People's Democratic Republic of Algeria

Ministry of Higher Education and Scientific Research

University of Mohamed Boudiaf -M'sila

Faculty of Mathematics and Computer Sciences

Second National Conference on Mathematics and Applications

M'sila, Algeria - 27-28 Nov. 2024.

CERTIFICATE OF PARTICIPATION

The organizing committee of the Second National Conference on Mathematics and Applications, certifies that:

ABDELAZIZ HELLAL

*Presented an **ORAL COMMUNICATION** entitled:*

REGULARIZING EFFECT IN SOME ANISOTROPIC NONLINEAR DIRICHLET PROBLEM.

Chairman of the NCMA'2024:

Pr. Noureddine BENHAMIDOUCHE

جامعة محمد بوضياف - المسيلة
Université Mohamed Boudiaf - M'sila

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Faculty of Mathematics and Informatics

Laboratory of Pures and Applied Mathematics

Second National Conference on Mathematics and Applications
M'sila, Algeria - 27-28 Nov. 2024

NCMA'2024 Program

About and Topics

About :

The second edition of the Conference on Mathematics and Applications serves as a platform for researchers and scientists, to exchange ideas and explore the latest advances in the field of mathematics and their applications. Our conference includes keynote speeches, paper presentations, and posters.

Topics :

- **Ordinary and Partial Differential Equations**
- **Algebra, Number theory, and applications**
- **Numerical Analysis and Applied mathematics: biology, image processing, fluid mechanics, ...**

First Day: Wednesday 27th November 2024

8h30 – 9h: Registration

9h – 9h 20: Opening Ceremony (Ibn El Haithem Conference Room)

Time	Plenary Session: (First Day : Wednesday 27th November 2024)
9h20-10h 45	**Chairman: Pr A. Medegheri - Dj. Bentorki**
9h20 – 10 h10	**Speaker**: Pr Lamnouar Noui Univ. Batna , Outils algébriques et applications.
10h10 – 10h 45	**Speaker**: Badreddine Benhellal - Univ. Oldenburg – Germany, On Schrödinger operators with oblique transmission conditions on non-smooth curve.
10h 45- 11h05	**Break Coffee**
11h 05-12 h30	**Chairman: Pr M. Hachama - A. Mansour**
11h 05 – 11h55	**Speaker**: Pr Nouri Fatma Zohra Univ. Annaba, A study of the dynamic of a fluid by an Atmospheric Pressure.
11h 55 – 12 h30	**Speaker**: Dr Bilel Selikh ENS- Bou Saada, Applications of non-commutative algebra in cryptography and security.
13h	**Lunch**

Oral parallel Sessions 1 (First Day : Wednesday 27th November 2024) - Afternoon		
Time	**Workshop 1:** Algebra, Number theory, and applications **Room 1** **Chairman**: Pr A. Boudaoud - L. Zedam	**Workshop 2:** fractional PDEs and ODEs **Room 2** **Chairman**: Pr Benyettou Benabderahmane - A. Yacine
14h – 14h 20	**Speaker**: Kheir Saadaoui, On a fuzzy lattice Instead to ,like computer sciences, chemistry	**Speaker**: Memou Ameur, An inverse source boundary value problem for a fractional parabolic differential equation of second order.
14h20 – 14h 40	**Speaker**: Hakim Moussaoui, Déformation in algebraic structures.	**Speaker**: Amouria Hamou, EXISTENCE RESULTS FOR FRACTIONAL DIFFERENTIAL INCLUSIONS.
14h40 – 15h 00	**Speaker**: Imane Douadi, Kinds of ideals and filters in hyperlattices semigroups.	**Speaker**: Ouagni Noura, Self-similar solutions of space-time fractional partial differential equation with Hilfer- Katugampola's Derivative.
15h – 15h 20	**Speaker**: Sylia Abdoun, A study on the impact information in Markovian queue with strategic customers.	**Speaker**: Bouthina Sabah Hammou, Existence and Uniqueness results for a Fractional Differential Equation with Integral Conditions
15h20 – 15h 40	**Coffee Break**	**Coffee Break**
15h40 – 16h 00	**Speaker**: Ahmed Benkahla, Construction of Bihom group	**Speaker**: Zakaria Malki, Combination of Hadamard derivative and boundary random fractional differential equations.
16h00 – 16h 20	**Speaker**: Oussama Zehani, Linearly multifunctions based construction of order relation.	**Speaker**: Imane Aouina, Analysis of a System of n-Nonlinear Fractional q-Differential Equations Involving Caputo

		q-Derivatives.
16h20 – 16h 40	**Speaker:** Adem Aikous, On the structure of ordered semi-hypergroup.	**Speaker**: Imane Boudrissa, A novel modification of the ROF model for image processing.
16h40 – 17h 00	**Speaker**: Saad Mohamed, Some properties of picture fuzzy subgroups on a group.	**Speaker**: Rami Amira, Chaos control in a fractional remanufacturing duopoly game.

Second Day : Thursday 28th November 2024

Oral parallel Sessions 2 (2nd day: Thursday 28th November 2024**)**		
Time	**Workshop 1:** Algebra, Number theory, and applications **Room 1** **Chairman**: D. Mihoubi - A. Amroune	**Workshop 2:** PDES and ODEs **Room 2** **Chairman**: Pr N. Bensalem - A. Gasmi
8h40 – 9h 00	**Speaker**: Heboob Lakhdar, Some cyclic codes of length 2p.	**Speaker**: Mimia Benhadri, Positive periodic solutions of of functional delay diferential equations with parameter.
9h00 – 9h 20	**Speaker**: Fares Mezrag, Implementation and Performance Evaluation of Public-Key Algorithms in Constrained Devices.	**Speaker**: Houd Kheireddine, Analysis of frictionless contact problem with friction.
9h20 – 9h 40	**Speaker**: Zahra Amroune, On factoring of unlimited generalized Fibonacci numbers.	**Speaker**: Haroune Lamrad, On The existence of the solution to an elliptic non-local problem involving critical Sobolev exponent.
9h40 – 10h 00	**Speaker**: Nasser Ghedbane, New Public Key Cryptosystem Using the Isomorphism Problem on Matrix Representations of finite Groups.	**Speaker**: Chahnaz Zakia Timimoun, Stability of the nonlinear Korteweg-de Vries equation.
10h00 – 11h 00	**Coffee Break and Poster Session**	
11h00 – 11h 20	**Speaker**: Frahtia Nassim, Localization of some functional spaces.	**Speaker**: Abdelaziz Hellal, REGULARIZING EFFECT IN SOME ANISOTROPIC NONLINEAR DIRICHLET PROBLEM.
11h20 – 11h 40	**Speaker**: Sara Boudaoud, A study on fuzzy graphs.	**Speaker**: Meriem Araour, NEW RESULTS FOR SOLVING LOGARITHMIC FUZZYINTEGRO-DIFFERENTIAL EQUATION:EXISTENCEAND UNIQUENESS OF THE SOLUTION
12h00 – 12h 20	**Speaker**: Ahlem Hamani, Some Properties Of Minimal Non-Finite-By-(Locally Nilpotent) Groups.	**Speaker**: H. Ait Mohammed, Existence and uniqueness results of solutions for a multi-point Riemann-Liouville fractional boundary value problem (RL-FBVP).

	Speaker: Rachid Chergui, La théorie du codage sur l'extension des nombres de Lucas.	**Speaker**: Bilel Madjour, Asymptotic behavior of memory-type thermoelastic problem with a polynomial source.
12h30	**Closing and Lunch**	**Closing and Lunch**

Oral parallel Sessions 2 (2nd day: Thursday 28th November 2024**)**		
Time	**Workshop 3:** Algebra, Number theory, and applications **Room 3** **Chairman** F.Z. Nouri - A. Merzougui	
8h40 – 9h 00	**Speaker**: Amina Khirani, Lucas Collocation Method for Solving linear Fredholm Integral Equations of the second kind.	
9h00 – 9h 20	**Speaker**: Abdelouahab Mani, A Comparative Study of Numerical Methods for Solving Integral Delay Equations.	
9h20 – 9h 40	**Speaker**: Bounouiga Souad, Mathematical Modeling of Malaria Transmission: Analysis and Numerical Simulation.	
9h40 – 10h 00	**Speaker**: Bochra Amroune, Application of the Hilbert method to the analysis of free surface flow phenomena.	
10h00 – 11h 00	**Coffee Break and Poster Session**	
11h00 – 11h 20	**Speaker**: Chouder Rafaa, Fast difference scheme for a general mean curvature Flow.	
11h20 – 11h 40	**Speaker**: Lachache Mohamed, Theoretical and Numerical results for a shallow water model coupled with a transport equation.	
11h 40– 12h 00	**Speaker**: Hemici Youcef Elhamam, Study of a diagonalconjugate gradient-like method using different line searches.	
12h00 – 12h 20	**Speaker**: Hossemddine Achour, Self-similar solutions for free boundary problem and contour enhancement in image.	
12h30	**Closing and Lunch**	

Poster Session1 (2nd day: Thursday 28th November 2024)

10h00 – 11h 00

P1 : Bounab Noura , Two-dimensional Potential Flow through a Nozzle.
P2 : Nadir Hana, Existence results for nonlinear problems when the source term is small.
P3 : Saqd Abdelkebir, Approximate Solution for Time-Conformable Fractional Heat Equations Using Adomian Decomposition Method.
P4 : Khayra Djerioui, coefficient problem for a time-fractional diffusion-wave equation with nonlocal boundary conditions.
P5 : Barkahoum Chebabhi, Shifted fifth-kind Chebyshev spectral approach for solving Volterra-Fredholm integro differential Equations.
P6 : Fatma seraiche, SUPERCONVERGENCE OF THE ITERATED GALERKIN METHOD FOR INTEGRAL EQUATIONS OF THE SECOND KIND.
P7 : Abderrazak Mehllou, numerical solution of fredholm integral equations using galerkin-legendre-wavelets method.
P8 : Lamine Salaheddine, An inverse problem for a time-fractional reaction-diffusion equation with involution and periodic boundary conditions.
P9 : Nour el imane Khadidja CHERIET, The hyper order and fixed points of solutions of a class of linear differential equations.
P10 : Chahinez Chhi ; Existence and uniqueness of positive solution for some nonlocal elliptic problem.
P11 : HIBATERRAHMANE BENMESSAOUD , PROPERTY OF LAMINAR FLOW OF BINGHAM FLUID.
P12 : Benaissi Brahim , THE FUNDAMENTAL EXISTENCE OF A SOLUTION TO THE GENERALIZED LINEAR FRACTIONAL DIFFUSION EQUATIO.
P13 : Aissa Amour, Numerical simulation of cement particle dispersion in a Cement-Mill Fan for erosion prediction.

N.B : It is preferred to present posters in A0 format

Regularizing effect in some anisotropic nonlinear Dirichlet problem
Oral communication

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Abstract

In this work, we establish the existence of weak solutions for a particular class of nonlinear double phase problems with irregular data and variable exponents. Our results can be viewed as a natural generalization of some results given in the constant exponents case.

Keywords: Nonlinear elliptic equations; weak solutions; double phase problems; variable exponents.

1 Introduction

We consider the homogeneous Dirichlet problem associated to the model equation:

$$\begin{cases} -\text{div}\left(A(x)\,|Du|^{p(x)-2}\,Du\right) - \text{div}\left(F(u)\,|Du|^{q(x)-2}\,Du\right) = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded open domain in $\mathbb{R}^N$ $(N > 2)$ with Lipschitz boundary $\partial\Omega$ and $A(\cdot)$ is a measurable function such that:

$$\alpha \leq A(x) \leq \beta, \quad \text{a.e. } x \in \Omega, \quad \text{with } \alpha,\ \beta > 0. \tag{2}$$

Suppose that $F : \mathbb{R} \longrightarrow \mathbb{R}$ such that:

$$F(u) = |u|^{(r(\cdot)-1)q(\cdot)+1}, \tag{3}$$

and f is a function belongs to $L^{m(\cdot)}(\Omega)$.
Here, the variable exponents $p : \overline{\Omega} \longrightarrow (1,\infty)$, $q : \overline{\Omega} \longrightarrow (1,\infty)$, $r : \overline{\Omega} \longrightarrow (1,\infty)$, and $m : \overline{\Omega} \longrightarrow (1,\infty)$ are continuous functions for all $x \in \overline{\Omega}$ such that:

$$r(x)q'(x) > 1, \quad 1 < q(x) < N, \text{ and } \quad p(x) < q(x) < p^\star(x), \tag{4}$$

where $q'(\cdot) = \frac{q(\cdot)}{q(\cdot)-1}$ and $p^\star(\cdot) = \frac{Np(\cdot)}{p(\cdot)-N}$. Moreover, $p(\cdot) > 1$ is logarithm Hölder continuous function, that is

$$\exists C > 0 :\ |p(x) - p(y)| \leq -\frac{C}{\ln(|x-y|)}, \quad \forall x \neq y \in \Omega :\ |x-y| \leq 1/2. \tag{5}$$

Problem 1 can be esteemed as the stationary problem associated with the parabolic equation

$$\partial_t u - \text{div}\left(A(x)\,|Du|^{p(x)-2}\,Du\right) - \text{div}\left(F(u)\,|Du|^{q(x)-2}\,Du\right) = f$$

So, it can be seen as the Darcy law of fluid filtration (in a turbulent regime) in a porous media simulating the pressure u.

Our results depend on the presence of the additional term $-\text{div}\left(F(u)\,|Du|^{q(\cdot)-2}\,Du\right)$, which has a regularizing effect on the summability of the solution of the problem 1. In the particular case $f \in L^1 \log L^1(\Omega)$, the regularizing effect of this term has been recently investigated in [4]. In point of fact, we shall demonstrate that the solutions of problem 1 are more regular than the ones of the problems (see [3] and [9]) under an appropriate balance of the parameters $p(\cdot)$, $q(\cdot)$, $r(\cdot)$, and the summability of f. Inspired by [1, 2, 8], we will provide additional regularity results based on the summability of the source term f. To achieve this, we need to define the notion of weak solution we will use in accordance with the summability of the right-hand side. For simplify our investigation, we will only discuss the case when f belongs to $L^{m(\cdot)}(\Omega)$ with $m(\cdot) > \frac{N}{p(\cdot)}$.

2 Main result:

Definition 2.1. *If $f \in L^m(\Omega)$ with $m > 1$, by a weak solution of the problem 1 we mean a function $u \in W_0^{1,p(\cdot)}(\Omega)$ such that*

$$F(u)\,|Du|^{q(\cdot)-1} \in L^{p'(\cdot)}(\Omega) \tag{6}$$

and the integral identity

$$\int_\Omega A(x)\,|Du|^{p(x)-2}\,Du\cdot D\varphi\,dx + \int_\Omega F(u)\,|Du|^{q(x)-2}\,Du\cdot D\varphi\,dx = \int_\Omega f\varphi\,dx \tag{7}$$

holds for every $\varphi \in W_0^{1,p(\cdot)}(\Omega)$.

We have the following theorem:

Theorem 2.2. *Under the assumptions* (2)-(5). *If $f \in L^{m(\cdot)}(\Omega)$ with $m(\cdot) > \frac{N}{p(\cdot)}$ then the problem 1 has at least one solution $u \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$.*

The idea of the proof of Theorem 2.2: The proof is based on three steps: First, we introduce a suitable sequence of problems approximating. In addition, we establish some priori estimates for the solutions of the problem 1. Finally, we pass to the limit in the approximate problems.

References

[1] H. Abdelaziz; *Singular Elliptic Equations with Variable Exponents.* Int. J. Math. And Appl., 11(4): (2023) 141-168.

[2] H. Abdelaziz and F. Mokhtari; *Nonlinear anisotropic degenerate parabolic equations with variable exponents and irregular data*, J. Ellip. Para. Equa. 8 (2022) 513-532.

[3] F. Achhoud and G.R. Cirmi; *Regularity results for an anisotropic nonlinear Dirichlet problem.* Complex Variables and Elliptic Equations, (2024) 1-22.

[4] L. Boccardo and G.R. Cirmi ; *Regularizing effect in some Mingione's double phase problems with very singular data.* Math Eng. 5(3): (2022) 1-15.

[5] Y. Chen, S. Levine, and M. Rao; *Variable exponent, linear growth functionals in image restoration.* SIAM J. Appl. Math. 66 (2006) 1383-1406.

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[9] G. Stampacchia, *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus.* Ann. Inst. Fourier (Grenoble), 15 (1965) 189-258.

Regularizing effect in some anisotropic nonlinear Dirichlet problem

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Second National Conference on Mathematics and Applications, (NCMA'2024)

November $27^{th} - 28^{th}$, 2024

Outline:

Mathematical theory

$$\begin{cases} -\mathrm{div}\left(A(x)\,|Du|^{p(x)-2}\,Du\right) - \mathrm{div}\left(F(u)\,|Du|^{q(x)-2}\,Du\right) = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded open domain in $\mathbb{R}^N$ $(N > 2)$ with Lipschitz boundary $\partial\Omega$ and $A(\cdot)$ is a measurable function such that:

$$\alpha \leq A(x) \leq \beta, \quad \text{a.e. } x \in \Omega, \quad \text{with } \alpha,\ \beta > 0. \tag{2}$$

Suppose that $F : \mathbb{R} \longrightarrow \mathbb{R}$ such that:

$$F(u) = |u|^{(r(\cdot)-1)q(\cdot)+1}, \tag{3}$$

and

$$f \in L^{m(\cdot)}(\Omega).$$

Here, the variable exponents $p : \overline{\Omega} \longrightarrow (1, \infty)$, $q : \overline{\Omega} \longrightarrow (1, \infty)$, $r : \overline{\Omega} \longrightarrow (1, \infty)$, and $m : \overline{\Omega} \longrightarrow (1, \infty)$ are continuous functions for all $x \in \overline{\Omega}$ such that:

$$r(x)q'(x) > 1 \quad \text{and} \quad 1 < q(x) < p(x) < N \tag{4}$$

where $q'(\cdot) = \frac{q(\cdot)}{q(\cdot)-1}$. Moreover, $p(\cdot) > 1$ is logarithm Hölder continuous function, that is

$$\exists C > 0 : \ |p(x) - p(y)| \leq -\frac{C}{\ln(|x - y|)}, \quad \forall x \neq y \in \Omega : \ |x - y| \leq 1/2. \tag{5}$$

Some previous results and primary aim

- The presence of the additional term $-\mathrm{div}\left(F(u)\,|Du|^{q(\cdot)-2}\,Du\right)$ which has a regularizing effect on the summability of the solution of the problem 1.
- In the particular case $f \in L^1 \log L^1(\Omega)$, the regularizing effect of this term has been recently investigated by L. Boccardo and G.R. Cirmi (2022) [4].
- The solutions of problem 1 are more regular than the ones of the problems by F. Achhoud and G.R. Cirmi (2024) [3].

Physical interpretation-Material science

One important application of elliptic double phase problems is in modeling nonlinear elasticity for composite materials with varying properties. Composite materials can have regions where the behavior is governed by different physical laws or growth conditions.

$$-\mathrm{div}\left(A(x)\,|Du|^{p(x)-2}\,Du\right)-\mathrm{div}\left(F(u)\,|Du|^{q(x)-2}\,Du\right)=f$$

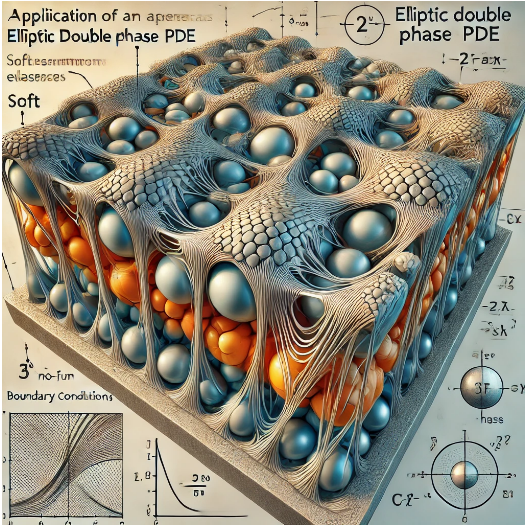
- It describes the equilibrium state of the material.
- u is the displacement field.
- A and F indicate the phase transition between the soft and stiff (hard) material.
- f represents external forces (stress).
- Depending on the spatial region, the different phases show distinct elastic responses in the equilibrium state of the material.

Remark for the physical interpretation

- if F is small (soft material), the equation reduces to a $p(\cdot)$-Laplace type equation (behavior of the softer material).
- if F is large (stiff material), the equation behaves like a $q(\cdot)$-Laplace equation, modeling the stiffer material's response.

An illustrated example

- The image shows a composite material with two phases soft and hard under stress.



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Mathematical theory-Main result

The notion of weak solution

If $f \in L^m(\Omega)$ with $m > 1$, by a weak solution of the problem 1 we mean a function $u \in W_0^{1,p(\cdot)}(\Omega)$ such that

$$F(u)\,|Du|^{q(\cdot)-1} \in L^{p'(\cdot)}(\Omega) \tag{6}$$

and the integral identity

$$\int_\Omega A(x)\,|Du|^{p(x)-2}\,Du\cdot D\varphi + \int_\Omega F(u)\,|Du|^{q(x)-2}\,Du\cdot D\varphi = \int_\Omega f\varphi \tag{7}$$

holds for every $\varphi \in W_0^{1,p(\cdot)}(\Omega)$.

Theorem

Under the assumptions (2)-(5). If $f \in L^{m(\cdot)}(\Omega)$ with $m(\cdot) > \frac{N}{p(\cdot)}$ then the problem 1 has at least one solution

$$u \in W_0^{1,p(\cdot)}(\Omega) \cap L^{\infty}(\Omega)$$

- Lebesgue-Sobolev spaces with variable exponents (look at the book [2]).
- The case $p(\cdot) = p,\ q(\cdot) = q,\ r(\cdot) = r$, and $m(\cdot) = m$ have recently been studied by F. Achhoud and G.R. Cirmi (2024) [3].

Strategy of the proof of the Theorem

1. Approximate problem
2. A priori estimate
3. Passage to the limit

Step 01: Approximate problem

The approximate problem

$$\begin{cases} -\mathrm{div}\left(A(x)\,|Du_n|^{p(x)-2}\,Du_n\right) - \mathrm{div}\left(F(T_n(u_n))\,|Du_n|^{q(x)-2}\,Du_n\right) = f_n, \\ u_n = 0, \end{cases} \tag{8}$$

where $(f_n)_n$ is a sequence of bounded functions such that

$$f_n \longrightarrow f \quad \text{strongly in } L^{m(\cdot)}(\Omega), \tag{9}$$

and

$$\|f_n\|_{L^{m(\cdot)}(\Omega)} \leq \|f\|_{L^{m(\cdot)}(\Omega)}, \quad \text{for all } n \in \mathbb{N}$$

Theorem: L. Boccardo and G.R. Cirmi (2022); [4]

Assume that (2)-(5) hold and let $f_n \in L^{\infty}(\Omega)$. Then there exists a weak solution $u_n \in W_0^{1,p(\cdot)}(\Omega) \cap L^{\infty}(\Omega)$ which solves the problem 8 in the following weak sense

$$\int_{\Omega} A(x)\,|Du_n|^{p(x)-2}\,Du_n\cdot Dv + \int_{\Omega} F(T_k(u_n))\,|Du_n|^{q(x)-2}\,Du_n\cdot Dv = \int_{\Omega} f_n v \tag{10}$$

holds for every $v \in W_0^{1,p(\cdot)}(\Omega)$.

Moreover, due to $f_n \in L^{\infty}(\Omega)$ and adapting the well known method used by G. Stampacchia; [5], each u_n is a bounded function and there exists $C > 0$, independent on n, such that:

$$\|u_n\|_{L^{\infty}(\Omega)} \leq C \tag{11}$$

Step 02: A priori estimate

- Suppose that the assumptions (2)-(5) are satisfied. Then, we have

$$\int_\Omega |DT_k(u_n)|^{p(x)}\, dx \le C_1, \quad \text{for all } n \in \mathbb{N}. \tag{12}$$

- Let $f \in L^{m(\cdot)}(\Omega)$ with $m(\cdot) > \frac{N}{p(\cdot)}$ and assume that the assumptions (2)-(5) are satisfied. Then, there exists $C_2 > 0,\ C_3 > 0$, independent on n, such that

$$\|u_n\|_{W_0^{1,p(\cdot)}(\Omega)} \le C_2, \quad \text{for all } n \in \mathbb{N}. \tag{13}$$

and

$$\int_\Omega \left(|u_n|^{(r(\cdot)-1)q(\cdot)+1}\, |Du_n|^{q(x)-1}\right)^{p'(x)} dx \le C_3, \quad \text{for all } n \in \mathbb{N}. \tag{14}$$

Step 03: Passage to the limit

As a consequence of the estimates (12) and (13), there exists a function $u_n \in W_0^{1,p(\cdot)}(\Omega)$ such that, up to a subsequence still denoted by (u_n) one has

$$
\begin{aligned}
&T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in} \quad W_0^{1,p(\cdot)}(\Omega),\\
&T_k(u_n) \rightarrow T_k(u) \quad \text{strongly in} \quad L^{p(\cdot)}(\Omega) \text{ and a.e. in } \Omega\\
&u_n \rightharpoonup u \quad \text{weakly in} \quad W_0^{1,p(\cdot)}(\Omega),\\
&u_n \rightarrow u \quad \text{strongly in} \quad L^{p(\cdot)}(\Omega)\\
&u_n \rightarrow u \quad \text{a.e in } \Omega
\end{aligned}
$$

Using the uniform estimate (14), we proceed by

$$
T_k(u_n) \rightarrow T_k(u) \quad \text{strongly in} \quad W_0^{1,p(\cdot)}(\Omega)
$$
$$
Du_n \rightarrow Du \quad \text{a.e in} \quad \Omega
$$

Putting the previous convergences together, we prove that

$$u_n \to u \quad \text{strongly in} \quad W_0^{1,p(\cdot)}(\Omega) \tag{15}$$

Now, thanks to the previous estimates and convergences, we can pass to the limit in the approximating problems. As a matter of the fact, due to (2) and the last strong convergence (15) we deduce that

$$A(x)\,|Du_n|^{p(\cdot)-2}\,Du_n \to A(x)\,|Du|^{p(\cdot)-2}\,Du \quad \text{strongly in} \quad \left(L^{p'(\cdot)}(\Omega)\right)^N$$

Therefore, given $\varphi \in W_0^{1,p(\cdot)}(\Omega)$,

$$\int_\Omega A(x)\,|Du_n|^{p(x)-2}\,Du_n \cdot Dv\,dx \to \int_\Omega A(x)\,|Du|^{p(x)-2}\,Du \cdot Dv\,dx$$

Moreover,

$$F(u_n)\,|Du_n|^{q(x)-2}\,Du_n \to F(u)\,|Du|^{q(x)-2}\,Du \quad \text{a.e in} \quad \Omega$$

By Vitali's Theorem, we deduce that

$$F(u_n)\,|Du_n|^{q(x)-2}\,Du_n \to F(u)\,|Du|^{q(x)-2}\,Du \quad \text{strongly in} \quad \left(L^{p'(\cdot)}(\Omega)\right)^N$$

At last, we are able to pass to the limit in the approximating problems thanks to the last convergences and the proof of the theorem is complete.

Conclusion and perspectives

Problem 1 addresses complex mathematical models with variable growth conditions, enabling the analysis of materials with mixed properties, such as variable elasticity or viscosity. By focusing on the existence, regularity, and uniqueness of solutions, this problem contributes to advancements in material science, engineering, and mathematical theory, offering insights into real-world applications.

- Problem 1 can be regarded as the stationary problem associated with the parabolic equation

$$\partial_t u - \operatorname{div}\left(A(x)\,|Du|^{p(x)-2}\,Du\right) - \operatorname{div}\left(F(u)\,|Du|^{q(x)-2}\,Du\right) = f$$

This is an open problem worth studying.

Some references

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Thank you for your attention

Questions are welcome