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CERTIFICATE OF PARTICIPATION

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participated and presented a talk titled:

EXISTENCE RESULTS FOR DOUBLE PHASE PROBLEMS

at the 8th National M'sila Conference on Mathematical Analysis (Banach spaces and operator theory), held on October 23rd, 2024 at M'sila.

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8^{th} M'sila Conference on Mathematical Analysis

" Banach spaces & Operator theory" October 23^{rd}, 2024. M'sila, Algeria

Program Schedule

Room 03

Topic 03: Functional analysis of PDEs

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10:20-10:40	MOHAMED EL FAROUK OUNANE	INFINITELY MANY SOLUTIONS FOR A Q-CURVATURE TYPE PROBLEM ON A CLOSED RIEMANNIAN MANIFOLD
10:40-11:00	BRAHIMI SIHAM	ÈTUDE DU CARACTÈRE FAIBLEMENT BIEN POS´E DANS L2 POUR DES PROBLÉMES AUX LIMITES CARACTÈRISTIQUES
11:00-11:20	AISSA BENSEGHIR	TRANSMISSION SYSTEM FOR WAVES WITH NONLINEAR WEIGHTS AND DELAY
11:20-11:40	Coffee Break	

11:40-12:00	ABDELAZIZ HELLAL	EXISTENCE RESULTS FOR DOUBLE PHASE PROBLEMS
12:00-12:15	MOHAMED AMINE ZOUATINI	REGULARITY RESULTS FOR ANISOTROPIC DEGENERATE PARABOLIC EQUATIONS WITH A SINGULARITY
12:15-12:30	BENZATAT RIM	VARIATIONAL ANALYSIS OF A DYNAMIC VISCOELASTIC PROBLEM
12:30-12:45	GUIDOUM SABRINA	THE EXISTENCE AND UNIQUENESS OF SOLUTION FOR AN ELECTRO-ELASTIC PROBLEM
12:45-13:00	AMARA KADDOURI JOUINT	OPTIMAL CONTROL OF SOLUTION OF A NONLINEAR HISTORY-DEPENDENT BOUNDARY VALUE PROBLEM
13:00-13:15	BILEL MADJOUR	ASYMPTOTIC BEHAVIOR OF THE ENERGY FOR THE KIRCHHOFF EQUATION WITH NONLINEAR FEEDBACK IN BEAM DYNAMICS
13:15-13:30	ASMA SAHRAOUIL	QUASI-EXACTLY SOLVABLE SCHRODINGER EQUATION FOR THE INVERSE QUARTIC POWER POTENTIAL
13:30	Closing and Lunch	

UNIVERSITY OF M'SILA
8^{th} M'SILA CONFERENCE ON MATHEMATICAL ANALYSIS (8MCMA'2024)
Banach spaces and operator theory
October 23^{rd}, 2024. M'sila, Algeria

EXISTENCE RESULTS FOR DOUBLE PHASE PROBLEMS

ABDELAZIZ HELLAL

ABSTRACT. In this paper, we prove the existence of weak solutions for a specific class of nonlinear double-phase problems involving irregular data and variable exponents. Our findings serve as a natural extension of previous results obtained in the constant exponents case.

1. INTRODUCTION:

In this work, we consider the homogeneous Dirichlet problem associated to the model equation:

$$\begin{cases} -\text{div}\left(A(x)\,|Du|^{p(x)-2}\,Du\right) - \text{div}\left(F(u)\,|Du|^{q(x)-2}\,Du\right) = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open domain in $\mathbb{R}^N$ $(N > 2)$ with Lipschitz boundary $\partial\Omega$ and $A(\cdot)$ is a measurable function such that:

$$\alpha \leq A(x) \leq \beta, \quad \text{a.e. } x \in \Omega, \quad \text{with } \alpha,\ \beta > 0. \tag{1.2}$$

Suppose that $F : \mathbb{R} \longrightarrow \mathbb{R}$ such that:

$$F(u) = |u|^{(r(\cdot)-1)q(\cdot)+1}, \tag{1.3}$$

and f is a function belongs to $L^{m(\cdot)}(\Omega)$.

Here, the variable exponents $p : \overline{\Omega} \longrightarrow (1,\infty)$, $q : \overline{\Omega} \longrightarrow (1,\infty)$, $r : \overline{\Omega} \longrightarrow (1,\infty)$, and $m : \overline{\Omega} \longrightarrow (1,\infty)$ are continuous functions for all $x \in \overline{\Omega}$ such that:

$$r(x)q'(x) > 1, \quad \text{and} \quad 1 < q^- \leq q^+ < p^- \leq p^+ < N \tag{1.4}$$

where $p^- := \min_{x\in\overline{\Omega}} p(x)$, $p^+ := \max_{x\in\overline{\Omega}} p(x)$, $q^- := \min_{x\in\overline{\Omega}} q(x)$, $q^+ := \max_{x\in\overline{\Omega}} q(x)$ and $q'(\cdot) = \frac{q(\cdot)}{q(\cdot)-1}$. Moreover, $p(\cdot) > 1$ is logarithm Hölder continuous function, that is

$$\exists C > 0 :\ |p(x) - p(y)| \leq -\frac{C}{\ln(|x-y|)}, \quad \forall x \neq y \in \Omega :\ |x-y| \leq 1/2. \tag{1.5}$$

Problem 1.1 can be esteemed as the stationary problem associated with the parabolic equation

$$\partial_t u - \text{div}\left(A(x)\,|Du|^{p(x)-2}\,Du\right) - \text{div}\left(F(u)\,|Du|^{q(x)-2}\,Du\right) = f$$

1991 *Mathematics Subject Classification.* 35J60; 35D30; 35J66.
Key words and phrases. Nonlinear elliptic equations; weak solutions; double phase problems; variable exponents.

So, it can be seen as the Darcy law of fluid filtration (in a turbulent regime) in a porous media simulating the pressure u.

Our results depend on the presence of the additional term $-\text{div}\Big(F(u)\,|Du|^{q(\cdot)-2}\,Du\Big)$, which has a regularizing effect on the summability of the solution of the problem 1.1. In the particular case $f \in L^1(\Omega)$, the regularizing effect of this term has been recently investigated in [4]. In point of fact, we shall demonstrate that the solutions of problem 1.1 are more regular than the ones of the problems (see [3] and [9]) under an appropriate balance of the parameters $p(\cdot)$, $q(\cdot)$, $r(\cdot)$, and the summability of f. Inspired by [1, 2, 8], we will provide additional regularity results based on the summability of the source term f. To achieve this, we need to define the notion of weak solution we will use in accordance with the summability of the right-hand side. For simplify our investigation, we will only discuss the case when f belongs to $L^{m(\cdot)}(\Omega)$ with $m(\cdot) > \frac{N}{p(\cdot)}$.

2. Preliminaries:

We recall some definitions, facts, and basic properties of Lebesgue-Sobolev spaces with variable exponents. We refer to [6] and references therein. In this work we denote

$$p^+ := \max_{x\in\overline{\Omega}} p(x) \quad \text{and} \quad p^- := \min_{x\in\overline{\Omega}} p(x).$$

We define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable functions $u : \Omega \to \mathbb{R}$ for which the convex modular

$$\rho_{p(\cdot)}(u) = \int_\Omega |u|^{p(x)}\,dx,$$

is finite.

The space $L^{p(\cdot)}(\Omega)$ equipped with the norm

$$\|u\|_{p(\cdot)} := \|u\|_{L^{p(\cdot)}(\Omega)} = \inf\big\{k > 0 : \rho_{p(\cdot)}\big(\frac{u}{k}\big) \le 1\big\},$$

which called the Luxemburg norm. The space $(L^{p(\cdot)}(\Omega), \|u\|_{p(\cdot)})$ is a separable Banach space. Moreover, if $1 < p^- \le p^+ < +\infty$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$ where $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$.

For all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, the Hölder type inequality

$$\Big|\int_\Omega uv\,dx\Big| \le \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big)\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)} \le 2\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)}, \tag{2.1}$$

holds.

We define also the Banach space

$$W^{1,p(\cdot)}(\Omega) = \big\{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\big\},$$

endowed with the norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

The space $(W^{1,p(\cdot)}(\Omega), \|u\|_{1,p(\cdot)})$ is a Banach space. while

$$W_0^{1,p(\cdot)}(\Omega) = \big\{u \in W^{1,p(\cdot)}(\Omega) : u = 0 \text{ on } \partial\Omega\big\},$$

is Sobolev space with zero boundary values endowed with the norm $\|\cdot\|_{1,p(\cdot)}$. The space $W_0^{1,p(\cdot)}(\Omega)$ is separable and reflexive provided that $1 < p^- \le p^+ < +\infty$.

The smooth functions (i.e. $\mathcal{D}(\Omega)$) are in general not dense in $W_0^{1,p(\cdot)}(\Omega)$, but if the exponent variable $p(\cdot) > 1$ satisfy the logarithmic Hölder continuity condition (1.5) then $\mathcal{D}(\Omega)$ are dense in $W_0^{1,p(\cdot)}(\Omega)$.

For $u \in W_0^{1,p(\cdot)}(\Omega)$ with $p \in C_+(\overline{\Omega})$, the Poincaré inequality holds

$$\|u\|_{p(\cdot)} \leq C\|\nabla u\|_{p(\cdot)}, \tag{2.2}$$

for some $C > 0$ which depends on Ω and $p(\cdot)$. Therefore, $\|\nabla u\|_{p(\cdot)}$ and $\|u\|_{1,p(\cdot)}$ are equivalent norms.

Our treatment requires to use the following results:

Proposition 1 ([6])**.** If $(u_n), u \in L^{p(\cdot)}(\Omega)$ and $p^+ < +\infty$, then the following properties hold:

- $\|u\|_{p(\cdot)} < 1$ (resp. $= 1, > 1$) $\Longleftrightarrow \rho_{p(\cdot)}(u) < 1$ (resp. $= 1, > 1$),
- $\min\left(\rho_{p(\cdot)}(u)^{\frac{1}{p^+}}, \rho_{p(\cdot)}(u)^{\frac{1}{p^-}}\right) \leq \|u\|_{p(\cdot)} \leq \max\left(\rho_{p(\cdot)}(u)^{\frac{1}{p^+}}, \rho_{p(\cdot)}(u)^{\frac{1}{p^-}}\right)$,
- $\min\left(\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right) \leq \rho_{p(\cdot)}(u) \leq \max\left(\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right)$,
- $\|u\|_{p(\cdot)} \leq \rho_{p(\cdot)}(u) + 1$,
- $\|u_n - u\|_{p(\cdot)} \to 0 \Longleftrightarrow \rho_{p(\cdot)}(u_n - u) \to 0$.

Remark 2.1. As in [7], the following inequality

$$\int_\Omega |u|^{p(x)}\,dx \leq C\int_\Omega |Du|^{p(x)}\,dx,$$

in general does not hold. So, thanks to Proposition 1 and (2.2), we get the following inequality which will be used later

$$\min\{\|Du\|_{p(\cdot)}^{p^-}; \|Du\|_{p(\cdot)}^{p^+}\} \leq \int_\Omega |u(x)|^{p(x)}\,dx \leq \max\{\|Du\|_{p(\cdot)}^{p^-}; \|Du\|_{p(\cdot)}^{p^+}\}. \tag{2.3}$$

We need an important embedding as follows

Lemma 2.2 (Sobolev embedding)**.** *Let $\Omega \in \mathbb{R}^N$ be an open bounded set with Lipschitz boundary and assume that $p : \overline{\Omega} \to (1, N)$ satisfy the logarithm Hölder continuity condition* (1.5)*. Then we have the following continuous embedding:*

$$W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p^\star(\cdot)}(\Omega),$$

where $p^\star(\cdot) = \frac{Np(\cdot)}{N-p(\cdot)}$.

For the notion of weak solutions to problem 1.1, we will use through this work, the truncation function T_k at height k ($k > 0$) and the associated function which denoted by

$$T_k(t) = \min\Big\{k, \max\{-k, t\}\Big\}, \quad G_k(t) = t - T_k(t) \tag{2.4}$$

It is obvious that T_k is Lipschitz functions satisfying $|T_k(t)| \leq k$.

Throughout this work, C always indicate any non-negative constant which depends only on data and whose value may change from line to line.

3. Main result:

Definition 3.1. If $f \in L^m(\Omega)$ with $m > 1$, by a weak solution of the problem 1.1 we mean a function $u \in W_0^{1,p(\cdot)}(\Omega)$ such that

$$F(u)\,|Du|^{q(\cdot)-1} \in L^{p'(\cdot)}(\Omega) \tag{3.1}$$

and the integral identity

$$\int_\Omega A(x)\,|Du|^{p(x)-2}\,Du\cdot D\varphi\,dx + \int_\Omega F(u)\,|Du|^{q(x)-2}\,Du\cdot D\varphi\,dx = \int_\Omega f\varphi\,dx \tag{3.2}$$

holds for every $\varphi \in W_0^{1,p(\cdot)}(\Omega)$.

We have the following theorem:

Theorem 3.2. *Under the assumptions* (1.2)-(1.5). *If* $f \in L^{m(\cdot)}(\Omega)$ *with* $m(\cdot) > \frac{N}{p(\cdot)}$ *then the problem 1.1 has at least one solution* $u \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

The idea of the proof of Theorem 3.2: The proof is based on three steps: First, we introduce a suitable sequence of problems approximating. In addition, we establish some priori estimates for the solutions of the problem 1.1. Finally, we pass to the limit in the approximate problems.

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