)} = \frac{1}{p(x)} - \frac{1}{n}$

Example 1: Elliptic Equations

$p(x)$-Laplace Equation

$$-\Delta_{p(x)} u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

where $\Delta_{p(x)} u = \text{div}(|\nabla u|^{p(x)-2}\nabla u)$

Weak Formulation: Find $u \in W_0^{1,p(\cdot)}(\Omega)$ such that:

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla v dx = \int_\Omega f v dx$$

for all $v \in W_0^{1,p(\cdot)}(\Omega)$

Energy Functional:

$$J(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)} dx - \int_\Omega f u dx$$

Existence & Regularity

- **Existence**: If p log-Hölder, $1 < p^- \leq p^+ < \infty$, $f \in L^{p'(\cdot)}(\Omega)$
- **Regularity**: Additional conditions yield $u \in C^{1,\alpha}(\Omega)$
- **Uniqueness**: Guaranteed if $p(x) \geq 2$ or $1 < p(x) \leq 2\ \forall x$

Eigenvalue Problem

Find $\lambda \in \mathbb{R}$, $u \in W_0^{1,p(\cdot)}(\Omega) \setminus \{0\}$:

$$-\Delta_{p(x)} u = \lambda|u|^{p(x)-2}u$$

First eigenvalue: $\lambda_1 = \inf\left\{\frac{\int_\Omega |\nabla u|^{p(x)} dx}{\int_\Omega |u|^{p(x)} dx}\right\}$

Physical Applications

- **Electrorheological fluids**: $p(x)$ depends on electric field
- **Image processing**: Adaptive regularization
- **Non-Newtonian fluids**: Shear-dependent viscosity

Example 2: Parabolic Equations

$p(x)$-Heat Equation

$$u_t - \Delta_{p(x)} u = f(x,t) \quad \text{in } \Omega \times (0,T)$$
$$u = 0 \quad \text{on } \partial\Omega \times (0,T)$$
$$u(x,0) = u_0(x) \quad \text{in } \Omega$$

Weak Formulation: Find $u \in L^2(0,T; W_0^{1,p(\cdot)}(\Omega))$ with $u_t \in L^2(0,T; W^{-1,p'(\cdot)}(\Omega))$ such that:

$$\langle u_t, v\rangle + \int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla v dx = \int_\Omega f v dx$$

for all $v \in W_0^{1,p(\cdot)}(\Omega)$, a.e. $t \in (0,T)$

Energy Estimates

Basic Energy:

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \int_\Omega |\nabla u|^{p(x)} dx \leq \int_\Omega f u dx$$

Higher Integrability: Under suitable conditions:

$$|\nabla u|^{p(x)} \in L^{1+\delta}(\Omega \times (0,T)) \quad \text{for some } \delta > 0$$

Long-Time Behavior

- **Global existence** for appropriate initial data
- **Decay estimates** depending on p^- and p^+
- **Stationary solutions** as $t \to \infty$ solve elliptic problem
- **Extinction in finite time** possible for $p^- < 2$

Physical Contexts

- **Thermal processes** with temperature-dependent conductivity
- **Filtration in porous media** with variable porosity
- **Biological systems** with density-dependent diffusion

Mathematical Challenges

- **Degenerate/Singular**: Behavior depends on $p(x)$ relative to 2
- **Non-uniform ellipticity**: Growth conditions vary spatially
- **Compactness**: Requires careful treatment in evolution problems
- **Regularity**: Hölder continuity and gradient estimates

Recent Developments

- **Calderón-Zygmund theory** for variable exponents
- **Free boundary problems** with non-standard growth
- **Numerical methods** for $p(x)$-PDEs
- **Fractional $p(x)$-Laplacians**
- **Stochastic versions** of variable exponent PDEs

References

1. Diening et al. (2011) *Lebesgue and Sobolev Spaces with Variable Exponents*
2. Růžička (2000) *Electrorheological Fluids*
3. Fan (2015) *An overview of p(x)-Laplace equations*
4. Antontsev & Rodrigues (2006) *On stationary thermo-rheological viscous flows*
5. Acerbi & Mingione (2007) *Regularity results for parabolic systems*
6. Zhikov (2007) *On the density of smooth functions in Sobolev-Orlicz spaces*

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