Setif 1 University Ferhat Abbas
Faculty of Sciences
Department of Mathematics
Fundamental and Numerical Mathematics Laboratory
Laboratory of Applied Mathematics

جامعة سطيف 1 فرحات عباس
كلية العلوم
قسم الرياضيات
مخبر الرياضيات الأساسية و العددية
مخبر الرياضيات التطبيقية

CERTIFICATE OF PARTICIPATION

International Conference on Mathematics and its Applications in Science and Technology

ICMAST'2024 – Setif, December 15-16, 2024

This is to certify that

Abdelaziz Hellal

has participated in an online presentation titled:

DOUBLE PHASE PROBLEMS WITH VARIABLE EXPONENTS

at the International Conference on Mathematics and its Applications in Science and Technology (ICMAST'2024), held on December 15-16, 2024, at the University Setif 1 Ferhat Abbas, Algeria.

Conference Chairman
Dr. Mohamed Kara

Setif 1 University Ferhat ABBAS – International Conference on Mathematics and its Application in Science and Technology – ICMAST'2024 – Faculty of Sciences

رئيس الملتقى
الأستاذ: قارة محمد

Setif, December 16, 2024

People's Democratic Republic of Algeria
Ministry of Higher Education and Scientific Research
Setif 1 University Ferhat Abbas
Faculty of Sciences
Department of Mathematics

INTERNATIONAL CONFERENCE ON MATHEMATICS AND ITS APPLICATIONS IN SCIENCE AND TECHNOLOGY

ICMAST' 2024

Setif, December 15-16, 2024

Auditorium : Mouloud Kacem Nait Belkacem- El Bez

PROGRAM (ONLINE)

PROGRAM OF THE CONFERENCE (Online)

Sunday, December 15th, 2024

Session 1: Differential Equations and its Applications Chair: A. Saadallah Google Meet Link: https://meet.google.com/qge-wfpo-uep		
Hours	**Speakers**	**Presentation Title**
13h 15 – 13h 30	**Mohamed Houasni**	On the stability of a nonlinear micro temperatures damped porous-elastic system with nonlinear distributed delay
13h 30 – 13h 45	**Abdelaziz Hellal**	Double phase problems with variable exponents
13h 45 – 14h 00	**Hibaterrahmane Benmessaoud**	Dynamic evolution of thermo-viscoelastic materials
14h 00 – 14h 15	**Mohammed El Amine Riahi**	Optimal control for stationary Marguerre-Von Karman equation
14h 15 – 14h 30	**Khelifa Daoudi**	Mild of solutions for integro-differential impulsive equations with infinite interval
14h 30 – 14h 45	**Ishak Kettaf**	Existence and uniqueness of solution for a class of superlinear Kirchhoff-type equations
14h 45 – 15h 00	**Abeer Hasek**	Optical solitons governed by the stochastic RKL equation with multiplicative white noise using the generalized (G'/G)-expansion approach
15h 00 – 15h 15	**Amina Chaili**	Investigating decay behaviors in system of coupled biharmonic Schrodinger equations with internal fractional damping
15h 15 – 15h 30	**Chaima Boulkheloua**	Exponential stability of Lord Shulman thermoelastic system with porous damping and delay term

Session 2: Differential Equations and its Applications Chair: A. Bachmar Google Meet Link: https://meet.google.com/cqn-jncp-rzp		
Hours	**Speakers**	**Presentation Title**
13h 15 – 13h 30	**Abir Mechaouf**	New results for the Marguerre–Von Karman equations with long memory
13h 30 – 13h 45	**Fatima Ezahra Bentata**	Existence of solution and numerical simulation to a hyperbolic Kirchhoff type problem modeling the free vibration of an elastic string
13h 45 – 14h 00	**Abdelatif Kainane Mezadek**	Some existence results to weakly coupled system of k semi-linear fractional $\sigma-$evolution models
14h 00 – 14h 15	**Fatima Zahra Arioui**	Caputo fractional stochastic differential inclusions with impulses
14h 15 – 14h 30	**Hayet Bouzeraieb**	Numerical approach for solving neutral linear differential equations with variable coefficients
14h 30 – 14h 45	**Douha Saadi**	Numerical solution of nonlocal boundary value problems for partial differential equations
14h 45 – 15h 00	**Bouthina Sabah Hammou**	Solvability of the second order fractional diffusion equation with non-local condtions
15h 00 – 15h 15	**Oussama Khaldi**	Well-posedness and exponential stability of Timoshenko system of second sound with time-varying delay and forcing terms
15h 15 – 15h 30	**Brahim Kilani**	On the stability of swelling soils system with strong damping and neutral delay term

Session 3: Dynamic Systems and Mathematical Biology

Chair: A. Leulmi

Google Meet Link: https://meet.google.com/tmy-vsvr-dgo

Hours	Speakers	Presentation Title
13h 15 – 13h 30	Nadjiba Abdi	A study of an evolution problem driven by maximal monotone operators
13h 30 – 13h 45	Yacine Elhadj Moussa	Analysis of Caputo fractional order derivative of covid-19 transmission dynamics
13h 45 – 14h 00	Rima Chouader	A class of Kolmogorov differential system with one hyperbolic algebraic limit cycle
14h 00 – 14h 15	Ahlam Belfar	On the limit cycles of discontinuous piecewise linear differential systems formed by centers or hamiltonian without equilibria separated by irreducible conic
14h 15 – 14h 30	Hamidi Nabil	A mathematical model of the competition between plasmid-bearing and plasmid-free organisms in a chemostat with substrate inhibition
14h 30 – 14h 45	El Ouahma Bendib	Limit cycles for a certain class of generalized Kukles differential systems
14h 45 – 15h 00	Ali Kessouri	Combinatorics and log-concavity of over Mahonian numbers
15h 00 – 15h 15	Nora Dekkiche	A study on diffusion models for population dynamics incorporating individual behavior at boundaries

Session 4: Algebra, Cryptography and Information Security

Chair: Z. Kebaili

Google Meet Link: https://meet.google.com/tjt-bngf-rux

Hours	Speakers	Presentation Title
13h 15 – 13h 30	Rachid Chergui	Les matrices ”QL” et un nouveau type de cryptographie
13h 30 – 13h 45	Adem Aikous	Rota-Baxter operators on Hom-group algebra
13h 45 – 14h 00	Nasser Ghedbane	A new public key cryptosystem using space representations of finite groups
14h 00 – 14h 15	Amina Daoui	New identities for complex bivariate polynomials with some special numbers by symmetric functions
14h 15 – 14h 30	Akila Djoumakh	Some new identities concerning bi-periodic Fibonacci and Lucas sequences
14h 30 – 14h 45	Nadia Amal Messaoudi	A new discrete chao map developped through the combination of sine function and linear function
14h 45 – 15h 00	Safia Seffah	Repdigits expressed as products of two k-Lucas numbers
15h 00 – 15h 15	Yacine Briedj	The $d(1)$-extensibility of the $d(-1)$-triple $\{1,5,c\}$ and integer solutions on the associated elliptic curves

Session 5: Numerical Analysis and Simulation Chair: R. Ziadi Google Meet Link: https://meet.google.com/ksd-cpxz-ysi		
Hours	**Speakers**	**Presentation Title**
13h 15 – 13h 30	**Meryem Benssaad**	Analytic and numerical study intégro-differential Frefholm-Schander equation
13h 30 – 13h 45	**Hanane Chinoune**	Application of the modified homotopy perturbation method for solving fractional differential equation
13h 45 – 14h 00	**Yasmine Fourar**	Resolution of Bezout Equations in the synthesis of non-stationary biorthogonal wavelets using cardinal Chebyshev B-splines
14h 00 – 14h 15	**Walid Remili**	A rational spectral Galerkin method for solving Fredholm integral equation on the whole line
14h 15 – 14h 30	**Abdelouahab Mani**	Comparative analysis of numerical methods for solving integral delay equations
14h 30 – 14h 45	**Mohamed Dellal**	Modeling microbial competition with allelopathy and substrate inhibition in chemostat
14h 45 – 15h 00	**Khaoula Meansri**	Improving large-scale optimization: a three-term enhancement of the conjugate gradient method
15h 00 – 15h 15	**Samar Chebbah**	Restricted additive Schwarz method for variational inequalities

Session 6: Numerical Analysis and Simulation Chair: H. Grar Google Meet Link: https://meet.google.com/utu-ohrz-bfo		
Hours	**Speakers**	**Presentation Title**
13h 15 – 13h 30	**Isma Debbah**	Nonparametric estimation of a regression function using the gamma kernel method
13h 30 – 13h 45	**Nacira Allaoua**	A language shift simulation based on asynchronous cellular automata
13h 45 – 14h 00	**Samia Khelladi**	Study of a modified Liu-story method based on different line searches
14h 00 – 14h 15	**Fatma Kaci**	Independence polynomial approach for solving high-order linear Fredholm-Volterra integro-differential equations
14h 15 – 14h 30	**Imene Laribi**	Stability result of Timoshenko system with thermoelasticity, diffusion effect and memory term
14h 30 – 14h 45	**Ameur Lounes**	A numerical solution for the single-server queue with phase type distribution service times
14h 45 – 15h 00	**Soumia Aici**	Lower bounds for the Euclidean operator radius
15h 00 – 15h 15	**Linda Menniche**	A penalty method for linear programming

Session 7: Fractional Calculus

Chair: A. Djabi

Google Meet Link: https://meet.google.com/izh-ukiw-pmk

Hours	Speakers	Presentation Title
13h 15 – 13h 30	**Zineb Bellabes**	ψ-Caputo equations through weak topology analysis
13h 30 – 13h 45	**Ahlem Sidi Yekhlef**	Stabilization of a hyperbolic system with fractional damping, memory term and time delay
13h 45 – 14h 00	**Moufida Guechi**	Adaptation of homotopy perturbation method for solving nonlinear fractional integro-differential equations
14h 00 – 14h 15	**Nedjoua Dria**	Homotopy perturbation method for the solution of system of nonlinear fractional order partial differential equations
14h 15 – 14h 30	**Souad Guedim**	New results for the integral equation with Hadamard derivatives of variable order
14h 30 – 14h 45	**Said Rafa**	Well-posedness and stabilization of the wave equation by the boundary dissipation with fractional derivative
14h 45 – 15h 00	**Ahmed Bouchenak**	Fractional initial matrix-value problems solved via conformable Laplace transform
15h 00 – 15h 15	**Hammou Benmehidi**	Existence, uniqueness and stability analysis of coupled hybrid pantograph systems

Monday, December 16th, 2024

Session 1: Differential Equations and its Applications Chair: A. Saadallah Google Meet Link: https://meet.google.com/poe-uknq-mos		
Hours	**Speakers**	**Presentation Title**
13h 15 – 13h 30	**Somia Tamouza**	An optimization problem for a fractional differential equation
13h 30 – 13h 45	**Farida Meghatria**	Predictive mathematical analysis to mitigating obesity diseases
13h 45 – 14h 00	**Saiah Seyyid Ali**	Well-posedness results for weakly coupled systems of some semi-linear fractional σ -evolution models
14h 00 – 14h 15	**Tayeb Mahrouz**	Smoothness of solutions of differential equations of constant strength in roumieu spaces
14h 15 – 14h 30	**Mahdi Rakah**	New uniqueness results for fractional differential equations and some travelling waves
14h 30 – 14h 45	**Marwa Saci**	On the stability of linear porous elastic materials with microtemperatures effect
14h 45 – 15h 00	**Aziza Bachmar**	A problem electro-viscoelastic materials
15h 00 – 15h 15	**Bochra Amroune**	Analytical solution of two dimensional free surface flow via the hodograph method
15h 15 – 15h 30	**Aimen Daoudi**	Well-posedness of fourth-order nonlinear Caputo viscoelastic equations with relaxation and interaction terms

Session 2: Differential Equations and its Applications Chair: R. Ziadi Google Meet Link: https://meet.google.com/ecz-bykm-bhg		
Hours	**Speakers**	**Presentation Title**
13h 15 – 13h 30	**Fatiha Ghejdemis**	Application of flower pollination algorithm on differential equations
13h 30 – 13h 45	**Ouidad Boulakour**	The residual power series method for solving nonlinear time-fractional partial differential equations
13h 45 – 14h 00	**Asma Bougherara**	Existence and non-existence results for non linear elliptic system involving gradient terms
14h 00 – 14h 15	**Raouane Mouedenne**	Stability of a Schrodinger equation with internal fractional damping
14h 15 – 14h 30	**Fatima Hamani**	Class of self similar solution for the time-fractional porous media equation
14h 30 – 14h 45	**Rebeiha Allaoua**	Integrability and limit cycles surrounding a non-elementary singular point of a class of polynomial differential system
14h 45 – 15h 00	**Naziha Belkhir**	Well posedness and decay estimate for a porous thermoelastic system free of second spectrum
15h 00 – 15h 15	**Chahira Guenoune**	Existence and uniqueness results for thermo-elasto viscoplastic contact problem
15h 15 – 15h 30	**Sakina Benkaddour**	The study of stochastic differential equation driven by a non Gaussian process

Session 3: Dynamic Systems and Mathematical Biology + Algebra, Cryptography and Information Security Chair: Z. Kebaili Google Meet Link: https://meet.google.com/ydk-apzz-kgo		
Hours	**Speakers**	**Presentation Title**
13h 15 – 13h 30	**Ines Tababouchet**	ON the limit cycles of a family of discontinuous piecewise differential systems separated by a straight line, and formed by an arbitrary linear system and a quadratic center
13h 30 – 13h 45	**Sarra Gaouir**	Semi-discritisation of nonconvex integro-differential inclusions
13h 45 – 14h 00	**Souad Bounouiga**	Analysis of malaria transmission dynamics using fractional models and numerical simulations
14h 00 – 14h 15	**Noureddine Touati Brahim**	On the study of parabolic degenerate P-biharmonic problem with memory
14h 15 – 14h 30	**Khadidja Moussaoui**	Certain $D(\ell)$ diophantine triples
14h 30 – 14h 45	**Ahmed Benkahla**	The tree isomorphism theorem in bihom groups
14h 45 – 15h 00	**Nacima Rosa Ait-Amrane**	Algebraic results on certain dual generalized complex numbers
15h 00 – 15h 15	**Chahinaze Djadi**	Congruences concerning quadrinomial coefficients
15h 15 – 15h 30	**Imane Merzougui**	Isogenies based cryptography

Session 4: Numerical Analysis and Simulation Chair: S. Khalladi Google Meet Link: https://meet.google.com/pxy-sikm-yyn		
Hours	**Speakers**	**Presentation Title**
13h 15 – 13h 30	**Sabrina Benzamouche**	Kernel estimator of relative error regression function for twice censored and dependent data
13h 30 – 13h 45	**OumKeltoum Benhamouda**	Scheme approximation equation of the one-dimensional variable-order time-fractional diffusion equation
13h 45 – 14h 00	**Ahlem Bennani**	An adaptive penalty algorithm for quadratic fractional programming
14h 00 – 14h 15	**Narimen Hafsi**	Cryptanalysis of Merkle-Hellman cipher based on hybrid firefly-genetic algorithms
14h 15 – 14h 30	**Saloua Aliouche**	Asymptotic normality of the heteroscedastic regression function for left truncated data
14h 30 – 14h 45	**Fares Bekhouche**	The Laplace residual power series method for solving fractional initial value problems
14h 45 – 15h 00	**Othman Yakhlef**	Fluid structure evolution interaction problem

Session 5: Fractional Calculus

Chair: A. Khalouta

Google Meet Link: https://meet.google.com/umr-rxpp-htn

Hours	Speakers	Presentation Title
13h 15 – 13h 30	**Amine Loumi**	A new method for solving in parallel linear systems by using a fractional linear decomposition
13h 30 – 13h 45	**Reda Soufiane Bousserhane**	Finite time blow up of solutions for coupled system of wave equations with nonlinear memory terms
13h 45 – 14h 00	**Aboubaker El-Saddik Bouziane**	Applications of a fixed point results for solving fractional Caputo-differential problem with integral boundary conditions
14h 00 – 14h 15	**Amira Abdenebi**	Numerical solution of fractional linear boundary value problems and beam deflection application
14h 15 – 14h 30	**Mohammed Khirani**	On boundedness and compactness of Hadamard operators in morrey spaces
14h 30 – 14h 45	**Wissame Selmani**	A new approximating solution for an abstract fractional Elliptic Cauchy problem
14h 45 – 15h 00	**Djamel-eddine Hettadj**	Existence of solutions for fractional differential inclusion with multipoint boundary conditions

Session 6: Data Science and Analytics + Mathematical Methods in Engineering

Chair: A. Leulmi

Google Meet Link: https://meet.google.com/vwc-xhwq-wgk

Hours	Speakers	Presentation Title
13h 15 – 13h 30	**Roumaissa Elbay**	Plug in method for selection an optimal bandwidth for distribution functions with double truncated data
13h 30 – 13h 45	**Abida Zahnit**	A new robust estimation procedure for the tail index of Heavy-Tailed-Pareto-type distributions under random truncation
13h 45 – 14h 00	**Mohammed Bachir Bederina**	Enumerating fair solutions for multi-agent problems
14h 00 – 14h 15	**Razika Grine**	Novel two parameter model: statistical properties
14h 15 – 14h 30	**Dihia Belaiza**	Analysis of an unreliable M/M/1 retrial G-queue
14h 30 – 14h 45	**Nabil Boumedine**	An improved simulated annealing algorithm for maximizing the coverage area in wireless sensor networks
14h 45 – 15h 00	**Samia Djemai**	Presolving a convex quadratic minimization problem

Session 7: Mathematical Finance and Economics + History of science **Chair: A. Djabi** **Google Meet Link: https://meet.google.com/ftp-rytx-kdx**		
Hours	**Speakers**	**Presentation Title**
13h 15 – 13h 30	**Sylia Abdoun**	Effects of stractegic customers on an M/M/1 queue under two distinct vacations
13h 30 – 13h 45	**Nour El Houda Djemoui**	Estimation bayesienne des parametres et caracteristiques de fiabilite de la distribution exponentielle exponentiee odd lindley : analyse comparative et simulation
13h 45 – 14h 00	**Salima Doubbakh**	$\mathbb{L}^p$-Holder continuity of the solutions of Q-BSDES $\left(2 \leq p < \frac{q}{2}\right)$
14h 00 – 14h 15	**Mourad Azioune**	Chaos control and dynamics analysis of a Bertrand Duopoly team game
14h 15 – 14h 30	**Elias Taki Eddine Mohammed Chikouche**	The one-default model in finance: a mathematical framework for risk assessment
14h 30 – 14h 45	**Hadia Messaoudene**	History of the number π
14h 45 – 15h 00	**Mansour Belarbi**	From Poincare to Perelman the fabulous story of the geometrization conjecture

SETIF 1 UNIVERSITY FERHAT ABBAS
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IN SCIENCE AND TECHNOLOGY (ICMAST'2024)
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DOUBLE PHASE PROBLEMS WITH VARIABLE EXPONENTS

ABDELAZIZ HELLAL

ABSTRACT. In this article, we prove the existence of weak solutions for a specific class of nonlinear double phase problems involving variable exponents and irregular data.

1. INTRODUCTION

In this work, we consider the homogeneous Dirichlet problem associated to the model equation:

$$\begin{cases} -\mathrm{div}\left(A(x)\,|Du|^{p(x)-2}\,Du\right) - \mathrm{div}\left(F(u)\,|Du|^{q(x)-2}\,Du\right) = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open domain in $\mathbb{R}^N$ $(N > 2)$ with Lipschitz boundary $\partial\Omega$ and $A(\cdot)$ is a measurable function such that:

$$\alpha \leq A(x) \leq \beta, \quad \text{a.e. } x \in \Omega, \quad \text{with } \alpha,\ \beta > 0. \tag{1.2}$$

Suppose that $F : \mathbb{R} \longrightarrow \mathbb{R}$ such that:

$$F(u) = |u|^{(r(\cdot)-1)q(\cdot)+1}, \tag{1.3}$$

and f is a function belongs to $L^{m(\cdot)}(\Omega)$.

Here, the variable exponents $p : \overline{\Omega} \longrightarrow (1,\infty)$, $q : \overline{\Omega} \longrightarrow (1,\infty)$, $r : \overline{\Omega} \longrightarrow (1,\infty)$, and $m : \overline{\Omega} \longrightarrow (1,\infty)$ are continuous functions for all $x \in \overline{\Omega}$ such that:

$$r(x)q'(x) > 1, \quad \text{and} \quad 1 < q^- \leq q^+ < p^- \leq p^+ < N \tag{1.4}$$

where $p^- := \min_{x\in\overline{\Omega}} p(x)$, $p^+ := \max_{x\in\overline{\Omega}} p(x)$, $q^- := \min_{x\in\overline{\Omega}} q(x)$, $q^+ := \max_{x\in\overline{\Omega}} q(x)$ and $q'(\cdot) = \frac{q(\cdot)}{q(\cdot)-1}$. Moreover, $p(\cdot) > 1$ is logarithm Hölder continuous function, that is

$$\exists C > 0 :\ |p(x) - p(y)| \leq -\frac{C}{\ln(|x-y|)}, \quad \forall x \neq y \in \Omega :\ |x-y| \leq 1/2. \tag{1.5}$$

Problem 1.1 can be esteemed as the stationary problem associated with the parabolic equation

$$\partial_t u - \mathrm{div}\left(A(x)\,|Du|^{p(x)-2}\,Du\right) - \mathrm{div}\left(F(u)\,|Du|^{q(x)-2}\,Du\right) = f$$

2010 *Mathematics Subject Classification.* 35J60; 35D30; 35J66.

Key words and phrases. Nonlinear elliptic equations; weak solutions; double phase problems; variable exponents.

So, it can be seen as the Darcy law of fluid filtration (in a turbulent regime) in a porous media simulating the pressure u.

Our results depend on the presence of the additional term $-\text{div}\left(F(u)\,|Du|^{q(\cdot)-2}\,Du\right)$, which has a regularizing effect on the summability of the solution of the problem 1.1. In the particular case $f \in L^1(\Omega)$, the regularizing effect of this term has been recently investigated in [4]. In point of fact, we shall demonstrate that the solutions of problem 1.1 are more regular than the ones of the problems (see [3] and [9]) under an appropriate balance of the parameters $p(\cdot)$, $q(\cdot)$, $r(\cdot)$, and the summability of f. Inspired by [1, 2, 8], we will provide additional regularity results based on the summability of the source term f. To achieve this, we need to define the notion of weak solution we will use in accordance with the summability of the right-hand side. For simplify our investigation, we will only discuss the case when f belongs to $L^{m(\cdot)}(\Omega)$ with $m(\cdot) > \frac{N}{p(\cdot)}$.

2. Preliminaries

We recall some definitions, facts, and basic properties of Lebesgue-Sobolev spaces with variable exponents. We refer to [6] and references therein. In this work we denote

$$p^+ := \max_{x\in\overline{\Omega}} p(x) \quad \text{and} \quad p^- := \min_{x\in\overline{\Omega}} p(x).$$

We define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable functions $u : \Omega \to \mathbb{R}$ for which the convex modular

$$\rho_{p(\cdot)}(u) = \int_\Omega |u|^{p(x)}\,dx,$$

is finite.

The space $L^{p(\cdot)}(\Omega)$ equipped with the norm

$$\|u\|_{p(\cdot)} := \|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{k > 0 : \rho_{p(\cdot)}\left(\frac{u}{k}\right) \le 1\right\},$$

which called the Luxemburg norm. The space $(L^{p(\cdot)}(\Omega), \|u\|_{p(\cdot)})$ is a separable Banach space. Moreover, if $1 < p^- \le p^+ < +\infty$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$ where $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$.

For all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, the Hölder type inequality

$$\left|\int_\Omega uv\,dx\right| \le \left(\frac{1}{p^-} + \frac{1}{p'^-}\right)\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)} \le 2\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)}, \tag{2.1}$$

holds.

We define also the Banach space

$$W^{1,p(\cdot)}(\Omega) = \left\{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\right\},$$

endowed with the norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

The space $(W^{1,p(\cdot)}(\Omega), \|u\|_{1,p(\cdot)})$ is a Banach space. while

$$W_0^{1,p(\cdot)}(\Omega) = \left\{u \in W^{1,p(\cdot)}(\Omega) : u = 0 \text{ on } \partial\Omega\right\},$$

is Sobolev space with zero boundary values endowed with the norm $\|\cdot\|_{1,p(\cdot)}$. The space $W_0^{1,p(\cdot)}(\Omega)$ is separable and reflexive provided that $1 < p^- \le p^+ < +\infty$.

The smooth functions (i.e. $\mathcal{D}(\Omega)$) are in general not dense in $W_0^{1,p(\cdot)}(\Omega)$, but if the exponent variable $p(\cdot) > 1$ satisfy the logarithmic Hölder continuity condition (1.5) then $\mathcal{D}(\Omega)$ are dense in $W_0^{1,p(\cdot)}(\Omega)$.

For $u \in W_0^{1,p(\cdot)}(\Omega)$ with $p \in C_+(\overline{\Omega})$, the Poincaré inequality holds

$$\|u\|_{p(\cdot)} \leq C\|\nabla u\|_{p(\cdot)}, \tag{2.2}$$

for some $C > 0$ which depends on Ω and $p(\cdot)$. Therefore, $\|\nabla u\|_{p(\cdot)}$ and $\|u\|_{1,p(\cdot)}$ are equivalent norms.

Our treatment requires to use the following results:

Proposition 1 ([6])**.** If $(u_n), u \in L^{p(\cdot)}(\Omega)$ and $p^+ < +\infty$, then the following properties hold:

- $\|u\|_{p(\cdot)} < 1$ (resp. $= 1, > 1$) $\Longleftrightarrow \rho_{p(\cdot)}(u) < 1$ (resp. $= 1, > 1$),
- $\min\left(\rho_{p(\cdot)}(u)^{\frac{1}{p^+}}, \rho_{p(\cdot)}(u)^{\frac{1}{p^-}}\right) \leq \|u\|_{p(\cdot)} \leq \max\left(\rho_{p(\cdot)}(u)^{\frac{1}{p^+}}, \rho_{p(\cdot)}(u)^{\frac{1}{p^-}}\right)$,
- $\min\left(\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right) \leq \rho_{p(\cdot)}(u) \leq \max\left(\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right)$,
- $\|u\|_{p(\cdot)} \leq \rho_{p(\cdot)}(u) + 1$,
- $\|u_n - u\|_{p(\cdot)} \to 0 \Longleftrightarrow \rho_{p(\cdot)}(u_n - u) \to 0$.

Remark 2.1. As in [7], the following inequality

$$\int_\Omega |u|^{p(x)}\,dx \leq C\int_\Omega |Du|^{p(x)}\,dx,$$

in general does not hold. So, thanks to Proposition 1 and (2.2), we get the following inequality which will be used later

$$\min\{\|Du\|_{p(\cdot)}^{p^-}; \|Du\|_{p(\cdot)}^{p^+}\} \leq \int_\Omega |u(x)|^{p(x)}\,dx \leq \max\{\|Du\|_{p(\cdot)}^{p^-}; \|Du\|_{p(\cdot)}^{p^+}\}. \tag{2.3}$$

We need an important embedding as follows

Lemma 2.2 (Sobolev embedding)**.** *Let $\Omega \in \mathbb{R}^N$ be an open bounded set with Lipschitz boundary and assume that $p : \overline{\Omega} \to (1, N)$ satisfy the logarithm Hölder continuity condition* (1.5)*. Then we have the following continuous embedding:*

$$W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p^\star(\cdot)}(\Omega),$$

where $p^\star(\cdot) = \frac{Np(\cdot)}{N-p(\cdot)}$.

For the notion of weak solutions to problem 1.1, we will use through this work, the truncation function T_k at height k $(k > 0)$ and the associated function which denoted by

$$T_k(t) = \min\Big\{k, \max\{-k, t\}\Big\}, \quad G_k(t) = t - T_k(t) \tag{2.4}$$

It is obvious that T_k is Lipschitz functions satisfying $|T_k(t)| \leq k$.

Throughout this work, C always indicate any non-negative constant which depends only on data and whose value may change from line to line.

3. Main result:

Definition 3.1. If $f \in L^m(\Omega)$ with $m > 1$, by a weak solution of the problem 1.1 we mean a function $u \in W_0^{1,p(\cdot)}(\Omega)$ such that

$$F(u)\,|Du|^{q(\cdot)-1} \in L^{p'(\cdot)}(\Omega) \tag{3.1}$$

and the integral identity

$$\int_\Omega A(x)\,|Du|^{p(x)-2}\,Du\cdot D\varphi\,dx + \int_\Omega F(u)\,|Du|^{q(x)-2}\,Du\cdot D\varphi\,dx = \int_\Omega f\varphi\,dx \tag{3.2}$$

holds for every $\varphi \in W_0^{1,p(\cdot)}(\Omega)$.

We have the following theorem:

Theorem 3.2. *Under the assumptions* (1.2)-(1.5). *If* $f \in L^{m(\cdot)}(\Omega)$ *with* $m(\cdot) > \frac{N}{p(\cdot)}$ *then the problem 1.1 has at least one solution* $u \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

The idea of the proof of Theorem 3.2: The proof is based on three steps: First, we introduce a suitable sequence of problems approximating. In addition, we establish some priori estimates for the solutions of the problem 1.1. Finally, we pass to the limit in the approximate problems.

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Double phase problems with variable exponents

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International Conference on Mathematics and its Applications in Science and Technology,

(ICMAST'2024)

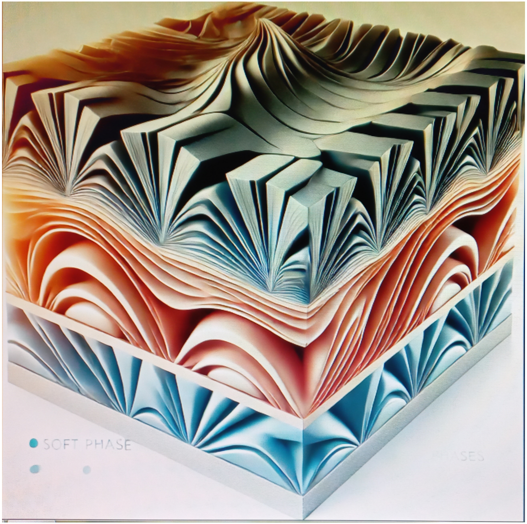
Setif 1 University Ferhat Abbas

December $15^{th} - 16^{th}$, 2024

Outline:

Introduction: An example (Material science)

- The image shows a composite material with two phases soft and hard under stress.



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Physical interpretation

Modeling nonlinear elasticity for composite materials with different properties is an important application of elliptic double phase problems.

$$-\operatorname{div}\left(A(x)\,|Du|^{p(x)-2}\,Du\right)-\operatorname{div}\left(F(u)\,|Du|^{q(x)-2}\,Du\right)=f$$

- The state of balance of the composite material.
- u is the displacement field while Du indicates the rate and direction of change of u at any point.
- A and F indicate the phase transition between the soft and stiff (hard) material.
- f represents external forces (stress).
- Depending on the spatial region, the different phases show distinct elastic responses in the state of balance of the material.

Discussion

- if F is small (soft material), then the composite material exhibits the behavior of the softer material.

$$-\mathrm{div}\left(A(x)\,|Du|^{p(x)-2}\,Du\right) = f$$

- if F is large (stiff material), then the modeling focuses on the stiffer material's response.

$$-\mathrm{div}\left(F(u)\,|Du|^{q(x)-2}\,Du\right) = f$$

Mathematical models

$$\begin{cases} -\operatorname{div}\left(A(x)\,|Du|^{p(x)-2}\,Du\right) - \operatorname{div}\left(F(u)\,|Du|^{q(x)-2}\,Du\right) = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded open domain in $\mathbb{R}^N$ $(N > 2)$ with Lipschitz boundary $\partial\Omega$ and $A(\cdot)$ is a measurable function such that:

$$\alpha \leq A(x) \leq \beta, \quad \text{a.e. } x \in \Omega, \quad \text{with } \alpha,\ \beta > 0. \tag{2}$$

Suppose that $F : \mathbb{R} \longrightarrow \mathbb{R}$ such that:

$$F(u) = |u|^{(r(\cdot)-1)q(\cdot)+1}, \tag{3}$$

and

$$f \in L^{m(\cdot)}(\Omega).$$

Here, the variable exponents $p : \overline{\Omega} \longrightarrow (1, \infty)$, $q : \overline{\Omega} \longrightarrow (1, \infty)$, $r : \overline{\Omega} \longrightarrow (1, \infty)$, and $m : \overline{\Omega} \longrightarrow (1, \infty)$ are continuous functions for all $x \in \overline{\Omega}$ such that:

$$r(x)q'(x) > 1 \quad \text{and} \quad 1 < q(x) < p(x) < N \tag{4}$$

where $q'(\cdot) = \frac{q(\cdot)}{q(\cdot)-1}$. Moreover, $p(\cdot) > 1$ is logarithm Hölder continuous function, that is

$$\exists C > 0 : \; |p(x) - p(y)| \leq -\frac{C}{\ln(|x-y|)}, \quad \forall x \neq y \in \Omega : \; |x-y| \leq 1/2. \tag{5}$$

Some previous results and primary aim

- The presence of the additional term $-\mathrm{div}\left(F(u)\,|Du|^{q(\cdot)-2}\,Du\right)$ which has a regularizing effect on the summability of the solution of the problem 1.
- In the particular case $f \in L^1 \log L^1(\Omega)$, the regularizing effect of this term has been recently investigated by L. Boccardo and G.R. Cirmi (2022) [4].
- The solutions of problem 1 are more regular than the ones of the problems by F. Achhoud and G.R. Cirmi (2024) [3].

Main result

The notion of weak solution

If $f \in L^m(\Omega)$ with $m > 1$, by a weak solution of the problem 1 we mean a function $u \in W_0^{1,p(\cdot)}(\Omega)$ such that

$$F(u)\,|Du|^{q(\cdot)-1} \in L^{p'(\cdot)}(\Omega) \tag{6}$$

and the integral identity

$$\int_\Omega A(x)\,|Du|^{p(x)-2}\,Du \cdot D\varphi + \int_\Omega F(u)\,|Du|^{q(x)-2}\,Du \cdot D\varphi = \int_\Omega f\varphi \tag{7}$$

holds for every $\varphi \in W_0^{1,p(\cdot)}(\Omega)$.

Theorem

Under the assumptions (2)-(5). If $f \in L^{m(\cdot)}(\Omega)$ with $m(\cdot) > \frac{N}{p(\cdot)}$ then the problem 1 has at least one solution

$$u \in W_0^{1,p(\cdot)}(\Omega) \cap L^{\infty}(\Omega)$$

- Lebesgue-Sobolev spaces with variable exponents (look at the book [2]).
- The case $p(\cdot) = p,\ q(\cdot) = q,\ r(\cdot) = r$, and $m(\cdot) = m$ have recently been studied by F. Achhoud and G.R. Cirmi (2024) [3].

Strategy of the proof of the Theorem

1. Approximate problem
2. A priori estimate
3. Passage to the limit

Step 01: Approximate problem

The approximate problem

$$\begin{cases} -\mathrm{div}\left(A(x)\,|Du_n|^{p(x)-2}\,Du_n\right) - \mathrm{div}\left(F(T_n(u_n))\,|Du_n|^{q(x)-2}\,Du_n\right) = f_n, \\ u_n = 0, \end{cases} \tag{8}$$

where $(f_n)_n$ is a sequence of bounded functions such that

$$f_n \longrightarrow f \quad \text{strongly in } L^{m(\cdot)}(\Omega), \tag{9}$$

and

$$\|f_n\|_{L^{m(\cdot)}(\Omega)} \leq \|f\|_{L^{m(\cdot)}(\Omega)}, \quad \text{for all } n \in \mathbb{N}$$

Theorem: L. Boccardo and G.R. Cirmi (2022); [4]

Assume that (2)-(5) hold and let $f_n \in L^\infty(\Omega)$. Then there exists a weak solution $u_n \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ which solves the problem 8 in the following weak sense

$$\int_\Omega A(x)\,|Du_n|^{p(x)-2}\,Du_n \cdot Dv + \int_\Omega F(T_k(u_n))\,|Du_n|^{q(x)-2}\,Du_n \cdot Dv = \int_\Omega f_n v \tag{10}$$

holds for every $v \in W_0^{1,p(\cdot)}(\Omega)$.

Moreover, due to $f_n \in L^\infty(\Omega)$ and adapting the well known method used by G. Stampacchia; [5], each u_n is a bounded function and there exists $C > 0$, independent on n, such that:

$$\|u_n\|_{L^\infty(\Omega)} \leq C \tag{11}$$

Step 02: A priori estimate

- Suppose that the assumptions (2)-(5) are satisfied. Then, we have

$$\int_\Omega |DT_k(u_n)|^{p(x)}\,dx \leq C_1, \quad \text{for all } n \in \mathbb{N}. \tag{12}$$

- Let $f \in L^{m(\cdot)}(\Omega)$ with $m(\cdot) > \frac{N}{p(\cdot)}$ and assume that the assumptions (2)-(5) are satisfied. Then, there exists $C_2 > 0,\ C_3 > 0$, independent on n, such that

$$\|u_n\|_{W_0^{1,p(\cdot)}(\Omega)} \leq C_2, \quad \text{for all } n \in \mathbb{N}. \tag{13}$$

and

$$\int_\Omega \left(|u_n|^{(r(\cdot)-1)q(\cdot)+1}\,|Du_n|^{q(x)-1}\right)^{p'(x)} dx \leq C_3, \quad \text{for all } n \in \mathbb{N}. \tag{14}$$

Step 03: Passage to the limit

As a consequence of the estimates (12) and (13), there exists a function $u_n \in W_0^{1,p(\cdot)}(\Omega)$ such that, up to a subsequence still denoted by (u_n) one has

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in} \quad W_0^{1,p(\cdot)}(\Omega),$$
$$T_k(u_n) \to T_k(u) \quad \text{strongly in} \quad L^{p(\cdot)}(\Omega) \text{ and a.e. in } \Omega$$
$$u_n \rightharpoonup u \quad \text{weakly in} \quad W_0^{1,p(\cdot)}(\Omega),$$
$$u_n \to u \quad \text{strongly in} \quad L^{p(\cdot)}(\Omega)$$
$$u_n \to u \quad \text{a.e in } \Omega$$

Using the uniform estimate (14), we proceed by

$$T_k(u_n) \to T_k(u) \quad \text{strongly in} \quad W_0^{1,p(\cdot)}(\Omega)$$
$$Du_n \to Du \quad \text{a.e in} \quad \Omega$$

Putting the previous convergences together, we prove that

$$u_n \to u \quad \text{strongly in} \quad W_0^{1,p(\cdot)}(\Omega) \tag{15}$$

Now, thanks to the previous estimates and convergences, we can pass to the limit in the approximating problems. As a matter of the fact, due to (2) and the last strong convergence (15) we deduce that

$$A(x)\,|Du_n|^{p(\cdot)-2}\,Du_n \to A(x)\,|Du|^{p(\cdot)-2}\,Du \quad \text{strongly in} \quad \left(L^{p'(\cdot)}(\Omega)\right)^N$$

Therefore, given $\varphi \in W_0^{1,p(\cdot)}(\Omega)$,

$$\int_\Omega A(x)\,|Du_n|^{p(x)-2}\,Du_n \cdot Dv\,dx \to \int_\Omega A(x)\,|Du|^{p(x)-2}\,Du \cdot Dv\,dx$$

Moreover,

$$F(u_n)\,|Du_n|^{q(x)-2}\,Du_n \to F(u)\,|Du|^{q(x)-2}\,Du \quad \text{a.e in} \quad \Omega$$

By Vitali's Theorem, we deduce that

$$F(u_n)\,|Du_n|^{q(x)-2}\,Du_n \to F(u)\,|Du|^{q(x)-2}\,Du \quad \text{strongly in} \quad \left(L^{p'(\cdot)}(\Omega)\right)^N$$

At last, we are able to pass to the limit in the approximating problems thanks to the last convergences and the proof of the theorem is complete.

Conclusion and Perspectives

Problem 1 looks at complex mathematical models with changing growth conditions. This helps us understand materials that have mixed properties, like varying elasticity or viscosity. By studying the existence, regularity, and uniqueness of solutions, this problem contributes to advancements in material science, engineering, and mathematics, offering insights into real-world applications.

- Problem 1 can be regarded as the stationary problem associated with the parabolic equation

$$\partial_t u - \mathrm{div}\left(A(x)\,|Du|^{p(x)-2}\,Du\right) - \mathrm{div}\left(F(u)\,|Du|^{q(x)-2}\,Du\right) = f$$

This is an open problem worth studying.

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Thank you for your attention

Questions are welcome