



CERTIFICATE



The Director of the **Laboratoire des Mathématiques Appliquées et de Modélisation (LMAM)** certifies that:

« Abdelaziz Hellal »

Participated with an Online Oral Presentation and presented a paper entitled:

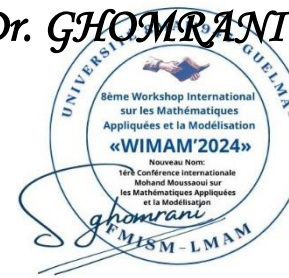
« Existence of weak solutions to double phase problems with irregular data »

at the **1st International Conference Mohand Moussaoui on Applied Mathematics and Modeling**, previously known as the **8th International Workshop on Applied Mathematics and Modelling (WIMAM'2024)**, held on **19-20 November 2024**, Université 8 Mai 1945 Guelma

The Laboratory Director
Pr. Hamza GUEBBAI



Chairman of the Organizing Committee
Dr. GHOMRANI Sarra



People's Democratic Republic of Algeria



Ministry of Higher Education and Scientific Research

University of 08 Mai 1945 – Guelma

Faculty of Mathematics and Computer Science and Material Sciences

Laboratory of Applied Mathematics and Modeling (LMAM)

Intervention Program

**8th International Workshop on
Applied Mathematics and Modeling
“WIMAM’2024”**

New Name:

**« 1st International Conference Mohand
Moussaoui on Applied Mathematics
and Modeling »**

Organized by

**Laboratory of Applied
Mathematics and Modeling
(LMAM)**

**University 8 Mai 1945
Guelma**

Guelma, 19-20th Nov, 2024



1st International Conference **Mohand Moussaoui on Applied
Mathematics and Modeling**



First Day Program

1st International Conference **Mohand Moussaoui** on Applied
Mathematics and Modeling



First day, 19th November 2024

08h30–09h30 **Opening ceremony** Google meeting link: <https://meet.google.com/wum-sirq-gih>

In memory of the late Mr. Moussaoui Mohand

Presented by: Prof. Ayadi Abdelhamid

09h30–14h30 **Plenary sessions** Google meeting link: <https://meet.google.com/wum-sirq-gih>

Plenary Sessions Program

	<u>Chairman: Prof. Hamza GUEBBAI</u>
09h30–10h30	Pr. Greshnov Alexandr Valerevich: Exact values of the constants in (q_1, q_2) -generalized triangle inequality for (q_1, q_2) -quasimetric spaces.
10h30–11h00	Break + Poster Session
	<u>Chairman: Prof. Ayadi Abdelhamid</u>
11h00–12h00	Pr. Kurulay Muhamet: Mathematical modeling of Dynamical Systems Epidemiology
12h00–13h00	Lunch Break
	<u>Chairman: Prof. Aissaoui Mohamed Zine</u>
13h00–14h00	Pr. Nasserddine Kechkar: Numerical methods for fractional PDEs: A State of the Art



Ordinary Sessions Program

Room 1: Integral equations, optimization and numerical methods

Chairwoman: Dr. Kaidouchi Wahida

Chairman: Dr. Bouchnak Ahmed

14h00 – 14h15	Meryem BENSSAAD: Analytical And Numerical Study of New Integro-Differential Fredholm Equation
14h15 – 14h30	Ahlem DJEBAR: Around Of The Bivariate Buhlamann's Optimization Problem And Its Applications
14h30 – 14h45	Allaoua MEHRI : Estimation D'une Erreur De La Solution D'un Problème D'un Obstacle Parabolique D'un Ordre Fractionnaire Avec Terme Source Non-Linéaire
14h45 – 15h00	Bahi Mostafa: Approximation solution for Noncoercive Parabolic Variational in equality using multigrid methods
15h00 – 15h30	Coffee break
15h30 – 15h45	Houria SELATNIA: Average Sentinel for a Parabolic Equation with Incompet Data
15h45 – 16h00	Sami SEGNI: Nonlinear Volterra Integro-Differential Equations with Delay Terms: An Application of the Picard Iteration Method
16h00 – 16h15	Mohamed Lamine MERIKHI : Modélisation Avancée Du Système De Brusselator Avec Vitesses De Réaction Variables : Une Approche Fractionnaire

Room 2: Differential equations and operator theory

Chairman: Prof. CHAOUI Abderrezak

14h00 – 14h15	Guebbai Hamza: Traitement numérique du Spectre généralisé
14h15 – 14h30	Ridha DIDA: Efficient Numerical Method for Solving Linear Fredholm Integro-Differential Equations Using Backward Finite Difference and Nyström Techniques
14h30 – 14h45	Hamza Brahim BOULARES: Weak type estimate of singular integral operators on variable weak Herz-type Hardy spaces
14h45 – 15h00	Roumaissa Khalfllaoui: On the solution of evolution p-Laplace equation with memory term and unknown boundary Dirichlet condition
15h00 – 15h30	Coffee break
15h30 – 15h45	Randa Chakar: Théorème Du Point Fixe Généralisé Pour Les Contractions F -Weak En Utilisant Une Fonction
15h45 – 16h00	Labadla Amel et Djaghout Manal: Parabolic p-biharmonic equation with variable exponent: existence results and full discretization

Room 3: Fixed point and fractional theory

Chairman: Pr. Nasserline Kechkar

14h00 – 14h15	Bouchenak Ahmed: Existence and Uniqueness Study of the Conformable Laplace Transform with Applications
14h15 – 14h30	Rebiai Ghania: URHS for Darboux problem
14h30 – 14h45	Merchela Wassim: The existence of solution for an operator equation on b -metric Space
14h45 – 15h00	Kerboua Mourad: Mild Solutions for Non Local Hybrid -Hilfer Fractional Systems: Novel Qualitative Results
15h00 – 15h30	Coffee break

Room 4: Other Topics

Chairwoman: Dr. Belhireche Hanane

14h00 – 14h15	Bilel Madjour: Theoretical Study of Decay Solution in Euler-Bernoulli Equation With a Source Term
14h15 – 14h30	Sarra Ghomrani: Modeling Seismic Activity: A Theoretical and Numerical Study of Nonlinear Volterra Systems
14h30 – 14h45	Abdallah Brik: Periodic orbits of a fourth dimensional hyperchaotic system
14h45 – 15h00	Rabah DEBBAR: Existence and uniqueness of solutions for the ψ - Caputo sequential fractional order neutral functional differential equations
15h00 – 15h30	Coffee break
15h30 – 15h45	Souhil Yekhlef: The difference between parametric and non-parametric data and how to deal with it
15h45 – 16h00	Bahloul Tarek: Exploring Asymptotic Properties and Growth Constraints of a Monotonically Decreasing Function

Poster Session

- **Mohamed-Salih ARISS:** An Innovative Numerical Approach for Addressing Weakly Singular Nonlinear Integro-Differential Volterra Equations.
- **Sara LABIDI:** On the decay rate for a neutral delayed porous elastic system.
- **Boutheina TAIR:** Approximation Of Higher-Order Nonlinear Volterra Integro-Differential Equations.
- **Wahida KAIDOUCHI:** The Study Of A Nonlinear Equations.
- **Nesrine KAMOUCHE:** Méthode De Laplace Pour L'estimation Asymptotique Des Intégrales De Type De Laplace Dans Le Cas Bidimensionnel.
- **Arafa Ghada:** The balanced quantile credibility estimator with dependent risk structure.
- **Imane ZOUAK:** On the Stability of a Discrete Computer Virus Model with Fractional-Order.
- **Sarra BENARAB:** The generalization of Arutyunov's theorem on (q_1, q_1) -quasimetric space.
- **Haneche NABIL:** An incommensurate fractional-order discrete-time SIR epidemic model for influenza A viruses: Bifurcation, chaos, and control.
- **Nouadri ROCHDI:** Le test de khi-deux modifié de Nikulin-Rao-Robson (NRR).
- **Khaoula AIDI:** Different non-Bayesian estimation methods under the Exponential Generalized Log-Logistic Model.
- **Sarra BELAKHDAR:** Clinical trials subjected to Bayesian Sequential Analysis.
- **Bouaziz CHAMSEDDINE:** Zero-Hopf bifurcation of the Lotka-Volterra systems in R^3 via averaging theory of third order.
- **Bounab NOURA:** Numerical study of a two-dimensional potential flow problem with surface tension.
- **BELILI MOHAMED CHERIF:** Dynamical analysis of a discrete-time SIR epidemic model with loss of immunity



Second Day Program

Second day, 20th November 2024

Online Day

08h00–08h15 Opening ceremony

In memory of the late Mr. Moussaoui Mohand



1st International Conference **Mohand Moussaoui** on Applied
Mathematics and Modeling



Room 1: Integral equations, optimization and numerical methods

Ordinary session (Morning 8h00-12h00)

Chairwoman: Dr. Benssaad Meryem

Chairman: Dr. Merchela Wassim

Google meeting link: <https://meet.google.com/yfx-kqdj-ygr>

08h15 – 08h30	Hanane CHINOUNE: Method DGJIM for Solving Differential Equations of Fractional Order
08h30 – 08h45	Imane BAZINE: An Analytical and Numerical Approach to a Nonlinear Fredholm Integro-Differential Equation with Weak Singularities
08h45 – 09h00	Hamza Youcef ETTOUMI: Homogenization of Quadratic Forms
09h00 – 09h15	Hanane BELHIRECHE: Mixed Volterra-Fredholm Nonlinear Integral Equations with Regular Kernels
09h15 – 09h30	Mohammed KHIRANI: On Integral Inequalities Related to The Extended Chebyshev Functional
09h30 – 09h45	Douha SAADI: Solving Nonlocal Initial-Boundary Value Problems for Parabolic Integro-Differential Equations by Numerical Method
09h45 – 10h00	Meriem ARAOUR: On The Existence and Uniqueness of The Solution of The Logarithmic Fuzzy Integro-Differential Equation
10h00 – 10h15	Abderrazak Mehellou: Numerical Solution of Fredholm Integral Equations Using Legendre-Galerkin Method
10h15 – 10h30	Sofiane BOUTARCHA: Spectral Collocation Method for Solving Volterra Integral Equation on the Half-Line
10h30 – 10h45	Safia BELLABACI: New Fractional Mask to Edge Detection Using Laplacian Method
10h45 – 11h00	Sabrina BENHANACHI: A New Class of Optimization Methods Based on Coefficient Conjugate Gradient
11h00 – 11h15	Ahlem ZAARAT: The Adaptive Method: Theory and Application
11h15 – 11h30	Bachir DEHDA: A New Quadratic Method to Approximate The First Order Caputo-Fabrizio Fractional Derivative
11h30 – 11h45	Chayma BOUKOUBA: Numerical Solutions for Multidimensional Parabolic Problems with Nonlocal Boundary Conditions
11h45 – 12h00	Aimen DAOUDI: Well Posedness for Viscoelastic Fourth-Order Problems with Relaxation Term Involving The Generalized Caputo Fractional Derivative

Ordinary session (Afternoon 13h30-16h30)

Chairwoman: Dr. HAFAIDIA Imane

Google meeting link: <https://meet.google.com/yfx-kqdj-ygr>

13h30 – 13h45	Ilyes SEDKA: Optimizing Numerical Solutions for Nonlinear Integral Equations via the T.L.D. Algorithm
13h45 – 14h00	Hadjer ZEROUALI: Numerical Technique for Solving Schrodinger Equation with Boundary Conditions.
14h00 – 14h15	Somia KAMOUICHE: Analytical and Numerical Treatment of a Specific Nonlinear Fredholm Integral Equation
14h15 – 14h30	Yasmina Bendaas: Numerical Study of An Infeasible Interior-Point Algorithm for Semi-Definite Optimization
14h30 – 14h45	Chahinaz CHENOUF: Analytical And Numerical Study of Covering Method For H"olderian Global Optimization
14h45 – 15h00	Fares BEKHOUCHE: The Residual Power Series Method For Solving The Time Fractional Nonlinear Reaction-Diffusion Equations



Room 2: Differential equations and operator theory

Room 2A: Ordinary session (Morning 8h00-12h00)

Chairwoman: Dr. Kaidouchi Wahida

Google meeting link: <https://meet.google.com/ecw-fctt-vdb>

08h15 – 08h30	Mokhtar KHAASSANI: Elliptic Differential Equation with Variable Operators Coefficients
08h30 – 08h45	Bahlali Dounia: Hamilton Bellman Jacobi systems in financial market problem
08h45 – 09h00	Abdelaziz HELLAL: Existence of Weak Solutions to Double Phase Problems with Irregular Data
09h00 – 09h15	Amina CHAILI: Decay Behavior of Solutions to Coupled Biharmonic Schrödinger Equations with Internal Fractional Damping
09h15 – 09h30	Mouhous AMIROUCHE: Existence Results for First-Order Boundary Value Problems
09h30 – 09h45	Oussama KHALIDI: Well-Posedness and Stability Result for A Timoshenko System with Thermodiffusion Effects and Time-Varying Delay Term
09h45 – 10h00	Achref Eddine TABET: Limit Cycles Generated from A Zero-Hopf Bifurcation in A Continuous Three-Dimensional Differential System via Averaging Theory
10h00 – 10h15	Meryem CHETTI: Growth of Analytic Solutions of Linear Differential Equations with Analytic Coefficients Near A Finite Singular Point
10h15 – 10h30	Fatima Zohra BENGHIA: Orthogonal polynomials corresponding to polynomial Szegő measure
10h30 – 10h45	Imad AIMENE: Dynamic Properties of a Kirchhoff-Type Plate Equation with Damping Terms and Acoustic Control Boundary Conditions
10h45 – 11h00	Asma BOUGHERARA: Critical Curve for Existence of Non-Linear Elliptic System Involving Hardy Potential and Potential-Gradient Terms
11h00 – 11h15	Meriem HAMEURLAINE: Solving the Resulting Differential Equations for the Bianchi I Cosmological Model
11h15 – 11h30	Ines GARTI: Exploring Solutions to the Cauchy Problem for the Generalized Double Dispersion Equation
11h30 – 11h45	Khadidja Aicha KADA: Modélisation Mathématique de la Dynamique de l'Insecte Ravageur de Vignoble (Lobesia Botrana): Interactions Adultes-Juvéniles
11h45 – 12h00	Nawal IRZI: On The Fourth-Order Leray-Lions Problem with Indefinite Weight and Nonstandard Growth Conditions

Room 2B: Ordinary session (Morning 8h00-12h00)

Chairwoman: Dr. Ghomrani Sarra

Google meeting link: <https://meet.google.com/byq-dutb-ors>

08h15 – 08h30	Ishak KETTAF: Existence of Solutions for A Second Order Problem On The Real Half-Line Via Ekeland's Variational Principle
08h30 – 08h45	Khaoula CHABANE: $N/2$ -Normal Operators on Hilbert Spaces
08h45 – 09h00	Farida MEKKAOU: A New Characterization of Some Classes of Operators
09h00 – 09h15	Safa MENKAD: The Generalized Mean Transform Of Closed Range Operators
09h15 – 09h30	Mustapha MAADANI: Class $A^\#$ In Semi-Hilbertian Space
09h30 – 09h45	Naima MEHENAOU: Exponential Decay For Nonlinear Wave Equations On A Star-Shaped-Network Using Kelvin-Voigt Mechanisms
09h45 – 10h00	Mohamed El Farouk OUNANE: Infinitely Many Solutions for A Perturbation Of The Prescribed Q-Curvature Problem On Closed Riemannian Manifolds
10h00 – 10h15	Abdellah MESKINE: Stability And Boundedness of Solutions For Some Second-Order Non-Autonomous Vector Delay Differential Equations Of Neutral Type
10h15 – 10h30	Aicha BENGUETAIB: Nonlinear $p(x)$ -Laplacian Elliptic Equations with Two Singularities and L^1 Data
10h30 – 10h45	Imane Boudrissa: Nonlinear diffusion model featuring the Caputo fractional derivative and its application in image processing
10h45 – 11h00	Hakima LASSAL: Growth of Solutions to Linear Differential Equations with $[p, q]$ -Order
11h00 – 11h15	Rima Chouader: A class of Kolmogorov differential system with three hyperbolic algebraic limit cycles.
11h15 – 11h30	Bellamouchi Chahinez: Positive solutions results for some nonlocal elliptic problem
11h30 – 11h45	Maatougui Belaala: Pietsch factorization theorem for Lipschitz ϕ -summing operators
11h45 – 12h00	Nadjet Bensebaa: Analysis of a bilateral contact problem for piezoelectric material

Room 2A: Ordinary session (Afternoon 13h30-16h30)

Chairwoman: Dr. Amel Labadla and Dr. Manal Djaghout

Google meeting link: <https://meet.google.com/ecw-fctt-vdb>

13h30 – 13h45	Samira LECHEHEB: Nontrivial Solutions for Nonlinear Problem with Non-Compact Resolvent Operators
13h45 – 14h00	Boudersa Ferial: Local existence of solutions in anisotropic Sobolev spaces for a coupled system of generalized Kadomtsev-Petviashvili I (KP-I) equations.
14h00 – 14h15	Bahlali Dounia: Non trivial weak solution of nonlinear problem involving a $p(x)$ -Laplace type operator
14h15 – 14h30	Smina DJENNADI: Application of a Regularization Method for Reconstructing a Stable Solution to an Inverse Backward Problem in Fractional Diffusion Equations
14h30 – 14h45	BRAIK Abdelkader: Existence and Asymptotic Stability of the Solution for the Timoshenko Transmission System with Distributed Delay
14h45 – 15h00	BENTATA Fatima Ezahra: Existence of solutions and numerical simulation for a hyperbolic Kirchhoff-type problem with a free boundary.
15h00 – 15h15	Mansouri Mouhamed: Theoretical Study of the Generalized Spectrum
15h15 – 15h30	Bengrine Fatma zohra: Existence and non-existence results with an Hardy-Leray potential

Room 3: Fixed Point and Fractional Theory

Ordinary session (Morning 8h00-12h00)

Chairman: Dr. Ilyes Sadka

Google meeting link: <https://meet.google.com/qzv-xpmo-xyt>

08h15 – 08h30	Roufaida Ketfi: The variational study of a dynamic problem with fractional derivative
08h30 – 08h45	Soufiane Bousserhane Reda: On the Global Non existence of a Solution for Wave Equations with Nonlinear Memory Term
08h45 – 09h00	AIT OUALI Nadia: A Novel Approach to Mild Solutions in Stochastic Fractional Equations
09h00 – 09h15	Malki Zakarya: Fractional differential equation system with random boundaries
09h15 – 09h30	Omar ALIMERINA: Results for the Blow-up of nonlinear wave equation with time-dependent damping
09h30 – 09h45	Leyla Soudani: Hybrid fractional differential equation in Banach algebra
09h45 – 10h00	Guedim Souad: Uniqueness and stability of Hadamard variable-order boundary value problem
10h00 – 10h15	Boutillis Mokhtaria: Existence of solutions for fractional impulsive neutral stochastic functional integrodifferential equations with infinite delay
10h15 – 10h30	Berrighi Fatma: Exponential stability of mild solutions for impulsive functional evolution equations
10h30 – 10h45	Abdelkader Hakim Benhana: Existence of solutions for a class of biharmonic equation with p-Laplacian
10h45 – 11h00	Haroun LAMRAD: The Existence of the Solution to a Fractional Schrödinger Equation with Doubly Criticality
11h00 – 11h15	Hacen Serrai: Existence of Solutions for an Ordinary Hybrid Version of Generalized Sturm-Liouville-Langevin Equation Using alpha-varphi Contraction and Psi-Hilfer Fractional Derivative
11h15 – 11h30	Hacene Gharout: Isoclines et Langues d'Arnold d'un endomorphisme tridimensionnel
11h30 – 11h45	Noredine Rezoug: Nonlocal second-order delay semilinear integro-differential time-varying evolution equation with convex-power condensing operators
11h45 – 12h00	Nouikas Azeddine: Existence of Solutions for a Boundary Value Problem of Hadamard Fractional Derivative Order Via Pseudomonotone Theory

Ordinary session (Afternoon 13h30-16h30)

Chairman: Dr. DIDA Ridha, Dr. DEBBAR Rabeh

Google meeting link: <https://meet.google.com/qzv-xpmo-xyt>

13h30 – 13h45	Abdelhamid Gouasmia: Fixed Point Theorem for Systems Involving Mixed Local and Non-Local Operators
13h45 – 14h00	Yacine Boulatiour: A Fractional Differential System of Beam Type With Two Generalised Nonlinearities
14h00 – 14h15	Naouel Khial: The existence and stability of coincidence point for mappings acting from b-metric space into generalized metric space
14h15 – 14h30	Bazine Safia: Some fixed point theorems
14h30 – 14h45	MEHDI SELMANI: THE EXISTENCE OF WEAK SOLUTIONS FOR A NEW FRACTIONAL PROBLEM VIA ψ -HILFER OPERATOR
14h45 – 15h00	BENTERKI Abdessalem: Recent Results on Data Dependence in Partial Metric Spaces
15h00 – 15h15	Arioui Fatima Zahra: Fractional Stochastic differential equations



Room 4: Other Topics

Room 4A: Ordinary session (Morning 8h00-12h00)

Chairwoman: Dr. Djennaoui Saliha

Google meeting link: <https://meet.google.com/jeh-fmqi-wkw>

08h15 – 08h30	Bennour Abdelaziz: Solutions for a nonlocal elliptic equation with critical Sobolev exponent and singular term
08h30 – 08h45	Djeniah Abd Elbasset: Inference on volatility estimation with missing data: a functional data approach
08h45 – 09h00	Khouloud Makhlouf: Model Uncertainty Stochastic Partial Differential Equations (SPDEs) Control and Application
09h00 – 09h15	Zineb Bellabes: On the study of Ψ -caputo fractional impulsive differential equations involving variational approach
09h15 – 09h30	Sara Hamaizia: Intermittency in fractional Duffing equation
09h30 – 09h45	Azzi Cheyma: Étude d'un modèle de pêche avec stock variable
09h45 – 10h00	Allaeddine Haddari: Prime de crédibilité dans le cas pondéré équilibré
10h00 – 10h15	BECHEROUL Amina: Non-parametric estimation of stochastic differential equations driven by mixed bi-fractional Brownian motion
10h15 – 10h30	Chahrazed Lellou: Application de l'algorithme MCEM sur la dynamique du VIH via le modèle espace d'état à effets mixtes
10h30 – 10h45	Chettih Nadjat: Autoencoders-Based Feature Extraction for Random Forest Classifier
10h45 – 11h00	Mohammed Chikouche Elias Taki Eddine: Modeling Risk in Finance: The One-Default Mathematical Model
11h00 – 11h15	Touati Fatima: Étude qualitative de quelques systèmes différentiels polynomiaux
11h15 – 11h30	Hamidi Nabil: The operating diagram of the competition model between plasmid bearing and plasmid-free organisms in the chemostat
11h30 – 11h45	Ouchen Imene: Prime Bayésienne du système Bonus-malus basée sur la fréquence et le montant du sinistre
11h45 – 12h00	Isma Debbah: Impact of Missing Data Percentage on the Consistency of a non-parametric Gamma Estimators



Room 4B: Ordinary session (Morning 8h00-12h00)

Chairwoman: Dr. Kamouche Somia

Google meeting link: <https://meet.google.com/onq-bvde-ryq>

08h15 – 08h30	Rachid Chergui: Matrices de Lucas et une nouvelle théorie du codage.
08h30 – 08h45	Saihi Ferial: Asymptotic Normality of the Conditional Hazard Rate Function Estimator
08h45 – 09h00	Mohamed Touati: Some results on the existence of nodal or positive solutions for Paneitz-Branson Equations on Non-Einstein Manifolds
09h00 – 09h15	Yfrah Abed: Energy decay of solutions for a linear wave equation with dynamical boundary control
09h15 – 09h30	Mechaouf Abir: Optimal Control of Evolution Equations
09h30 – 09h45	Abdellatif Bencherif Madani: Homogénéisation localement périodique
09h45 – 10h00	Saiah Seyyid Ali: On the Cauchy problem for a weakly coupled system of semi-linear fractional σ -evolution equation with different nonlinear memory
10h00 – 10h15	Amina Delhoum: DNA codes from skew cyclic codes over $F_4[\ell^2-]$
10h15 – 10h30	Amroune Zahra: On the Diophantine equation: $d(kn) = \varphi(\varphi(n))$
10h30 – 10h45	Djadi Chahineze: Quadrinomial-like versions for Wolstenholme and Morley congruences
10h45 – 11h00	Sara Dridi: Regional controllability of nonlinear cellular automata using a preimage generation algorithm
11h00 – 11h15	Khelifa Berkane: Stochastic Modelling of Epidemic Systems with Fractional Brownian Motion
11h15 – 11h30	Hadjadj Houari: Non-Markovian queue with two types of service, optional reservice, and general vacation distribution
11h30 – 11h45	Khaoula Meansri: New iterative modification of the conjugate gradient method for unconstrained problems
11h45 – 12h00	IMINE Chahinez: Vertex-distinguishing threshold of graph

Room 4C: Ordinary session (Morning 8h00-12h00)

Chairwoman: Dr. Bouacida Ichrak

Google meeting link: <https://meet.google.com/wyb-ubsu-ahv>

08h15 – 08h30	Merai Manel ElBatoul: The asymptotic properties for a conditional class of kernel estimators
08h30 – 08h45	Bernou Ismahen: Weak law of large numbers for dependent random variables with infinite means
08h45 – 09h00	Mohammed El Amine Riahi: Problem of Control for Stationary Marguerre-von Kàrmàn Equation Shallow shell
09h00 – 09h15	Achour Hossemddine: Free-boundary problem and image contour enhancement with self-similar solution
09h15 – 09h30	Ahlem Boucherikha: A numerical strategy for trapped waves in the β -plane
09h30 – 09h45	Bellout Aida: Analysis of bifurcation, chaos, and entropy in the fractional chaotic Burgers map
09h45 – 10h00	Akermi Seif Eddine: Modeling infectious disease incidence in Algeria: Comparative efficacy of SARIMA, NNAR, and hybrid models
10h00 – 10h15	Adel Boucetta: Exploring confidence bands for nonparametric relative error regression in survival analysis with functional covariables
10h15 – 10h30	Boukhalfa Asma: Estimation non paramétrique par la méthode des Sieves
10h30 – 10h45	Ghorab Elhouari: Generalized tempered ultradistributions
10h45 – 11h00	Si Bachir Fatima: Successive approximations for Caputo-Fabrizio fractional differential equations
11h00 – 11h15	Boubakr Lamouri: Analysis and dynamics of tuberculosis (TB) with vaccination and treatment
11h15 – 11h30	Hibaterrahmane Benmessaoud: Estimation of blocking force in stationary Herschel-Bulkley fluid regime
11h30 – 11h45	Imane Merzougui: Elliptic curves and their applications in cryptography
11h45 – 12h00	MOUSSAOUI Hakim: On Classification of Supermodules and Supermodule superalgebras

Room 4 A: Ordinary session (Afternoon 13h30-16h30)

Chairwoman: Dr. Imane Bazine

Google meeting link: <https://meet.google.com/jeh-fmqi-wkw>

13h30 – 13h45	Bousseba Fatma Zohra: A new quadratic exponential distribution with two parameters
13h45 – 14h00	Said Bensliman: PRODUCT OF ASYMMETRIC TOEPLITZ MATRICES
14h00 – 14h15	Abdoun Sylia: Investigation of full information availability in a Markovian queue with differentiated vacations
14h15 – 14h30	Souad Bounouiga: Analysis of the modified SIR model for infectious diseases with fractional derivatives
14h30 – 14h45	Chougar Wissem: Null controllability of linear heat equation with mixed boundary conditions
14h45 – 15h00	Zineb Douara: Analyse de la régularité d'un problème de transmission gouverné par une équation abstraite dans un cadre h ô l d é r i e n
15h00 – 15h15	Benkhaled Radhwane: Application du théorème de Tikhonov pour les systèmes lents-rapides à un modèle épidémiologique
15h15 – 15h30	Lazri Nouara: SZL Distribution: Properties and Simulation
15h30 – 15h45	Melouka Remil: Discretization for a 2D-Viscoelastic Wave Equation with Acoustic Dynamic Boundary Conditions
15h45 – 16h00	Makreloufi Abed: On Controllability for Linear Differential Algebraic Equations
16h00 – 16h15	Roumaissa Elbay: The weak convergence of the smooth estimator of the distribution function under double truncation data

Room 4B: Ordinary session (Afternoon 13h30-16h30)

Chairwoman: Dr. Bazine Safia and Dr. Benarab Sarra

Google meeting link: <https://meet.google.com/onq-bvde-ryq>

13h30 – 13h45	Khedidja Abba: Optimal Control in Partially Observed McKean-Vlasov FBSDEs with Random Jumps: A Stochastic Maximum Principal Approach
13h45 – 14h00	Samia Teghri: Two Parameter Lindley Frailty Model: Properties, Estimation, and Application
14h00 – 14h15	Manseur Sara : Principe des grandes d'écartiations pour la file d'attente `a serveur unique
14h15 – 14h30	Selma Kimouche: Non-relativistic limit for particle confined on a surface
14h30 – 14h45	BENALI Tinhinane: Étude des équations différentielles abstraites complète de type elliptique provenant de modèles physiques concrets, sur des domaines cylindriques hétérogènes dans un cadre höldérien: application au potentiel électrostatique
14h45 – 15h00	Chellouf Yassamine: Numerical solution of fractional Hopfield neural networks using reproducing kernel Hilbert space method
15h00 – 15h15	KACIMI Bouazza: Some Results on the Harmonicity of 1-forms
15h15 – 15h30	Grine Razika: Nouvelle Distribution zéro tronquée
15h30 – 15h45	MIME Rania: Counter-example of Park's necessary condition of realizability over $SL_2(R[x^+, \dots, x^+])$
15h45 – 16h00	Nassima Bayarassou: Nonparametric estimation of the relative regression function for left truncated and right censored data: Methodology and Applications
16h00 – 16h15	Sabrina Benzamouche: Consistency of a kernel estimator of the relative error regression function for incomplete and dependent data

Room 4C: Ordinary session (Afternoon 13h30-16h30)

Chairwoman: Dr. Hanane BELHIRECHE

Google meeting link: <https://meet.google.com/wyb-ubsu-ahv>

13h30 – 13h45	Abderrahmane Layati: Nonlinear Problem with Neumann Condition
13h45 – 14h00	Sara Youcef Achira: Hybrid Taylor and block-pulse functions operational matrix algorithm and its application to obtain the approximate solution of stochastic Itô-Volterra integral equations
14h00 – 14h15	Yahi Rachid: Multilinear p-th power factorable operators
14h15 – 14h30	Benzatat Rim: Analysis of a dynamic viscoelastic problem with long-term memory
14h30 – 14h45	Boughambouz Hamza: On ternary fuzzy preorder
14h45 – 15h00	Belaiza Dihia: M/M/1 Retrial Queuing System with Negative arrivals and two types of Breakdowns
15h00 – 15h15	Omar Elfarouk Cherifi: A remark on the Hájek-Rényi inequality and some limit theorems for a class of dependent random variables
15h15 – 15h30	Messabihi Nafissa: Almost complete convergence and asymptotic normality of non-parametric estimator
15h30 – 15h45	Salima Mansouri: Numerical study of fractional boundary value problem with a conformable fractional derivative of Caputo-Fabrizio and its fractional integral
15h45 – 16h00	BENARBIA Narimene: Modeling Chagas Disease Transmission: A Semigroup approach
16h00 – 16h15	Sadji Roufaida: Existence of solution for a Fractional Boundary Value Problem with higher fractionnal order.

Existence of weak solutions to double phase problems with irregular data

Abdelaziz Hellal

University of M'sila, University Pole

*First International Conference Mohand Moussaoui on Applied
Mathematics and Modeling,
(CIM'24)*

November 19th – 20th, 2024



Outline:

- 1 **Introduction-Mathematical theory**
- 2 **Physical interpretation-Material science**
- 3 **Mathematical theory-Main result**
- 4 **Conclusion-perspectives**
- 5 **Some references**

Mathematical theory

$$\begin{cases} -\operatorname{div} \left(A(x) |Du|^{p(x)-2} Du \right) - \operatorname{div} \left(F(u) |Du|^{q(x)-2} Du \right) = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where Ω is a bounded open domain in $\mathbb{R}^N$ ($N > 2$) with Lipschitz boundary $\partial\Omega$ and $A(\cdot)$ is a measurable function such that:

$$\alpha \leq A(x) \leq \beta, \quad \text{a.e. } x \in \Omega, \quad \text{with } \alpha, \beta > 0. \quad (2)$$

Suppose that $F: \mathbb{R} \rightarrow \mathbb{R}$ such that:

$$F(u) = |u|^{(r(\cdot)-1)q(\cdot)+1}, \quad (3)$$

and

$$f \in L^{m(\cdot)}(\Omega).$$

Here, the variable exponents $p : \overline{\Omega} \longrightarrow (1, \infty)$, $q : \overline{\Omega} \longrightarrow (1, \infty)$, $r : \overline{\Omega} \longrightarrow (1, \infty)$, and $m : \overline{\Omega} \longrightarrow (1, \infty)$ are continuous functions for all $x \in \overline{\Omega}$ such that:

$$r(x)q'(x) > 1 \quad \text{and} \quad 1 < q(x) < p(x) < N \quad (4)$$

where $q'(\cdot) = \frac{q(\cdot)}{q(\cdot)-1}$. Moreover, $p(\cdot) > 1$ is logarithm Hölder continuous function, that is

$$\exists C > 0 : |p(x) - p(y)| \leq -\frac{C}{\ln(|x - y|)}, \quad \forall x \neq y \in \Omega : |x - y| \leq 1/2. \quad (5)$$

Some previous results and primary aim

- ❑ The presence of the additional term $-\operatorname{div} \left(F(u) |Du|^{q(\cdot)-2} Du \right)$ which has a regularizing effect on the summability of the solution of the problem 1.
- ❑ In the particular case $f \in L^1 \log L^1(\Omega)$, the regularizing effect of this term has been recently investigated by L. Boccardo and G.R. Cirmi (2022) [4].
- ❑ The solutions of problem 1 are more regular than the ones of the problems by F. Achhoud and G.R. Cirmi (2024) [3].

Physical interpretation-Material science

One important application of elliptic double phase problems is in modeling nonlinear elasticity for composite materials with varying properties. Composite materials can have regions where the behavior is governed by different physical laws or growth conditions.

$$-\operatorname{div} \left(A(x) |Du|^{p(x)-2} Du \right) - \operatorname{div} \left(F(u) |Du|^{q(x)-2} Du \right) = f$$

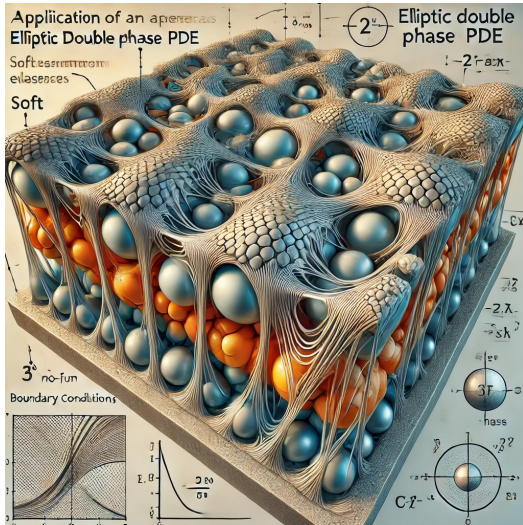
- ☐ It describes the equilibrium state of the material.
- ☐ u is the displacement field.
- ☐ A and F indicate the phase transition between the soft and stiff (hard) material.
- ☐ f represents external forces (stress).
- ☐ Depending on the spatial region, the different phases show distinct elastic responses in the equilibrium state of the material.

Remark for the physical interpretation

- if F is small (soft material), the equation reduces to a $p(\cdot)$ -Laplace type equation (behavior of the softer material).
- if F is large (stiff material), the equation behaves like a $q(\cdot)$ -Laplace equation, modeling the stiffer material's response.

An illustrated example

- ❑ The image shows a composite material with two phases soft and hard under stress.



10.64

Mathematical theory-Main result

The notion of weak solution

If $f \in L^m(\Omega)$ with $m > 1$, by a weak solution of the problem 1 we mean a function $u \in W_0^{1,p(\cdot)}(\Omega)$ such that

$$F(u) |Du|^{q(\cdot)-1} \in L^{p'(\cdot)}(\Omega) \quad (6)$$

and the integral identity

$$\int_{\Omega} A(x) |Du|^{p(x)-2} Du \cdot D\varphi + \int_{\Omega} F(u) |Du|^{q(x)-2} Du \cdot D\varphi = \int_{\Omega} f\varphi \quad (7)$$

holds for every $\varphi \in W_0^{1,p(\cdot)}(\Omega)$.

Theorem

Under the assumptions (2)-(5). If $f \in L^{m(\cdot)}(\Omega)$ with $m(\cdot) > \frac{N}{p(\cdot)}$ then the problem 1 has at least one solution

$$u \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$$

- Lebesgue-Sobolev spaces with variable exponents (look at the book [2]).
- The case $p(\cdot) = p$, $q(\cdot) = q$, $r(\cdot) = r$, and $m(\cdot) = m$ have recently been studied by F. Achhoud and G.R. Cirmi (2024) [3].

Strategy of the proof of the Theorem

1. Approximate problem
2. A priori estimate
3. Passage to the limit

Step 01: Approximate problem

The approximate problem

$$\begin{cases} -\operatorname{div} \left(A(x) |Du_n|^{p(x)-2} Du_n \right) - \operatorname{div} \left(F(T_n(u_n)) |Du_n|^{q(x)-2} Du_n \right) = f_n, \\ u_n = 0, \end{cases} \quad (8)$$

where $(f_n)_n$ is a sequence of bounded functions such that

$$f_n \longrightarrow f \quad \text{strongly in } L^{m(\cdot)}(\Omega), \quad (9)$$

and

$$\|f_n\|_{L^{m(\cdot)}(\Omega)} \leq \|f\|_{L^{m(\cdot)}(\Omega)}, \quad \text{for all } n \in \mathbb{N}$$

Theorem: L. Boccardo and G.R. Cirmi (2022); [4]

Assume that (2)-(5) hold and let $f_n \in L^\infty(\Omega)$. Then there exists a weak solution $u_n \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ which solves the problem 8 in the following weak sense

$$\int_{\Omega} A(x) |Du_n|^{p(x)-2} Du_n \cdot Dv + \int_{\Omega} F(T_k(u_n)) |Du_n|^{q(x)-2} Du_n \cdot Dv = \int_{\Omega} f_n v \quad (10)$$

holds for every $v \in W_0^{1,p(\cdot)}(\Omega)$.

Moreover, due to $f_n \in L^\infty(\Omega)$ and adapting the well known method used by G. Stampacchia; [5], each u_n is a bounded function and there exists $C > 0$, independent on n , such that:

$$\|u_n\|_{L^\infty(\Omega)} \leq C \quad (11)$$

Step 02: A priori estimate

- Suppose that the assumptions (2)-(5) are satisfied. Then, we have

$$\int_{\Omega} |DT_k(u_n)|^{p(x)} dx \leq C_1, \quad \text{for all } n \in \mathbb{N}. \quad (12)$$

- Let $f \in L^{m(\cdot)}(\Omega)$ with $m(\cdot) > \frac{N}{p(\cdot)}$ and assume that the assumptions (2)-(5) are satisfied. Then, there exists $C_2 > 0$, $C_3 > 0$, independent on n , such that

$$\|u_n\|_{W_0^{1,p(\cdot)}(\Omega)} \leq C_2, \quad \text{for all } n \in \mathbb{N}. \quad (13)$$

and

$$\int_{\Omega} \left(|u_n|^{(r(\cdot)-1)q(\cdot)+1} |Du_n|^{q(x)-1} \right)^{p'(x)} dx \leq C_3, \quad \text{for all } n \in \mathbb{N}. \quad (14)$$

Step 03: Passage to the limit

As a consequence of the estimates (12) and (13), there exists a function $u_n \in W_0^{1,p(\cdot)}(\Omega)$ such that, up to a subsequence still denoted by (u_n) one has

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in } W_0^{1,p(\cdot)}(\Omega),$$

$$T_k(u_n) \rightarrow T_k(u) \quad \text{strongly in } L^{p(\cdot)}(\Omega) \text{ and a.e. in } \Omega$$

$$u_n \rightharpoonup u \quad \text{weakly in } W_0^{1,p(\cdot)}(\Omega),$$

$$u_n \rightarrow u \quad \text{strongly in } L^{p(\cdot)}(\Omega)$$

$$u_n \rightarrow u \quad \text{a.e in } \Omega$$

Using the uniform estimate (14), we proceed by

$$T_k(u_n) \rightarrow T_k(u) \quad \text{strongly in } W_0^{1,p(\cdot)}(\Omega)$$

$$Du_n \rightarrow Du \quad \text{a.e in } \Omega$$

Putting the previous convergences together, we prove that

$$u_n \rightarrow u \quad \text{strongly in} \quad W_0^{1,p(\cdot)}(\Omega) \quad (15)$$

Now, thanks to the previous estimates and convergences, we can pass to the limit in the approximating problems. As a matter of the fact, due to (2) and the last strong convergence (15) we deduce that

$$A(x) |Du_n|^{p(\cdot)-2} Du_n \rightarrow A(x) |Du|^{p(\cdot)-2} Du \quad \text{strongly in} \quad \left(L^{p'(\cdot)}(\Omega) \right)^N$$

Therefore, given $\varphi \in W_0^{1,p(\cdot)}(\Omega)$,

$$\int_{\Omega} A(x) |Du_n|^{p(x)-2} Du_n \cdot Dv \, dx \rightarrow \int_{\Omega} A(x) |Du|^{p(x)-2} Du \cdot Dv \, dx$$

Moreover,

$$F(u_n) |Du_n|^{q(x)-2} Du_n \rightarrow F(u) |Du|^{q(x)-2} Du \quad \text{a.e in } \Omega$$

By Vitali's Theorem, we deduce that

$$F(u_n) |Du_n|^{q(x)-2} Du_n \rightarrow F(u) |Du|^{q(x)-2} Du \quad \text{strongly in } \left(L^{p'(\cdot)}(\Omega) \right)^N$$

At last, we are able to pass to the limit in the approximating problems thanks to the last convergences and the proof of the theorem is complete.

Conclusion and perspectives

Problem 1 addresses complex mathematical models with variable growth conditions, enabling the analysis of materials with mixed properties, such as variable elasticity or viscosity. By focusing on the existence, regularity, and uniqueness of solutions, this problem contributes to advancements in material science, engineering, and mathematical theory, offering insights into real-world applications.

- ❑ Problem 1 can be regarded as the stationary problem associated with the parabolic equation

$$\partial_t u - \operatorname{div} \left(A(x) |Du|^{p(x)-2} Du \right) - \operatorname{div} \left(F(u) |Du|^{q(x)-2} Du \right) = f$$

This is an open problem worth studying.

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Thank you for your attention
Questions are welcome



PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA
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In the name of Allah, the Most Gracious, the Most Merciful,

We are pleased to extend our heartfelt gratitude and appreciation to the invited speakers who have honored us with their valuable participation in this scientific event. Your presence and contributions have added a distinctive dimension to our activities, enriching knowledge and fostering the exchange of ideas in our areas of expertise.

We firmly believe that science progresses through constructive dialogue and active participation, and your efforts have served as a cornerstone for further collaboration and innovation. For this, we are deeply grateful for the insights and expertise you have shared, which greatly enrich this book and make it a valuable reference for researchers and enthusiasts alike.

In conclusion, we hope that this participation has reflected the spirit of academic collaboration that unites us, and we look forward to achieving more shared accomplishments in the future.

WIMAM 24
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Invited Speakers



Mathematical modeling of Dynamical Systems Epidemiology

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Abstract. In this study, Mathematical modeling of dynamic systems includes the study of equilibrium points within dynamical systems and evaluating the stability of these points to understand the system's long-term behaviour. In modeling the dynamics of B-cell chronic lymphocytic leukaemia (B-CLL) with interactions from Natural Killer (NK) cells, Cytotoxic T cells (such as CD8+ T cells), and chemotherapy, we construct a system of differential equations to represent the interactions and influences each component has on the others. Parameter u is the concentration of chemotherapy. A quantitative understanding of the kinetics of the immune response to treatment is crucial in designing treatment strategies, such as dosing, timing, and predicting the response to a specific treatment

Keywords: Mathematical modeling, Dynamic systems, Equilibrium points, Differential equations.

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Exact values of the constants in (q_1, q_2) -generalized triangle inequality for (q_1, q_2) -quasimetric spaces

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Abstract. In our talk we discuss the problem to find exact values of the constant q_1, q_2 for the most important cases of (q_1, q_2) -quasimetric spaces that are of Carnot groups equipped with Box-quasimetrics.

Keywords: (q_1, q_2) -quasimetric, Carnot group, Box-quasimetric

Introduction

Consider a set X consisting of at least two points. Function $\rho_X : X \times X \rightarrow \mathbb{R}^+$, $\rho_X(x, y) = 0 \Leftrightarrow x = y$, is called (q_1, q_2) -quasimetric if the following (q_1, q_2) -generalized triangle inequality takes place:

$$\rho_X(x, z) \leq q_1 \rho_X(x, y) + q_2 \rho_X(y, z) \quad \forall x, y, z \in X,$$

¹The work is supported by the Project FWNF-2022-0006



where q_1, q_2 are some positive numbers. Pair (X, ρ_X) is called (q_1, q_2) -quasimetric space [1]–[10]. The expression $\rho_X(x, y)$ denotes a (q_1, q_2) -quasi-distance exactly from the point x to the point y . If $q_1 = q_2 = 1$, then (X, ρ_X) is a quasimetric space [11].

If for a (q_1, q_2) -quasimetric ρ_X the following condition holds

$$\rho_X(x, y) \leq q_0 \rho_X(y, x) \quad \forall x, y \in X$$

for some $q_0 > 0$ then we refer to a (q_1, q_2) -quasimetric space (X, ρ_X) as a q_0 -symmetric one; for the case when $q_0 = 1$, we use the notion of symmetric (q_1, q_2) -quasimetric space. The (q_2, q_1) -quasimetric $\bar{\rho}_X(x, y) = \rho_X(y, x)$ is said to be conjugate to $\rho_X(x, y)$. If ρ_X is symmetric then $\bar{\rho}_X$ is symmetric too; also every symmetric (q_1, q_2) -quasimetric is (q_2, q_1) -quasimetric.

For a (q_1, q_2) -quasimetric space (X, ρ_X) we denote by $Q = Q(\rho_X)$ the set of points $(q'_1, q'_2) \in \mathbb{R}^2$, such that for ρ_X (q'_1, q'_2) -generalized triangle inequality holds. Directly from the definition of the set Q follows the

Proposition 0.1 ([1, 2]). ^{1°} The set $Q = Q(\rho_X)$ is convex and closed, and, moreover,

$$Q \subseteq \{(x, y) \in \mathbb{R}^2 \mid x \geq 1, y \geq 1\} = K,$$

^{2°} the condition $(1, 1) \in Q$ is equivalent to the fact that ρ_X is a quasimetric; ^{3°} if (q_1, q_2) -quasimetric is symmetric, then the set Q is symmetric with respect to the bisector of the right upper coordinate angle of the Euclidean plane.

If $(q'_1, q'_2) \in Q$ and $\tilde{q}_i \geq q'_i$, $i = 1, 2$, then $(\tilde{q}_1, \tilde{q}_2) \in Q$. Drawing supporting lines through the boundary points of the closed convex set Q we obtain that Q has extreme points. (Recall that a point $x_0 \in A$ is called an *extreme point* of a set A , if there are no points $x_1, x_2 \in A$, such that $x_0 \in (x_1, x_2)$, that is, $x_0 = tx_1 + (1 - t)x_2$ for some $0 < t < 1$.) It is easy to see that every extreme point of the set Q is a Pareto optimal point of the set Q (in the sense of minimisation of components). Extreme point $(q_1^0, q_2^0) \in Q$ is called *best*, if for every $(q'_1, q'_2) \in Q$ we have that $q'_i \geq q_i^0$, $i = 1, 2$. From the definition of the best point, directly follows the next

Proposition 0.2 ([1, 2]). If the best point exists, then it is unique; if the best point exists, then $Q(\rho_X) = \{(q_1, q_2) \in \mathbb{R}^2 \mid q_1 \geq a, \quad q_2 \geq b\}$ for some positive a, b .

See the examples of (q_1, q_2) -quasimetric spaces with the best points (q_1^0, q_2^0) such that $q_1^0 + q_2^0 > 2$, in [1], [5]–[7].

The best point of the (q_1, q_2) -quasimetric is unique; note that there are (q_1, q_2) -quasimetrics that do not have the best points [1]. We will call the set $Q(\rho_X)$ the domain of admissible parameters of (q_1, q_2) -quasimetric ρ_X . So the boundary $\partial Q(\rho_X)$ coincides with the set of all extreme points of (q_1, q_2) -quasimetric ρ_X .

Note that b -metric spaces, introduced by Bakhtin in 1989 (see [12]) are a particular case of (q_1, q_2) -quasimetric spaces. A nontrivial example of a (q_1, q_2) -quasimetric space, is given by the spaces $L_p(E)$, where E is a measurable



bounded set in $\mathbb{R}^n$, for $0 < p < 1$. These spaces are quasinormed since we have $\|f_1 + f_2\|_p \leq 2^{\frac{1}{p'}} (\|f_1\|_p + \|f_2\|_p)$ for all $f_1, f_2 \in L_p(E)$, where $p' < 0$ is the number conjugate to p , i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. Moreover, for every $\epsilon > 0$, we have

$$\|f_1 + f_2\|_p \leq (1 + \epsilon)\|f_1\|_p + C(\epsilon, p)\|f_2\|_p, \quad \forall f_1, f_2 \in L_p(E),$$

where $C(\epsilon, p) = (1 - (1 + \epsilon)^{\frac{1}{p'}})^{\frac{1}{p'}}$. Thus, $L_p(E)$ with $0 < p < 1$ is a symmetric (q_1, q_2) -quasimetric space whose (q_1, q_2) -quasimetric $\rho_{L_p(E)}$ is given by the formula

$$\rho_{L_p(E)}(f_1, f_2) = \|f_1 - f_2\|_{L_p(E)}.$$

Moreover, for every $\epsilon > 0$ the (q_1, q_2) -generalized triangle inequality holds in $L_p(E)$ provided that $q_1 = 1 + \epsilon$, $q_2 = C(\epsilon, p)$. (q_1, q_2) -quasimetric $\rho_{L_p(E)}$ has no the best point.

An important special case of symmetric (q_1, q_2) -quasimetric spaces is the class of the symmetric $(1, q_2)$ -quasimetric spaces [1]; these include Carnot groups $\mathbb{G}$ and more general equiregular Carnot-Carathéodory spaces (M, ρ_{Box_M}) , equipped by Box-quasimetrics ρ_{Box_M} [7]–[16]. Moreover, in the general case, the constant q_2 does not equal 1 [17]. Box-quasimetrics were introduced in [18]. $(1, q_2)$ -generalized triangle inequality plays a crucial role in obtaining the divergence estimates of the equiregular Carnot-Carathéodory space (M, ρ_{Box_M}) from its nilpotent tangential cone, see, for example, [19, 20].

The values of the extreme and the best points are important in obtaining sharp estimates in the theorems of the coincidence points of the covering and Lipschitz mappings in (q_1, q_2) -quasimetric spaces [1, 2, 3]. So the the problem of determining the set $\partial Q(\rho_X)$ for a given (q_1, q_2) -quasimetric ρ_X is relevant.

Below we present the results related to determining the domains $Q(\rho_{\text{Box}_\mathbb{G}})$ for some Carnot groups $\mathbb{G}$.

Carnot groups and Box-quasimetrics

A Lie algebra is called graduated [24], if it decomposes into a direct sum of vector subspaces $V = \bigoplus_{i=1}^r V_i$, and, moreover, $[V_i, V_k] \subset V_{i+k}$, if $i + k \leq r$, and $[V_i, V_k] = 0$, if $i + k > r$. Note that a graduated algebra is always nilpotent of degree r . r -step stratified Lie algebra V [25] is a Lie algebra nilpotent of degree r , that has a stratification, that is,

$$V = \bigoplus_{i=1}^r V_i, \quad V_{i+1} = [V_1, V_i], \quad [V_1, V_r] = \{0\}.$$

An r -step Carnot algebra [25] is a graduated Lie algebra V , which has a stratification; a simply connected Lie group $\mathbb{G}$, corresponding to an r -step Carnot algebra V , is called an r -step Carnot group. Let $N = \sum_{i=1}^r n_i$, $n_i = \dim V_i$, and the basis of the left-invariant vector fields $\{X_1, \dots, X_N\}$ of the Carnot group $\mathbb{G}$



is ordered such that the values of the first n_1 of them form at every point $u \in \mathbb{G}$ the basis of the subspace $V_1(u)$, the values of the next n_2 of them form at every point $u \in \mathbb{G}$ the basis of the subspace $V_2(u)$, and so on. We assign to every vector field X_k a natural number $j = \deg X_k$, defined by the inclusion $X_k \in V_j$. A Box-quasimetric $\rho_{\text{Box}\mathbb{G}}$ is defined as

$$\rho_{\text{Box}\mathbb{G}}(u, w) = \max\{|a_i|^{\frac{1}{\deg X_i}} \mid i = 1, \dots, N\}, \quad w = \exp\left(\sum_{i=1}^N a_i X_i\right)(u) \quad \forall u, w \in \mathbb{G}, \quad (0.1)$$

where $\exp$ is standard exponential mapping that is identical on canonical Lie group. The definition implies that $\rho_{\text{Box}\mathbb{G}}$ satisfies the identity and symmetry axioms.

Further in our consideration we will use only the notion of canonical Carnot group. (See the definition of canonical finite-dimensional Lie group in [22], [21].)

Exact values of the constant q_2 for some 2-step Carnot groups

Further we will use the term *exact value* that implies such value of the constant q_2 that for every number q' , $q' < q_2$, the $(1, q_2)$ -generalized triangle inequality does not hold.

A canonical 2-step Carnot group $\mathbb{D}_n$ with a one-dimensional centre is defined in the standard Euclidean space $\mathbb{R}^{n+1}$ with the coordinate system $(x_1, \dots, x_n, t)$ and the coordinate frame $(O, e_1, \dots, e_n, e_{n+1})$ with the help of the following commutator

$$[e_i, e_j] = \alpha_{ij} e_{n+1}, \quad \sum_{i,j=1}^n \alpha_{ij}^2 \neq 0,$$

the rest of possible commutators $e_1, \dots, e_{n+1}$ are equal 0; if we put $n = 2m$, $m \in \mathbb{N}$, $\sum_{i=1}^{m-1} \alpha_{2i, 2i+1}^2 = 0$ and $\alpha_{2j-1, 2j} = \alpha \neq 0$, $j = 1, \dots, m$, then we obtain a commutator table that defines the canonical Heisenberg group $\mathbb{H}_\alpha^m$ [17].

Let $C(n) = \{(x_1, \dots, x_n) \mid \max_{i=1, \dots, n} |x_i| \leq 1\}$. Denote

$$M_{\mathbb{D}_n} = \sup_{(x'_1, \dots, x'_n), (x''_1, \dots, x''_n) \in C(n)} \left| \sum_{i,j=1, \dots, n, i < j} \frac{\alpha_{ij}}{2} (x'_i x''_j - x'_j x''_i) \right|,$$

where $(x'_1, \dots, x'_n), (x''_1, \dots, x''_n)$ are the vertices of the cube $C(n)$.

Theorem 0.1 ([21]). *Exact value of constant q_2 in the $(1, q_2)$ -generalized triangle inequality for canonical Carnot group $\mathbb{D}_n$ is defined by the formula*

$$q_2 = \begin{cases} 1, & M_{\mathbb{D}_n} \leq 2, \\ \frac{M_{\mathbb{D}_n}}{2}, & M_{\mathbb{D}_n} > 2. \end{cases}$$



Note that $M_{\mathbb{D}_3} = \{|a_{12} \pm a_{23}|, |a_{31} \pm a_{12}|, |a_{23} \pm a_{31}|\}$.

Corollary 0.1. *Consider the 2-step Carnot group G_1 , which is defined in $\mathbb{R}^6$ by the following commutator table:*

$$\begin{cases} [e_1, e_2] = a_{12}e_4, \\ [e_2, e_3] = a_{23}e_5, \\ [e_3, e_1] = a_{31}e_6. \end{cases}$$

Exact value of constant q_2 in the $(1, q_2)$ -generalized triangle inequality for canonical Carnot group G_1 is defined by the formula

$$q_2 = \begin{cases} 1, & a \leq 2, \\ \frac{a}{2}, & a > 2, \end{cases}$$

where $a = \max\{|a_{12}|, |a_{23}|, |a_{31}|\}$.

Corollary 0.2. *Consider the 2-step Carnot group G_2 , which is defined in $\mathbb{R}^5$ by the following commutator table:*

$$\begin{cases} [e_1, e_2] = a_{12}e_4, \\ [e_2, e_3] = a_{23}e_4, \\ [e_3, e_1] = a_{31}e_5. \end{cases}$$

Exact value of constant q_2 in the $(1, q_2)$ -generalized triangle inequality for canonical Carnot group G_1 is defined by the formula

$$q_2 = \begin{cases} 1, & b \leq 2, \\ \frac{a}{2}, & b > 2, \end{cases}$$

where $b = \max\{|a_{31}|, |a_{12} \pm a_{23}|\}$.

From all the $\mathbb{D}_n$ we select the groups D_n for which we have $\alpha_{ij} = 1 \ \forall i < j$.

Theorem 0.2 ([26]). *Exact value of constant in the $(1, q_2)$ -generalized triangle inequality for canonical Carnot group D_n is defined by the formula $q_2 = \frac{M_n}{2}$, where*

$$M_n = \begin{cases} 2k^2, & n = 2k, \\ 2k^2 + 2k, & n = 2k + 1. \end{cases}$$

Theorem 0.3 ([17]). *Exact value of constant q_2 in the $(1, q_2)$ -generalized triangle inequality for canonical Heisenberg group $\mathbb{H}_\alpha^n$ is defined by the formula*

$$q_2 = \begin{cases} 1, & \alpha \leq \frac{2}{n}, \\ \frac{\alpha n}{2}, & \alpha > \frac{2}{n}. \end{cases}$$



The domain of admissible parameters for the first Heisenberg group and Engel group

The first Heisenberg group H_α^1 is defined in the standard Euclidean space $\mathbb{R}^3$ with the coordinate system (x, y, t) induced by the coordinate frame (O, e_1, e_2, e_3) with the use of the table of commutators $[e_1, e_2] = \alpha e_3$, $\alpha > 0$, $[e_1, e_3] = [e_2, e_3] = 0$.

Theorem 0.4. *The set $\partial Q(\mathbb{H}_\alpha^1, \rho_{\text{Box}_{\mathbb{H}_\alpha^1}})$ is: $1^0 \gamma \cup p_1 \cup p_2$, where γ is the arc of the parabola $4q_1^2 + 4q_2^2 - 4q_1q_2\alpha + \alpha^2 - 4 = 0$ restricted by the points $M_{q_1} = (1, \frac{\alpha}{2})$, $M_{q_2} = (\frac{\alpha}{2}, 1)$, $p_1 = \{(1, q_2) \mid q_2 \geq \frac{\alpha}{2}\}$, $p_2 = \{(q_1, 1) \mid q_1 \geq \frac{\alpha}{2}\}$, in the case $\alpha > 2$, $2^0 \partial K$ in the case $\alpha \leq 2$.*

Engel group $\mathbb{E}_{\alpha, \beta}$ is defined in the standard Euclidean space $\mathbb{R}^4$ with the coordinate system (x, y, t, z) induced by the coordinate frame (O, e_1, e_2, e_3, e_4) with the use of the table of commutators $[e_1, e_2] = \alpha e_3$, $\alpha > 0$, $[e_1, e_3] = \beta e_4$, $\beta > 0$; the rest of possible commutators $e_1, \dots, e_4$ are equal 0.

Define the set $Q_{\alpha, \beta}$ as

$$\{(q_1, q_2) \in K \mid 1 + (\frac{\beta}{2} + \frac{\alpha\beta}{6})\varepsilon + (\frac{\beta}{2} + \frac{\alpha\beta}{6})\varepsilon^2 + \varepsilon^3 \leq (q_1 + \varepsilon q_2)^3, \quad \varepsilon \geq 0\}.$$

Theorem 0.5.

$$Q(\mathbb{E}_{\alpha, \beta}, \rho_{\text{Box}_{\mathbb{E}_{\alpha, \beta}}}) = K \cap Q(\mathbb{H}_\alpha^1, \rho_{\text{Box}_{\mathbb{H}_\alpha^1}}) \cap Q_{\alpha, \beta}.$$

Note that in the general case the set $\partial Q_{\alpha, \beta}$ consist of the points (q_1, q_2) such that

$$(4(a - 3q_1q_2^2)^3 - 3D(a - 3q_1q_2^2) + 108(q_1^3 - 1)(q_2^3 - 1)^2)^2 = D^3,$$

where

$$D = 4(3q_1q_2^2 - a)^2 - 12(q_2^3 - 1)(3q_1^2q_2 - a), \quad a = \frac{\beta}{2} + \frac{\alpha\beta}{6},$$

and the points $(1, t)$, $(t, 1)$, $t \geq \frac{a}{3}$.

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Numerical methods for fractional PDEs: A State of the Art

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Abstract. Partial differential equations (PDE)s play an important role in the simulation of systems in applied sciences and technology. A lot of numerical methods have been developed for classes of PDEs arising in biology and medicine. Additionnally to the classical numerical techniques, other computational approaches also exist. Fractional calculus (FC) has been playing a crucial role in various fields of applied sciences and technology. It has some potential to change the model approach and control the results in order to tune with the realistic data. Beside its impact on problems with anomalous behavior, it was observed that FC can provide efficient numerical models having better consistencies with known data. The objective of the talk is to give a brief state of the art on existing numerical methods using a simple model problem for a PDE with integer and/or fractional orders. A swift review of some existing related works is also provided.

Keywords: Fractional calculus, Fractional PDEs, Finite difference, Finite element, Finite volume.



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Stability results for a class of proportional fractional stochastic differential equations

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Abstract. This work explores fractional stochastic differential equations, emphasizing their existence, uniqueness, and Hyers-Ulam Stability. By using the Banach Fixed Point Theorem, Itô's isometry formula, and some classical inequalities, we obtain the theoretical findings. Finally, we present an example to illustrate the effectiveness of the results.

Keywords: Ulam-Hyers stability, proportional fractional derivative, stochastic differential equation.

Mathematics Subject Classification: 60H20.



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Fractional Calculus: From Abstract Mathematics to Practical Engineering

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Abstract. Fractional calculus, a field once confined to theoretical mathematics, has emerged as a powerful tool in modern engineering applications. By extending the concept of integer-order derivatives to arbitrary orders, fractional calculus provides a nuanced framework for modeling systems with memory and hereditary properties. This presentation bridges the gap between abstract mathematical theory and its practical engineering implications. Key topics include the foundational principles of fractional-order systems, their stability and control, and their application in diverse domains such as signal processing, robotics, and energy systems. Drawing on insights from recent works, including theoretical advancements and real-world implementations, we will explore how fractional calculus enhances system modeling accuracy and robustness. This comprehensive overview aims to inspire innovative applications, fostering interdisciplinary collaboration in this dynamic field.

Keywords: Fractional calculus, engineering applications, stability, control systems, system modeling.

Mathematics Subject Classification: 60H20.



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Traitement numérique du Spectre généralisé

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Abstract. Dans ce travail, nous présentons une méthode novatrice pour l'approximation spectrale généralisée de deux opérateurs bornés, basée sur l'introduction d'un nouveau mode de convergence. Cette approche étend la notion classique de ν -convergence, en proposant un cadre plus général avec des hypothèses assouplies par rapport aux méthodes existantes. En plus de sa robustesse théorique, cette nouvelle méthodologie garantit la propriété U, permettant ainsi de résoudre efficacement le problème de pollution spectrale dans l'approximation spectrale des opérateurs non bornés. Ces résultats élargissent les perspectives de l'analyse spectrale et ouvrent la voie à des applications dans des contextes où les approches traditionnelles échouent.

Keywords: Théorie des opérateurs, Approximation spectrale, Pollution spectrale, ν -convergence



Introduction

L'approximation spectrale est un outil crucial dans l'analyse des opérateurs linéaires, qu'ils soient bornés ou non bornés, avec des applications allant des équations différentielles aux modèles quantiques. Cependant, cette approximation est souvent entravée par des défis théoriques et numériques, notamment la pollution spectrale, qui survient lorsque des valeurs spectrales parasites apparaissent en raison d'une mauvaise convergence des méthodes utilisées.

Le problème étudié dans cet article concerne l'approximation spectrale généralisée de deux opérateurs bornés, dans le but de fournir une approche robuste qui surmonte les limitations des méthodes classiques, en particulier dans des situations où les opérateurs sont définis sur des espaces de Hilbert infinis. Nous cherchons à répondre à la question suivante : comment étendre la notion de ν -convergence pour traiter efficacement les cas complexes tout en garantissant la résolution du problème de pollution spectrale ?

Pour répondre à cette question, nous introduisons un nouveau mode de convergence, basé sur des hypothèses plus faibles que celles nécessaires dans les travaux antérieurs.

- (P1) $\|G_n\|$ et $\|S_n\|$ sont bornés indépendamment de n ,
pour tout $n \in \mathbb{N}$ $\|G_n\| < \infty$ et $\|S_n\| < \infty$.
- (P2) $\lim_{n \rightarrow \infty} \|(G_n - G)G_n\| = \lim_{n \rightarrow \infty} \|(G_n - G)S_n\| = \lim_{n \rightarrow \infty} \|(S_n - S)S_n\| = \lim_{n \rightarrow \infty} \|(S_n - S)G_n\| = 0$.
- (P3) $\lim_{n \rightarrow \infty} \|(G_n - G)G\| = \lim_{n \rightarrow \infty} \|(G_n - G)S\| = \lim_{n \rightarrow \infty} \|(S_n - S)S\| = \lim_{n \rightarrow \infty} \|(S_n - S)G\| = 0$.

Contrairement aux méthodes traditionnelles, notre approche n'exige pas que les opérateurs étudiés satisfassent des contraintes rigides telles que la normalité ou certaines conditions spectrales spécifiques. Cette flexibilité rend notre méthode applicable à une classe plus large d'opérateurs.

Nous montrons également que cette approche garantit la propriété U, une caractéristique essentielle qui permet de filtrer les valeurs spectrales indésirables, résolvant ainsi efficacement le problème de pollution spectrale. Ce résultat est fondé sur une analyse théorique rigoureuse, démontrant que notre méthode est à la fois généralisante et fiable.

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Approximation solution for Noncoercive Parabolic Variational inequality using multigrid methods

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Abstract. In this article, our work is focused around proof of the uniform convergence of the Multigrid method for parabolic variational inequalities with an noncoercive operator and its numerical solution, we use two numerical methods to study the uniform convergence of the Multigrid method for parabolic variational inequalities with a noncoercive operator. To discretize the problem, we utilize a finite element method for the operator and Euler scheme for the time. To obtain the system discretization of the problem, we reformulate the parabolic variational inequality as a Hamilton-Jacobi-Bellman equation. On the smooth grid, we apply the multigrid method as an interior iteration on the linear system.

Keywords: Multigrid method, Parabolic Variational Inequality, non coersive operator, Hamilton-Jacobi-Bellman equation .



Introduction

In the beginning, we will present our problem. Let $\mathbb{L}$ be an operator non elliptic, and Ω is an open domain in $\mathbb{R}^d$, where $d \geq 1$, with sufficiently regular bounds Γ . We introduce the definition of our proposed problem: find $\mathbf{u} \in L^2(H_0^1(\Omega); [0, T])$ solution of

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} + \mathbb{L}\mathbf{u} \leq f & ; \quad \mathbf{u} - \varphi \leq 0, \\ \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbb{L}\mathbf{u} - f \right) (\mathbf{u} - \varphi) = 0 & \text{in } \mathbf{Q}_t, \\ \mathbf{u}(x, t) = \mathbf{u}_0 & \text{in } \Omega \\ \mathbf{u}(x, t) = 0 & \text{in } \Sigma. \end{cases} \quad (0.2)$$

where $\mathbf{Q}_T$ defined as: $\mathbf{Q}_T = \Omega \times [0, T]$, $\Sigma = \Gamma \times [0, T]$, $T < \infty$, and the obstacle $\varphi \in W^{2,\infty}$. such that $\varphi \geq 0$.

We have organized this work as follows:

We formulate our problem and provide the necessary and sufficient conditions for studying the uniform convergence of Multigrid methods. In §2, we introduce important symbols and hypotheses that will be used in our research. Subsequently, we present the theorem that proves the existence and uniqueness of the solution for the continuous problem.

In §3, we define the space $\mathbb{V}_h$ that

$$\mathbb{V}_h = \{\mathbf{v}_h \in L^2(H_0^1(\Omega); [0, T]) \cap C^1(H_0^1(\overline{\Omega}); [0, T]), \mathbf{v}_h(., t) \in P_1\}.$$

with P_1 is a Lagrange polynomial with a degree ≤ 1 .

To discretize, we employ the Euler scheme for time discretization and use the finite element method on the operator $\mathbb{L}$, resulting in the following discrete problem. Find $\mathbf{u}_h^n \in \mathbb{V}_h$ as a solution of:

$$\begin{cases} \langle \mathbb{C}_h \mathbf{u}_h^n, \mathbf{v}_h - \mathbf{u}_h^n \rangle \geq \langle \mathbb{F}^n, \mathbf{v}_h - \mathbf{u}_h^n \rangle, \\ \mathbf{u}_h^n \leq r_h \varphi, \\ \mathbf{u}_h(x) = \mathbf{u}_h(x, 0) & \text{in } \Omega \\ \mathbf{u}(x, t) = 0. & \text{in } \Sigma \end{cases} \quad (0.3)$$

In §4, to solve the Parabolic Variational Inequality (PVI) using Multigrid methods, we discretize the domain through the finite element method and employ the Euler scheme for time discretization. Subsequently, we reformulate the PVI as the Hamilton-Jacobi-Bellman (HJB) equation, expressed as follows:

$$\max_{1 \leq i \leq N} (\mathbb{C}_{k,i} \mathbf{u}_{k,i}^v - \mathbb{F}_{k,i}^v; \mathbf{u}_{k,i}^v - r_h \varphi) = 0$$



We employ the iterative procedure introduced by Hoppe (see [12][13][?]) to solve the obtained system. Subsequently, Following this, we employ the multi-grid method on the linear system in all iterations.

In §5 ,to establish the uniform convergence of the Multigrid method, we rely on the approximation and smoothing suggestions proposed by Huckbach (see[10][11]). In this research, we demonstrate the uniform convergence in the addressed problem based on the techniques presented in Huckbach's work.

Here, we give a numerical example that confirms the results obtained in the theoretical part.

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Analytical and numerical study of new integro-differential Fredholm equation

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Abstract. In this work we give an analytical and numerical study of the nonlinear integro-differential Fredholm-Chandrasekhar equation. We present the conditions ensuring the existence and uniqueness of the solution. Then we construct a numerical approach based on the well known Nyström method. The numerical examples illustrate the efficiency of the proposed method in connection with the existence and uniqueness conditions.

Keywords: Fredholm equation, Nyström method, Nonlinear equation.



Introduction

Throughout many fields of science, the mathematical models for diverse phenomena are generally constructed with integro-differential equations. Many studies have been carried out in this area, for example [1-3]. In recent paper [4], the authors investigated the following equation:

$$\varphi(t) = f(t) + \int_a^b K(t, s, \varphi(s), \varphi'(s))ds, \quad a, b \in \mathbb{R}.$$

which is a nonlinear integro-differential equation of Fredholm type of the second kind.

Our present work concerns a general form of the previous equation which is given by:

$$\begin{cases} u(t) = K_1(t, u(t)) \int_a^b K_2(s, u(s), u'(s))ds + f(t), & \forall t \in [a, b], \\ |u'(t)| \leq d, & \forall t \in [a, b], \quad \text{where } d \in \mathbb{R}_+^*. \end{cases} \quad (0.4)$$

This equation is a nonlinear integro-differential Fredholm-Chandrasekhar equation.

Main results

In this work, we investigate the equation (0.4) in analytical sense and in numerical sense. We prove that the existence and uniqueness of the solution by using ideas based on the Banach fixed point theorem [5-6]. Thereafter, In numerical sense we use the Nyström method [7-8] to approximate the exact solution. Finally, we present a numerical application to show the accuracy of this method.

Theorem 0.6. *If*

$$\begin{cases} (M_2 \max(A_2, B_2) + A_2 M_2) \leq \gamma < \frac{1}{2}, \\ (M_3 \max(A_2, B_2) + M_2 \max(A_3, B_3)) \leq \gamma < \frac{1}{2}, \end{cases} \quad (0.5)$$

then the equation (0.4) has a unique solution in $\varphi \in C^1([0, 1], \mathbb{R})$, such that

$$\|\varphi_{n+1} - \varphi_n\|_{C^1([0,1],\mathbb{R})} \leq (2\gamma)^{n+1}C,$$

where $C = (M_1 + M_3)M_2 + 2\|f\|_{C^1([0,1],\mathbb{R})}$ and that

$$\|\varphi_n - \varphi\|_{C^1([0,1],\mathbb{R})} \rightarrow 0, \quad n \rightarrow +\infty.$$

Now, according to Eqs. (1.1) and (1.2), we can write:

$$\varphi(t_i) = K_1(t_i, \varphi(t_i)) \int_0^1 K_2(s, \varphi(s), \varphi'(s))ds + f(t_i), \quad (0.6)$$



$$\varphi'(t_i) = H(t_i, \varphi(t_i), \varphi'(t_i)) \int_0^1 K_2(s, \varphi(s), \varphi'(s)) ds + f'(t_i). \quad (0.7)$$

Then, using the quadrature formula on the previous equations, we obtain for $i = 0, \dots, n$:

$$\varphi_i = f_i + K_1(t_i, \varphi_i) \sum_{j=0}^n w_j K_2(t_j, \varphi_j, \varphi'_j) + K_1(t_i, \varphi_i) R(h, i), \quad (0.8)$$

$$\varphi'_i = f'_i + H(t_i, \varphi_i, \varphi'_i) \sum_{j=0}^n w_j K_2(t_j, \varphi_j, \varphi'_j) + H(t_i, \varphi_i, \varphi'_i) R(h, i), \quad (0.9)$$

where $\varphi(t_i) = \varphi_i$, $\varphi'(t_i) = \varphi'_i$, $f(t_i) = f_i$ and $f'(t_i) = f'_i$ and $R(h, i)$ is the residual which depends on the quadrature rule.

We recall that the used of quadrature rule is said to be consistent if,

$$\lim_{n \rightarrow +\infty} \max_{0 \leq i \leq n} R(h, i) = 0$$

Now, by neglecting the local consistency error $R(h, i)$, we obtain the following algebraic system:

$$\begin{cases} \alpha_i = f_i + K_1(t_i, \alpha_i) \sum_{j=0}^n w_j K_2(t_j, \alpha_j, \beta_j) \\ \beta_i = f'_i + H(t_i, \alpha_i, \beta_i) \sum_{j=1}^n w_j K_2(t_j, \alpha_j, \beta_j), \end{cases} \quad (0.10)$$

where $i = 0, \dots, n$. The next theorem proves that the system (0.10) has a unique solution $(\alpha, \beta) \in \mathbb{R}^{2n+2}$

Theorem 0.7. *The system (0.10) has a unique vector solution.*

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**parabolic p -biharmonic equation with variable exponent:
existence results and full discretization**

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Abstract. A high-order parabolic $p(\cdot)$ -Bilaplace equation with variable exponent is studied. The solvability of the proposed problem as well as some regularity results are shown using Roth-Galerkin method.

Keywords: parabolic $p(\cdot)$ – Bilaplace equation, Galerkin method, Time discretization, Weak solution, A priori estimates.



Introduction

Let Ω be a simply connected bounded domain of $\mathbf{R}^d$, ($d = 1, 2, 3$) with Lipschitz continuous boundary $\partial\Omega$. Consider the following non local hyperbolic problem

$$\frac{\partial u}{\partial t} + \Delta \left(|\Delta u|^{p(x)-2} \Delta u \right) = f \quad \text{in } I \times \Omega \quad (0.11)$$

$$u = 0, \quad \nabla u = 0 \quad \text{on } \Sigma = I \times \partial\Omega \quad (0.12)$$

$$u(0, x) = u_0, \quad \text{on } \Omega \quad (0.13)$$

where f is continuous function satisfies

$$|f(\gamma, x)| \leq g(x) + c|\gamma|^{r-1}$$

Semi discretized problem

Let us divide the interval $I = [0, T]$ to n subintervals where $\tau = \frac{T}{n}$, $t_i = i\tau$, $u^i(x) = u(t_i, x)$ and $\delta u^i(x) = \frac{u^i(x) - u^{i-1}(x)}{\tau}$. Then recurrent approximation scheme for $i = 1, \dots, n$ is

$$(u^i - u^{i-1}, v) + \tau \left(\left(|\Delta u^i|^{p(x)-2} \Delta u^i \right), \Delta v \right) = \tau (f^i, v), \quad \forall v \in W_0^{2,p(\cdot)}(\Omega) \quad (0.14)$$

Let us show that problem(3.1) admits a weak solution at each time step t_i .

Theorem 0.8. *Let $f^i \in L^{q(\cdot)}(\Omega)$, then for $i = 1, \dots, n$ problem (3.1) admits a unique weak solution u^i in $W_0^{2,p(\cdot)}(\Omega)$*

Galerkin-Finit element discretization and existence results

We consider a triangulation Υ_h made of triangles T whose edges are denoted by e . We assume that the intersection of two different elements is either empty, a vertex, or a whole edge e and we assume also that this triangulation is regular in Ciarlet sens i. e.

$$\exists \sigma > 0; \frac{h_T}{\rho_T} \leq \sigma, \quad \forall T \in \Upsilon_h$$

where h_T is the diameter of T and ρ_T is the diameter of its largest inscribed ball. We define $h = \max_{T \in \Upsilon_h} h_T$.

For $h > 0$, we introduce the following spaces

$$X^h = \{ \phi \in C^0(\overline{\Omega}); \forall T \in \Upsilon_h; \phi|_T \in P^k(T), \phi|_{\partial\Omega \cap \Upsilon_h} = 0 \text{ and } \nabla \phi|_{\partial\Omega \cap \Upsilon_h} = 0 \} \quad (0.15)$$



Let $u_h^0 = \Pi_{L^2} u^0$. For $i = 1, \dots, n$, the discrete formulation of the regularized problem is to seek solution $u_h^i \in X^h$ such that

$$(\delta u_h^i, v) + ((k_\epsilon (|\Delta u_h^i|) \Delta u_h^i), \Delta v) = (f_h^i, v), \quad \forall v \in X^h \quad (0.16)$$

where

$$k_\epsilon(r) = (r^2 + \epsilon)^{\frac{p(x)-2}{2}}, \text{ for some } \epsilon \succ 0$$

Let us define Rothe function by the piecewise linear interpolation with respect to the time t

$$u_h^n(t) \equiv u_{\tau, h, \epsilon} =: u_h^{i-1} + (t - t_{i-1}) \delta u_h^i, \quad i = 1, \dots, n$$

together with step functions

$$\overline{u_h^n(t)} = u_h^i, \quad i = 1, \dots, n$$

and regularized energy functional

$$J_\epsilon(u^i) = \int_{\Omega} \left(\frac{1}{2} |u_h^i|^2 + \frac{\tau}{p(x)} \left[|\Delta u_h^i|^2 + \epsilon \right]^{\frac{p(x)}{2}} + \int_0^{u_h^i} f(s, x) ds \right) dx \quad (0.17)$$

Now, we are able to announce our main result

Theorem 0.9. *Suppose that $p^- \geq 2$ and $1 \prec r \prec p^-$, then problem (1.1)–(1.3) admits a weak solution u in the sense of the definition 2.5.*

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Efficient Numerical Method for Solving Linear Fredholm Integro-Differential Equations Using Backward Finite Difference and Nyström Techniques

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Abstract. This paper introduces a novel numerical method for solving linear Fredholm integro-differential equations. By combining the backward finite difference technique with the Nyström method, we reduce the system size by 50 % compared to the approach proposed by Tair et al. [2]. This reduction significantly improves computational efficiency, especially for large integration intervals. To ensure convergence, we develop a new norm for the space $\mathbb{R}^{N+1}$. Numerical experiments highlight the method's advantages in both execution speed and accuracy. Our results provide a valuable contribution to solving integro-differential equations more efficiently, offering a powerful tool for a range of scientific and engineering applications.

Keywords: Fredholm integro-differential equation, Nyström method, backward finite difference

Introduction

The study of integro-differential equations plays a crucial role in applied mathematics and physics, as they can model a wide range of complex phenomena. Linear Fredholm integro-differential equations, in particular, are highly valuable



due to their applications in fields such as engineering, biological systems, and financial mathematics. Despite their importance, numerically solving these equations—especially over large integration intervals—poses significant challenges.

In this article, we address a specific linear Fredholm integro-differential equation of the form:

$$f(x) = \int_a^b \eta(x, y) f(y) dy + \int_a^b \mu(x, y) f'(y) dy + g(x), \quad \forall x \in [a, b],$$

where η , μ , and g are given real functions that satisfy $\frac{\partial \eta}{\partial x}, \frac{\partial \mu}{\partial x} \in C^0([a, b]^2)$, and $g \in C^1[a, b]$.

Previous analytical studies, such as those referenced in Tair et al.[2], have demonstrated the existence and uniqueness of the solution $f \in C^1[a, b]$ for this equation under the given assumptions. Additionally, they derived the form of its derivative $f'(x)$:

$$f'(x) = \int_a^b \frac{\partial \eta}{\partial x}(x, y) f(y) dy + \int_a^b \frac{\partial \mu}{\partial x}(x, y) f'(y) dy + g'(x).$$

In their approach, Tair et al. [2] applied the Nyström method [10, 11, 12] to solve the coupled system of equations (1) and (2). However, their technique produced a large linear system, making it inefficient for extensive integration intervals.

Our method presents a notable improvement by reducing the system size by 50% compared to the approach of Tair et al. [2]. This is accomplished by combining the finite difference derivative method [13] with the Nyström method [10, 11]. This reduction not only increases computational efficiency but also preserves the accuracy and stability of the solution. Numerical experiments highlight the superiority of our method, particularly in terms of reduced execution time, when compared to Tair et al.'s method.

Main results

Problem Position

We are focused on the numerical analysis of the linear Fredholm integro-differential equation given by:

$$\text{Find } f \in C^1[a, b] \text{ such that} \quad (0.18)$$

$$\forall x \in [a, b], \quad f(x) = \int_a^b \eta(x, y) f(y) dy + \int_a^b \mu(x, y) f'(y) dy + g(x), \quad (0.19)$$

where η , μ , and g satisfy the following conditions:



- (A1) $\frac{\partial \eta}{\partial x}, \frac{\partial \mu}{\partial x} \in C^0([a, b]^2),$
- (A2) $g \in C^1[a, b],$
- (A3)

$$\Omega := \max_{a \leq x, y \leq b} \left(|\eta(x, y)|, \left| \frac{\partial \eta}{\partial x}(x, y) \right| \right) + \max_{a \leq x, y \leq b} \left(|\mu(x, y)|, \left| \frac{\partial \mu}{\partial x}(x, y) \right| \right) < (2(b-a))^{-1}.$$

An extensive analytical study was conducted in [2], demonstrating that under the assumptions (A), the equation (1) has a unique solution $f \in C^1[a, b]$. The norms used for $C^1[a, b]$ and $C^0[a, b]$ are defined as follows:

$$\forall k \in C^0[a, b], \quad \|k\|_{C^0[a, b]} = \max_{a \leq x \leq b} |k(x)|,$$

$$\forall k \in C^1[a, b], \quad \|k\|_{C^1[a, b]} = \|k\|_{C^0[a, b]} + \|k'\|_{C^0[a, b]}.$$

Additionally, utilizing the assumptions (A1) through (A3), we can derive further information about f :

$$\forall x \in [a, b], \quad f'(x) = \int_a^b \frac{\partial \eta}{\partial x}(x, y) f(y) dy + \int_a^b \frac{\partial \mu}{\partial x}(x, y) f'(y) dy + g'(x). \quad (0.20)$$

In [2], Tair et al. introduced a novel Nyström method to address both equations (1) and (2) simultaneously. However, this approach resulted in a large linear system, complicating its application for extensive integration intervals.

The objective of our paper is to implement a finite difference derivative method in conjunction with the Nyström method to effectively reduce the size of the system.

Backward finite difference method

For $N \geq 2$, we introduce a subdivision of $[a, b]$ defined as follows:

$$h = \frac{b-a}{N-1}, \quad x_j = a + (j-1)h, \quad 1 \leq j \leq N.$$

We also define a numerical integration formula for all $\varphi \in C^0[a, b]$ given by

$$\int_a^b \varphi(x) dx \approx \sum_{j=1}^N \rho_j \varphi(x_j),$$

where the weights $\{\rho_j\}_{j=1}^N$ are required to satisfy:



$$\Xi := \sup_{N \geq 2} \sum_{j=1}^N |\rho_j| < \infty,$$

$$\forall \varphi \in C^0[a, b], \quad \lim_{N \rightarrow +\infty} \left| \int_a^b \varphi(x) dx - \sum_{j=1}^N \rho_j \varphi(x_j) \right| = 0. \quad (0.21)$$

Applying the Nyström method [10, 11, 12] with the above numerical integration formula to equation (1), we obtain, for all $1 \leq j \leq N$:

$$F_i = \sum_{j=1}^N \rho_j \eta(x_i, x_j) F_j + \sum_{j=1}^N \rho_j \mu(x_i, x_j) F'_j + g(x_i), \quad (0.22)$$

where $F_j \approx f(x_j)$ and $F'_j \approx f'(x_j)$. We now apply the backward finite difference method [13] to approximate F'_j for $2 \leq j \leq N$:

$$F'_j \approx h^{-1}(F_j - F_{j-1}). \quad (0.23)$$

Incorporating this approximation into system (3), we obtain, for all $1 \leq i \leq N$:

$$F_i = \sum_{j=1}^N \rho_j \eta(x_i, x_j) F_j + \sum_{j=2}^N \rho_j \mu(x_i, x_j) h^{-1}(F_j - F_{j-1}) + \rho_1 \mu(x_i, x_1) F'_1 + g(x_i). \quad (0.24)$$

At this point, we note the need to add an equation for F'_1 . To achieve this, we apply the Nyström method with the same numerical integration formula to equation (2), which yields:

$$F'_1 = \sum_{j=1}^N \rho_j \frac{\partial \eta}{\partial x}(x_1, x_j) F_j + \sum_{j=1}^N \rho_j \frac{\partial \mu}{\partial x}(x_1, x_j) F'_j + g'(x_1). \quad (0.25)$$

Using (4) and following the same steps, we obtain:

$$F'_1 = \sum_{j=1}^N \rho_j \frac{\partial \eta}{\partial x}(x_1, x_j) F_j + \sum_{j=2}^N \rho_j \frac{\partial \mu}{\partial x}(x_1, x_j) h^{-1}(F_j - F_{j-1}) + \rho_1 \frac{\partial \mu}{\partial x}(x_1, x_1) F'_1 + g'(x_1). \quad (0.26)$$

Our new method involves solving the linear system consisting of equations (6) and (7), which results in a system of size $N + 1$, a significant improvement compared to the method developed by Tair et al. [2], which yields a system of size $2N$.



Numerical Test

To compute $\{F_i\}_{1 \leq i \leq N}$ and F'_1 , we rewrite (6) as

$$\begin{aligned} F_i &= (\rho_N \eta(x_i, x_N) + \rho_N h^{-1} \mu(x_i, x_N)) F_N \\ &+ \sum_{j=2}^{N-1} (\rho_j \eta(x_i, x_j) + \rho_j h^{-1} \mu(x_i, x_j) - \rho_{j+1} h^{-1} \mu(x_i, x_{j+1})) F_j \\ &+ (\rho_1 \eta(x_i, x_1) + \rho_2 h^{-1} \mu(x_i, x_2)) F_1 + \rho_1 \mu(x_i, x_1) F'_1 + g(x_i), \end{aligned}$$

and (7) as

$$\begin{aligned} F'_1 &= \left(\rho_N \frac{\partial \eta}{\partial x}(x_1, x_N) + \rho_N h^{-1} \frac{\partial \mu}{\partial x}(x_1, x_N) \right) F_N \\ &+ \sum_{j=2}^{N-1} \left(\rho_j \frac{\partial \eta}{\partial x}(x_1, x_j) + \rho_j h^{-1} \frac{\partial \mu}{\partial x}(x_1, x_j) - \rho_{j+1} h^{-1} \frac{\partial \mu}{\partial x}(x_1, x_{j+1}) \right) F_j \\ &+ \left(\rho_1 \frac{\partial \eta}{\partial x}(x_1, x_1) - \rho_2 h^{-1} \frac{\partial \mu}{\partial x}(x_1, x_2) \right) F_1 + \rho_1 \frac{\partial \mu}{\partial x}(x_1, x_1) F'_1 + g'(x_1). \end{aligned}$$

This results in a linear system of size $N + 1$. To compare our method with that developed by Tair et al. [2], we introduce the following example for all $x \in [-5, 5]$:

$$\begin{aligned} f(x) &= \int_{-5}^5 \frac{1}{5 + x^2 + y^2} f(y) dy + \int_{-5}^5 0.1 \cos(x^2 + y) f'(y) dy \\ &+ x + 0.1 \sin(-5 + x^2) - 0.1 \sin(5 + x^2), \end{aligned}$$

where $f(x) = x$ for all $x \in [-5, 5]$.

Table 1 shows the numerical results obtained for various values of N .

Table 1: Comparison of our method with Tair et al. [2]

N	Backward with Nyström			Full Nyström (Tair et al.)		
	Error	Matrix Conditioning	Time	Error	Matrix Conditioning	Time
20	5.35e-2	9.64e+0	4.45e-3	8.43e-1	3.41e+1	5.54e-3
50	2.07e-2	7.41e+0	2.08e-2	7.22e-2	1.46e+1	2.52e-2
100	1.04e-2	7.08e+0	1.20e-1	5.48e-2	1.46e+1	9.97e-2
200	5.33e-3	7.85e+0	4.11e-1	4.76e-2	1.46e+1	4.19e-1
500	2.18e-3	1.06e+1	2.97e+0	3.83e-2	1.46e+1	3.42e+0
750	1.46e-3	1.29e+1	6.74e+0	1.84e-2	1.46e+1	9.29e+0
1000	1.10e-3	1.30e+1	1.10e+1	8.54e-3	1.46e+1	2.96e+1



Conclusion

In this paper, we present a novel numerical method for solving linear Fredholm integro-differential equations. By combining the backward finite difference method with the Nyström method, we have successfully reduced the size of the resulting system by half compared to the approach proposed by Tair et al. This reduction has significantly enhanced computational efficiency, making our method particularly advantageous for large integration intervals.

Additionally, we have developed a new norm for $\mathbb{R}^{N+1}$ to ensure the convergence of our method. Numerical tests demonstrate that our approach outperforms that of Tair et al. in both execution time and accuracy. Specifically, we compared our method with the one that utilizes only the Nyström method as proposed by Tair et al. Our method achieved better numerical accuracy, improved conditioning of the associated matrix, and faster execution times.

Our results confirm that the combination of the backward finite difference method and the Nyström method provides a robust and efficient tool for solving Fredholm integro-differential equations, with potential applications in various scientific and engineering fields. Furthermore, our approach offers a practical and effective solution that could be adapted to other types of integro-differential equations.

Future work will focus on extending our method to systems of nonlinear integro-differential equations and incorporating computational acceleration techniques for even faster computations. We anticipate that this method will significantly advance the numerical solutions for integro-differential equations and their numerous applications.

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Around of the bivariate Bühlmann's optimization problem and its applications

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Abstract. We propose in this paper new credibility premiums, which is based on a relationship between the number and the amount of claims of a contract for that year and based on an optimization problem, under the irreducible random variables, which helps us ensure the covariance matrix inversion. And then we calculate bivariate Bühlmann and Bühlmann Straub estimators with application. Thus we arrive at the new estimators of the individual premium by using additional sources of data. We conclude with structure parameters estimators and the application of a simulation.

Keywords: Credibility premium, Irreducible random variable, Bühlmann model, Bühlmann Straub model, Structure parameters.



Introduction

Credibility theory is an actuarial pricing method. Due to this theory, the insurer may calculate the premium of a portfolio of contracts. Credibility theory is to combine the available data on the portfolio and each individual (the contract) see [11], [12]. Several models of credibility have been established [1], [2], [3]. For that we propose in this paper to present new premiums credibility of the insurance contract. We extend the univariate credibility model to the bivariate credibility theory by using additional sources of data. In many cases we find that insurance there is a relationship between the number and the number of claims of a contract for a single period see [5]. We write new estimators of individual premiums. The indispensable problem of linear credibility is the determined estimator of the individual premium for a contract or a portfolio of insurance contracts [7],[8]. The credibility estimator is linear in the observations of the portfolio. To determine a precise form of the estimator of credibility, we must solve the problem of minimizing the least squares sense:

$$\Psi = \min_{\zeta^1, \zeta^2, \xi^1, \xi^2} E \left(\mu^1(\theta) - \widehat{\mu^1}(\zeta^1, \zeta^2) \right)^2 + E \left(\mu^2(\theta) - \widehat{\mu^2}(\xi^1, \xi^2) \right)^2 \quad (0.27)$$

With μ^1 and μ^2 are respectively, the conditional expectation of the number of claims, and the conditional expectation of number of claims, and θ is a risk profile. Here, $\mu^1(\zeta^1, \zeta^2)$ and $\mu^2(\xi^1, \xi^2)$ are linear forms which will be specified in the various models.

Main results

We consider an homogeneous insurance portfolio consisting of $n \geq 2$ contracts. Each contract i has been activated in the last t periods at least for one period. By $T(i) \subseteq \{1, \dots, t\}$ we denote the set of the activated past periods of contract i , assuming that its cardinality . We set $t_0 := \sum_{1 \leq i \leq n} t_i$. In order to be able to include the future period $t+1$, we set $T^+(i) := T(i) \cup \{t+1\}$. Now we have for each contract i the set of pairs of random variables:

$$(Z_i^1, Z_i^2) = (Z_{i,s}^1, Z_{i,s}^2)_{s \in T^+(i)}, \quad (0.28)$$

$i = 1, \dots, n$. For example, $Z_{i,s}^1$ and $Z_{i,s}^2$ can be interpreted as respectively the amount and the number of claims of the contract i in the period s . The random variables of the whole model are denoted by

$$(Z^1, Z^2) = (Z_i^1, Z_i^2)_{i=1, \dots, n}. \quad (0.29)$$

We make the following model assumptions:

- (B1)** The random structure variables $(\Theta_i)_{i=1, \dots, n}$ have the identical distribution $\Pr_{\Theta}$.



- (B2) The families of random variables (Θ_i, Z_i^1, Z_i^2) , $i = 1, \dots, n$, are mutually non-correlated with $Z_{i,s}^1, Z_{i,s}^2 \in L^2$ for all $s \in T^+(i)$, $i = 1, \dots, n$. The complete set of random variables $\{Z_{i,s}^1, Z_{i,s}^2 | s \in T^+(i), i = 1, \dots, n\}$ is assumed to be Pr-irreducible.
- (B3) For fixed i and conditioned by Θ_i , the pairs of random variables $(Z_{i,s}^1, Z_{i,s}^2)_{s \in T^+(i)}$ are mutually non-correlated with the first and second conditional moments depending only on Θ_i :

$$E[Z_{i,s}^1 | \Theta_i] := \mu^1(\Theta_i), \quad (0.30)$$

$$\text{var}[Z_{i,s}^1 | \Theta_i] := \alpha^1(\Theta_i), \quad (0.31)$$

$$\text{cov}(Z_{i,s}^1, Z_{i,s}^2 | \Theta_i) := \kappa(\Theta_i). \quad (0.32)$$

$\tau = 1, 2$. Moreover, on a non-negligible set the pair $(Z_{i,s}^1, Z_{i,s}^2)$ is assumed to be $\text{Pr}(\cdot | \Theta_i)$ -irreducible:

$$\Pr_{\Theta} \{ \omega | (Z_{i,s}^1, Z_{i,s}^2) \text{ is } \text{Pr}(\cdot | \Theta_i(\omega))\text{-irreducible} \} > 0. \quad (0.33)$$

For non-conditional moments we use the following notations, $\tau = 1, 2$:

$$E[\mu^1(\Theta)] := M^1 \quad (0.34)$$

$$E[\alpha^1(\Theta)] := A^1 \quad (0.35)$$

$$\text{var}[\mu^1(\Theta)] := B^1 \quad (0.36)$$

$$E[\kappa(\Theta)] := K \quad (0.37)$$

$$\text{cov}[\mu^1(\Theta), \mu^2(\Theta)] := L \quad (0.38)$$

A first result from the assumptions made is the following:

Proposition 0.3. *Under the model assumptions (B1) - (B3) we have*

$$D_{i,1} := (t_i B^1 + A^1)(t_i B^2 + A^2) - (t_i L + K)^2 > 0 \quad \text{and} \quad (0.39)$$

$$D_2 := A^1 A^2 - K^2 > 0. \quad (0.40)$$

proof To show (3), we know by assumption (B1) that the sequence of random variables (Z_i^1, Z_i^2) is Pr-irreducible, and so is the pair vector $(\sum_{s \in T(i)} Z_{i,s}^1, \sum_{s \in T(i)} Z_{i,s}^2)$. Proposition tells us that their covariance matrix is positive definite:

$$\text{cov}_{\text{Pr}} \left(\left(\sum_{s \in T(i)} Z_{i,s}^1, \sum_{s \in T(i)} Z_{i,s}^2 \right) \right) = \begin{pmatrix} t_i B^1 + A^1 & t_i L + K \\ t_i L + K & t_i B^2 + A^2 \end{pmatrix}.$$

This implies that the determinant $D_{i,1} := (t_i B^1 + A^1)(t_i B^2 + A^2) - (t_i L + K)^2$ of the last matrix is strictly positive which is the statement in (4).

For the inequality (10), the assumption (B3) implies with A Proposition that on a set of positive Pr_{Θ} -measure the covariance matrix with respect to the conditional probability $\text{Pr}(\cdot | \Theta_i)$ of the pair $(Z_{i,s}^1, Z_{i,s}^2)$ is positive definite:

$$\text{cov}_{\text{Pr}(\cdot | \Theta_i)}((Z_{i,s}^1, Z_{i,s}^2)) = \begin{pmatrix} \alpha^1(\Theta_i) & \kappa(\Theta_i) \\ \kappa(\Theta_i) & \alpha^2(\Theta_i) \end{pmatrix}.$$



This implies that for any pair $(\alpha_1, \alpha_2) \neq 0$ the conditional variance is strictly positive on a set of positive $\Pr_{\Theta}$ -measure. Therefore also

$$E_{\Theta} [\text{var}_{\Pr(\cdot|\Theta_i)} (\alpha_1 Z_{i,s}^1 + \alpha_2 Z_{i,s}^2)] = \begin{pmatrix} \alpha_1 & \alpha_2 \end{pmatrix} \begin{pmatrix} A^1 & K \\ K & A^2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} > 0$$

This shows that the matrix $\begin{pmatrix} A^1 & K \\ K & A^2 \end{pmatrix}$ is positive definite and consequently its determinant $D_2 = A^1 A^2 - K^2$ is strictly positive.

We want to find a linear credibility estimator of the bivariate system $(Z_{i,t+1}^1, Z_{i,t+1}^2)$. Consequently, it is assumed that such estimators of $Z_{i,t+1}^1$, $\tau = 1, 2$, have the form

$$\hat{\mu}_i^1(Z^1, Z^2) = \zeta_{i,0}^1 + \sum_{j=1}^n \sum_{s \in T(j)} (\zeta_{i,j,s}^{1\tau} Z_{j,s}^1 + \zeta_{i,j,s}^{1com\tau} Z_{j,s}^2) \quad (0.41)$$

with real coefficients $\zeta_i^1 = (\zeta_{i,0}^1, \zeta_{i,j,s}^{1\tau}, \zeta_{i,j,s}^{1com\tau})$.

Theorem 0.10. *Under the assumptions (B1) – (B3) the optimal solution to the linear credibility problem is:*

$$\hat{\mu}_i^{1*}(Z_i^1, Z_i^2) = (1 - \zeta_i^{1\tau}) M^1 + \frac{1}{t_i} \sum_{s \in T(i)} (\zeta_i^{1\tau} Z_{i,s}^1 + \zeta_i^{1com\tau} (Z_{i,s}^2 - M^2)) \quad (0.42)$$

$\tau = 1, 2$, where the **credibility coefficients** $\zeta_i^{1\tau}, \zeta_i^{1com\tau}$ are given by the following formulas:

$$\zeta_i^{1\tau} = \frac{t_i}{D_{i,1}} (B^1(t_i B^2 + A^2) - L(t_i L + K)) \quad (0.43)$$

$$\zeta_i^{1com\tau} = \frac{t_i}{D_{i,1}} (A^1 L - B^1 K). \quad (0.44)$$

Remark 0.1. In contrast to the standard credibility formula, here the credibility factors $\zeta_i^{1\tau}, \zeta_i^{1com\tau}$ are not necessarily in $[0, 1]$.

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The Average Sentinel of the heat Equation

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Abstract. In this work, we analyse the problem of identification of the pollution term in a heat system when the dynamics of the state is governed by a parameterized operator. In this way, we introduce a notion of average sentinel. First, we prove the existence of such sentinels introduced by Lions by solving a problem of null average controllability given witch is given by Zuazua and secondly, we identify the information for pollution terms by using the average sentinel witch is the nouveauty of our work.

Keywords: Average sentinel, Averaged observability, Gradient method, Numerical application.



Introduction

In the modelling of the evolution system type, the source terms as well as the initial or boundary conditions may be unknown.

We analyse the problem of identification of pollution terms (the source) in a systems governed by a heat equation depending on unknown parameters in a deterministic manner [1]. We look for control, independent of the values of these parameters that need to introduce the sentinel in an averaged sense to be made precise [3]. The notion of averaged control for parameter-dependent family of the parabolic system is introduced by Lions [2] and the sentinel method introduced by Lions [2] is adapted to the estimation of this incomplete or unknown data in the problems governed by parabolic system in general, for example, pollution in rivers or a lake. So since the introduction of the sentinel method many authors developed several applications, such as in the environment, in ecology [4].

The notion of average sentinel, as formulated here, has not been analysed until now, this notion is very interested in the identification of the missing data when the system depends on unknown parameters [5].

Setting of the problem

Let $\Omega \subset R^d$, $d \geq 1$, be a bounded domain with smooth boundary Γ and ω be an open non-empty subset of Ω [61]. Denote by $Q = \Omega \times [0, T]$ the space time cylinder where the equation holds and $\Sigma = \Gamma \times]0, T[$ for the lateral boundary. Let $y = y(x, t, \theta)$ be the solution of the following system

$$\begin{cases} \frac{\partial y}{\partial t}(x, t) - \operatorname{div}(a(x, \theta) \nabla y(x, t)) = f(x, t) & \text{in } Q, \\ y_\theta = 0 & \text{on } \Sigma, \\ y_\theta(x, 0) = y_0(x) & \text{in } \Omega, \end{cases} \quad (0.45)$$

Definition 0.1. Let S_m be a real function in equation (3.4) depending only on the parameters λ and τ . S_m is said a average sentinel defined by h if the following conditions are satisfied:

i)

$$\frac{\partial S_m}{\partial \tau}(\lambda, \tau) \Big|_{\lambda=0, \tau=0} = 0. \quad (0.46)$$

ii) there exists a average control $u \in L^2(\omega)$ such that:

$$\|u\|_{L^2(\omega)} = \min_{v \in U} \|v\| \quad (0.47)$$

$$\text{where } U = \left\{ u \in L^2(\omega), \text{ such that } \frac{\partial S_m}{\partial \tau}(\lambda, \tau) \Big|_{\lambda=0, \tau=0} = 0 \right\}.$$



Average controllability problem

We introduce the classical adjoint system :

$$\begin{cases} -\frac{\partial q_\theta}{\partial t} + \operatorname{div}(a(x, \theta) \nabla q_\theta) = h\chi_o + u\chi_\omega & \text{in } Q, \\ q_\theta = 0 & \text{on } \Sigma, \\ q_\theta(x, T) = \hat{g} & \text{in } \Omega, \end{cases} \quad (4)$$

with $q_\theta = q(x, t, \theta)$.

Theorem 0.11. *Let q_θ be the solution to the backward problem (3), then the existence of an average sentinel insensitive to the missing data is equivalent to the average null-controllability problem*

$$q_m(x, 0) = \int_0^1 q_\theta(x, 0) d\theta = 0, \quad (4.1)$$

Theorem 0.12. *Solving a problem of the average null-controllability is equivalent to finding a control u so that the solution of system (4) satisfies equation (4.1) such that*

$$u(x, t) = - \int_0^1 \rho_\theta(x, t) \chi_\omega d\theta \quad (4.2)$$

Theorem 0.13. *The pollution term is identified as follows:*

$$\int_0^1 \int_\Omega q(h) \hat{f} d\Omega = S_{obs}(\lambda, \tau) - S(0, 0).$$

In this work, we have introduced the problem of identification of the pollution term in a parabolic system when the dynamics of the state is governed by a parameterized operator; we have introduced a notion of average sentinel to obtain information about the pollution term. The notion of average sentinel, as formulated here, has not been analysed until now, this notion is very interested in the identification of the missing data when the system is depending on unknown parameters.

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Traitement analytique et numérique d'une équation intégrale de Fredholm non linéaire spécifique

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Abstract. Dans cet article, nous effectuons une analyse approfondie d'une équation intégrale de Fredholm non linéaire du deuxième type dans l'espace de Banach $C([a, b])$. L'équation implique une fonction noyau et un terme non linéaire qui inclut un opérateur intégral de Fredholm, les deux satisfaisant des conditions spécifiques de continuité et de bornitude. Nous établissons l'existence et l'unicité de la solution en utilisant les théorèmes classiques de point fixe. De plus, la méthode de Nyström est employée pour obtenir une approximation numérique de la solution. Enfin, plusieurs exemples numériques sont fournis pour démontrer l'efficacité et la précision de la méthode.

Keywords: Équation intégrale de Fredholm, équation non linéaire, méthode de Nyström, théorème du point fixe.



Introduction

Les **équations intégrales de Fredholm** sont une classe d'équations intégrales où la fonction inconnue apparaît sous le signe d'intégrale. Elles se présentent sous deux formes principales : du premier type et du deuxième type. Dans le premier type, la fonction solution apparaît uniquement sous l'intégrale, tandis que dans le deuxième type, l'équation implique à la fois la fonction solution et une intégrale de cette fonction. Ces équations sont exprimées sous la forme suivante :

- Premier type :

$$\int_a^b K(x, s)f(s) ds = g(x)$$

- Deuxième type :

$$f(x) - \lambda \int_a^b K(x, s)f(s) ds = g(x)$$

où $K(x, s)$ est le noyau, $f(s)$ est la fonction inconnue, $g(x)$ est une fonction donnée, et λ est une constante.

Les **équations intégrales de Fredholm** sont fondamentales tant en physique mathématique qu'en mathématiques appliquées, car elles modélisent un large éventail de phénomènes. Ces équations apparaissent dans divers domaines, tels que :

- **La mécanique quantique**, où elles sont utilisées pour décrire les processus de diffusion et les systèmes quantiques.
- **La théorie du potentiel**, où elles modélisent des problèmes liés à l'électrostatique, la magnétostatique et la gravitation.
- **Le traitement du signal**, où les équations de Fredholm aident à reconstruire des images et des signaux à partir de données incomplètes ou bruyantes.

De plus, en **économie**, ces équations sont utilisées pour analyser les problèmes d'équilibre dans les systèmes économiques. Les équations intégrales de Fredholm apparaissent également fréquemment dans les problèmes aux limites, jouant un rôle crucial dans la simplification de la solution des équations différentielles partielles en les convertissant en une forme intégrale plus abordable.

Dans ce travail, nous visons à réaliser une analyse approfondie d'une équation intégrale de Fredholm non linéaire du deuxième type dans l'espace de Banach $C([a, b])$. L'équation implique une fonction noyau et un terme non linéaire incluant un opérateur intégral de Fredholm, les deux satisfaisant des conditions



spécifiques de continuité et de bornitude. Cette équation prend la forme suivante :

$$\forall t \in [a, b], \quad x(t) = G \left(t, x(t), \int_a^b K(t, s)x(s) ds \right), \quad (0.48)$$

où x est l'inconnu à trouver dans $C^0([a, b])$, et K est une fonction noyau continue définie par

$$K : [a, b] \times [a, b] \rightarrow \mathbb{R},$$

et G est une fonction non linéaire donnée par

$$G : [a, b] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}.$$

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Estimation d'erreur de la solution d'un problème d'obstacle parabolique d'ordre fractionnaire avec terme source non-linéaire

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Abstract. Dans ce travail, nous considérons la méthode des éléments finis pour un problème d'obstacle parabolique d'ordre fractionnaire en temps avec terme source non linéaire. Une estimation d'erreur du problème semi-discret spatialement par la méthode des éléments finis (MEF) est discutée. Un Problème totalement discret est proposé en introduisant l'approximation par différences finies pour la variable temps. La L^2 -stabilité inconditionnelle et la convergence de la solution approchée ont été prouvées.

Keywords: Problème d'obstacle fractionnaire, Dérivée Grunwald-Letnikov, stabilité, estimation d'erreur.

Introduction

Le traitement numérique et l'analyse des équations aux dérivées partielles d'ordre fractionnaire sont devenus un sujet de recherche important qui offre un grand potentiel. Les méthodes des différences finies et des éléments finis (MEF) sont l'une des méthodes numériques efficaces pour résoudre les équations aux dérivées partielles fractionnaires [2, 3, 4]. Pour autant que nous le sachions, les méthodes d'éléments finis n'ont pas été prises en compte pour les inéquations variationnelles paraboliques d'ordre fractionnaire. Dans ce travail,



on présente une approche de MEF pour estimer l'erreur en norme L^2 d'un problème d'obstacle parabolique de dérivée temporelle fractionnaire avec un terme source non linéaire. Cette approche est basée sur la propriété L^2 -stabilité de la solution approchée.

Plus précisément, considérons le problème fractionnaire suivant: trouver $u \in L^2(0, T; H_0^1(\Omega)) \cap H^\alpha(0, T; L^2(\Omega))$ telle que

$$\begin{cases} \langle {}_0D_t^\alpha u(t), v - u(t) \rangle + a(u(t), v - u(t)) & \geq \langle f(u(t)), v - u(t) \rangle, \forall v \in H_0^1(\Omega) \\ u(t=0) & = u_0, \quad \text{dans } \Omega \\ u & = 0, \quad \text{sur } \partial\Omega \times]0, T[\\ u \leq \psi, \quad v \leq \psi & \text{dans } \Omega \times]0, T[. \end{cases} \quad (0.49)$$

où Ω est un ouvert borné convexe de l'espace $\mathbb{R}^2$, avec frontière $\partial\Omega$ suffisamment régulière, $\langle \cdot, \cdot \rangle$ désigne le produit scalaire dans l'espace $L^2(\Omega)$, $H^\alpha(0, T; L^2(\Omega))$ est un espace de Sobolev fractionnaire d'ordre $0 < \alpha < 1$, $a(\cdot, \cdot)$ est une forme bilinéaire continue et coercive, définie par,

$$a(u, v) = \int_{\Omega} \left(\sum_{i,j=1}^2 a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + a_0(x) uv \right) dx, \quad (0.50)$$

avec $a_{ij}(x), a_0(x)$ sont des coefficients supposés suffisamment réguliers et satisfaisant les conditions d'ellipticités, tel que $a_0(x) \geq \beta > 0$, $\forall x \in \bar{\Omega}$, β constante. ψ est une fonction régulière (appelée obstacle) définie sur Ω (voir [1]) telle que $\psi \in L^2(0, T; W^{2,\infty}(\Omega))$, et $\psi \geq 0$ sur Γ .

On suppose que le second membre $f(\cdot)$ est non linéaire, non décroissant et Lipschitz continue sur $\mathbb{R}$ de constante L strictement supérieure à β , $L > \beta$. ${}_0D_t^\alpha u(t)$, pour $0 < \alpha < 1$ sont les dérivées fractionnaires de Grunwald-Letnikov à gauche.

Semi-Discretisation

Considérons V_h l'espace d'éléments finis linéaires par morceaux sur chaque triangle K de $\mathcal{T}^h$, plus précisément soit,

$$V_h = \{v_h \in \mathbb{C}(\bar{\Omega}) \cap V : v_h|_K \in P_1, \forall K \in \mathcal{T}^h, \}, \quad (\text{ici } V = H_0^1(\Omega))$$

Alors l'analogie semi-discret du problème fractionnaire (0.50) est donné par,

$$\begin{cases} \langle {}_0D_t^\alpha u_h(t), v_h - u_h(t) \rangle + a(u_h(t), v_h - u_h(t)) & \geq \langle f(u_h(t)), v_h - u_h(t) \rangle, \forall v_h \in V_h \\ u_h(t=0) & = u_{0h}, \quad \text{dans } \Omega \\ u_h & = 0, \quad \text{sur } \partial\Omega \times]0, T[\\ u_h \leq r_h \psi, \quad v_h \leq r_h \psi & \text{dans } \Omega \times]0, T[. \end{cases} \quad (0.51)$$



Discretisation totale

Divisons l'intervalle $[0, T]$ en N parties égales, en posant le pas $\tau = \frac{T}{N}$, et $t_n = n\tau$, pour $n = 0, 1, \dots, N$. Soit u_h^n la valeur approchée de la solution semi-discrète $u_h(t_n)$ au point $t = t_n$. En introduisant l'approximation par différences finies d'ordre 1 de la dérivée fractionnaire de Grunwald-Letnikov [?], alors le problème totalement discret est défini par: étant donnée $u_h^0 \in V_h$, trouver $u_h^n \in V_h$; $n = 1, 2, \dots$, telle que,

$$\begin{cases} \langle u_h^n, v_h - u_h^n \rangle + \tau^\alpha a(u_h^n, v_h - u_h^n) \geq \tau^\alpha \langle f(u_h^n), v_h - u_h^n \rangle - \sum_{j=0}^{n-1} \mathcal{W}_{\alpha, n-j} \langle u_h^j, v_h - u_h^n \rangle, \forall v_h \in V_h \\ u_h^n \leq r_h \psi, \quad v_h \leq r_h \psi \quad \text{dans } \Omega. \end{cases} \quad (0.52)$$

Analyse de l'erreur

Cette section a pour objet l'étude de la convergence et l'ordre de convergence du problème totalement discret d'ordre fractionnaire (0.52).

Theorem 0.14. (Main result) Soit $u(t_n), u_h^n$ les solutions des problèmes fractionnaires (0.530), (0.52) à l'instant $t = t_n$ respectivement. Alors il existe une constante C indépendante de τ et h tel qu'on a l'estimation suivante,

$$\|u_h^n - u(t_n)\| \leq C \left[\left(\frac{1}{1 + \tau^\alpha(\beta - L)} + 1 \right) h^2 + \frac{1}{\beta - L} \tau \right].$$

Proof Pour plus de détails voir le travail de l'auteur [5].

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Théorème du point fixe généralisé pour les contractions F-weak en utilisant une fonction ρ .

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Abstract.

Dans le cadre de la théorie des points fixes, Dariusz Wardowski a introduit les notions de F-contraction et de F-weak contraction pour étendre le théorème de contraction de Banach. Une F-contraction est définie par une application $T : X \rightarrow X$ sur un espace métrique complet, satisfaisant une inégalité qui permet de garantir l'existence d'un point fixe. La contraction F-weak assouplit cette définition en permettant des relations plus flexibles entre les distances. Récemment, des recherches ont élargi ces concepts en introduisant une fonction ρ pour généraliser les F-contractions. Cette approche permet de moduler la contraction de manière plus souple, garantissant l'existence et l'unicité des points fixes sous des hypothèses élargies. Dans ce travail, on propose la notion de ρ -F-weak-contraction, une généralisation de toutes les notions précédentes. Nous démontrons que sous certaines conditions, il est possible d'assurer l'existence et l'unicité d'un point fixe pour l'application $T : X \rightarrow X$, tout en englobant les théorèmes précédents et en élargissant le champ d'applications possibles.

?



Keywords: Point fixe, Espace métrique complet, Suite de Cauchy, contraction.

Introduction

La théorie des points fixes joue un rôle central dans de nombreux domaines des mathématiques, en particulier dans l'analyse et la résolution d'équations. Dariusz Wardowski a élargi ce domaine en introduisant les concepts de F-contraction et de F-weak contraction. La F-contraction est définie pour une application $T : X \rightarrow X$ dans un espace métrique complet (X, d) , satisfaisant une inégalité qui permet d'établir l'existence d'un point fixe $x^* \in X$ vers lequel les itérations de T convergent, quelle que soit la condition initiale choisie. Cette définition repose sur une fonction F qui module la distance, permettant ainsi de traiter des cas où les conditions de contraction strictes ne sont pas vérifiées.

La contraction F-weak assouplit encore cette notion, en remplaçant la stricte diminution des distances par une relation plus flexible, décrite par une inégalité impliquant le maximum des distances entre les points. Cette approche ouvre la voie à de nouvelles applications, en permettant de considérer des cas plus variés où la dynamique de la contraction peut être soutenue.

Dans une perspective d'élargissement des concepts de contraction, des travaux récents, notamment ceux de Chakar et al., ont proposé une généralisation des F-contractions en introduisant une fonction ρ . Cette nouvelle condition permet de moduler la contraction de manière plus flexible et est définie par des propriétés spécifiques de la fonction ρ , telles que la croissance et la continuité, ainsi qu'un comportement asymptotique à l'infini. En définissant une ρ -F-contraction, on ouvre un nouveau champ d'étude pour garantir l'existence et l'unicité des points fixes.

Dans cette étude, nous proposons d'introduire la notion de ρ -F-weak-contraction, qui généralise toutes les notions précédentes. Nous montrons que, sous certaines hypothèses, il est toujours possible de garantir l'existence et l'unicité d'un point fixe pour une application $T : X \rightarrow X$. Plus précisément, nous considérons T comme une ρ -F-weak-contraction si, pour tout $x, y \in X$, l'inégalité suivante est satisfaite :

$$F(d(Tx, Ty)) \leq \rho \left(F \left(\frac{1}{2} \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty) + d(y, Tx)\} \right) \right),$$

où F satisfait les conditions F1 F2 F3 et les propriétés de $\rho(1 - 4)$. Nous démontrons que ces hypothèses généralisées permettent d'obtenir des résultats en matière de points fixes qui englobent les théorèmes antérieurs, tout en étant applicables à une classe plus large d'applications.

De plus, il est essentiel de noter que la ρ -F-contraction représente une généralisation des contractions classiques, y compris la F-contraction, tandis que la F-weak-contraction constitue également une extension de la F-contraction.



Cependant, il n'existe pas de lien direct entre la ρ -F-contraction et la F-weak-contraction. En revanche, la ρ -F-weak-contraction englobe toutes ces notions, offrant ainsi un cadre théorique plus large et flexible pour l'étude des points fixes. Ce concept est illustré dans un diagramme, démontrant clairement la hiérarchie et les relations entre ces différentes classes de contractions, tout en soulignant leur interconnexion et leur complémentarité.

Main results

Theorem 0.15. *Soit (X, d) un espace métrique complet et $T : X \rightarrow X$ une ρ -F-weak-contraction telle que ρ vérifie les conditions suivantes:*

- ρ est une fonction croissante,
- ρ est continue,
- $\lim_{t \rightarrow \infty} \rho(t) = -\infty$,
- $\rho(t) < t$ pour tout $t \in \mathbb{R}$.

Si T ou F est continue, alors:

- T admet un unique point fixe $x^* \in X$,
- pour tout $x \in X$, la suite $\{T^n x\}$ converge vers x^* .

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Nonlinear Volterra Integro-Differential Equations with Delay Terms: An Application of the Picard Iteration Method

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Abstract. In this study, we investigate the Volterra integro-differential equation that incorporates a delay term. By employing a Picard iteration scheme, we establish the existence and uniqueness of the solution and present the quadrature method as an effective tool for numerically approximating the analytical solution.

Our research enhances the understanding of nonlinear Volterra integro-differential equations with delay terms. We demonstrate the convergence of our numerical approximation method and provide two numerical examples to validate the accuracy of our results. By varying the delay term, we emphasize its significant impact on the solution of the equation.

Keywords: Volterra integro-differential equation, quadrature method, delay term.

Introduction

Integral and integro-differential equations of Volterra are essential in various fields of applied mathematics, particularly in solving differential equations [21, 22, 23, 24] and modeling physical phenomena. A comprehensive study of these



equations enhances our understanding of complex dynamic systems and facilitates the resolution of practical problems across diverse domains, including physics, economics, engineering, and medicine [10, 14, 12]. Over the years, many researchers have explored the properties of these equations and developed effective approximation methods for solving them [1, 2, 1, 2, 3, 4, 8, 9, 7, 12, 13, 14, 15, 16].

Significant contributions to the analysis and resolution of these equations have been made by researchers such as Aissaoui, Bounaya, Guebbai, Lemita, Segni, Merchela, Salah, Ghat, Debbar, and others [1, 2, 1, 2, 3, 4, 8, 9, 7, 12, 13, 14, 15, 16]. Notably, the works of Linz [7] and Salah [8] have been pivotal in advancing our understanding of Volterra integro-differential equations and developing new approximation methods.

Linz's seminal work, "Analytical and Numerical Methods for Volterra Equations," published in 1985 [7], introduced sophisticated analytical and numerical techniques for solving Volterra equations, including integro-differential equations. His research has laid the foundation for many contemporary approaches in this field, providing deeper insights into the fundamental properties of solutions [1].

Similarly, Salah's research [8] has significantly advanced the study of nonlinear Volterra integro-differential equations. In a 2020 article published in the Boletim da Sociedade Paranaense de Matematica, Salah, alongside Guebbai, Lemita, and Aissaoui, investigated the solution of a nonlinear Volterra integro-differential equation within the framework of a seismic model. Their work has deepened the understanding of complex dynamic phenomena associated with earthquakes and led to the development of specific methods for solving these equations.

In this paper, we focus on a new Volterra integro-differential equation that incorporates a delay term:

$$\forall t \in [a; b], \quad u(t) = \int_a^{\alpha(t)} N(t, s, u(s), u'(s)) ds + f(t), \quad (0.53)$$

This equation differs from those examined in [25, 26, 27, 28, 29], which involve various forms of delay terms compared to the one presented in equation (0.53).

Our primary objective is to establish the existence and uniqueness of the solution to this equation using the Picard iteration method, which is commonly employed in the analysis of integral and integro-differential equations. Additionally, we propose utilizing a Nyström method to construct a numerical approximation of the analytical solution.



Problem Position

Let $N : [a, b]^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ a non-linear function that satisfies the following conditions

$$(C) \left\{ \begin{array}{l} (C_1) : \partial_t N \in C^0([a, b]^2 \times \mathbb{R}^2) \\ (C_2) : \forall t, s \in [a, b], \quad \forall x, y \in \mathbb{R}; \forall \bar{x}, \bar{y} \in \mathbb{R} \\ |N(t, s, x, y) - N(t, s, \bar{x}, \bar{y})| \leq L_{11} |x - \bar{x}| + L_{12} |y - \bar{y}| \\ |\partial_t N(t, s, x, y) - \partial_t N(t, s, \bar{x}, \bar{y})| \leq L_{21} |x - \bar{x}| + L_{22} |y - \bar{y}| \\ (C_3) : bL_{12} < 1 \end{array} \right.$$

Our objective is to demonstrate the existence and uniqueness of the solution $u \in C^1[a, b]$ that satisfies the following problem:

$$\forall t \in [a, b], \quad u(t) = \int_a^{\alpha(t)} N(t, s, u(s), u'(s)) ds + f(t),$$

Where, f is a function defined in $C^1[a, b]$, and α is the assumed delay function that satisfies the following conditions:

$$(C_\alpha) \left\{ \begin{array}{l} (C_{\alpha,1}) : \forall t \in [a, b], \quad a \leq \alpha(t) \leq t, \\ (C_{\alpha,2}) : \alpha \in C^1[a, b]. \\ (C_{\alpha,3}) : \|\alpha'(t)\|_{C^0[a,b]} L_{1,2} < 1 \end{array} \right.$$

To better understand our equation (0.53), we need to derivate both sides of equation (0.53). For this purpose, we require the following proposition:

Proposition 0.4. Let ψ be a functional defined in $\mathbb{R}^2$ into $\mathbb{R}$ as

$$\begin{aligned} \psi : \quad \mathbb{R}^2 &\longrightarrow \mathbb{R}, \\ (t, s) &\longmapsto \psi(t, s), \end{aligned}$$

such that: $\psi'(t) \in C^0(\mathbb{R}^2)$, α satisfy the conditions (C_α) and the functional φ defined in $\mathbb{R}$ into $\mathbb{R}$ is given the following formula

$$\varphi(t) = \int_a^{\alpha(t)} \psi(t, s) ds, \quad \forall t \geq a.$$

Then,

$$\varphi'(t) = \alpha'(t) \psi(t, \alpha(t)) + \int_a^{\alpha(t)} \psi'(t)(t, s) ds, \quad \forall t \geq a.$$



We apply the derivative on the both side of our equation (0.53) and we use the previous proposition, we obtain

$$\begin{aligned} \forall t \in [a, b]; \quad u'(t) &= \alpha'(t)N(t, \alpha(t), u(\alpha(t)), u'(\alpha(t))) \\ &+ \int_a^{\alpha(t)} \partial_t N(t, s, u(s), u'(s))ds + f'(t) \end{aligned} \quad (0.54)$$

To achieve our goal of demonstrating the existence and uniqueness of the solution to our equation through analytical investigation, as well as developing an approximate method for this equation through numerical study, we will employ a change of variables as outlined below.

$$\forall t \in [a, b]; \quad u'(t) = v(t) \Rightarrow \int_a^t u'(s)ds = \int_a^t v(s)ds \Rightarrow u(t) = \int_a^t v(s)ds + u(a)$$

hence, $\alpha(a) = a$ then $u(a) = f(a)$, which gives

$$u(t) = \int_a^t v(s)ds + f(a) \quad (0.55)$$

resulting in the equivalent problem to (0.53)

Find $v \in C^0[a, b]$ such that

$\forall t \in [a, b]$

$$\begin{aligned} v(t) &= \alpha'(t)N(t, \alpha(t), \int_a^{\alpha(t)} v(s)ds + f(a), v(\alpha(t))) \\ &+ \int_a^{\alpha(t)} \partial_t N(t, s, \int_a^s v(o)do + f(a), v(s))ds + f'(t). \end{aligned} \quad (0.56)$$

In the subsequent discussion, in order to establish both the existence and uniqueness of the solution $u \in C^1[a, b]$ for our equation (0.53), we proceed by demonstrating the existence and uniqueness of the solution $v \in C^0[a, b]$ for its corresponding equivalent equation presented in (0.158).

System study

Before delving into the analysis of convergence and conducting numerical tests, it is imperative to establish the existence of a unique solution for the system (S) composed of equations (0.255) and (0.434). Where,



$$(S) \begin{cases} V_1 = \alpha'(a)N(a, a, f(a), V_1) + f'(t_1) \\ V_i = \alpha'(t_i)N(t_i, t_{p(i)}, h \sum_{j=1}^{p(i)} w_j V_j + f(a), V_{p(i)}) \\ \quad + h \sum_{j=1}^{p(i)} w_j \partial_t N(t_i, t_j, h \sum_{q=1}^j w_q V_q + f(a), V_j) + f'(t_i), \quad 2 \leq i \leq n \end{cases}$$

Theorem 0.16. For h sufficiently small (or n sufficiently large), the system (S) possesses a unique solution.

Proof We start with the first equation of system (S) , where we put

$$\forall x \in \mathbb{R}, \quad \varphi_1(x) = \alpha'(a)N(a, a, f(a), x),$$

Therefore, (S_1) is equivalent to

$$v_1 = \varphi_1(v_1).$$

Now, $\forall x, y \in \mathbb{R}$, we have

$$\begin{aligned} |\varphi_1(x) - \varphi_1(y)| &= |\alpha'(a)N(a, a, f(a), x) - \alpha'(a)N(a, a, f(a), y)| \\ &\leq \|\alpha'\|_{C^0[a,b]} L_{12} |x - y| \end{aligned}$$

Under the condition $(C_w) - 3$, this result establishes that φ_1 is a contraction and admits a fixed point according to Banach's theorem.

Now, we pass to second equation of the system (S) for $2 \leq i \leq n$, we distinguish two cases:

On one hand, if $p(i) \leq 1$ it is clear that (S_2) is composed of explicit equations in terms of V_j , for all $1 \leq j \leq p$. These are proved in the previous steps.

On the other hand, if $p(i) = i$, we introduce the function φ_i as follows: For all $x \in \mathbb{R}$,

$$\begin{aligned} \varphi_i(x) &= \alpha'(t_i)N(t_i, t_i, h w_i x + h \sum_{j=1}^{i-1} w_j V_j + f(a), x) \\ &\quad + h w_i \partial_t N(t_i, t_i, h w_i x + h \sum_{j=1}^{i-1} w_j V_j + f(a), x) \\ &\quad + h \sum_{j=1}^{i-1} w_j \partial_t N(t_i, t_j, h \sum_{q=1}^j w_q V_q + f(a), V_j) + f'(a). \end{aligned}$$



This implies that the solution of (S_2) is a fixed point of φ_i , i.e.,

$$V_i = \varphi_i(V_i)$$

We have, $\forall x, y \in \mathbb{R}$,

$$\begin{aligned} & |\varphi_i(x) - \varphi_i(y)| \\ & \leq |\alpha'(t_i)| |N(t_i, t_i, hw_i x + h \sum_{j=1}^{i-1} w_j V_j + f(a), x) - N(t_i, t_i, hw_i y + h \sum_{j=1}^{i-1} w_j V_j + f(a), y)| \\ & + h |w_i| |\partial_t N(t_i, t_i, hw_i x + h \sum_{j=1}^{i-1} w_j V_j + f(a), x) - \partial_t N(t_i, t_i, hw_i y + h \sum_{j=1}^{i-1} w_j V_j + f(a), y)| \\ & \leq \|\alpha'\|_{C^0[a,b]} (hW L_{11} |x - y| + L_{12} |x - y|) + (h^2 W^2 L_{21} |x - y| + hW L_{22} |x - y|) \\ & \leq (\|\alpha'\|_{C^0[a,b]} hW L_{11} + \|\alpha'\|_{C^0[a,b]} L_{12} + h^2 W^2 L_{21} + hW L_{22}) |x - y|, \end{aligned}$$

this demonstrates that for a sufficiently small h , the function φ_i is a contraction. Applying Banach's theorem, we conclude that (S_2) has a unique solution for h sufficiently small.

In the end, we conclude that the system (S) has a unique solution. ■

Below is the algorithm for our newly modified Nyström method, designed to solve system (S) .

Step 0: Input the initial functions N, dN, f, f', α and α' .

Step 1: Give $n > 1, \varepsilon = 10^{-7}$ and define w_i, t_i .

Step 2: Initiate $V_1 = f'(t_1)$

Step 3: While $|V_1^{New} - V_1^{Old}| \geq \varepsilon$ Do

$$V_1^{New} = \alpha'(a) N(a, a, f(a), V_1^{Old}) + f'(t_1)$$

Step 4: For $i = 2 : n$

Step 5: If $p(i) \leq i - 1$ Then V_i is given explicitly by

$$\begin{aligned} V_i &= \alpha'(t_i) N(t_i, t_{p(i)}, h \sum_{j=1}^{p(i)} w_j V_j + f(a), V_{p(i)}) \\ &+ h \sum_{j=1}^{p(i)} w_j \partial_t N(t_i, t_j, h \sum_{q=1}^j w_q V_q + f(a), V_j) + f'(t_i). \end{aligned}$$



Step 6: If $p(i) = i$, While $|V_i^{New} - V_i^{Old}| \geq \varepsilon$ Do

$$\begin{aligned}
V_i^{New} &= \alpha'(t_i)N(t_i, t_i, hw_i V_i^{Old} + h \sum_{j=1}^{i-1} w_j V_j + f(a), V_i^{Old}) \\
&+ hw_i \partial_t N(t_i, t_i, hw_i V_i^{Old} + h \sum_{j=1}^{i-1} w_j V_j + f(a), V_i^{Old}) \\
&+ h \sum_{j=1}^{i-1} w_j \partial_t N(t_i, t_j, h \sum_{q=1}^j w_q V_q + f(a), V_j) + f'(a).
\end{aligned}$$

Numerical examples

In this section, we present two numerical examples to demonstrate the effectiveness and precision of our proposed method. The calculations in the following illustrations are based on the algorithm presented above. Furthermore, we compare our method with another numerical method to ensure that our study is more precise than the other.

Example 1:

We present the initial example featuring the following equation:

$$u(t) = \int_a^{t^{1.5}} \frac{(t-s)}{1+u(s)^2+u'(s)^2} ds + 1 + 0.25((t-t^{1.5})^2 - t^2),$$

Where the functions $N, f, \alpha(t), u$ satisfy all the assumptions $(C), (C_w)$ mentioned above.

$$\begin{aligned}
N(t, s, x, y) &= \frac{(t-s)}{1+x^2+y^2}, \\
f(t) &= 1 + 0.25((t-t^{1.5})^2 - t^2), \\
\alpha(t) &= t^{1.5}, \\
u(t) &= 1.
\end{aligned}$$

In the associated visual representations, we depict the exact solution, the approximate solution, and, notably, the error between both solutions for various values of n .



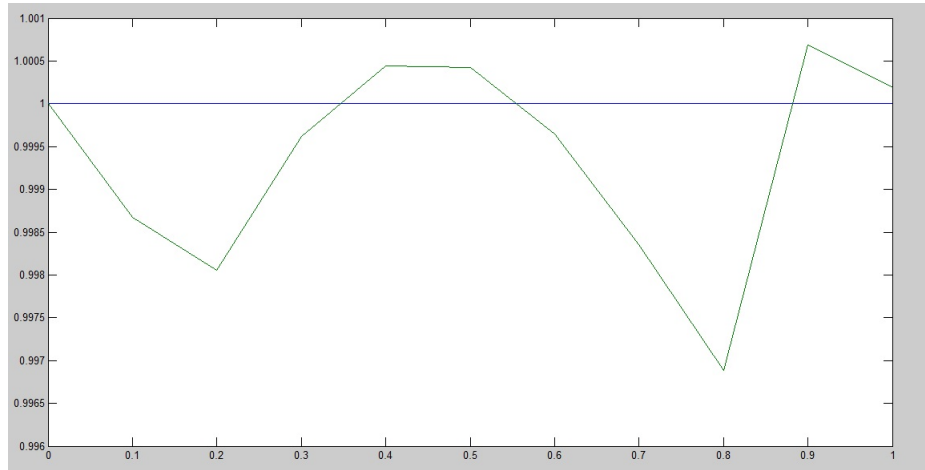


Figure 1: Graphs of the exact solution of example 1 (U_{ex} =Blue) and the approximated solution (U_{ap} =Green)for $N = 10$.

`N=100,uex=Bleu,Uap=Vert.jpg`

Figure 2: Graphs of the exact solution of example 1 (U_{ex} =Blue) and the approximated solution (U_{ap} =Green)for $N = 100$.



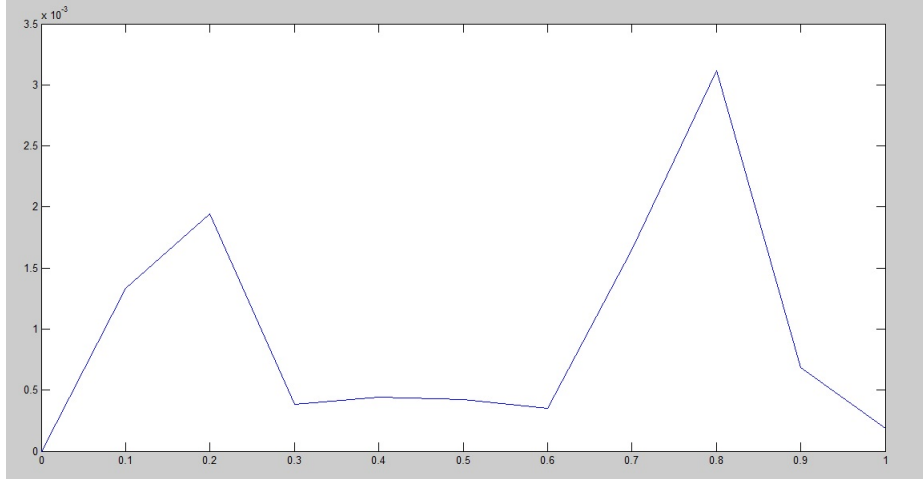


Figure 3: Error between the exact solution of the first example and its approximation for $N = 10$.

Additionally, in Table 1, we present a numerical comparison between our method and one based on Taylor's series expansion [20] for Example 1. Here, EMN_n denotes the error of our method (Modified Nyström method), while ETS_n represents the error of the Taylor's series expansion.

n	EMN_n	ETS_n
10	1.2E-3	9.1E-1
20	3.3E-4	7.9E-1
100	8.2E-5	3.2E-1
200	1.1E-6	8.1E-2

Table 2: Comparison between MN method and TS method : Example 1

Example 2:

In this example, we vary the delay term and we present the equation:

$$u(t) = \int_a^{-t^2+t} \frac{(t-s)}{1+u(s)^2+u'(s)^2} ds + 1 + 0.25(t^4 - t^2),$$

where the functions $N, f, \alpha(t)$, and u satisfy the outlined assumptions (C) and



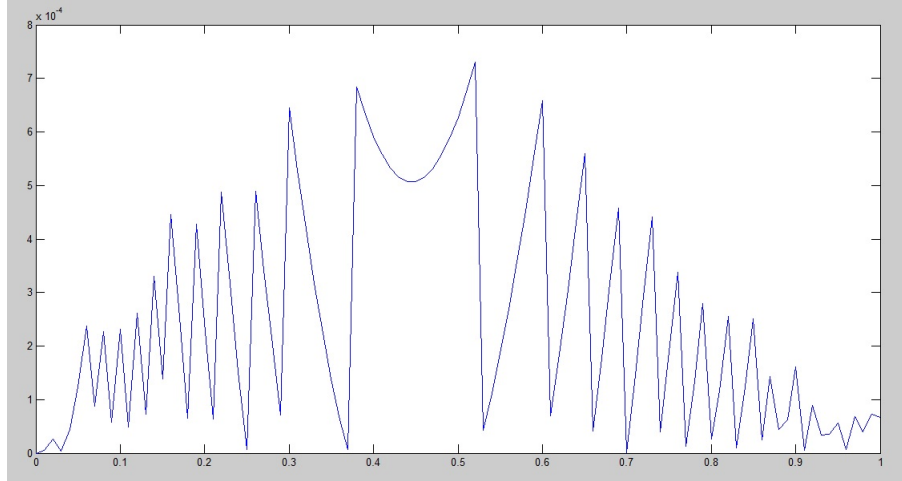


Figure 4: Error between the exact solution of the first example and its approximation for $N = 100$.

C_w). The corresponding components for this example are defined as:

$$\begin{aligned} N(t, s, x, y) &= \frac{(t-s)}{1+x^2+y^2}, \\ f(t) &= 1+0.25(t^4-t^2), \\ \alpha(t) &= -t^2+t, \\ u(t) &= 1. \end{aligned}$$

Once again, in the accompanying visual representations, we illustrate the exact solution, the approximate solution, and, of significant importance, the error between both solutions for various values of n .



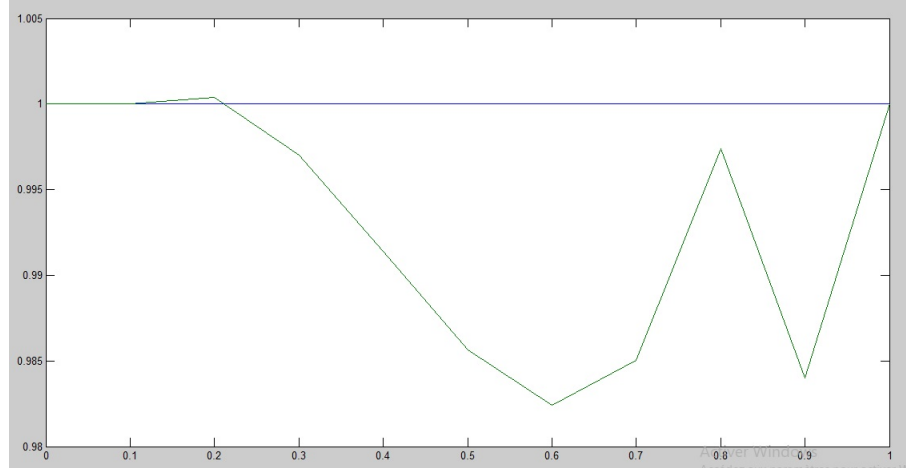


Figure 5: Graphs of the exact solution of example 2 (U_{ex} =Blue) and the approximated solution (U_{ap} =Green) for $N = 10$.

Furthermore, Table 2 illustrates the numerical comparison between our method and one based on Taylor's series expansion [20] for Example 2.

n	EMN_n	ETS_n
10	2.2E-3	8.9E-1
20	2.9E-4	7.7E-1
100	7.2E-5	4.2E-1
200	1.9E-6	7.9E-2

Table 3: Comparison between MN method and TS method : Example 2

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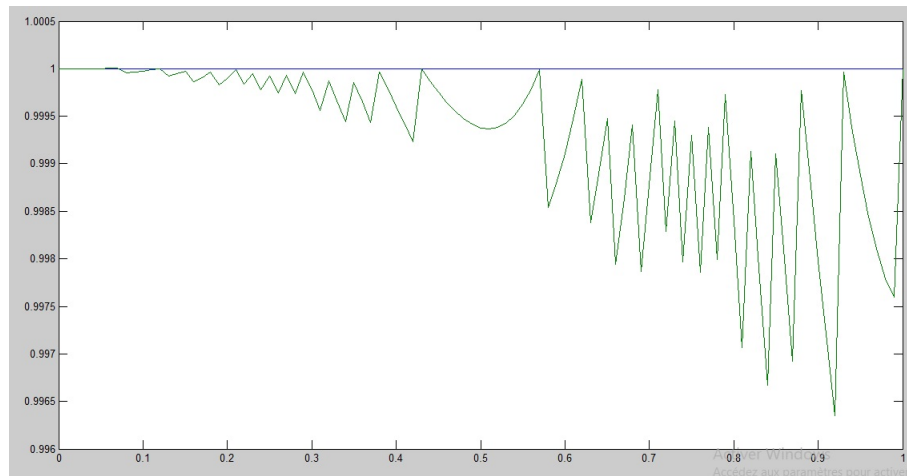


Figure 6: Graphs of the exact solution of example 2 (U_{ex} =Blue) and the approximated solution (U_{ap} =Green) for $N = 100$.

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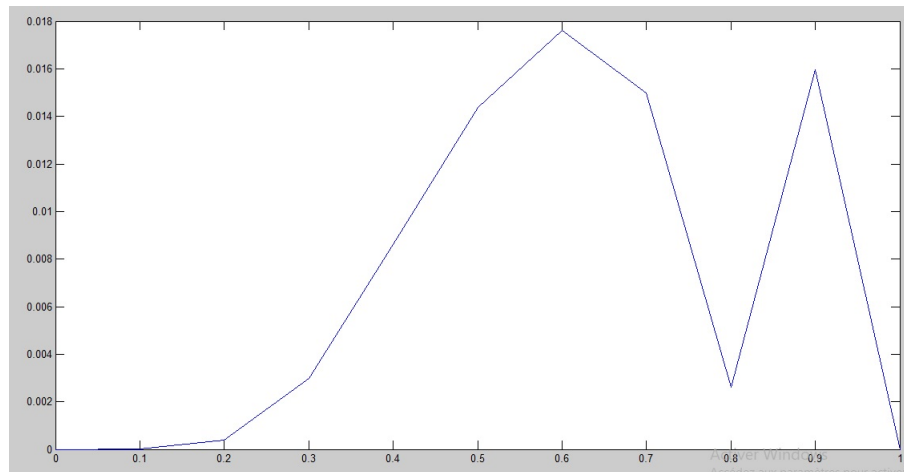


Figure 7: Error between the exact solution of the second example and its approximation for $N = 10$.

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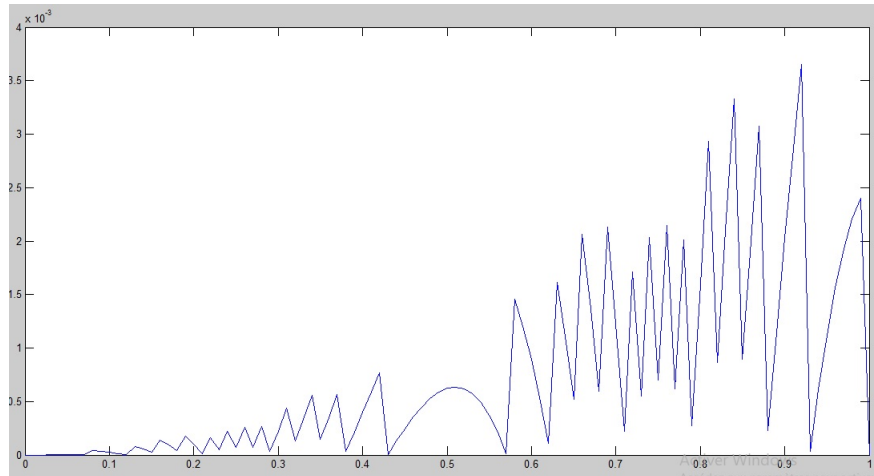


Figure 8: Error between the exact solution of the second example and its approximation for $N = 100$.

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WEAK TYPE ESTIMATE OF SINGULAR INTEGRAL OPERATORS ON VARIABLE WEAK HERZ-TYPE HARDY SPACES

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Abstract. This paper is concerned with boundedness properties of singular integral operators on variable homogeneous weak Herz spaces $WK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ and variable inhomogeneous weak Herz spaces $WK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$. Allowing our parameters to vary from point to point will raise extra difficulties which, in general, are overcome by imposing regularity assumptions on these exponents, either at the origin or at infinity. Our results cover the classical results on weak Herz-type Hardy spaces with fixed exponents.

Keywords: Herz Hardy space, weak Herz spaces, variable exponent, singular integral operators.



Introduction

It is well known that Herz spaces play an important role in Harmonic Analysis. After they have been introduced in [8], the theory of these spaces had a remarkable development in part due to its usefulness in applications. For instance, they appear in the characterization of multipliers on Hardy spaces [3], in the summability of Fourier transforms [7] and in regularity theory for elliptic equations in divergence form [16].

In recent years, there has been growing interest in generalizing classical spaces such as Lebesgue, Herz spaces and Sobolev spaces to the case with either variable integrability or variable smoothness. The motivation for the increasing interest in such spaces comes not only from theoretical purposes, but also from applications to fluid dynamics [17], image restoration [5] and PDE with non-standard growth conditions.

Herz spaces $K_{p(\cdot),q}^\alpha(\mathbb{R}^n)$ and $\dot{K}_{p(\cdot),q}^\alpha(\mathbb{R}^n)$ with variable exponent p but fixed $\alpha \in \mathbb{R}$ and q were recently studied by Izuki [10, 11]. These spaces with variable exponents $\alpha(\cdot)$ and $p(\cdot)$ were studied in [2], where they gave the boundedness results for a wide class of classical operators on these function spaces. The spaces $K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ and $\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$, were first introduced by Izuki and Noi in [9].

Recently, [20] introduced variable weak Herz spaces $W\dot{K}_{p(\cdot),q}^\alpha(\mathbb{R}^n)$ and variable weak Herz-type Hardy spaces $WH\dot{K}_{p(\cdot),q}^\alpha(\mathbb{R}^n)$, including the inhomogeneous version with variable p , but with fixed α and q , and gave the boundedness for a class of singular integral operators on such spaces. For fixed exponents this problem has been provided in [14].

Preliminaries

As usual, for $x \in \mathbb{R}^n$ and $r > 0$ we denote by $B(x, r)$ the open ball in $\mathbb{R}^n$ with center x . For any measurable set $A \subset \mathbb{R}^n$, then $|A|$ stands for the Lebesgue measure of A and χ_A denotes its characteristic. The expression $f \lesssim g$ means that $f \leq cg$ for some independent constant c (and non-negative functions f and g), and $f \approx g$ means $f \lesssim g \lesssim f$.

The set $\mathcal{P}_0(\mathbb{R}^n)$ consists of all measurable functions $p(\cdot) : \mathbb{R}^n \rightarrow [c, \infty[$ for some $c > 0$. The subset of variable exponents with range $[1, \infty)$ is denoted by $\mathcal{P}$. For $p \in \mathcal{P}(\mathbb{R}^n)$, we use the notation

$$p^+ = \operatorname{ess\,sup}_{x \in \mathbb{R}^n} p(x) \quad \text{and} \quad p^- = \operatorname{ess\,inf}_{x \in \mathbb{R}^n} p(x).$$

Let $p \in \mathcal{P}_0(\mathbb{R}^n)$. The *variable exponent Lebesgue space* $L^{p(\cdot)}(\mathbb{R}^n)$ is the class of all measurable functions f on $\mathbb{R}^n$ such that for some $\lambda > 0$ the modular

$$\varrho_{p(\cdot)}(f) = \int_{\mathbb{R}^n} |f(x)|^{p(x)} dx$$



is finite. This is a quasi-Banach function space equipped with the norm

$$\|f\|_{p(\cdot)} = \inf \left\{ \mu > 0 : \varrho_{p(\cdot)} \left(\frac{1}{\mu} f \right) \leq 1 \right\}.$$

If $p(x) \equiv p$ is constant, then $L^{p(\cdot)}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ is the classical Lebesgue space.

A useful property is that $\varrho_{p(\cdot)}(f) \leq 1$ if and only if $\|f\|_{p(\cdot)} \leq 1$ (*unit ball property*). This property is clear for constant exponents due to the obvious relation between the norm and the modular in that case.

We say that a function $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is *locally log-Hölder continuous*, if there exists a constant $c_{\log} > 0$ such that

$$|g(x) - g(y)| \leq \frac{c_{\log}}{\ln(e + 1/|x - y|)}$$

for all $x, y \in \mathbb{R}^n$. If

$$|g(x) - g(0)| \leq \frac{c_{\log}}{\ln(e + 1/|x|)}$$

for all $x \in \mathbb{R}^n$, then we say that g is *log-Hölder continuous at the origin* (or has a *log decay at the origin*). If, for some $g_{\infty} \in \mathbb{R}$ and $c_{\log} > 0$, there holds

$$|g(x) - g_{\infty}| \leq \frac{c_{\log}}{\ln(e + |x|)}$$

for all $x \in \mathbb{R}^n$, then we say that g is *log-Hölder continuous at infinity* (or has a *log decay at infinity*).

By $\mathcal{P}_0^{\ln}(\mathbb{R}^n)$ and $\mathcal{P}_{\infty}^{\ln}(\mathbb{R}^n)$ we denote the class of all exponents $p \in \mathcal{P}(\mathbb{R}^n)$ which have a log decay at the origin and at infinity, respectively.

The notation $\mathcal{P}^{\ln}(\mathbb{R}^n)$ is used for all those exponents $p \in \mathcal{P}(\mathbb{R}^n)$ which are locally log-Hölder continuous and have a log decay at infinity, with $p_{\infty} := \lim_{|x| \rightarrow \infty} p(x)$. Obviously we have $\mathcal{P}^{\ln}(\mathbb{R}^n) \subset \mathcal{P}_0^{\ln}(\mathbb{R}^n) \cap \mathcal{P}_{\infty}^{\ln}(\mathbb{R}^n)$. Note that $p \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ if and only if $p' \in \mathcal{P}^{\ln}(\mathbb{R}^n)$, and since $(p')_{\infty} = (p_{\infty})'$ we write only p'_{∞} for any of these quantities.

Let $p, q \in \mathcal{P}_0(\mathbb{R}^n)$. The mixed Lebesgue-sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$ is defined on sequences of $L^{p(\cdot)}$ -functions by the modular

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) = \sum_v \inf \left\{ \lambda_v > 0 : \varrho_{p(\cdot)} \left(\frac{f_v}{\lambda_v^{1/q(\cdot)}} \right) \leq 1 \right\}.$$

The (quasi)-norm is defined from this as usual:

$$\|(f_v)_v\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} = \inf \left\{ \mu > 0 : \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})} \left(\frac{1}{\mu} (f_v)_v \right) \leq 1 \right\}. \quad (0.57)$$

Since $q^+ < \infty$, then we can replace (0.57) by the simpler expression $\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) = \sum_v \| |f_v|^{q(\cdot)} \|_{\frac{p(\cdot)}{q(\cdot)}}$. Furthermore, if p and q are constants, then $\ell^{q(\cdot)}(L^{p(\cdot)}) = \ell^q(L^p)$. It is known, cf. [1] and [12], that $\ell^{q(\cdot)}(L^{p(\cdot)})$ is a norm if $q(\cdot) \geq 1$



is constant almost everywhere (a.e.) on $\mathbb{R}^n$ and $p(\cdot) \geq 1$, or if $\frac{1}{p(x)} + \frac{1}{q(x)} \leq 1$ a.e. on $\mathbb{R}^n$, or if $1 \leq q(x) \leq p(x) < \infty$ a.e. on $\mathbb{R}^n$.

We end this section by technical lemmas which will be used in the throughout this paper. The next lemma is a Hardy-type inequality which is easy to prove.

Lemma 0.1. *Let $0 < a < 1$ and $0 < q \leq \infty$. Let $\{\varepsilon_k\}_{k \in \mathbb{Z}}$ be a sequence of positive real numbers, such that*

$$\|\{\varepsilon_k\}_{k \in \mathbb{Z}}\|_{\ell^q} = I < \infty.$$

Then the sequences $\{\delta_k : \delta_k = \sum_{j \leq k} a^{k-j} \varepsilon_j\}_{k \in \mathbb{Z}}$ and $\{\eta_k : \eta_k = \sum_{j \geq k} a^{j-k} \varepsilon_j\}_{k \in \mathbb{Z}}$ belong to ℓ^q , and

$$\|\{\delta_k\}_{k \in \mathbb{Z}}\|_{\ell^q} + \|\{\eta_k\}_{k \in \mathbb{Z}}\|_{\ell^q} \leq cI,$$

with $c > 0$ only depending on a and q .

The proof of the following results are given in [2].

Lemma 0.2. *Let $\alpha \in L^\infty(\mathbb{R}^n)$ and $r_1 > 0$. If α is log-Hölder continuous both at the origin and at infinity, then*

$$r_1^{\alpha(x)} \lesssim r_2^{\alpha(y)} \times \begin{cases} \left(\frac{r_1}{r_2}\right)^{\alpha^+} & \text{if } 0 < r_2 \leq \frac{r_1}{2} \\ 1 & \text{if } \frac{r_1}{2} < r_2 \leq 2r_1 \\ \left(\frac{r_1}{r_2}\right)^{\alpha^-} & \text{if } r_2 > 2r_1 \end{cases}$$

for any $x \in B(0, r_1) \setminus B(0, \frac{r_1}{2})$ and $y \in B(0, r_2) \setminus B(0, \frac{r_2}{2})$, with the implicit constant not depending on x, y, r_1 and r_2 .

Lemma 0.3. *Let $p \in \mathcal{P}_\infty^{\text{ln}}(\mathbb{R}^n)$ and let $R = B(0, r) \setminus B(0, \frac{r}{2})$. If $|R| \geq 2^{-n}$, then*

$$\|\chi_R\|_{p(\cdot)} \approx |R|^{\frac{1}{p(x)}} \approx |R|^{\frac{1}{p_\infty}}$$

with the implicit constants independent of r and $x \in R$.

The left-hand side equivalence remains true for every $|R| > 0$ if we assume, additionally, $p \in \mathcal{P}_0^{\text{ln}}(\mathbb{R}^n) \cap \mathcal{P}_\infty^{\text{ln}}(\mathbb{R}^n)$.

Sublinear operators on variable weak Herz spaces

In this section, we present our first result of this paper. For convenience, we set

$$B_k = B(0, 2^k), \quad R_k = B_k \setminus B_{k-1} \quad \text{and} \quad \chi_k = \chi_{R_k}, \quad k \in \mathbb{Z}.$$

Now we present the definition of Herz spaces of variable smoothness and integrability were introduced and investigated in [6].



Definition 0.2. Let $p, q \in \mathcal{P}_0(\mathbb{R}^n)$ and $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$. The *inhomogeneous Herz space* $K_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ consists of all $f \in L_{oc}^{p(\cdot)}(\mathbb{R}^n)$ such that

$$\|f\|_{K_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} = \|f \chi_{B_0}\|_{p(\cdot)} + \left\| \left(2^{k\alpha(\cdot)} f \chi_k \right)_{k \geq 1} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty.$$

Similarly, the *homogeneous Herz space* $\dot{K}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ is defined as the set of all $f \in L_{oc}^{p(\cdot)}(\mathbb{R}^n \setminus \{0\})$ such that

$$\|f\|_{\dot{K}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} = \left\| \left(2^{k\alpha(\cdot)} f \chi_k \right)_{k \in \mathbb{Z}} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty.$$

If α and p, q are constant, then $K_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n) = K_{p, q}^\alpha(\mathbb{R}^n)$ and $\dot{K}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n) = \dot{K}_{p, q}^\alpha(\mathbb{R}^n)$ are the classical Herz spaces.

Let us denote

$$\|\{g_k\}\|_{\ell_{\geq}^q(L^{p(\cdot)})} = \left(\sum_{k=0}^{\infty} \|g_k\|_{p(\cdot)}^q \right)^{1/q} \quad \text{and} \quad \|\{g_k\}\|_{\ell_{<}^q(L^{p(\cdot)})} = \left(\sum_{k=-\infty}^{-1} \|g_k\|_{p(\cdot)}^q \right)^{1/q}$$

for sequences $\{g_k\}_{k \in \mathbb{Z}}$ of measurable functions (with the usual modification when $q = \infty$).

The following proposition plays a fundamental role in this paper.

Proposition 0.5. Let $\alpha \in L^\infty(\mathbb{R}^n)$, $p, q \in \mathcal{P}_0(\mathbb{R}^n)$. If α and q are log-Hölder continuous at infinity, then

$$K_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n) = K_{p(\cdot), q_\infty}^{\alpha_\infty}(\mathbb{R}^n).$$

Additionally, if α and q have a log decay at the origin, then

$$\|f\|_{\dot{K}_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} \approx \|\{2^{k\alpha(0)} f \chi_k\}\|_{\ell_{<}^{q(0)}(L^{p(\cdot)})} + \|\{2^{k\alpha_\infty} f \chi_k\}\|_{\ell_{\geq}^{q_\infty}(L^{p(\cdot)})}.$$

For the proof see [6, Proposition 1]. We refer the reader to the paper [18], where the continuous variable Herz spaces are studied.

Let $k \in \mathbb{Z}$ and $\lambda > 0$. We set $A_k(\lambda, f) = \{x \in R_k : |f(x)| > \lambda\}$ and $\tilde{A}_0(\lambda, f) = \{x \in B(0, 1) : |f(x)| > \lambda\}$.

Now we will give the definition of the weak Herz space.

Definition 0.3. Let $p, q \in \mathcal{P}_0(\mathbb{R}^n)$ and $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$. The *inhomogeneous weak Herz space* $WK_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ consists of all measurable functions f such that

$$\|f\|_{WK_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} = \sup_{\lambda > 0} \lambda \left\| \left(2^{k\alpha(\cdot)} \chi_{A_k(\lambda, f)} \right)_{k \geq 0} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty,$$

where A_0 is replaced by $\tilde{A}_0$. Similarly, the *homogeneous weak Herz space* $\dot{W}K_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ is defined as the set of all measurable functions f such that

$$\|f\|_{\dot{W}K_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} = \sup_{\lambda > 0} \lambda \left\| \left(2^{k\alpha(\cdot)} \chi_{A_k(\lambda, f)} \right)_{k \in \mathbb{Z}} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty.$$



If α, p and q are constant, then $WK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} = WK_{p,q}^{\alpha}$ and $WK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n) = WK_{p,q}^{\alpha}(\mathbb{R}^n)$ are the classical weak Herz spaces. Now we recall the variable weak $L^{p(\cdot)}(\mathbb{R}^n)$ spaces.

Definition 0.4. Let $p \in \mathcal{P}_0(\mathbb{R}^n)$. The weak Lebesgue space with variable exponent $L^{p(\cdot),\infty}(\mathbb{R}^n)$ consists of all Lebesgue measurable function f satisfying

$$\|f\|_{L^{p(\cdot),\infty}} = \sup_{\lambda>0} \lambda \|\chi_{\{x \in \mathbb{R}^n : |f(x)| > \lambda\}}\|_{p(\cdot)} < \infty.$$

It is obvious that if p is constant, then $WK_{p,p}^0(\mathbb{R}^n) = WK_{p,p}^0(\mathbb{R}^n) = L^{p,\infty}(\mathbb{R}^n)$. We present the main result of this section.

Proposition 0.6. Let $\alpha \in L^\infty(\mathbb{R}^n)$, $p, q \in \mathcal{P}_0(\mathbb{R}^n)$. If α and q are log-Hölder continuous at infinity, then

$$WK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n) = WK_{p(\cdot),q_\infty}^{\alpha_\infty}(\mathbb{R}^n).$$

Additionally, if α and q have a log decay at the origin, then

$$\|f\|_{WK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} \approx \sup_{\lambda>0} (\lambda \|\{2^{k\alpha(0)} \chi_{A_k}(\lambda, f)\}\|_{\ell_{<}^{q(0)}(L^{p(\cdot)})} + \lambda \|\{2^{k\alpha_\infty} \chi_{A_k}(\lambda, f)\}\|_{\ell_{>}^{q_\infty}(L^{p(\cdot)})}). \quad (0.58)$$

We consider sublinear operators satisfying the size condition

$$|Tf(x)| \lesssim \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n} dy, \quad x \notin \text{supp } f, \quad (0.59)$$

for integrable and compactly supported functions f . Condition (0.59) was first considered in [19] and it is satisfied by several classical operators in Harmonic Analysis, such as Calderón-Zygmund operators, the Carleson maximal operator and the Hardy-Littlewood maximal operator (see [19], [13]).

We have the following results:

Theorem 0.17. Let $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}$ and $q \in \mathcal{P}_0(\mathbb{R}^n)$ are log-Hölder continuous at infinity, $p \in \mathcal{P}_\infty^{\text{ln}}(\mathbb{R}^n)$, $0 < q^- \leq q^+ < 1$ and $1 < p^- \leq p^+ < \infty$ with

$$-\frac{n}{p_\infty} < \alpha_\infty < n\left(1 - \frac{1}{p_\infty}\right). \quad (0.60)$$

Suppose that T is a sublinear operator satisfying estimate (0.59). If T is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{p(\cdot),\infty}(\mathbb{R}^n)$, then T is bounded on from $WK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ to $WK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.

For homogeneous spaces, we have the following statement:

Theorem 0.18. Let $q \in \mathcal{P}_0(\mathbb{R}^n)$, $p \in \mathcal{P}_0^{\text{ln}}(\mathbb{R}^n) \cap \mathcal{P}_\infty^{\text{ln}}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$ and $0 < q^- \leq q^+ < 1$. Let $\alpha \in L^\infty(\mathbb{R}^n)$ and q are log-Hölder continuous, both at the origin and at infinity, such that

$$-\frac{n}{p^+} < \alpha^- \leq \alpha^+ < n\left(1 - \frac{1}{p^-}\right). \quad (0.61)$$



Then every sublinear operator T satisfying (0.59) which is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{p(\cdot),\infty}(\mathbb{R}^n)$ is also bounded from $\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ to $W\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.

Remark 0.2. Theorems 0.17 and 0.18 are proved in [20] in the case of constant $\alpha > 0$. In [20] the author has assumed the boundedness of the maximal operator $\mathcal{M}$ on $L^{p(\cdot)}(\mathbb{R}^n)$ both in the homogeneous and in the inhomogeneous case. Although we assume the slightly stronger assumption $p \in \mathcal{P}_0^{\text{ln}}(\mathbb{R}^n) \cap \mathcal{P}_\infty^{\text{ln}}(\mathbb{R}^n)$ in the homogeneous case, we stress that only the log-decay of p is assumed when we deal with inhomogeneous spaces. In the classical Herz spaces Theorems 0.17 and 0.18 can be found in [14]. We refer the reader to [15] for the boundedness of Riesz potential operator in continual variable exponents Herz spaces.

Conclusion

In this work, we presented the most important basic properties of variable exponent spaces and the Mixed variable Lebesgue's-sequence spaces. We also gave some key technical lemmas needed to prove the main statements. We also introduced weak Herz spaces $W\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ with variable exponents including the inhomogeneous version, and we presented an equivalent quasi-norm of such spaces. Finally, based on these equivalent quasi-norms and the results of [6], we proved the boundedness properties of sublinear operators on $W\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ and $WK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.

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Existence and Uniqueness Study of the Conformable Laplace Transform with Applications

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Abstract. This work tackles the topic of conformable Laplace transform. The authors aim at discussing its existence by exploring and providing the kind of functions that possess a conformable Laplace transform. Furthermore, the comparison theorem of conformable improper integrals is presented to further explain and justify the existence of conformable Laplace transform for some functions. The uniqueness is also established in order to determine the inverse of conformable Laplace transform for functions. Moreover, we present a table of the conformable Laplace transform of the usual functions. Finally, as an application, we use the conformable Laplace transform for solving some fractional differential equations.

Keywords: Conformable Derivative 1, Conformable Integral 2, Conformable Laplace Transform 3, Fractional Differential Equation 4.

Preliminaries

In 2015, the conformable Laplace transform was introduced by T. Abdeljawad in his paper entitled (On conformable fractional calculus) [2].



Definition 0.5. Let $y : [0, \infty) \rightarrow \mathbb{R}$ be a real valued function and $0 < \gamma \leq 1$. Then the conformable Laplace transform of y is defined as:

$$\mathfrak{L}_\gamma \{y(t)\} = \mathcal{Y}_\gamma(\zeta) = \int_0^\infty y(t) e^{-\zeta \frac{t^\gamma}{\gamma}} d_\gamma t = \int_0^\infty y(t) e^{-\zeta \frac{t^\gamma}{\gamma}} t^{\gamma-1} dt,$$

provided the integral exists.

One of the nice results is the relation between the usual and the conformable Laplace transforms, which is given in the theorem below.

Theorem 0.19. Let $y : [0, \infty) \rightarrow \mathbb{R}$ be a function such that $\mathfrak{L}_\gamma \{y(t)\}(\zeta) = \mathcal{Y}_\gamma(\zeta)$ exists. Then

$$\mathfrak{L}_\gamma \{y(t)\}(\zeta) = \mathcal{Y}_\gamma(\zeta) = \mathfrak{L} \left\{ y \left((\gamma t)^{\frac{1}{\gamma}} \right) \right\}(\zeta), \quad 0 < \gamma \leq 1,$$

where $\mathfrak{L}$ is the usual Laplace transform.

Theorem 0.20. Let $y : [0, \infty) \rightarrow \mathbb{R}$, $z : [0, \infty) \rightarrow \mathbb{R}$ and let $\lambda, \mu, a \in \mathbb{R}$ and $0 < \gamma \leq 1$. Then

- (1) $\mathfrak{L}_\gamma \{ \lambda y(t) + \mu z(t) \} = \lambda \mathcal{Y}_\gamma(\zeta) + \mu \mathcal{Z}_\gamma(\zeta), \zeta > 0.$
- (2) $\mathfrak{L}_\gamma \left\{ e^{-a \frac{t^\gamma}{\gamma}} y(t) \right\}(\zeta) = \mathcal{Y}_\gamma(\zeta + a), \zeta > |a|.$
- (3) $\mathfrak{L}_\gamma \{ I^\gamma y(t) \}(\zeta) = \frac{\mathcal{Y}_\gamma(\zeta)}{\zeta}, \zeta > 0.$
- (4) $\mathfrak{L}_\gamma \left\{ \frac{t^{n\gamma}}{\gamma^n} y(t) \right\}(\zeta) = (-1)^n \frac{d^n}{d\zeta^n} \mathcal{Y}_\gamma(\zeta), \zeta > 0.$
- (5) $\mathfrak{L}_\gamma \{ (y * z)(t) \} = \mathcal{Y}_\gamma(\zeta) \mathcal{Z}_\gamma(\zeta), \zeta > 0.$

where $\mathcal{Y}_\gamma$ and $\mathcal{Z}_\gamma$ are the conformable Laplace transforms of the functions y and z respectively, $y * z$ is the convolution product of y and z and $I^\gamma y(t)$ is the conformable integral.

Existence and Uniqueness

When we go back to Definition 1.1, we notice that the integral defining a conformable Laplace transform represents an improper integral. The latter have the possibility of converging or diverging, based on the integral. The function $y(t)$ is said to have a conformable Laplace transform if the improper integral converge. This leads us to wonder about the types of functions which have conformable Laplace transforms.

Example 0.1. We need to investigate the existence of conformable Laplace transform of the following functions :

$$(a) y(t) = e^{a \frac{t^\gamma}{\gamma}}, \quad (b) y(t) = 1, \quad (c) y(t) = \frac{t^\gamma}{\gamma}.$$



(a) First, using Definition 1., we see that

$$\begin{aligned}\mathfrak{L}_\gamma \left[e^{a \frac{t^\gamma}{\gamma}} \right] &= \int_0^\infty e^{a \frac{t^\gamma}{\gamma}} e^{-\zeta \frac{t^\gamma}{\gamma}} d_\gamma t = \int_0^\infty e^{(a-\zeta) \frac{t^\gamma}{\gamma}} t^{\gamma-1} dt = \lim_{T \rightarrow \infty} \int_0^T e^{(a-\zeta) \frac{t^\gamma}{\gamma}} t^{\gamma-1} dt \\ &= \lim_{T \rightarrow \infty} \left[\frac{1}{(a-\zeta)} e^{(a-\zeta) \frac{t^\gamma}{\gamma}} \right]_0^T = \begin{cases} \frac{T^\gamma}{\gamma}, & \text{if } \zeta = a \\ \frac{1}{(\zeta-a)} + \frac{1}{(\zeta-a)} e^{-(\zeta-a) \frac{T^\gamma}{\gamma}}, & \text{otherwise.} \end{cases}\end{aligned}$$

For the convergence of the improper integral $\zeta > a$ is needed, in this case,

$$\mathfrak{L}_\gamma \left[e^{a \frac{t^\gamma}{\gamma}} \right] = \frac{1}{(\zeta - a)}, \quad \zeta > a.$$

(b) Similarly to (a), we obtain

$$\mathfrak{L}_\gamma [1] = \int_0^\infty e^{-\zeta \frac{t^\gamma}{\gamma}} d_\gamma t = \lim_{T \rightarrow \infty} \int_0^T e^{-\zeta \frac{t^\gamma}{\gamma}} t^{\gamma-1} dt = \lim_{T \rightarrow \infty} \left[-\frac{1}{\zeta} e^{-\zeta \frac{t^\gamma}{\gamma}} \right]_0^T = \frac{1}{\zeta}, \quad \zeta > 0.$$

(c) We have $\mathfrak{L}_\gamma \left[\frac{t^\gamma}{\gamma} \right] = \int_0^\infty \frac{t^\gamma}{\gamma} e^{-\zeta \frac{t^\gamma}{\gamma}} d_\gamma t = \int_0^\infty \frac{t^\gamma}{\gamma} (t^{\gamma-1} e^{-\zeta \frac{t^\gamma}{\gamma}}) dt$.

By integration by part we get $\mathfrak{L}_\gamma \left[\frac{t^\gamma}{\gamma} \right] = \frac{1}{\zeta^2}, \quad \zeta > 0$.

To be presised, if $y(t)$ is a continuous function subject to where $M \geq 0$ and a and C are constants, then this condition yields

$$\int_0^\infty y(t) e^{-\zeta \frac{t^\gamma}{\gamma}} d_\gamma t \leq \int_0^C y(t) e^{-\zeta \frac{t^\gamma}{\gamma}} d_\gamma t + M \int_C^\infty e^{-(\zeta-a) \frac{t^\gamma}{\gamma}} d_\gamma t.$$

Since $y(t)$ is continuous in $0 \leq t \leq C$, by letting $A = \max\{|y(t)| : 0 \leq t \leq C\}$ we have

$$\int_0^C y(t) e^{-\zeta \frac{t^\gamma}{\gamma}} d_\gamma t \leq A \int_0^C e^{-\zeta \frac{t^\gamma}{\gamma}} d_\gamma t = A \left(\frac{1}{\zeta} - \frac{e^{-\zeta \frac{C^\gamma}{\gamma}}}{\zeta} \right) < \infty.$$

On the other hand, by Example 4.(a), the integral $\int_0^\infty e^{-(\zeta-a) \frac{t^\gamma}{\gamma}} d_\gamma t$ is convergent for $\zeta > a$. By the comparison theorem of conformable improper integrals (see Theorem 9. below), the integral on the left is also convergent. That is, $y(t)$ have a conformable Laplace transform.

Definition 0.6. We say that a continuous function $y(t)$ has a conformable exponential order at infinity if it satisfies the following condition

$$|y(t)| \leq M e^{a \frac{t^\gamma}{\gamma}}, \quad t \geq C,$$

where $M \geq 0$ and a and C are constants.

Theorem 0.21. (Existence)

Let $y(t)$ be a piecewise continuous on $t \geq 0$ and has a conformable exponential



order at infinity with $|y(t)| \leq Me^{a\frac{t^\gamma}{\gamma}}$ for $t \geq C$. Hence the conformable Laplace transform

$$\mathfrak{L}_\gamma \{y(t)\} = \mathcal{Y}_\gamma(\zeta) = \int_0^\infty e^{-\zeta \frac{t^\gamma}{\gamma}} y(t) d_\gamma t = \int_0^\infty e^{-\zeta \frac{t^\gamma}{\gamma}} y(t) t^{\gamma-1} dt$$

exists as long as $\zeta > a$.

Proof: The integral in the definition of $\mathfrak{L}_\gamma \{y(t)\}$ can be splitted into two integrals as follows

$$\int_0^\infty e^{-\zeta \frac{t^\gamma}{\gamma}} y(t) d_\gamma t = \int_0^C e^{-\zeta \frac{t^\gamma}{\gamma}} y(t) d_\gamma t + \int_C^\infty e^{-\zeta \frac{t^\gamma}{\gamma}} y(t) d_\gamma t.$$

Since $y(t)$ is piecewise continuous in $0 \leq t \leq C$, it is bounded there. By letting $A = \max\{|y(t)| : 0 \leq t \leq C\}$ we have

$$\int_0^C y(t) e^{-\zeta \frac{t^\gamma}{\gamma}} d_\gamma t \leq A \int_0^C e^{-\zeta \frac{t^\gamma}{\gamma}} d_\gamma t = A \left(\frac{1}{\zeta} - \frac{e^{-\zeta \frac{C^\gamma}{\gamma}}}{\zeta} \right) < \infty.$$

Now, from Example 2.1(a), the integral $\int_0^C y(t) e^{-\zeta \frac{t^\gamma}{\gamma}} d_\gamma t$ is convergent for $\zeta > a$. By Comparison theorem of conformable improper integrals, the integral on the left is also convergent. That is, $y(t)$ have a conformable Laplace transform.

Theorem 0.22. (Uniqueness)

Let $y(t)$ and $z(t)$ are two piecewise continuous functions with conformable exponential order at infinity with conformable Laplace transforms $Y_\gamma(\zeta)$ and $Z_\gamma(\zeta)$ such that $Y_\gamma(\zeta) = Z_\gamma(\zeta)$ for some $\zeta > a$. Then $y(t) = z(t)$ for all $t \geq 0$ where both functions are continuous.

For a piecewise continuous function y of conformable exponential order at infinity whose conformable Laplace transform is Y_γ , we call y the inverse conformable Laplace transform of Y_γ and write $y = \mathfrak{L}_\gamma^{-1}[Y(\zeta)]$. Symbolically,

$$y(t) = \mathcal{L}_\gamma^{-1}[Y_\gamma(\zeta)] \iff Y_\gamma(\zeta) = \mathcal{L}_\gamma[y(t)].$$

Application

In order to study the solution of the most difficult problems, like a non-homogeneous fractional differential equation with variables coefficients and fractional initial value problems; we are going to explore the following theorems :

Theorem 0.23. Let $y : [0, \infty) \rightarrow \mathbb{R}$ be a continuous real valued differentiable function and $0 < \gamma \leq 1$. Then

$$\mathfrak{L}_\gamma \{D^\gamma y(t)\} = \zeta y_\gamma(\zeta) - y(0), \quad \zeta > 0.$$

$$\mathfrak{L}_\gamma \{D^{(2\gamma)} y(t)\} = \zeta^2 y_\gamma(\zeta) - y^{(\gamma)}(0) - \zeta y(0), \quad \zeta > 0.$$



Example 0.2. Consider the following conformable differential problem :

$$y^{(2\gamma)}(t) + \frac{t^\gamma}{\gamma} y^{(\gamma)}(t) - 2y(t) = 4, \quad y^{(\gamma)}(0) = y(0) = 0$$

Let us solve such a problem using conformable Laplace transform.

$$\mathcal{L}_\gamma \left\{ y^{(2\gamma)}(t) \right\} + \mathcal{L}_\gamma \left\{ \frac{t^\gamma}{\gamma} y^{(\gamma)}(t) \right\} - \mathcal{L}_\gamma \{ 2y(t) \} = \mathcal{L}_\gamma \{ 4 \} \quad (1)$$

Let $\mathcal{L}_\gamma \{ y(t) \} = y_\gamma(\zeta)$. Then equation (1) is reduced to $\zeta^2 y_\gamma(\zeta) - \frac{d}{d\zeta} [\zeta y_\gamma(\zeta)] - 2y_\gamma(\zeta) = \frac{4}{\zeta^3}$,
 which can be simplified to

$$y'_\gamma(\zeta) + \left(\frac{3}{\zeta} - \zeta \right) y_\gamma(\zeta) = -\frac{4}{\zeta^2} \quad (2)$$

Since $\int \zeta^3 e^{-\frac{\zeta^2}{2}} d\zeta = \zeta^3 e^{-\frac{\zeta^2}{2}}$, the solution of equation (2) is given by

$$y_\gamma(\zeta) = \frac{e^{\frac{\zeta^2}{2}}}{\zeta^3} \left[\int \zeta^3 e^{-\frac{\zeta^2}{2}} \frac{-4}{\zeta^2} d\zeta + c \right] = \frac{e^{\frac{\zeta^2}{2}}}{\zeta^3} \left[4e^{-\frac{\zeta^2}{2}} + c \right].$$

Hence

$$y_\gamma(\zeta) = \frac{4}{\zeta^3} + \frac{c}{\zeta^3} e^{\frac{\zeta^2}{2}}, \quad c \text{ is constant} \quad (3)$$

since $\lim_{\zeta \rightarrow \infty} y_\gamma(\zeta) = 0$, the constant c in the second term should be zero, i.e.,

$$y_\gamma(\zeta) = \frac{4}{\zeta^3} \quad (4)$$

Applying the inverse conformable Laplace transform to the both sides of equation (4) we obtain

$$y(x) = \mathcal{L}_\gamma^{-1} \left\{ \frac{4}{\zeta^3} \right\} = \frac{2}{\gamma^2} t^{2\gamma}.$$

Hence a result as required.

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UHRS for Darboux problem

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Abstract. In this paper, we investigate the Darboux problem of conformable partial differential equations (DPCDEs) using fixed point theory. We focus on the existence and Ulam–Hyers–Rassias stability (UHRS) of the solutions to the problem, which requires finding solutions to nonlinear partial differential equations that satisfy certain boundary conditions.

Keywords: Ulam–Hyers–Rassias stability, Darboux problem, generalized conformable derivative.

Introduction

Our article aims to generalize the UHRS for PCDE. The main contributions of our work can be summarized as follows:



1. Existence and Uniqueness of Solution: We provide a rigorous proof of the existence and uniqueness of the solution for DPCDE.
2. UHRS of DPCDE: Our study delves into the UHRS of DPCDE. We explore the behavior and stability characteristics of solutions to DPCDE under perturbations, taking into account the principles and methodologies established in the UHRS framework.

Main results

In this part, we present the definitions of UHR and proceed to showcase our main results.

Definition 0.7. Let $\phi : [w, d) \longrightarrow \mathbb{R}$. The generalized conformable derivative of ϕ is defined by

$$T_w^{\delta, \psi} \phi(y) = \lim_{\sigma \rightarrow 0} \frac{\phi(y + \sigma \psi_w(y, \delta)) - \phi(y)}{\sigma}, \quad (0.62)$$

for every $y > w$, where $\delta \in (0, 1)$ and $\psi_w(y, \delta)$ is continuous and nonnegative with

$$\psi_w(y, 1) = 1,$$

$$\psi_w(., \delta_1) \neq \psi_w(., \delta_2), \text{ where } \delta_1 \neq \delta_2 \text{ and } \delta_1, \delta_2 \in (0, 1].$$

If $T_w^{\delta, \psi} \phi(y)$ exists, for every $y \in (w, a)$, for some $a > w$ and $\lim_{y \rightarrow w^+} T_w^{\delta, \psi} \phi(y)$ exists, therefore

$$T_w^{\delta, \psi} \phi(w) := \lim_{y \rightarrow w^+} T_w^{\delta, \psi} \phi(y).$$

Theorem 0.24. Suppose that $\mathcal{H}_1$ holds. If $V \in AC(J, \mathbb{R})$ satisfies

$$\left| T_{c_1}^{\theta_1, \psi_{c_1}} T_{c_2}^{\theta_2, \psi_{c_2}} V(\lambda_1, \lambda_2) - f(\lambda_1, \lambda_2, V(\lambda_1, \lambda_2)) \right| \leq \epsilon \pi(\lambda_1, \lambda_2), \quad (0.63)$$

$\forall (\lambda_1, \lambda_2) \in J$, where $\epsilon > 0$ and $\pi \in C(J, \mathbb{R})$ is nondecreasing w.r.t λ_1 and λ_2 , then there is a unique solution U^* of (??) such that

$$|V(\lambda_1, \lambda_2) - U^*(\lambda_1, \lambda_2)| \leq \epsilon \frac{\bar{K} + \varrho}{\varrho} \int_{c_1}^{d_1} \frac{ds_1}{\psi_{c_1}(s_1, \theta_1)} \int_{c_2}^{d_2} \frac{ds_2}{\psi_{c_2}(s_2, \theta_2)} \beta(d_1, d_2) \pi(\lambda_1, \lambda_2), \forall (\lambda_1, \lambda_2) \in J,$$

for any positive constant ϱ , where

$$\beta(\lambda_1, \lambda_2) = \mathbb{E}_{\theta_1}^{\psi_{c_1}} \left(\sqrt{\bar{K} + \varrho}, \lambda_1, c_1 \right) \times \mathbb{E}_{\theta_2}^{\psi_{c_2}} \left(\sqrt{\bar{K} + \varrho}, \lambda_2, c_2 \right).$$



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Theoretical Study of Decay Solution in Euler-Bernoulli Equation With a Source Term

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Abstract. This study focuses on a viscoelastic Euler-Bernoulli equation with a logarithmic source term in a bounded domain $\Omega \subset \mathbb{R}^2$. By selecting an appropriate relaxation function, we analyze the asymptotic behavior of the system using a suitable Lyapunov functional and properties of convex functions.

Keywords: Logarithmic nonlinearity, Decay solution, Convex function, Viscoelasticity; Relaxation function.



Introduction

It is well known that viscoelastic materials have memory effects, which is due to the mechanical response influenced by the history of the materials themselves. As these materials have a wide application in the natural sciences, their dynamics are interesting and of great importance. From the mathematical point of view, their memory effects are modeled by integro-differential equations. Hence, questions related to the behavior of the solutions for the PDE system have attracted considerable attention in recent years. In the present work, Ω be an open and bounded subset of $\mathbb{R}^2$ with a smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$, where $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$. Denote $n = (\eta_1, \eta_2)$ as the unit outward normal to Γ and $\tau = (-\eta_2, \eta_1)$ as the unit tangent vector positively oriented along $\partial\Omega$, with $\text{meas}(\Gamma_0) > 0$. We then consider the following physical model, which describes the transverse displacement of a thin vibrating plate subject to internal viscoelastic damping:

$$(P) \begin{cases} u_{tt} + \Delta^2 u - \int_0^t g(t-s) \Delta^2 u(s) ds + u = ku \ln(|u|), & \text{in } \Omega \times (0, \infty), \\ u(x, t) = 0, & \text{on } \Gamma_0 \times (0, \infty), \\ \mathcal{B}_1(u) - \mathcal{B}_1 \left(\int_0^t g(t-s) u(s) ds \right) = 0, & \text{on } \Gamma_1 \times (0, \infty), \\ \mathcal{B}_2(u) - \mathcal{B}_2 \left(\int_0^t g(t-s) u(s) ds \right) = 0, & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & x \in \Omega. \end{cases}$$

In this new model, k represent a small positive number. The integral terms in $(P)_3$ and $(P)_4$ represent memory effects responsible for the viscoelastic damping, where g is assumed to be a positive and non-increasing function. The differential operators

$$\mathcal{B}_1 u = \Delta u + (1 - \nu) B_1 u, \text{ and } \mathcal{B}_2 u = \frac{\partial \Delta u}{\partial n} + (1 - \nu) \frac{\partial B_2 u}{\partial \tau}$$

where ν (with $0 < \nu < \frac{1}{2}$) denotes Poisson's ratio and

$$B_1 u = 2\eta_1 \eta_2 \frac{\partial^2 u}{\partial x \partial y} + \eta_1^2 \frac{\partial^2 u}{\partial x^2} + \eta_2^2 \frac{\partial^2 u}{\partial y^2}, \quad B_2 u = (\eta_1^2 - \eta_2^2) \frac{\partial^2 u}{\partial x \partial y} + \eta_1 \eta_2 \left(\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2} \right),$$

Our aim in this work is to obtain a general relation between the decay rate of the energy and that of the relaxation function g without imposing restrictive assumptions on the behavior of g at infinity.

Main results

To state and prove our main result we first introduce the following:

(H1) The kernel $g \in C^1(\mathbb{R}_+)$ satisfying:

$$1 - \int_0^{+\infty} g(s) ds = l > 0, \text{ and } g(0) > 0.$$



(H2) The function $H \in C^1(\mathbb{R}_+)$, is strictly increasing and strictly convex or linear in $C^2((0, r]; \mathbb{R}_+)$, ($r < 1$) such that

$$H^{(i)}(0) = 0, \quad \forall i \in \{0, 1\} \quad \text{and} \quad g'(t) \leq -\xi(t) H(g(t)),$$

where the function $g(t)$ is positive and nonincreasing.

We introduce the following functionals

$$I(t, u) = \left(1 - \int_0^t g(s) ds\right) a(u, u) + (g \star u)(t) + \|u(t)\|_{L^2(\Omega)}^2 - k \int_{\Omega} |u|^2 \ln |u| d\Omega, \quad (0.64)$$

where

$$(g \star u)(t) = \int_0^t g(t-s) a(u(t) - u(s), u(t) - u(s)) ds,$$

and

$$J(t, u) = \frac{1}{2} I(t, u) + \frac{k}{4} \|u(t)\|_{L^2(\Omega)}^2, \quad (0.65)$$

then

$$E(t, u) = J(t, u) + \frac{1}{2} \|u_t(t)\|_{L^2(\Omega)}^2, \quad (0.66)$$

where $E(t, u)$ represents the energy function, and

$$a(u, v) = \int_{\Omega} \Delta u \Delta v dx - (1 - \nu) \int_{\Omega} \left\{ \frac{\partial^2 u}{\partial x_1^2} \frac{\partial^2 v}{\partial x_1^2} - 2 \frac{\partial^2 u}{\partial x_1 \partial x_2} \frac{\partial^2 v}{\partial x_1 \partial x_2} + \frac{\partial^2 u}{\partial x_2^2} \frac{\partial^2 v}{\partial x_2^2} \right\} dx.$$

is a bilinear form on $V = \{u \in H^2(\Omega) : u = 0 \text{ on } \Gamma_0\}$.

Theorem 0.25. Consider $(u_0, u_1) \in V \times L^2(\Omega)$ and $\eta \in (0, 1)$. Assuming that condition (A) is met. Then there exist four positive constants k'_0, k'_1 and k'_2 such that the solution of (P) satisfies

$$E(t) \leq k'_2 t^{\frac{1}{1+\epsilon}} H_1^{-1} \left(\frac{k'_1}{t^{\frac{1}{1+\epsilon}} \int_{t_2}^t \mu(s) ds} \right), \quad \forall t \geq t_2, \quad (0.67)$$

provided that $H_1(t) = t H'_0(\varepsilon_0 t)$ and $H_0 = \left([\overline{H}^{-1}]^{\frac{1}{1+\epsilon}} \right)^{-1}$.

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Modélisation de l'activité sismique : une étude théorique et numérique des systèmes de Volterra non linéaires

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Abstract. Dans cette recherche, nous introduisons une approche scientifique pionnière pour comprendre l'activité sismique, en intégrant le concept de résistance variable. Notre investigation se concentre sur un modèle complet de tremblement de terre défini par un système d'équations intégral-différentielles non linéaires de Volterra. La solution de ce modèle est étroitement liée aux facteurs de vitesse et de résistance. Sous un ensemble de conditions raisonnables, nous démontrons l'existence d'une solution unique. En utilisant la méthode de Nyström, nous réalisons une série de tests numériques qui valident la praticité et l'efficacité de notre étude, offrant une compréhension approfondie des complexités de ce phénomène naturel.

Keywords: Équation de Volterra non linéaire, Intégral-différentielle, Tremblements de terre, Point fixe



Introduction

Dans leur étude, Salah et al.[1] ont présenté un modèle physique déterministe complet expliquant les phénomènes sismiques. Ce modèle reflète l'engagement des auteurs à améliorer notre compréhension des processus sismiques, en s'inspirant du modèle de démonstration des Instituts de Recherche Intégrés en Sismologie (IRIS) (www.iris.edu). Leur réalisation dans le développement d'approximations numériques étroitement alignées avec les observations empiriques est remarquable.

Dans ce travail, nous considérons un modèle comprenant deux oscillateurs masse-ressort couplés. Le premier oscillateur inclut deux masses, M_1 et m_1 , reliées par un ressort de longueur naturelle L_1 lorsque $t = 0$. De même, le second oscillateur comprend deux masses, M_2 et m_2 , reliées par un ressort de longueur naturelle L_2 lorsque $t = 0$. Les deux oscillateurs sont libres de se mouvoir. L'objectif est de concevoir un système cohérent qui conserve les propriétés essentielles du phénomène en question.

Résultat Numérique

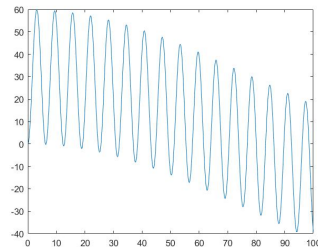


Figure 9: Fonction Sismique $u_1(t)$

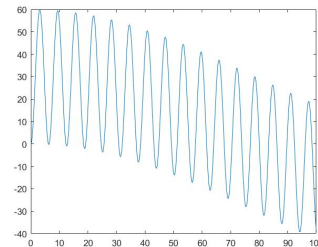


Figure 10: Fonction Sismique $u_2(t)$

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Periodic orbits of a fourth dimensional hyperchaotic system

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Abstract. We study the zero-Hopf bifurcation of a modified hyperchaotic Chen system

$$\begin{aligned}\dot{x} &= \alpha(a_1y - a_2x + w), \\ \dot{y} &= dx + cy - xz, \\ \dot{z} &= xy - bz, \\ \dot{w} &= yz + rw,\end{aligned}$$

where $\alpha, a_1, a_2, b, c, d,$ and r are real arbitrary parameters. The dot denotes the derivative with respect to an independent variable t . Using the averaging theory, we prove analytically the existence of two periodic orbits for this system that born from a zero-Hopf bifurcation.

Keywords: Hyperchaotic Chen system, periodic orbit, averaging theory, zero-Hopf bifurcation.



Introduction

Chaos has been broadly investigated in mathematics and physics since the discovery of the first chaotic attractor in a third-order autonomous system by Lorenz in 1963. Through this investigation, hyperchaos also became a center of interest when in 1979 Otto Rossler proposed one of the first hyperchaotic attractors. A hyperchaotic system is typically defined as a chaotic system with at least two positive Lyapunov exponents, implying that its chaotic dynamics are extended in many different directions simultaneously. This system is more complex than the traditional chaotic systems, and with such complexity, hyperchaos became a feature in enhancing security in chaotic communication systems. Also, it can be useful in encryption, fluid mixing, nonlinear circuits, and many other fields. Furthermore, The minimal dimension for a system to be hyperchaotic is four. Hyperchaotic systems such as the hyperchaotic Lorenz-Hakan system, the hyperchaotic attractor of Rossler, the hyperchaotic Lü system, and Chua's circuit are known systems that are extended from third-order chaotic systems, which is a typical way of constructing hyperchaotic systems.

We consider a modified hyperchaotic Chen system

$$\begin{aligned}\dot{x} &= \alpha(a_1y - a_2x + w), \\ \dot{y} &= dx + cy - xz, \\ \dot{z} &= xy - bz, \\ \dot{w} &= yz + rw,\end{aligned}$$

in which we will prove analytically the existence of periodic orbits bifurcating from the zero-Hopf equilibrium located at the origin of the system, by using the averaging method of first order.

Main results

The main result of this work is summed up in the following theorem.

Theorem 0.26. *Let $(r, b) \rightarrow (\epsilon r, \epsilon b)$ where ϵ is a small parameter and $c = \alpha a_2$. Then, the hyperchaotic Chen system (1) has a zero-Hopf bifurcation at the equilibrium point localized at the origin. Assume that $\alpha(a_2^2\alpha + a_1d) < 0$ and $\alpha a_2rb > 0$. Then, when $\epsilon = 0$ the system exhibits two periodic orbits bifurcating from the zero-Hopf equilibria as parameters vary.*

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Existence and uniqueness of solutions for the ψ - Caputo sequential fractional order neutral functional differential equations

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Abstract. This work is concerned with the existence and uniqueness of solutions to the following initial value problem (IVP) of ψ -Caputo sequential fractional order neutral functional differential equations . To obtain the desired results, we employ the Banach fixed point theorem , a nonlinear variation of the Leray-Schauder fixed point theorem , and the Krasnoselski fixed point theorem . Additionally, we provide illustrative examples that demonstrate the key findings. Furthermore, we address a scenario where an initial value integral condition is considered.

Keywords: Fractional differential equations; existence; fixed point; functional fractional differential equations; Caputo-Hadamard fractional differential equations.



Introduction

This work is concerned with the existence and uniqueness of solutions to the following initial value problem (IVP) of ψ -Caputo sequential fractional order neutral functional differential equations

$$\mathcal{D}_a^{\alpha_1; \psi} [\mathcal{D}_a^{\alpha_2; \psi} y(t) - g(t, y_t)] = f(t, y_t), \quad t \in J := [a, b], \quad (0.68)$$

$$y(t) = \phi(t), \quad t \in [a - r, a], \quad (0.69)$$

$$\mathcal{D}_a^{\alpha_2; \psi} y(a) = \eta \in \mathbb{R}, \quad (0.70)$$

where $\mathcal{D}_a^{\alpha_1; \psi}, \mathcal{D}_a^{\alpha_2; \psi}$ are the ψ -Caputo fractional derivatives, $0 < \alpha, \beta < 1$, $f, g : J \times C([-r, 0], \mathbb{R}) \rightarrow \mathbb{R}$ are given functions and $\phi \in C([a - r, a], \mathbb{R})$. For any function y defined on $[a - r, b]$ and any $t \in J$, we denote by y_t the element of $C_r := C([-r, 0], \mathbb{R})$ defined by

$$y_t(\theta) = y(t + \theta), \quad \theta \in [-r, 0].$$

Functional differential equations are found to be of central importance in many disciplines such as control theory, neural networks, epidemiology, etc. [3]. In analyzing the behavior of real populations, delay differential equations are regarded as effective tools. Since the delay terms can be finite as well as infinite in nature, one needs to study these two cases independently. Moreover, the delay terms may appear in the derivatives involved in the given equation. As it is difficult to formulate such a problem, an alternative approach is followed by considering neutral functional differential equations. On the other hand, fractional derivatives are capable to describe hereditary and memory effects in many processes and materials. So the study of neutral functional differential equations in presence of fractional derivatives constitutes an important area of research. For more details, see the text [6].

In this paper, we investigate a new class of Hadamard-type sequential fractional neutral functional differential equations. Our study is based on fixed point theorems due to Banach and Krasnoselskii [3], and nonlinear alternative of Leray-Schauder type [2].

The rest of this paper is organized as follows: in Section 2 we recall some useful preliminaries. In Section 3 we discuss the existence and uniqueness of solutions for the problem (0.63)-(0.489). Examples are constructed in Section 4 for illustrating the obtained results.

Preliminaries

In this section, we introduce notation, definitions, and preliminary facts that we need in the sequel.

By $C(J, \mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty := \sup\{|y(t)| : t \in J\}.$$



Also C_r is endowed with norm

$$\|\phi\|_C := \sup\{|\phi(\theta)| : -r \leq \theta \leq 0\}.$$

Definition 0.8 ([1, 4]). The ψ -Riemann-Liouville fractional integral (ψ -RLFI) of order $\alpha > 0$ for a CF $\varphi : [a, \Im] \rightarrow \mathbb{R}$ is referred to as

$$\mathcal{I}_a^{\alpha;\psi} \varphi(\varsigma) = \frac{1}{\Gamma(\alpha)} \int_a^\varsigma (\psi(\varsigma) - \psi(s))^{\alpha-1} \psi'(s) \varphi(s) ds.$$

Definition 0.9 ([1, 4]). The ψ -Caputo fractional derivative (ψ -CFD) of order $\alpha > 0$ for a CF $\varphi : [a, \Im] \rightarrow \mathbb{R}$ is the aim of

$$\mathcal{D}_a^{\alpha;\psi} \varphi(\varsigma) = \frac{1}{\Gamma(n-\alpha)} \int_a^\varsigma (\psi(\varsigma) - \psi(s))^{n-\alpha-1} \psi'(s) \partial_\psi^n \varphi(s) ds, \quad \varsigma > a, \quad n-1 < \alpha < n,$$

where $\partial_\psi^n = \left(\frac{1}{\psi'(\varsigma)} \frac{d}{d\varsigma}\right)^n$, $n \in \mathbb{N}$

Lemma 0.4 ([1, 4]). Let $q, \ell > 0$, and $\varphi \in \mathcal{C}([a, b], \mathbb{R})$. Then, $\forall \varsigma \in [a, b]$, and by assuming

$F_a(\varsigma) = \psi(\varsigma) - \psi(a)$, we have

1. $\mathcal{I}_a^{q;\psi} \mathcal{I}_a^{\ell;\psi} \varphi(\varsigma) = \mathcal{I}_a^{q+\ell;\psi} \varphi(\varsigma)$,
2. $\mathcal{D}_a^{q;\psi} \mathcal{I}_a^{q;\psi} \varphi(\varsigma) = \varphi(\varsigma)$,
3. $\mathcal{I}_a^{q;\psi} (F_a(\varsigma))^{\ell-1} = \frac{\Gamma(\ell)}{\Gamma(\ell+q)} (F_a(\varsigma))^{\ell+q-1}$,
4. $\mathcal{D}_a^{q;\psi} (F_a(\varsigma))^{\ell-1} = \frac{\Gamma(\ell)}{\Gamma(\ell-q)} (F_a(\varsigma))^{\ell-q-1}$,
5. $\mathcal{D}_a^{q;\psi} (F_a(\varsigma))^k = 0$, for $k \in \{0, \dots, n-1\}$, $n \in \mathbb{N}$, $q \in (n-1, n]$.

Lemma 0.5 ([1, 4]). Let $n-1 < \alpha < n$, $\beta > 0$, $a > 0$, $\varphi \in L(a, \mathcal{T})$, $\mathcal{D}_a^{\alpha;\psi} \varphi \in L(a, \mathcal{T})$. Then, the differential equation

$$\mathcal{D}_a^{\alpha;\psi} \varphi = 0$$

has the unique solution

$$\varphi(\varsigma) = c_0 + c_1(\psi(\varsigma) - \psi(a)) + c_2(\psi(\varsigma) - \psi(a))^2 + \dots + c_{n-1}(\psi(\varsigma) - \psi(a))^{n-1},$$

and

$$\mathcal{I}_a^{\alpha;\psi} \mathcal{D}_a^{\alpha;\psi} \varphi(\varsigma) = \varphi(\varsigma) + c_0 + c_1(\psi(\varsigma) - \psi(a)) + c_2(\psi(\varsigma) - \psi(a))^2 + \dots + c_{n-1}(\psi(\varsigma) - \psi(a))^{n-1},$$

with $c_\ell \in \mathbb{R}$, $\ell = 0, 1, \dots, n-1$.

Furthermore,

$$\mathcal{D}_a^{\alpha;\psi} \mathcal{I}_a^{\alpha;\psi} \varphi(\varsigma) = \varphi(\varsigma),$$

and

$$\mathcal{I}_a^{\alpha;\psi} \mathcal{I}_a^{\beta;\psi} \varphi(\varsigma) = \mathcal{I}_a^{\beta;\psi} \mathcal{I}_a^{\alpha;\psi} \varphi(\varsigma) = \mathcal{I}_a^{\alpha+\beta;\psi} \varphi(\varsigma).$$



Lemma 0.6. *The function $y \in C^2([a - r, b], \mathbb{R})$ is a solution of the problem*

$$\begin{aligned} \mathcal{D}_a^{\alpha_1; \psi} [\mathcal{D}_a^{\alpha_2; \psi} y(t) - g(t, y_t)] &= f(t, y_t), \quad t \in J := [a, b], \\ y(t) &= \phi(t), \quad t \in [a - r, a], \\ \mathcal{D}_a^{\alpha_2; \psi} y(a) &= \eta \in \mathbb{R} \end{aligned} \quad (0.71)$$

if and only if

$$y(t) = \begin{cases} \phi(t), & \text{if } t \in [a - r, a], \\ \phi(a) + (\eta - g(a, \phi(a))) \frac{(\psi(t) - \psi(a))^{\alpha_2}}{\Gamma(\alpha_2 + 1)} \\ + \frac{1}{\Gamma(\alpha_2)} \int_a^t (\psi(t) - \psi(s))^{\alpha_2 - 1} \psi'(s) g(s, y_s) ds \\ + \frac{1}{\Gamma(\alpha_1 + \alpha_2)} \int_a^t (\psi(t) - \psi(s))^{\alpha_1 + \alpha_2 - 1} \psi'(s) f(s, y_s) ds, & \text{if } t \in [a, b]. \end{cases} \quad (0.72)$$

Main results

In this section, we establish the existence and uniqueness of a solution for the IVP (0.63)–(0.489).

Definition 0.10. A function $y \in C^2([a - r, b], \mathbb{R})$, is said to be a solution of (0.63)–(0.489) if y satisfies the equation $\mathcal{D}_a^{\alpha_1; \psi} [\mathcal{D}_a^{\alpha_2; \psi} y(t) - g(t, y_t)] = f(t, y_t)$ on J , the condition $y(t) = \phi(t)$ on $[a - r, a]$ and $\mathcal{D}_a^{\alpha_2; \psi} y(a) = \eta$.

The next theorem gives us a uniqueness result using the assumptions

(A1) there exists $\ell > 0$ such that

$$|f(t, u) - f(t, v)| \leq \ell \|u - v\|_C, \quad \text{for } t \in J \text{ and every } u, v \in C_r;$$

(A2) there exists a nonnegative constant k such that

$$|g(t, u) - g(t, v)| \leq k \|u - v\|_C, \quad \text{for } t \in J \text{ and every } u, v \in C_r.$$

Theorem 0.27. *Assume that (A1), (A2) hold. If*

$$\frac{k(F_a(b))^{\alpha_2}}{\Gamma(\alpha_2 + 1)} + \frac{\ell(F_a(b))^{\alpha_1 + \alpha_2}}{\Gamma(\alpha_1 + \alpha_2 + 1)} < 1, \quad (0.73)$$

then there exists a unique solution for IVP (0.63)–(0.489) on the interval $[a - r, b]$.



Examples

In this section we give an example to illustrate the usefulness of our main results. Let us consider the fractional functional differential equation,

$$\mathcal{D}_1^{\frac{2}{5}; \log_e} \left[\mathcal{D}_1^{\frac{1}{5}; \log_e} y(t) - \frac{e^{-t}}{\sqrt{5}} \frac{\|y_t\|_C}{(1 + \|y_t\|_C)} \right] = \frac{\|y_t\|_C}{\sqrt{7}(1 + \|y_t\|_C)} + t, \quad (0.74)$$

$$t \in J := [1, e],$$

$$y(t) = \phi(t), \quad t \in [1 - r, 1], \quad (0.75)$$

$$\mathcal{D}_1^{\frac{1}{5}; \log_e} y(1) = 1/2. \quad (0.76)$$

Let

$$f(t, x) = \frac{x}{\sqrt{7}(1+x)} + t, \quad g(t, x) = \frac{e^{-t}}{\sqrt{5}} \left(\frac{x}{1+x} \right), \quad (t, x) \in [1, e] \times [0, \infty).$$

For $x, y \in [0, \infty)$ and $t \in J$, we have

$$|f(t, x) - f(t, y)| = \frac{1}{\sqrt{7}} \left| \frac{x}{1+x} - \frac{y}{1+y} \right| = \frac{|x-y|}{\sqrt{7}(1+x)(1+y)} \leq \frac{1}{\sqrt{7}} |x-y|,$$

and

$$\begin{aligned} |g(t, x) - g(t, y)| &= \frac{e^{-t}}{\sqrt{5}} \left| \frac{x}{1+x} - \frac{y}{1+y} \right| = \frac{e^{-t}}{\sqrt{5}} \frac{|x-y|}{(1+x)(1+y)} \\ &\leq \frac{1}{\sqrt{5}} |x-y|. \end{aligned}$$

Hence conditions (A1) and (A2) hold with $\ell = \frac{1}{\sqrt{7}}$ and $k = \frac{1}{\sqrt{5}}$ respectively. Since $\frac{k(F_1(b))^{\alpha_2}}{\Gamma(\alpha_2+1)} + \frac{\ell(F_1(b))^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} \approx 0.910079666 < 1$, therefore, by Theorem 0.27, problem (0.74)-(0.76) has a unique solution on $[1-r, e]$.

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Exploring Asymptotic Properties and Growth Constraints of a Monotonically Decreasing Function

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Abstract. We analyze a positive, decreasing function $h(t)$ that satisfies the inequality

$$G \left(\int_0^t e^s h(s) ds \right) \leq e^{\xi t} h(t),$$

which leads to the condition $(h^{-\alpha})'' \geq 0$. This condition further implies the inequality

$$E \left(\vartheta \sqrt{\psi} + \psi t \right) \leq \left(1 + \tau \sqrt{\psi t} \right)^{-\frac{1}{\psi}}.$$

Our investigation focuses on the behaviour of $h(t)$ alongside another positive, decreasing function $E(t)$. The results offer key insights into these functions, with meaningful implications for differential equations and dynamical systems. This work introduces a framework for generating new inequalities, advancing both the theoretical understanding and practical applications of these mathematical concepts.

Keywords: Decreasing Function, convexity, concavity, Memory term, Stabilization, Frictional damping



Introduction

This study delves into the analysis of non-increasing functions and their integral properties, forming the basis of our investigation titled "Bounds, Decay, and Integrals of Non-Increasing Functions." We aim to explore the intricate behaviour of such functions and uncover the underlying patterns that govern their behaviour. Our analysis begins by examining the integral expression $1 > \int_0^\infty h(s)ds > 0$, which sheds light on the significance of these functions and provides valuable insights.

We also investigate the function $h(t) = \int_t^\infty E^{\alpha+1}(s)ds$ in relation to $E(t)$ and a constant $T > 0$, establishing the key inequality $G\left(\int_0^t e^s h(s) ds\right) \leq e^{\xi t} h(t)$. This inequality emphasizes the influence of $E(t)$ on $h(t)$ and highlights a power-law decay for $t \geq T$ in $E^{\alpha+1}(t)$.

Furthermore, our goal is to identify the conditions on the function $h(t)$ to ensure uniform exponential decay of $E(t)$. Specifically, we seek to find constants $C > 1$ such that the inequality holds for all $t \geq 0$:

$$E\left(\vartheta\sqrt{\psi} + \psi t\right) \leq \left(1 + \tau\sqrt{\psi t}\right)^{-\frac{1}{\psi}}.$$

Our analysis also presents three key lemmas that provide critical insights into the behavior of non-increasing functions. These lemmas form the foundation for the subsequent results in our investigation.

Additionally, we prove two theorems that further our understanding of the bounds, decay behavior, and integral characteristics of non-increasing functions. These theorems formally state our conclusions and their broader implications.

Through detailed analysis, we establish a set of inequalities that reveal deeper relationships between non-increasing functions, their decay, bounds, and integral properties. These findings have potential applications in fields such as control theory, differential equations, and dynamical systems.

The field of viscoelasticity has been extensively studied (see [1],[13]). Early work focused on kernels of the form $h(t) = \exp(-\beta t)$, where $\beta > 0$. Later studies investigated kernels that satisfied $-\lambda_1 h(t) \leq h'(t) \leq -\lambda_2 h(t)$ for all $t \geq 0$, with λ_1 and λ_2 as positive constants, along with conditions on the second derivative.

Over time, some of these conditions were relaxed, considering $h'(t) \leq -\lambda_2 h(t)$ or $h'(t) \leq -\lambda_2 h^\varsigma(t)$ for $t \geq 0$, where $\lambda > 0$. Eventually, these conditions evolved to $h'(t) \leq -\theta(t)h(t)$, where $\theta(t)$ is a positive function, allowing for decay rates beyond exponential or polynomial forms.

It's worth noting that other works have demonstrated exponential decay results under the condition $h'(t) \leq 0$ and $\exp(\alpha t)h(t) \in L^1(0, \infty)$ for some $\alpha > 0$.

The structure of this paper is as follows: first, we present several important lemmas that serve as the basis for our analysis. We then use these lemmas in proving the theorems, which ultimately lead to our main result regarding decay behavior.



Finally, we conclude by summarizing our findings and discussing potential avenues for future research. This work contributes to the broader understanding of non-increasing functions and their relevance across various mathematical and scientific disciplines.

Theorem 0.28. *Let $E : R^+ \rightarrow R^+$ be a non-increasing function. If*

$$h(t) = e^{\frac{t}{T}} \int_t^\infty E(s) ds, \quad t \in R^+, \quad T = \frac{\sqrt{\delta(1+\alpha)}}{\delta}, \quad (0.77)$$

then

$$E(t) = E_0 \exp \left(\sqrt{\alpha} \frac{\sqrt{\alpha} - 2\sqrt{(1+\alpha)}}{T} t \right). \quad (0.78)$$

Proof: *We start with the expression*

$$h(t) = e^{\frac{t}{T}} \int_t^\infty E(s) ds.$$

Differentiating $h(t)$ with respect to t , we obtain

$$h'(t) = \frac{1}{T} h(t) - e^{\frac{t}{T}} E(t), \quad (0.79)$$

and

$$h''(t) = \frac{1}{T} h'(t) - \frac{1}{T} e^{\frac{t}{T}} E(t) - e^{\frac{t}{T}} E'(t). \quad (0.80)$$

Substitute $h'(t)$ back into $h''(t)$ to get

$$h''(t) = \frac{1}{T^2} h(t) - \frac{2}{T} e^{\frac{t}{T}} E(t) - e^{\frac{t}{T}} E'(t). \quad (0.81)$$

We now multiply both sides of the equation by $h(t)$ and simplify:

$$h(t)h''(t) = \frac{1}{T^2} h^2(t) - \frac{2}{T} e^{\frac{t}{T}} E(t)h(t) - e^{\frac{t}{T}} E'(t)h(t).$$

Next, we also consider the expression for $(1+\alpha)(h'(t))^2$:

$$(1+\alpha)(h'(t))^2 = \frac{(1+\alpha)}{T^2} h^2(t) + (1+\alpha)e^{\frac{2t}{T}} E^2(t) - 2(1+\alpha)e^{\frac{t}{T}} E(t)h(t).$$

Subtracting the two expressions gives:

$$h(t)h''(t) - (1+\alpha)(h'(t))^2 = -\frac{\alpha}{T^2} h^2(t) - (1+\alpha)e^{\frac{2t}{T}} E^2(t) + \left(\frac{\alpha}{T} E(t) - E'(t) \right) e^{\frac{t}{T}} h(t).$$

Using this result and rearranging terms, we obtain the following first-order differential equation:

$$E'(t) = \sqrt{\alpha} \frac{\sqrt{\alpha} - 2\sqrt{(1+\alpha)}}{T} E(t).$$



This is a linear differential equation of the form:

$$E'(t) = kE(t),$$

where

$$k = \sqrt{\alpha} \frac{\sqrt{\alpha} - 2\sqrt{(1+\alpha)}}{T}.$$

The general solution to this equation is:

$$E(t) = E_0 \exp(kt),$$

where E_0 is an arbitrary constant. Substituting the value of k gives the final solution:

$$E(t) = E_0 \exp\left(\sqrt{\alpha} \frac{\sqrt{\alpha} - 2\sqrt{(1+\alpha)}}{T} t\right).$$

Lemma 0.7. If a C^2 -function $h(t) > 0$ satisfies for $t \geq 0$,

$$h(t) \geq ae^{-\gamma't} + b \int_0^t e^{-\gamma(t-s)} h(s) ds, \quad (0.82)$$

with $a, b, \gamma, \gamma' > 0$, for $\gamma > \gamma' + b$, $t > 0$,
 $\delta = \gamma'(\gamma' - b) - b(\gamma - b) > 0$, $\beta > \sqrt{\delta}$,
then

$$h(t)h''(t) - (1+\alpha)(h'(t))^2 \leq 0.$$

Lemma 0.8. Let $E: R^+ \rightarrow R^+$ be C^1 a non-increasing function.
Then

$$\frac{E'}{E} \leq 2\beta(1+\alpha)e^{-\beta t}.$$

By invoking Lemma 2.1, we derive the following expression:

$$\left[(h')^2 E\right]' = 2h''h'E + (h')^2 E'. \quad (0.83)$$

Given the inequality

$$h(t)h''(t) \leq (1+\alpha)(h'(t))^2,$$

we can further infer:

$$2h'h''(t)E \geq 2(1+\alpha) \frac{(h'(t))^3}{h(t)} E.$$

Combining the above, we obtain:

$$2h'h''(t)E + (h')^2 E' \geq 2(1+\alpha) \frac{(h'(t))^3}{h(t)} E + (h')^2 E'.$$



Therefore:

$$0 \geq 2(1 + \alpha) \frac{(h'(t))^3}{h(t)} E + (h')^2 E'.$$

Dividing through by $(h')^2 E$, we get:

$$\frac{E'}{E} \leq -2(1 + \alpha) \frac{h'(t)}{h(t)}.$$

Considering the condition $h' + \beta h \geq 0$ and the fact that $\beta e^{-\beta t} < \beta$, it follows that:

$$0 \leq h' + \beta e^{-\beta t} h \leq h' + \beta h.$$

This implies:

$$0 \geq -(1 + \alpha)h' - (1 + \alpha)\beta e^{-\beta t} h \geq -(1 + \alpha)h' - (1 + \alpha)\beta h,$$

and consequently:

$$0 \geq -(1 + \alpha) \frac{h'}{h} - (1 + \alpha)\beta e^{-\beta t} \geq -(1 + \alpha) \frac{h'}{h} - (1 + \alpha)\beta.$$

Thus:

$$\frac{E'}{E} \leq 2\beta(1 + \alpha)e^{-\beta t}.$$

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On the solution of evolution p-Laplace equation with memory term and unknown boundary Dirichlet condition

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Abstract. In the present paper, we are interested in investigating a classe of nonlinear parabolic integro-differential equation with unknown flux on the part of Dirichlet boundary. A discrete scheme for the time approximations is introduced. Existence and uniqueness of a weak solution at each time step are proved. Convergence of the approximate solution to the weak solution is shown with the help of some a priori estimates. At the end, our proposed theoretical approach is supported by computational experiments.

Keywords: Parabolic equation, p-Laplace equation, Weak solution, A priori estimate, Unknown Dirichlet condition.



Introduction

We consider a connected bounded open domain Ω of $\mathbb{R}^n$, with a Lipschitz-continuous boundary $\Gamma = \overline{\Gamma_N} \cup \overline{\Gamma_D}$ and $I = [0, T]$, $T > 0$. Here $\Gamma_N \cap \Gamma_D = \emptyset$, $|\Gamma_D| > 0$, where Γ_N and Γ_D are two open subsets of Γ . We denote η the outward unit normal vector on Γ . Our main purpose here is to establish the reconstruction of the unknown Dirichlet condition via an additional integral measurement to the following problem:

$$\begin{cases} \frac{\partial \beta(u)}{\partial t} - \operatorname{div}(|\nabla u|^{p-2} \nabla u) = f(t, x) + \int_0^t M(s, u(s, x)) ds & \text{in } I \times \Omega, \\ -\frac{\partial u}{\partial \eta} = \alpha(t, x) & \text{on } I \times \Gamma_N, \\ u = \rho(t) & \text{on } I \times \Gamma_D, \\ u(0, x) = u_0 & \text{on } \Omega \end{cases} \quad (1.1)$$

Where $2 < p < \infty$, M is globally Lipschitz continuous function satisfying the growth condition

$$|M(t, s)| \leq C(1 + |s|^{p-1}), \quad \forall s \in \mathbb{R} \quad (1.2)$$

β is a monotonically increasing continuous functions verifying the growth condition

$$|\beta(s)| \leq C(1 + |s|^{p-1}), \quad \forall s \in \mathbb{R} \quad (1.3)$$

$$\text{With } \beta(0) = 0 \text{ and } 0 < \lambda \leq \beta'(s). \quad (1.4)$$

The addition integral measurement is given by

$$\int_{\Omega} \beta(u(t, x)) dx = \sigma(t). \quad (1.5)$$

Main results

Semi-Discretized problem

We subdivide the time interval $I = [0, T]$ to n subintervals, $\tau = \frac{T}{n}$, $i = 0, 1, \dots, n$ where $u_i = u(x, t_i)$, $t_i = i\tau$ and let

$$\delta u_i = \frac{u_i - u_{i-1}}{\tau}, \quad \delta \beta(u_i) = \frac{\beta(u_i) - \beta(u_{i-1})}{\tau}, \quad \rho_i = \rho(t_i)$$

For $i = 1, \dots, n$, the recurrent semi discretized problem is



$$\left\{ \begin{array}{l} \text{Find } (u_i, \rho_i) \cong (u(t_i, \cdot), \rho(t_i)), \quad i = 1, 2, \dots, n \text{ such that } , \\ (\beta(u_i) - \beta(u_{i-1}), v) + \tau(|\nabla u_i|^{p-2} \nabla u_i, \nabla v) = -\tau(\alpha_i, v)_{\Gamma_N} + \tau(f_i, v) \\ + v|_{\Gamma_D} \tau \left[\sigma'_i + (\alpha_i, 1)_{\Gamma_N} - (f_i, 1) - \left(\sum_{j=0}^{i-1} M(t_j, u_j) \tau, 1 \right) \right], \quad \forall v \in V \\ + \tau \left(\sum_{j=0}^{i-1} M(t_j, u_j) \tau, v \right). \end{array} \right. \quad (2.1)$$

Theorem 0.29. Let $\alpha(t) \in L^q(\Gamma_N)$, $f(t) \in L^q(\Omega)$ and $\sigma'(t) \in \mathbb{R}$, for all $t \in I$, the problem (2.1) admits a unique weak solution $u_i \in V, \forall i = 1, \dots, n$.

Necessary a priori estimates

Now we are going to show some a priori estimates which will help us in the existence part,

Lemma 0.9. There exists a positive constant C , such that

$$\sum_{i=1}^m \tau \|\delta u_i\|_{L^2(\Omega)}^2 \leq C, \quad \sum_{i=1}^m \|\nabla u_i\|_{L^p(\Omega)}^p \leq C, \quad (2.2)$$

Lemma 0.10. There exists a positive constant C such that

$$\|\delta \beta(u_i)\|_{V'} \leq C. \quad (2.3)$$

Existence

In this part, we will show the existence of a weak solution to the problem (1.1).

To this end, let us introduce Rothe functions

$$\left\{ \begin{array}{l} u_n(0) = u_0 \\ u_n(t) = u_{i-1} + (t - t_{i-1})\delta u_i, \quad 1 \leq i \leq n, \quad \forall t \in (t_{i-1}, t_i], \\ \beta_n(0) = \beta_0 \\ \beta_n(t) = \beta(u_{i-1}) + (t - t_{i-1})\delta \beta(u_i), \quad 1 \leq i \leq n, \quad \forall t \in (t_{i-1}, t_i], \end{array} \right.$$

Together with step functions,

For $1 \leq i \leq n$ and $\forall t \in (t_{i-1}, t_i]$,

$$\left\{ \begin{array}{l} \bar{u}_n(0) = u_0, \bar{u}_n(t) = u_i, \\ \bar{f}_n(0) = f(0), \bar{f}_n(t) = f(t_i), \\ \bar{\alpha}_n(0) = \alpha(0), \bar{\alpha}_n(t) = \alpha(t_i), \\ \bar{M}_n(0, \bar{u}_n(0)) = M(0, u_0), \bar{M}_n(t, \bar{u}_n(t)) = M(t_i, u_i), \\ \bar{\sigma}'_n(0) = \sigma'(0), \bar{\sigma}'_n(t) = \sigma'(t_i). \end{array} \right.$$

Then, (2.1) may be written as

$$\begin{aligned} & (\partial_t \beta_n(t), v) + (|\nabla \bar{u}_n|^{p-2} \nabla \bar{u}_n, \nabla v) + (\bar{\alpha}_n(t), v)_{\Gamma_N} = (\bar{f}_n(t), v) + \left(\int_0^{t_i} \bar{M}_n(s, \bar{u}_n(s)) ds, v \right) \\ & + v|_{\Gamma_D} \left[\bar{\sigma}'_n(t) + (\bar{\alpha}_n(t), 1)_{\Gamma_N} - (\bar{f}_n(t), 1) - \left(\int_0^{t_i} \bar{M}(s, \bar{u}_n(s)) ds, 1 \right) \right]. \end{aligned}$$



(2.4)

Uniqueness of weak solution

Now we will show the uniqueness of a weak solution to (1.1) in the case

$$m(x, t) = \int_0^t M(s, u(x, s)) ds = \int_0^t k(t-s) \Delta_p u(x, s) ds, \quad (2.5)$$

Where

$$k \in C^1(0, T), \quad \Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u), \quad 2 < p < \infty.$$

The first equation of the problem (1.1) becomes

$$\partial_t \beta(u) - \operatorname{div}(|\nabla u|^{p-2} \nabla u) = f(x, t) + \int_0^t k(t-s) \Delta_p u(x, s) ds \quad (2.6)$$

A simple calculation gives us

$$\begin{aligned} \int_0^t k(t-s) \Delta_p u(x, s) ds &= - \int_0^t k(t-s) \int_0^s k(s-z) \Delta_p u(x, z) dz ds \\ &\quad + k(0) \beta(u(x, t)) - k(t) \beta(u_0(x)) + \int_0^t k'(t-s) u(x, s) ds \\ &\quad - \int_0^t k(t-s) f(x, s) ds, \end{aligned} \quad (6.7)$$

Which proves that m can be written as

$$m(x, t) = - \int_0^t k(t-s) m(x, s) ds + g(x, t, u), \quad (2.8)$$

Such that

$$g(x, t, u) = \int_0^t k'(t-s) u(x, s) ds - \int_0^t k(t-s) f(x, s) ds + k(0) \beta(u(x, t)) - k(t) \beta(u_0(x)). \quad (2.9)$$

Therefore, (1.1) is equivalent to the following system

$$\begin{cases} \text{Find } (u, m) \text{ such that ,} \\ (\partial_t \beta(u), v) + (|\nabla u|^{p-2} \nabla u, \nabla v) = -(\alpha, v)_{\Gamma_N} + (f, v) \\ + (m, v) + v|_{\Gamma_D} [\sigma' + (\alpha, 1)_{\Gamma_N} - (f, 1) - (m, 1)] \\ (m, v) = (- \int_0^t k(t-s) m(x, s) ds, v) + (g, v) \end{cases}, \forall v \in H_0^1(\Omega) \quad (2.10)$$

Theorem 0.30. *The problem (2.10) has a unique weak solution.*



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Modélisation Avancée du Système de Brusselator avec Vitesses de Réaction Variables : Une Approche Fractionnaire

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Abstract. Dans cet exposé, nous proposons une approche innovante pour modéliser le système de Brusselator, un modèle chimique, en intégrant des vitesses de réaction variables. Contrairement aux méthodes classiques, cette approche capture de manière plus réaliste les fluctuations des réactions chimiques. En utilisant les dérivées fractionnaires conformes, les équations différentielles sont reformulées en



une équation non linéaire de type Volterra, ce qui permet une étude approfondie des oscillations chimiques au fil du temps. Les résultats obtenus montrent des oscillations dans les concentrations chimiques, similaires à celles des systèmes réels, ouvrant des perspectives pour des applications dans divers systèmes dynamiques complexes.

Keywords: Système Brusselator 1, vitesses variables 2, Équation non linéaire de type Volterra 3, Dérivées fractionnaires conformables 4, Méthode Nyström 5.

Introduction

Aujourd'hui, nous allons explorer la modélisation du système de Brusselator, un modèle utilisé en chimie pour étudier les réactions dynamiques complexes. Dans cette étude, nous avons développé une approche novatrice en intégrant des vitesses de réaction variables, contrairement aux modèles classiques qui utilisent des vitesses constantes. Cette approche permet de mieux capturer les variations observées dans les réactions chimiques réelles, où la vitesse de réaction peut être influencée par des facteurs tels que la température, la concentration des réactifs ou l'action de catalyseurs.

Pourquoi les vitesses variables ? Les réactions chimiques dans la nature sont rarement constantes ; elles varient en fonction des conditions du milieu. En intégrant ces vitesses de réaction variables, nous avons construit un modèle qui correspond plus fidèlement aux processus réels.

La modélisation : Pour représenter ces variations de vitesse, nous avons utilisé les dérivées fractionnaires conformes, qui permettent de capturer les phénomènes dynamiques en tenant compte des états passés du système. En utilisant ces dérivées, nous avons reformulé le modèle de Brusselator classique en une équation non linéaire de type Volterra. Cette approche, bien que plus complexe, est plus réaliste et adaptée aux systèmes réels.

Démonstration mathématique : Une étape clé de notre étude a consisté à démontrer l'existence d'une solution unique pour ce système sous certaines conditions. Pour cela, nous avons employé la méthode de Picard, qui est une technique mathématique permettant d'assurer la convergence des solutions dans des systèmes complexes.



La méthode numérique : Pour obtenir des résultats concrets, nous avons appliqué la méthode de Nyström, très efficace pour résoudre les équations intégrales non linéaires. Elle nous a permis de simuler avec précision le comportement dynamique du modèle.

Résultats et simulations : Les simulations effectuées ont montré comment les concentrations de réactifs varient en fonction des vitesses de réaction. Ce qui est particulièrement intéressant, c'est que les résultats révèlent des oscillations similaires à celles observées dans les systèmes chimiques réels, offrant ainsi une vue plus précise sur les mécanismes de réaction.

Conclusion : En conclusion, cette étude démontre que l'intégration des dérivées fractionnaires et des vitesses de réaction variables permet une modélisation plus réaliste des systèmes chimiques. Cette approche ouvre des perspectives pour l'étude de systèmes dynamiques dans divers domaines scientifiques, permettant une compréhension approfondie de phénomènes complexes.

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The existence of solution for an operator equation on b -metric space

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Abstract. The existence of solution for an operator defined from metric space into an other one was demonstrated by A. V. Arutyunov [1]. Authors in [2] found its generalization, where operator defined from metric space into space with generalized distance (distance satisfies only the axiom of identity).

Keywords: Metric space, Covering mapping, b -metric space, Lipschitz mapping.



Introduction

In this work, we will state a new result of the existence of solution for operator equation for mappings of two variables acting from b -metric space to space equipped by a generalized distance. the mapping is α -covering and closed for one variable, and β -Lipschitz for the other one.

Main results

The following definition is the definition of b -metric space.

Definition 0.11. Let X be a nonempty set, we define the application $\rho : X \times X \rightarrow \mathbb{R}_+$, and let $s \geq 1$. If the following relations are satisfied for all $x, y, z \in X$:

- (b₁) $\rho(x, y) = 0$ iff $x = y$,
- (b₂) $\rho(x, y) = \rho(y, x)$,
- (b₃) $\rho(x, z) \leq s(\rho(x, y) + \rho(y, z))$,

so ρ is called a b -metric on X and (X, ρ) is a b -metric space.

Now, the space Y is a generalized metric space (that's mean distance satisfied only the axiom of identity). (Y, d) is a generalized metric space where Y is a nonempty set and d is a distance satisfied only the axiom of identity i.e.:

$$\forall y_1, y_2 \in Y \quad d(y_1, y_2) = 0 \quad \Leftrightarrow \quad y_1 = y_2.$$

The following definitions are the definition of covering mapping and Lipschitz mapping, respectively.

Definition 0.12. Let $\alpha > 0$, the mapping $\psi : X \rightarrow Y$ is called α -covering, if satisfying the relation :

$$\forall x_0 \in X \quad \forall y \in Y \quad \exists x \in X : \psi(x) = y \quad \text{and} \quad \rho(x, x_0) \leq \frac{1}{\alpha} d(\psi(x), \psi(x_0)).$$

Definition 0.13. : Let $\beta > 0$, the mapping $\varphi : X \rightarrow Y$ is called β -Lipschitz, if satisfying the relation:

$$d(\varphi(x), \varphi(x_0)) \leq \beta \rho(x, x_0) \quad \forall x, x_0 \in X$$

Given a mapping $F : X \times X \rightarrow Y$, an element $\tilde{y} \in Y$. We define a mapping $\Phi : X \rightarrow Y$ by the identity $\Phi(x) = F(x, x)$, and consider the equation

$$\Phi(x) = \tilde{y}. \tag{1}$$

We have the following theorem.



Theorem 0.31. *Let a b -metric space (X, ρ) be complete and (Y, d) is space with a generalized distance d . If the following conditions are satisfied:
An application $F(x, \cdot) : X \rightarrow Y$ is α -covering and closed, and an application $F(\cdot, x) : X \rightarrow Y$ is β -Lipschitz with $\alpha > \beta$.
Then, for all $x_0 \in X$ there exists ξ solution for (1) such that:*

$$\rho(x_0, \xi) \leq \frac{s^2 \alpha^{t_0-1} R(s\beta/\alpha, t_0 - 1) + s(s\beta)^{t_0-1}}{\alpha^{t_0} - s\beta^{t_0}} d(F(x_0, x_0), \tilde{y}),$$

where, $t_0 = \min\{j \in \mathbb{N} / s\beta^j < \alpha^j\}$, and for $\theta \in [0, 1)$ and $n \in \mathbb{N}$, we have $R(\theta, n) = \frac{1-\theta^n}{1-\theta}$.

This result is a generalization of the results published in [1] and [2].

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Poster Presentations



The study of a nonlinear equations

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Abstract. Nonlinear functional equations are a fundamental tool for mathematical modeling derived from physical, chemical, or other discipline processes. In this paper, we propose to solve nonlinear functional equations given in an infinite-dimensional Banach space by discretizing and solving a finite-dimensional linear problem. We establish new sufficient conditions which provide new criteria for dealing with convergence results.

Keywords: Nonlinear equation, Iterative methods, Convergence, Kantorovich Projection approximate.

Introduction

In this work, we study a functional nonlinear equation defined generally as:

$$\text{Find } \psi \in \mathcal{O} \subset X : F(\psi) = 0, \quad (0.84)$$



where F is a nonlinear operator defined over an infinite dimensional Banach space, denoted by X , and $\mathcal{O}$ is a nonempty open set of X . The solution of problem (0.470) can be characterized as the limite of the sequence $(\psi^{(k)})_{k \geq 0}$ which is defined through the relation

$$F'(\psi^{(k)})(\psi^{(k+1)} - \psi^{(k)}) = -F(\psi^{(k)}), \quad \psi^{(0)} \in \mathcal{O}. \quad (0.85)$$

If $F'(x)$ is invertible operator for all $x \in \mathcal{O}$, then equation (0.351) can be rewritten as:

$$\psi^{(k+1)} = \psi^{(k)} - F'(\psi^{(k)})^{-1} F(\psi^{(k)}), \quad \psi^{(0)} \in \mathcal{O}. \quad (0.86)$$

Solving Eq.(0.457) in exact arithmetics (i.e. using analytical method) is typically far out to perform in an infinite dimensional Banach space. Therefore, we discretize the Eq.(0.457) then we solve numerically the corresponding finite dimensional problem.

If we denote by $\Sigma_n(x)$ the numerical approximation of $F'(x)^{-1}$ for $x \in \mathcal{O}$, then the solution of problem (0.470) can be characterized by a sequence $(\psi_n^{(k)})_{k \geq 0}$ defined as:

$$\psi_n^{(k+1)} = \psi_n^{(k)} - \Sigma_n(\psi_n^{(k)}) F(\psi_n^{(k)}), \quad \psi^{(0)} \in \mathcal{O}, \quad (0.87)$$

where n is fixed in $\mathbb{N}^*$.

In [1] the authors studied the convergence results of scheme (0.458), where they show that under appropriate conditions on $\Sigma_n(x)$ for $x \in \mathcal{O}$, the sequence $\psi_n^{(k)}$ converges to ψ the solution of Eq.(0.470) when k goes to infinity and n is fixed in $\mathbb{N}^*$.

In our work, we propose a new scheme defined as:

$$\psi_{n,m}^{(k+1)} = \psi_{n,m}^{(k)} - \Theta_n^m(\psi_{n,m}^{(k)}) F(\psi_{n,m}^{(k)}), \quad \psi^{(0)} \in \mathcal{O}, \quad (0.88)$$

where n, m are fixed in $\mathbb{N}^*$ and for all $x \in \mathcal{O}$,

$$\Theta_n^m(x) = \sum_{j=0}^n \left(I - \Sigma_m(\psi^{(0)}) F'(x) \right)^j \Sigma_m(\psi^{(0)}),$$

and recall that $\Sigma_m(\psi^{(0)})$ is the numerical approximation of $F'(\psi^{(0)})^{-1}$ for a fixed integer $m \in \mathbb{N}^*$.

Main results

Theorem 0.32. *A priori convergence theorem. Suppose that $F, \mathcal{O}, \psi \in \mathcal{O}, \mu > 0, R > 0, l > 0$ and $\alpha \in]0, 1]$ are such that:*

- 1) $F(\psi) = 0$
- 2) *The closed ball $B_R(\psi)$ is included in the open set $\mathcal{O}$, and F' is (l, α) -Holder continuous on $B_R(\psi)$ where $b = \sup_{x \in B_R(\psi)} \|F'(x)\|$.*



3) There exists $\psi^{(0)} \in B_r$ such that

$$r = \min \left\{ R, \frac{1}{(4\mu l)^{\frac{1}{\alpha}}} \right\} \quad \text{and} \quad \|F'(\psi^{(0)})^{-1}\| \leq \mu$$

there exists $\beta_m \in]0, 1[$ such that

$$\|I - \Sigma_m(\psi^{(0)})F'(x)\| \leq \beta_m. \quad (0.89)$$

4) The starting approximation

$$\rho_{n,m} := \min \left\{ r, \left(\frac{(1 - \gamma_{n,m})(1 - \beta_m)}{4l\mu(1 + \beta_m)(1 - (\beta_m)^{n+1})} \right)^{\frac{1}{\alpha}} \right\}.$$

Then, for all k , $\psi_{n,m}^{(k)} \in B_{\rho_{n,m}}(\psi)$, and we have

$$\|\psi_{n,m}^{(k)} - \psi\| \leq \rho_{n,m} \left(\frac{1 + \gamma_{n,m}}{2} \right)^k \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty \quad (0.90)$$

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Méthode de Laplace pour l'estimation asymptotique des intégrales de type de Laplace dans le cas bidimensionnel

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Abstract. Dans ce travail, nous considérons le développement asymptotique d'intégrales doubles de type de Laplace au cas de points minimaux non stationnaires, situés sur la frontière du domaine d'intégration, on démontre dans ce travail que l'ordre de contact entre la courbe du frontière du domaine d'intégration et la courbe niveau de la phase à travers le point minimum gère le développement asymptotique en question. Cette idée permettra de construire des développements asymptotiques complets dans des contextes plus généraux. En particulier, le problème sera complètement résolu si la phase et la courbe du frontière sont analytiques au voisinage du point minimum.

Keywords: Intégrale de Laplace, développement asymptotique, lemme de Watson, point minimum.



Introduction

Considérons l'intégrale du type de Laplace

$$I(\lambda) = \int_D g(x_1, x_2) e^{-\lambda f(x_1, x_2)} dx_1 dx_2, \quad (0.91)$$

où D est un domaine borné de $\mathbb{R}^2$, f et g sont deux fonctions à valeurs réelles, de classe C^∞ sur la fermeture $\overline{D}$ de D , et λ un grand paramètre positif. Le problème du développement asymptotique de telles intégrales est fréquemment rencontré en mathématiques appliquées, en particulier pour approximer des solutions intégrales de certaines équations différentielles, on cite par exemple l'application de ce concept à l'étude du temps de sortie d'un système dynamique perturbé par un bruit aléatoire ([8], Ch. 10). Concernant les principaux résultats sur ce concept, on note qu'il est bien connu depuis l'époque de Laplace que la majeure contribution dans le développement asymptotiques de telles intégrales provient de l'intégration dans un voisinage de l'ensemble des points sur la fermeture $\overline{D}$ du domaine d'intégration D où la phase f atteint son minimum absolu (voir e.g., [3], Ch. 8).

Résultats principaux

Théorème.

Si la phase f a un point minimum non stationnaire unique $(x_1^0, x_2^0) \in \partial D$, et p est fini, alors p est pair et

$$I(\lambda) \sim \sum_{k=1}^{\infty} \frac{\Psi^{(k)}}{k!} (0) \Gamma\left(\frac{k}{p} + 1\right) \lambda^{-\left(\frac{k}{p} + 1\right)}, \quad (\lambda \rightarrow \infty) \quad (0.92)$$

où le terme principal est

$$\begin{cases} 2g(x_1^0, x_2^0) \left(\frac{1}{p!} \frac{\partial^p \beta}{\partial s^p}(0)\right)^{-\frac{1}{p}} (\nabla f(x_1^0, x_2^0) \cdot \nu_\gamma(0))^{-\left(\frac{1}{p} + 1\right)} \\ \times \Gamma\left(\frac{1}{p} + 1\right) \lambda^{-\left(\frac{1}{p} + 1\right)}. \end{cases} \quad (0.93)$$

Conclusion

Dans ce papier on construit des développements asymptotiques d'intégrales doubles du type de Laplace, quand la phase atteint son minimum en un point, non stationnaire pour la phase, situé sur la frontière du domaine d'intégration. Ces développements, qui sont gouvernés par l'ordre de contact entre la courbe niveau passant par le point critique et la frontière du domaine d'intégration, sont de l'ordre de $\lambda^{-\frac{p+2}{p+1}}$, p étant l'ordre de contact. On note que dans les travaux antérieurs, on n'a obtenu des développements que dans un cas particulier, où l'ordre de contact est minimal ($= 1$).



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The general stability of porous-elastic system with a neutral delay

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Abstract. In this article, we analyze a one-dimensional porous-elastic system consisting of distributed delays of neutral type and a nonlinear damping term. In the first paragraph, we demonstrate the existence and uniqueness results of the solution using the Faedo–Galerkin method. By employing the multiplier method, with some assumptions about the kernel of the neutral delay term and some characteristics of convex functions, we show that the dissipation given by the nonlinear damping term is strong enough to control the effects produced from this type of delay, which is a more general class than the classical delay. We establish a general stability of the solution for the case of equal speeds of wave propagation.

Keywords: General decay, Porous-elastic system, Neutral delay, Faedo–Galerkin approximations.



Introduction

In the present work, we consider the following porous-elastic system with nonlinear damping term and subject to a distributed delay of neutral type

$$\left\{ \begin{array}{l} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, \quad x \in (0, 1), \quad t > 0, \\ J \left(\phi_t + \int_0^t k(t-s) \phi_t(s) ds \right)_t - \delta \phi_{xx} + bu_x + \xi \phi \\ + \alpha(t) g(\phi_t) = 0, \quad x \in (0, 1), \quad t > 0, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in (0, 1), \\ \phi(x, 0) = \phi_0(x), \quad \phi_t(x, 0) = \phi_1(x), \quad x \in (0, 1), \\ u_x(0, t) = u_x(1, t) = \phi(0, t) = \phi(1, t) = 0, \quad t > 0, \end{array} \right. \quad (0.94)$$



Main results

We prove the global existence and the uniqueness of the solution of the problem (1.1) by using the classical Faedo-Galerkin approximations.

Definition 0.14. The demonstration of the existence and uniqueness of solution is based on the Faedo-Galerkin method, which consists of the following three steps:

- Search for approachable solutions .
- Based on these approximations, a priori estimates are made.
- We go to the limit, thanks to compactness properties (in non-linear terms).

Theorem 0.33. *Assume that some hypothesis are satisfied, and the initial data*

$$\begin{aligned}(u_0, u_1) &\in H_*^1(0, 1) \times L_*^2(0, 1), \\ (\phi_0, \phi_1) &\in H_0^1(0, 1) \times L^2(0, 1),\end{aligned}\tag{0.95}$$

problem (0.94) has a unique global weak solution

$$\begin{aligned}u &\in C(\mathbb{R}_+, H_*^2(0, 1) \cap H_*^1(0, 1)) \cap C^1(\mathbb{R}_+, H_*^1(0, 1)) \cap C^2(\mathbb{R}_+, L_*^2(0, 1)), \\ \phi &\in C(\mathbb{R}_+, H^2(0, 1) \cap H_0^1(0, 1)) \cap C^1(\mathbb{R}_+, H_0^1(0, 1)) \cap C^2(\mathbb{R}_+, L^2(0, 1)).\end{aligned}\tag{0.96}$$

In addition, the solution (u, ϕ) depends continuously on the initial data.

Lemma 0.11. *The energy $E(t)$ of the system (0.94) given by*

$$\begin{aligned}E(t) &= \frac{1}{2} \int_0^1 (\rho u_t^2 + \mu u_x^2 + J \phi_t^2 + 2b u_x \phi + \xi \phi^2 + \delta \phi_x^2) dx \\ &+ \frac{J}{2} \int_0^1 \left(\int_0^t k(t-s) \phi_t^2(s) ds \right) dx, \quad t \geq 0.\end{aligned}\tag{0.97}$$

satisfies

$$E'(t) \leq -\alpha(t) \int_0^1 \phi_t g(\phi_t) dx.\tag{0.98}$$

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Approximation of higher-order nonlinear Volterra integro-differential equations

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Abstract. Our main goal is to develop an effective approach for addressing a high-degree nonlinear Volterra integro-differential equation. We begin with a thorough literature review to highlight the importance of such mathematical equations in scientific research. Following this, we establish sufficient conditions to prove the existence and uniqueness of the exact solution. We then apply two distinct discretization methods: Nyström and collocation. Each method leads to an approximate system that converges to the exact solution. The convergence of these systems is rigorously validated through two theorems. Finally, we conduct numerical tests to assess the effectiveness and simplicity of the proposed numerical techniques.

Keywords: Integro-differential equations · Nonlinear integral equation · Volterra equation · Numerical integration · Collocation method

Introduction

The history of integro-differential equations began in the 18th century with Leonhard Euler's work on the vibrating string problem and continued with



Jean-Baptiste Fourier's application to the heat equation. However, it was not until the 20th century that Vito Volterra popularized these equations by modeling bacterial reproduction, which significantly advanced applied mathematics. Throughout the 20th century, contributions from mathematicians like Abel, Verhulst, and Henri Poincaré expanded their applications to areas such as population dynamics, infectious diseases, energy, and finance. Over time, various solution methods have evolved, from early analytical techniques to modern numerical methods and machine learning approaches. This article specifically focuses on nonlinear Volterra integral equations with continuous kernels, examining a general form of these higher-order equations to explore their properties and solution techniques.

There are different types of the integro-differential equation and these types can vary depending on the scientific or engineering application. In this manuscript, we are interested in solving the following general form of a higher-order nonlinear Volterra integro-differential equation:

$$\forall x \in [a, b], \lambda u^{(N)}(x) = \int_a^x K(x, t, u(t)) dt + f(x), \quad \text{with } u^{(i)}(a) = \alpha_i, \quad i = 0, \dots, N-1, \quad (0.99)$$

where λ is a real number, f is a given continuous function, α_i are constant, u is an unknown function to be found in $C^N[a, b]$ which is the Banach space of N differential function and the N -th derivative is continuous, $u^{(N)}$ denotes the N -th derivative of the unknown function u , and the kernel K is a real-valued function of three variables $(x, t, w) \in [a, b]^2 \times \mathbb{R} \rightarrow K(x, t, w) \in \mathbb{R}$.

In the next theorem, we present a sufficient condition that guarantees the existence and uniqueness of the solution.

Theorem 0.34. *If $\frac{(b-a)^{N+1}}{(N-1)!}L < |\lambda|$. The equation (0.351) has a unique solution v .*

Numerical study

The equation (0.470) is considered the general case for non-linear Volterra differential equations. Therefore, it was necessary for us to ensure the solution and its existence. After confirming the existence and uniqueness of the solution and setting a fundamental condition for it. It is time to apply digital methods to solve it. Initially, many approaches crossed our minds, and while we could compare between them, this paper will not suffice. Hence, we opted for only two methods which are Collocation and Nystrom methods. We start by applying Collocation method.

Collocation method

In this section, we need to search an approximation of (0.351) by applying of the collocation method. This method can lead to a reduction in the dimen-



sionality of the equation (0.351). By selecting an appropriate set of collocation points, the integral equation can be transformed into a system of algebraic equations. It can achieve higher accuracy by increasing the choosing of the number of the discretisation n . We get the following approximation system

$$\left\{ \begin{array}{l} \lambda v_0 = f(x_0), \\ \lambda v_{n,i} = \int_a^{x_i} K\left(x_i, t, g(t) + \frac{1}{(N-1)!} \sum_{j=0}^i v_{n,j} \int_a^t (t-y)^{N-1} e_j(y) dy\right) dt + f(x_i), \\ u_{n,i} = g(x_i) + \frac{1}{(N-1)!} \sum_{j=0}^i v_{n,j} \int_a^{x_i} (x_i - y)^{N-1} e_j(y) dy. \end{array} \right. \quad (0.100)$$

We have identified the approximate system; now, we are going to demonstrate the convergence of the solution in the subsequent section.

The next theorem states that under suitable conditions, the approximation $v_{n,i}$ approaches the exact value $v(x_i)$ for each collocation point x_i .

Theorem 0.35. *Let define $\varepsilon_{n,i}$ by $\forall 0 \leq i \leq n, \quad \varepsilon_{n,i} = |v(x_i) - v_{n,i}|$. Then,*

$$\lim_{n \rightarrow +\infty} \left(\max_{0 \leq i \leq n} \varepsilon_{n,i} \right) = 0.$$

After finding the first numerical solution by using collocation method and proving its convergence, it's time to transition to the second approach which is Nyström solution.

Nyström method

To approximate the numerical solution of the equation (0.351), we use the Nyström method, which is widely recognized as one of the most popular techniques for solving such mathematical equations. The advantage of the Nyström method is its ability to strike a balance between computational efficiency and accuracy. Provides reliable numerical solutions by defining integral terms and using appropriate numerical integration techniques. Using the Nyström method,



we get the following algebraic system

$$\left\{ \begin{array}{l} \lambda v_0 = f(x_0), \\ \lambda v_i = \omega_i K \left(x_i, x_i, g(x_i) + \frac{1}{(N-1)!} \sum_{j=0}^{i-1} \omega_j (x_i - x_j)^{N-1} v_j \right) + S_i + f(x_i), \\ S_i = \sum_{j=0}^{i-1} \omega_j K \left(x_i, x_j, g(x_j) + \frac{1}{(N-1)!} \sum_{k=0}^{j-1} \omega_k (x_j - x_k)^{N-1} v_k \right), \\ u_i = g(x_i) + \frac{1}{(N-1)!} \sum_{j=0}^{i-1} \omega_j (x_i - x_j)^{N-1} v_j. \end{array} \right. \quad (0.101)$$

where v_i is an approximation of $v(x_i)$ to be find, u_i is an approach of $u(x_i)$ and $\{\omega_i\}_{i=0}^n$ are called weight. We can compute them by using Gauss quadrature, simpson rule or trapezoidal rule. In this paper, we choose $\{\omega_i\}_{i=0}^n$ by the trapezoidal rule this mean $\omega_0 = \omega_n = \frac{h}{2}$ and $\omega_j = h$ for all $j = 1, \dots, n$ and $n \geq 1$.

Therefore $\sum_{j=0}^n \omega_j = b - a$.

Theorem 0.36. Let $(v_0, v_1, \dots, v_n) \in \mathbb{R}^{n+1}$ and $(u_0, u_1, \dots, u_n) \in \mathbb{R}^{n+1}$ be solutions of the system (0.457). Let $\varepsilon_{n,i}$ and $\bar{\varepsilon}_{n,i}$ be the errors defined by

$$\forall 0 \leq i \leq n, \quad \varepsilon_{n,i} = |v(x_i) - v_i|, \quad \text{and} \quad \bar{\varepsilon}_{n,i} = |u(x_i) - u_i|.$$

Then

$$\lim_{n \rightarrow +\infty} \left(\max_{0 \leq i \leq n} \varepsilon_{n,i} \right) = 0, \quad \text{and} \quad \lim_{n \rightarrow +\infty} \left(\max_{0 \leq i \leq n} \bar{\varepsilon}_{n,i} \right) = 0.$$

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Dynamical analysis of a discrete-time SIR epidemic model with loss of immunity

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Abstract. This study examines the dynamics of a discrete-time SIR epidemic model with loss of immunity, focusing on equilibrium points and their stability. Using Carcasses method [2], the research analytically and numerically proves the existence of various bifurcations, investigating two-parameter bifurcations and detecting flip and transcritical bifurcations. Graphical representations verify the results.

Keywords: Flip bifurcation, Transcritical bifurcation, bifurcation curve, Discrete epidemic model, Stability, Numerical simulation .



Introduction

In recent years, understanding the dynamics of infectious diseases has become increasingly vital, particularly in light of global health challenges. This study delves into a discrete-time SIR (Susceptible-Infected-Recovered) epidemic model that incorporates the critical aspect of immunity loss (see Figure 11), a phenomenon that significantly affects the spread of infectious diseases. The SIR model serves as a foundational framework in epidemiology, providing insights into how diseases propagate through populations and the impact of various interventions. One of the central objectives of this research is to analyze equilibrium points within the model and assess their stability. Equilibrium points are crucial in determining the long-term behavior of the epidemic. To validate our findings, we employ graphical representations that illustrate the relationships between parameters and the corresponding equilibrium states. These visualizations not only serve as a verification tool but also provide a clearer understanding of the underlying dynamics within the model.

In summary, this study contributes to the existing literature by elucidating the behavior of a discrete-time SIR model with immunity loss, providing valuable insights into equilibrium points and their stability, and enhancing our understanding of bifurcations in epidemic models.

The discrete-time SIRS epidemic model:

$$\begin{cases} S_{n+1} = F(S_n, I_n, R_n,) := S_n + h(A - \mu S_n - \lambda S_n I_n + \beta R_n), \\ I_{n+1} = G(S_n, I_n, R_n,) := I_n + h(\lambda S_n I_n - (\mu + r)I_n), \\ R_{n+1} = H(S_n, I_n, R_n,) := R_n + h(rI_n - (\mu + \beta)R_n), \\ N_{n+1} = (1 - h\mu)N_n + hA, \end{cases} \quad (0.102)$$

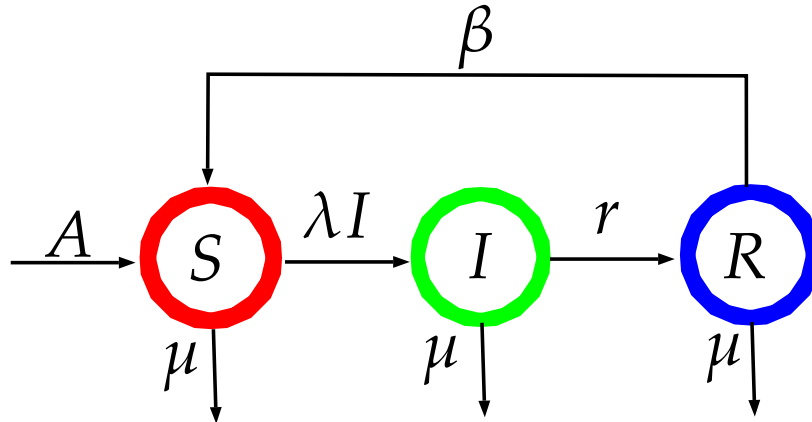


Figure 11: Transfer diagram for an SIR model with loss of immunity.



Main results

Equilibrium Points and Stability

The study identifies two key equilibrium points in the SIRS model: the **disease-free equilibrium** and the **endemic equilibrium**. When $R_0 < 1$, the disease-free equilibrium is stable, meaning the infection will eventually die out. When $R_0 > 1$, the endemic equilibrium becomes stable, meaning the disease persists in the population.

The stability of these equilibria is influenced by parameters such as the population recruitment rate A and the infection rate λ .

Definition 0.15. The basic reproduction number:

$$\mathcal{R}_0 = \frac{A \lambda}{\mu(\mu + r)},$$

which represents the average amount of subsequent infections produced by an initial group of infected individuals over their lifetime.

Concerning the existence of non-negative equilibrium points for system (0.102), we present the following results.

Theorem 0.37. System (0.102) always has a disease-free equilibrium $E_0 = (\frac{A}{\mu}, 0, 0)$. Moreover,

- if $\mathcal{R}_0 < 1$, then model (0.102) has only disease-free equilibrium E_0 ,
- if $\mathcal{R}_0 > 1$, then model (0.102) also has an endemic equilibrium, given by

$$E_1 = \left(\frac{\mu + r}{\lambda}, \frac{(A\lambda - \mu(\mu + r))(\beta + \mu)}{\mu\lambda(\beta + \mu + r)}, \frac{r(A\lambda - \mu(\mu + r))}{\mu\lambda(\beta + \mu + r)} \right).$$

Bifurcation Analysis

Two main types of bifurcations are observed

- **Flip (Period-Doubling) Bifurcation:** Occurs when oscillations in the system double their period, potentially leading to chaotic behavior. This explains how small changes in model parameters can lead to unpredictable disease outbreaks.
- **Transcritical Bifurcation:** Happens when the disease-free equilibrium and the endemic equilibrium exchange their stabilities, marking the critical threshold where the disease can become endemic or disappear.

Bifurcation curves are generated in the parameter space (A, λ) , showing how these bifurcations affect the system's stability.



Numerical Simulations

Numerical simulations confirm the theoretical bifurcation analysis. They illustrate stability regions and the transition from regular behavior to chaotic dynamics, showing how changes in key parameters lead to different types of disease spread patterns. The simulations help visualize the effects of bifurcations on the overall behavior of the disease, including chaotic dynamics in certain parameter ranges.

These results provide insights into epidemic dynamics, showing how bifurcations can lead to complex disease behavior of infections like COVID-19 [1].

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Zero-Hopf bifurcation of the Lotka-Volterra systems in $\mathbb{R}^3$ via averaging theory of third order

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Abstract. Here we study the Lotka-Volterra systems in $\mathbb{R}^3$, i.e. the differential systems of the form:

$$\frac{dx_i}{dt} = x_i \left(r_i - \sum_{j=1}^3 a_{ij} x_j \right), \quad i = 1, 2, 3.$$

It is known that some of these differential systems can have at least four periodic orbits bifurcating from one of their equilibrium points. Here we prove that there are some of these differential systems exhibiting at least six periodic orbits bifurcating from one of their equilibrium points. The tool for proving this result is the averaging theory of third order.

Keywords: Lotka-Volterra polynomial differential systems, periodic orbit, Hopf bifurcation, averaging theory.



Introduction

A Lotka-Volterra system in $\mathbb{R}^3$ with coordinates (x_1, x_2, x_3) is a quadratic polynomial differential system of the form:

$$\frac{dx_i}{dt} = x_i \left(r_i - \sum_{j=1}^3 a_{ij} x_j \right), \quad i = 1, 2, 3, \quad (0.103)$$

where the dot denotes derivative with respect to the independent variable t , usually called the time, and the r_i 's and the a_{ij} 's are parameters. Many natural phenomena can be modeled by the Lotka-Volterra systems, starting in biology with the time evolution of conflicting species, which continue to be studied intensively (see [9]). Later, such systems have also been applied to problems of plasma physics [15], and hydrodynamics [3].

Main results

We prove that some of these differential systems (0.63) exhibit at least six periodic orbits bifurcating from one of their equilibrium points, using third-order averaging theory.

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Numerical study of a two-dimensional potential flow problem with surface tension

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Abstract. The present work consider a two dimensional flow with a free surface of an incompressible and inviscid fluid above an obstacle formed an angle $\beta = \frac{\pi}{4}$ with the horizontal and with a width $2H$. The effect of surface tension is taken account but the gravity force is neglected. The shape of the free surface is determined numerically via series truncation technique for various values of Weber number α .

Keywords: Free surface, Potential flow , Series truncation, Weber number.



Introduction

Free surface flow has been studied in many engineering problems and flow over different forms of obstacles is one of example of these problems. This kind of problem is difficult to solve analytically due to the shape of the free surface which is unknown and the nonlinear boundary condition.

In this work, we present a numerical study of two dimensional jet flow above an obstacle which forms an angle β with the horizontal by taking in consideration the effect of the surface tension and supposing that the gravity influence is negligible. The flow is assumed to be steady, potential and irrotational. The fluid is considered as inviscid and incompressible. A mathematical solution is obtained numerically by applying the truncation series method used by several authors: Vanden-Broeck, F.Dias, H.Mekias, A.Gasmi and others.

Main results

Our study is of a problem which is characterized by the nonlinear boundary condition on the unknown shape of free surface equation. We solved this problem using the truncation method. We have numerically obtained the shape of free surface flow for various values of Weber number.

$$\alpha \geq \alpha^*, \text{ where } \alpha^* = 0.5$$

It is noted that from this critical point α^* , the numerical process diverges and some waves appear on the free surface. In this study, we have also obtained the same shape of the free surface determined for $\alpha \rightarrow \infty$ (i.e. exact solution) by N.Bounab and A.Djafer in other work about finding the exact solution.

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On the Stability of a Discrete Computer Virus Model with Fractional-Order

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Abstract.

The goal of this research is to create a novel fractional discrete-time computer virus model by generalizing the integer-order computer virus model. It is constructed to describe the propagation of computer viruses across the Internet and examine its dynamic properties. The paper shows that the proposed compartment model, described by difference equations, has two fixed points: a virus-free fixed point and a viral fixed point. Stability analyses of the disease-free fixed point are performed, with theoretical results linking the stability of fixed points to the basic reproduction number R_0 . Numerical simulations are conducted to verify the theoretical findings.

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Keywords: Fractional Discrete Model, Computer Virus, Basic Reproduction Number, Stability.



Introduction

Discrete fractional calculus extends traditional discrete calculus by incorporating non-integer order differences, enabling the study of systems with memory and hereditary properties. It generalizes integer-order difference equations to fractional orders, providing a powerful tool for modeling complex dynamic processes. This framework is widely applied in various fields, including physics, engineering, and biology.

These models allow researchers to simulate and predict the dynamics of computer virus propagation across networks, similar to how biological viruses spread within populations.

Computer viruses are malicious software programs designed to replicate themselves and spread from one computer to another. Understanding their behavior and how they spread is crucial for safeguarding digital systems and sensitive information from potential harm.

The main objective of our study is to present a fractional discrete-time computer virus model based on the Caputo like difference operator and analyse its stability.

Main results

In this study, a fractional-order discrete-time computer virus propagation model has been developed and examined, which is based on the Caputo-like difference operator. The suggested model more accurately captures several characteristics of computer viruses. Both commensurate and incommensurate fractional orders were used to comprehensively examine the dynamical features of this model. The stability of the fixed points is investigated.

Definition 0.16. For $0 < v < 1$, and $X(t)$ defined on $\mathbb{N}_a$, the Caputo-like delta difference is defined by:

$${}^c\Delta_a^v X(t) = \frac{1}{\Gamma(n-v)} \sum_{s=a}^{t-(n-v)} (t-s-1)^{-v} \Delta X(s), t \in \mathbb{N}_{a+1-v}, \quad (0.104)$$

where $X(t) : \mathbb{N}_a \rightarrow \mathbb{R}$ and the term $t^{(v)}$ defined in terms of the Gamma function Γ as: $t^{(v)} = \frac{\Gamma(t+1)}{\Gamma(t+1-v)}$.

Theorem 0.38. *The fractional difference equation*

$${}^c\Delta_a^v X(t) = f(t+v-1, X(t+v-1)),$$

has an equivalent discrete integral equation

$$X(t) = X(a) + \frac{1}{\Gamma(v)} \sum_{s=a+n-v}^{t-v} (t-s-1)^{v-1} f(t+v-1, X(s+v-1)), t \in \mathbb{N}_{a+1}, \quad (0.105)$$



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The balanced quantile credibility estimator with dependent risk structure.

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Abstract. This work presents a model in the methodology of the highest precision approach in credibility theory, we extend the quantile credibility model with dependent risks structure and an inflation factor by including the loss function weighted balanced, below in order to establish a new quantile credibility premium under the weighted balanced loss function. More precisely, the inhomogeneous credibility estimator is derived .

Keywords: Quantiles, Balanced Loss Function, Credibility estimator, Insurance risks .

Introduction

Credibility theory is a set of quantitative methods which allows an insurer to adjust premium based on the policyholder's experience and the experience of the entire group of policyholders. It has been widely used in commercial property of liability insurance and group health or life insurance. The well-known formulas in credibility theory take premium as a weighted sum of the average



experience of the policyholder and the average of the entire collection of policyholders. These formulas are easy to understand and simple to apply due to their linear properties. The modern credibility theory is believed to be attributed to the remarkable contribution by Bühlmann (1967), which is the first one that based the distribution free credibility theory on modern Bayes statistics. For the recent detailed introduction, see Herzog (2010) and Bühlmann and Gisler (2005), which describe modern credibility theory comprehensively.

In classical decision theory, the loss function usually focus on precision of estimation. However, goodness of fit is also a very important criterion. Thus there is a need to provide of estimation formally. Zellner (1994) introduced a general class of balanced loss functions of the form

$$L_1(x, \theta) = \omega(\delta_0(x) - \theta)^2 + (1 - \omega)(x - \theta)^2$$

where $0 \leq \omega \leq 1$ and $\delta_0(x)$ is chosen a prior target estimator of θ , which can be obtained for instance by means of maximum likelihood, least-squares or unbiased among others. The balanced loss function has been the subject of many theoretical and applied studies, for instance, Asgharzadeh and Sanjari Farsipour (2008), Mohammad et al. (2012), Sanjari Farsipour and Asgharzadeh (2004). Gómez-Déniz (2006, 2008) generalized the result in Bühlmann (1967) by using the balanced loss function. This present paper is an extension of the work of Wen and Deng (2011), while they addressed squared loss function, this study addresses balanced loss function with using quantiles. We aim at extending the Bühlmann–Straub credibility models to account for not only a certain inflation factor for claim amounts each individual risk, but also equal correlation between risks. Inspired by these papers, we will establish the credibility formulae for the model with correlation risk and inflation factor.

Main results

Definition 0.17. Let X_j be a random variable with cumulative distribution function $F_j(x)$, ($F_j(x)$ is differentiable and continuous everywhere), and p be a real number comprised between 0 and 1, i.e. $0 < p < 1$. We can write the p -quantile function ξ_p as follows:

$$\xi_{pj} = F_j^{-1}(p) = \inf \{x : F_j(x) \geq p\}$$

Given a sample $X_1, X_2, \dots, X_n$ of a continuous random variable X , with the corresponding orders statistics $X^{(1)}, X^{(2)}, \dots, X^{(n)}$, let $\hat{\xi}_{pj}$ denote the sample p -quantile which has the property that if

$$-\infty < X < +\infty \quad \text{and} \quad X \in [0, 1[: F_j(x) \geq p \quad \text{if and only if} \quad \xi_{pj} \leq p,$$

Theorem 0.39. *From the assumptions, the quantile credibility estimator of future claim Ξ_{i, n_i+1}^p , $i = 1, \dots, K$ in inhomogenous case is given by*



$$\widehat{\Xi}_{i,n_i+1}^{P \text{ inhom}} = r^{n+1} \left[Z_{i1} \widehat{\xi}^{T_i} + Z_{i2} \widehat{\xi}^d + Z_{i3} \widehat{\xi}^m + (1 - Z_{i1} - Z_{i2} - Z_{i3}) \Xi_p \right]$$

where Z_{i1} , Z_{i2} and Z_{i3} are credibility factors.

Lemma 0.12. *The inhomogeneous credibility estimator are actually the orthogonal projection of a random variable Y on $L(X,1)$ and $L_E(X)$. The following formulae hold true:*

$$PROJ(Y|L(X,1)) = E(Y) + Cov(Y, X)Var(X)^{-1}(X - E(X)),$$

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An incommensurate fractional-order discrete-time SIR epidemic model for influenza A viruses: Bifurcation, chaos, and control

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Abstract. In this paper, a three-dimensional incommensurate fractional discrete SIR epidemic model for influenza A viruses is presented. Adopting the Caputo-like difference operator, the fractional-order model is constructed, and the numerical scheme is obtained. The detailed dynamic analysis of the SIR model is performed using powerful nonlinear tools, such as phase plots, bifurcation diagrams, and the maximum Lyapunov exponent spectrum. We have confirmed that the proposed epidemic model can exhibit chaotic behavior for certain system parameters and fractional orders. Furthermore, based on the stability theory of discrete-time fractional-order systems, the stabilization of chaotic dynamics in the proposed model is realized via an active controller. MATLAB simulations were implemented in this paper to validate the theoretical results.

Keywords: SIR model, Fractional order, Chaotic dynamics, Chaos control.

Introduction

Influenza, often referred to as the flu, is a contagious viral disease caused by three virus types: A, B, and C influenza viruses are the most perilous, capable



of inducing serious disease and even death. They have also been associated with pandemics and epidemics. The viruses possess the capability to alter the antigenic properties of their surface, facilitating evasion of immunity established by prior infection. Influenza The subtypes of a virus are defined by two surface proteins: hemagglutinin (HA) and neurominidase (NA). Currently, there are 18 subtypes of hemagglutinin and 11 subtypes of neurominidase [1]. Mathematical models are essential tools in the analysis of infectious disease spread and control. It results in more effective techniques for mitigating disease transmission. Numerous epidemic models have been developed for analyzing the dynamics of A(H1N1) influenza virus transmission, including the SIR, SIS, SEIR, and SIRC models [2]. Due to changes in the antigenic features of influenza viruses, the SIR model is the best to describe this type of infectious diseases.

Fractional calculus is an important tool for the mathematical modeling of many problems in mathematics, physics, and engineering [3]. Discrete fractional calculus has received significant interest from several researchers in recent years, especially due to its potential uses in epidemiology, biology, finance, neural networks, secure communication, and various other fields. Because of the memory effect and hereditary properties, fractional derivatives are more suitable for modeling a variety of real-world phenomena. In particular, fractional discrete operators can capture the complete history of states in an epidemic model, allowing us to better describe and predict the disease's future behavior [1]. As a result, we can implement effective strategies for controlling its spread, such as new medicaments or vaccination programs.

In the current paper, a discrete-time fractional-order SIR model is presented. The SIR model's dynamic analysis reveals that it can exhibit chaotic dynamics. Thus, a chaos control scheme is constructed to stabilize the fractional-order SIR model.

Mathematical model

Recently, Yu et al. [2] proposed a three-dimensional dynamical system which describe the dynamics of SIR model for influenza A viruses as follows:

$$\begin{cases} \frac{dS}{dt} = A - dS - \lambda SI \\ \frac{dI}{dt} = \lambda SI - (d + r)I \\ \frac{dR}{dt} = rI - dR \\ N(t) = S(t) + I(t) + R(t), \end{cases} \quad (0.1)$$

where $S(t)$, $I(t)$, $R(t)$, and $N(t)$ represent the numbers of susceptible, infected, recovered individuals and total number of the individuals at time t , respectively. The parameter A represents the rate at which individuals are recruited into the population, d is the constant rate of death, r is the rate of recovery, λ is the rate of transmission. In the rest, we assume that $S(0) > 0$, $I(0) \geq 0$, $R(0) \geq 0$, and all the parameters are positive. In order to obtain the discretization of model (0.530), we apply the forward Euler scheme to model (0.530), then we get the



following discrete-time SIR epidemic model

$$\begin{cases} S_{m+1} = S_m + h(A - dS_m - \lambda S_m I_m) \\ I_{m+1} = I_m + h(\lambda S_m I_m - (d + r)I_m) \\ R_{m+1} = R_m + h(rI_m - dR_m), \end{cases} \quad (0.2)$$

where h is the step size.

Based on the Caputo-like difference operator [3], the incommensurate fractional-order discrete-time SIR model for influenza A viruses is obtained as

$$\begin{cases} {}^C\Delta_a^{\gamma_1} S(t) = h(A - dS(t-1+\gamma_1) - \lambda S(t-1+\gamma_1)I(t-1+\gamma_1)) \\ {}^C\Delta_a^{\gamma_2} I(t) = h(\lambda S(t-1+\gamma_2)I(t-1+\gamma_2) - (d+r)I(t-1+\gamma_2)) \\ {}^C\Delta_a^{\gamma_3} R(t) = h(rI(t-1+\gamma_3) - dR(t-1+\gamma_3)), \end{cases} \quad (0.3)$$

where a is the starting point, $t \in \mathbb{N}_{a+1-\gamma}$, and $\gamma_i \in (0, 1]$ ($i = 1, 2, 3$) are the fractional orders. Assuming that $a = 0$ and using the theorem in [4], the numerical formula of the discrete-time incommensurate fractional-order SIR model (0.544) is given by

$$\begin{cases} S(n) = S(0) + \frac{1}{\Gamma(\gamma_1)} \sum_{j=1}^n \frac{\Gamma(n-j+\gamma_1)}{\Gamma(n-j+1)} (h(A - dS(j-1) - \lambda S(j-1)I(j-1))) \\ I(n) = I(0) + \frac{1}{\Gamma(\gamma_2)} \sum_{j=1}^n \frac{\Gamma(n-j+\gamma_2)}{\Gamma(n-j+1)} (h(\lambda S(j-1)I(j-1) - (d+r)I(j-1))) \\ R(n) = R(0) + \frac{1}{\Gamma(\gamma_3)} \sum_{j=1}^n \frac{\Gamma(n-j+\gamma_3)}{\Gamma(n-j+1)} (h(rI(j-1) - dR(j-1))) \end{cases} \quad (0.4)$$

where $S(0)$, $I(0)$, and $R(0)$ are the initial conditions.

Stability of equilibria

The fractional-order discrete-time model (0.544) has the disease-free equilibrium $E_0 = (\frac{A}{d}, 0, 0)$, and the endemic fixed point $E_1 = (\frac{d+r}{\lambda}, \frac{A\lambda-d^2-dr}{\lambda(d+r)}, \frac{r(A\lambda-d^2-dr)}{\lambda d(d+r)})$. In order to establish the stability of fixed points of model (0.544), we will recall the following theorem.

Theorem 0.40. [8] Let $\phi(t) = (\phi_1(t), \phi_2(t), \dots, \phi_n(t))^T$, $\gamma \in (0, 1]$ be a fractional order, and $M \in \mathbb{R}^{n \times n}$. The zero equilibrium point of the fractional-order discrete-time system

$${}^C\Delta_a^\gamma \phi(t) = M\phi(t+\gamma-1), \quad t \in \mathbb{N}_{a+1-\gamma}, \quad (0.5)$$

is asymptotically stable if

$$\xi_j \in \left\{ z_0 \in \mathbb{C} : |z_0| < \left(2 \cos \frac{|\arg z_0| - \pi}{2 - \gamma} \right)^\gamma \quad \text{and} \quad |\arg z_0| > \gamma \frac{\pi}{2} \right\}, \quad j = 1, 2, \dots, n \quad (0.6)$$

where ξ_j are the eigenvalues of the matrix M .

Lemma 0.13. Suppose that $\mathcal{R}_0 = \frac{A\lambda}{d(d+r)}$ is the basic reproduction number for model (0.544), then



1. If $\mathcal{R}_0 < 1$, then model (0.544) has only disease-free equilibrium E_0 .
2. If $\mathcal{R}_0 > 1$, then model (0.544) has two equilibria: disease-free equilibrium E_0 and endemic equilibrium E_1 .

Using theorem 0.40, we will determine the stability conditions for the fixed point E_0 in the following theorem.

Theorem 0.41. *The disease-free equilibrium E_0 of the fractional-order discrete-time SIR model (0.544) is asymptotically stable if the following conditions are satisfied*

$$hd < 2^\gamma, \quad \frac{\lambda A}{d} < d + r, \quad |h(\frac{\lambda A}{d} - (d + r))| < 2^\gamma. \quad (0.7)$$

Now, the following theorem provides the stability conditions of the endemic fixed point E_1 .

Theorem 0.42. *The endemic equilibrium E_1 of the fractional-order discrete-time SIR model (0.544) is asymptotically stable if either one of the following two sets of conditions are met*

$$hd < 2^\gamma, \quad \Delta \geq 0, \quad \mathcal{R}_0 > 1, \quad \left| \frac{-B + \sqrt{\Delta}}{2} \right| < 2^\gamma, \quad (0.8)$$

where $B = d(\mathcal{R}_0 - 1)$, $\Delta = d(\mathcal{R}_0 - 1)(d(\mathcal{R}_0 - 1) - 4(d + r))$.
or

$$hd < 2^\gamma, \quad \Delta < 0, \quad |\xi_{2,3}| < \left(2 \cos \frac{|\arg \xi_{2,3}| - \pi}{2 - \gamma} \right)^\gamma, \quad |\arg \xi_{2,3}| > \gamma \frac{\pi}{2}, \quad (0.9)$$

where $\xi_{2,3} = \frac{-d(\mathcal{R}_0 - 1) \pm \sqrt{d(\mathcal{R}_0 - 1)(d(\mathcal{R}_0 - 1) - 4(d + r))}}{2}$.

Dynamics of the discrete-time fractional-order SIR model

In this section, the dynamics of the discrete-time incommensurate fractional-order SIR model (0.544) are analyzed by varying both the system parameters and fractional orders.

Dynamics with A varying

Fix the system parameters as $h = 0.78$, $\lambda = 0.81$, $r = 0.7$, $d = 0.3$, and the fractional orders as $(\gamma_1, \gamma_2, \gamma_3) = (0.98, 0.99, 0.96)$. The initial states are selected as $(S(0), I(0), R(0)) = (1.23, 4.58, 10.72)$. Let the parameter A range from 4.8 to 5.65. The bifurcation diagram and maximum Lyapunov exponent (MLE) are shown in Figs. 0.470(a) and 0.470(b), respectively. It should be noted that for the discrete-time fractional-order systems, the MLE can be approximated



using the Jacobian matrix algorithm; see [9]. As we can see, the MLE is strictly positive in the range $[5.44, 5.65]$, meaning that the system (0.544) is chaotic. However, the MLE takes negative values when A ranges between $4.8 \leq A < 5.44$, which indicates that the system is in a periodic state. Here, the route leading to chaos is a period-doubling bifurcation.

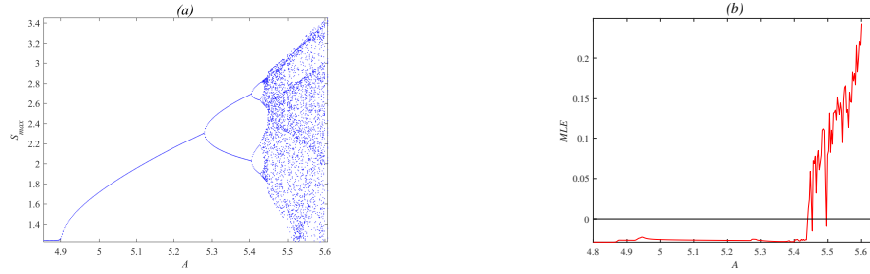


Figure 12: Dynamic analysis of the discrete-time fractional-order SIR model (0.544): (a) Bifurcation diagram as A varies; (b) Maximum Lyapunov exponent as A varies.

Dynamics with γ_1 varying

Here, we fix the system parameters as $h = 0.78$, $\lambda = 0.81$, $r = 0.7$, $d = 0.3$, $A = 5.55$, and the fractional orders $\gamma_2 = 0.99$, $\gamma_3 = 0.92$, and we select γ_1 as a bifurcation parameter varied in the range $[0.965, 1]$. The results of bifurcation diagram and maximum Lyapunov exponent are shown in Figs. 0.351(a) and 0.351(b), respectively. As can be shown, the effect of the fractional order γ_1 on the global dynamics of the SIR epidemic model is confirmed. Various epidemiological observed states are displayed, including, stable, periodic, and chaotic behaviors. Hence, the incommensurate fractional-order system (0.544) has rich dynamics.

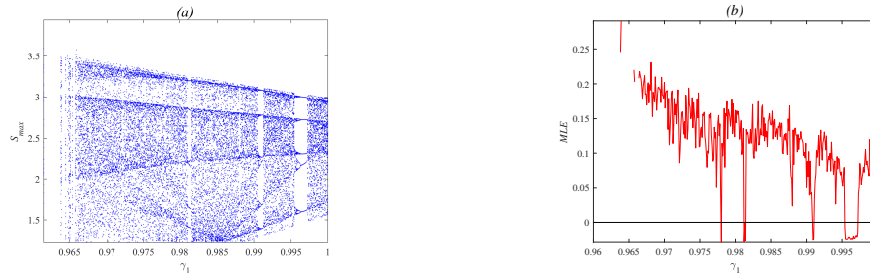


Figure 13: Dynamic analysis of the discrete-time fractional-order SIR model (0.544): (a) Bifurcation diagram as γ_1 varies; (b) Maximum Lyapunov exponent as γ_1 varies.



In order to provide further clarification, phase portrait of the incommensurate fractional-order SIR model (0.544) with different fractional orders in 2-D and 3-D projections is depicted in Fig. 0.457.

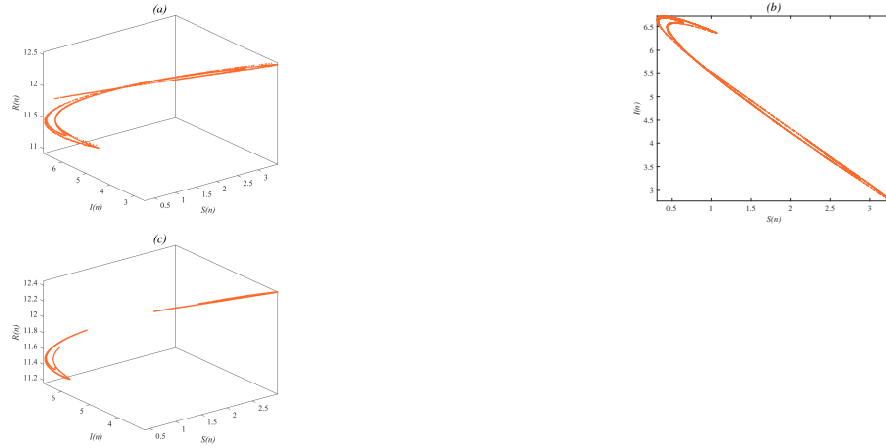


Figure 14: Phase portrait of the discrete-time incommensurate fractional-order SIR model (0.544): (a) for $(\gamma_1, \gamma_2, \gamma_3) = (0.967, 0.99, 0.92)$; (b) for $(\gamma_1, \gamma_2, \gamma_3) = (0.98, 0.99, 0.92)$; (c) for $(\gamma_1, \gamma_2, \gamma_3) = (1, 0.99, 0.92)$.

Dynamics with γ_2 varying

Now, we fix the system parameters as $h = 0.78$, $\lambda = 0.81$, $r = 0.7$, $d = 0.3$, $A = 5.55$, and the fractional orders $\gamma_1 = 0.985$, $\gamma_3 = 0.92$, and we select γ_2 as a bifurcation parameter varied in the range $[0.94, 1]$

.

Chaos control for the proposed SIR model

In this section, we design an active controller to stabilize the chaotic dynamics of the proposed SIR model when $\gamma_i = \gamma$ ($i = 1, 2, 3$). All system states will converge asymptotically toward zero when the controller is applied.

The states of the fractional-order discrete-time SIR model (0.544) with controller $(u_1, u_2, u_3)^T$ are described as

$$\begin{cases} {}^C\Delta_a^\gamma S(t) = h(A - dS(t-1+\gamma) - \lambda S(t-1+\gamma)I(t-1+\gamma)) + u_1(t-1+\gamma) \\ {}^C\Delta_a^\gamma I(t) = h(\lambda S(t-1+\gamma)I(t-1+\gamma_2) - (d+r)I(t-1+\gamma)) + u_2(t-1+\gamma) \\ {}^C\Delta_a^\gamma R(t) = h(rI(t-1+\gamma) - dR(t-1+\gamma)) + u_3(t-1+\gamma), \end{cases} \quad (0.10)$$

where $t \in \mathbb{N}_{a+1-\gamma}$.



Theorem 0.43. *The fractional-order discrete-time SIR model (0.544) can be stabilized under the following control law*

$$\begin{cases} u_1(t) = h(\lambda S(t)I(t) - A) \\ u_2(t) = -h\lambda S(t)I(t) \\ u_3(t) = -hrI(t). \end{cases} \quad (0.11)$$

Proof Substituting (0.547) into (0.546) yields

$$\begin{cases} {}^C\Delta_a^\gamma S(t) = -hdS(t-1+\gamma) \\ {}^C\Delta_a^\gamma I(t) = -h(d+r)I(t-1+\gamma) \\ {}^C\Delta_a^\gamma R(t) = -hdR(t-1+\gamma), \end{cases} \quad (0.12)$$

which can be expressed in compact form as

$${}^C\Delta_a^\gamma (S(t), I(t), R(t))^T = M \times (S(t-1+\gamma), I(t-1+\gamma), R(t-1+\gamma))^T, \quad (0.13)$$

where

$$M = \begin{pmatrix} -hd & 0 & 0 \\ 0 & -h(d+r) & 0 \\ 0 & 0 & -hd \end{pmatrix} \quad (0.14)$$

Obviously, the matrix M has the eigenvalues $\xi_1 = -hd$, $\xi_2 = -h(d+r)$, and $\xi_3 = -hd$. For $h = 0.78$, $\lambda = 0.81$, $r = 0.7$, $d = 0.3$, $A = 5.55$, the following conditions

$$|\xi_i| < \left(2 \cos \frac{|\arg \xi_i| - \pi}{2 - \gamma}\right)^\gamma \quad \text{and} \quad |\arg \xi_i| = \pi > \gamma \frac{\pi}{2}, \quad i = 1, 2, 3. \quad (0.15)$$

are satisfied, then the zero equilibrium of the SIR model (0.546) is asymptotically stable. Therefore, the states of the fractional-order discrete-time SIR model (0.546) converge towards zero asymptotically.

■ For numerical simulations, we select the parameter values as the chaotic case and the fractional order as $\gamma = 0.967$. The initial state is chosen as $(S(0), I(0), R(0)) = (1.23, 4.58, 10.72)$. Fig. 0.458 shows the evolution of states and phase portrait of the controlled fractional-order discrete-time SIR model (0.546).

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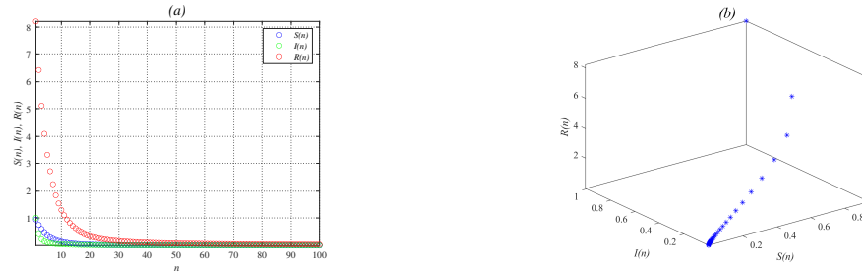


Figure 15: Stabilization of discrete-time fractional-order SIR model (0.544): (a) Evolution of states $S(n)$, $I(n)$, and $R(n)$; (b) Phase portrait in $(S - I - R)$ space.

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Different non-Bayesian estimation methods under the Exponential Generalized Log-Logistic Model

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Abstract. A modified Bagdonavicius-Nikulin chi-square goodness-of-fit is defined and studied. The lymphoma data is analyzed using the modified goodness-of-fit test statistic. Different non Bayesian estimation methods under uncensored scheme are considered and discussed. A numerical simulation studies are performed for comparing these estimation methods. The potentiality of the new model is illustrated using three real data sets and compared with many other well-known generalizations.

Keywords: Bagdonavicius-Nikulin Test; Barzilai-Borwein; Anderson Darling Estimation; Log-Logistic Model; Cramér-von-Mises Estimation.



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Le test de khi-deux modifié de Nikulin-Rao-Robson (NRR)

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Abstract. La validation du modèle choisi pour décrire les données collectées pour l'analyse est très importante. Les tests d'adéquation les plus couramment utilisés sont ceux basés sur les fonctions de distribution empirique (EDF) tels que le test de Kolmogorov-Smirnov et le célèbre test du chi-carré basé sur la statistique de Pearson X^2 . Néanmoins, la théorie montre que le test d'adéquation du chi-carré ne peut pas être appliqué lorsque la distribution n'est pas spécifique et que la distribution de la statistique du test dépend de la méthode utilisée pour l'estimation des paramètres inconnus. Par conséquent, différentes modifications ont été proposées dans la littérature statistique. Dans ce travail, nous utilisons la technique introduite par Nikulin (1973) et Rao et Robson (1974) pour construire un critère de test Y^2 afin de valider la distribution exponentielle étendue.

Keywords: test de khi-deux, estimateur de maximum de vraisemblance, test de khi-deux modifié.



Introduction

Le test du khi-deux modifié de Nikulin-Rao-Robson (NRR) est une extension du test du khi-deux classique, connu pour surmonter certaines de ses limitations dans des contextes pratiques. Ce test est principalement utilisé pour vérifier l'adéquation d'un modèle statistique à un ensemble de données empiriques, particulièrement lorsque les hypothèses classiques du test du khi-deux ne sont pas complètement respectées.

L'intérêt principal de la statistique NRR réside dans sa capacité à être plus flexible et robuste que le test classique, en permettant d'incorporer des pondérations adaptées au type de données et à la structure des échantillons disponibles. Là où le test de khi-deux standard peut perdre en efficacité, notamment pour des tailles d'échantillons petites ou moyennes, le test NRR ajuste la contribution de chaque catégorie à la statistique globale, ce qui le rend plus sensible aux écarts réels entre les données observées et le modèle théorique.

Dans un cadre pratique, le test NRR est particulièrement utile dans des domaines tels que la fiabilité, où les distributions de données sont souvent complexes ou hétérogènes. Il est utilisé pour évaluer la qualité de l'ajustement d'une distribution paramétrique, que ce soit pour des temps de vie, des taux de défaillance, ou d'autres mesures critiques dans l'étude de la fiabilité. En offrant un cadre plus souple pour estimer l'adéquation, le test NRR permet aux chercheurs de mieux identifier les écarts significatifs qui pourraient passer inaperçus avec d'autres tests d'ajustement plus simples.

L'une des forces majeures de la statistique NRR est sa capacité à s'adapter aux situations où les hypothèses du modèle ne sont pas strictement respectées. Cette adaptabilité est rendue possible grâce à des ajustements qui prennent en compte les spécificités des données, en améliorant l'efficacité des tests d'ajustement tout en réduisant les erreurs de type I ou II. De cette manière, le test NRR constitue un outil précieux pour les analyses statistiques avancées, en particulier dans des situations où la fiabilité des systèmes ou des processus est au centre des préoccupations.

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CLINICAL TRIALS SUBJECTED TO BAYESIAN SEQUENTIAL ANALYSIS

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Abstract.

This research presents a unique approach to Bayesian sequential analysis and forecasting methods for clinical trials. We offer a unified technique for sequential clinical trials using a Bayesian framework.

Keywords: Predictive methods, Bayesian analysis, test hypotheses.

Introduction

The group sequential design is probably one of the most commonly used clinical trial designs in clinical research and development. The primary reasons for conducting interim analyses of accrued data are probably ethical considerations, administrative reasons, and economic constraints (Jennison and Turnbull, 2000). Group sequential design has the advantage of early trial termination for safety, futility, or effectiveness reasons. In practice, because clinical trials involve human subjects, it is ethical to monitor the trials to ensure that individual subjects are not exposed to unsafe or ineffective treatment regimens; therefore, we consider clinical trial simulation to mimic the conduct of a clinical trial by creating virtual patients and predicting clinical outcomes for each virtual patient based on pre-specified models (Choi and Chang, 2011). We find in (Giovagnoli



and Zagoraoui, 2012) some challenger questions on the scientific rigour, ethics and effectiveness of simulation in clinical research and the interesting research problem of how to integrate virtual and physical experiments in a clinical context.

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An innovative Numerecal Approach for Addressing Weakly integro-Differential Volterra Équations

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Abstract. This work addresses the numerical treatment of a weakly singular Volterra integro-differential equation, where the singularity arises due to the derivative of the kernel $q'(t-s)$ diverging as $s \rightarrow t$. To handle this challenge, we reformulate the problem by introducing an auxiliary function $y(t) = x'(t)$, allowing the solution $x(t)$ to be reconstructed from $y(t)$. Using piece-wise projection and numerical quadrature, particularly the trapezoidal rule, we approximate the integral terms while mitigating the effects of the kernel's singularity. Additionally, the Nyström method is employed for efficient numerical approximation of the solution. The proposed approach effectively resolves the weak singularity and provides a robust framework for approximating solutions to such equations. The methodology is validated and compared with existing techniques.

Keywords: Volterra equation , integrodifferential equation , piece-wise projection, Nystrom method



Introduction

The weak singularity **appears** when the both sides of (1) one derived to give $\forall t \in [a, b]$,

$$x'(t) = \int_0^t q'(t-s)N(t, s, x(s), x'(s)) ds + \int_0^t q(t-s) \frac{\partial N(t, s, x(s), x'(s))}{\partial t} ds + f'(t), \quad (1)$$

Because,

$$\lim_{s \rightarrow t} q'(t-s) = +\infty.$$

We are going to use a methode developped basicaly by **Segni** and all [ref] to deal with regular case.

We introduce for all $t \in [a, b]$

$$y(t) = x'(t) \iff x(t) = f(a) + \int_0^t y(\theta) d\theta.$$

Then, (2) because for all $t \in [a, b]$,

At this stage, we have changed our goal, we will approximate numerically $y \in C^0[a, b]$ and x will be constructed from this approximation.

For that, we introduce a subdivision of $[a, b]$ given, for a natural number $N \geq 2$ by:

$$h = \frac{b-a}{N-1}, \quad t_j = a + (j-1)h, \quad 1 \leq j \leq N.$$

This allours us to use, trapazoidal rules to get for all $2 \leq j \leq N$,

$$\begin{aligned} \int_0^{t_i} q(t_i-s) \frac{\partial N}{\partial t}(t_i, s, f(a) + \int_a^s y(\theta) d\theta, y(s)) ds \approx \\ h \sum_{j=2}^{i-1} q(t_i-t_j) \frac{\partial N}{\partial t}(t_i, t_j, f(a) + h \sum_{p=2}^j y(t_p) + \frac{h}{2} y(t_1), y(t_1) \\ + \frac{h}{2} q(t_i-t_1) \frac{\partial N}{\partial t}(t_i, t_1, f(a), y(t_1)), \end{aligned}$$

with the convention $\sum_{j=2}^1$.

To deal with the second integral in (3), we have to use piece wise projection which gives us,

for all $s \in [t_{j-1}, t_j]$, $2 \leq j \leq N$,

$$\begin{aligned} N(t, s, f(a) + \int_a^s y(\theta) d\theta, y(s)) \approx \frac{s-t_{j-1}}{h} N\left(t, t_j, f(a) + \int_a^{t_j} y(\theta) d\theta, y(t_j)\right) \\ + \frac{t_j-s}{h} N\left(t, t_{j-1}, f(a) + \int_a^{t_{j-1}} y(\theta) d\theta, y(t_{j-1})\right) \\ \approx \frac{s-t_{j-1}}{h} N\left(t, t_j, f(a) + \frac{h}{2} y(t_1) + h \sum_{p=2}^j y(t_p) + y(t_j)\right) \\ + \frac{t_j-s}{h} N\left(t, t_{j-1}, f(a) + \frac{h}{2} y(t_1) + h \sum_{p=2}^{j-1} y(t_p) + y(t_{j-1})\right). \end{aligned}$$



This approach allows us to treat the weak singularity of p^λ as

$$\begin{aligned} & \int_0^{t_i} q^\lambda(t_i - s) N(t_i, s, f(a) + \int_a^s y(\theta) d\theta, y(s)) ds \approx \\ & \sum_{j=2}^i \int_{t_{j-1}}^{t_j} q^\lambda(t_i - s) N(t_i, s, f(a) + \int_a^s y(\theta) d\theta, y(s)) ds \\ & \approx \sum_{j=2}^i \left\{ \begin{array}{l} N\left(t_i, t_j, f(a) + \frac{h}{2}y(t_1) + h \sum_{p=2}^j y(t_p) + y(t_j)\right) \int_{t_{j-1}}^{t_j} q^\lambda(t_i - s) \frac{s-t_{j-1}}{h} ds \\ N\left(t_i, t_{j-1}, f(a) + \frac{h}{2}y(t_1) + h \sum_{p=2}^{j-1} y(t_p) + y(t_{j-1})\right) \int_{t_{j-1}}^{t_j} q^\lambda(t_i - s) \frac{t_{j-1}-s}{h} ds \end{array} \right\} \\ & \approx A_{ii} N\left(t_i, t_j, f(a) + \frac{h}{2}y(t_1) + h \sum_{p=2}^i y(t_p) + y(t_i)\right) + \\ & \sum_{j=2}^{i-1} (A_{ij} + B_{ij+1}) N\left(t_i, t_j, f(a) + \frac{h}{2}y(t_1) + h \sum_{p=2}^j y(t_p) + y(t_j)\right) \\ & + B_{i2} N\left(t_i, t_1, f(a) + \frac{h}{2}y(t_1) + y(t_1)\right), \end{aligned}$$

where, for $2 \leq j \leq N$,

$$\begin{aligned} A_{ij} &= \int_{t_{j-1}}^{t_j} q^\lambda(t_i - s) \frac{s-t_{j-1}}{h} ds, \\ B_{ij} &= \int_{t_{j-1}}^{t_j} q^\lambda(t_i - s) \frac{t_{j-1}-s}{h} ds. \end{aligned}$$

On the other hand, if we replace t by $t_1 = a$ in (3) we obtain $y(t_1) = f^\lambda(a)$. Now, using the previous steps, we build a numerical solution of y given by

$$Y_1 = f^\lambda(a), \quad (S_1)$$

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titre The generalization of Arutyunov's theorem on (q_1, q_1) -quasimetric space

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Abstract. The Arutyunov's theorem [1] showed the existence of solution for an operator defined from metric space into an other one. In our work we will present the same result for an operator defined from (q_1, q_2) -quasimetric space into space with generalized distance.

Keywords: Metric space, Covering mapping, (q_1, q_2) -quasimetric space, Lipschitz mapping.

Introduction

In this work, we will present the existence of solution for operator equation for mappings of two variables acting from (q_1, q_2) -quasimetric space to space with a generalized distance satisfied only the first condition of distance.



Main results

The following definition is the definition of (q_1, q_2) -quasimetric space.

Definition 0.18. Let $X \neq \emptyset$, q_1 and q_2 are positive numbers. Let the application $d_X : X \times X \rightarrow \mathbb{R}_+$ satisfied the following properties:

1. $d_X(x, y) = 0 \Leftrightarrow x = y, \quad \forall x, y \in X$
2. $d_X(x, z) \leq q_1 d_X(x, y) + q_2 d_X(y, z), \quad \forall x, y, z \in X,$

so d_X is called a (q_1, q_2) -quasimetric on X and (X, d_X) is a (q_1, q_2) -quasimetric space.

Let $q_0 > 0$,

- If $d_X(x, y) \leq q_0 d_X(y, x)$ for all $x, y \in X$, so d_X is called q_0 -symmetric (q_1, q_2) -quasimetric and (X, d_X) is a q_0 -symmetric (q_1, q_2) -quasimetric space.
- If $q_0 = 1$, so (X, d_X) is called symmetric (q_1, q_2) -quasimetric.
- If $q_1 = q_2 = 1$, so d_X is called quasimetric and (X, d_X) is called quasimetric space.
- If $q_0 = q_1 = q_2 = 1$, so d_X is called a metric (distance) and (X, d_X) is a metric space.

Now, the space Y is a generalized metric space (that's mean distance satisfied only the axiom of identity). (Y, d) is a generalized metric space where Y is a nonempty set and d is a distance satisfied only the axiom of identity i.e.:

$$\forall y_1, y_2 \in Y \quad d(y_1, y_2) = 0 \Leftrightarrow y_1 = y_2.$$

For mappings acting from (q_1, q_2) -quasimetric space to space with distance, we have the following definitions of covering mapping and Lipschitz mapping, respectively.

Definition 0.19. Let $\alpha > 0$, the mapping $\psi : X \rightarrow Y$ is called α -covering, if satisfying the relation :

$$\forall x_0 \in X \quad \forall y \in Y \quad \exists x \in X : \psi(x) = y \quad \text{and} \quad \rho(x, x_0) \leq \frac{1}{\alpha} d(\psi(x), \psi(x_0)).$$

Definition 0.20. : Let $\beta > 0$, the mapping $\varphi : X \rightarrow Y$ is called β -Lipschitz, if satisfying the relation:

$$d(\varphi(x), \varphi(x_0)) \leq \beta \rho(x, x_0) \quad \forall x, x_0 \in X$$

Now, we define the equation studied on this abstract. Given a mapping $F : X \times X \rightarrow Y$, an element $\tilde{y} \in Y$. We define a mapping $\Phi : X \rightarrow Y$ by the identity $\Phi(x) \rightarrow F(x, x)$, and consider the equation

$$\Phi(x) = \tilde{y}. \tag{1}$$



For $\theta \in [0, 1)$ and for $n \in \mathbb{N}$, we will denote by $S(\theta, n)$ the sum of the n terms of a geometric sequence $\sum_{i=0}^{n-1} \theta^i$ and therefore, $S(\theta, n) = \frac{1-\theta^n}{1-\theta}$. We consider that $S(\theta, 0) = 0$ and $\beta^0 = 1$ when $\beta = 0$. For arbitrary values $q_0, q_1, q_2 \geq 1$, we set $m_0 = \min\{j \in \mathbb{N} / q_2 \beta^j < \alpha^j\}$, in the hypothesis that $q_0^2 \beta < \alpha$, we set

$$n_0 = \min\{j \in \mathbb{N} / q_1 (q_0^2 \beta)^j < \alpha^j\}.$$

Note that the existence of m_0 follows from the hypothesis $\beta < \alpha$.

We have the following theorem.

Theorem 0.44. *Let a (q_1, q_2) -quasimetric space (X, d_X) be complete and (Y, d) is space with a generalized distance d . If the following conditions are satisfied: An application $F(x, \cdot) : X \rightarrow Y$ is α -covering and closed, and an application $F(\cdot, x) : X \rightarrow Y$ is β -Lipschitz with $\alpha > \beta$. Then, for all $x_0 \in X$ there exists ξ solution for (1) and if (X, d_X) is a q_0 -symmetric space and $q_0^2 \beta < \alpha$, so for ξ , the following estimation is verified:*

$$\bar{d}_X(x_0, \xi) \leq q_0 q_2 \frac{q_2 \alpha^{n_0-1} S(q_1 q_0^2 \beta / \alpha, n_0 - 1) + (q_1 q_0^2 \beta)^{n_0-1}}{\alpha^{n_0} - q_1 (q_0^2 \beta)^{n_0}} d_Y(F(x_0, x_0), \tilde{y}),$$

where $\bar{d}_X(\xi, x_i) = d_X(x_i, \xi)$.

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Fixed Point Theorem for Systems Involving Mixed Local and Non-Local Operators

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Abstract. This article presents results on the existence, uniqueness, and regularities properties, as well as the boundary behavior of solutions for singular systems involving mixed local and non-local elliptic operators. More precisely, we first establish a new weak comparison principle for a singular equation. Afterward, we discuss constructing suitable ordered pairs of sub-solutions and super-solutions. This allows us to obtain the existence of a pair of positive weak solutions for System (S) by employing Schauder's fixed-point theorem in the associated conical shell. Finally, we adapt a method of Krasnoselsky to establish the uniqueness of such a positive pair of solutions.

Keywords: Singular systems; local and non-local operator; regularity results; Schauder's fixed-point Theorem.



Introduction

Let $0 < s_1 < 1$, $0 < s_2 < 1$, and $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$. Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be an open bounded domain with $C^{1,1}$ boundary $\partial\Omega$.

In this article, we deal with the existence, non-existence, uniqueness, and other qualitative properties of solutions to the following singular system:

$$\begin{cases} \mathcal{L}_1 u = k_1(x) u^{-\alpha_1} v^{-\beta_1}, & u > 0 \quad \text{in } \Omega; \quad u = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega, \\ \mathcal{L}_2 v = k_2(x) v^{-\alpha_2} u^{-\beta_2}, & v > 0 \quad \text{in } \Omega; \quad v = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (\text{S})$$

Here $\mathcal{L}_i$, $i = 1, 2$ is the mixed operator, defined as:

$$\mathcal{L}_i = -\Delta + (-\Delta)^{s_i}. \quad (0.16)$$

On the right-hand side, $k_j : \Omega \rightarrow (0, +\infty)$, $j = 1, 2$ are continuous functions and satisfy the following growth condition: for some $a_j > 0$ and any $x \in \Omega$

$$C_1 d^{-a_j}(x) \leq k_j(x) \leq C_2 d^{-a_j}(x), \quad (0.17)$$

where $C_1, C_2 > 0$ and $d(x) := \text{dist}(x, \partial\Omega) = \inf_{y \in \partial\Omega} |x - y|$, for any $x \in \Omega$. The word "**mixed**" above refers to the type of the operator combining the classical Laplacian $(-\Delta)$ and the fractional Laplacian $(-\Delta)^s$, which for a fixed parameter $s \in (0, 1)$, is defined as

$$(-\Delta)^s u(x) := C(N, s) \mathbf{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy,$$

where $\mathbf{P.V.}$ denotes the Cauchy principal value, and $C(N, s)$ is an appropriate normalizing constant, whose explicit expression is given by

$$C(N, s) := \left(\int_{\mathbb{R}^N} \frac{1 - \cos(\xi_1)}{|\xi|^{N+2s}} d\xi \right)^{-1}.$$

We refer the reader to [1, 2, 3] for a comprehensive discussion of the main properties of this fractional operator, which includes various real-world applications as well as results pertaining to continuous and compact embeddings and other important properties. For recent advancements in obtaining precise estimates for the best constants in fractional subcritical Sobolev embeddings, we direct the reader to the work by [1]. Furthermore, significant results have been explored in the context of fractional problems involving mixed Dirichlet-Neumann boundary conditions or Choquard problems. For additional relevant works and valuable insights on these topics, we refer to [2, 3].

The following is our result regarding the **existence** and **uniqueness** of a pair of positive weak solutions to (0.344):

Theorem 0.45. *Assume that $\alpha_1, \alpha_2, \beta_1, \beta_2$ are positive numbers. Additionally, assume that k_i , $i = 1, 2$ satisfies (0.309).*



- (1) If $a_1 + \beta_1 + \alpha_1 \leq 1$ and $a_2 + \beta_2 + \alpha_2 \leq 1$. Then system (0.344) possesses a unique positive weak solution $(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$ satisfying the following inequalities for some $C > 0$:

$$C^{-1}d \leq u, v \leq Cd \text{ hold in } \Omega, \text{ if } a_1 + \beta_1 + \alpha_1 < 1 \text{ and } a_2 + \beta_2 + \alpha_2 < 1,$$

and for some $\kappa \in (0, 1)$

$$C^{-1}d \leq u, v \leq Cd^{1-\kappa} \text{ hold in } \Omega, \text{ if } a_1 + \beta_1 + \alpha_1 = 1 \text{ and } a_2 + \beta_2 + \alpha_2 = 1.$$

- (2) Let

$$\gamma = \frac{(2 - a_1)(\alpha_2 + 1) - \beta_1(2 - a_2)}{(\alpha_1 + 1)(\alpha_2 + 1) - \beta_1\beta_2} \text{ and } \xi = \frac{(2 - a_2)(\alpha_1 + 1) - \beta_2(2 - a_1)}{(\alpha_1 + 1)(\alpha_2 + 1) - \beta_1\beta_2}.$$

Now, assume that $a_1 + \xi\beta_1 + \alpha_1 > 1$ with $a_1 + \xi\beta_1 < \frac{3}{2}$ and $a_2 + \gamma\beta_2 + \alpha_2 > 1$ with $a_2 + \gamma\beta_2 < \frac{3}{2}$. Then, the problem (0.344) has a unique weak solution (u, v) according to Definition ?? and satisfies the following inequalities with a constant $C > 0$:

$$C^{-1}d^\gamma \leq u \leq Cd^\gamma \text{ and } C^{-1}d^\xi \leq v \leq Cd^\xi \text{ hold in } \Omega.$$

Moreover, we have the following Sobolev regularity:

- $(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$ if and only if $\nu_1^* < 1$ and $\nu_2^* < 1$.
- $(u^{\nu_1}, v^{\nu_2}) \in H_0^1(\Omega) \times H_0^1(\Omega)$ if and only if $\nu_i > \nu_i^* \geq 1, i = 1, 2$,

$$\text{where } \nu_1^* := \frac{\alpha_1 + 1}{2(2 - a_1 - \xi\beta_1)} \text{ and } \nu_2^* := \frac{\alpha_2 + 1}{2(2 - a_2 - \gamma\beta_2)}.$$

- (3) Let

$$\gamma = \frac{2 - a_1 - \beta_1}{\alpha_1 + 1}.$$

If $a_1 + \alpha_1 + \beta_1(1 - \kappa_2) > 1$ for some $\kappa_2 \in (0, 1)$, with $a_1 + \beta_1 < \frac{3}{2}$, and $\gamma\beta_2 + a_2 \leq 1$ hold, then the problem (0.344) possesses a unique weak solution (u, v) in the sense of Definition ??, satisfying the following inequalities for some constant $C > 0$:

$$C^{-1}d^\gamma \leq u \leq Cd^\gamma \text{ and } C^{-1}d \leq v \leq Cd \text{ hold in } \Omega, \text{ if } \gamma\beta_2 + a_2 < 1,$$

$$C^{-1}d^{\gamma + \kappa_2} \leq u \leq Cd^\gamma \text{ and } C^{-1}d \leq v \leq Cd^{1 - \kappa_2} \text{ hold in } \Omega, \text{ if } \gamma\beta_2 + a_2 = 1.$$

Furthermore, $v \in H_0^1(\Omega)$ and:

- $u \in H_0^1(\Omega)$ if and only if $\nu_1^* < 1$.
- If $\gamma\beta_2 + a_2 < 1$, then $u^{\nu_1} \in H_0^1(\Omega)$ if and only if $\nu_1 > \nu_1^* \geq 1$.
- If $\gamma\beta_2 + a_2 = 1$, then $u^{\nu_1} \in H_0^1(\Omega)$ if and only if $\nu_1 > \nu_1^{**} \geq 1$.

$$\text{where } \nu_1^* := \frac{\alpha_1 + 1}{2(2 - a_1 - \beta_1)} \text{ and } \nu_1^{**} := \frac{\alpha_1 + 1}{2(2 - a_1 - \beta_1(1 - \kappa_2))}.$$



(4) Similarly to **Part 3** mentioned above, let

$$\xi = \frac{2 - a_2 - \beta_2}{\alpha_2 + 1}.$$

If $a_2 + \alpha_2 + \beta_2(1 - \kappa_1) > 1$ for some $\kappa_1 \in (0, 1)$, with $a_2 + \beta_2 < \frac{3}{2}$, and $\xi\beta_1 + a_1 \leq 1$ hold, then the problem (0.344) possesses a unique weak solution (u, v) in the sense of Definition ??, satisfying the following inequalities for some constant $C > 0$:

$C^{-1}d^\xi \leq v \leq Cd^\xi$ and $C^{-1}d \leq u \leq Cd$ hold in Ω , if $\xi\beta_1 + a_1 < 1$,

$C^{-1}d^{\xi+\kappa_1} \leq v \leq Cd^\xi$ and $C^{-1}d \leq u \leq Cd^{1-\kappa_1}$ hold in Ω , if $\xi\beta_1 + a_1 = 1$.

Furthermore, $u \in H_0^1(\Omega)$ and:

- $v \in H_0^1(\Omega)$ if and only if $\nu_2^* < 1$.
- If $\xi\beta_1 + a_1 < 1$, then $u^{\nu_2} \in H_0^1(\Omega)$ if and only if $\nu_2 > \nu_2^* \geq 1$.
- If $\xi\beta_1 + a_1 = 1$, then $u^{\nu_2} \in H_0^1(\Omega)$ if and only if $\nu_2 > \nu_2^{**} \geq 1$.

where $\nu_2^* := \frac{\alpha_2+1}{2(2-a_2-\beta_2)}$ and $\nu_2^{**} := \frac{\alpha_2+1}{2(2-a_2-\beta_2(1-\kappa_1))}$.

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Existence of solutions for a class of biharmonic equation with p-Laplacian

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Abstract. In the paper, we study the multiplicity of solutions for a class of biharmonic equation with p-Laplacian of the form:

$$\begin{cases} \Delta^2 u - \Delta_p u + \lambda V(x)u = f(x, u) + \mu \xi(x)|u|^{q-2}u, & x \in \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{cases} \quad (0.18)$$

where $N \geq 5$, $\Delta^2 := \Delta(\Delta)$ is the biharmonic operator, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ with $p > 2$, $V \in \mathcal{C}(\mathbb{R}^N)$, $f \in \mathcal{C}(\mathbb{R}^N \times \mathbb{R})$, $\xi \in L^{\frac{2}{2-q}}(\mathbb{R}^N, \mathbb{R}^+)$, $\lambda > 0, \mu > 0$ and $1 < q < 2$.

Keywords: Variational methods, p-Laplacian, mixed nonlinearity, Gagliardo–Nirenberg inequality.



Introduction

We consider the following biharmonic equation:

$$\begin{cases} \Delta^2 u - \Delta_p u + \lambda V(x)u = f(x, u) + \mu \xi(x)|u|^{q-2}u, & x \in \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{cases} \quad (0.19)$$

Unlike most other works, we replace the Laplacian with a p-Laplacian. By using the mountain pass theorem and Ekeland's variational principle. To overcome the difficulty of the convergence of the subsequences for the Palais-Smale sequences of the Euler-Lagrange functional, we consider Cerami sequences.

We make the following assumptions:

- (A1) $V(x) \in \mathcal{C}(\mathbb{R}^N, \mathbb{R}^+)$ and there exists a constant $b > 0$ such that the set $\{V < b\} := \{x \in \mathbb{R}^N | V(x) < b\}$ has finite positive Lebesgue measure;
- (A2) $\Omega = \text{int } V^{-1}(0)$ is nonempty and has smooth boundary with $\bar{\Omega} = V^{-1}(0)$.
- (A3) $f \in \mathcal{C}(\mathbb{R}^N \times \mathbb{R})$ and $|f(x, u)| \leq c(1 + |u|^{r-1})$ for some $r \in (2, 2^*)$, where $2^* = \frac{2N}{N-4}$;
- (A4) $f(x, u) = o(|u|)$ as $|u| \rightarrow 0$ uniformly for $x \in \mathbb{R}^N$;
- (A5) there exists $\theta > p$ such that $0 < \theta F(x, u) \leq u f(x, u)$ for every $x \in \mathbb{R}^N$ and $u \neq 0$, where $F(x, u) = \int_0^u f(x, t) dt$.

Main results

Here is the main result of this work:

Theorem 0.46. *Assume that conditions (A1) – (A5) hold, $2 < p \leq \frac{2N}{N-2}$, and $\xi \in L^{\frac{2}{2-q}}(\mathbb{R}^N, \mathbb{R}^+)$ ($1 < q < 2$). Then there exist two positive constants Λ_0 and $\mu_0 > 0$ such that for every $\lambda > \Lambda_0$ and $0 < \mu < \mu_0$, problem (0.73) has at least two nontrivial solutions u_λ^i ($i = 1, 2$).*

Under some mild assumptions and $p = 2$ in problem (0.73), Zhang, Tang, J Zhang, Luo [1] have established the existence of two nontrivial solution



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A Novel Approach to Mild Solutions in Stochastic Fractional Equations

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Abstract. This paper examines the existence of mild solutions for fractional differential equations of neutral type, incorporating non-local conditions and infinite delay within Hilbert spaces. We utilize fractional calculus and the Krasnoselskii-Schaefer fixed point theorem to derive our results. Additionally, we provide an example to illustrate the practical application of these findings.

Keywords: Infinite delay, Stochastic fractional differential equations, mild solution, fixed point theorem.



Introduction

The main purpose of this paper is to prove the existence of mild solutions for fractional differential equations of neutral type with infinite delay in Hilbert spaces of the form:

$$\begin{cases} {}^c D_t^\alpha [x(t) - h(t, x_t)] = A[x(t) - h(t, x_t)] + f(t, x_t) + \int_{-\infty}^t \sigma(t, s, x_s) dW(s) & t \in J = [0, b] \\ x(0) + \mu(x) = x_0 = \phi(t) & t \in (-\infty, 0], \end{cases} \quad (0.20)$$

Here, $x(\cdot)$ takes values in a real separable Hilbert space $\mathbb{H}$ with inner product $(\cdot, \cdot)_{\mathbb{H}}$ and the norm $\|\cdot\|_{\mathbb{H}}$. The fractional derivative ${}^c D^\alpha, \alpha \in (0, 1)$, is understood in the Caputo sense. The operator A generates a strongly continuous semigroup of bounded linear operators $S(t), t \geq 0$, on $\mathbb{H}$. Let $\mathbb{K}$ be another separable Hilbert space with inner product $(\cdot, \cdot)_{\mathbb{K}}$ and the norm $\|\cdot\|_{\mathbb{K}}$. W is a given $\mathbb{K}$ -valued Wiener process with a finite trace nuclear covariance operator $Q \geq 0$ defined on a filtered complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. The histories $x_t : \Omega \rightarrow \mathcal{C}_v$ defined by $x_t = \{x(t + \theta), \theta \in (-\infty, 0]\}$ belong to the phase space $\mathcal{C}_v$, which will be defined in section* 2. The initial data $\phi = \{\phi(t), t \in (-\infty, 0]\}$ is an $\mathcal{F}_0$ -measurable, $\mathcal{C}_v$ -valued random variable independent of W with finite second moments, and $h : J \times \mathcal{C}_\theta \rightarrow \mathbb{H}, h : J \times \mathcal{C}_v \rightarrow \mathbb{H}, \sigma : J \times J_1 \times \mathcal{C}_v \rightarrow \mathcal{L}_2^0(\mathbb{K}, \mathbb{H})$ are appropriate functions, where $J_1 = (-\infty, b]$ and $\mathcal{L}_2^0(\mathbb{K}, \mathbb{H})$ denotes the space of all Q -Hilbert Schmidt operators from $\mathbb{K}$ into $\mathbb{H}$. $\mu : C(J, \mathbb{H}) \rightarrow \mathbb{H}$ is bounded and the initial data x_0 is an $\mathcal{F}$ adapted $\mathbb{H}$ -valued random variable independent of Wiener process W .

Main results

Lemma 0.14. 4 *Let H be a Hilbert space and Φ_1, Φ_2 two operators on H such that*

- i) Φ_1 is a contraction and
- ii) Φ_2 is completely continuous.

Then either

- a) *the operator equation $\Phi_1 x + \Phi_2 x = x$ has a solution or*
- b) *$G = \{x \in \mathbb{H} : \lambda \Phi_1(\frac{x}{\lambda}) + \lambda \Phi_2 x = x\}$ is unbounded for $\lambda \in (0, 1)$.*

Lemma 0.15. [12] *Let $v(\cdot), w(\cdot) : [0, b] \rightarrow [0, \infty)$ be continuous function. If $w(\cdot)$ is nondecreasing and there exist two constants $\theta \geq 0$ and $0 < \alpha < 1$ such that*

$$v(t) \leq w(t) + \theta \int_0^t \frac{v(s)}{(t-s)^{1-\alpha}} ds, \quad t \in J$$



then

$$v(t) \leq e^{\theta^n (\Gamma(\alpha))^n t^{n\alpha} / \Gamma(n\alpha)} \sum_{j=0}^{n-1} \left(\frac{\theta b^\alpha}{\alpha} \right)^j w(t),$$

for every $t \in [0, b]$ and every $n \in \mathbb{N}$ such that $n\alpha > 1$.

Definition 0.21. An $\mathbb{H}$ - valued stochastic process $\{x(t), t \in (-\infty, b]\}$ is a mild solution of the system **0.20** if $x(0) + \mu(x) = x_0 = \phi(t)$ on $(-\infty, 0]$ satisfying $\|\phi\|_{\mathcal{C}_v}^2 < +\infty$, the process x satisfies the following integral equation

$$\begin{aligned} x(t) = & S_\alpha(t)[\phi(0) - \mu(x) - h(0, \phi)] + h(t, x_t) + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s, x_s) ds \\ & + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) \left[\int_{-\infty}^s \sigma(s, \tau, x_\tau) dW(\tau) \right] ds \end{aligned}$$

where

$$S_\alpha(t)x = \int_0^\infty \zeta_\alpha(\theta) S(t^\alpha \theta) x d\theta, \quad T_\alpha(t)x = \alpha \int_0^\infty \theta \zeta_\alpha(\theta) S(t^\alpha \theta) x d\theta$$

and ζ_α is a probability density function defined on $(0, \infty)$

The following properties of $S_\alpha(t)$ and $T_\alpha(t)$ appeared in [13] are useful.

Lemma 0.16. *The operators $S_\alpha(t)$ and $T_\alpha(t)$ have the following properties*

i) *For any fixed $t \geq 0$, $S_\alpha(t)$ and $T_\alpha(t)$ are linear and bounded operators such that for any $x \in \mathbb{H}$*

$$\|S_\alpha(t)x\|_{\mathbb{H}} \leq M \|x\|_{\mathbb{H}} \quad \text{and} \quad \|T_\alpha(t)x\|_{\mathbb{H}} \leq \frac{M_\alpha}{\Gamma(1+\alpha)} \|x\|_{\mathbb{H}}$$

ii) *$S_\alpha(t)$ and $T_\alpha(t)$ are strongly continuous and compact.*

To study existence of mild solutions of **0.20**, we introduce the following hypotheses.

(H_1) : The function $h, f : J \times \mathcal{C}_v \rightarrow \mathbb{H}$ are continuous and there exist some constants M_h, M_f , such that

$$E \|h(t, x) - h(t, y)\|_{\mathbb{H}}^2 \leq M_h \|x - y\|_{\mathcal{C}_v}^2, \quad x, y \in \mathcal{C}_v, \quad t \in J$$

$$E \|h(t, x)\|_{\mathbb{H}}^2 \leq M_h (1 + \|x\|_{\mathcal{C}_v}^2)$$

$$E \|f(t, x) - f(t, y)\|_{\mathbb{H}}^2 \leq M_f \|x - y\|_{\mathcal{C}_v}^2, \quad x, y \in \mathcal{C}_v, \quad t \in J$$

$$E \|f(t, x)\|_{\mathbb{H}}^2 \leq M_f (1 + \|x\|_{\mathcal{C}_v}^2)$$



(H₂) : μ is continuous and there exists some positive constants M_μ such that

$$E \|\mu(x) - \mu(y)\|_{\mathbb{H}}^2 \leq M_\mu \|x - y\|_{\mathcal{C}_v}^2, x, y \in \mathcal{C}_v, t \in J$$

$$E \|\mu(x)\|_{\mathbb{H}}^2 \leq M_\mu (1 + \|x\|_{\mathcal{C}_v}^2)$$

(H₃) : For each $\varphi \in \mathcal{C}_v$,

$$k(t) = \lim_{a \rightarrow \infty} \int_{-a}^0 \sigma(t, s, \varphi) dW(s)$$

exists and is continuous. Further, there exists a positive constant M_k such that

$$E \|k(t)\|_{\mathbb{H}}^2 \leq M_k$$

(H₄) The function $\sigma : J \times J_1 \times \mathcal{C}_v \rightarrow L(\mathbb{K}, \mathbb{H})$ satisfies the following:

- i) for each $(t, s) \in J \times J$, $\sigma(t, s, \cdot) : \mathcal{C}_v \rightarrow L(\mathbb{K}, \mathbb{H})$ is continuous and for each $x \in \mathcal{C}_v$, $\sigma(\cdot, \cdot, x) : J \times J \rightarrow L(\mathbb{K}, \mathbb{H})$ is strongly measurable;
- ii) there is a positive integrable function $m \in L^1([0, b])$ and a continuous non-decreasing function $M_\sigma : [0, \infty) \rightarrow (0, \infty)$ such that for every $(t, s, x) \in J \times J \times \mathcal{C}_v$, we have

$$\int_0^t E \|\sigma(t, s, x)\|_{\mathcal{L}_2^0}^2 ds \leq m(t) M_\sigma(\|x\|_{\mathcal{C}_v}^2), \quad \liminf_{r \rightarrow \infty} \frac{M_\sigma(r)}{r} ds = \Delta < \infty$$

iii) For any $x, y \in \mathcal{C}_v, t \geq 0$, there exists a positive constant L_σ such that

$$\int_0^t E \|\sigma(t, s, x) - \sigma(t, s, y)\|_{\mathcal{L}_2^0}^2 ds \leq L_\sigma \|x - y\|_{\mathcal{C}_v}^2$$

(H₅) :

$$N_0 = 2l^2 \{12M^2 M_\mu + 4M_h\} \quad (0.21)$$

$$N_1 = 2 \|\phi\|_{\mathcal{C}_v}^2 + 2l^2 \bar{F} \quad (0.22)$$

$$N_2 = 8l^2 \left(\frac{M_\alpha}{\Gamma(1+\alpha)} \right)^2 \frac{b^\alpha}{\alpha} M_f \quad (0.23)$$

$$N_3 = 16bl^2 \left(\frac{M_\alpha}{\Gamma(1+\alpha)} \right)^2 \frac{b^\alpha}{\alpha} Tr(Q) \quad (0.24)$$

$$K_1 = \frac{N_1}{1 - N_0}, \quad K_2 = \frac{N_2}{1 - N_0}, \quad K_3 = \frac{N_3}{1 - N_0} \quad (0.25)$$

$$\bar{F} = 12M^2(C_1 + C_2) + 12M^2 M_\mu + 4M_h + 4 \left(\frac{M_\alpha}{\Gamma(1+\alpha)} \right)^2 \frac{b^{2\alpha}}{\alpha^2} M_f + 8b \left(\frac{M_\alpha}{\Gamma(1+\alpha)} \right)^2 \frac{b^{2\alpha}}{\alpha^2} M_k \quad (0.26)$$



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Some fixed point theorems

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Keywords: Common Fixed point, matrices convergent to zero, weakly compatible.

Main results

Fixed point theory is currently one of the most important tools in many scientific fields, such as engineering, computer science, applied science, and the development of nonlinear analysis. One of the most useful tools in this area is the Banach contraction theorem, which states that suppose T is a mapping on a complete metric space (X, d) into itself satisfying $d(Tx, Ty) \leq Kd(x, y)$, where $K \in [0, 1) \forall x, y \in X$, then T has a unique fixed point on X which was introduced by Banach in 1922. Then, using various contractions and mappings in various metric spaces, this result was expanded by a large number of authors.

The Banach contraction principle's counterpart in b -metric space, proposed by Bakhtin as a generalization of metric space, was established in 1989. Various problems of the convergence of measurable functions with respect to measure, Czerwik first presented as a generalization of Banach fixed point theorem in b -metric spaces. In 2009, Boriceanu investigated fixed point theory for multivalued generalized contraction on a set with two b -metrics. A class of contractive mappings for b -metric space was used by S. Agrawal, K. Qureshi and J. Nema



in 2016 to study the fixed point theorem. In 2006, I. A. Rus primarily studied contractions in metric space with fixed point properties and first proposed a new contraction known as the Rus contraction.

In our work, a version modified of the contraction Hardy-Rogers type is proved. Moreover, we apply this modified version to investigate the existence of unique fixed point for mappings satisfying generalized contractive condition through the help of the properties of the matrices convergent to zero and weakly compatible maps. These results extend various comparable results from the literature.

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Recent Results on Data Dependence in Partial Metric Spaces

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Abstract. The aim of this work is to show the last result mentioned in [6] which is extended the several results such as [4]

Keywords: $\mathcal{C}$ -class function, Data dependence, Multifunction, Partial metric space

Introduction

The fixed point theory is a powerful tool with numerous applications in various fields such as biology, chemistry, economics, engineering, game theory, computer science, and mathematical modeling. Recent developments in fixed point theory have focused on extending classical results to more general abstract spaces like b-metric spaces, partial metric spaces, and fuzzy metric spaces. One notable advancement is the introduction of $\mathcal{C}$ -class functions [1], which have been used to prove fixed point theorems in different abstract spaces, particularly in the context of partial metric spaces. $\mathcal{C}$ -class functions provide a unified framework for studying various types of contractions and have applications in solving differential equations, integral equations, and variational inequalities [2].

The study of partial metric spaces began in 1994, when Matthews introduced this generalization of traditional metric spaces in [3]. Since then, partial metric spaces have found applications across many fields because of their ability to represent asymmetric distance relationships.

During the same period, there was significant progress in multivalued mapping research, which greatly contributed to the development of generalized fixed point theory. Multifunctions emerged as a natural approach to addressing problems that involved non-uniqueness or set-valued constraints.



However, as people became more interested in using these methods in real-life situations modeled by partial metrics, it became necessary to come up with new ways to think about data dependence for multifunctions in this new setting [4]. Data dependence properties play a crucial role in examining how perturbations in the domain affect or propagate to the range sets. In the case of single-valued mappings on metric spaces, classic results have established strong connections between input and output distances. [4]

Main results

Let us proceed to review the essential characteristics and definitions of partial metric spaces.

Definition 0.22. [3] The function $p : X \times X \rightarrow \mathbb{R}^+$ defines a partial metric on a nonempty set X , where $\mathbb{R}^+$ includes all nonnegative real numbers. If the following four conditions are satisfied for every $x, y, z \in X$, we call the pair (X, p) a partial metric space:

$$\mathfrak{P}_1: p(x, x) = p(y, y) = p(x, y) \Leftrightarrow x = y.$$

$$\mathfrak{P}_2: p(x, x) \leq p(x, y).$$

$$\mathfrak{P}_3: p(x, y) = p(y, x)$$

$$\mathfrak{P}_4: p(x, y) + p(z, z) \leq p(x, z) + p(z, y).$$

The partial metric space represented by the pair (X, p) .

Definition 0.23 ($\mathcal{C}$ -class functions). [1, 2] Assume that there is a continuous mapping $\mathfrak{F} : J \times J' \rightarrow \mathbb{R}$. If $\mathfrak{F}$ satisfies these requirements, we will classify it as a $\mathcal{C}$ -class function.

$$(\mathfrak{F}_1) \text{ For any } (s, t) \in J \times J', \text{ we have } s \geq \mathfrak{F}(s, t).$$

$$(\mathfrak{F}_2) \text{ If } \mathfrak{F}(s, t) = s, \text{ then the product } st = 0.$$

In addition, note that $\mathfrak{F}(0, 0) = 0$ and that $\mathcal{C}$ is the set of all functions of the $\mathcal{C}$ -class on $J \times J'$.

In the work [2], the authors introduced the following collections of $\mathcal{C}$ -class functions:

Definition 0.24. [2] The set of functions of the $\mathcal{C}$ -class that satisfy these criteria is called $\mathcal{C}_I$:

- $\mathfrak{F}(s, t)$ is non-decreasing for both s and t when $(s, t) \in J \times J'$.
- For any fixed $t \in J'$, the series

$$\tilde{\mathfrak{w}}(s, t) := \sum_{n=0}^{\infty} \mathfrak{F}^n(s, t)$$



converges for all $s \in J$. The function $\mathfrak{F}$ is defined as follows, and $\mathfrak{F}^n$ represents the n -th iteration of this function:

$$\mathfrak{F}^0(s, t) = s, \mathfrak{F}^1(s, t) = \mathfrak{F}(s, t), \text{ and } \mathfrak{F}^{n+1}(s, t) = \mathfrak{F}(\mathfrak{F}^n(s, t), t).$$

Definition 0.25. [2] $\mathcal{C}_{II}$ comprises a set of $\mathcal{C}$ -class functions that adhere to the following specifications:

- $\mathfrak{F}(s, t)$ exhibits non-decreasing behavior in s and non-increasing behavior in t .
- For any given $t \in J'$, the series

$$\tilde{\mathfrak{w}}(s, t) := \sum_{n=0}^{\infty} \mathfrak{F}^n(s, t)$$

converges for every $s \in J$, where the n -th iteration of the function $\mathfrak{F}$ with the following recurrence relation is represented as $\mathfrak{F}^n$:

$$\mathfrak{F}^0(s, t) = s, \mathfrak{F}^1(s, t) = \mathfrak{F}(s, t), \text{ and } \mathfrak{F}^{n+1}(s, t) = \mathfrak{F}(\mathfrak{F}^n(s, t), \mathfrak{F}^n(s, t))$$

A class of functions Ξ that were mentioned in [2] are recalled in the following. These functions, represented as $\tau : X^2 \times (2^X)^2 \rightarrow J'$, satisfy a crucial condition. To be more precise, for any $x, y \in X$ and $A, C \in 2^X$, $\tau(x, y, A, C) = 0$ implies that $x = y$ or $p(x, y) = 0$ is true. Additionally, we establish the nondecreasing property of $\tau \in \Xi$ within the (X, p) space, as indicated by the following inequality:

$$p(x, y) \leq p(a, b) \Rightarrow \tau(x, y, A_x, C_y) \leq \tau(a, b, A_a, C_b) \quad \forall A_x, A_a, C_y, C_b \in 2^X.$$

Theorem 0.47. Consider a partial metric space (X, p) , where $\bar{x} \in X$, $\lambda \in [0, 1]$, and $r > 0$ such that the subspace $\overline{\mathbb{B}_{p,r}}(\bar{x})$ is complete. Let $T, F : \overline{\mathbb{B}_{p,r}}(\bar{x}) \rightarrow C^p(X)$ be multivalued mappings. Assuming that $\tau \in \Xi$, $\alpha \in J$, and $\mathfrak{F} \in \mathcal{C}$, which is upper semicontinuous with respect to the first variable, satisfy either of the following conditions:

- $\mathfrak{F} \in \mathcal{C}_I$ and τ is nondecreasing,
- $\mathfrak{F} \in \mathcal{C}_{II}$ and $\tau(x, y, F(x), F(y)) \geq \alpha$ for $x, y \in \overline{\mathbb{B}_{p,r}}(\bar{x})$.

We will establish our assumptions based on the satisfaction of the following two conditions:

- (a) $p(z, F(z)) < \alpha$ where $\tilde{\mathfrak{w}}(\alpha, \cdot) \leq r(1 - \lambda)$, $\forall z \in \overline{\mathbb{B}_{p,\lambda r}}(\bar{x})$.
- (b) $\delta_p(F(x) \cap \overline{\mathbb{B}_{p,r}}(\bar{x}), F(y)) \leq \mathfrak{F}(M_F(x, y), \tau(x, y, F(x), F(y)))$, $\forall x, y \in \overline{\mathbb{B}_{p,r}}(\bar{x})$.

Then, for any given $K \subseteq \overline{\mathbb{B}_{p,r}}(\bar{x})$, and for every $y \in \text{Fix}(T)$ and $w \in F(y)$, satisfying $p(y, w) < \alpha$, we can establish the following inequality:

$$\delta_p(\text{Fix}(T) \cap \overline{\mathbb{B}_{p,\lambda r}}(\bar{x}) \cap K, \text{Fix}(F)) \leq \tilde{\mathfrak{w}}(\mathfrak{M}, \tau(y, w, F(y), F(w))) \quad (0.27)$$

where $\mathfrak{M} := \sup_{x \in \overline{\mathbb{B}_{p,r}}(\bar{x})} \delta_p(T(x) \cap \overline{\mathbb{B}_{p,\lambda r}}(\bar{x}) \cap K, F(x))$.



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Exponential stability of mild solutions for Impulsive functional evolution equations

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Abstract. In this study, we explore mild solutions to first-order impulsive functional evolution equations defined by local conditions within a separable Banach space. Our main objective is to demonstrate the exponential stability of these solutions. This research is crucial in illuminating the behavior of systems that experience abrupt changes or impulses over time. By proving exponential stability, we enhance the understanding of how such systems can retain stability despite sudden shifts. Our findings not only advance the theoretical framework but also offer practical insights into the analysis and control of real-world systems affected by impulsive dynamics.

Keywords: Impulsive functional differential equations, mild solutions, exponential stability, fixed point, semigroups, separable Banach space.



Introduction

Impulsive equations play a crucial role in modeling dynamics across numerous scientific disciplines. Recent progress, especially in fixed-moment equations, is comprehensively explored in the monographs by Bainov and Simeonov [1], Benchohra et al. [2], and Lakshmikantham et al. [3]. In this paper, our goal is to establish the necessary conditions for the exponential stability of mild solutions to first-order impulsive functional evolution equations under local conditions in a separable Banach space $(E, |\cdot|)$. These equations are expressed as follows:

$$y'(t) - Ay(t) = f(t, y_t), \quad \text{a.e. } t \in J := [0, +\infty), \quad t \neq t_k, \quad k \in \mathbb{N}^*; \quad (0.28)$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k \in \mathbb{N}^*; \quad (0.29)$$

$$y(t) = \varphi(t), \quad t \in H := [-r, 0], \quad \text{where } 0 < r < +\infty; \quad (0.30)$$

Where $f : J \times D \rightarrow E$ is a given function, $\varphi \in D$, $I_k \in C(E, E)$, $t_k < t_{k+1}$, $\forall k \in \mathbb{N}^*$, $A : D(A) \subset E \rightarrow E$ is the infinitesimal generator of a C_0 -semigroup $T(t)$, $t \geq 0$, and E is a real separable Banach space with norm $|\cdot|$. Such that $D = \{\psi : [-r, 0] \rightarrow E : \psi|_{I_t} \text{ is continuous and } \psi(t_k^-), \psi(t_k^+) \text{ exist with } \psi(t_k^-) = \psi(t_k^+)\}$ and $I_t = [-r, +\infty) \setminus \{t_k, k \in \mathbb{N}^*\}$.

We denote by y_t the element of D defined by: $y_t(\theta) = y(t + \theta)$, $\theta \in [-r, 0]$, here $y_t(\cdot)$ represents the history of the state from $t - r$ up to the present time t .

In this framework, N. Abada et al. [?] provided criteria guaranteeing the existence of both mild and extremal solutions over a bounded interval. To conclude, we will demonstrate an application that highlights the practical relevance of our theoretical findings.

Main results

Definition 0.26. (Mild solution) A function $y \in BPC([-r, +\infty), E)$ is said to be a mild solution of problem (0.63) – (0.33) if $y(t) = \varphi(t)$, $t \in H := [-r, 0]$ and y is a solution of the following impulsive integral equation:

$$y(t) = T(t) \varphi(0) + \int_0^t T(t-s) f(s, y_s) ds + \sum_{0 < t_k < t} T(t-t_k) I_k(y(t_k^-)), \quad \text{for each } t \in J.$$

We'll need to propose the following hypotheses, which will be assumed later:

- (H1) $A : D(A) \rightarrow E$ is the infinitesimal generator of a strongly continuous semigroup $T(t)$, $t \in J$ which is compact for $t > 0$ in the Banach space E . Let $\|T(t)\|_{B(E)} \leq M$ for all $t \geq 0$ such that $M > 0$;
- (H2) The function $f : J \times D \rightarrow E$ is caratheodory;
- (H3) There exists a continuous function $k : J \rightarrow [0, +\infty)$ such that:

$$|f(t, u) - f(t, v)| \leq k(t) \|u - v\|, \quad t \in J, \quad u, v \in D, \quad f(t, 0) = 0$$

$$\text{and } k^* := \sup_{t \in J} \int_0^t k(s) ds;$$



(H4) There exist constants $d_k > 0$, $k \in \mathbb{N}^*$ with $\sum_{k \in \mathbb{N}^*} d_k = d < \infty$ such that for each $x, y \in E$, we have:

$$|I_k(y) - I_k(x)|_E \leq d_k |y - x|_E \text{ and } I_k(0) = 0;$$

Theorem 0.48. *Assume that (H1) – (H4) hold. If $M(k^* + d) < 1$. Then the mild solutions of (0.63) – (0.33) is exponentially stable.*

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Existence of solutions For Fractional impulsive neutral stochastic Functional integro-differential equations with infinite delay

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Abstract. This paper is mainly concerned with the existence of mild solutions for Fractional impulsive neutral stochastic Functional integro-differential equations infinite delay in Hilbert spaces. The results are obtained by the fixed point theorems. An Example is provided to illustrate the obtained theory.

Keywords: Fractional impulsive neutral stochastic Functional integrodifferential equations, Fixed point theorems, infinite delay.



Introduction

Fractional Calculus deals with the generalization of integrals and derivatives of noninteger order. Fractional calculus involves a wide area of applications by bringing into a broader paradigm concepts of physics, mathematics and engineering [12,13]. Fractional differential equations draw a great application in many physical phenomenas such as seepage flow in porous media and in fluid dynamic traffic models. The most important advantage of using fractional differential equations in these and other applications is their nonlocal property. This means that the next state of a system depends not only upon its current state but also upon all of its historical states. This is probably the most relevant feature for making this fractional tool useful from an applied standpoint and interesting from a mathematical standpoint and in turn led to the sustained study of the theory of fractional differential equations.

Differential equations with impulsive conditions constitute an important field of research due to their numerous applications in ecology, medicine biology, electrical engineering and other areas of science. Many physical phenomena in evolution processes are modeled as impulsive fractional differential equations and existence results for such equations have been studied by several authors [28,33]. Feckan [11] studied the existence of a solution for a class of impulsive differential equations with fractional derivative. The existence of mild solutions for a class of impulsive fractional partial semilinear differential equations has been discussed in [28]. The existence results for a class of impulsive fractional order semilinear evolution equations with infinite delay has been reported in [8]. Moreover, it is known that the development of the theory of functional differential equations with infinite delay depends on a choice of a phase space. The existence of a mild solution for fractional semilinear differential equations with infinite delay has been discussed in [19]. Wang et al. [30] studied the solvability and optimal controls for a class of fractional integrodifferential evolution systems with infinite delay in Banach spaces. The stochastic differential equations with infinite delay have become important in recent years as mathematical models of phenomena in both physical and social sciences [31,24]. Wei and Wang [31] considered a class of stochastic functional differential equations with infinite delay in which the initial value belongs to the phase space. The existence, uniqueness and stability of mild solutions for time-dependent stochastic evolution equations with Poisson jumps and infinite delay has been investigated in [25].

Since impulsive effects also widely exist in fractional stochastic differential systems, it is important and necessary to discuss the qualitative properties for stochastic fractional equations with impulsive perturbations and infinite delay. However, to the authors knowledge, no result has been reported on the existence problem of impulsive fractional stochastic differential equations with infinite delay and the aim of this paper is to fill this gap.

We consider the following impulsive fractional stochastic differential equations with infinite delay in the form



$$D^\alpha[x(t)-g(t, x_t)] = A[x(t)+e(t, s, \int_0^t h_1(t, s, x_s)ds)]dt+f(t, x_t)dt+\sigma(t, x_t, \int_0^t h_2(t, s, x_s)ds)dw(t),$$

$$t \in J = [0, b], \quad t \neq t_k, k = 1, 2, \dots, m, \quad (0.31)$$

$$\Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_K(x(t_k^-)), \quad k = 1, 2, \dots, m, \quad (0.32)$$

$$x_t = \phi(t), \quad \phi(t) \in \mathcal{B}_h \quad (0.33)$$

where D_t^α is the Caputo fractional derivative of order $\alpha, 0 < \alpha < 1$; $x(\cdot)$ takes the value in the separable Hilbert space H ; $A : D(A) \subset H \rightarrow H$ is the infinitesimal generator of an α -resolvent family $S_\alpha(t)_{t \geq 0}$. The history $x_t : (-\infty, 0] \rightarrow H, x_t(\theta) = x(t+\theta), \theta \leq 0$, belongs to an abstract phase space B_h ; $f, g : J \times B_h \rightarrow H, e : J \times B_h \times H \rightarrow H, h_1, h_2 : J \times J \times H \rightarrow H$ and $\sigma : J \times B_h \times L_0^2 \rightarrow H$ are appropriate functions to be specified later; $I_k : B_h \rightarrow H, (k = 1, 2, \dots, m)$, are appropriate functions. Here $0 = t_0 \leq t_1 \leq \dots \leq t_m \leq t_{m+1} = T, \Delta x(t_k) = x(t_k^+) - x(t_k^-), x(t_k^+) = \lim_{h \rightarrow 0} x(t_k + h)$ and $x(t_k^-) = \lim_{h \rightarrow 0} x(t_k - h)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively. The initial data $\varphi = \{\varphi(t), t \in (-\infty, 0]\}$ is an $\mathcal{F}_0$ -measurable, B_h -valued random variable independent of w with finite second moments. Now, we present the abstract space phase B_h . Assume that $h : (-\infty, 0] \rightarrow (0, \infty)$ with $l = \int_{-\infty}^0 h(t)dt < \infty$ a continuous function. We define the abstract phase space B_h by

$$\mathcal{B}_h = \left\{ \varphi : (-\infty, 0] \rightarrow H, \text{ for any } a > 0, (E|\varphi(\theta)|^2)^{1/2} \text{ is bounded and measurable function on } [-a, 0] \right. \\ \left. \text{and } \int_{-\infty}^0 h(s) \sup_{s \leq \theta \leq 0} (E|\varphi(\theta)|^2)^{1/2} ds < \infty. \right\}$$

If B_h is endowed with the norm

$$\|\varphi\|_{B_h} = \int_{-\infty}^0 h(s) \sup_{s \leq \theta \leq 0} (E|\varphi(\theta)|^2)^{1/2} ds, \varphi \in B_h,$$

then $(B_h, \|\cdot\|_{B_h})$ is a Banach space [29, 18].

Now we consider the space

$$B_b = \left\{ x : (-\infty, T] \rightarrow H \text{ such that } x|_{J_k} \in C(J_k, H) \text{ and there exist } x(t+k) \text{ and } x(t-k) \text{ with} \right. \\ \left. x(t_k) = x(t-k), x_0 = \varphi \in B_h, k = 1, 2, \dots, m \right\},$$

where $x|_{J_k}$ is the restriction of x to $J_k = (t_k, t_{k+1}]$, $k = 0, 1, 2, \dots, m$. The function $\|B_b\|$ to be a seminorm in B_b , it is defined by $\|x\|_{B_b} = \|\varphi\|_{B_h} + \sup_{s \in [0, T]} (E\|x(s)\|^2)^{1/2}, x \in B_b$.



The rest of this paper is organized as follows. In section* 2, some preliminaries are presented. In section* 3, we study the existence and the uniqueness of solution for the impulsive fractional system (2.1) – (2.2) – (2.3). In section* 4, we give an example to illustrate our theory.

Main results

In this section*, we first establish the existence of mill solutions to nonlinear fractional stochastic equation (2.1) – (2.2) – (2.3). More precisely, we will formulate and prove sufficient conditions for the existence of solutions to (2.1) – (2.2) – (2.3) with infinite delay and impulses. In order to establish the results, we impose the following conditions.

(H1) The operators $S_\alpha(t)$ and $T_\alpha(t)$ have

- (i) If $\alpha \in (0, 1)$ and $\mathbf{A} \in A^a(\theta_0, \omega_0)$, then for any $x \in H$ and $t > 0$ we have $\|T^\alpha(t)\|Me^{\omega t}$ and $\|S^\alpha(t)\| \leq Ce^{\omega t}(1 + t^{\alpha-1})$, $\omega > \omega_0$. Thus, we have

$$\|T^\alpha(t)\| \leq \tilde{M}_T$$

and

$$\|S^\alpha(t)\| \leq t^{\alpha-1}\tilde{M}_S,$$

where $\tilde{M}_T = \sup_{0 \leq t \leq T} \|T_\alpha(t)\|$, and $\tilde{M}_S = \sup_{0 \leq t \leq T} Ce^{\omega t}(1 + t^{1-\alpha})$ (for more details, see [23]).

- (ii) The operators $\{S_\alpha(t)\}_{t \geq 0}$ and $\{T_\alpha(t)\}_{t \geq 0}$ are strongly continuous and compact.

- (iii) For any $x \in \mathbf{H}$, $\beta \in (0, 1)$ and $\eta \in (0, 1]$, we have

$$\mathbf{A}T_\alpha(t)x = A^{1-\beta}T_\alpha(t)A^\beta x, \quad 0 \leq t < b$$

and

$$\|\mathbf{A}^\eta T_\alpha(t)\| \leq \frac{\alpha C_\eta \Gamma(2 - \alpha)}{t^{\alpha\eta} \Gamma(1 + \alpha(1 - \eta))}, \quad 0 \leq t < b$$

(H2) There exist $\mu_1 > 0$ such that

$$E\|f(t, x) - f(t, y)\|_H^2 \leq \mu_1 \|x - y\|_{B_h}^2$$



(H3) There exists a constant $M_{h_1} \geq 0$, such that

$$\left\| \int_0^t [h_1(t, s, x) - h_1(t, s, y)] ds \right\|^2 \leq M_{h_1} \|x - y\|_{B_h}^2.$$

(H4) For each $(t, s) \in J \times J$, the function $h_2(t, s, \cdot) : B_h \rightarrow H$ is continuous for each $x \in B_h$, the function $h_2(\cdot, \cdot, x) : J \times J \rightarrow H$ is strongly measurable. There exists a constant $M_{h_2} \geq 0$, such that

$$\|h_2(t, s, x) - h_2(t, s, y)\|^2 \leq M_{h_2} \|x - y\|_{B_h}^2.$$

(H5) There exists constants $0 < \beta < 1$, such that e is H^β -valued, $(A)^\beta e : J \times B_h \rightarrow H$ is completely continuous,

(i) The function $e : J \times B_h \times H \rightarrow H$ for $t \in J, x_1, x_2 \in B_h$ and $y_1, y_2 \in H$ such that the function M_e - satisfies the Lipschitz condition:

$$E\|(A)^\beta e(t, x_1, y_1) - (A)^\beta e(t, x_2, y_2)\|^2 \leq M_e[\|x_1 - x_2\|_{B_h} + \|y_1 - y_2\|].$$

(ii) There exist constants $0 < \beta < 1, M_g$ such that g is H^β -valued, $(A)^\beta g$ is continuous, and

$$E\|(A)^\beta g(t, x) - (A)^\beta g(t, y)\|^2 \leq M_g \|x - y\|_{B_h}^2 \quad t \in J, x, y \in B_h$$

(H6) There exist $\nu_1, \nu_2 > 0$ such that

$$E\|\sigma(t, \gamma, x) - \sigma(t, \psi, y)\|_{L_0^2}^2 \leq \nu_1 \|\gamma - \psi\|_{B_h}^2 + \nu_2 E\|x - y\|_H^2$$

(H7) For each $k = 1, 2, \dots, m$, there exist $n_k > 0$ such that

$$E\|I_k(x) - I_k(y)\|_H^2 \leq n_k E\|x - y\|_H^2, \text{ for all } x, y \in H.$$

Now, we present the definition of mild solutions for the system (2.1)–(2.2)–(2.3) based on the paper [25].

Definition 0.27. An $\mathfrak{F}_t$ -adapted stochastic process $x : (-\infty, T] \rightarrow H$ is called a mild solution of the system (2.1) – (2.2) – (2.3) if $x_0 = \varphi \in B_h$ satisfying $x_0 \in L_0^2(\Omega, H)$ and the following conditions hold.

(i) $x(t)$ is B_h -valued and the restriction of $x(\cdot)$ to the interval $(t_k, t_{k+1}]$, $k = 1, 2, \dots, m$ is continuous.



(ii) for each $t \in J$, $x(t)$ satisfies the following integral equation

$$x(t) = \begin{cases} \phi(t); t \in]-\infty, 0] \\ \\ T_\alpha(t)[\phi(0) - g(0, \phi)] + g(t, X_t) + \int_0^t S_\alpha(t-s)f(s, x_s)ds + \int_0^t AT_\alpha(t-s)g(s, x_s)ds \\ + \int_0^t AT_\alpha(t-s)e(s, x_s, \int_0^s h_1(s, \tau, x_\tau)d\tau)ds + \int_0^t S_\alpha(t-s)\sigma(s, x_s, \int_0^s h_2(s, \tau, x_\tau)d\tau)d\omega(s); \quad t \in [0, t_1] \\ \\ T_\alpha(t)[x(t_1) - g(t_1, x(t_1))] + g(t, X_t) + \int_{t_1}^t S_\alpha(t-s)f(s, x_s)ds + \int_{t_1}^t AT_\alpha(t-s)g(s, x_s)ds \\ + \int_{t_1}^t AT_\alpha(t-s)e(s, x_s, \int_0^s h_1(s, \tau, x_\tau)d\tau)ds + \int_{t_1}^t S_\alpha(t-s)\sigma(s, x_s, \int_0^s h_2(s, \tau, x_\tau)d\tau)d\omega(s) \\ + T_\alpha(t-t_1)(x(t_1^-) + I_1(x(t_1^-))); \quad t \in [t_1, t_2] \\ \cdot \\ \cdot \\ \cdot \\ T_\alpha(t)[x(t_m) - g(t_m, x(t_m))] + g(t, X_t) + \int_{t_m}^t S_\alpha(t-s)f(s, x_s)ds + \int_{t_m}^t AT_\alpha(t-s)g(s, x_s)ds \\ + \int_{t_m}^t AT_\alpha(t-s)e(s, x_s, \int_0^s h_1(s, \tau, x_\tau)d\tau)ds + \int_{t_m}^t S_\alpha(t-s)\sigma(s, x_s, \int_0^s h_2(s, \tau, x_\tau)d\tau)d\omega(s) \\ + T_\alpha(t-t_m)(x(t_m^-) + I_m(x(t_m^-))); \quad t \in [t_m, T] \end{cases}$$

(iii) $\Delta x|_{t=t_k} = I_k(x(t_k^-))$, $k = 1, \dots, m$ the restriction of $x(\cdot)$ to the interval $[0, T] \setminus \{t_1, \dots, t_m\}$ is continuous.

Now, we are in a position to state the existence theorem. Our first theorem is based on the Banach contraction principle

Theorem 0.49. Assume that the conditions (H1)-(H7) hold. If $A \in A^\alpha(\theta(0), \omega(0))$, then the system (2.1) – (2.2) – (2.3) has a unique mild solution provided that

$$\max_{1 \leq i \leq m} \left(\left[9\tilde{M}_T^2(2 + n_i) + 9\frac{\alpha^2 C_{-\beta} \Gamma^2(2-\alpha)}{t^{-2\alpha\beta} \Gamma^2(1+\alpha(1+\beta))} M_g L \right] + 9 \left[\|A^{-\beta}\|^2 M_g L + \right. \right. \\ \left. \left. \tilde{M}_s^2(\mu_1 L + \nu_1 L + \nu_2 M_{h_2}) \frac{T^{2\alpha-1}}{2\alpha-1} + \frac{\alpha^2 C_{1-\beta}^2 \Gamma^2(2-\alpha)}{\Gamma^2(1+\alpha\beta)} L(M_g + M_e(1 + M_{h_1})) \frac{T^{1-2\alpha(1-\beta)}}{1-2\alpha(1-\beta)} \right] \right) < 1 \quad (3.1)$$

Proof: Define the operator $\Phi : B_b \rightarrow B_b$ by



$$(\Phi x)(t) = \begin{cases} \phi(t); t \in]-\infty, 0] \\ \\ T_\alpha(t)[\phi(0) - g(0, \phi)] + g(t, X_t) + \int_0^t S_\alpha(t-s)f(s, x_s)ds + \int_0^t AT_\alpha(t-s)g(s, x_s)ds \\ + \int_0^t AT_\alpha(t-s)e(s, x_s, \int_0^s h_1(s, \tau, x_\tau)d\tau)ds + \int_0^t S_\alpha(t-s)\sigma(s, x_s, \int_0^s h_2(s, \tau, x_\tau)d\tau)d\omega(s); \quad t \in \\ \\ T_\alpha(t)[x(t_1) - g(t_1, x(t_1))] + g(t, X_t) + \int_{t_1}^t S_\alpha(t-s)f(s, x_s)ds + \int_{t_1}^t AT_\alpha(t-s)g(s, x_s)ds \\ + \int_{t_1}^t AT_\alpha(t-s)e(s, x_s, \int_0^s h_1(s, \tau, x_\tau)d\tau)ds + \int_{t_1}^t S_\alpha(t-s)\sigma(s, x_s, \int_0^s h_2(s, \tau, x_\tau)d\tau)d\omega(s) \\ + T_\alpha(t-t_1)(x(t_1^-) + I_1(x(t_1^-))); \quad t \in [t_1, t_2] \\ \cdot \\ \cdot \\ \cdot \\ T_\alpha(t)[x(t_m) - g(t_m, x(t_m))] + g(t, X_t) + \int_{t_m}^t S_\alpha(t-s)f(s, x_s)ds + \int_{t_m}^t AT_\alpha(t-s)g(s, x_s)ds \\ + \int_{t_m}^t AT_\alpha(t-s)e(s, x_s, \int_0^s h_1(s, \tau, x_\tau)d\tau)ds + \int_{t_m}^t S_\alpha(t-s)\sigma(s, x_s, \int_0^s h_2(s, \tau, x_\tau)d\tau)d\omega(s) \\ + T_\alpha(t-t_m)(x(t_m^-) + I_m(x(t_m^-))); \quad t \in [t_m, T] \end{cases}$$

For $\phi \in B_h$, we defined $\tilde{\phi}$ by:

$$\tilde{\phi}(t) = \begin{cases} \phi(t); \quad t \in (-\infty, 0], \\ T_\alpha(t)\phi(0); \quad t \in [0, t_1], \\ 0; \quad t \in [t_2, T] \end{cases}$$

Next we define the function

$$\tilde{y}(t) = \begin{cases} 0; \quad t \in (-\infty, 0], \\ y(t); \quad t \in J \end{cases}$$



for each $y \in C(J, \mathbb{R})$ with $y(0) = 0$. If $x(\cdot)$ satisfies (3.1) then $x(t) = \tilde{\phi}(t) + \tilde{y}(t)$ for $t \in J$, which implies $x_t = \tilde{\phi}_t + \tilde{y}_t$ for $t \in J$ and the function $y(\cdot)$ satisfies

$$y(t) = \begin{cases} -T_\alpha(t)g(0, \phi) + g(t, \tilde{\phi}_t + \tilde{y}_t) + \int_0^t S_\alpha(t-s)f(s, \tilde{\phi}_s + \tilde{y}_s)ds + \int_0^t AT_\alpha(t-s)g(s, \tilde{\phi}_s + \tilde{y}_s)ds \\ + \int_0^t AT_\alpha(t-s)e(s, \tilde{\phi}_s + \tilde{y}_s, \int_0^s h_1(s, \tau, \tilde{\phi}_\tau + \tilde{y}_\tau)d\tau)ds \\ + \int_0^t S_\alpha(t-s)\sigma(s, \tilde{\phi}_s + \tilde{y}_s, \int_0^s h_2(s, \tau, \tilde{\phi}_\tau + \tilde{y}_\tau)d\tau)d\omega(s); \quad t \in [0, t_1] \\ \\ T_\alpha(t)[(\tilde{\phi}_{t_1} + \tilde{y}_{t_1}) - g(t_1, \tilde{\phi}_{t_1} + \tilde{y}_{t_1})] + g(t, \tilde{\phi}_t + \tilde{y}_t) + \int_{t_1}^t S_\alpha(t-s)f(s, \tilde{\phi}_s + \tilde{y}_s)ds + \int_{t_1}^t AT_\alpha(t-s)g(s, \tilde{\phi}_s + \tilde{y}_s)ds \\ + \int_{t_1}^t AT_\alpha(t-s)e(s, \tilde{\phi}_s + \tilde{y}_s, \int_0^s h_1(s, \tau, \tilde{\phi}_\tau + \tilde{y}_\tau)d\tau)ds + \int_{t_1}^t S_\alpha(t-s)\sigma(s, \tilde{\phi}_s + \tilde{y}_s, \int_0^s h_2(s, \tau, \tilde{\phi}_\tau + \tilde{y}_\tau)d\tau)d\omega(s) \\ + T_\alpha(t-t_1)((\tilde{y}(t_1^-) + \tilde{\phi}(t_1^-)) + I_1(\tilde{y}(t_1^-) + \tilde{\phi}(t_1^-))); \quad t \in [t_1, t_2] \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ T_\alpha(t)[(\tilde{\phi}_{t_m} + \tilde{y}(t_m)) - g(t_m, \tilde{\phi}_{t_m} + \tilde{y}(t_m))] + g(t, \tilde{\phi}_t + y(t_m)) + \int_{t_m}^t S_\alpha(t-s)f(s, y(s) + \tilde{\phi}_s)ds \\ + \int_{t_m}^t AT_\alpha(t-s)g(s, y(s) + \tilde{\phi}_s)ds \\ + \int_{t_m}^t AT_\alpha(t-s)e(s, \tilde{\phi}_s + \tilde{y}(s), \int_0^s h_1(s, \tau, \tilde{\phi}_\tau + \tilde{y}_\tau)d\tau)ds \\ + \int_{t_m}^t S_\alpha(t-s)\sigma(s, \tilde{\phi}_s + \tilde{y}(s), \int_0^s h_2(s, \tau, \tilde{\phi}_\tau + \tilde{y}_\tau)d\tau)d\omega(s) \\ + T_\alpha(t-t_m)((\tilde{\phi}(t_m^-) + \tilde{y}(t_m^-)) + I_m(\tilde{\phi}(t_m^-) + \tilde{y}(t_m^-))); \quad t \in [t_m, T] \end{cases}$$

Set $B_b^0 = \left\{ y \in B_b, \text{ such that } y_0 = 0 \right\}$ and for any $y \in B_b^0$, we have

$$\|y_b^0\| = \|y_0\|_{B_b} + \sup_{t \in J} (E\|y(t)\|^2)^{\frac{1}{2}} = \sup_{t \in J} (E\|y(t)\|^2)^{\frac{1}{2}},$$

thus $(B_b^0, \|\cdot\|_{B_b^0})$ is a Banach space.

Define the operator $\tilde{\Phi} : B_b^0 \rightarrow B_b^0$ by



$$(\tilde{\Phi}y)(t) = \begin{cases} -T_\alpha(t)g(0, \phi) + g(t, y_t) + \int_0^t S_\alpha(t-s)f(s, \tilde{\phi}_s + \tilde{y}_s)ds + \int_0^t AT_\alpha(t-s)g(s, \tilde{\phi}_s + \tilde{y}_s)ds \\ + \int_0^t AT_\alpha(t-s)e(s, \tilde{\phi}_s + \tilde{y}_s, \int_0^s h_1(s, \tau, \tilde{\phi}_\tau + \tilde{y}_\tau)d\tau)ds \\ + \int_0^t S_\alpha(t-s)\sigma(s, \tilde{\phi}_s + \tilde{y}_s, \int_0^s h_2(s, \tau, \tilde{\phi}_\tau + \tilde{y}_\tau)d\tau)d\omega(s); \quad t \in [0, t_1] \\ \left\{ \begin{array}{l} T_\alpha(t)[y_{t_1} - g(t_1, y_{t_1})] + g(t, y_t) + \int_{t_1}^t S_\alpha(t-s)f(s, \tilde{\phi}_s + \tilde{y}_s)ds + \int_{t_1}^t AT_\alpha(t-s)g(s, \tilde{\phi}_s + \tilde{y}_s)ds \\ + \int_{t_1}^t AT_\alpha(t-s)e(s, \tilde{\phi}_s + \tilde{y}_s, \int_0^s h_1(s, \tau, \tilde{\phi}_\tau + \tilde{y}_\tau)d\tau)ds \\ + \int_{t_1}^t S_\alpha(t-s)\sigma(s, \tilde{\phi}_s + \tilde{y}_s, \int_0^s h_2(s, \tau, \tilde{\phi}_\tau + \tilde{y}_\tau)d\tau)d\omega(s) + T_\alpha(t-t_1)(y(t_1^-) + I_1(y(t_1^-))); \quad t \in [t_1, t_2] \\ . \\ . \\ T_\alpha(t)[y(t_m) - g(t_m, y(t_m))] + g(t, y(t_m)) + \int_{t_m}^t S_\alpha(t-s)f(s, \tilde{\phi}_s + \tilde{y}(s))ds \\ + \int_{t_m}^t AT_\alpha(t-s)g(s, \tilde{\phi}_s + \tilde{y}(s))ds + \int_{t_m}^t AT_\alpha(t-s)e(s, \tilde{\phi}_s + \tilde{y}(s), \int_0^s h_1(s, \tau, \tilde{\phi}_\tau + \tilde{y}_\tau)d\tau)ds \\ + \int_{t_m}^t S_\alpha(t-s)\sigma(s, \tilde{\phi}_s + \tilde{y}(s), \int_0^s h_2(s, \tau, \tilde{\phi}_\tau + \tilde{y}_\tau)d\tau)d\omega(s) + T_\alpha(t-t_m)(y(t_m^-) + I_m(y(t_m^-))); \quad t \in [t_m, T] \end{array} \right. \end{cases}$$

In order to prove the existence result, it is enough to show that ψ has a unique fixed point. Let $y, y \in B_b^0$, then for all $t \in [0, t_1]$, we have



$$\begin{aligned}
E\|(\tilde{\Phi}y)(t) - (\tilde{\Phi}y^*)(t)\|_H^2 &= E\|(-T_\alpha(t)g(0, \phi) + g(t, y_t) + \int_0^t S_\alpha(t-s)f(s, \tilde{\phi}_s + \tilde{y}_s)ds \\
&\quad + \int_0^t AT_\alpha(t-s)g(s, \tilde{\phi}_s + \tilde{y}_s)ds \\
&\quad + \int_0^t AT_\alpha(t-s)e(s, \tilde{\phi}_s + \tilde{y}_s, \int_0^s h_1(s, \tau, \tilde{\phi}_\tau + \tilde{y}_\tau)d\tau)ds \\
&\quad + \int_0^t S_\alpha(t-s)\sigma(s, \tilde{\phi}_s + \tilde{y}_s, \int_0^s h_2(s, \tau, \tilde{\phi}_\tau + \tilde{y}_\tau)d\tau)d\omega(s) \\
&\quad - (-T_\alpha(t)g(0, \phi) + g(t, y_t^*) + \int_0^t S_\alpha(t-s)f(s, \tilde{\phi}_s + \tilde{y}_s^*)ds \\
&\quad + \int_0^t AT_\alpha(t-s)g(s, \tilde{\phi}_s + \tilde{y}_s^*)ds \\
&\quad + \int_0^t AT_\alpha(t-s)e(s, \tilde{\phi}_s + \tilde{y}_s^*, \int_0^s h_1(s, \tau, \tilde{\phi}_\tau + \tilde{y}_\tau^*)d\tau)ds \\
&\quad + \int_0^t S_\alpha(t-s)\sigma(s, \tilde{\phi}_s + \tilde{y}_s^*, \int_0^s h_2(s, \tau, \tilde{\phi}_\tau + \tilde{y}_\tau^*)d\tau)d\omega(s))\|_H^2 \\
&\leq 5E\|g(t, y_t) - g(t, y_t^*)\|_H^2 + 5E\|\int_0^t S_\alpha(t-s)(f(s, \tilde{\phi}_s + \tilde{y}_s) - f(s, \tilde{\phi}_s + \tilde{y}_s^*))ds\|_H^2 \\
&\quad + 5E\|\int_0^t AT_\alpha(t-s)(g(s, \tilde{\phi}_s + \tilde{y}_s) - g(s, \tilde{\phi}_s + \tilde{y}_s^*))ds\|_H^2 \\
&\quad + 5E\|\int_0^t AT_\alpha(t-s)(e(s, \tilde{\phi}_s + \tilde{y}_s, \int_0^s h_1(s, \tau, \tilde{\phi}_\tau + \tilde{y}_\tau)d\tau) \\
&\quad - e(s, \tilde{\phi}_s + \tilde{y}_s^*, \int_0^s h_1(s, \tau, \tilde{\phi}_\tau + \tilde{y}_\tau^*)d\tau))ds\|_H^2 \\
&\quad + 5E\|\int_0^t S_\alpha(t-s)(\sigma(s, \tilde{\phi}_s + \tilde{y}_s, \int_0^s h_2(s, \tau, \tilde{\phi}_\tau + \tilde{y}_\tau)d\tau) \\
&\quad - \sigma(s, \tilde{\phi}_s + \tilde{y}_s^*, \int_0^s h_2(s, \tau, \tilde{\phi}_\tau + \tilde{y}_\tau^*)d\tau)d\omega(s))\|_H^2
\end{aligned}$$



$$+5\tilde{M}_s^2 L \frac{T^{2\alpha-1}}{2\alpha-1} (\nu_1 \|y - y^*\|_{B_0^0}^2 + \nu_2 M_{h_2} \|y - y^*\|_{B_0^0}^2)$$

$$\leq 5[\|A^{-\beta}\|^2 M_g L + \tilde{M}_s^2 (\mu_1 L + \nu_1 L + \nu_2 M_{h_2}) \frac{T^{2\alpha-1}}{2\alpha-1}]$$

$$+ \frac{\alpha^2 C_{1-\beta}^2 \Gamma^2(2-\alpha)}{\Gamma^2(1+\alpha\beta)} L (M_g + M_e(1 + M_{h_1})) \frac{T^{1-2\alpha(1-\beta)}}{1-2\alpha(1-\beta)} \|y - y^*\|_{B_0^0}^2$$

For $t \in (t_1, t_2]$, we have:

$$\begin{aligned} E\|(\tilde{\Phi}y)(t) - (\tilde{\Phi}y^*)(t)\|_H^2 &\leq 9\|T_\alpha(t - t_1)\|_H^2 E\|(y(t_1^-) - y^*(t_1^-))\|_H^2 \\ &\quad + 9\|T_\alpha(t - t_1)\|_H^2 E\|I_1(y(t_1^-)) - I_1(y^*(t_1^-))\|_H^2 + 9\|T_\alpha(t)\|_H^2 E\|y(t_1) - y^*(t_1)\|_H^2 \\ &\quad + 9\|A^{-\beta}T_\alpha\|^2 E\|A^\beta g(t, y_{t_1}) - A^\beta g(t, y_{t_1}^*)\|_H^2 + 9\|A^{-\beta}\|^2 E\|A^\beta g(t, y_t) - A^\beta g(t, y_t^*)\|_H^2 \\ &\quad + 9\int_0^t \|S_\alpha(t - s)\|^2 E\|(f(s, \tilde{\phi}_s + \tilde{y}_s) - f(s, \tilde{\phi}_s + \tilde{y}_s^*))\|_H^2 ds \\ &\quad + 9\int_0^t \|A^{1-\beta}T_\alpha(t - s)\|^2 E\|(A^\beta g(s, \tilde{\phi}_s + \tilde{y}_s) - A^\beta g(s, \tilde{\phi}_s + \tilde{y}_s^*))\|_H^2 ds \\ &\quad + 9\int_0^t \|A^{1-\beta}T_\alpha(t - s)\|^2 E\|A^\beta e(s, \tilde{\phi}_s + \tilde{y}_s, \int_0^s h_1(s, \tau, \tilde{\phi}_\tau + \tilde{y}_\tau) d\tau) \\ &\quad - A^\beta e(s, \tilde{\phi}_s + \tilde{y}_s^*, \int_0^s h_1(s, \tau, \tilde{\phi}_\tau + \tilde{y}_\tau^*) d\tau)\|_H^2 ds \\ &\quad + 9\int_0^t \|S_\alpha(t - s)\| E\|\sigma(s, \tilde{\phi}_s + \tilde{y}_s, \int_0^s h_2(s, \tau, \tilde{\phi}_\tau + \tilde{y}_\tau) d\tau) \\ &\quad - \sigma(s, \tilde{\phi}_s + \tilde{y}_s^*, \int_0^s h_2(s, \tau, \tilde{\phi}_\tau + \tilde{y}_\tau^*) d\tau)\|_{\mathcal{L}_2^0}^2 ds \\ &\leq 9\tilde{M}_T^2 [E\|(y(t_1^-) - y^*(t_1^-))\|_H^2 + E\|I_1(y(t_1^-)) - I_1(y^*(t_1^-))\|_H^2] + 9\tilde{M}_T^2 E\|y(t_1) - y^*(t_1)\|_H^2 \\ &\quad + 9\frac{\alpha^2 C_{1-\beta}^2 \Gamma^2(2-\alpha)}{t^{-2\alpha\beta} \Gamma^2(1+\alpha(1+\beta))} M_g \|y_t - y_t^*\|_{B_h}^2 + 9\|A^{-\beta}\|^2 M_g \|y_t - y_t^*\|_{B_h}^2 \\ &\quad + 9\tilde{M}_s^2 \mu_1 \int_0^t (t - s)^{2(\alpha-1)} \|\tilde{y}_s - \tilde{y}_s^*\|_{B_h}^2 ds \\ &\quad + 9\frac{\alpha^2 C_{1-\beta}^2 \Gamma^2(2-\alpha)}{\Gamma^2(1+\alpha\beta)} M_g \int_0^t (t - s)^{-2\alpha(1-\beta)} \|\tilde{y}_s - \tilde{y}_s^*\|_{B_h}^2 ds \\ &\quad + 9\frac{\alpha^2 C_{1-\beta}^2 \Gamma^2(2-\alpha)}{\Gamma^2(1+\alpha\beta)} M_e \int_0^t (t - s)^{-2\alpha(1-\beta)} (\|\tilde{y}_s - \tilde{y}_s^*\|_{B_h}^2 + M_{h_1} \|\tilde{y}_s - \tilde{y}_s^*\|_{B_h}^2) ds \\ &\quad + 9\tilde{M}_s^2 \int_0^t (t - s)^{2(\alpha-1)} (\nu_1 \|\tilde{y}_s - \tilde{y}_s^*\|_{B_h}^2 + \nu_2 M_{h_2} E\|\tilde{y}_s - \tilde{y}_s^*\|_{B_h}^2) ds \end{aligned}$$



$$\begin{aligned}
&\leq 9\tilde{M}_T^2(1+n_1) \sup E \|y_t - y_t^*\|_H^2 + 9\tilde{M}_T^2 \sup E \|y(t) - y^*(t)\|_H^2 \\
&+ 9 \frac{\alpha^2 C_{-\beta} \Gamma^2(2-\alpha)}{t^{-2\alpha\beta} \Gamma^2(1+\alpha(1+\beta))} M_g L \sup E \|y_t - y_t^*\|_H^2 \\
&+ 9 \|A^{-\beta}\|^2 M_g L \sup E \|y_t - y_t^*\|_H^2 + 9\tilde{M}_s^2 \mu_1 \int_0^t (t-s)^{2(\alpha-1)} L \sup E \|y_s - y_s^*\|_H^2 ds \\
&+ 9 \frac{\alpha^2 C_{1-\beta}^2 \Gamma^2(2-\alpha)}{\Gamma^2(1+\alpha\beta)} M_g \int_0^t (t-s)^{-2\alpha(1-\beta)} L \sup E \|y_s - y_s^*\|_H^2 ds \\
&+ 9 \frac{\alpha^2 C_{1-\beta}^2 \Gamma^2(2-\alpha)}{\Gamma^2(1+\alpha\beta)} M_e \int_0^t (t-s)^{-2\alpha(1-\beta)} (L \sup E \|y_s - y_s^*\|_H^2 + M_{h_1} L \sup E \|y_s - y_s^*\|_H^2) ds \\
&+ 9\tilde{M}_s^2 \int_0^t (t-s)^{2(\alpha-1)} (\nu_1 L \sup E \|y_s - y_s^*\|_H^2 + \nu_2 M_{h_2} \sup E \|y_s - y_s^*\|_H^2) ds
\end{aligned}$$

So

$$\begin{aligned}
E \|(\tilde{\Phi}y)(t) - (\tilde{\Phi}y^*)(t)\|_H^2 &\leq \left(\left[9\tilde{M}_T^2(2+n_1) + 9 \frac{\alpha^2 C_{-\beta} \Gamma^2(2-\alpha)}{t^{-2\alpha\beta} \Gamma^2(1+\alpha(1+\beta))} M_g L \right] \right. \\
&+ 9 \left[\|A^{-\beta}\|^2 M_g L + \tilde{M}_s^2 (\mu_1 L + \nu_1 L + \nu_2 M_{h_2}) \frac{T^{2\alpha-1}}{2\alpha-1} \right. \\
&\left. \left. + \frac{\alpha^2 C_{1-\beta}^2 \Gamma^2(2-\alpha)}{\Gamma^2(1+\alpha\beta)} L (M_g + M_e(1+M_{h_1})) \frac{T^{1-2\alpha(1-\beta)}}{1-2\alpha(1-\beta)} \right] \right) \|y - y^*\|_{B_b^0}^2
\end{aligned}$$

Similarly, when $t \in (t_i, t_{i+1}]$, $i = 2, \dots, m$, we get

$$\begin{aligned}
E \|(\tilde{\Phi}y)(t) - (\tilde{\Phi}y^*)(t)\|_H^2 &\leq \left(\left[9\tilde{M}_T^2(2+n_i) + 9 \frac{\alpha^2 C_{-\beta} \Gamma^2(2-\alpha)}{t^{-2\alpha\beta} \Gamma^2(1+\alpha(1+\beta))} M_g L \right] \right. \\
&+ 9 \left[\|A^{-\beta}\|^2 M_g L + \tilde{M}_s^2 (\mu_1 L + \nu_1 L + \nu_2 M_{h_2}) \frac{T^{2\alpha-1}}{2\alpha-1} \right. \\
&\left. \left. + \frac{\alpha^2 C_{1-\beta}^2 \Gamma^2(2-\alpha)}{\Gamma^2(1+\alpha\beta)} L (M_g + M_e(1+M_{h_1})) \frac{T^{1-2\alpha(1-\beta)}}{1-2\alpha(1-\beta)} \right] \right) \|y - y^*\|_{B_b^0}^2
\end{aligned}$$

Thus for all $t \in [0, T]$, we have



$$\begin{aligned}
 E\|(\tilde{\Phi}y)(t) - (\tilde{\Phi}y^*)(t)\|_H^2 &\leq \max_{1 \leq i \leq m} \left(\left[9\tilde{M}_T^2(2 + n_i) + 9 \frac{\alpha^2 C_{-\beta} \Gamma^2(2-\alpha)}{t^{-2\alpha\beta} \Gamma^2(1+\alpha(1+\beta))} M_g L \right] \right. \\
 &\quad + 9 \left[\|A^{-\beta}\|^2 M_g L + \tilde{M}_s^2(\mu_1 L + \nu_1 L + \nu_2 M_{h_2}) \frac{T^{2\alpha-1}}{2\alpha-1} \right. \\
 &\quad \left. \left. + \frac{\alpha^2 C_{1-\beta}^2 \Gamma^2(2-\alpha)}{\Gamma^2(1+\alpha\beta)} L(M_g + M_e(1 + M_{h_1})) \frac{T^{1-2\alpha(1-\beta)}}{1-2\alpha(1-\beta)} \right] \right) \|y - y^*\|_{B_b^0}^2
 \end{aligned}$$

Hence, $\tilde{\Phi}$ is a contraction map by the condition (3.1) and therefore it has a unique fixed point $y \in B_b^0$, which is a mild solution of (2.1) – (2.2) – (2.3) on $(-\infty, T]$. The proof is complete. $\square$

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Uniqueness and stability of Hadamard variable -ordre boundary value problem

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Abstract. The existence, uniqueness, and stability of solutions to a nonlinear FDE boundary value problem of variable order are established in this paper. To do this, we first look into a few aspects of Hadamard-type variable order operators. Next, use the generalized, we translate the variable order Hadamard FBVP into an analogous standard Hadamard BVP of the fractional constant order using intervals and piecewise constant functions. Additionally, two fixed point theorems attributed to Schauder and Banach are employed, and ultimately, an analysis is conducted to determine the stability of the Ulam-Hyers-Rassias stability of the specified variable order Hadamard FBVP.

Keywords: boundary value problem; Hadamard derivatives of variable order; piecewise constant functions; fixed point theorems.



Introduction

Forming the rational numbers in the order of derivation operators with natural numbers is the fundamental principle behind fractional calculus. Despite being in its early stages, this concept has some amazing effects and results that accurately reflect several physical and natural events. We address certain qualitative elements of the following Hadamard variable order type FBVP solutions with terminal conditions in this paper.

$$\begin{cases} {}^H D_{1+}^{\nu(\tilde{t})} \bar{\mu}(\tilde{t}) = \varrho(t, \bar{\mu}(\tilde{t})), \tilde{t} \in I := [1, \bar{\Theta}], \\ \bar{\mu}(1) = \bar{\mu}(\bar{\Theta}) = 0, \end{cases} \quad (0.34)$$

where $1 < \bar{\Theta} < +\infty$, $1 < \nu(\tilde{t}) \leq 2$, $\varrho : I \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function and ${}^H D_{1+}^{\nu(\tilde{t})}$ is the Hadamard fractional derivative of variable-order $\nu(\tilde{t})$.

Main results

Definition 0.28. [1] Let $k \in \mathbb{N}$ and $\nu : [a, b] \rightarrow (k-1, k)$. The Hadamard derivative of variable order $\nu(\tilde{t})$ for ϱ is defined by

$$({}^H D_{a+}^{\nu(\tilde{t})} \varrho)(\tilde{t}) = \frac{1}{\Gamma(k - \nu(\tilde{t}))} \left(t \frac{d}{dt} \right)^k \int_a^{\tilde{t}} \left(\log \frac{\tilde{t}}{s} \right)^{k-\nu(\tilde{t})-1} \frac{\varrho(s)}{s} ds, \tilde{t} > a. \quad (0.35)$$

if the right-hand side integral exists.

Let us introduce the following assumptions:

(H1) For $n \in \mathbb{N}$ define,

$$P = \{I_1 := [0, \bar{\Theta}_1], I_2 := (\bar{\Theta}_1, \bar{\Theta}_2], I_3 := (\bar{\Theta}_2, \bar{\Theta}_3], \dots, I_n := (\bar{\Theta}_{n-1}, \bar{\Theta}]\}$$

as a partition of the interval I , and assume that $\nu(\tilde{t}) : I \rightarrow (1, 2]$ be a piecewise constant with respect to P ; in other words

$$\nu(\tilde{t}) = \sum_{i=1}^n \nu_i l_i(\tilde{t}) = \begin{cases} \nu_1, & \text{if } \tilde{t} \in I_1, \\ \nu_2, & \text{if } \tilde{t} \in I_2, \\ \vdots & \\ \nu_n, & \text{if } \tilde{t} \in I_n, \end{cases}$$

in which $1 < \nu_i \leq 2$ belong to $\mathbb{R}$, and l_i illustrates the indicator of $I_i := (\bar{\Theta}_{i-1}, \bar{\Theta}_i]$, $i \in \mathbb{N}_1^n$, (by assuming $\bar{\Theta}_0 = 1, \bar{\Theta}_n = \bar{\Theta}$) so that

$$l_i(\tilde{t}) = \begin{cases} 1, & \text{if } \tilde{t} \in I_i, \\ 0, & \text{o.w.} \end{cases}$$



(H2) For $0 \leq g \leq 1$, let $(\log \tilde{i})^g \varrho \in C(I \times \mathbb{R}, \mathbb{R})$ and $\exists l > 0$ so that $|(\log \tilde{i})^g \varrho(\tilde{i}, \bar{\mu}_1) - \varrho(\tilde{i}, \bar{\mu}_2)| \leq l |\bar{\mu}_1 - \bar{\mu}_2|$, for any $\bar{\mu}_1, \bar{\mu}_2 \in \mathbb{R}$ and $\tilde{i} \in I$. In addition, by $E_i = C(I_i, \mathbb{R})$, we denote the class of functions which form a Banach space via

$$\|\bar{\mu}\|_{E_i} = \sup_{\tilde{i} \in I_i} |\bar{\mu}(\tilde{i})|, \text{ where } i \in \{1, 2, \dots, n\}$$

By using the definition (0.28) and according to (H1), (H2) Going through several basic and necessary steps From now on, we follow our study on the equivalent standard Hadamard Fractional FBVP which takes the form

$$\begin{cases} ({}^H D_{T_{i-1}^+}^{w_i} \bar{\mu})(\tilde{i}) = \varrho(\tilde{i}, \bar{\mu}(\tilde{i})), & \tilde{i} \in I_i, \\ \bar{\mu}(\bar{\Theta}_{i-1}) = 0, & \bar{\mu}(\bar{\Theta}_i) = 0, \end{cases} \quad (0.36)$$

Proposition 0.7. *The function $\bar{\mu} \in E_i$ is a solution of the equivalent standard Hadamard Fractional FBVP (0.36) if $\bar{\mu}$ satisfies*

$$\bar{\mu}(\tilde{i}) = -(\log \frac{\bar{\Theta}_i}{\bar{\Theta}_{i-1}})^{1-\nu_i} (\log \frac{\tilde{i}}{\bar{\Theta}_{i-1}})^{\nu_i-1} {}^H I_{\bar{\Theta}_{i-1}^+}^{\nu_i} \varrho(\bar{\Theta}_i, \bar{\mu}(\bar{\Theta}_i)) + {}^H I_{\bar{\Theta}_{i-1}^+}^{\nu_i} \varrho(\tilde{i}, \bar{\mu}(\tilde{i})). \quad (0.37)$$

Theorem 0.50. *Consider (H1) and (H2).if*

$$\frac{2l \left[(\log \bar{\Theta}_i)^{1-g} - (\log \bar{\Theta}_{i-1})^{1-g} \right]}{(1-g)\Gamma(g)} (\log \frac{\bar{\Theta}_i}{\bar{\Theta}_{i-1}})^{e_i-1} < 1, \quad (0.38)$$

then the variable order Hadamard Fractional FBVP (0.36) involves a solution in E_i uniquely.

Theorem 0.51. *Consider the hypotheses (H1), (H2) and the inequality (0.549). Assume:*

(H3) $\exists \phi \in C(I_i, \mathbb{R}^+)$ as an increasing mapping and $\exists \lambda_\phi > 0$ so that $\forall \tilde{i} \in I_i$,

$${}^H I_{\bar{\Theta}_{i-1}^+}^{\varrho_i} \phi(\tilde{i}) \leq \lambda_{\phi(\tilde{i})} \phi(\tilde{i}).$$

Then, the variable order Hadamard Fractional FBVP (0.63) is Ulam-Hyers-Rassias stable with respect to ϕ .

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**Existence of Solutions for a Ordinary Hybrid Version of
Generalized Sturm-Liouville-Langevin Equation Using
 α - φ -Contraction
and Ψ -Hilfer Fractional Derivative**

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Abstract. The fractional Langevin equation provides a powerful mathematical framework for studying the motion of particles in complex environments where anomalous diffusion occurs, and the fractional Sturm-Liouville equation is a mathematical tool for studying particle behavior in complex systems, like fluid diffusion or string vibration, helping understand anomalous diffusion that deviates from classical rules. This paper intends to study the existence of solutions for an ordinary hybrid version of the generalized Sturm-Liouville-Langevin (GSSL) equation under boundary condition using a different approach that is based on the technique of α - φ -contraction via admissible mapping in the fixed point theorem. The Ψ -Hilfer fractional-order derivative (Ψ -Hilfer FOD) is used to model this problem, an example is given to illustrate the applicability of our work. The results are novel and extend some of the findings known in the literature.

Keywords: Ψ -Hilfer fractional-order derivative; fractional Sturm-Liouville equation; fractional Langevin equation; α - φ -contraction mapping; admissible



mapping; fixed point theorem.

Introduction

One of the most attractive topics for research is the metric-fixed-point theory. The first metric fixed point theorem was supposedly proven by Banach in 1922, according to the literature. The existence and uniqueness of a fixed point are implied to the existence and uniqueness of the differential equation solutions, and this is a core tool in the solution of certain differential equations. Indeed, this fact may be the reason why this theory has been investigated heavily. Thousands of new results have been announced in the framework of the metric fixed point theory in the last three decades. Most of them claimed that it was the generalization of existing results. Mostly, these announced results are slight generalizations or extensions of the existing ones. The fact that there are so many publications on this topic has encouraged researchers to organize and unify this scattered literature. One of the best examples of this trend was given by Samet et al. [4] in 2012 by involving the notion of admissible mappings. Few results concerning such a type of fractional differential equation in hybrid versions have been published before. To bridge this gap and be motivated by [3, 2], the existence of solutions by α - φ -contraction mappings for ordinary hybrid versions of nonlinear GSLL is discussed under the Ψ -Hilfer fractional-order derivative, which reads as follows: we investigate the ordinary hybrid version of Ψ -Hilfer type GSLL equation which reads as

$$\begin{cases} {}^H\mathfrak{D}_{0+}^{\alpha_1, \beta_1; \Psi} \left(\frac{\rho(\mathfrak{z}) {}^H\mathfrak{D}_{0+}^{\alpha_2, \beta_2; \Psi} u(\mathfrak{z}) + q(\mathfrak{z})u(\mathfrak{z})}{\mathcal{G}(\mathfrak{z}, u(\mathfrak{z}), \mathcal{I}_{0+}^{\varsigma_1, \Psi} u(\mathfrak{z}), \mathcal{I}_{0+}^{\varsigma_2, \Psi} u(\mathfrak{z}), \dots, \mathcal{I}_{0+}^{\varsigma_m, \Psi} u(\mathfrak{z}))} \right) = f(\mathfrak{z}, u(\mathfrak{z})) \\ u(0) = 0 \\ \frac{\rho(T) {}^H\mathfrak{D}_{0+}^{\alpha_2, \beta_2; \Psi} u(T) + q(T)u(T)}{\mathcal{G}(T, u(T), \mathcal{I}_{0+}^{\varsigma_1, \Psi} u(T), \mathcal{I}_{0+}^{\varsigma_2, \Psi} u(T), \dots, \mathcal{I}_{0+}^{\varsigma_m, \Psi} u(T))} = 0 \end{cases} \quad (0.39)$$

where $\mathcal{O} = [0, T]$, $f \in \mathcal{C}(\mathcal{O} \times \mathbb{R}, \mathbb{R})$, $\mathcal{G} \in \mathcal{C}(\mathcal{O} \times \mathbb{R}^{m+1}, \mathbb{R} - \{0\})$, $\rho \in \mathcal{C}(\mathcal{O}, \mathbb{R} - \{0\})$, $q \in \mathcal{C}(\mathcal{O}, \mathbb{R})$, ${}^H\mathfrak{D}_{0+}^{\alpha_i, \beta_i; \Psi}$ is the Ψ -HFOD depending on an increasing function Ψ of order $0 < \alpha_i < 1$ and type $0 \leq \beta_i \leq 1$ $i \in \{1, 2\}$, and $\mathcal{I}_{0+}^{\gamma, \Psi}$ is Ψ -Riemann-Liouville fractional-order integral depends on the function Ψ of order $\gamma \in \{\varsigma_1, \varsigma_2, \dots, \varsigma_m\}$.



Remark 0.3. The proposed problem (0.39) covers a set of problem for each function Ψ and type β_i . Of between it:

1. RL type problem: $\psi(z) = z$ and $\beta_i = 0 \quad i \in \{1, 2\}$.
2. Caputo type problem: $\psi(z) = z$ and $\beta_i = 1 \quad i \in \{1, 2\}$.
3. ψ -RL type problem: $\beta_i = 0 \quad i \in \{1, 2\}$.
4. ψ -Caputo type problem: $\beta_i = 1 \quad i \in \{1, 2\}$.
5. Hilfer type problem: $\psi(z) = z$.
6. Hilfer-Hadamard type problem: $\psi(z) = \log z$.
7. ψ -Hilfer type problem: $\psi(z) = z^\rho, \rho > 0$.

Prelimanries

In what follows, we'll review the (c)-comparison function, an auxiliary function. See e.g. [4] for more details and examples.

Definition 0.29. [4] Let Π is a family of functions $\psi : [0, +\infty) \rightarrow [0, +\infty)$ with

1. ψ is increasing;
2. there exists $k_0 \in \mathbb{N}, b \in (0, 1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{+\infty} v_k$ such that $\psi^{k+1}(\mathfrak{z}) \leq b\psi^k(\mathfrak{z}) + v_k$, for $k \geq k_0$ and any $\mathfrak{z} \in [0, \infty)$.

Further, (ii) yields

$$\sum_{n=1}^{+\infty} \psi^n(\mathfrak{z}) < \infty,$$

for all $\mathfrak{z} > 0$, where ψ^n is the n -th iterate of Π .

A function $\psi \in \Pi$ is called a (c)-comparison function.

Lemma 0.17. If $\psi : [0, \infty) \rightarrow [0, \infty)$ is a (c)-comparison function, then:

1. each iterate ψ^k of $\psi, k \geq 1$, is also a (c)-comparison function;
2. ψ is continuous at 0 ;
3. $\psi(t) < t$, for any $t > 0$.

We recall the admissible mapping defined that Samet et al.[4] defined.

Definition 0.30. [4] Let $E \neq \emptyset, \mathcal{N} : E \rightarrow E$ and $\alpha : E \times E \rightarrow \mathbb{R}$. We say that $\mathcal{N}$ is α -admissible if

$$x, y \in E \quad \alpha(x, y) \geq 1 \Rightarrow \alpha(\mathcal{N}x, \mathcal{N}y) \geq 1.$$



Definition 0.31. [4] Let $\mathcal{N} : E \rightarrow E$, $\mathcal{N}$ is called an α - φ -contraction if there exist a (c) -comparison function $\varphi \in \Pi$ and $\alpha : E \times E \rightarrow E$ with

$$\alpha(x, y)d(\mathcal{N}x, \mathcal{N}y) \leq \varphi(d(x, y)), \quad \text{for all } x, y \in E$$

Remark 0.4. [4] If $\mathcal{N} : E \rightarrow E$ satisfies the Banach contraction principle, then $\mathcal{N}$ is an α - φ -contraction mapping, where $\alpha(x, y) = 1$ for all $x, y \in E$ and $\varphi(\mathfrak{z}) = k\mathfrak{z}$ for all $\mathfrak{z} \geq 0$ and some $k \in [0, 1)$.

We recall the fixed point theorem of Samet et al.[4] is as follows.

Lemma 0.18. [4] Let (E, d) be a complete metric space and $\mathcal{N} : E \rightarrow E$ be an α - φ -contractive mapping satisfying the following conditions:

1. $\mathcal{N}$ is α -admissible;
2. there exists $x_0 \in E$ such that $\alpha(x_0, \mathcal{N}x_0) \geq 1$;
3. if $\{u_n\}$ is a sequence in E such that $\alpha(u_n, u_{n+1}) \geq 1$ for all n and $u_n \rightarrow u \in E$ as $n \rightarrow +\infty$, then $\alpha(u_n, u) \geq 1$ for all n .

Then, $\mathcal{N}$ has a fixed point.

Definition 0.32. [1] Assume that $\Psi \in \mathcal{C}^1(\mathcal{O}, \mathbb{R})$ is an increasing function with $\Psi'(\mathfrak{z}) \neq 0$ for each $\mathfrak{z} \in \mathcal{O}$.

The left-sided Ψ -Riemann-Liouville fractional-order integral of order α of function f depending on Ψ is defined by

$$\mathcal{I}_{a+}^{\alpha; \Psi} f(\mathfrak{z}) = \frac{1}{\Gamma(\alpha)} \int_a^{\mathfrak{z}} (\Psi(\mathfrak{z}) - \Psi(s))^{\alpha-1} \Psi'(s) f(s) ds, \quad \mathfrak{z} > a > 0, \quad \alpha > 0 \quad (0.40)$$

The left-sided Ψ -Riemann-Liouville fractional-order derivative of function f depending on Ψ is defined as

$$\begin{aligned} \mathbb{D}_{a+}^{\alpha; \Psi} f(\mathfrak{z}) &= \left(\frac{1}{\Psi'(\mathfrak{z})} \frac{d}{d\mathfrak{z}} \right)^n \mathcal{I}_{a+}^{n-\alpha; \Psi} f(\mathfrak{z}) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{\Psi'(\mathfrak{z})} \frac{d}{d\mathfrak{z}} \right)^n \int_a^{\mathfrak{z}} (\Psi(\mathfrak{z}) - \Psi(s))^{n-\alpha-1} \Psi'(s) f(s) ds, \quad \alpha > 0. \end{aligned}$$

Assume that $\gamma = \alpha + \rho(n - \alpha)$, $\alpha \in (n - 1, n)$ with $n \in \mathbb{N}$ and $f \in \mathcal{C}^n(\mathcal{O}, \mathbb{R})$.

The left-sided Ψ -HFOD of order α and type $\rho \in [0, 1]$ of function f depending on Ψ is defined as

$$\begin{aligned} {}^H\mathfrak{D}_{a+}^{\alpha, \rho; \Psi} f(\mathfrak{z}) &= \mathcal{I}_{a+}^{\rho(n-\alpha); \Psi} \left(\frac{1}{\Psi'(\mathfrak{z})} \frac{d}{d\mathfrak{z}} \right)^n \mathcal{I}_{a+}^{(1-\rho)(n-\alpha); \Psi} f(\mathfrak{z}) \\ &= \mathcal{I}_{a+}^{\gamma-\alpha; \Psi} \mathbb{D}_{a+}^{\gamma; \Psi} f(\mathfrak{z}) \end{aligned}$$

where $\mathbb{D}_{a+}^{\gamma; \Psi} f(\mathfrak{z}) = \left(\frac{1}{\Psi'(\mathfrak{z})} \frac{d}{d\mathfrak{z}} \right)^n \mathcal{I}_{a+}^{(1-\rho)(n-\alpha); \Psi} f(\mathfrak{z})$, $n = [\alpha] + 1$, $[\alpha]$ is an integer part of $\text{Re}(\alpha)$ and $\Gamma(\cdot)$ is the Euler's gamma function.



Theorem 0.52. [4] Let (E, d) be a complete metric space and $\mathcal{N} : E \rightarrow E$ be an α - φ -contractive mapping satisfying the following conditions:

1. $\mathcal{N}$ is α -admissible;
2. there exists $x_0 \in E$ such that $\alpha(x_0, \mathcal{N}x_0) \geq 1$;
3. if $\{u_n\}$ is a sequence in E such that $\alpha(u_n, u_{n+1}) \geq 1$ for all n and $u_n \rightarrow u \in E$ as $n \rightarrow +\infty$, then $\alpha(u_n, u) \geq 1$ for all n .

Then, $\mathcal{N}$ has a fixed point.

Lemma 0.19. Assume that $h \in \mathcal{L}^1(\mathcal{O}, \mathbb{R})$. Let $0 < \alpha_i < 1$, $0 < \varsigma_m \leq 1$, $0 \leq \beta_i \leq 1$, and $\gamma_i = \alpha_i + \beta_i(1 - \alpha_i)$ $i \in \{1, 2\}$. Consider the problem

$$\begin{cases} {}^H\mathfrak{D}_{0+}^{\alpha_1, \beta_1; \Psi} \left(\frac{\rho(\mathfrak{z}) {}^H\mathfrak{D}_{0+}^{\alpha_2, \beta_2; \Psi} u(\mathfrak{z}) + q(\mathfrak{z})u(\mathfrak{z})}{\mathcal{G}(\mathfrak{z}, u(\mathfrak{z}), \mathcal{I}_{0+}^{\varsigma_1, \Psi} u(\mathfrak{z}), \mathcal{I}_{0+}^{\varsigma_2, \Psi} u(\mathfrak{z}), \dots, \mathcal{I}_{0+}^{\varsigma_m, \Psi} u(\mathfrak{z}))} \right) = h(\mathfrak{z}) \\ u(0) = 0 \\ \frac{\rho(T) {}^H\mathfrak{D}_{0+}^{\alpha_2, \beta_2; \Psi} u(T) + q(T)u(T)}{\mathcal{G}(T, u(T), \mathcal{I}_{0+}^{\varsigma_1, \Psi} u(T), \mathcal{I}_{0+}^{\varsigma_2, \Psi} u(T), \dots, \mathcal{I}_{0+}^{\varsigma_m, \Psi} u(T))} = 0 \end{cases} \quad (0.41)$$

Then, the function $u \in \mathcal{C}(\mathcal{O}, \mathbb{R})$ is a solution of the problem (0.41) whenever

$$\begin{aligned} u(\mathfrak{z}) = & \frac{\mathcal{G}(\mathfrak{z}, u(\mathfrak{z}), \mathcal{I}_{0+}^{\varsigma_1, \Psi} u(\mathfrak{z}), \mathcal{I}_{0+}^{\varsigma_2, \Psi} u(\mathfrak{z}), \dots, \mathcal{I}_{0+}^{\varsigma_m, \Psi} u(\mathfrak{z})) (\mathcal{I}_{0+}^{\alpha_1, \Psi} h(\mathfrak{z}) - \frac{\mathcal{I}_{0+}^{\alpha_1, \Psi} h(T) (\Psi(\mathfrak{z}) - \Psi(0))^{\gamma_1 - 1}}{(\Psi(T) - \Psi(0))^{\gamma_1 - 1}})}{\rho(\mathfrak{z})} \\ & - \mathcal{I}_{0+}^{\alpha_2, \Psi} \left(\frac{q(\mathfrak{z})u(\mathfrak{z})}{\rho(\mathfrak{z})} \right) + c_1 \frac{(\Psi(\mathfrak{z}) - \Psi(0))^{\gamma_2 - 1}}{\Gamma(\gamma_2)} \end{aligned} \quad (0.42)$$

Main results

Theorem 0.53. Assume the following hypotheses hold:

(H1) There exists $w \in \mathcal{C}(\mathcal{O}, \mathbb{R}_+)$ such that

$$|f(\mathfrak{z}, u_1) - f(\mathfrak{z}, u_2)| \leq w(\mathfrak{z}) |u_1 - u_2| \quad \text{for all } (\mathfrak{z}, u_1, u_2) \in \mathcal{O} \times \mathbb{R} \times \mathbb{R}$$

(H2) There exists $\Theta \in \mathcal{C}(\mathcal{O}, \mathbb{R}_+)$, we have the following inequality

$$|\mathcal{G}(\mathfrak{z}, x_0, x_1, x_2, \dots, x_m) - \mathcal{G}(\mathfrak{z}, y_0, y_1, y_2, \dots, y_m)| \leq \Theta(\mathfrak{z}) \sum_{i=0}^m |x_i - y_i|$$

$$\text{for all } (\mathfrak{z}, x_0, x_1, \dots, x_m, y_0, y_1, \dots, y_m) \in \mathcal{O} \times \mathbb{R}^{2m+2},$$



(H3) There exists $R > 0$ such that

$$\frac{2 \left(\|\Theta\| \sum_{i=0}^m \frac{(\Psi(T) - \Psi(0))^{\varsigma_i}}{\Gamma(\varsigma_i + 1)} R + M_G \right) (\|w\| R + M_f) (\Psi(T) - \Psi(0))^{\alpha_1 + \alpha_2}}{\rho \Gamma(\alpha_1 + 1) \Gamma(\alpha_2 + 1)} + \frac{\bar{q} R (\Psi(T) - \Psi(0))^{\alpha_2}}{\rho \Gamma(\alpha_2 + 1)} \leq R$$

and

$$\frac{2(\Psi(T) - \Psi(0))^{\alpha_1 + \alpha_2}}{\rho \Gamma(\alpha_1 + 1) \Gamma(\alpha_2 + 1)} \left((2\|w\| R + M_f) \|\Theta\| \sum_{i=0}^m \frac{(\Psi(T) - \Psi(0))^{\varsigma_i}}{\Gamma(\varsigma_i + 1)} + M_G \|w\| \right) + \frac{\bar{q}(\Psi(T) - \Psi(0))^{\alpha_2}}{\rho \Gamma(\alpha_2 + 1)} < 1$$

Then the problem (0.39) has at least one solution defined on $\mathcal{O}$.

Example 0.3. We start with the ordinary hybrid version of the Ψ -Hilfer type GSLL problem:

$$\begin{cases} {}^H\mathfrak{D}_{0+}^{\frac{\sqrt{5}}{7}, 0; x} \left(\frac{{}^H\mathfrak{D}_{0+}^{\frac{5}{6}, 0; x} u(\mathfrak{z}) + 10^{-3} \mathfrak{z} u(\mathfrak{z})}{\frac{(\mathfrak{z}+1)^2}{6} + \frac{(\mathfrak{z}+1)^2}{6} u + \frac{1}{\mathfrak{z}^2 + e^3} \mathcal{I}_{0+}^{\frac{\pi}{3}, x} u(\mathfrak{z})} \right) = \frac{e^{\mathfrak{z}^2}}{400(1 + \mathfrak{z}^2)(|u| + 1)} \\ u(0) = 0 \\ {}^H\mathfrak{D}_{0+}^{\frac{\sqrt{5}}{7}, 0; x} \left(\frac{{}^H\mathfrak{D}_{0+}^{\frac{5}{6}, 0; x} u(1) + 10^{-3} u(1)}{\frac{4}{6} + \frac{4}{6} u + \frac{1}{1+e} \mathcal{I}_{0+}^{\frac{\pi}{3}, x} u(1)} \right) = 0 \end{cases} \quad (0.43)$$

For $R = 1$ such that

$$\frac{\left(\frac{e}{200} \right) \left(\frac{2}{3} + \frac{1}{\Gamma(\frac{\pi}{3} + 1)} + 1 \right)}{\Gamma(\frac{\sqrt{5}+7}{7}) \Gamma(\frac{11}{6})} + \frac{1}{10^3 \Gamma(\frac{11}{6})} = 0.043797 < 1$$

and

$$\frac{e}{400 \Gamma(\frac{\sqrt{5}+7}{7}) \Gamma(\frac{11}{6})} \left[3 \left(1 + \frac{1}{\Gamma(\frac{\pi}{3} + 1)} \right) + \frac{4e}{4800} \right] + \frac{1}{10^3 \Gamma(\frac{11}{6})} = 0.049033 < 1.$$

Hence, from Theorem 0.53, we deduce that the ordinary hybrid version of Riemann–Liouville type GSLL problem (0.43) has at least one solution defined on $[0, 1]$.

Conclusion

The results of this paper provide a study that associates the crucial role of the notion of admissible mapping in the fixed point theorem with the theory of



fractional differential equations. We have presented some sufficient conditions dealing with the existence of solutions using a different approach, unlike that used in the literature, which was based on classical fixed point theorems. We utilized the α - φ -contraction via admissible mapping to prove the existence of solutions for the ordinary hybrid version of the Ψ -Hilfer-type GSSL equation (0.39). In this context, many promising topics could be discussed in the future, such as inclusion boundary value problems involving Ψ -Hilfer-type GSSL differential equations and a system of the Ψ -Hilfer-type GSSL differential equations by using common fixed point theorems with contractions via admissible mapping.

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Isoclines et Langues d'Arnold d'un endomorphisme tridimensionnel

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Abstract. L'objectif assigné de ce travail est l'étude de la période des cycles correspondants aux langues d'Arnold obtenues dans le diagramme de bifurcation simulé en utilisant les lignes isoclines, d'un système dynamique non-linéaire et non inversible de dimension trois à deux paramètres réels en utilisant la jacobienne du système.

Keywords: endomorphisme, cycles, langue d'Arnold, isoclines.



Introduction

Dans notre on s'intéressera ? la transformations ponctuelles (récurrentes) tridimensionnelles non linéaires et non inversibles (endomorphismes) suivante:

$$T \begin{cases} x_{n+1} = y_n, \\ y_{n+1} = z_n, \\ z_{n+1} = x_n^2 + ay_n(x_n + z_n) + b, \end{cases} \quad (0.44)$$

où a et b sont deux paramètres réels.

Definition 0.33. Une ligne isocline est formée par les valeurs des paramètres du point fixe pour chaque argument θ spécifiés [2, 3, 1]. Elle montre la phase instantanée autour du point fixe; le long de cette ligne, la stabilité est indexée par ρ ; entre 0 et 1 il est stable et $\rho = 0$ indique la super stabilité. Pour calculer et retrouver la ligne isocline, on utilise le système suivant:

$$\begin{cases} T(X) = X; \\ \Re_s = 0; \\ \Im_s = 0, \end{cases} \quad (0.45)$$

où $\Re_s$ et $\Im_s$ sont la partie réelle et imaginaire de l'équation caractéristique, définie par:

$$\det[J_c - \rho e^{i\theta} I] = \Re_s + i\Im_s = 0, \quad (0.46)$$

avec J_c est la jacobienne de T et I est la matrice identité.

Main results

*

Calcul du polynôme caractéristique La jacobienne de T est:

$$J_{(a,b)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2x + ay & a(x + z) & ay \end{pmatrix} \quad (0.47)$$

Une simulation numérique a été effectuée dans le plan des paramètres (a, b) (figure (16)). Cette figure permet de mettre en évidence les valeurs des paramètres pour lesquelles il existe au moins un attracteur (points fixes, cycles d'ordre k , attracteurs chaotiques,...) à distance finie. Cette figure joue le rôle d'une loupe de stabilité pour les cycles d'ordre k , $k = \overline{1..14}$, où ? chaque cycle d'ordre k est associé une couleur. Les zones de couleur bleu représentent les valeurs de paramètres (a, b) , pour lesquelles il existe un point fixe stable (attractif); les zones rouges représentent l'existence des cycles attractifs d'ordre 2, ainsi de suite pour les cycles jusqu'à l'ordre $k = 14$. Dans la zone noire, les phénomènes chaotiques peuvent être observés. En observant la figure (16),



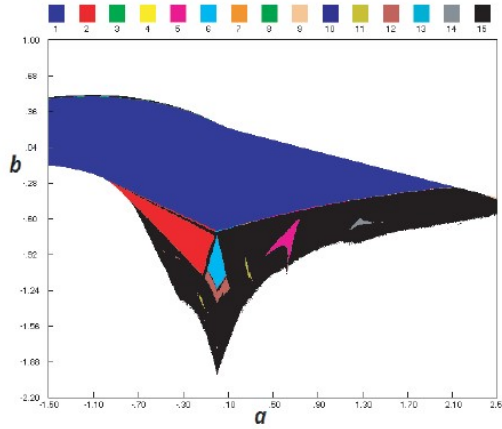


Figure 16: Domaines de stabilité des cycles attractifs dans le plan des paramètres (a, b) .

on peut voir le phénomène d'une certaine zone colorée en rose et en forme de langue (couleur désignant la période cinq). Ce phénomène se caractérise par la structure des langues d'Arnold [4] dans le plan de bifurcation. La méthode numérique utilisée, nous permet d'obtenir l'emplacement précis et la valeur de paramètre du point fixe correspondant à un argument spécifique $\theta = 2\pi \frac{p}{q}$ (où p et q sont des entiers premiers entre eux) des valeurs propres complexes conjuguées $S = e^{\pm i\theta}$. Lorsque le paramètre θ change, une telle solution dessine une isocline dans l'espace des paramètres. Le point d'intersection* de l'isocline $\theta = 2\pi \frac{p}{q}$ et la courbe de bifurcation Neimark-Sacker coïncide sur la cuspide de la langue d'Arnold de période- q . La recherche des lignes isoclines se ramène à la recherche des racines du polynôme caractéristique de la matrice en tout point fixe (x, y, z) en fonction de l'argument θ , ainsi localiser les paramètres de régions de l'existence des points périodiques.

$$J(x, y, z) = J_{(a,b)}(x, y, z) - \rho e^{i\theta} I_3 = \begin{pmatrix} -\rho e^{i\theta} & 1 & 0 \\ 0 & -\rho e^{i\theta} & 1 \\ 2x + ay & a(x + z) & ay - \rho e^{i\theta} \end{pmatrix}, \quad (0.48)$$

avec $J_{(a,b)}$ est la jacobienne de T .

Le polynôme caractéristique obtenu est:

$$P_{cara} = -\rho^3 (e^{i\theta})^3 + \rho^2 (e^{i\theta})^2 ay + \rho e^{i\theta} (ax + az) + 2x + ay. \quad (0.49)$$

En remplaçant $e^{i\theta}$ par sa forme trigonométrique $\cos(\theta) + i \sin(\theta)$, on aura:

$$\begin{aligned} P_{cara} = & -\rho^3 \cos(\theta)^3 - (3i)\rho^3 \cos(\theta)^2 \sin(\theta) + 3\rho^3 \cos(\theta) \sin(\theta)^2 + i\rho^3 \sin(\theta)^3 + \rho^2 ay \cos(\theta)^2 \\ & + 2i\rho^2 ay \cos(\theta) \sin(\theta) - \rho^2 ay \sin(\theta)^2 + \rho ax \cos(\theta) + i\rho ax \sin(\theta) + \rho az \cos(\theta) \\ & + i\rho az \sin(\theta) + 2x + ay \end{aligned}$$



D'où? P_{cara} s'écrit sous la forme complexe $P_{cara} = Re(P_{cara}) + iIm(P_{cara})$, où $Re(P_{cara})$ et $Im(P_{cara})$ sont la partie réelle et la partie imaginaire, respectivement, de P_{cara} :

$$Re(P_{cara}) = \rho^3(-\cos(\theta)^3 + 3\cos(\theta)\sin(\theta)^2) + \rho^2(ay\cos(\theta)^2 - ay\sin(\theta)^2) + \rho(ax\cos(\theta) + az\cos(\theta)) + 2x + ay; \quad (0.50)$$

$$Im(P_{cara}) = \rho^3(-3\cos(\theta)^2\sin(\theta) + \sin(\theta)^3) + 2\rho^2ay\cos(\theta)\sin(\theta) + \rho(ax\sin(\theta) + az\sin(\theta)). \quad (0.51)$$

La recherche des lignes isoclines maintenant revient à résoudre le système (0.53) de deux équations pour tout point fixe (x, y, z) de T en posant $\rho = 1$.

$$\begin{cases} Re(P_{cara}) = 0; \\ Im(P_{cara}) = 0. \end{cases} \quad (0.52)$$

Sachant que tout point fixe de T est tel que $x = y = z$ et la valeur de x est déterminée connue (voir formule ??), alors la résultante des deux équations du système sera de la forme:

$$F(\theta) = 0, \quad (0.53)$$

avec,

$$\begin{aligned} F(\theta) = & -(2+a)\sin(\theta)^3(-16a^3\cos(\theta)^4\sin(\theta)^2 + 4a^4\sin(\theta)^6 - 8a^3\cos(\theta)^2\sin(\theta)^4 \\ & + 8a^4\cos(\theta)^2\sin(\theta)^4 - 8a^3\cos(\theta)^6 + 4a^4\cos(\theta)^4\sin(\theta)^2)x^5 - (2+a)\sin(\theta)^3(36a^3\cos(\theta)^2\sin(\theta)^4 \\ & - 72\cos(\theta)^6a^2 + 8a^2\sin(\theta)^4\cos(\theta)^2 - 72\sin(\theta)^2a^2\cos(\theta)^4 + 28a^3\cos(\theta)^4\sin(\theta)^2 - 4a^3\cos(\theta)^6 \\ & + 8a^2\sin(\theta)^6 + 4a^3\sin(\theta)^6)x^4 - (2+a)\sin(\theta)^3(108\sin(\theta)^2\cos(\theta)^4a - 27\cos(\theta)^6a^2 \\ & + 27\sin(\theta)^2a^2\cos(\theta)^4 - 9a^2\sin(\theta)^4\cos(\theta)^2 - 108\cos(\theta)^6a - 36\sin(\theta)^4a\cos(\theta)^2 - 108\cos(\theta)^6 \\ & + 4\sin(\theta)^6 - 36\sin(\theta)^4\cos(\theta)^2 + a^2(\sin(\theta))^6 + 4(\sin(\theta))^6a + 108\sin(\theta)^2\cos(\theta)^4)x^3. \end{aligned}$$

L'équation (0.53) admet deux solutions, qui dépendent du paramètre θ , et chacune correspond à l'un des deux points fixes de T . Vu leurs complexités, le reste du calcul se fera numériquement et les lignes isoclines seront illustrées dans le plan des paramètres (a, b) , pour différentes valeurs de θ , dans la figure 17.

Dans la figure 18, la cuspide de la langue d'Arnold de point fixe de période 5 coïncide avec les points de croisement de la bifurcation de Neimark-Sacker et une des lignes isoclines de $\theta = \frac{2\pi}{5}$.

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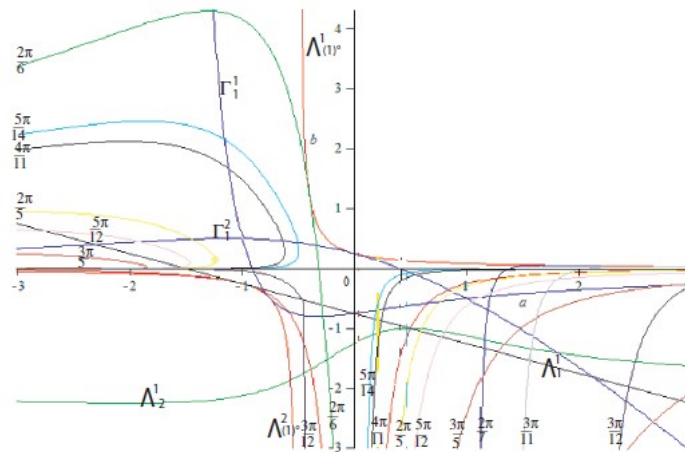


Figure 17: Lignes isoclines de T .

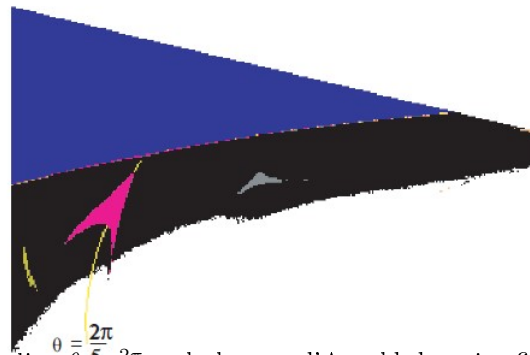


Figure 18: L'isocline $\theta = \frac{2\pi}{5}$ et la langue d'Arnold de point fixe de p période 5.

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Mild Solutions for Non Local Hybrid ψ -Hilfer Fractional Systems: Novel Qualitative Results

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Abstract. This work explores the existence of mild solutions for a class of Sobolev type non-local hybrid fractional differential equations. These equations involve the ψ -Hilfer fractional derivative, a generalization of fractional calculus, and are defined in Hilbert spaces. The study leverages fixed point theorems and semigroup theory to establish conditions for the existence of solutions. By introducing novel hypotheses and utilizing key fractional integral operators, we present a framework to solve these equations.

Keywords: Hybrid fractional differential equations, ψ -Hilfer fractional derivative, Mild solutions, Fixed point theorems.



Introduction

In this work, we investigate novel qualitative results of the existence of mild solutions of the following Sobolev type non local hybrid fractional differential equations (HDEs) involving ψ -Hilfer fractional derivative of the form:

$$\begin{cases} {}^H\mathbb{D}_{0+}^{\alpha,\beta;\psi} \left(\frac{Vx(t)}{f(t,x(t))} \right) + Ax(t) = g(t, x(t)), & t \in (0, b], \\ \mathbb{I}_{0+}^{1-\gamma;\psi} \left[\frac{Ex(t)}{f(t,x(t))} \right]_{t=0} + h(x) = x_0, \end{cases} \quad (1.1)$$

where ${}^H\mathbb{D}_{0+}^{\alpha,\beta;\psi}$ is the ψ -Hilfer fractional derivative of order $0 < \alpha < 1$ and type $0 < \beta < 1$. $\mathbb{I}_{0+}^{1-\gamma;\psi}$ is the ψ -Riemann-Liouville fractional integral of order $1 - \gamma$, ($\gamma = \alpha + \beta(1 - \alpha)$) and $J = [0, b]$. The operators $A : D(A) \subseteq X \rightarrow X$, $V : D(V) \subseteq X \rightarrow X$ and $E : D(E) \subseteq X \rightarrow X$. $x(t)$ takes value in Hilbert spaces X . The functions $f \in C(J \times X, X \setminus \{0\})$ and $g, h \in C(J \times X, X)$ are appropriate functions satisfying certain conditions.

Main results

Definition 0.34. For each $t \in J$ and $\mathbb{I}_{0+}^{1-\gamma;\psi} \left[\frac{Ex(0)}{f(0,x(0))} \right] = x_0$, a solution $x \in \tilde{\mathcal{C}} = \mathcal{C}_{1-\gamma,\psi}(J, X)$ is called a mild solution of (1.1) if satisfies the integral equation

$$x(t) = f(t, x(t)) \left[\begin{aligned} & E^{-1} \mathcal{S}_{\psi}^{\alpha,\beta}(t, 0) [x_0 - h(x)] \\ & + \int_0^t \mathcal{K}_{\alpha}^{\psi}(t, s) \mathcal{T}_{\psi}^{\alpha}(t, s) g(s, x(s)) ds \end{aligned} \right] \quad (2.1)$$

To prove the main result, we need the following hypotheses:

(H1) **a)** There exist a constant $L_1 > 0$ such that

$$\|g(t, x) - g(t, y)\| \leq L_1 Q_{\psi}^{1-\gamma}(t, 0) \|x - y\|_X,$$

for any $x, y \in X$.

b) For any $k > 0$, there exist $p > \frac{1}{\alpha}$ and function $\mu_k \in L^p(J, \mathbb{X})$, such that for any $x \in X$ satisfying $\|x\| \leq k$,

$$\|g(t, x)\| \leq \mu_k(t), \quad a.e. t \in J,$$

and there exists $\eta > 0$ such that

$$\liminf_{k \rightarrow \infty} \frac{\|\mu_k\|_{L^p}}{k} = \eta.$$



(H2) The function $f : J \times X \rightarrow X \setminus \{0\}$ is continuous and there exists there exists a function $\vartheta \in L^1(J; \mathbb{R}_+)$ such that

$$\|f(t, x) - f(t, y)\| \leq \|\vartheta\|_{L^1} \|x - y\|_X,$$

for a.e. $t \in J$ and all $x, y \in X$.

(H3) The nonlocal function $h : C(J; X) \rightarrow X$ is continuous and there exists a positive constant L_2 such that

$$\|h(x) - h(y)\| \leq L_2 \|x - y\|,$$

for $x, y \in X$.

Theorem 0.54. *Assume hypotheses (H1)–(H3) are satisfied. Then the Sobolev type nonlinear ψ -Hilfer hybrid fractional control system (1.1) admits a unique mild solution in $\tilde{\mathcal{C}} = \mathcal{C}_{1-\gamma, \psi}(J, X)$, provided*

$$K [\tau_1 + \tau_2 \|\mu_k\|_{L^p}] < 1, \quad (3.1)$$

where

$$\begin{aligned} \tau_1 &= Q_\psi^{1-\gamma}(b, 0) C_1 M_1 \|x_0\|, \\ \tau_2 &= Q_\psi^{1-\gamma}(b, 0) M_2 \sup_{0 \leq t \leq b} |\psi'(t)|^{\frac{1}{p}} \left(\frac{p-1}{\alpha p - 1} \right)^{\frac{p-1}{p}} \left\{ Q_\psi^{\frac{\alpha p - 1}{p-1}}(b, 0) \right\}^{\frac{p-1}{p}}. \end{aligned}$$

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The Existence of The Solution To a Fractional Schrödinger Equation With Doubly Criticality.

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Abstract. In this work, we prove the existence of the solution to a fractional elliptic Schrödinger equation with Hardy potential and critical nonlinearity. We prove our result using variational arguments and the minimax method.

Keywords: Fractional Laplacian, Linking theorem, Critical problem, Schrödinger equation.



Introduction

Let $\Omega \in \mathbb{R}^n (n \geq 2s)$ contains 0 with Lipschitz boundaries, we consider the following problem,

$$\begin{cases} (-\Delta)^s u - \mu \frac{u}{|x|^{2s}} = \lambda u + |u|^{2_s^*-2} u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \quad (0.54)$$

Where $\lambda \geq 0$, $0 < s < 1$, $0 < \mu < \bar{\mu} = 2^{2s} \frac{\Gamma^2(\frac{n+2s}{4})}{\Gamma^2(\frac{n-2s}{4})}$ the latter being the best constant of Hardy Sobolev inequality

$$\bar{\mu}_H \int_{\mathbb{R}^n} \frac{u^2(x)}{|x|^{2s}} dx \leq \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dxdy, \quad \forall u \in C_0^\infty(\mathbb{R}^n),$$

and $2_s^* = \frac{2n}{n-2s}$ be the critical fractional Sobolev exponent.

In recent years, much attention has been paid to fractional calculus, especially fractional PDEs. In the paper [1] Dipierro et al. investigated the existence and qualitative properties of the solution to the equation

$$(-\Delta)^s u - \mu \frac{u}{|x|^{2s}} = |u|^{2_s^*-2} u \text{ in } \mathbb{R}^n,$$

and gave us a paramount result concerning the asymptotic behavior of the solution near the origin and infinity using a ground-state representation, where authors proved that,

$$\frac{C}{(|x|^{1-\eta} + |x|^{1+\eta})^{\frac{n-2s}{2}}} \leq u \leq \frac{c}{(|x|^{1-\eta} + |x|^{1+\eta})^{\frac{n-2s}{2}}}, \text{ such that } \eta \in (0,1).$$

and in the paper [2] Ghoussoub et al. using a complicated analysis, they proved a similar result, such as the previous equation but with a critical singularity, moreover the existence of the solution to the problem

$$(-\Delta)^s u - \mu \frac{u}{|x|^{2s}} = \lambda u + \frac{|u|^{2_s^*(t)-2} u}{|x|^t} \text{ in } \mathbb{R}^n,$$

where $\lambda \in (0, \lambda_{\mu,1})$, the latter being the first eigenvalue of the fractional Hardy Schrödinger equation.

In our work, we have proved the existence of the solution to the problem (0.56) using the Linking theorem. The fact that we have a critical Sobolev exponent in our equation caused a problem of lack of compactness. We overcame this difficulty by pressing a range of compactness such that a Palais-Smale sequence converges in this range.



Main results

We consider the eigenvalue problem with Dirichlet boundaries,

$$\begin{cases} (-\Delta)^s \varphi_k - \mu \frac{\varphi_k}{|x|^{2s}} = \lambda_k \varphi_k & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \quad (0.55)$$

First of all, we begin with the next lemma concerning the asymptotic behavior of eigenfunctions associated to the fractional Hardy-Schrödinger operator,

Lemma 0.20. *There exists a constant C depending on λ, Ω, N, s , such that*

$$|\varphi_k| \leq C |x|^{-\gamma}, k > 0.$$

where φ_k satisfy (0.57), $\gamma = \frac{n-2s}{2} (1 - \eta_\gamma)$.

Then the major outcome of our work is the following,

Theorem 0.55. *Let $\Omega \in \mathbb{R}^n$ an open bounded domain which contains 0 in its interior, and $0 \leq \mu < \bar{\mu}_H$, then for $n > 4s$ and $\forall \lambda > 0$ the problem (0.56) has a nontrivial solution u .*

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The Existence of The Solution To a Fractional Schrödinger Equation With Doubly Criticality.

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Abstract. In this work, we prove the existence of the solution to a fractional elliptic Schrödinger equation with Hardy potential and critical nonlinearity. We prove our result using variational arguments and the minimax method.

Keywords: Fractional Laplacian, Linking theorem, Critical problem, Schrödinger equation.



Introduction

Let $\Omega \in \mathbb{R}^n (n \geq 2s)$ contains 0 with Lipschitz boundaries, we consider the following problem,

$$\begin{cases} (-\Delta)^s u - \mu \frac{u}{|x|^{2s}} = \lambda u + |u|^{2_s^*-2} u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \quad (0.56)$$

Where $\lambda \geq 0$, $0 < s < 1$, $0 < \mu < \bar{\mu} = 2^{2s} \frac{\Gamma^2(\frac{n+2s}{4})}{\Gamma^2(\frac{n-2s}{4})}$ the latter being the best constant of Hardy Sobolev inequality

$$\bar{\mu}_H \int_{\mathbb{R}^n} \frac{u^2(x)}{|x|^{2s}} dx \leq \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dxdy, \quad \forall u \in C_0^\infty(\mathbb{R}^n),$$

and $2_s^* = \frac{2n}{n-2s}$ be the critical fractional Sobolev exponent.

In recent years, much attention has been paid to fractional calculus, especially fractional PDEs. In the paper [1] Dipierro et al. investigated the existence and qualitative properties of the solution to the equation

$$(-\Delta)^s u - \mu \frac{u}{|x|^{2s}} = |u|^{2_s^*-2} u \text{ in } \mathbb{R}^n,$$

and gave us a paramount result concerning the asymptotic behavior of the solution near the origin and infinity using a ground-state representation, where authors proved that,

$$\frac{C}{(|x|^{1-\eta} + |x|^{1+\eta})^{\frac{n-2s}{2}}} \leq u \leq \frac{c}{(|x|^{1-\eta} + |x|^{1+\eta})^{\frac{n-2s}{2}}}, \text{ such that } \eta \in (0,1).$$

and in the paper [2] Ghoussoub et al. using a complicated analysis, they proved a similar result, such as the previous equation but with a critical singularity, moreover the existence of the solution to the problem

$$(-\Delta)^s u - \mu \frac{u}{|x|^{2s}} = \lambda u + \frac{|u|^{2_s^*(t)-2} u}{|x|^t} \text{ in } \mathbb{R}^n,$$

where $\lambda \in (0, \lambda_{\mu,1})$, the latter being the first eigenvalue of the fractional Hardy Schrödinger equation.

In our work, we have proved the existence of the solution to the problem (0.56) using the Linking theorem. The fact that we have a critical Sobolev exponent in our equation caused a problem of lack of compactness. We overcame this difficulty by pressing a range of compactness such that a Palais-Smale sequence converges in this range.



Main results

We consider the eigenvalue problem with Dirichlet boundaries,

$$\begin{cases} (-\Delta)^s \varphi_k - \mu \frac{\varphi_k}{|x|^{2s}} = \lambda_k \varphi_k & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \quad (0.57)$$

First of all, we begin with the next lemma concerning the asymptotic behavior of eigenfunctions associated to the fractional Hardy-Schrödinger operator,

Lemma 0.21. *There exists a constant C depending on λ, Ω, N, s , such that*

$$|\varphi_k| \leq C |x|^{-\gamma}, k > 0.$$

where φ_k satisfy (0.57), $\gamma = \frac{n-2s}{2} (1 - \eta_\gamma)$.

Then the major outcome of our work is the following,

Theorem 0.56. *Let $\Omega \in \mathbb{R}^n$ an open bounded domain which contains 0 in its interior, and $0 \leq \mu < \bar{\mu}_H$, then for $n > 4s$ and $\forall \lambda > 0$ the problem (0.56) has a nontrivial solution u .*

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Hybrid fractional differential equation in Banach Algebra

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Abstract. We study the existence of solutions for hybrid fractional differential equations with boundary hybrid conditions. Our results are based on fixed point theorem for two operators in a Banach algebra due to Dhage.

Keywords: Hybrid fractional equation, Banach algebra, Fixed point theorem, Existence of solutions.

Introduction

Fractional calculus can be defined as the generalization of classical calculus to orders of integration and differentiation not necessarily integer. The theory of fractional differential equations is an important branch of differential equation theory, which has an extensive physical, chemical and biological. Fixed point theory provides the tools to have existence theorems for many different nonlinear problems. So, we are interested in the existence of solutions for



the following hybrid fractional boundary value problem

$$\begin{cases} (\lambda {}^C\mathcal{D}^\alpha + {}^R\mathcal{D}^\alpha) \left[\frac{x(t)}{h(t, x(t))} \right] = f(t, x(t)), & 0 < t < T, \alpha \in (1, 2], \\ \left[\frac{x(t)}{h(t, x(t))} \right]_{t=0} = 0, \\ \mu_1 {}^R\mathcal{D}^{\beta_1} \left[\frac{x(t)}{h(t, x(t))} \right]_{t=T} + {}^R\mathcal{D}^{\beta_2} \left[\frac{x(t)}{h(t, x(t))} \right]_{t=T} = \delta_1, \\ \mu_2 \mathcal{I}^{\gamma_1} \left[\frac{x(t)}{h(t, x(t))} \right]_{t=T} + \mathcal{I}^{\gamma_2} \left[\frac{x(t)}{h(t, x(t))} \right]_{t=T} = \delta_2. \end{cases} \quad (0.58)$$

Where

- ${}^C\mathcal{D}$, ${}^R\mathcal{D}$ denotes the Caputo fractional derivative and the Riemann-Liouville fractional derivative respectively.
- $\mathcal{I}^{\gamma_i}$ denote the Riemann Liouville fractional integral of order $\gamma_i \in \mathbb{R}_+$ ($i = 1, 2$).
- $h : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}^*$ and $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ are a given functions continuous.
- $\lambda, \mu_1, \mu_2, \delta_1, \delta_2 \in (0, 1]$ are real constants and $0 \leq \beta_1, \beta_2 < \alpha$.

Main results

Let $X = C([0, T], \mathbb{R})$ denotes the Banach space of all continuous functions from $x : [0, T] \rightarrow \mathbb{R}$ with

$$\|x\| = \sup_{t \in [0, T]} |x(t)|.$$

First, let us define what we mean by a solution to the problem (0.58).

Definition 0.35. A function $x \in AC([0, T], \mathbb{R})$ is said to be a solution of the problem (0.58) if x satisfies the equation $(\lambda {}^C\mathcal{D}^\alpha + {}^R\mathcal{D}^\alpha) \left[\frac{x(t)}{h(t, x(t))} \right] = f(t, x(t))$ on $[0, T]$ and the conditions

$$\begin{cases} \left[\frac{x(t)}{h(t, x(t))} \right]_{t=0} = 0, \\ \mu_1 {}^R\mathcal{D}^{\beta_1} \left[\frac{x(t)}{h(t, x(t))} \right]_{t=T} + {}^R\mathcal{D}^{\beta_2} \left[\frac{x(t)}{h(t, x(t))} \right]_{t=T} = \delta_1, \\ \mu_2 \mathcal{I}^{\gamma_1} \left[\frac{x(t)}{h(t, x(t))} \right]_{t=T} + \mathcal{I}^{\gamma_2} \left[\frac{x(t)}{h(t, x(t))} \right]_{t=T} = \delta_2. \end{cases}$$

The integral form that is equivalent to problem (0.58) is given by the following:



Lemma 0.22. *Let $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. A function x is a solution of the fractional integral equation*

$$\begin{aligned} x(t) = h(t, x(t)) & \left(\frac{1}{(\lambda + 1)\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} f(r, x(r)) dr + \frac{t}{Q} \left[K_4 \mu_1 \mathcal{I}^{\alpha-\beta_1} f(T, x(T)) \right. \right. \\ & - K_2 \mu_2 \mathcal{I}^{\alpha+\gamma_1} f(T, x(T)) + (\lambda + 1)(K_2 \delta_2 - K_4 \delta_1) + K_4 \mathcal{I}^{\alpha-\beta_2} f(T, x(T)) \\ & - K_2 \mathcal{I}^{\alpha+\gamma_2} f(T, x(T)) \left. \right] + \frac{t^{\alpha-1}}{Q} \left[K_1 \mu_2 \mathcal{I}^{\alpha+\gamma_1} f(T, x(T)) - K_3 \mu_1 \mathcal{I}^{\alpha-\beta_1} f(T, x(T)) \right. \\ & \left. \left. + (\lambda + 1)(K_3 \delta_1 - K_1 \delta_2) + K_1 \mathcal{I}^{\alpha+\gamma_2} f(T, x(T)) - K_3 \mathcal{I}^{\alpha-\beta_2} f(T, x(T)) \right] \right). \end{aligned} \quad (0.59)$$

if and only if x is a solution of the fractional boundary value problem (0.58).

In the following, we prove existence results, for the boundary value problem (0.58) by fixed point theorem, under the following hypotheses:

(H1) There exist a bounded function $\theta : [0, T] \rightarrow \mathbb{R}_+$ such that

$$|h(t, x) - h(t, y)| \leq \theta(t)|x - y|, \quad \forall x, y \in \mathbb{R} \text{ and } t \in [0, T].$$

(H2) There exist a nonnegative continuous functions $p : [0, T] \rightarrow \mathbb{R}_+$ and a continuous nondecreasing function $\psi : [0, \infty) \rightarrow (0, \infty)$ such that

$$|f(t, x)| \leq p(t)\psi(\|x\|) \text{ for } x \in \mathbb{R} \text{ and } t \in [0, T].$$

(H3) There exist a constant $r \in \mathbb{R}_+$ such that

$$r > \frac{h^* p^* \psi(\|x\|) M + h^* N}{1 - \theta^* p^* \psi(\|x\|) M - \theta^* N}, \quad (0.60)$$

where

$$h^* = \sup_{t \in [0, T]} |h(t, 0)|, \quad p^* = \sup_{t \in [0, T]} |p(t)|, \quad \theta^* = \sup_{t \in [0, T]} |\theta(t)|.$$

and

$$\begin{aligned} M = & \frac{T^\alpha}{(\lambda + 1)\Gamma(\alpha + 1)} + \frac{T}{|Q|} \left[\frac{K_4 \mu_1 T^{\alpha-\beta_1}}{\Gamma(\alpha - \beta_1 + 1)} + \frac{K_2 \mu_2 T^{\alpha+\gamma_1}}{\Gamma(\alpha + \gamma_1 + 1)} + \frac{K_4 T^{\alpha-\beta_2}}{\Gamma(\alpha - \beta_2 + 1)} \right. \\ & \left. + \frac{K_2 T^{\alpha+\gamma_2}}{\Gamma(\alpha + \gamma_2 + 1)} \right] + \frac{T^{\alpha-1}}{|Q|} \left[\frac{K_1 \mu_2 T^{\alpha+\gamma_1}}{\Gamma(\alpha + \gamma_1 + 1)} + \frac{K_3 \mu_1 T^{\alpha-\beta_1}}{\Gamma(\alpha - \beta_1 + 1)} + \frac{K_1 T^{\alpha+\gamma_2}}{\Gamma(\alpha + \gamma_2 + 1)} \right. \\ & \left. + \frac{K_3 T^{\alpha-\beta_2}}{\Gamma(\alpha - \beta_2 + 1)} \right], \\ N = & \frac{T}{|Q|} (\lambda + 1)(|K_2 \delta_2| + |K_4 \delta_1|) + \frac{T^{\alpha-1}}{|Q|} (\lambda + 1)(|K_3 \delta_1| + |K_1 \delta_2|). \end{aligned}$$



Now, we give an existence result for the boundary problem (0.58).

Theorem 0.57. Assume that conditions (H1) – (H3) hold.
If

$$\theta^* p^* \psi(\|x\|)M + \theta^* N < 1,$$

Then the problem (0.58) has at least one solution on $[0, T]$.

For the proof of theorem 0.190, we need the fixed point theorem that we recall

Theorem 0.58. Let $\bar{B}(0, r)$ be closed balls in a Banach algebra X centered at origin 0 of radius r , for some real number $r > 0$.
Let $A, B : \bar{B}(0, r) \rightarrow X$ be two operators satisfying

- i. A is Lipschitz with Lipschitz constant l^* ,
- ii. B is continuous and compact,
- iii. $l^* \Delta < 1$, where

$$\Delta = \|B(\bar{B}(0, r))\| = \sup\{\|Bx\| : x \in \bar{B}(0, r)\}.$$

Then either

- a. the equation $AxBx = x$ has a solution in $\bar{B}(0, r)$, or
- b. there is an element $u \in X$ and $\alpha_0 \in (0, 1)$ such that : $\|u\| = r$ and $\alpha_0(AuBu) = u$.

Finally, we give example to illustrate the usefulness of the main result.

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Fractional differential equation system with random boundaries

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Abstract. We study the existence of solutions for random system of fractional differential equations with boundary nonlocal initial conditions. Our approach based on random fixed point principles of Schaefer and Perov, combined with a vector approach that uses matrices that converge to zero. We prove existence and uniqueness results for these systems. Some examples are presented to illustrate the theory.

Keywords: andom fractional differential equation, Hadamard fractional differential equation, existence, fixed point, vector metric space.



Introduction

In this paper, we consider the systems of random fractional differential equations with boundary conditions in the following form:

$$\left\{ \begin{array}{lcl} D^\alpha(D^\beta + \lambda_1)x(t, \omega) & = & f(t, x(t, \omega), y(t, \omega), \omega) \\ D^\gamma(D^\sigma + \lambda_2)y(t, \omega) & = & g(t, x(t, \omega), y(t, \omega), \omega) \\ \sum_{i=1}^m \theta_i I^{\mu_i} x(\eta_i, \omega) & = & \sum_{j=1}^n \phi_j I^{\gamma_j} x(\xi_j, \omega) \\ \sum_{k=1}^p \varepsilon_k I^{\varsigma_k} x(\psi_k, \omega) & = & \sum_{l=1}^q \nu_l I^{\tau_l} x(\varphi_l, \omega) \\ \sum_{i=1}^m \bar{\theta}_i I^{\bar{\mu}_i} y(\bar{\eta}_i, \omega) & = & \sum_{j=1}^n \bar{\phi}_j I^{\bar{\gamma}_j} y(\bar{\xi}_j, \omega) \\ \sum_{k=1}^p \bar{\varepsilon}_k I^{\bar{\varsigma}_k} y(\bar{\psi}_k, \omega) & = & \sum_{l=1}^q \bar{\nu}_l I^{\bar{\tau}_l} y(\bar{\varphi}_l, \omega) \end{array} \right. \quad (0.61)$$

where D^ρ denotes the Hadamard Caputo-type fractional derivative of order ρ , $\rho \in \{\alpha, \beta, \gamma, \sigma\}$ with $0 < \alpha, \beta, \gamma, \sigma < 1$, $1 < \alpha + \beta < 2$, $1 < \gamma + \sigma < 2$, λ_1, λ_2 are given constants, I^r is the Hadamard fractional integral of order $r > 0$, $r \in \{\mu_i, \gamma_j, \varsigma_k, \tau_l, \bar{\mu}_i, \bar{\gamma}_j, \bar{\varsigma}_k, \bar{\tau}_l\}$ the constants $\eta_i, \xi_j, \psi_k, \varphi_l, \bar{\eta}_i, \bar{\xi}_j, \bar{\psi}_k, \bar{\varphi}_l \in (1, e)$ and $\theta_i, \phi_j, \varepsilon_k, \nu_l, \bar{\theta}_i, \bar{\phi}_j, \bar{\varepsilon}_k, \bar{\nu}_l \in \mathbb{R}$, for all $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$, $k = 1, 2, \dots, p$, $l = 1, 2, \dots, q$ and $f, g : [1, e] \times \mathbb{R}^m \times \mathbb{R}^m \times \Omega \rightarrow \mathbb{R}^m$, $(\Omega, \mathcal{A})$ is a measurable space.

Main results

Theorem 0.59. Let $f, g : [1, e] \times \mathbb{R}^m \times \mathbb{R}^m \times \tilde{\Omega} \rightarrow \mathbb{R}^m$ are two Carathéodory functions. Assume that the following condition

(H) There exists random variables $p_1, p_2, p_3, p_4 : \tilde{\Omega} \rightarrow \mathbb{R}_+$ such that

$$\|f(t, x, y, \omega) - f(t, \tilde{x}, \tilde{y}, \omega)\| \leq p_1(\omega) \|x - \tilde{x}\| + p_2(\omega) \|y - \tilde{y}\|, \forall x, y, \tilde{x}, \tilde{y} \in \mathbb{R}^m$$

and

$$\|g(t, x, y, \omega) - g(t, \tilde{x}, \tilde{y}, \omega)\| \leq p_3(\omega) \|x - \tilde{x}\| + p_4(\omega) \|y - \tilde{y}\|, \forall x, y, \tilde{x}, \tilde{y} \in \mathbb{R}^m$$

holds.

If for every $\omega \in \tilde{\Omega}$, $\tilde{M}(\omega)$ converge to 0, where

$$\tilde{M}(\omega) = \begin{pmatrix} \Lambda_1(\alpha) p_1(\omega) + |\lambda_1| \Lambda_1(0) & \Lambda_1(\alpha) p_2(\omega) \\ \Lambda_2(\gamma) p_3(\omega) & \Lambda_2(\gamma) p_4(\omega) + |\lambda_2| \Lambda_2(0) \end{pmatrix},$$

then problem (1.1) has unique random solution.



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THE EXISTENCE OF WEAK SOLUTIONS FOR A NEW FRACTIONAL PROBLEM VIA ψ -HILFER OPERATOR

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Abstract. In this paper, by using variational methods and the critical point theory we give some results for the existence of three weak solutions to new fractional nonlinear problem with a fractional p -Laplacian via ψ -Hilfer operator.

Keywords: Critical point theory, Fractional BVP, p -laplacian, Three weak solution, Fractional calculus.



Introduction

The importance of fractional differential equations (FDEs) is that they model phenomena in various fields. We find some of their applications in the field of physics and chemistry, see [3, 1].

Authors in reference [3], gave us some new results on the multiplicity of solutions of the following Sturm-Liouville boundary value problem.

$$\begin{cases} {}_t D_T^\alpha \left(\frac{1}{h(t)^{p-2}} \Phi_p(h(t) {}_0^c D_t^\alpha u(t)) \right) + a(t) \Phi_p(u(t)) = \lambda f(t, u(t)), & a.e. \ t \in [0, T], \\ \alpha_1 \Phi_p(u(0)) - \alpha_2 {}_t D_T^{\alpha-1} (\Phi_p({}_0^c D_t^\alpha u(0))) = 0, \\ \beta_1 \Phi_p(u(T)) + \beta_2 {}_t D_T^{\alpha-1} (\Phi_p({}_0^c D_t^\alpha u(T))) = 0, \end{cases}$$

where ${}_0^c D_t^\alpha$ and ${}_t D_T^\alpha$ are the left Caputo and right Riemann-Liouville fraction derivatives of order $\alpha \in (\frac{1}{2}, 1]$.

In [4] Sousa, J.V.C., Zuo, J., O'Regan, D discussed results of the existence and non-existence of weak solutions to the non-linear problem with a fractional p-Laplacian involving by the ψ -Hilfer operator

$$\begin{cases} {}^H D_T^{\alpha, \beta; \psi} \left(\left| {}^H D_{0+}^{\alpha, \beta; \psi} \xi(x) \right|^{p-2} {}^H D_{0+}^{\alpha, \beta; \psi} \xi(x) \right) = \lambda |\xi(x)|^{p-2} \xi(x) + b(x) |\xi(x)|^{q-1} \xi(x), \\ I_{0+}^{\beta(\beta-1); \psi} \xi(0) = I_T^{\beta(\beta-1); \psi} \xi(T) = 0, \end{cases}$$

where ${}^H D_{0+}^{\alpha, \beta; \psi}(\cdot)$, ${}^H D_T^{\alpha, \beta; \psi}(\cdot)$ are ψ -Hilfer fractional derivatives left-sided and right-sided of order $\frac{1}{p} < \alpha < 1$, type $0 \leq \beta \leq 1$, $1 < q < p-1 < \infty$, $b \in L^\infty(\Omega)$, $I_{0+}^{\beta(\beta-1); \psi}(\cdot)$ and $I_T^{\beta(\beta-1); \psi}(\cdot)$ are ψ -Riemann-Liouville fractional integrals left-sided and right-sided, for all $x \in \Omega = [0, T]$, $\xi \in \mathbb{H}_p^{\alpha, \beta; \psi}$ and $\lambda > 0$.

Motivated by recent studies, we will study the following nonlinear equation

$$\begin{cases} {}^{\mathbb{H}} D_T^{\alpha, \beta; \psi} \left(\frac{1}{h(t)^{p-2}} \phi_p(h(t) {}_0^c D_t^{\alpha, \beta; \psi} u(t)) \right) = e(t) \phi_p(u(t)) + \lambda q(t, u(t)), & t \in I = [0, T] \\ u(0) = u(T) = 0 \end{cases} \quad (0.62)$$

Where $\phi_p(m) = |m|^{p-2}m$ and $\phi(0) = 0$, λ is positive parameter ($\lambda > 0$) and $h(t) \in L^\infty([0, T], \mathbb{R})$ with $h_0 = \text{ess inf}_{[0, T]} h(t) > 0$, $e \in C([0, T], \mathbb{R})$ and there exist $\tilde{e} \in \mathbb{R}$ such that $0 < e_0 < e(t) < \tilde{e}$, $q : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and continuously differentiable function, i.e, $q(t, u(t)) \in C(I)$ and $q(t, \cdot) \in C^1(\mathbb{R}^2)$.

Definition 0.36. [1] Let $\alpha \in (0, 1)$ and $\beta \in [0, 1]$, $L = [a, b]$ an interval such that $-\infty \leq a < b \leq +\infty$, the tow function $u, \psi \in C([a, b], \mathbb{R})$ such that ψ is increasing and $\psi'(x) \neq 0$ for all $t \in L$, the left and right ψ -Hilfer fractional derivatives ${}^{\mathbb{H}} D_{a+}^{\alpha, \beta; \psi}(\cdot)$ and ${}^{\mathbb{H}} D_{b-}^{\alpha, \beta; \psi}(\cdot)$ of order α and type β are defined by

$${}^{\mathbb{H}} D_{a+}^{\alpha, \beta; \psi} u(t) = I_{a+}^{\beta(1-\alpha); \psi} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right) I_{a+}^{(1-\beta)(1-\alpha); \psi} u(t),$$



and

$${}^{\mathbb{H}}D_{b^-}^{\alpha,\beta;\psi}u(t) = I_{b^-}^{\beta(1-\alpha);\psi} \left(-\frac{1}{\psi'(t)} \frac{d}{dt} \right) I_{b^-}^{(1-\beta)(1-\alpha);\psi}u(t),$$

Definition 0.37. [2] Let $\alpha \in \left(\frac{1}{p}, 1\right]$, $\beta \in [0, 1]$ and $p \in (1, \infty)$. The ψ -fractional derivatives space $\mathbb{H}_p^{\alpha,\beta;\psi} = \mathbb{H}_p^{\alpha,\beta;\psi}([0, T], \mathbb{R})$ is defined by the closure of $C_0^\infty([0, T], \mathbb{R})$, which is give by

$$\mathbb{H}_p^{\alpha,\beta;\psi} = \{u \in L^p([0, T], \mathbb{R}), {}^{\mathbb{H}}D_{0+}^{\alpha,\beta;\psi}u \in L^p([0, T], \mathbb{R}), I_{0+}^{\beta(\beta-1)}u(0) = I_T^{\beta(\beta-1)}u(T) = 0\}$$

with the following norm

$$\|u\|_{\mathbb{H}_p^{\alpha,\beta;\psi}} = \left(\|u\|_{L^p}^p + \|{}^{\mathbb{H}}D_{0+}^{\alpha,\beta;\psi}u\|_{L^p}^p \right)^{\frac{1}{p}}, \text{ for all } u \in \mathbb{H}_p^{\alpha,\beta;\psi},$$

and ${}^{\mathbb{H}}D_{0+}^{\alpha,\beta;\psi}(\cdot)$ is the ψ -Hilfer fractional derivative with all $\alpha \in (0, 1)$ and $\beta \in [0, 1]$.

Lemma 0.23. [3] If $\alpha \geq \frac{1}{p}$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\|u\|_\infty \leq \frac{(\psi(T) - \psi(0))^{\alpha - \frac{1}{p}}}{\Gamma(\alpha)(1 + q(\alpha - 1))^{\frac{1}{q}}} \|{}^{\mathbb{H}}D_{0+}^{\alpha,\beta;\psi}u\|_{L^p},$$

where $\|u\|_\infty = \sup_{t \in I} |u(t)|$

We have also,

$$\|u\|_\infty \leq \frac{(\psi(T) - \psi(0))^{\alpha - \frac{1}{p}}}{\Gamma(\alpha)(1 + q(\alpha - 1))^{\frac{1}{q}}} \|{}^{\mathbb{H}}D_{0+}^{\alpha,\beta;\psi}u\|_{\mathbb{H}_p^{\alpha,\beta;\psi}}$$

Remark 0.5. The norm in (0.468) is equivalent to

$$\|u\|_\alpha = \|{}^{\mathbb{H}}D_{0+}^{\alpha,\beta;\psi}u\|_{L^p}, \quad u \in \mathbb{H}_p^{\alpha,\beta;\psi}$$

We consider the space $E = \mathbb{H}_p^{\alpha,\beta;\psi}$ equipped with the norm $\|u\|_E = \|u\|_\alpha$

Theorem 0.60. [2] Let E be a reflexive real Banach space, suppose that $\mu : E \rightarrow \mathbb{R}$ be a coercive, continuously Gateaux differentiable and sequentially weakly lower semi continuous functional whose Gateaux derivative admits a continuous inverse on E^* , $\zeta : E \rightarrow \mathbb{R}$ be a sequentially weakly upper semi continuous and continuously Gateaux differentiable functional in which its Gateaux derivative is compact such that $\zeta(0) = \mu(0) = 0$.

Assume that there exist $r \in \mathbb{R}$ and $\bar{u} \in E$ with $0 < r < \mu(\bar{u})$ such that

$$(i) \sup_{u \in \mu^{-1}(-\infty, r]} \zeta(u) < r \frac{\zeta(\bar{u})}{\mu(\bar{u})},$$

$$(ii) \text{ The functional } \mu - \lambda\zeta \text{ is coercive for each } \lambda \in \Lambda = \left(\frac{\mu(\bar{u})}{\zeta(\bar{u})}, \frac{r}{\sup_{u \in \mu^{-1}(-\infty, r]} \zeta(u)} \right)$$

Then for any $\lambda \in \Lambda$, the functional $\mu - \lambda\zeta$ admits at least three critical points in E .



Main results

In order to prove the existence of at least three solution for nonlinear equation (0.468), we assume the following

$$\begin{aligned}\tau &= h_0 - \frac{\tilde{e}T(\psi(T) - \psi(0))^{p\alpha-1}}{\Gamma(\alpha)^p(1 + q(\alpha - 1))^{\frac{p}{q}}} \\ S &= \frac{(\psi(T) - \psi(0))^{\alpha p-1}}{h_0\Gamma(\alpha)^p(1 + q(\alpha - 1))^{\frac{p}{q}} - \tilde{e}T(\psi(T) - \psi(0))^{\alpha p-1}} \\ H(d) &= \left\{ \nu \in \mathbb{R}; \quad \frac{1}{p}\nu^p \leq d \right\} \\ \tau k &= \frac{(\psi(T) - \psi(0))^{\alpha p}}{(\Gamma(1 + \alpha))^p}\end{aligned}$$

Lemma 0.24. *Considering $Q : I \times \mathbb{R} \rightarrow \mathbb{R}$ to be a function that satisfies $Q(., u) \in C(I)$, $Q(t, .) \in C^1(\mathbb{R})$ and $Q(t, 0) = 0$ for all $t \in [0, T]$, fix*

$$\lambda_1 = \frac{\int_0^T h(t)|^{\mathbb{H}} D_{0+}^{\alpha_i, \beta_i; \psi} y(t)|^p dt - \int_0^T e(t)|y(t)|^p dt}{p \int_0^T Q(t, y(t)) dt}$$

and

$$\lambda_2 = \frac{r}{\int_0^T \sup_{\nu \in H(sr)} Q(t, \nu) dt}$$

For any $\lambda \in (\lambda_1, \lambda_2)$, nonlinear problem (0.468) admit at least three different weak solutions in E , if there exist $r \in \mathbb{R}$ which $r > 0$ and a function $y(t) = \bar{u}(t)$ such that the following hypotheses are satisfied

$$\begin{aligned}(A_0) : & \frac{1}{p} < \alpha < 1 \\ (A_1) : & \frac{\tilde{e}T(\psi(T) - \psi(0))^\alpha}{\Gamma(\alpha)^p(1 + q(\alpha - 1))^{\frac{p}{q}}} < h_0 \\ (A_2) : & \int_0^T h(t)|^{\mathbb{H}} D_{0+}^{\alpha, \beta; \psi} y(t)|^p dt \geq p \left(r + \frac{1}{p} \int_0^T e(t)|y(t)|^p dt \right) \\ (A_3) : & \lambda_1 < \lambda_2 \\ (A_4) : & \liminf_{|\nu| \rightarrow +\infty} \frac{Q(t, \nu)}{|\nu|^p} < \frac{1}{pk\lambda_2}\end{aligned}$$

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Nonlocal second order delay semilinear integro-differential time varying evolution equation with convex-power condensing operators

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Abstract. In this paper, we present findings concerning the existence of solutions to a particular type of nonlocal second-order delay semilinear integro-differential equation with time-varying evolution. We utilize the theory of the resolvent family, the Kuratowski measure of noncompactness, and fixed point theorems in conjunction with a convex-power condensing operator to establish novel and sufficient conditions for ensuring the existence of mild solutions. Finally, we provide an illustrative example to demonstrate the validity of the proposed results.

Keywords: Existence, nonlocal condition, second order semilinear equations, integro-differential equations, evolution equation, delay, measure of noncompactness..



Introduction

The aim of this paper is to discuss of the following nonlocal second Order delay semilinear integro-differential time varying evolution equation

$$\begin{cases} \frac{d^2 y(t)}{dt^2} = \mathcal{A}(t)y(t) + \mathcal{N}\left(t, y_t, \int_0^t \mathcal{M}(t, s, y_s)ds\right), & t \in [0, d], \\ y(t) = \phi(t) + (\hbar y)(t), & y'(0) = y_1. \end{cases} \quad (0.63)$$

In this text, E is a Banach space endowed with a norm $|\cdot|$, $\mathcal{I} = [0, d]$, $\{\mathcal{A}(t)\}_{0 \leq t \leq d} : D(\mathcal{A}) \subset E \rightarrow E$ is a linear closed operator, that generate an evolution system of linear bounded operators $\{\mathcal{G}(t, s)\}_{(t,s) \in \Delta}$. $\mathcal{N}$ is a given function $\mathcal{I} \times \Omega \times E \rightarrow E$ and $\mathcal{M} : \Delta \times \Omega \rightarrow R$, $\hbar : \mathcal{C}(\mathcal{I}, E) \rightarrow \Omega$, $\Omega = \mathcal{C}([-\eta, 0], E)$, $\Delta = \{(t, s) \in \mathcal{I} \times \mathcal{I} : s \leq t\}$.

Main results

In this section* we will establish the existence results by using the Kuratowski measure of noncompactness (0.63). Firstly, let us propose the definition of the mild solution of system (0.63).

Definition 0.38. A function $y \in \mathcal{C}([\sigma, d], E)$ is said to be a mild solution of system (0.63) if

- (1) for $t \in [\sigma, 0]$, $y(t) = \phi(t) + (\hbar y)(t)$,
- (2) for each $0 \leq t \leq d$, $y(t)$ satisfies the following integral equation:

$$(\mathbb{Q}y)(t) = -\frac{\partial}{\partial s} \mathcal{G}(t, 0) (\varphi(0) + \hbar(y)(0)) + \mathcal{G}(t, 0)y_1 + \int_0^t \mathcal{G}(t, s) \mathcal{N}\left(s, y_s, \int_0^s \mathcal{M}(s, \tau, y_\tau) d\tau\right) ds, \quad (0.64)$$

Before stating and proving the main results, we introduce the following hypotheses:

- (H₁) There exist a pair of constants $N_1 > 0, N_2 > 0$ and $\lambda > 0$, such that

$$\|\mathcal{G}(t, s)\|_{B(E)} \leq N_1 e^{-\lambda(t-s)} \quad \left\| \frac{\partial}{\partial s} \mathcal{G}(t, s) \right\|_{B(E)} \leq N_2, (t, s) \in \Delta.$$

- (H₂) The function $\mathcal{N} : \mathcal{I} \times \Omega \times E \rightarrow E$ satisfies:

- (i) For every $t \in \mathcal{I}$, the function $\mathcal{N}(t, \cdot, \cdot) : \mathcal{I} \times E \times E \rightarrow E$ is continuous, and for each $(y_1, y_2) \in E \times E$, the function $\mathcal{N}(\cdot, y_1, y_2) : \mathcal{I} \rightarrow E$ is strongly measurable,



- (ii) There exist $\Phi_{\mathcal{N}} \in L^{\frac{1}{p}}(\mathcal{I}, \mathbb{R}^+)$, $p \in [1, \infty)$ and a continuous nondecreasing function $\Psi_{\mathcal{N}} : [0, \infty) \rightarrow (0, \infty)$ such that:

$$|\mathcal{N}(t, y_1, y_2)| \leq \Phi_{\mathcal{N}}(t) \Psi_{\mathcal{N}}(|y_1| + |y_2|) \text{ for a.e } t \in \mathcal{I} \text{ and each } y_1, y_2 \in E.$$

$$\text{where } \Psi_{\mathcal{N}} \text{ satisfies } \lim_{r \rightarrow +\infty} \inf \frac{\Psi_{\mathcal{N}}(r)}{r} = \Lambda_{\mathcal{N}}$$

- (iii) There exist functions $\rho \in L^\infty(\mathcal{I}, \mathbb{R}^+)$, such that $\|\rho\|_{L^\infty} \in \left(0, \frac{1}{2N_1}\right]$ and for any bounded and countable set $V_1, V_2 \subset E$

$$\beta_{\mathcal{Z}}(\mathcal{N}(t, V_1, V_2)) \leq \rho(t) \left(\sup_{t \in [-\eta, 0]} \beta_E(V_1(t)) + \beta_E(V_2(t)) \right) \text{ for a.e } t \in \mathcal{I}.$$

- (H₃) The function $\mathcal{M} : \Delta \times E \rightarrow E$ satisfies the following:

- (i) There exists a positive function $\xi(t, s) \in L^1(\Delta, \mathbb{R}^+)$ such that:

$$|\mathcal{M}(t, s, y)| \leq \xi(t, s) |y|, \xi^* = \sup_{t \in \mathcal{I}} \int_0^t \xi(t, s) dt < +\infty, \text{ for a.e } t \in \mathcal{I} \text{ and each } y \in E.$$

- (ii) There exists positive number K^* , such that for any bounded and countable set $V \subset E$

$$\beta_{\mathcal{Z}}(\mathcal{M}(t, s, V)) \leq K^* \sup_{t \in [-\eta, 0]} \beta_E(V(t)) \text{ for a.e } t \in J.$$

- (H₄) The operator $\hbar; \mathfrak{C}(\mathcal{I}, E) \rightarrow \Omega$, satisfies the following:

- (i) For each $t \in [-\eta, 0]$, the operator $\Gamma : \mathfrak{C}(\mathcal{I}, E)$ defined by $\Gamma_t(y) = (\hbar y)(t)$ is continuous and the subset $\hbar(\Lambda) \subset \Omega$ is equicontinuous for each bounded set $\Lambda \subset \mathfrak{C}(\mathcal{I}, E)$.

- (ii) There exists positive number $N_{\hbar}$ and a continuous nondecreasing function $\Psi_{\hbar} : [0, \infty) \rightarrow (0, \infty)$ such that:

$$\|\hbar(y)\|_{\Omega} \leq N_{\hbar} \Psi_{\hbar}(\|y\|),$$

$$\text{where } \Psi_{\hbar} \text{ satisfies } \lim_{r \rightarrow +\infty} \inf \frac{\Psi_{\hbar}(r)}{r} = \Lambda_{\hbar} \in \left[0, \frac{1}{N_{\hbar}}\right].$$

- (iii) There exists positive number $K_{\hbar} \in \left[0, \frac{1}{\max(N_2, 2)}\right]$ such that for any bounded and countable set $V \subset E$

$$\beta_E(\hbar(V)) \leq K_{\hbar} \beta_E(V(t)) \text{ for a.e } t \in [-\eta, 0].$$



Theorem 0.61. Assume that the conditions $(\mathbb{H}_1) - (\mathbb{H}_4)$ hold. Then the nonlocal Cauchy problem (0.63) has at least a mild solution on $[-\eta, d]$, if

$$N_2 N_h \Lambda_h + \frac{(1-p)^{1-p} \|\Phi_N\|_{L^{\frac{1}{p}}}}{\lambda^{1-p}} \Lambda_N \leq 1 \quad (0.65)$$

is satisfied.

Proof. Let us consider the operator $\mathbb{Q} : \mathfrak{C}[-\sigma, d], E) \rightarrow \mathfrak{C}[-\sigma, d], E)$, defined by

$$\begin{cases} (\mathbb{Q}y)(t) = \phi(t) + h(y)(t), & t \in [-\sigma, 0]. \\ (\mathbb{Q}y)(t) = -\frac{\partial}{\partial s} \mathcal{G}(t, 0) (\varphi(0) + h(y)(0)) + \mathcal{G}(t, 0) y_1 + \int_0^t \mathcal{G}(t, s) \mathcal{N} \left(s, y_s, \int_0^s \mathcal{M}(s, \tau, y_\tau) d\tau \right) ds, & t \in \mathfrak{J}, \end{cases} \quad (0.66)$$

It is obvious that the fixed point of $\mathbb{Q}$ is the mild solution of (0.63). We now show that $\mathbb{Q}$ satisfies all the assumptions of fixed point theorem with respect to a convex-power condensing operator (see [8]).

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Existence of Solutions For a Boundary Value Problem of Hadamard Fractional Derivative Order Via Pseudomonotone Theory

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Abstract. In this work we discuss existence of a weak solution of a fractional problem on an bounded interval. By making use of a Pseudomonotone Theory.

Keywords: Fractional Differential Equations, Hadamard Fractional, Boundary Value Problem, Pseudomonotone Theory.



Introduction

The aim of this paper is to investigate the existence of at least one solution for the fractional Hadamard boundary value problem defined on a bounded interval, which is formulated as follows:

$$\begin{cases} {}^H D_{b-}^{\alpha} {}^H D_{a+}^{\alpha} u(t) + u(t) = f(t, u(t)), & t \in [a, b], \\ {}^H I_{a+}^{1-\alpha} u(a) = u(b) = 0, \end{cases}$$

where $0 < a < b < \infty$, and $\frac{1}{2} < \alpha < 1$, $f : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function. ${}^H D_{a+}^{\alpha}$, ${}^H D_{b-}^{\alpha}$ are respectively the left and right fractional Hadamard derivative. By using the variational method .

Main results

Definition 0.39. (Hadamard fractional integral [10]) Let $0 < a < b < \infty$, $\alpha > 0$, the left and right Hadamard fractional integrals of order α of $u \in L^p(a, b)$ are respectively defined by

$${}^H I_{a+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s} \right)^{\alpha-1} \frac{u(s)}{s} ds \quad (a < t < b),$$

and

$${}^H I_{b-}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \left(\ln \frac{s}{t} \right)^{\alpha-1} \frac{u(s)}{s} ds \quad (a < t < b),$$

provided that the right-hand side is pointwise defined on (a, b) , where $\Gamma(\alpha)$ is the gamma function.

Definition 0.40. (Hadamard fractional derivative [10]) Let $0 < a < b < \infty$, $\delta = t \frac{d}{dt}$, $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the real number α . The left and right Hadamard fractional derivatives of order α of $u \in L^p(a, b)$ are respectively defined by:

$$\begin{aligned} {}^H D_{a+}^{\alpha} u(t) &= \delta^n ({}^H I_{a+}^{n-\alpha} u(t)) \\ &= \left(t \frac{d}{dt} \right)^n \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\ln \frac{t}{s} \right)^{n-\alpha-1} \frac{u(s)}{s} ds \quad a < t < b, \end{aligned}$$

and

$$\begin{aligned} {}^H D_{b-}^{\alpha} u(t) &= (-\delta)^n ({}^H I_{b-}^{n-\alpha} u(t)) \\ &= \left(-t \frac{d}{dt} \right)^n \frac{1}{\Gamma(n-\alpha)} \int_t^b \left(\ln \frac{s}{t} \right)^{n-\alpha-1} \frac{u(s)}{s} ds \quad a < t < b. \end{aligned}$$

Definition 0.41. [1] Let X be a Banach space. An operator $A : X \rightarrow X^*$ which satisfies

$$\langle Au - Av, u - v \rangle \geq 0 \quad (0.67)$$



for any $u, v \in X$ is called a monotone operator. An operator A is called strictly monotone if for $u \neq v$, strict inequality holds in (0.67). An operator A is called strongly monotone if there exists $C > 0$ such that

$$\langle Au - Av, u - v \rangle \geq C\|u - v\|^2$$

for any $u, v \in X$. It is clear that a strongly monotone is strictly monotone.

Definition 0.42. [11] Let $A : X \rightarrow X^*$ be an operator on the real reflexive Banach space is pseudomonotone if

A is bounded;

and

$$u_n \rightharpoonup u \text{ and } \limsup_{n \rightarrow \infty} \langle A(u_n), u_n - u \rangle \leq 0$$

implies

$$\langle A(u), u - u \rangle \leq \liminf_{n \rightarrow \infty} \langle A(u_n), u_n - u \rangle, \quad \forall u \in X.$$

Definition 0.43. [1] Let $A : X \rightarrow X^*$ be an operator on the real Banach space X .

(a) A is said to be demicontinuous if

$$u_n \rightarrow u \text{ lorsque } n \rightarrow +\infty \text{ implique } Au_n \rightarrow Au \text{ lorsque } n \rightarrow +\infty.$$

(b) A is said to be hemicontinuous if the real function

$$t \mapsto \langle A(u + tv), w \rangle \text{ est continue sur } [0, 1] \text{ pour tous } u, v, w \in X.$$

(c) A is said to be coercive if

$$\lim_{\|u\| \rightarrow +\infty} \frac{\langle Au, u \rangle}{\|u\|} = +\infty.$$

(d) A is said to be strongly continuous at $u \in X$ if for every sequence $u_n \subset X$ for which $u_n \rightarrow u$ in X it follows that $A(u_n) \rightarrow A(u)$ in X^* .

Proposition 0.8. [11] Let: $A, B : X \rightarrow X^*$ be operators on the real reflexive Banach space X Then:

(a) If A is monotone and hemicontinuous, then A is pseudomonotone.

(b) If A is strongly continuous, then A is pseudomonotone.

(c) If A is demicontinuous with

$$\text{if } u_n \rightharpoonup u \text{ and } \limsup_{n \rightarrow \infty} \langle Au_n - Au, u_n - u \rangle \leq 0 \text{ implies } u_n \rightarrow u,$$

then A is pseudomonotone.



- (d) If A is continuous and $\dim X < \infty$, then A is pseudomonotone.
- (e) Additivity. If A and B are pseudomonotone, then $A + B$ is pseudomonotone.
- (f) If A is monotone and hemicontinuous and B is strongly continuous, then $A + B$ is pseudomonotone.
- (j) If A is pseudomonotone and locally bounded, then A is demicontinuous.
- (h) If A is pseudomonotone and B is monotone and hemicontinuous, then $A + B$ is pseudomonotone.
- (i) If A is pseudomonotone and B is strongly continuous, then $A + B$ is pseudomonotone.

Theorem 0.62. [11] Let X be a separable reflexive real Banach space. Any operator $A : X \rightarrow X^*$, which is pseudomonotone and coercive is surjective. Stated differently, for any $f \in X^*$, there is at least one solution to the equation $A(u) = f$.

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Results for the Blow-up of nonlinear wave equation with time-dependent damping

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Abstract. In this research, we concentrate on the Cauchy problem for viscoelastic damped wave models with a time-dependent effective damping term, and nonlinear memory components on the right-hand side. Our main goal is to examine the blow-up phenomena and identify the critical exponent in the context of Fujita's criterion. The model under investigation shows parabolic-like characteristics, as suggested by the energy decay estimates of the corresponding linear Cauchy problem. Consequently, we utilize the test function method to derive our results.

Keywords: Wave equation, Time-dependent damping, Blow-up, Critical exponent, Test function method.



Introduction

Firstly, the effectively damped wave models with nonlinear memory

$$\begin{cases} u_{tt} - \Delta u + b(t)u_t = \int_0^t (t-\tau)^{-\gamma} |u(\tau, \cdot)|^p d\tau, & (t, x) \in (0, \infty) \times \mathbb{R}^n, \\ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), & x \in \mathbb{R}^n, \end{cases}$$

Were examined in [2] and [3]. In [2], the authors established the global existence of small data solutions using Matsumura-type estimates. Meanwhile, in [3], they demonstrated the finite-time blow-up of weak solutions for a family of effectively $\{b(t) = (1+t)^r, r \in (-1, 1)\}$ models by applying the test function method.

Recently, the authors proved in [1] that weak solution of Cauchy problem

$$\begin{cases} u_{tt} - \Delta u + u_t - \Delta u_t = \int_0^t (t-\tau)^{-\gamma} |u(\tau, \cdot)|^p d\tau, & (t, x) \in (0, \infty) \times \mathbb{R}^n, \\ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \quad (0.68)$$

blows in finite time too. They, also, prove that the viscoelastic term does not influence the critical exponent in Fujita sense. They showed that the critical exponent of this problem is :

$$p^*(\gamma) = \max\{p_\gamma, \gamma^{-1}\},$$

where $p_\gamma = 1 + \frac{2(2-\gamma)}{(n-2+2\gamma)_+}$.

In this work, we study the nonexistence of global weak solutions to the Cauchy problem for the effectively damped wave models with viscoelastic term and nonlinear memory

$$\begin{cases} u_{tt} - \Delta u + b(t)u_t - \Delta u_t = \int_0^t (t-\tau)^{-\gamma} |u(\tau, \cdot)|^p d\tau, & (t, x) \in (0, \infty) \times \mathbb{R}^n, \\ u(0, x) = f(x), \quad u_t(0, x) = g(x), & x \in \mathbb{R}^n, \end{cases} \quad (0.69)$$

where $\gamma \in (0, 1)$, $p > 1$ and $\{b(t) = (1+t)^r, r \in (-1, 0)\}$ is a non negative decreasing effective function in the sense of Corollary (). For this reason we apply the test function method for proving our results.

Next, we provide adequate conditions that render a given function effective. In a given hyperbolic PDE, the dissipative term $b(t)u_t$ is *effective*, if $b = b(t)$ satisfies the following properties:

- (1) $b(t) > 0$ for any $t > 0$.
- (2) b is monotone and $tb(t) \rightarrow \infty$ as $t \rightarrow \infty$.
- (3) $((1+t)^2 b(t))^{-1} \in L^1([0, \infty))$.
- (4) $b \in C^3$ and $|b^{(k)}(t)|(1+t)^k \lesssim b(t)$, for any $k = 1, 2, 3$.
- (5) $\frac{1}{b} \notin L^1(0, \infty)$.



Main results

Theorem 0.63. *Let $0 < \gamma < 1$, $p \in (1, \infty)$ for $n = 1, 2$ and $1 < p < \frac{n}{n-2}$ for $n \geq 3$. Assume that $(f, g) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ and satisfy*

$$\int_{\mathbb{R}^n} f(x)dx > 0, \quad \int_{\mathbb{R}^n} g(x)dx > 0. \quad (0.70)$$

Then,

If p satisfies the condition $p \leq p^*$. Then weak solutions of Cauchy problem (0.69) do not exist globally in time.

where

$$\tilde{p}_\gamma(n) := 1 + \frac{2(1-\gamma)}{n+2\gamma} \quad \text{and} \quad p_\gamma^*(n) = \max\{\tilde{p}_\gamma(n), \frac{1}{\gamma}\}$$

Conclusion

In [1], the authors demonstrated that the critical exponent for the Cauchy problem of the viscoelastic damped wave equation for model (0.68) is

$$p_\gamma(n) = 1 + \frac{2(2-\gamma)}{(n-2+2\gamma)_+}.$$

In this study, we explored the impact of effective dissipation on the critical exponent by examining the Cauchy problem (0.69). By comparison, it is evident that $\tilde{p}_\gamma(n) < p_\gamma(n)$. We therefore concluded that effective damping affects the critical exponent of the problem under study.

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The variational study of a dynamic problem with fractional derivatives

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Abstract. In this study, we extended previous research by examining a generalized model of a dynamic frictionless contact problem between a viscoelastic body and a rigid foundation . The constitutive behavior is governed by the fractional Kelvin-Voigt law, and the contact is characterized by the normal compliance condition. We developed a weak formulation of the problem and established the existence of a weak solution. The proofs rely on the theory of abstract monotone operators, Caputo derivatives, the Galerkin method, and Banach's fixed point theorem

Keywords: fixed point, Caputo derivatives, frictionless contact, Kelvin-Voigt law



Introduction

In general, systems of partial differential equations associated with boundary conditions are governed by elliptic, parabolic, and evolutional variational equations and inequalities. On the other hand, the theory of monotone operators is a branch of functional analysis that studies the properties of monotone operators in vector spaces. These operators play a crucial role in various fields of applied mathematics, particularly in partial differential equations, nonlinear analysis, and convex optimization. Caputo's derivative is a generalization of the classical derivative, introduced by Italian mathematician Michele Caputo in the 1960s. It is used to describe non-local differentiation phenomena such as anomalous diffusion, fractal deformation, or non-local behavior in various scientific domains such as physics, engineering, and applied sciences. The Galerkin method was developed by mathematician Boris Galerkin in the 1920s. It is a numerical approach used to solve partial differential equations (PDEs). This method is based on the variational formulation of the PDE and allows for an approximate solution by projecting the PDE onto a finite subspace of test functions. In this work, a new model is presented for a dynamic contact problem without friction between a viscoelastic body and a rigid foundation. The constitutive relation is modeled using the fractional Kelvin-Voigt law. The contact is described by the normal compliance condition. The key aspects discussed are: Caputo's derivative is used to model the viscoelastic behavior of the material. The Galerkin method is employed to obtain a weak formulation of the problem and prove the existence of its weak solution. Banach's fixed-point theorem is also used in the proofs. The mathematical formulation of the problem includes the equation of motion with a fractional time derivative.

Main results

Problem (PV): Find a displacement field $u : \Omega \times]0, T[\longrightarrow \mathbb{R}^d$ and a stress tensor $\sigma : \Omega \times]0, T[\longrightarrow \mathbb{S}^d$ for $v \in V$ and $\alpha \in (0, 1]$ such that

$$\sigma(t) = \mathcal{B}\varepsilon \left({}^C_0 D_t^\alpha u(t) \right) + \mathcal{A}\varepsilon(u(t)) \left({}^C_0 D_t^{2\alpha} u(t), v \right)_V \quad (0.71)$$

$$+ b \left({}^C_0 D_t^\alpha u(t), v \right) + a(u(t), v) + j(u(t), v) = (f(t), v)_V \quad (0.72)$$

$$u(0, x) = u_0, \quad {}^C_0 D_t^\alpha u(0, x) = w_0$$

Our main result on existence, which will be proven in the next section*, can be stated as follows:

Theorem 3.1 Assuming that assumptions (HP1) to (HP4) hold, we can conclude that Problem (PV) has at least one solution (u, σ) that satisfies the following conditions:

$$u \in W^{2,2}(0, T; V), \text{ and } \sigma \in L^2(0, T; L^2(\Omega, \mathbb{S}^d))$$



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On the Global Non existence of a Solution for Wave Equations with Nonlinear Memory Term

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Abstract. The paper is devoted to the problem of the local existence for a solution to a nonlinear wave equation, with the dissipation given by a nonlinear form with the presence of a nonlinear memory term. Moreover, the global nonexistence of a solution is established using the test function method. We combine the Fourier transform and fractional derivative calculus to achieve our goal.

Keywords: Lyapunov functions, wave equations, well-posedness, nonlinear memory term, multiplier method



Introduction

Many researchers investigate how the different terms that make up the PDE affect the existence of the solution and its stability and determine whether the solution exists in general with respect to time. The terms vary according to the physical phenomenon being studied and the modeled problem. The linear memory term is considered among the most famous terms, which effect studied problems. It often establishes the existence of the solution under some conditions. Researchers have recently turned to studying this term nonlinearly and choose the initial idea in a way that allows them to rewrite it in terms of fractional derivative.

We look at the following Cauchy problem with a nonlinear memory term

$$\begin{cases} u_{tt} - \Delta u + (1+t)^r |u|^{m-1} u_t = \int_0^t (t-s)^{-\gamma} |u|^p ds & (t, x) \in (0, \infty) \times \mathbb{R}^n \\ u(0, x) = g(x), u_t(0, x) = h(x) & x \in \mathbb{R}^n \end{cases} \quad (0.73)$$

where u is the unknown real-valued function, $n > 0$, $m > 1$, $p > 1$, $r, \gamma \in \Omega := (0, 1)$ and $g(x)$, $h(x)$ are the initial data.

Main results

*

Existence and uniqueness Now we are ready to prove the existence and the uniqueness result of the local solution to our problem, the result is given in the next theorem

Theorem 0.64. $n > 0$, $s > \frac{n}{2} + 1$, $r \in (0, 1)$, $m, p \in (1, +\infty) \cap (s-1; +\infty)$ then for $g \in H^s(\mathbb{R}^n)$ and $h \in H^{s-1}(\mathbb{R}^n)$ the problem (0.73) has a unique solution

$$u \in C([0, T]; H^s(\mathbb{R}^n)) \cap C^1([0, T]; H^{s-1}(\mathbb{R}^n))$$

with some positive T , which depends only on $\|g\|_{H^s} + \|h\|_{H^{s-1}}$

*

Blow-up In this section* we will proof the blow-up of solution of the problem (0.73). Before that, let us introduce the weak solution of (0.73) definition

Definition 0.44. Let $T > 0$, $r \in (0, 1)$, $0 < \gamma < 1$ and $g \in L^1_{loc}(\mathbb{R}^n) \cap L^m_{loc}(\mathbb{R}^n)$, $h \in L^1_{loc}(\mathbb{R}^n)$

We call u a weak solution if $u \in L^p((0, T), L^p_{loc}(\mathbb{R}^n)) \cap L^m((0, T), L^m_{loc}(\mathbb{R}^n))$



and satisfies

$$\begin{aligned}
& \Gamma(\alpha) \int_0^T \int_{\mathbb{R}^n} J_{0|t}^\alpha (|u|^p) \varphi dx dt + \int_{\mathbb{R}^n} h(x) \varphi(0, x) dx \\
& - \int_{\mathbb{R}^n} g(x) \varphi_t(0, x) dx + \frac{1}{m} \int_{\mathbb{R}^n} |g|^{m-1} g \varphi(0, x) dx \\
& = \int_0^T \int_{\mathbb{R}^n} u \varphi_{tt} dx dt - \frac{1}{m} \int_0^T \int_{\mathbb{R}^n} b'(t) |u|^{m-1} u \varphi dx dt \\
& - \frac{1}{m} \int_0^T \int_{\mathbb{R}^n} b(t) |u|^{m-1} u \varphi_t dx dt - \int_0^T \int_{\mathbb{R}^n} u \Delta \varphi dx dt
\end{aligned} \tag{0.74}$$

For all nonnegative test function $\varphi \in C^2([0, T] \times \mathbb{R}^n)$ such that $\varphi(T, x) = \varphi_t(T, x) = 0$ where $\alpha = 1 - \gamma$ and $b(x) = (1 + t)^r$

In all the paper we write $f \prec g$ when there exists a constant $C > 0$ such that $f \leq Cg$, the Nonnegative constants C is always supposed to be independent of $T > 0$.

Theorem 0.65. Let $n > 0$ and $0 < \gamma < 1$ and let $m > p > 1$. Assume that the initial data (g, h) satisfy

$$\begin{aligned}
& \int_{\mathbb{R}^n} g(x) dx > 0 \\
& \int_{\mathbb{R}^n} h(x) dx > 0 \\
& \int_{\mathbb{R}^n} |g(x)|^{m-1} g(x) dx > 0
\end{aligned} \tag{0.75}$$

Then if

$$\frac{p}{p-1} > \inf_{d>0} \max \left\{ \frac{\frac{nd}{2} + 1}{\alpha + d}, \sqrt{\left(\frac{B}{2A}\right)^2 - \frac{C}{A} - \frac{B}{2A}} \right\} \tag{0.76}$$

with

$$\begin{aligned}
A &= (1 - m)(r - 2 + m)(1 + \rho) - \alpha(r - 2 + m) - (1 - r)(1 - m) \\
B &= m[(2m - 3 + r)(1 + \rho) + \alpha + (1 - r)] \\
C &= -m^2(1 + \rho)
\end{aligned}$$

Then the weak solution in the sense of definition (0.44) of the Cauchy problem (0.73) do not exist globally in time.

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A Fractional Differential System of Beam Type With Two Generalised Nonlinearities

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Abstract. This study explores a nonlinear coupled system related to beam deflection, using Caputo derivatives. The system includes new derivative parameters $0 < \alpha_1, \alpha_2 \leq 1$, $2 < \beta_1$ and $\beta_2 \leq 3$ that are used for sequential derivatives and do not follow the standard semi-group and commutativity rules, it includes also two generalised nonlinearities in its right hand sides. In some cases, our system is simplified to a fourth-order ordinary coupled system like beam dynamics. We use the Banach contraction principle and the Schauder fixed point theorem to prove two main results: the uniqueness of solutions and the existence of at least one solution. We also provide an example to demonstrate the first main result. Additionally, we discuss Ulam-Hyers stability with some examples. At the end, the Tanh method is applied to study travelling wave solutions for a coupled evolution system that has some connection to the sequential system.

Keywords: Caputo derivative, existence of solution, beam deflection, Schauder fixed point, travelling waves



Introduction

The elastic beam, also known as Euler-Bernoulli equation, plays a crucial role in material science and engineering. It models the behavior of a beam under bending loads and can be used to determine the stress distribution, deflection and strain in a beam made of elastic material; this equation is given by:

$$\frac{d^2}{dx^2}(EI \frac{d^2}{dx^2}u(x)) = -dl(x),$$

where u is the deflection of the beam, E is the Young's modulus of the material, I represents the moment of inertial of the cross-sectional area of the beam and dl represents the distributed load on the beam. In mathematical analysis this equation motivated the study of equations of the type:

$$\begin{cases} u^4(t) = g(t, u(t), u''(t)), & 0 < t < 1 \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases}$$

Inspired by the work of K. Bensassa et al. [2], we want to study the following generalized elastic beam system in which with fractional derivatives in the sense of Caputo of order $\delta - 1$; $1 < \delta < 2$ and two general nonlinearities F_1, F_2 :

$$\begin{aligned} D^{\alpha_1} D^{\beta_1} u_1(x) &= F_1(x, u_1(x), u_2(x), D^{\delta-1} u_1(x), D^{\delta} u_1(x)), \\ D^{\alpha_2} D^{\beta_2} u_2(x) &= F_2(x, u_1(x), u_2(x), D^{\delta-1} u_2(x), D^{\delta} u_2(x)), \end{aligned} \quad (0.77)$$

under the conditions

$$\begin{cases} u_1(0) = u_1(1) = \theta_1, \\ u_1'(0) = u_1'(1) = 0, \\ u_2(0) = u_2(1) = \theta_2, \\ u_2'(0) = u_2'(1) = 0, \end{cases} \quad (0.78)$$

where, for $i = 1, 2$, $\theta_i \in \mathbb{R}^+$, $F_i \in C([0, 1] \times \mathbb{R}^4, \mathbb{R})$ and $D^{\alpha_i}, D^{\beta_i}, D^{\delta}$ are Caputo fractional derivatives, with $0 < \alpha_1, \alpha_2 \leq 1$, $2 < \beta_1, \beta_2 \leq 3$. Using the Banach contraction principle and the Schauder fixed point theorem, we investigate the existence and the uniqueness of a solution for this system. We also discuss its Ulam-Hyers stability.

Main results

We proved the following existence theorem.

Theorem 0.66. *Assume that the following conditions are satisfied:*

(G₁) *The functions $F_1, F_2 : [0, 1] \times \mathbb{R}^4 \rightarrow \mathbb{R}$ are continuous.*



(G₂) There exist positive constants h_{ij} , $i = 1, 2$, $j = 1, 2, 3, 4$, such that, for all $x \in [0, 1]$ and $(y, z, t, w), (y', z', t', w') \in \mathbb{R}^4$, we have

$$|F_i(x, y, z, t, w) - F_i(x', y', z', t', w')| \leq h_{i1}|y - y'| + h_{i2}|z - z'| + h_{i3}|t - t'| + h_{i4}|w - w'|,$$

where

$$H_i = \max(h_{i1}, h_{i2}, h_{i3}, h_{i4}), \quad \text{for } i = 1, 2.$$

(G₃) There exist positive constants z_1, z_2 :

$$\begin{aligned} |F_1(x, y, z, t, w)| &\leq z_1, \\ |F_2(x, y, z, t, w)| &\leq z_2. \end{aligned}$$

Then, the system (0.530) – (0.476) has at least one solution.

We also give a sufficient condition for the uniqueness of this solution. Under this same condition, we showed that the system (1)-(2) is stable in the sense of Ulam-Hyers.

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Numerical solution of Fredholm Integral Equations using Legendre-Galerkin Method

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Abstract. Legendre-Galerkin Method is used for solving linear Fredholm integral equations.

This method reduces the integral equations to a linear system of algebraic equations, which can be solved by wolfram mathematica program.

Illustrative Examples show that the proposed Method is accurate and efficient.

Keywords: linear Fredholm integral equations, Legendre-Galerkin Method, Legendre polynomials.



Introduction

Fredholm integral equations are very important equations in mathematics they can be found in various fields which include mathematics, physics and engineering. The process of solving the Fredholm integral equations analytically is very complicated and sometimes it is impossible, so for application purposes it will be sufficient to solve it numerically. The problem that finding the numerical solutions for integral equations are often a complicated process and requires a large number of arithmetic computations.

In this work we present a new technic to approximate the solution of Fredholm integral equations (FIEs) using Legendre-Galerkin Method with higher order convergence and higher accuracy.

Consider the FIEs of the form:

$$u(x) = f(x) + \int_{-1}^1 k(x, t)u(t)dt, \quad x, t \in [-1, 1], \quad (0.79)$$

where the functions $f(x)$ and $k(x, t)$ are given, the function $u(x)$ is to be determined.

First, we approximate solution of FIEs with the form:

$$u(x) \cong u_N(x) = \sum_{i=0}^n a_i P_i(x), \quad (0.80)$$

where P_i Legendre polynomials of degree i , and $a_i, i = 0, 1, \dots, n$ denoted the coefficients to be determined. By substituting the Eq. 0.457 in Eq.(0.351) and applying Legendre-Galerkin Method we have linear algebraic system, which can be solved by wolfram mathematica program.

Main results

The present work have lead to highlight the following results:

1. A novel proposed method for solving linear Fredholm integral equations.
2. The obtained results show that the maximum absolute error decreases with the increasing values of n using this method for different examples.

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Well-Posedness for Viscoelastic Fourth-Order Problems with Relaxation Term Involving The Generalized Caputo Fractional Derivative

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Abstract. This research examines the well-posedness of viscoelastic fourth-order problems with a relaxation term. The governing equation is

$$u_{ttt}(t, x) + \gamma_1 \Delta^2 u(t, x) + \gamma_2 \int_0^t g(t - \tau) \Delta u(\tau, x) d\tau + \partial_t^{\eta, \alpha} u(t, x) = 0,$$

where $\partial_t^{\eta, \alpha}$ denotes the generalized Caputo fractional derivative of order $0 < \alpha < 1$, and $(x, t) \in \Omega \times \mathbb{R}_+$. The aim is to establish requirements for well-posedness, guaranteeing the local existence of solutions that depend continuously on g .

Keywords: Bilaplacian operator, generalized Caputo fractional derivative, viscoelastic.



Introduction

Viscoelastic materials display both viscous and elastic properties during deformation. The mathematical modeling of these materials frequently entails higher-order partial differential equations, which consider intricate characteristics such as time-dependent stress and strain.

Fractional partial differential equations are employed to elucidate scientific phenomena across various fields, (see, [1, 2, 3, 4]).

In recent years, the study of viscoelasticity has gained significant attention, particularly in the context of fractional calculus. Adding fractional derivatives, like the generalized Caputo derivative, makes it possible to show the memory effects of viscoelastic materials more accurately [5, 6]. The well-posedness of such models is crucial for ensuring the existence, uniqueness, and stability of solutions. Earlier research has looked at different aspects of well-posedness in PDEs, such as the conditions for initial and boundary value problems [7, 8]. However, the specific challenges posed by fourth-order viscoelastic problems with relaxation terms and fractional derivatives necessitate further investigation.

This paper aims to fill this gap by establishing the well-posedness of the proposed viscoelastic model (0.81):

$$\begin{cases} u_{tt}(t, x) + \gamma_1 \Delta^2 u(t, x) + \gamma_2 \int_0^t g(t - \tau) \Delta u(\tau, x) d\tau + \partial_t^{\alpha, \eta} u(t, x) = 0, \\ u(t, x) = 0, \quad \Delta u(t, x) = 0, \quad \text{for all } (t, x) \in \mathbb{R}_+ \times \partial\Omega, \\ u(0, x) = u_0(x) ; \quad u_t(0, x) = u_1(x). \end{cases} \quad (0.81)$$

Following [1], the fractional damping term $\partial_t^{\alpha, \eta} u$ is expressed using the generalized Caputo fractional derivative of order $0 < \alpha < 1$ as

$$\partial_t^{\alpha, \eta} u(x, t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - s)^{-\alpha} e^{-\eta(t-s)} u_s(s, x) ds.$$

Essential Preliminaries

This section* outlines some fundamental ideas and lemmas required for this study. [11] For any $y \in D := \{z \in \mathbb{C} : \operatorname{Re}(z) \geq 0\} \cap \{z \in \mathbb{C} : z \neq 0\}$, the following estimation holds

$$\frac{1}{|y + \lambda|} \leq \sqrt{2} \max \left(1, \frac{1}{|y|} \right) \frac{1}{1 + \lambda}, \quad \text{for all } \lambda \geq 0. \quad (2)$$

[16] In the following system

$$\begin{cases} \partial_t \phi(x, \xi, t) + (\xi^2 + \eta) \phi(x, \xi, t) - \mu(\xi) w(x, t) = 0, \\ \phi(x, \xi, 0) = 0, \\ o(x, t) = \frac{\sin(\pi\alpha)}{\pi} \int_{-\infty}^{+\infty} \mu(\xi) \phi(x, \xi, t) d\xi, \end{cases}$$

the relationship between the input u and the output of for $(x, t) \in (0, L) \times (0, +\infty)$ and $\eta > 0$ can be equivalently written as $o = I^{1-\alpha, \eta} u$. Taking into



account the identity $\partial_t^{\alpha,\eta} w = I^{1-\alpha,\eta} w_t$ and using Theorem , it is easy to deduce the equivalence between the generalized Caputo fractional derivative $\partial_t^{\alpha,\eta} w$ and the following system

$$\begin{cases} \partial_t \phi(x, \xi, t) + (\xi^2 + \eta) \phi(x, \xi, t) - \mu(\xi) w_t(x, t) = 0, \\ \phi(x, \xi, 0) = 0 \\ \partial_t^{\alpha,\eta} w(x, t) = \frac{\sin(\alpha\pi)}{\pi} \int_{-\infty}^{+\infty} \mu(\xi) \phi(x, \xi, t) d\xi. \end{cases}$$

This together with problem (P), the augmented system (P') is obtained as $\forall(x, t) \in \Omega \times \mathbb{R}$.

$$\begin{cases} u_{tt}(t, x) + \gamma_1 \Delta^2 u(t, x) + \gamma_2 \int_0^t g(t - \tau) \Delta u(\tau, x) d\tau + \frac{\sin(\alpha\pi)}{\pi} \int_{-\infty}^{+\infty} \mu(\xi) \phi(x, \xi, t) d\xi = 0 \\ \partial_t \phi(x, \xi, t) + (\xi^2 + \eta) \phi(x, \xi, t) - \mu(\xi) u_t(x, t) = 0 \end{cases}, \quad (P')$$

with the same boundary conditions

$$u(t, x) = 0 ; \Delta u(t, x) = 0 \quad \forall(t, x) \in \mathbb{R}_+ \times \partial\Omega,$$

and the updated initial conditions

$$\begin{cases} u(x, 0) = u_0(x), \\ u_t(x, 0) = u_1(x), \\ \phi(x, \xi, 0) = 0 \end{cases} \quad x \in \Omega, \xi \in (-\infty, +\infty)$$

Next, we discuss a few fundamental components needed to confirm our main conclusions. To do this, we assume the following assumptions for the relaxation function g .

(A1) $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a differentiable function such that:

$$g(0) > 0; 1 - \lambda \int_0^\infty g(s) ds = I > 0,$$

where λ constant such that

$$\|\nabla u\|_2^2 \leq \lambda \|\Delta u\|_2^2 \quad \forall u \in D(u);$$

(A2) there exist a differentiable function γ satisfying:

$$g'(t) \leq -\gamma(t)g(t) \quad t \geq 0$$

$$\gamma(t) > 0 ; \gamma'(t) \leq 0 \quad \forall t > 0.$$

Well-Posedness of the main problem

If we consider the notation $U := (u, u_t, \phi)^T$, then the augmented system (P') can be rewritten as a first order Cauchy problem

$$\begin{cases} U_t + \mathcal{A}U = \mathcal{B}U, \\ U(0) = (w_0, w_1, 0)^T. \end{cases} \quad (0.82)$$



$$\mathcal{A}U = \begin{pmatrix} \gamma_1 \Delta^2 u(t, x) + \frac{\sin(\alpha\pi)}{\pi} \int_{-\infty}^{+\infty} \mu(\xi) \phi(x, \xi, t) d\xi, \\ (\xi^2 + \eta) \phi - \mu u_t, \end{pmatrix} \quad (0.83)$$

and

$$\mathcal{B} \begin{pmatrix} u \\ u_t \\ \phi \end{pmatrix} := \begin{pmatrix} 0 \\ -\gamma_2 \int_0^t g(t - \tau) \Delta u(\tau, x) d\tau \\ 0 \end{pmatrix} \quad (0.84)$$

The state space associated to U is given by the following product Hilbert space

$$\mathcal{H} := H_0^2(\Omega) \times H_0^2 \times L^2(\Omega \times (-\infty, +\infty)),$$

with

$$L^2(\Omega \times (-\infty, +\infty)) := \left\{ f : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \text{ such that } \int_{\Omega} \int_{-\infty}^{+\infty} |f|^2 d\xi dx < +\infty \right\},$$

and

$$H_0^2(\Omega) = \{ w \in H^2(\Omega) \mid \Delta w|_{x \in \partial\Omega} = 0, w|_{x \in \partial\Omega} = 0 \}.$$

The inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ is defined for two elements of $\mathcal{H}, U = (w, w_t, \phi)^T$ and $\tilde{U} = (\tilde{w}, \tilde{w}_t, \tilde{\phi})^T$, by

$$\langle U, \tilde{U} \rangle_{\mathcal{H}} := \int_{\Omega} w_t \tilde{w}_t dx + \gamma_1 \int_{\Omega} \Delta w \Delta \tilde{w} dx + \frac{\sin(\alpha\pi)}{\pi} \int_{\Omega} \int_{-\infty}^{+\infty} \phi \tilde{\phi} d\xi dx. \quad (0.85)$$

For the domain of the operator $\mathcal{A}$, we have

$$\mathcal{D}(\mathcal{A}) := \left\{ \begin{pmatrix} (w, w_t, \phi)^T \in \mathcal{H} : w \in H_0^2(\Omega), w_t \in H_0^2(\Omega), \\ -(\xi^2 + \eta) \phi + \mu w_t \in L^2(\Omega \times (-\infty, +\infty)), \\ |\xi| \phi \in L^2(\Omega \times (-\infty, +\infty)). \end{pmatrix} \right\} \quad (0.86)$$

Additionally, let us provide the energy functional related to the problem (0.81)

$$E(t) = \frac{1}{2} \|u_t\|_2^2 + \frac{1}{2} \|\Delta u(t)\|_2^2 - \left(\int_0^t g(s) ds \right) \|\nabla u(t)\|_2^2 + \frac{1}{2} (g \circ \nabla u)(t, x) + \frac{\sin(\alpha\pi)}{\pi} \int_{\Omega} \int_{-\infty}^{+\infty} |\phi|^2 d\xi dx, \quad (0.87)$$

and the auxiliary energy functional associated to the problem

$$\begin{aligned} J(t) &= \frac{1}{2} \|\nabla u_t\|_2^2 + \|\nabla(\Delta u(t))\|_2^2 \\ &- \left(\int_0^t g(s) ds \right) \|\Delta u(t)\|_2^2 \\ &+ \frac{1}{2} (g \circ \Delta u)(t, x) \\ &+ \frac{\sin(\alpha\pi)}{\pi} \int_{\Omega} \int_{-\infty}^{+\infty} |\phi|^2 d\xi dx, \end{aligned} \quad (0.88)$$

where

$$g \circ v(t, x) = \int_0^t g(t - \tau) \|v(t) - v(\tau)\|_2^2 d\tau.$$



The primary findings of the work are the analysis of the energy equation and the application of the ensuing existence and uniqueness theorem.

Following the approach in [12, 13], we adopt a two-stage argument. Initially, we demonstrate that the operator $\mathcal{A}$ is maximal accretive on the given Hilbert space $\mathcal{H}$. Subsequently, we establish the local Lipschitz continuity of $\mathcal{B}$ on the same space. Therefore, our attention is directed towards examining the properties of the operator $\mathcal{A}$, for which we introduce the following two lemmas.

The linear unbounded operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$ is accretive. proof. We have for $U = (w, w_t, \phi)^T \in \mathcal{H}$

$$\begin{aligned} \langle \mathcal{A}U, U \rangle_{\mathcal{H}} &= \int_{\Omega} \left[-\gamma_1 \Delta^2 w + \frac{\sin(\alpha\pi)}{\pi} \int_{-\infty}^{+\infty} \mu(\xi) \phi(x, \xi, t) d\xi \right] w_t dx \\ &\quad + \gamma_1 \int_{\Omega} \Delta w_t \Delta w dx \\ &\quad + \frac{\sin(\alpha\pi)}{\pi} \int_{\Omega} \int_{-\infty}^{+\infty} [(\xi^2 + \eta) \phi - \mu w_t] \phi d\xi dx. \end{aligned}$$

Using Green's integration formula, we obtain after simplification

$$\langle \mathcal{A}U, U \rangle_{\mathcal{H}} = \frac{\sin(\alpha\pi)}{\pi} \int_{\Omega} \int_{-\infty}^{+\infty} (\xi^2 + \eta) |\phi|^2 d\xi dx.$$

Since $a > 0$ and $\alpha \in (0, 1)$, then we have $\langle \mathcal{A}U, U \rangle_{\mathcal{H}} \geq 0$ i.e. the operator $\mathcal{A}$ is accretive.

The operator $\mathcal{A}$ is maximal. proof. To establish the maximality of $\mathcal{A}$, our objective is to demonstrate the surjectivity of $\mathbb{I} + \mathcal{A}$. Thus, we begin by considering $F = (f_1, f_2, f_3)^T \in \mathcal{H}$ and seek $U = (w, w_t, \phi)^T \in \mathcal{D}(\mathcal{A})$ as a solution to the resolvent equation $(\mathbb{I} + \mathcal{A})U = F$. The key steps of the proof are outlined as follows.

Step 1. Solving the associated weak formulation with respect to the component w .

The equation $(\mathbb{I} + \mathcal{A})U = F$ can be detailed as

$$\begin{cases} w - w_t = f_1 \\ w_t - \gamma_1 \Delta^2 w + \frac{\sin(\alpha\pi)}{\pi} \int_{-\infty}^{+\infty} \mu(\xi) \phi(x, \xi, t) d\xi = f_2 \\ \phi + (\xi^2 + \eta) \phi - \mu u_t = f_3. \end{cases} \quad (0.89)$$

We substitute the expressions for w_t and ϕ in (0.89)₂ with their corresponding forms involving w , f_1 , and f_3 as given in (0.89)₁ and (0.89)₃. Subsequently, we perform an inner product with $v \in H_0^2(\Omega)$, followed by the application of Green's integration formula. This process leads to the formulation in terms of variational principles.

$$\tilde{a}(w, v) = \tilde{\ell}(v), \quad (0.90)$$



where

$$\begin{aligned} \tilde{a} : H_0^2(\Omega) \times H_0^2(\Omega) &\longrightarrow \mathbb{R}, \\ \tilde{a}(w, v) &= \left[1 + \frac{\sin(\alpha\pi)}{\pi} \int_{-\infty}^{+\infty} \frac{\mu^2}{1 + \eta + \xi^2} d\xi \right] \int_{\Omega} w v dx \\ &+ \gamma_1 \int_{\Omega} \Delta w \Delta v dx \end{aligned}$$

and

$$\begin{aligned} \tilde{\ell} : H_0^2(\Omega) &\longrightarrow \mathbb{R} \\ \tilde{\ell}(v) &= \int_{\Omega} [f_2 + f_1] v dx - \frac{\sin(\alpha\pi)}{\pi} \int_{\Omega} \int_{-\infty}^{+\infty} \frac{\mu f_3 v}{1 + \eta + \xi^2} d\xi dx, \\ &+ \frac{\sin(\alpha\pi)}{\pi} \int_{\Omega} \int_{-\infty}^{+\infty} \frac{\mu^2 f_1 v}{1 + \eta + \xi^2} d\xi dx. \end{aligned}$$

It's straightforward to verify that the operator $\tilde{a}$ constitutes a bilinear, continuous, and coercive form, while the operator $\tilde{\ell}$ represents a linear and continuous form. Given that all the conditions of the Lax-Milgram theorem are satisfied, we conclude that the equation $\tilde{a}(w, v) = \tilde{\ell}(v)$ possesses a unique solution $w \in H_0^2(\Omega)$ for every $v \in H_0^2(\Omega)$. Once w is found, the remaining components of the solution $U = (w, w_t, \phi)^T$ can also be determined.

Step 2. Verifying that the solution $U = (w, w_t, \phi)^T \in \mathcal{D}(\mathcal{A})$.

Clearly, we have $w_t = w - f_1 \in L^2(\Omega)$ because $w, f_1 \in L^2(\Omega)$. For the component ϕ , it follows from the application of Young inequality that

$$\int_{\Omega} \int_{-\infty}^{+\infty} |\phi|^2 d\xi dx \leq \frac{(1-\alpha)\pi}{\sin(\alpha\pi)} \|w_t\|_{L^2(\Omega)}^2 + 2 \int_{\Omega} \int_{-\infty}^{+\infty} \frac{1}{(1 + \eta + \xi^2)^2} |f_3|^2 d\xi dx$$

. At this level, we make use of Lemma with $y = 1 + \eta \in D$ and $\lambda = \xi^2 \geq 0$ to get the estimate

$$\int_{\Omega} \int_{-\infty}^{+\infty} |\phi|^2 d\xi dx \leq \frac{(1-\alpha)\pi}{\sin(\alpha\pi)} \|w_t\|_{L^2(\Omega)}^2 + 4 \int_{\Omega} \int_{-\infty}^{+\infty} |f_3|^2 d\xi dx$$

. Since $w_t \in L^2(\Omega)$ and $f_3 \in L^2(\Omega \times (-\infty, +\infty))$, we have $\phi \in L^2(\Omega \times (-\infty, +\infty))$. Accordingly, the solution $U = (w, w_t, \phi)^T \in \mathcal{H}$.

Now, we need to check that constraints within the operator domain $\mathcal{D}(\mathcal{A})$ are fulfilled. We show that $-(\xi^2 + \eta) \phi + \mu w_t \in L^2(\Omega \times (-\infty, +\infty))$ by invoking Young inequality, where

$$\int_{\Omega} \int_{-\infty}^{+\infty} |-(\xi^2 + \eta) \phi + \mu w_t|^2 d\xi dx \leq 2 \int_{\Omega} \int_{-\infty}^{+\infty} |\phi|^2 d\xi dx + 2 \int_{\Omega} \int_{-\infty}^{+\infty} |f_3|^2 d\xi dx$$

. The result $-(\xi^2 + \eta) \phi + \mu w_t \in L^2(\Omega \times (-\infty, +\infty))$ holds as $\phi \in L^2(\Omega \times (-\infty, +\infty))$ and $f_3 \in L^2(\Omega \times (-\infty, +\infty))$.

Further, the evaluation of $\int_{\Omega} \int_{-\infty}^{+\infty} |\xi \phi|^2 d\xi dx$ using Young inequality and Lemma provides

$$\int_{\Omega} \int_{-\infty}^{+\infty} |\xi \phi|^2 d\xi dx \leq 2 \frac{\pi\alpha}{\sin(\pi\alpha)} \|w_t\|_{L^2(\Omega)}^2 + 4 \int_{\Omega} \int_{-\infty}^{+\infty} |f_3|^2 d\xi dx$$



. Since $w_t \in L^2(\Omega)$ and $f_3 \in L^2(\Omega \times (-\infty, +\infty))$ then $\int_{\Omega} \int_{-\infty}^{+\infty} \|\xi\| \phi_1^2 d\xi dx$ is finite and $|\xi| \phi \in L^2(\Omega \times (-\infty, +\infty))$. Lastly, we have $w_t = w - f_1 \in H_0^2(\Omega)$ thanks to $w, f_1 \in H_0^2(\Omega)$.

The problem $\tilde{a}(w, v) = \tilde{\ell}(v)$ admits a unique solution $w \in H_0^2(\Omega)$ for all $v \in H_0^2(\Omega)$, which allows writing

$$\begin{aligned} \int_{\Omega} \Delta w \Delta v dx &= - \left\{ 1 + \frac{\sin(\alpha\pi)}{\pi} \int_{-\infty}^{+\infty} \frac{\mu^2}{1+\eta+\xi^2} d\xi \right\} \int_{\Omega} w v dx \\ &+ \int_{\Omega} [f_2 + f_1] v dx - \frac{\sin(\alpha\pi)}{\pi} \int_{\Omega} \int_{-\infty}^{+\infty} \frac{\mu f_3 v}{1+\eta+\xi^2} d\xi dx \\ &+ \frac{\sin(\alpha\pi)}{\pi} \int_{\Omega} \int_{-\infty}^{+\infty} \frac{\mu^2 f_1}{1+\eta+\xi^2} v d\xi dx. \end{aligned} \quad (0.91)$$

Due to the fact that $w \in H_0^2(\Omega)$ and the right hand side of Eq. (0.553) belongs to $L^2(\Omega)$, then the application of the elliptic regularity theorem leads to $u \in H_0^2(\Omega)$.

By performing Green's integration theorem on the variational formula valid for all $v \in H_0^2(\Omega)$, it follows that

$$w - \gamma_1 \Delta^2 w + \frac{\sin(\alpha\pi)}{\pi} \int_{-\infty}^{+\infty} \mu(\xi) \phi(x, \xi, t) d\xi = f_2 + f_1, \quad (0.92)$$

with $\phi = \frac{f_3 + \mu w - \mu f_1}{1+\eta+\xi^2}$. Hence, we have $U = (w, w_t, \phi)^T \in \mathcal{D}(\mathcal{A})$ a unique solution to the resolvent equation $(\mathbb{I} + \mathcal{A})U = F$ and the operator $\mathcal{A}$ is maximal.

If g verify the assumption (A1) and (A2) then, the operator $\mathcal{B}$ is locally Lipschitz on the state space $\mathcal{H}$. proof. let $f(x) = -\gamma_2 \int_0^t g(t-\tau) \Delta u(\tau, x) d\tau$. To prove that f is locally Lipschitz in the Sobolev space $H^2(\Omega)$, we need to show that for any $u_1, u_2 \in H^2(\Omega)$, there exists a constant $C > 0$ such that

$$\|f(u_1) - f(u_2)\|_{H^2(\Omega)} \leq C \|u_1 - u_2\|_{H^2(\Omega)}.$$

Step 1: Expression for $f(u_1) - f(u_2)$
for u_1 and u_2 , we have

$$f(u_1) - f(u_2) = -\gamma_2 \int_0^t g(t-\tau) (\Delta u_1(\tau, x) - \Delta u_2(\tau, x)) d\tau.$$

Step 2: Norm Estimate

We estimate the difference $f(u_1) - f(u_2)$ in the $H^2(\Omega)$ -norm. Since $f(u)$ involves the Laplacian Δu , we are dealing with second derivatives. In Sobolev space $H^2(\Omega)$, the norm is given by

$$\|v\|_{H^2(\Omega)}^2 = \|v\|_{L^2(\Omega)}^2 + \|\nabla v\|_{L^2(\Omega)}^2 + \|\Delta v\|_{L^2(\Omega)}^2.$$

We will focus on the $L^2(\Omega)$ -norm of $f(u_1) - f(u_2)$, since the higher-order norms will follow similarly by considering derivatives of $f(u)$.

Using the properties of g from assumptions (A1) and (A2), we can estimate the $L^2(\Omega)$ -norm of the difference:



$$\|f(u_1) - f(u_2)\|_{L^2(\Omega)} = \left\| \gamma_2 \int_0^t g(t-\tau) (\Delta u_1(\tau, x) - \Delta u_2(\tau, x)) d\tau \right\|_{L^2(\Omega)}.$$

By the triangle inequality and boundedness of the relaxation function g , we get

$$\|f(u_1) - f(u_2)\|_{L^2(\Omega)} \leq \gamma_2 \int_0^t |g(t-\tau)| \|\Delta u_1(\tau, x) - \Delta u_2(\tau, x)\|_{L^2(\Omega)} d\tau.$$

Now, from the regularity of g given by (A2), we know that g is exponentially decaying. Specifically, since $g'(t) \leq -\gamma(t)g(t)$, we have

$$g(t) \leq Ce^{-\gamma_0 t} \quad \text{for some constant } C > 0.$$

Thus, the integral involving g can be controlled by

$$\int_0^t |g(t-\tau)| d\tau \leq \int_0^t Ce^{-\gamma_0(t-\tau)} d\tau = \frac{C}{\gamma_0}.$$

Therefore,

$$\|f(u_1) - f(u_2)\|_{L^2(\Omega)} \leq \frac{C\gamma_2}{\gamma_0} \|\Delta u_1 - \Delta u_2\|_{L^2(\Omega)}.$$

Step 3: Conclusion

Since $\|\Delta u\|_{L^2(\Omega)}$ is equivalent to the $H^2(\Omega)$ -norm for functions in $H^2(\Omega)$, we conclude that

$$\|f(u_1) - f(u_2)\|_{H^2(\Omega)} \leq C\|u_1 - u_2\|_{H^2(\Omega)},$$

where C is a constant depending on γ_2, γ_0 , and the properties of the relaxation function g .

Thus, f is locally Lipschitz in $H^2(\Omega)$.

Thus,

$$\|\mathcal{B}(U) - \mathcal{B}(\tilde{U})\|_{\mathcal{H}} \leq C_{Lip}\|U - \tilde{U}\|_{\mathcal{H}},$$

which is enough to warrant that the nonlinear operator $\mathcal{B}$ is locally Lipschitz on $\mathcal{H}$.

Based on Lemmas , , , and , the operators $\mathcal{A}$ and $\mathcal{B}$ are identified as maximal accretive and locally Lipschitz operators on $\mathcal{H}$. After these initial steps, the local well-posedness outcome is derived as a direct consequence of Theorem 1.4 on page 185 in [14], and can be stated as follows.

Assume that g verify the assumption (A1) and (A2) Then, for any $U_0 = (w_0, w_1, 0)^T \in \mathcal{H}$, there exists $T_{Loc} > 0$ such that the problem (0.311) has a unique weak solution $U = (w, w_t, \phi)^T \in C([0, T_{Loc}), \mathcal{H})$.

The following corollary provides a more precise insight into the regularity of the local solution obtained.



Let $U = (w, w_t, \phi)^T \in C([0, T_{\text{Loc}}), \mathcal{H})$ be the local weak solution obtained from Theorem . If $U_0 = (w_0, w_1, 0)^T \in \mathcal{D}(\mathcal{A})$, then $U = (w, w_t, \phi)^T$ constitutes a classical solution to problem (0.311). Moreover, U belongs to $C^1([0, T_{\text{Loc}}), \mathcal{H}) \cap \mathcal{C}([0, T_{\text{Loc}}), \mathcal{D}(\mathcal{A}))$. proof. Given that $\mathcal{H}$ is a reflexive Banach space, the conclusion follows directly from the application of Theorem 2.5.4 page 53 in [9] (also refer to Proposition 4.3.9 p. 60 [10]).



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On the existence and uniqueness of the solution of the logarithmic fuzzy integro-differential equation

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Abstract. The objective of this study is to examine one specific type of fuzzy integro-differential equation: the fuzzy logarithmic integro-differential equation. Initially, we demonstrate that the solution to this equation is distinct and exists. We employ Picard's theorem.

Keywords: Fuzzy integro-differential equations, Picard's Theorem, existence and uniqueness.



Introduction

Given a real constant λ and a fuzzy continuous function ψ , consider the problem of determining a fuzzy function φ that satisfies the equation below:

$$\psi'(s) - \frac{\lambda}{\pi} \oint_{-1}^1 \psi(t) \ln |s - t| dt = \psi(s), \quad \psi(-1) = 0, \quad -1 < s < 1. \quad (0.93)$$

The above equation is called the fuzzy Fredholm integro-differential equation with a logarithmic kernel. We assume λ is a real throughout the paper. We will use Picard's Theorem to prove that the solution of (0.93) exists and is unique.

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A NEW QUADRATIC METHOD TO APPROXIMATE THE FIRST ORDER CAPUTO-FABRIZIO FRACTIONAL DERIVATIVE

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Abstract. In this paper, we propose an efficient numerical method to approximate the first order fractional derivative in the Caputo-Fabrizio sense. This method develops a new quadratic formula using Haar wavelet integration method. Error analysis of our proposed method gives an exponential convergence rate of $\mathcal{O}(2^{-J})$. To check the effectiveness of the proposed method, we examine some examples with different fractional orders. The quantitative results demonstrated the stability and efficiency of the proposed method.

Keywords: Fractional calculus, Caputo-Fabrizio fractional derivative, Haar wavelet integration formula, Quadratic integration method.

Introduction

In recent years, fractional calculus has received great interest and wide popularity in many scientific fields, especially mathematics, due to the fact that it models many physical, engineering and biological phenomena. Generally, these



models include ordinary fractional differential equations, space-time fractional differential equations and integral fractional differential equations in the one of Riemann Liouville, Caputo or Caputo-Fabrizio fractional derivative sense. Due to the non-singular kernel of the Caputo-Fabrizio fractional derivative, it has been preferred and widely used by many researchers. As is well known, the most of fractional problems do not have exact solutions. Therefore different numerical methods for approximating the exact solutions have been provided in the literature. Some of these methods use quadratic formulas to approximate the fractional derivative in the Caputo-Fabrizio sense as L1 and L2 quadratic formulas.

However, All previous quadratic methods are based on the finite differences method and therefore they converge at a polynomial rate. To improve the convergence rate and get more efficiency, we propose in this paper a new quadratic method to approximate the Caputo-Fabrizio fractional derivative using the Haar wavelet integration formula. A theoretical study shows that our proposed method has an exponential convergence rate and it is more efficient compared to other quadratic methods.

Theorem 0.67. *Let $0 < \rho < 1$ and $u \in H^2(0, b)$, $b > 0$. Then the first order approximation of the Caputo-Fabrizio fractional derivative is given by:*

$${}^{CF}D_0^\rho u(t) = \sum_{k=1}^{2M} \lambda_k u(t_k) + \mathcal{O}\left(\frac{1}{M}\right),$$

where $t_k = (k - 0.5) \Delta t$, $\Delta t = \frac{t}{2M}$, $k = 1, \dots, 2M$ and

$$\lambda_k = \begin{cases} -\frac{e^{-\frac{\rho}{1-\rho}(t-t_1)}}{1-\rho} & \text{if } k = 1 \\ \frac{\left[e^{-\frac{\rho}{1-\rho}\Delta t} - 1\right] e^{-\frac{\rho}{1-\rho}(t-t_k)}}{1-\rho} & \text{if } 2 \leq k \leq 2M-2 \\ \frac{\left[e^{-\frac{\rho}{1-\rho}\Delta t} - e^{-\frac{\rho}{1-\rho}\Delta t} - 1\right] e^{-\frac{\rho}{1-\rho}(t-t_{2M-1})}}{1-\rho} & \text{if } k = 2M-1 \\ \frac{\left[e^{-\frac{\rho}{1-\rho}\Delta t} + 1\right] e^{-\frac{\rho}{1-\rho}(t-t_{2M})}}{1-\rho} & \text{if } k = 2M \end{cases}$$

Example 0.4. *Let us consider the function $u(t) = \sinh(t)$, then its exact fractional derivative of order $\rho = 0.5$ in Caputo-Fabrizio sense is given by ${}^{CF}D_0^{0.5}u(t) = (1-t)\sinh(t) + t\cosh(t)$.*



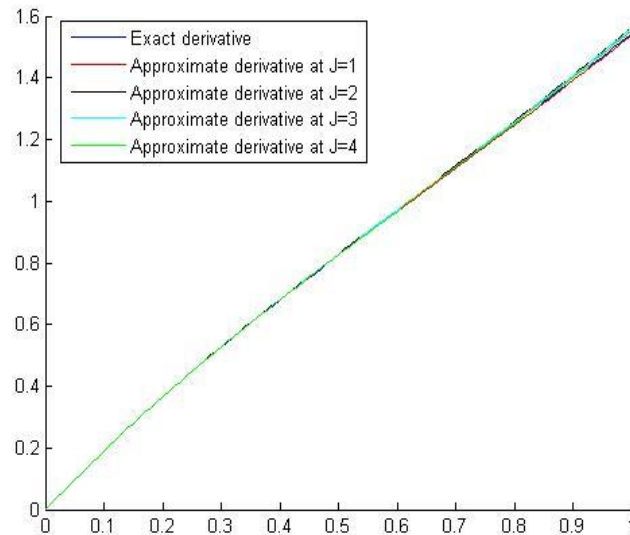


Figure 19: Exact and Approximate derivative of example 1 for $J = 1, 2, 3, 4$

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An Analytical and Numerical Approach to a Nonlinear Fredholm Integro-Differential Equation with Weak Singularities

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Abstract. This work presents an analytical and numerical study of a nonlinear Fredholm integro-differential equation with weakly singular kernels. The main challenge is managing the kernel's weak singularity, addressed using a hybrid approach combining the Nyström method for solution approximation and product integration to handle the singularity. We also prove the existence and uniqueness of the solution in relevant functional spaces. Numerical examples confirm the accuracy and efficiency of the proposed methods, demonstrating their effectiveness in tackling weakly singular kernels.

Keywords: Fredholm integro-differential equation, Banach fixed point theorem, Nyström method, product integration method, weakly singular kernels.



Introduction

The solvability of integral and integro-differential equations is a crucial research focus due to their extensive applications in areas like physics, biology and finance. To address these problems, numerous numerical methods have been introduced, such as the Galerkin-Chebyshev-wavelets method, product integration, and hat functions. In their study, Touati et al. [2] explored both the numerical and analytical properties of a nonlinear Fredholm integro-differential equation with a weakly singular kernel. Our research, however, addresses a similar equation, but with the added complexity that the kernel depends not only on the unknown function and its first derivative but also on its second derivative. Specifically, we analyze the following equation:

$$u(t) = \int_0^1 g(|t-s|)K(s, u(s), u'(s), u''(s))ds + f(t), \quad (0.94)$$

where the unknown function u belongs to $\mathcal{C}^2([0, 1])$ and f is a given function in the same space. We assume that:

Main results

1. **Analytical Study:** The existence and uniqueness of solutions for the nonlinear Fredholm integro-differential equation with weakly singular kernels are rigorously established.
2. **Numerical Study:** A hybrid approach combining the Nyström method and product integration is developed to approximate solutions effectively.
3. **Convergence:** The proposed numerical methods demonstrate convergence in relevant functional spaces.
4. **Examples:** Numerical examples validate the accuracy and efficiency of the methods, showcasing their effectiveness in handling weakly singular kernels.

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The residual power series method for solving the time-fractional nonlinear reaction-diffusion equations

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Abstract. This study focuses on the analysis and solution of the time-fractional nonlinear reaction diffusion equations. We employed the residual power series method (RPSM) to solve these equations, where the fractional derivative is used in the Caputo sense. We derived the analytical solutions using the Caputo Taylor series expansion method, substituting the truncated Caputo series solutions into the residual error functions. We demonstrated the efficacy, reliability, and usefulness of the proposed approach by examining a numerical application. The tabular and graphic results demonstrated that the method is superior, precise, and suitable for solving these fractional partial differential models. We can use the RPSM as an alternative method to solve various nonlinear problems.

Keywords: Fractional Partial Differential Equation, Residual Power Series Method, Approximate Solutions, Caputo Derivative Operator, reaction-Diffusion Equation.



Introduction

Fractional calculus theory [1] is very popular in many areas of science and engineering because it can be used in so many situations and is very useful for explaining complex dynamical behavior in real-world situations like fluid flow, traffic, biological populations, and diffusion systems. Compared to the classical order, the fractional state offers numerous advantages, allowing for streamlined control of nature's modeling without compromising genetic features or memory capacity. The fractional operator is a potent mathematical tool that is crucial in simulating various nonlinear issues, such as electrical circuits, electromagnetic waves, damping laws and signal processing.

Recent years have seen extensive investigation into nonlinear reaction-diffusion (RD) equations. Various evolution problems in the real world, such as those in chemistry and biology, commonly use these equations as descriptive models.

Lately, there has been a growing interest in fractional RD equations [2]-[3] due to their manifestation of self-organization phenomena and the introduction of a novel parameter.

In this work, we apply the residual power series method (RPSM)[1]-[2] to find series solution for nonlinear one dimensional time-fractional reaction-diffusion equations [3]-[8] of the form:

$$D_t^\alpha u(x, t) = dD_x^2 u(x, t) + G(u), \quad 0 < \alpha \leq 1, x \in \mathbb{R}, \quad (0.95)$$

where d is the diffusion coefficient and $G(u)$ is a nonlinear function representing reaction kinetics. The RPSM is a semi-analytical technique that combines Taylor's series and the residual error function. The software offers solutions to both linear and nonlinear differential equations in the form of a convergent series. This method's primary benefit is its simplicity in determining the coefficients of the terms in the series solution. It achieves this by employing differential operators, as opposed to other commonly used analytic techniques that require integration operators, which is difficult in fractional cases.

Main results

Definition 0.45.

The power series of the form

$$\sum_{k=0}^{\infty} g_k(x) (t - t_0)^{k\alpha}, \quad x \in I \quad (0.96)$$

is said to be a multiple fractional power series (MFPS) about t_0 , where $0 \leq n - 1 < \alpha \leq n$, t is a variable, such that $t \in [t_0, t_0 + R^{1/\alpha}]$, $R^{1/\alpha}$ is the radius of the convergence. Functions $g_k(x)$ are called the coefficients of the series.

Theorem 0.68.



Let $v(x, t) : I \times [t_0, t_0 + R^{1/\alpha}] \rightarrow \mathbb{R}$ can be represented as the MFPS at $t = t_0$ of the form

$$v(x, t) = \sum_{k=0}^{\infty} g_k(x) (t - t_0)^{k\alpha}, \quad x \in I, \quad t \in [t_0, t_0 + R^{1/\alpha}], \quad 0 \leq n-1 < \alpha \leq n. \quad (0.97)$$

If $v(x, t)$ is continuous on $I \times [t_0, t_0 + R^{1/\alpha}]$ and $\mathfrak{D}_t^{k\alpha} v(x, t) \in C(t_0 \leq t < t_0 + R^{1/\alpha})$, for $k = 0, 1, 2, \dots$, then $g_k(x) = \frac{\mathfrak{D}_t^{k\alpha} v(x, t_0)}{\alpha^k (k)!}$.

Theorem 0.69.

Let $0 \leq n-1 < \alpha \leq n$, $\mathfrak{D}_t^{k\alpha} v(x, t)$ exist at a neighborhood of a point t_0 for $k = 0, 1, 2, \dots, n+1$, and $v(x, t)$ can be expressed by the MFPS (0.96) about $t = t_0$ such that $|\mathfrak{D}_t^{(n+1)\alpha} v(x, t)| \leq M(x)$ for some $n \in \mathbb{N}$. Then, for all $(t_0, t_0 + R^{1/\alpha})$, the reminder $\mathcal{R}_n(t)$ of the MFPS satisfies

$$|\mathcal{R}_n(x, t)| \leq \frac{M(x)}{\alpha^{(n+1)} (n+1)!} (t - t_0)^{(n+1)\alpha}, \quad (0.98)$$

where

$$\mathcal{R}_n(x, t) = v(x, t) - \sum_{k=0}^{\infty} \frac{\mathfrak{D}_t^{k\alpha} v(x, t_0)}{\alpha^k (k)!} (t - t_0)^{k\alpha}.$$

Theorem 0.70. (Theorem of convergence)

Let $u(x, t)$, be exact solutions of the problem (0.95). Suppose that there exists a fixed constant $0 < \lambda < 1$, such that $\|u_{n+1}(x, t)\| \leq \lambda \|u_n(x, t)\|$, for all $n \in \mathbb{N}$, $x \in I$ and $t \in [t_0, t_0 + R^{1/\alpha}]$. Then the n -term approximate solutions $u_n(x, t)$ converge to $u(x, t)$ as $n \rightarrow \infty$



In this study we get:

- At $\alpha = 1$, the approximate solutions are nearly identical to the exact solutions, with negligible errors.
- At lower values of α , the error increases, but the LRPSM still provides reasonable approximations.
- The method demonstrates flexibility and applicability to fractional differential equations, with higher accuracy for integer orders of α .

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Mixed Volterra-Fredholm nonlinear integral equations with regular kernels

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Abstract. In this work we study two coupled integral equations. The type of equations is of Volterra-Fredholm equations. We put acceptable hypotheses to ensure the result. In the theoretical study we use the Picard method to ensure the existence and the uniqueness of the solution, whereas in the numerical study we use numerical methods based on the approximation of the integral as Nyström method and where the convergence order will depend on the order of regularity of the kernel. Finally, we include illustrative examples to show the validity and applicability of our method.

Keywords: Volterra integral equation, Fredholm integral equation, nonlinear equation, Nyström method.



Introduction

In this work, we will study the following coupled equations: for all $t \in [a, b]$

$$\begin{cases} \lambda_1 u(t) = f_1(t) + \int_a^t p_1(t, s) \int_a^b K_1(s, \theta, u(\theta), v(\theta)) d\theta ds \\ \lambda_2 v(t) = f_2(t) + \int_a^t p_2(t, s) \int_a^b K_2(s, \theta, u(\theta), v(\theta)) d\theta ds, \end{cases} \quad (0.99)$$

where λ_1 and $\lambda_2 \in \mathbb{R}^*$, f_1, f_2, p_1, p_2, K_1 and K_2 are given functions and the functions u, v are the unknowns.

To put the system in functional framework we need to assume that the functions f_1, f_2 and the kernel p_1, p_2 verify the hypotheses

$$(H1) \left\{ \begin{array}{l} i) f_1, f_2 \in \mathcal{C}^0([a, b]) \\ ii) p_1, p_2 \in \mathcal{C}^0([a, b]^2). \end{array} \right.$$

The hypothesis $H1(ii)$ implies that

$$\exists M_1, M_2 > 0 \quad \forall t, s \in [a, b], \quad |p_1(t, s)| \leq M_1, \quad |p_2(t, s)| \leq M_2$$

The kernel K_1 and K_2 are defined by

$$\begin{aligned} K_1 : [a, b]^2 \times \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (s, \theta, x, y) &\longmapsto K_1(s, \theta, x, y) \end{aligned}$$

$$\begin{aligned} K_2 : [a, b]^2 \times \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (s, \theta, x, y) &\longmapsto K_2(s, \theta, x, y) \end{aligned}$$

and verify the following hypotheses

$$(H2) \left\{ \begin{array}{l} i) K_1, K_2 \in \mathcal{C}([a, b]^2 \times \mathbb{R}^2) \\ ii) \exists \alpha_1, \alpha_2, \beta_1, \beta_2 > 0 \text{ tel que } \forall s, \theta \in [a, b], \forall X, Y, \bar{X}, \bar{Y} \in \mathbb{R}, \\ |K_1(s, \theta, x, y) - K_1(s, \theta, \bar{x}, \bar{y})| \leq \alpha_1 |x - \bar{x}| + \beta_1 |y - \bar{y}| \\ |K_2(s, \theta, x, y) - K_2(s, \theta, \bar{x}, \bar{y})| \leq \alpha_2 |x - \bar{x}| + \beta_2 |y - \bar{y}| \end{array} \right.$$

The purpose of this work is to prove the existence and uniqueness of the solution $(u, v) \in \mathcal{C}^0([a, b]) \times \mathcal{C}^0([a, b])$ and then build a numerical technique to approach it.

Theoretical study

To prove the existence and uniqueness of the solution for the system (0.99), we will use the method of successive approximation, also called the Picard method (see [2, 3]). We have the following theorem.

Theorem 0.71. *Suppose that the hypotheses (H1) and (H2) are verified, then the equations system (0.99) has unique solution $(u, v) \in \mathcal{C}^0([a, b]) \times \mathcal{C}^0([a, b])$.*



Numerical study

We are going to construct a numerical approximation of the solution using numerical methods based on the approximation of the integral.

We consider the subdivision $\{t_i\}_{0 \leq i \leq n}$, $n \in \mathbb{N}^*$ of $[a, b]$ with $t_i = a + ih$ and $h = \frac{b-a}{n}$. For the Fredholm integral part we use the following numerical integration formula (see [1])

$$\int_a^b \xi(t) dt \simeq \sum_{i=0}^n w_i \xi(t_i), \quad (0.100)$$

where, w_i are real such that $W = \max_{n \geq 1} \sum_{j=0}^n |w_j| < \infty$. For the Volterra integral part, we use the formula

$$\int_a^{t_i} \xi(s) ds \simeq h \sum_{j=1}^i \bar{w}_j \xi(t_j), \quad (0.101)$$

where, $\bar{w}_j$ are real such that there exists $\bar{W} > 0$, $\forall n \in \mathbb{N}^*$, $\max_{0 \leq j \leq n} |\bar{w}_j| \leq \bar{W} < \infty$. Applying these quadratures on equations system (0.99), we obtain the following system

$$\begin{cases} u_0 = \frac{1}{\lambda_1} f_1(t_0) \\ v_0 = \frac{1}{\lambda_2} f_2(t_0) \\ u_i = \frac{1}{\lambda_1} \left(f_1(t_i) + h^2 \sum_{j=0}^i w_j p_1(t_i, t_j) \sum_{k=0}^n w_k K_1(t_j, t_k, u_k, v_k) \right) \\ v_i = \frac{1}{\lambda_2} \left(f_2(t_i) + h^2 \sum_{j=0}^i w_j p_2(t_i, t_j) \sum_{k=0}^n w_k K_2(t_j, t_k, u_k, v_k) \right) \end{cases} \quad (0.102)$$

where, u_i approaches $u(t_i)$ and v_i approaches $v(t_i)$.

We have the following theorem.

Theorem 0.72. *Let $|\lambda_1| > 2W^2 M_1(b-a)^2 \max(\alpha_1, \beta_1)$ and $|\lambda_2| > 2W^2 M_2(b-a)^2 \max(\alpha_2, \beta_2)$. For n Large enough, h small enough, the system (0.550) has a unique solution.*

*

Convergence analysis of the proposed method For $0 \leq i \leq n$, we define the error of descritisation:

$$E_i = \begin{pmatrix} \varepsilon_i \\ \hat{\varepsilon}_i \end{pmatrix}$$

with

$$\begin{cases} \varepsilon_i = u(t_i) - U_i \\ \hat{\varepsilon}_i = v(t_i) - V_i \end{cases}$$



we denote by

$$\|E_i\|_1 = |\varepsilon_i| + |\hat{\varepsilon}_i|$$

We say that the method is convergent if

$$\lim_{h \rightarrow 0} \max_{1 \leq i \leq n} \|E_i\|_1 = 0$$

Moreover, we define the local consistency error by

$$\Delta(h, s) = \begin{pmatrix} \delta(K_1, h, s) \\ \delta(K_2, h, s) \end{pmatrix}, \quad \tilde{\Delta}(h, t_i) = \begin{pmatrix} \tilde{\delta}(p_1, h, t_i) \\ \tilde{\delta}(p_2, h, t_i) \end{pmatrix}$$

where

$$\begin{cases} \delta(K_1, h, s) = \int_a^b K_1(s, \theta, u(\theta), v(\theta)) d\theta - h \sum_{k=0}^n w_k K_1(s, t_k, u(t_k), v(t_k)) \\ \tilde{\delta}(p_1, h, t_i) = \int_a^b p_1(t_i, s) K_1(s, t, x, y) d\theta - h \sum_{j=0}^i w_j p_1(t_i, t_j) K_1(t_j, t, x, y) \end{cases}$$

with

$$\|\Delta(h, \cdot)\|_{C^0([a,b]) \times C^0([a,b])} = \max_{a \leq s \leq b} |\delta(K_1, h, s)| + \max_{a \leq s \leq b} |\delta(K_2, h, s)|$$

and

$$\|\tilde{\Delta}(h, \cdot)\|_{C^0([a,b]) \times C^0([a,b])} = \max_{a \leq s \leq b} |\tilde{\delta}(p_1, h, s)| + \max_{a \leq s \leq b} |\tilde{\delta}(p_2, h, s)|$$

We say that the approximation method is consistent if

$$\lim_{h \rightarrow 0} \max(\|\Delta(h, \cdot)\|_{C^0([a,b]) \times C^0([a,b])}, \|\tilde{\Delta}(h, \cdot)\|_{C^0([a,b]) \times C^0([a,b])}) = 0$$

We have the following theorem.

Theorem 0.73. *Let*

$$\begin{cases} |\lambda_1| > 2WM_1(b-a)^2 \max(\alpha_1, \beta_1) \\ |\lambda_2| > 2WM_2(b-a)^2 \max(\alpha_2, \beta_2) \end{cases}$$

If our method is consistant then

$$\lim_{h \rightarrow 0} \max_{1 \leq i \leq n} \|E_i\|_1 = 0$$

Numerical example

Let the equations system

$$\begin{cases} \lambda_1 u(t) = \int_0^t (t^2 + s) \int_0^1 \frac{s^2(2e^{-\theta} + 1)}{e^{-u(\theta)} + e^{-v(\theta)} + 1} d\theta ds + f_1(t) \\ \lambda_2 v(t) = \frac{1}{3} \int_0^t (1 + 5(t-s)^2) \int_0^1 \frac{\theta + s}{1 + (\sin(u(\theta)))^2 + (\cos(v(\theta)))^2} d\theta ds + f_2(t) \end{cases} \quad (0.103)$$



The kernels verify the hypotheses (H1) and (H2).

We take

$$f_1(t) = \lambda_1 t - \frac{1}{3}t^5 - \frac{1}{4}t^4$$

and

$$f_2(t) = \lambda_2 t - \frac{1}{2} \left(\frac{1}{2}t + \frac{1}{2}t^2 + \frac{5}{6}t^3 + \frac{5}{12}t^4 \right)$$

we obtain the solution $(u(t), v(t))$ given by

$$\begin{cases} u(t) = t \\ v(t) = t. \end{cases}$$

According to the convergence condition of the numerical system, the coefficient of the solution λ_1 and λ_2 verify

$$|\lambda_1| \geq 13 \text{ et } |\lambda_2| \geq 17.$$

The terms U_n and V_n will not be exactly calculated, they will be approached using the Banach iteration method with a stop condition of type

$$\|(U_{new}, V_{new}) - (U_{old}, V_{old})\| \leq 10^{-7}$$

the following table gives the error estimation with $\lambda_1 = 13$ et $\lambda_2 = 17$

Méthode de trapèzes	
n	$E = \max(\ u - U_i\ , \ v - V_i\)$
4	2.010673086343839e-03
10	3.216445554199421e-04
20	8.040863372804630e-05
50	1.286495416841227e-05
100	3.215941837719782e-06
200	8.036920635667855e-07

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A NUMERICAL STUDY OF AN INFEASIBLE INTERIOR-POINT ALGORITHM FOR SEMI-DEFINITE OPTIMIZATION.

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Abstract. The focus of this research is to apply primal-dual interior-point path-following methods, specifically those derived from Newton's method for solving semidefinite optimization (SDO) problems. In this paper, we present a numerical study of an infeasible primal-dual interior-point method for tackling this class of optimization problems. Unlike the feasible interior-point algorithms, the proposed algorithm can be start with any initial positive definite matrix and does not require the strictly feasible initial points. Under certain conditions, the Newton system is well defined and its Jacobian is nonsingular at the solution. For computing an iteration throughout the algorithm, a Newton direction and a step-size are determined. Here, our search direction is based on Alizadeh-Haeberly-Overton (AHO) symmetrization. However, for the step size along this direction an efficient procedure is suggested. Preliminary numerical results demonstrate the efficiency of our algorithm.

Keywords: Semidefinite optimization, Interior-point methods, Primal-dual algorithm, Newton method.



Introduction

Introduction

Let $\mathcal{S}^n$ and $\mathcal{S}_+^n$ denote the cones of $n \times n$ real symmetric and symmetric positive semi-definite matrices, respectively. The convex quadratic semidefinite optimization (CQSDO) problem is defined as follows

$$(\mathcal{P}) \quad \bar{p} = \min_{X \in \mathcal{S}^n} f(X) \quad \text{s.t.} \quad A_i \bullet X = b_i, i = 1, \dots, m, X \succeq 0,$$

and its dual

$$(\mathcal{D}) \quad \bar{d} = \max_{y \in \mathcal{R}^m, Z \in \mathcal{S}^n} g(y, Z) \quad \text{s.t.} \quad \sum_{i=1}^m y_i A_i + Z = C + \mathcal{Q}(X), Z \succeq 0,$$

where

$$f(X) = C \bullet X,$$

$$g(y, Z) = b^\top y,$$

$b, y \in \mathcal{R}^m$, $X, Z \in \mathcal{S}_+^n$, $C, A_i \in \mathcal{S}^n, i = 1, \dots, m$, $\mathcal{Q}(X)$ is a given linear transformation on $\mathcal{S}^n$ and the inequality $M \succeq 0$ means that $M \in \mathcal{S}_+^n$. Recall that the notation $A \bullet B = \text{Tr}(AB)$, $A, B \in \mathcal{S}^n$ denotes the trace inner product.

The CQSDO problem has wide applications such as the computation of nearest correlation matrix problem (NCMP) also it arises in nearest Euclidean distance matrix problems and other matrix least square problems (SDLS) (see [3]). Many problems in metric embeddings, covariance estimations, and molecular conformations can also be restated as CQSDO. It also includes the semidefinite optimization (SDO) as a special case when $\mathcal{Q}(X) = 0$.

Over the last decades, many feasible primal-dual interior-point algorithms (FIPA) for solving CQSDO have been proposed (see.g., [1], [2]). These algorithms are of Newton type methods and which enjoy certain property such as the locally quadratic convergence and have polynomial complexity. Further they are highly efficient in practice. It is well-known that FIPAs are iterative methods and start with a strictly feasible starting point and follow the central-path with keeping feasibility and centrality during the solution process. However, experimental issue of these algorithms show that getting a strictly feasible centered point is a hard task and even impossible. Therefore, to overcome this drawback many remedies are suggested. Among them, infeasible interior-point (IFIPA) algorithms which require that the starting initial point is any arbitrary symmetric positive definite where feasibility is reached as optimality is approached [4].



Preliminaries

Lemma 0.25. *If $X, Z \in \mathcal{S}_+^n$. Then the following statements*

1. $X \bullet Z = 0$,
2. $XZ = 0$,
3. $\frac{XZ + ZX}{2} = 0$

are equivalent.

the set of primal and dual optimal solutions consists of all solutions $(X, y, Z) \in \mathcal{S}_+^n \times \mathcal{R}^m \times \mathcal{S}_+^n$ of the following Karush-Khun-Tucker optimality system:

$$\begin{cases} A_i \bullet X - b_i = 0, \forall i = 1, \dots, m, X \succeq 0, \\ C - \sum_{i=1}^m y_i A_i - Z = 0_{\mathcal{S}^n}, Z \succeq 0, \\ X \bullet Z = 0. \end{cases} \quad (0.104)$$

Therefore, by Lemma 0.25 finding an optimal solution for $\mathcal{P}$ and $\mathcal{D}$ is equivalent to solving the system

$$\begin{cases} A_i \bullet X = b_i, i = 1, \dots, m, X \succeq 0, \\ C - Z - \sum_{i=1}^m y_i A_i = 0_{\mathcal{S}^n}, Z \succ 0, \\ \frac{XZ + ZX}{2} = 0. \end{cases} \quad (0.105)$$

*

The central-path of CQSDO The basic idea behind primal-dual IPAs is to replace the third equation in (2) by the parameterized equation

$$\frac{XZ + ZX}{2} = \sigma \mu I,$$

with $\mu > 0$. Thus we consider the system

$$\begin{cases} A_i \bullet X - b_i = 0, \forall i = 1, \dots, m, X \succ 0, \\ C - Z - \sum_{i=1}^m y_i A_i = 0_{\mathcal{S}^n}, Z \succ 0, \\ \frac{XZ + ZX}{2} = \sigma \mu I, \mu > 0, \end{cases} \quad (0.106)$$

*

The Newton search directions for CQSDO Next, we want to define search directions $(\Delta X, \Delta y, \Delta Z)$ that move in the direction of the central-path $\mathcal{C}$. Applying *Newton's method* for (0.351), for a given $\mu > 0$ and an infeasible point $(X \succ 0, y, Z \succ 0)$. Then the Newton direction $(\Delta X, \Delta y, \Delta Z)$ at this point is the unique symmetric solution of the linear system:



$$\begin{cases} A_i \bullet \Delta X = \mathcal{R}_P, \quad i = 1, \dots, m, \\ \sum_{i=1}^m (\Delta y)_i A_i + \Delta Z = \mathcal{R}_D, \\ \Delta X Z + X \Delta Z + \Delta Z X + Z \Delta X = \mathcal{R}_C, \quad X \succ 0, \quad Z \succ 0, \end{cases} \quad (0.107)$$

with

$$\mathcal{R}_P = b_i - A_i \bullet X, \quad i = 1, \dots, m,$$

$$\mathcal{R}_D = C - (Z + \sum_{i=1}^m y_i A_i),$$

$$\mathcal{R}_C = 2\sigma\mu I - (XZ + ZX).$$

We will refer to the assignment:

$$X^+ = X + \alpha \Delta X, \quad y^+ = y + \alpha \Delta y, \quad Z^+ = Z + \alpha \Delta Z$$

as a damped Newton step with $\alpha > 0$ is the step-size along this direction.

*

Algorithm The generic IFIPA for CQSDO is described in Figure 1 as follows.

Input:

An accuracy parameter $\epsilon > 0$;
initial guesses $(X^0, Z^0 \succ 0, y^0 \in \mathcal{R}^m)$ and $\mu > 0$;

begin

$k := 0$;

While $\max(\|\mathcal{R}_D^k\|_F, \|\mathcal{R}_P^k\|_2, n\mu^k) > \epsilon$ **do**

- Compute $(\Delta X^k, \Delta y^k, \Delta Z^k)$ by solving system (0.311);
- Determine a step-size $\alpha > 0$ s.t $X^k + \alpha \Delta X^k \succ 0$ and $Z^k + \alpha \Delta Z^k \succ 0$;
- Update $X^{k+1} := X^k + \alpha \Delta X^k$, $y^{k+1} := y^k + \alpha \Delta y^k$, $Z^{k+1} := Z^k + \alpha \Delta Z^k$;

$k := k + 1$;

endWhile

end.

Fig 1. Algorithm . Primal-dual IFIPA for CQSDO.

Numerical results

Example 1. Consider the CQSDO problem whose primal-dual pair $\mathcal{P}$ and $\mathcal{D}$ have the following data:



$$A_1 = \begin{bmatrix} 0.9 & 0 & 0 & -1.5 \\ 0 & 3 & 0.75 & 0 \\ 0 & 0.75 & 1.098 & 0 \\ -1.5 & 0 & 0 & -0.5 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1.414 & 1.386 & 0 & 0 \\ 1.386 & 1.732 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}, C = \begin{bmatrix} 1.7071 & 0.6931 & -0.1 & 0 \\ 0.6931 & 1.366 & -0.5 & 0.02 \\ -0.1 & -0.5 & 2 & 0 \\ 0 & 0.02 & 0 & 0 \end{bmatrix}$$

$$A_3 = \text{diag}(1, 0.5, 1.333, -0.333), b = [7, 4, 2]^\top.$$

The initial point is defined as follows

$$X^0 = \begin{bmatrix} 1 & 0 & -0.1 & 0 \\ 0 & 0.5 & 0 & 0.02 \\ -0.1 & 0 & 1 & 0 \\ 0 & 0.02 & 0 & 1 \end{bmatrix}, y^0 = \begin{bmatrix} 0 \\ 0.5 \\ 0 \end{bmatrix}, Z^0 = I.$$

The obtained primal-dual optimal solution is

$$X^* = \begin{bmatrix} 0.4419 & -1.0091 & 0.3849 & -0.2336 \\ -1.0091 & 2.3044 & -0.8790 & 0.5335 \\ 0.3849 & -0.8790 & 0.3353 & -0.2035 \\ -0.2336 & 0.5335 & -0.2035 & 0.1235 \end{bmatrix}, y^* = \begin{bmatrix} 0.2015 \\ 0.1656 \\ 0.1603 \end{bmatrix},$$

$$Z^* = \begin{bmatrix} 1.1314 & 0.4636 & 0.1000 & 0.3023 \\ 0.4636 & 0.3946 & 0.5144 & 0.0200 \\ 0.1000 & 0.5144 & 1.2338 & 0 \\ 0.3023 & 0.0200 & 0 & 0.4853 \end{bmatrix}.$$

$$p^* = d^* = 2.3933., \text{ CPU} = 0.0046, \text{ ITER} = 10$$

Conclusion

The advantage of this work based on this algorithm is that it can be started with any initial symmetric positive definite matrix which is not necessarily strictly feasible. The preliminary numerical results showed its performances for solving this problem. The computation of search directions based on other symmetrization scheme remains a good topic of research in the future.

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Spectral collocation method for solving Volterra integral equation on the half-line.

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Abstract. In this work, we discuss efficient collocation method for solving Volterra integral equation on the semi-infinite interval. The method is based upon modified legendre rational functions and development of the Legendre-Gauss quadrature formula for Volterra integral equation, the solution of the integral equation reduces to the solution of a simple linear equations of algebraic system. The error between the exact and approximate solution in the usual L_2 -norm is estimated. Finally some ideal numerical examples selected to illustrate the application and effectiveness of the proposed work are presented and discussed.

Keywords: Volterra integral equation, Legendre rational functions, Legendre polynomials, The half-line.



Introduction

This work aims to approximate the solution of the linear Volterra integral equation on the half-line of the form

$$u(x) = f(x) + \int_0^x g(x, t)u(t)dt, \quad x \in [0, \infty[, \quad (0.108)$$

where g and f are given sufficiently smooth functions, and u is the unknown function to be determined. We write Eq.(1) in the more compact form

$$(\mathcal{I} - \mathcal{G})u = f, \quad (0.109)$$

where $\mathcal{I}$ is the identity and $\mathcal{G}$ is defined by

$$(\mathcal{G}u)(x) = \int_0^x g(x, t)u(t)dt, \quad (0.110)$$

Linear Volterra integral equations appear in many scientific and various fields of engineering, such as Energy physics, medicine, radiative transfer, population dynamics, elastic waves and electromagnetic. However, the analytical resolution of this integrals equations are practically difficult, if not impossible. It is necessary to move to numerical resolution using approximation methods. In recent years there are many researchers who have used and developed a lot of methods. Despite these results in bounded and unbounded interval, there are still valuable and great efforts to make more progress, especially for those who work in unbounded domain.

Main results

Consider the following linear Volterra integral equation:

$$u(x) = \frac{|x-2|^{\frac{7}{2}}}{(x^2+2)^3} + \int_0^x \frac{e^{-\frac{t}{2}} \sin(\log(t^2+x+1))}{(x^2+t^2+2)^4} u(t)dt, \quad x \in \mathbb{R}_+. \quad (0.111)$$

whose exact solution is unknown. In table 4, we display the numerical errors by computing the L^∞ -norm of the difference between u_{256}^s and u_N^s , $\tilde{u}_{256}^s$ and $\tilde{u}_N^s$ by using the MJFs-scheme with $s = 3$. These results are compared with those obtained by using a Nyström type method (see [3]), the results from this comparison show that the proposed method is both accurate and efficient.

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Table 4: Comparison of the L^∞ -errors.

N	$s = 3$		Method in [3]
	$\ u_{256}^s - u_N^s\ _{L^\infty}$	$\ u_{256}^s - \tilde{u}_N^s\ _{L^\infty}$	
8	1.29e-01	1.06e-04	1.1135e-04
16	2.05e-03	3.62e-07	3.7856e-05
32	1.49e-05	1.96e-09	1.5291e-06
64	1.09e-06	6.31e-11	7.9058e-08
128	1.03e-07	7.20e-13	5.7575e-10

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Analytical and numerical study of covering method for Hölderian global optimization

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Abstract. In this paper, the one-dimensional unconstrained global optimization problem of continuous functions satisfying a Hölder condition is considered. We would like to study some efficient methods create to solve the Hölderian global optimization problem, and other interesting procedures we have proposed to accelerate the convergence to the global minimum and overcome some situation presented in the existing method. The numerical experiments of the proposed algorithms on some test functions are encouraging.

Keywords: Global optimization, Hölder condition, Affine lower bounding functions , Piyavskii's algorithm.



Introduction

The objective of this paper is to investigate the global optimization problem of minimizing a one-dimensional multiextremal function $f(x)$ over an interval $[a, b]$ of $\mathbb{R}$, i.e., the finding of a point $x^* \in [a, b]$, such that

$$f(x^*) \leq f(x), \forall x \in [a, b].$$

In this work, we assume that the objective function f satisfy a Hölder condition, i.e.

$$|f(x) - f(y)| \leq h |x - y|^\alpha, \quad \forall x, y \in [a, b].$$

Where the Hölder constant $0 < h < +\infty$ and the Hölder exponent $0 < \alpha < 1$. Global optimization problems arise in many disperse fields of science and technology. The existence of multiple local minima of a general nonconvex and non-differentiable objective function makes global optimization a great challenge. Note that the Hölderian functions such as α is small enough are much more irregular than those where α is large enough. The problem (P) appears in a variety of real-life situations, for example, the simple plant location problem under a uniform delivered price policy, see Hanjoul et al. [3] and infinite horizon optimization problems, see Kiatsupaibul et al. [1]. For that there are many authers interested by this problem like Gourdin E. Jaumard B. and Ellaia R. [2], Lera D. and Sergeyev Ya. D. [2], Rahal M. and Ziadi A. [3], Chenouf Ch. and Rahal M. [1]

In this work, we shall study some efficient techniques to minimize a function depending on a single variable over an interval of $\mathbb{R}$ and satisfying the Hölder condition which are an extended version of the Piyavskii's method , and to illustrate the efficiency of each method we perform some numerical tests.

Main results

In this paper, methods for solving non-convex univariate global optimization problems is studied where the objective function is Hölderian over an interval of $\mathbb{R}$ Recently, Gourdin et al. (see [2]) propose an extension of the method of Piyavskii to case of a univariate Hölder function. They used a technique which is based on the construction of piecewise parabolic lower bounding functions of f on $[a, b]$. At each iteration, the method necessitates the determination of the unique global minimizer of a convex nondifferentiable function. This amounts to solving a nonlinear equation of order α , which is not always easy to treat explicitly. In [2], the authors have suggested at each iteration, an approximation of the global minimizer of lower bounding functions by the intersection* point of two secantes linked to the piecewise parabols. In [3], the authors extended the Piyavskii's algorithm by constructing a lower bounding functions which are piecewise linear, the key idea is essentially based on a result proved in [4] for continuous functions on a compact C of $\mathbb{R}^n$. But, the drawback of this



method is that of the slope (L_ε) of the bounding functions as a function of the accuracy ε , such that if the accuracy ε is small enough, the constant (L_ε) is large enough, then convergence of the algorithm will become so slow. In [1] We suggest two algorithms as an extended version of the Piyavskii's method. The first one depends on almost the same idea as the method presented in [1], just our proposed method is based on the use of the tangent at the midpoint of each subinterval of $[a, b]$ of the parabolic lower bounding functions of f . The second one is a combination of two successive phases. In phase 1, we obtain with a few iterations, a subinterval which often contains the point of the global minimum using a K_α -Lipschitz lower bounding functions with a tolerance δ much larger than the given accuracy ε with the advantage that the bounding functions independent the accuracy ε . In phase 2, we apply the first algorithm on the interval obtained in phase 1. To illustrate the efficiency of each method we propose a series of numerical experiment where the test functions have different properties like non-convex, non-differentiable, Lipschitz and Hölder, many of them have more than at least two local minima. Most of these functions are drawn from the Lipschitz and Hölder optimization literature.

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Numerical Solutions for Multidimensional Parabolic Problems with Nonlocal Boundary Conditions

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Abstract. The development of numerical methods for solving partial differential equations (PDEs) with non local boundary conditions is crucial, as these problems frequently arise in the mathematical modeling of real-world processes. In this work, we focus on the numerical solution of a multidimensional linear parabolic equation with non local boundary conditions. We employ the spectral Galerkin method, which is highly efficient, straightforward to implement, and capable of delivering significant numerical accuracy.

Keywords: Multidimensional, Spectral method, Non local conditions, Parabolic equation.

Introduction

We consider the nonlocal boundary value problem for a multidimensional elliptic equation

$$\partial_t u(x, y) + \Delta u(x, y) = f(x, y), \quad (x, y) \in \Lambda, \quad t \in (0, T).$$

with the following integral boundary conditions

$$u(t, \pm 1, y) = \mu^\pm \int_{-1}^1 u(x, y) dx, \quad \text{on } y \in (-1, 1), \quad t \in (0, T),$$



$$u(t, x, \pm 1) = \gamma^\pm \int_{-1}^1 u(x, y) dy, \quad \text{on } x \in (-1, 1), \quad t \in (0, T),$$

and initial condition

$$u(0, x, y) = u_0(x, y)$$

where $x \in (-1, 1)$ and f is a given function sufficiently regular.

Note that this kind of non-classical boundary-value problem has many important applications in chemical diffusion, thermoelasticity, heat conduction processes, population dynamics, vibration problems, nuclear reactor dynamics, inverse problems, control theory, medical science. The proposed problems can be solved using a variety of numerical methods such as a Domain decomposition [1], Alternating direction method [2]. In our presentation, we will describe the way to implement the spectral Galerkin method to approximate the solution of the nonlocal boundary value problem considered in this paper. Then, we give the error estimate (convergence, stability).

Main results

we shall briefly describe the way to implement the Galerkin Legendre spectral method to approximate the solution of the nonlocal boundary value problem considered in this paper. First, we shall call the weak formulation of the problem: find $u \in C^1(0, T, H^1(\Lambda))$ such that $v \in H^1(\Lambda)$

$$\begin{cases} (\partial_t u, v) + (\nabla_x u, \nabla_x u) + (\nabla_x u, \nabla_x v) = (f, v). \\ u(t, x, \pm 1) = \gamma^\pm \int_{-1}^1 u(x, y) dy, \quad u(t, \pm 1, y) = \mu^\pm \int_{-1}^1 u(x, y) dx. \\ u(0, x, y) = u_0 \end{cases} \quad (0.112)$$

Based on the weak formulation, the Legendre-Galerkin variational approximation is formulated as follows: find $u \in C^1(0, T, P_N(\Lambda))$ such that $v \in P_N(\Lambda)$

$$\begin{cases} (\partial_t u_N, v) + (\nabla_x u_N, \nabla_x u) + (\nabla_y u_N, \nabla_y v) = (f, v). \\ u_N(t, x, \pm 1) = \gamma^\pm \int_{-1}^1 u_N(x, y) dy, \quad u_N(t, \pm 1, y) = \mu^\pm \int_{-1}^1 u_N(x, y) dy. \\ u_N(0, x, y) = I_N^C u_0 \end{cases} \quad (0.113)$$

where I_N^C the operator of interpolation at the Chebyshev-Gauss-Lobatto nodes $x_j = \cos \frac{j\pi}{N}$, $0 \leq j \leq N$. We set

$$u_N(x, t) = \sum_{k=0}^N \sum_{j=0}^N \bar{u}_k(t) \phi_k(x) \phi_j(y). \quad (0.114)$$

After defining the appropriate basis functions, and applying the Crank-Nicolson method for time discretization. we arrive at the following system form using the Kronecker matrix algebra

$$(2M_X \otimes M_Y + \Delta t M_x \otimes S_y + \Delta t S_x \otimes M_y) U^{i+1} = (2M_X \otimes M_Y + \Delta t M_x \otimes S_y + \Delta t S_x \otimes M_y) U^i + \Delta T (F^{i+1} - F^i).$$



M,S,U is matrix will define later.

Stability

Lemma 0.26. *Let $u_0 \in H^1(\Lambda)$ and $f \in C^1(0, T, L^2(\Lambda))$. Then there exists a positive constant C independent on N such that the solution u_N of the semi-discrete approximation satisfies the following estimate*

$$\|u_N\|_{H^1(\Lambda)}^2 \leq C \left(\int_0^t \|f\|^2 + \|u_0\|^2 \right).$$

Convergence Theorem. Let $r \leq 1$ be a positive real number. Assume that $u \in H^1(0, T, H^r(\Lambda))$, $f \in C^1(0, T, H^r(\Lambda))$ and $\beta_K \leq 1$. Then the following estimate holds

$$\|u(t) - u_N(t)\|_{H^1} \leq CN^{2(r-1)}.$$

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A method DGJIM for solving differential equations of fractional order

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Abstract. In this work we present the existence and uniqueness of the solution for a fractional boundary value problem involving the Ψ – Riemann-Liouville operators where we used Banach's fixed point theorem, after that, we applied an algorithm of DGJIM (Daftardar-Gejji and Jafari Method) with which we can obtain an approximate solution to our problem and ensure its convergence. Finally we present some examples illustrating our method.

Keywords: The existence and uniqueness, Banach's fixed point theorem, the Ψ – Riemann-Liouville operator, convergence,



Introduction

A large class of problems in the physical, chemical and biological sciences are expressed in their mathematical framework by linear or non-linear integral equations, which has attracted the attention of several researchers to develop numerical methods among these methods we will present in this work an iterative method to a large class of boundary value problems of integer or non-integer order, the advantage of this method is rapid, avoid extreme accounts and giving high accuracy in the calculation allowing to solve the following fractional differential problem of nonlinear functional equation.

$$\begin{cases} \mathcal{D}_{0+}^{\sigma_1, \Psi} (\mathcal{D}_{0+}^{\sigma_2, \Psi} u)(t) = h(t, u(t)), & t \in \mathcal{O} = [0, 1], \\ u(0) = 0, & u(1) = \lambda \mathcal{I}_{0+}^{\sigma_3, \Psi} g(\xi, u(\xi)), \end{cases} \quad (0.115)$$

The next section* of this paper are arranged as follows: Is reserved for some definitions and properties of ψ -Riemann-Liouville operators, then proved the existence and the uniqueness of solution to our main problem. Then the formulation and the analysis of an iterative method with an illustrative example of an approximate solution it is obtained and plotted.

Main results

Proposition 0.9. *Let u an integrable function, $\Psi \in C^n(\mathcal{O})$ and $\Psi'(t) \neq 0$ we have*

$$\mathcal{I}_{a+}^{\sigma, \Psi} u(t) = \frac{1}{\Gamma(\sigma)} \int_a^t \Psi'(s) (\Psi(t) - \Psi(s))^{\sigma-1} u(s) ds,$$

Lemma 0.27. *[1] Let $\sigma > 0$ and $\mathcal{D}_{a+}^{\sigma; \Psi} \phi \in AC^{n; \Psi}[a, b] \cap L^1[a, b]$, then*

$$\begin{aligned} \mathcal{I}_{a+}^{\sigma; \Psi} \mathcal{D}_{a+}^{\sigma; \Psi} \phi(t) &= \phi(t) + k_1 (\Psi(t) - \Psi(a))^{\sigma-1} + k_2 (\Psi(t) - \Psi(a))^{\sigma-2} \\ &+ \cdots + k_n (\Psi(t) - \Psi(a))^{\sigma-n}, \end{aligned}$$

where $k_1, \dots, k_n \in \mathbb{R}$ and $n = [\sigma] + 1$.
 and

$$\mathcal{D}_{a+}^{\sigma; \Psi} \mathcal{I}_{a+}^{\sigma; \Psi} \phi(t) = \phi(t).$$



Theorem 0.74. Assume that $0 < \sigma_1, \sigma_2, \xi < 1$, $\sigma_1 + \sigma_2 > 1$, $\lambda > 0$ and $\mathfrak{h}, \mathfrak{g} : \mathcal{O} \times \mathbb{R} \rightarrow \mathbb{R}$, are two continuous functions. Then $\mathbf{u}$ is a solution of the problem (0.530) if and only if

$$\begin{aligned} \mathbf{u}(\mathfrak{t}) = & \frac{1}{\Gamma(\sigma_1 + \sigma_2)} \int_0^{\mathfrak{t}} \Psi'(\mathfrak{s}) (\Psi(\mathfrak{t}) - \Psi(\mathfrak{s}))^{\sigma_1 + \sigma_2 - 1} \mathfrak{h}(\mathfrak{s}, \mathbf{u}(\mathfrak{s})) \mathbf{d}\mathfrak{s} \\ & - [\zeta(\mathfrak{t})]^{\sigma_1 + \sigma_2 - 1} \left[\frac{1}{\Gamma(\sigma_1 + \sigma_2)} \int_0^1 \Psi'(\mathfrak{s}) (\Psi(1) - \Psi(\mathfrak{s}))^{\sigma_1 + \sigma_2 - 1} \mathfrak{h}(\mathfrak{s}, \mathbf{u}(\mathfrak{s})) \mathbf{d}\mathfrak{s} \right. \\ & \left. - \frac{\lambda}{\Gamma(\sigma_3)} \int_0^\xi \Psi'(\mathfrak{s}) (\Psi(\xi) - \Psi(\mathfrak{s}))^{\sigma_3 - 1} \mathfrak{g}(\mathfrak{s}, \mathbf{u}(\mathfrak{s})) \mathbf{d}\mathfrak{s} \right], \end{aligned} \quad (0.116)$$

where $\zeta(\mathfrak{t}) = \left(\frac{\Psi(\mathfrak{t}) - \Psi(0)}{\Psi(1) - \Psi(0)} \right)$.

Theorem 0.75. Assume that the following assertions hold

(AS₁) There exists a real constant $\Lambda > 0$ such that

$$|\mathfrak{h}(\mathfrak{t}, \mathbf{u}) - \mathfrak{h}(\mathfrak{t}, \mathbf{v})| \leq \Lambda |\mathbf{u} - \mathbf{v}|, \quad \mathfrak{t} \in [0, 1], \quad \mathbf{u}, \mathbf{v} \in \mathbb{R}.$$

(AS₂) There exists an Ψ -RL integrable function $\Theta : [0, 1] \rightarrow \mathbb{R}^+$ such that

$$|\mathfrak{g}(\mathfrak{t}, \mathbf{u}) - \mathfrak{g}(\mathfrak{t}, \mathbf{v})| \leq \Theta(\mathfrak{t}) |\mathbf{u} - \mathbf{v}|, \quad \mathfrak{t} \in [0, 1], \quad \mathbf{u}, \mathbf{v} \in \mathbb{R}.$$

(AS₃) The real constant κ satisfies

$$0 < \kappa = \frac{2\Lambda(\Psi(1) - \Psi(0))^{\sigma_1 + \sigma_2}}{\Gamma(\sigma_1 + \sigma_2 + 1)} + \lambda \mathcal{I}_{0+}^{\sigma_3, \Psi} \Theta(\xi) < 1.$$

Then, the FBVP (0.530) admits a unique solution.



Lemma 0.28. *the following numerical algorithm called DGJIM:*

$$\left\{ \begin{array}{l} u_0(t) = g(t), \\ u_1(t) = \mathcal{L}(u_0(t)) + \mathcal{N}(u_0(t)), \\ u_2(t) = \mathcal{L}(u_1(t)) + \mathcal{N}(u_0(t) + u_1(t)) - \mathcal{N}(u_0(t)), \\ h_3(t) = \mathcal{L}(u_2(t)) + \mathcal{N}(u_0(t) + u_1(t) + u_2(t)) - \mathcal{N}(u_0(t) + u_1(t)), \\ \vdots = \vdots \\ u_n(t) = \mathcal{L}(u_{n-1}(t)) + \mathcal{N}\left(\sum_{i=0}^{n-1} u_i(t)\right) - \mathcal{N}\left(\sum_{i=0}^{n-2} u_i(t)\right), \\ \vdots = \vdots \end{array} \right. \quad (0.117)$$

where L is linear operator, N is nonlinear operator and g is known function.
This method is convergence.

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Solving Nonlocal Initial-Boundary Value Problems for Parabolic Integro-Differential Equations by numerical method

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Abstract. In this work, we will examine how to find the approximate solution to Nonlocal Initial-Boundary Value Problems for Parabolic Integro-Differential Equations by applying in numerical method (reproducing kernel Hilbert space method). This research analyzes how well the method converges. Specific assumptions are made to establish a theoretical foundation for the method. Additionally, the accuracy of the numerical approach within a reproducing kernel Hilbert space is evaluated by estimating the errors.

Keywords: Parabolic Integro-Differential, reproducing kernel Hilbert space method, Nonlocal Initial-Boundary Value Problems, the approximate solution.



Introduction

Non-local initial-boundary value problems for parabolic integro-differential equations are a class of mathematical problems that involve partial differential equations with integral terms. These types of equations arise in various fields such as mathematical physics, engineering, and biology, among others. Solving non-local initial-boundary value problems for parabolic integro-differential equations often requires advanced mathematical techniques such as functional analysis, operator theory, and numerical methods. This work looks at a way to Solving Non-local initial-boundary value problems for parabolic integro-differential equations in reproducing kernel Hilbert Space method (RKHSM) [1,2,3]. The approach uses reproducing kernel Hilbert space method.

Main results

In this section*, we first give some basic definitions of the reproducing kernel and reproducing kernel Hilbert space, and subsequently, we introduce several reproducing kernel Hilbert spaces needed in this work.

Definition 0.46. Let W be a real Hilbert space of functions on a set E . Denote by $\langle u, v \rangle_W$ the inner product and let $\|u\|_W = \sqrt{\langle u, u \rangle_W}$ be the norm in W , for $u, v \in W$. The real valued function $R : E \times E \rightarrow \mathbb{R}$ is called a reproducing kernel of W if the followings are satisfied

- For every x , $R_x(y) = R(x, y)$ as a function of y belongs to E .
- The reproducing property: $u(x) = \langle u(\cdot), R_x(\cdot) \rangle_W$, for all $u \in W$, and all $x \in E$.

Theorem 0.76. Let $(x_i, t_i)_{i=1}^\infty = 1$ be a dense subset in D , then $\partial_x^m \partial_t^p u_n(x, t)$, $m = 0, 1, 2, p = 0, 1$ converge uniformly to $\partial_x^m \partial_t^p u(x, t)$

Lemma 0.29. if $u(x, t) \in W_{(1,2)}(D)$, then there exists a constant $M > 0$, such that

$$|\partial_x^m \partial_t^p u(x, t)| \leq M \|u(x, t)\|_{W_{(3,1)}}, m = 0, 1, 2, p = 0, 1.$$

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Homogenization of quadratic forms

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Abstract.

In this paper, we examine the process of homogenization of energy within a quadratic form in 3 dimension, which is represented by the following integral functional: $F_\epsilon : L^2(0,1)^3 \rightarrow \mathbb{R}^+ \cup \{+\infty\}$

$$F_\epsilon(u) = \begin{cases} \int_0^1 \langle A_\epsilon(t). \nabla u, \nabla u \rangle dx & \text{if } u \in H_0^3(0,1)^3, \\ +\infty & \text{else.} \end{cases} \quad (0.118)$$

where: $A_\epsilon : \mathbb{R} \rightarrow \mathbb{R}^{3 \times 3}$ invertible, measurable and

$$0 < b \leq A_\epsilon^{i,j} \leq c < +\infty \quad i, j \in \{1, 2, 3\} \quad (0.119)$$

Keywords: Homogenization, Γ -Limit, Elliptic equations, Quadratic forms.



Introduction

Homogenization is a mathematical method used when a material with a heterogeneous component and a periodic structure is being studied physically. A small, positive parameter is defined, representing the size of the domain before the size of the parameter. When ε tends towards 0, this corresponds, from a practical point of view, to a transition from a non-homogeneous medium to a so-called 'homogenized' (i.e. a problem posed in a homogeneous medium). Homogenization problems are often addressed by studying minimization problems and associated derivative equations. The minimization problems and the partial differential equations that govern the physical properties being studied are key considerations. Γ -convergence is used to homogenize the energetic, ensuring that the minimizers converge to the minimum value of the energy associated with the problem. This concept was first introduced in 1975 by E. De Giorgi and T. Franzoni [7]. The objective of homogenization is to provide the macroscopic properties of the composite by considering the properties of the microscopic structure. In the first part of this document, we will review some fundamental definitions and theorems, including the classical theorems of integration theory, the application of functional analysis tools, the concept of the sub-differential, and the notion of Γ -convergence. In this section*, we will present some of the fundamental tools that will be used throughout the remainder of this document. The second part of this work will focus on functional homogenization results under the convexity condition. After proposing a condition that is more straightforward than the convexity condition, we utilize De Giorgi's slicing technique. The final objective of our work is to homogenize an elliptic equation of Dirichlet type associated with the minimization problem using the tools of convexity and Γ -convergence.

Main results

Definition 0.47. Let E be a complete metric space. The functional $F^{hom} : E \rightarrow \mathbb{R} \cup +\infty$ is called the (sequential) Γ -limit of the functionals $F_\varepsilon : E \rightarrow \mathbb{R} \cup +\infty$, $\varepsilon > 0$ if the following two conditions are satisfied:

i) (lim inf-inequality)

For all family $(u_\varepsilon) \subset X$ such that $u_\varepsilon \rightarrow u$ (resp $u_\varepsilon \rightharpoonup u$), we have

$$F_{hom}(u) \leq \liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon),$$



ii) (lim sup-inequality)

For all $u \in E$ there exists a recovery sequence $(u_\varepsilon) \in E$, that is, $u_\varepsilon \rightarrow u$ ($u_\varepsilon \rightharpoonup u$) in E and

$$F_{hom}(u) \geq \limsup_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon)$$

Theorem 0.77. Let F_{hom} a functional defined by

$$F_{hom}(u) = \begin{cases} \int_0^1 \langle A \nabla u, \nabla u \rangle dx & \text{if } u \in H_0^3(0,1)^3, \\ +\infty & \text{else.} \end{cases} \quad (0.120)$$

Assume that A fulfills condition (0.2) and the following assertions hold: $A_\varepsilon^{-1} \rightharpoonup A^{-1}$ in the weak sense of $L^1 - L^\infty$, then F_ε Γ -converge to the integral functional F defined on $H_0^3(0,1)^3$ by

$$F_{hom}(u) = \begin{cases} \int_{(0,1)^3} \langle A(x) \nabla u, \nabla u \rangle dx & \text{if } u \in H_0^3(0,1)^3, \\ +\infty & \text{sinon.} \end{cases} \quad (0.121)$$

Sketch of proof

i) inner bond:

Let $u_\varepsilon \rightarrow u$ in $L^2(0,1)^3$, without loss of generality, we assume that $\liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) < +\infty$, by the equi-coercivity of F_ε , by an extraction family $\{u_{g(\varepsilon)}\}_{\varepsilon > 0}$ bonded in



$H_0^3(0, 1)$ in one hand, and by other hand $u_{g(\varepsilon)} \rightarrow u$, it follow that:

$$\begin{aligned} F_{g(\varepsilon)}(u_{g(\varepsilon)}) &= \int_{(0,1)^3} \langle A_\varepsilon(t) \cdot \nabla u, \nabla u \rangle = \int_{(0,1)^3} \langle A_{g(\varepsilon)}(\nabla u_{g(\varepsilon)} - A \cdot A_{g(\varepsilon)}^{-1} \nabla u), (\nabla u_{g(\varepsilon)} - A \cdot A_{g(\varepsilon)}^{-1} \nabla u) \rangle dx \\ &+ 2 \int_{(0,1)^3} \langle A \nabla u, \nabla u_{g(\varepsilon)} \rangle dx - \int_{(0,1)^3} \langle A \nabla u, A \cdot A_{g(\varepsilon)}^{-1} \nabla u \rangle dx \\ &\geq 2 \int_{(0,1)^3} \langle A \nabla u_{g(\varepsilon)}, \nabla u_{g(\varepsilon)} \rangle dx - \int_{(0,1)^3} \langle A \nabla u, A \cdot A_{g(\varepsilon)}^{-1} \nabla u \rangle \end{aligned}$$

Hence,

$$F_{hom}(u) \leq \lim_{\varepsilon \rightarrow 0} F_{g(\varepsilon)}(u_{g(\varepsilon)}) = \liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon)$$

ii) *The upper bond:*

Let $u \in L^2(0, 1)^3$, without loss of generality assume that $F_{hom}(u) < +\infty$, then $u \in H_0^3(0, 1)^3$, thanks to Sobolev immersion, there exist $\tilde{u} \in C([0, 1])$:

$$\tilde{u} = u \quad a.e. x \in (0, 1)^3$$

We pose:

$$\begin{aligned} \tilde{u}_\varepsilon(x) &= \int A_\varepsilon(A \nabla u) dx. \\ |\tilde{u}_\varepsilon(x)| &\leq \frac{c}{b} \int_{(0,1)^3} |\nabla u| dx \\ &\leq \frac{c}{b} \|\nabla u\|_2 = Cte \in L^2(0, 1) \end{aligned} \tag{0.122}$$

Hence, by the dominant convergence theorem $\tilde{u}_\varepsilon \xrightarrow{L^2} u$.
 We take the limit and we find:

$$\lim_{\varepsilon \rightarrow 0} F_\varepsilon(\tilde{u}_\varepsilon) = \limsup_{\varepsilon \rightarrow 0} F_\varepsilon(\tilde{u}_\varepsilon) = F_{hom}(u).$$

Application to the Dirichlet Problem

The basic property of Γ -convergence, see e.g.[2] imply that for every $g \in L^2(\Omega)$, the solutions u_ε of the minimizer of the problem

$$\min \left\{ \int_\Omega \frac{1}{2} \left(\left\langle A \left(\frac{x}{\varepsilon} \right) \nabla u, \nabla u \right\rangle - g(x)u \right) dx : u \in H_0^1(\Omega) \right\} \tag{0.123}$$

converge in $L^2(\Omega)$, as $\varepsilon \rightarrow 0$, to the solution u_{hom} of the minimization problem

$$\min \left\{ \int_\Omega \frac{1}{2} (\langle A_{hom} \nabla u, \nabla u \rangle - g(x)u) dx : u \in H_0^1(\Omega) \right\} \tag{0.124}$$



By considering the Euler-Lagrange equation's, see e.g.[3] of the problems (1) and (2) in this special case, from the previous result we obtain the homogenization theorem for symmetric elliptic operators in divergence form, which asserts that for every $g \in L^2(\Omega)$ the solutions u_ε of the Dirichlet problems

$$\begin{cases} -\operatorname{div} \left(A\left(\frac{x}{\varepsilon}\right) \nabla u_\varepsilon \right) = g & \text{on } \Omega \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

converge in $L^2(\Omega)$ to the solution u_{hom} of the Dirichlet problem

$$\begin{cases} -\operatorname{div} (A_{hom} \nabla u) = g & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

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On integral inequalities related to the extended Chebyshev functional

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Abstract. In this work by applying the extended Chebyshev functional for monotone and quasi- monotone functions, we obtain some new weighted integral inequalities for weighted Hardy operators. Moreover some known integrals are generalized.

Keywords: weighted integral inequalities, Chebyshev functional, Hardy operator.



Introduction

Inequalities play a very important role in different fields of mathematics and present a active and attractive area of research. As examples, we cite the field of integration which is dominated by inequalities involving operators and integrals (see [10]). Among the inequalities, we recall two famous integral inequalities, first, the Chebyshev inequality (see [2]), second the Hardy integral inequality. G.H. Hardy stated in 1920 (see [4]) and proved in 1925 (see [5]) his famous inequality if $f(x)$ is non-negative and measurable on $(0, \infty)$, then

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(t) dx \right)^p dx \leq \left(\frac{p}{p-1} \right)^p \int_0^\infty f^p(x) dx, \quad p > 1. \quad (0.125)$$

The well known Chebyshev functional is defined as follows

$$T(h_1, h_2) \geq 0, \quad (0.126)$$

where

$$T(h_1, h_2) = \frac{1}{b-a} \int_a^b h_1(x) h_2(x) dx - \frac{1}{b-a} \int_a^b h_1(x) dx \frac{1}{b-a} \int_a^b h_2(x) dx \quad (0.127)$$

and f and g are two integrable functions which are synchronous on $[a, b]$ (i.e. $(f(x) - f(y))(g(x) - g(y)) \geq 0$), for any $x, y \in [a, b]$). If the functions h_1 and h_2 are asynchronous $[a, b]$ (i.e. $(f(x) - f(y))(g(x) - g(y)) \leq 0$), for any $x, y \in [a, b]$).

Main results

We first give some results for monotone functions.

Lemma 0.30. *Let h_1 and h_2 are two synchronous functions on (a, b) for each $t_1, t_2 \in (a, b)$, then*

$$\int_a^b p(t) h_1(t) dt \int_a^b p(t) h_2(t) dt \leq \int_a^b p(t) dt \int_a^b p(t) h_1(t) h_2(t) dt, \quad (0.128)$$

where p is weight function defined on (a, b) . If the functions h_1 and h_2 are asynchronous on (a, b) for each $t_1, t_2 \in (a, b)$, then

$$\int_a^b p(t) h_1(t) dt \int_a^b p(t) h_2(t) dt \geq \int_a^b p(t) dt \int_a^b p(t) h_1(t) h_2(t) dt. \quad (0.129)$$

Let

$$H_{1,p,q}(x) = \int_0^x h_1(t) p(t) q(t) dt, \quad H_{2,p,q}(x) = \int_0^x h_2(t) p(t) q(t) dt$$



and

$$H_{1,p,q}^*(x) = \int_x^\infty h_1(t)p(t)q(t)dt, \quad H_{2,p,q}^*(x) = \int_x^\infty h_2(t)p(t)q(t)dt,$$

where $t \in (a, b)$, $0 < a < b \leq \infty$, h_1 and h_2 are non-negative Lebesgue measurable functions on (a, b) and p, q are positive weight functions defined on (a, b) .

In this work, we assume that ψ is a non-negative function defined on (a, b) .

Theorem 0.78. *Let h_1, h_2, q and ψ be non-decreasing functions on (a, b) and p be a positive weight function on (a, b) . If the function*

$$\psi \left[h_1(x)h_2(x) \left(\frac{q(x)}{x} \int_0^x p(t)dt \right)^2 \right]$$

is integrable on (a, b) , then

$$\int_a^b \psi \left(\frac{H_{1,p,q}(x)H_{2,p,q}(x)}{x^2} \right) dx \leq \int_a^b \psi \left[h_1(x)h_2(x) \left(\frac{q(x)}{x} \int_0^x p(t)dt \right)^2 \right] dx. \quad (0.130)$$

Theorem 0.79. *Let h_1, h_2, q be non-increasing functions on (a, b) , ψ be a non-decreasing function on (a, b) and p be a positive weight function on (a, b) . If the function*

$$\psi \left[h_1(x)h_2(x) \left(\frac{q(x)}{x} \int_x^\infty p(t)dt \right)^2 \right]$$

is integrable on (a, b) , then

$$\int_a^b \psi \left(\frac{H_{1,p,q}^*(x)H_{2,p,q}^*(x)}{x^2} \right) dx \leq \int_a^b \psi \left[h_1(x)h_2(x) \left(\frac{q(x)}{x} \int_x^\infty p(t)dt \right)^2 \right] dx. \quad (0.131)$$

Now we turn our attention to the case of quasi-monotone functions.

The following definition was introduced in [1].

Definition 0.48. We say that a non-negative function h_1 is quasi-monotone on $]0, \infty[$ if for some real number α , $x^\alpha h_1$ is a decreasing or an increasing function of x . More precisely, given $\beta \in \mathbb{R}$ we say that $h_1 \in Q_\beta$ if $x^{-\beta} h_1(x)$ is non-increasing and $h_1 \in Q^\beta$ if $x^{-\beta} h_1(x)$ is non-decreasing.

Let $h_{1,\beta}(x) = x^{-\beta} h_1(x)$, $h_{2,\beta}(x) = x^{-\beta} h_2(x)$ on (a, b) , where $0 < a < b \leq \infty$.

Lemma 0.31. *If $h_{1,\beta}$ and $h_{2,\beta}$ are two synchronous functions on (a, b) , i.e $h_1, h_2 \in Q_\beta$ or $h_1, h_2 \in Q^\beta$ and p is a positive weight function on (a, b) , then*

$$\int_a^b p(x)x^{-\beta} h_1(x)dx \int_a^b p(x)x^{-\beta} h_2(x)dx \leq \int_a^b p(x)dx \int_a^b p(x)x^{-2\beta} h_1(x)h_2(x)dx. \quad (0.132)$$



If $h_{1,\beta}$ and $h_{2,\beta}$ are asynchronous functions on (a, b) , then

$$\int_a^b p(x)x^{-\beta}h_1(x)dx \int_a^b p(x)x^{-\beta}h_2(x)dx \geq \int_a^b p(x)dx \int_a^b p(x)x^{-2\beta}h_1(x)h_2(x)dx. \quad (0.133)$$

proof

We replace h_1 and h_2 in Lemma 0.42 by $h_{1,\beta}$ and $h_{2,\beta}$ respectively, then we get the inequality (0.528). In the same way, we have inequality (0.529).

Let

$$H_{1,\beta,p,q}(x) = \int_0^x t^{-\beta}h_1(t)p(t)q(t)dt, \quad H_{2,\beta,p,q}(x) = \int_0^x t^{-\beta}h_2(t)p(t)q(t)dt$$

and

$$H_{1,\beta,p,q}^*(x) = \int_x^\infty t^{-\beta}h_1(t)p(t)q(t)dt, \quad H_{2,\beta,p,q}^*(x) = \int_x^\infty t^{-\beta}h_2(t)p(t)q(t)dt,$$

where $t \in (a, b)$, $0 < a < b \leq \infty$, $\beta \in \mathbb{R}$, h_1 and h_2 are non-negative Lebesgue measurable functions on (a, b) and p, q are positive weight functions on (a, b) .

Theorem 0.80. Let $h_1, h_2 \in Q^\beta$, q and ψ be non-decreasing functions on (a, b) and p be a positive weight function defined on (a, b) . If the function

$$\psi \left[h_1(x)h_2(x) \left(x^{-(1+\beta)}q(x) \int_0^x p(t)dt \right)^2 \right]$$

is integrable on (a, b) , then

$$\int_a^b \psi \left(\frac{H_{1,\beta,p,q}(x)H_{2,\beta,p,q}(x)}{x^2} \right) dx \leq \int_a^b \psi \left[h_1(x)h_2(x) \left(x^{-(1+\beta)}q(x) \int_0^x p(t)dt \right)^2 \right] dx. \quad (0.134)$$

Theorem 0.81. Let $h_1, h_2 \in Q_\beta$, ψ be a non-decreasing function on (a, b) , q be a non-increasing function on (a, b) and p be a positive weight function on (a, b) .

If the function $\psi \left[h_1(x)h_2(x) \left(x^{-(1+\beta)}q(x) \int_x^\infty p(t)dt \right)^2 \right]$ is integrable on (a, b) , then

$$\int_a^b \psi \left(\frac{H_{1,\beta,p,q}^*(x)H_{2,\beta,p,q}^*(x)}{x^2} \right) dx \leq \int_a^b \psi \left[h_1(x)h_2(x) \left(x^{-(1+\beta)}q(x) \int_x^\infty p(t)dt \right)^2 \right] dx. \quad (0.135)$$



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A new class of optimization methods based on coefficient conjugate gradient

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Abstract. We know many conjugate gradient algorithms (CG) for solving unconstrained optimization problems. In this paper, based on the three famous Liu-Storey (LS), Fletcher-Reeves (FR) and Polak-Ribiere-Polyak (PRP) conjugate gradient methods, a new hybrid CG method is proposed. Furthermore, the search direction satisfies the sufficient descent condition independent of the line search. Likewise, we prove, under the strong Wolfe line search, the global convergence of the new method. In this respect, numerical experiments are performed and reported, which show that the proposed method is efficient and promising. In virtue of this, the application of the proposed method for solving regression models of COVID-19 is provided.

Keywords: Unconstrained optimization, hybrid conjugate gradient method, sufficient descent, convex combination, global convergence..



Introduction

Unconstrained optimization is a branch of optimization in which we minimize an objective function that depends on real variables with the total absence of restrictions on their values of those variables. Thus, we consider the general unconstrained optimization problems, as follows:

$$\min\{f(x), x \in \mathbb{R}^n\}, \quad (0.136)$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is the continuously differentiable function and its first derivative is represented by $g(x) = \nabla f(x)$. Though several robust optimization algorithms with rapid convergence have shown to be available to solve the above nonlinear optimization model, many researchers still refer to the conjugate gradient algorithm (CG) as it uses low memory and good convergence properties. Generally, the iterates of the CG methods are usually determined through the following recursive computational scheme:

$$x_{k+1} = x_k + \alpha_k d_k, \quad k = 0, 1, \dots, n, \quad (0.137)$$

and

$$d_k = \begin{cases} -g_k, & \text{if } k = 0, \\ -g_k + \beta_k d_{k-1}, & \text{if } k \geq 1, \end{cases} \quad (0.138)$$

where x_k is the current iteration, g_k is the gradient of f at the point x_k , d_k is the search direction, $\beta_k \in \mathbb{R}$ is the conjugate parameter which characterizes different versions of the CG methods and $\alpha_k > 0$ is the step size many line search techniques can obtain. In the midst of which, exact line search, weak Wolfe line search, or strong Wolfe line search, but we use, in our research, strong Wolfe line search, which is defined by the following conditions:

$$f(x_k + \alpha_k d_k) - f(x_k) \leq \delta \alpha_k g_k^T d_k, \quad (0.139)$$

$$|g_{k+1}^T d_k| \leq -\sigma g_k^T d_k, \quad (0.140)$$

where scalars δ and σ satisfy $0 < \delta \leq \sigma < 1$.

As we said, the CG algorithms differ by choosing the coefficient β_k . The most common standard CG methods are the Hestenes-Stiefel (HS), the Fletcher-Reeves (FR) method, the Conjugate-Descent (CD) method, the Dai-Yuan (DY) method, the Liu-Storey (LS) method and the Polak-Ribiere-Polyak (PRP) method, respectively, which are defined as:

$$\begin{aligned} \beta_k^{HS} &= \frac{g_k^T y_{k-1}}{y_{k-1}^T d_{k-1}} & \beta_k^{FR} &= \frac{\|g_k\|^2}{\|g_{k-1}\|^2} & \beta_k^{CD} &= \frac{\|g_k\|^2}{-g_{k-1}^T d_{k-1}} \\ \beta_k^{DY} &= \frac{\|g_k\|^2}{y_{k-1}^T d_{k-1}} & \beta_k^{LS} &= \frac{g_k^T y_{k-1}}{-d_{k-1}^T g_{k-1}} & \beta_k^{PRP} &= \frac{g_k^T y_{k-1}}{\|g_{k-1}\|^2}, \end{aligned}$$



while behaves differently for general non quadratic functions.

where $y_{k-1} = g_k - g_{k-1}$ and $||\cdot||$ stands for the Euclidean norm. If f is a strictly convex quadratic function, all the above methods are equivalent, while they behave differently for general non-quadratic functions. One of the essential classes of CG methods is the hybrid conjugate gradient algorithms. Further, the hybrid schemes have better computational performances and more robust convergence properties than conventional CG methods as they take advantage of the two parameters used to build them. For this reason, many researchers cared about hybrid or mixed conjugate gradient methods. Djordjevic [1], proposed the following hybrid method: $\beta_k^{hyb} = \theta_k \beta_k^{FR} + (1 - \theta_k) \beta_k^{HS}$, Andrei [2], proposed the following hybrid method: $\beta_k^c = \theta_k \beta_k^{DY} + (1 - \theta_k) \beta_k^{HS}$, Li and Sun [3], proposed the following hybrid method: $\beta_k^N = \theta_k \beta_k^{MWU} + (1 - \theta_k) \beta_k^{FR}$, Liu, J.K. and Li, S.J. In addition, Sabrina and al [1] proposed a new hybrid CG method based on combination of FR, PRP and DY conjugate gradient algorithms in which

$$\beta_k^{hyb} = \delta_k \beta_k^{FR} + \gamma_k \beta_k^{PRP} + (1 - \delta_k - \gamma_k) \beta_k^{DY},$$

where

$$\gamma_k = -\frac{g_k^T g_{k+1} ||g_k||^2 + \delta_k (y_k^T s_k - ||g_k||^2) ||g_{k+1}||^2}{(g_{k+1}^T y_k)(y_k^T s_k) - ||g_{k+1}||^2 ||g_k||^2}, \quad \text{where, } 0 < \delta_k < 1.$$

Inspired by this research, we propose in this study a new hybrid CG method based on combination of LS, FR and PRP conjugate gradient algorithms for solving unconstrained optimization problems. In addition, we alike apply, in this study, the new method for solving a model of COVID-19 outbreak around the globe in which the data is taken from January to September 2020.

Main results

In this paper, we propose another combination of LS, FR and PRP conjugate gradient algorithms. We use the following conjugate gradient parameter:

$$\beta_k^{New} = \delta_k \beta_k^{LS} + \gamma_k \beta_k^{FR} + (1 - \delta_k - \gamma_k) \beta_k^{PRP}. \quad (0.141)$$

where $\delta_k, \gamma_k \in [0, 1]$ are named the hybridization parameters. As a consequence, the direction d_k is given by:

$$d_k = \begin{cases} -g_k, & \text{if } k = 0, \\ -g_k + \beta_k^{New} d_{k-1}, & \text{if } k \geq 1, \end{cases} \quad (0.142)$$

The hybridization parameters are given by the following form:

$$\delta_k = \frac{(g_{k+1}^T y_k (||g_k||^2 - d_k^T y_k) - \gamma_k (g_{k+1}^T g_k) (d_k^T y_k))}{(||g_k||^2 + d_k^T g_k) (-g_{k+1}^T y_k) (d_k^T y_k)} d_k^T g_k, \quad \text{where } 0 < \gamma_k < 1 \quad (0.143)$$



The following Theorem proved that the search direction d_k generated by the selected method satisfies the sufficient descent condition.

Theorem 0.82. *Let the sequences $\{g_k\}$ and $\{d_k\}$ be generated by the new method. Then, the search direction d_k satisfies the sufficient descent condition:*

$$g_k^T d_k \leq -K \|g_k\|^2, \quad \forall k \geq 0, \quad (0.144)$$

where $K > 0$.

Now, we give the next theorem which establishes the global convergence of our proposed method with strong Wolfe line searches.

Theorem 0.83. *Consider the iterative method, defined by the new method and let all conditions of Theorem (0.191) hold. Then*

$$\liminf_{k \rightarrow \infty} \|g_k\| = 0. \quad (0.145)$$

Numerical Results

The results are given in two Figures which are represented the performance profiles of our new hybrid method versus FR, PRP and LS based on the CPU time and number of iterations, respectively. They shown that the performance profiles of our method was higher than the other CG methods on the testing problems.

Conclusion

In light of the facts above, the hybrid CG methods are usually obtained based on the classical CG methods by integrating their advantages. In this paper, we proposed a new hybrid conjugate gradient algorithm in which the famous parameter β_k is computed as a convex combination of β_k^{LS} , β_k^{FR} and β_k^{PRP} algorithms. Based on some conditions, we show that the proposed algorithm enjoys sufficient descent condition and converges globally under strong Wolfe line search. Further, numerical experiments are considered to illustrate the performance of the proposed method. The results shown that the selected method is faster and more efficient compared to the methods used. More to the point, our proposed method can solve the COVID-19 case model.

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New fractional mask to edge detection using Laplacian

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Abstract. Edge detection is crucial in image processing and computer vision problems. Currently, two methods, gradient and Laplacian methods, are used. However, these methods are sensitive to noise and can produce false edges. This paper proposes an efficient edge detection method based on the fractional Laplacian operator in the Caputo-Fabrizio sense. The method uses a new approximation formula and fractional mask to compute Laplacian convolution. The proposed method outperforms classical Laplacian methods in terms of peak signal to noise ratio and fine details extraction.

Keywords: Edge detection, Laplacian method, Gradient method, Caputo-Fabrizio fractional derivative, Traditional edge detectors, Fractional calculus.

Introduction

In fields such as image processing and computer vision, edge detection is the mathematical process of identifying discontinuous points in an image using a range of mathematical techniques, including gradient and Laplacian methods. The Sobel, Roberts, and Prewitt algorithms are some of the oldest and most popular algorithms for detecting discontinuous edges. The principle of gradient methods is to find the maximum in the first derivative of the image, where pixels with a high gradient are considered as edges, while Laplacian methods aim to identify zero intersection*s in the second derivative. Although Laplacian methods are less sensitive to noise compared to gradient methods, they do not give satisfactory results in some cases due to the use of the derivative based on the correct order. In this context, the use of the fractional derivative instead of



the integer derivative is common in many mathematical problems.

in this paper we propose a new more adaptive propose an efficient edge detection method based on the fractional Laplacian operator using the second order Caputo-Fabrizio fractional derivative.

The obtained results prove the efficiency and robustness of our method .

PROPOSED EDGE DETECTION METHOD

we use fractional derivatives that are useful for our study:

Theorem 1: Let $0 < \alpha \leq 1$ and $u \in H^2(0, b)$. Then the approximation of the Caputo-Fabrizio second-order fractional derivative is given by:

$${}^{CF}D_0^{(1+\alpha)}u(t) = \sum_{k=0}^N d_k u(t_k) + O(\Delta t) \quad (1)$$

where $t_k = k\Delta t$; $\Delta t = \frac{t}{N}$; $k = 0, \dots, N$ and

$$d_k = \begin{cases} \frac{e^{-\frac{\alpha}{1-\alpha}t_{N-1}}}{(1-\alpha)\Delta t} & ; \text{ for } k = 0 \\ \frac{e^{-\frac{\alpha}{1-\alpha}t} e^{\frac{\alpha}{1-\alpha}\Delta t} \left(e^{\frac{\alpha}{1-\alpha}\Delta t} - 2 \right)}{(1-\alpha)\Delta t} & ; \text{ for } k = 1 \\ \frac{\left(1 - e^{\frac{\alpha}{1-\alpha}\Delta t} \right) \left(e^{\frac{\alpha}{1-\alpha}\Delta t} - 1 \right) e^{-\frac{\alpha}{1-\alpha}(t-t_k)}}{(1-\alpha)\Delta t} & ; \text{ for } 2 \leq k \leq N-3 \\ \frac{\left(e^{-\frac{3\alpha}{1-\alpha}\Delta t} - 2e^{-\frac{2\alpha}{1-\alpha}\Delta t} + e^{-\frac{\alpha}{1-\alpha}\Delta t} \right)}{(1-\alpha)\Delta t} & ; \text{ for } k = N-2 \\ \frac{\left(e^{-\frac{2\alpha}{1-\alpha}\Delta t} - 2e^{-\frac{\alpha}{1-\alpha}\Delta t} \right)}{(1-\alpha)\Delta t} & ; \text{ for } k = N-1 \\ \frac{\left(e^{-\frac{\alpha}{1-\alpha}\Delta t} \right)}{(1-\alpha)\Delta t} & ; \text{ for } k = N \end{cases}$$



Construct the new fractional Laplacian mask

By looking at Theorem 1, the second-order fractional derivative in the Caputo-Fabrizio sense can be approximated as follows:

$${}^{CF}D_0^{(1+\alpha)}u(t) \approx \sum_{k=0}^N d_k u(t_k) \quad (2)$$

To construct a 3×3 fractional mask, we take only the three last terms of the Eq. 2 as follows:

$${}^{CF}D_0^{(1+\alpha)}u(t) \approx d_{N-2}u(t_{N-2}) + d_{N-1}u(t_{N-1}) + d_N u(t_N) \quad (3)$$

In practice, the points step in the image is taken as 1 ; i.e ; $\Delta t = 1$. Therefore:

$${}^{CF}D_0^{(1+\alpha)}u(t) \approx d_{N-2}u(t-2) + d_{N-1}u(t-1) + d_N u(t) \quad (4)$$

Now, we suggest the second-order fractional derivative approximation with respect to x and y respectively as follows:

$$\frac{\partial^2 u}{\partial x^2} = u * \begin{pmatrix} 0 & 0 & 0 \\ d_{N-2} & d_{N-1} & d_N \\ 0 & 0 & 0 \end{pmatrix} \quad ; \quad \frac{\partial^2 u}{\partial y^2} = u * \begin{pmatrix} 0 & d_{N-2} & 0 \\ 0 & d_{N-1} & 0 \\ 0 & d_N & 0 \end{pmatrix} \quad (5)$$

Finally, our fractional mask is taken by:

$$h = \begin{pmatrix} 0 & 0 & 0 \\ d_{N-2} & d_{N-1} & d_N \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & d_{N-2} & 0 \\ 0 & d_{N-1} & 0 \\ 0 & d_N & 0 \end{pmatrix} = \begin{pmatrix} 0 & d_{N-2} & 0 \\ d_{N-2} & d_{N-1} & d_N \\ 0 & d_N & 0 \end{pmatrix} \quad (6)$$

Where

$$d_{N-2} = \frac{1}{(1-\alpha)} \left(e^{-\frac{3\alpha}{1-\alpha}} - 2e^{-\frac{2\alpha}{1-\alpha}} + e^{-\frac{\alpha}{1-\alpha}} \right) \quad ; \quad d_{N-1} = \frac{1}{(1-\alpha)} \left(e^{-\frac{2\alpha}{1-\alpha}} - 2e^{-\frac{\alpha}{1-\alpha}} \right) \quad ; \quad d_N = \frac{1}{(1-\alpha)} \left(e^{-\frac{\alpha}{1-\alpha}} \right)$$

EXPERIMENTAL RESULTS AND DISCUSSION

Here, we will examine the accuracy and efficiency of our proposed edge detection method compared to the traditional Laplacian detector via the PSNR criterion. For this, we take some grayscale images which are affected by different levels of Gaussian noise:

Figure 1: Edge detection using the classical Laplacian method at $\sigma = 1$ and $T = 10\%$ with $psnr = 5.7497$

Figure 2: Edge detection using the proposed Laplacian method at $\sigma = 1$ and $T = 10\%$ with $\alpha = 0.85$ and $psnr = 5.7807$



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The Adaptive method : Theory and Application

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Abstract. In this research, we introduce the Adaptive Method, a thoughtfully designed algorithm for optimizing linear variables. Our approach navigates through complex numerical environments, adjusting fluidly to critical points, central nodes, and challenging boundaries. We are guided by strict criteria for both optimal and suboptimal solutions. Additionally, we explore the theoretical foundations of this method, enhancing our understanding of its capabilities. The real highlight, however, is our collaboration with MATLAB simulations: tackling over 2000 distinct problems, each representing its own unique challenge. The outcome? A tool that showcases both efficiency and reliability, ready to tackle real-world challenges.

Keywords: linear optimization, adaptive method theory, Interior point methods.



Introduction

In this work, we propose to solve a linear program with hybrid decision variables presented in the following form:

$$\max \quad z(x, y) = c'x + q'y, \quad (0.146)$$

$$\text{sous} \quad Ax + Hy = b, \quad (0.147)$$

$$d^- \leq x \leq d^+, \quad (0.148)$$

$$y \geq 0. \quad (0.149)$$

Using the Dual problems, we introduce a novel method inspired by principles from adaptive linear programming and constructive optimization techniques pioneered by Gabasov and Kirillova [2]. In 1977, they developed the primal support method a generalization of the simplex method. Their approach leverages support feasible solutions, which are more versatile than basic feasible solutions. Notably, the support method includes both an optimality criterion and valuable sub-optimality conditions, particularly beneficial for tackling large-scale linear programming problems.

We construct an algorithm based on the choice of the new improvement direction, step along this direction and the procedure to change the support (basic set). Then, we describe a scheme of the method. Finally, a numerical example is provided and a comparison has been made with others linear programming methods.

Main results

We outline the theory behind our new adaptive approach and provide computational results that highlight its efficiency on random linear programs. Our implementation was carried out using MATLAB version R2018a, which provides an ideal environment for rapidly programming such algorithms and comparing our new method (ADP) with the built-in Linprog solvers, including the interior-point legacy (INP-LG), the interior point (INP), and the dual-simplex method. However, we have chosen to focus this paper on comparing our method with the interior-point legacy and the interior point, as the dual-simplex method tends to be slower for this type of problem.

In our computational study, we evaluated a total of 2000 random problems across three different dimensional scenarios. The plot below illustrates that as the problem dimension increases, the advantages of ADP over the Linprog methods become more pronounced.

Figure 20: CPU time



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Numerical Technique for solving Schrodinger equation with boundary conditions

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Abstract. In this work, we will examine how to find the approximate solution to schrodinger equation with boundary conditions by applying a new numerical method (reproducing kernel Hilbert space method). This research analyzes how well the new method converges. Specific assumptions are made to establish a theoretical foundation for the method. Additionally, the accuracy of the numerical approach within a reproducing kernel Hilbert space is evaluated by estimating the errors. In addition to giving examples to show fi how accurate this method is.

Keywords: schrodinger equations , boundary conditions , approximate solution , numerical method .



Introduction

The Schrodinger equation is used to describe the state function of a quantum-mechanical system.

Reproducing kernels were used for the first time at the beginning of the twentieth century. A reproducing kernel Hilbert space is a useful framework for constructing approximate solutions of partial differential equations (PDE). Many numerical methods have been proposed for solving linear and nonlinear PDEs, but we did not find a method that uses reproducing kernels.

The work is organized as follows some properties of reproducing kernel spaces. , the exact and approximate solution of the problem . exemple The RKHSM systematically produces analytic and approximate solutions in the form of series. The convergence analysis . prove that the approximate solution converges to the exact solution uniformly. Some numerical experiments are illustrated and Some conclusions. the example that we used in this work was taken from an article [2] ,This example was previously applied to the method of Reproducing kernel Hilbert Space by the author F. Z. Geng¹.

Main results

Definition 0.49.

Lemma 0.32. *If $\|\mathcal{U}_n - \mathcal{U}\|_{A(\omega)} \rightarrow 0$, $\|\mathcal{U}\|_{a(\omega)}$ is bounded , $\mathcal{F}(x, t)$ is continous and $(x_k, t_k) \rightarrow (x, t)$ then [1] $\mathcal{F}(x_k, t_k) \rightarrow \mathcal{F}(x, t)$*

Theorem 0.84. *For each $\mathcal{U}(x, t) \in A(\omega)$, let $\varepsilon_n^2 = \|\mathcal{U}(x, t) - \mathcal{U}_n(x, t)\|^2$ then sequences ε_n is monotone decreasing and $\varepsilon_n \rightarrow 0, (n \rightarrow \infty)$ where the exact solution is:*

$$\mathcal{U}(x, t) = \sum_{i=1}^{\infty} \sum_{j=1}^i \gamma_{ij} \mathcal{F}(x_j, t_j) \bar{\psi}(x, t)$$

and the approximate solution

$$\mathcal{U}_n(x, t) = \sum_{i=1}^N \sum_{j=1}^i \gamma_{ij} \mathcal{F}(x_j, t_j) \bar{\psi}(x, t)$$



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Existence of weak solutions to double phase problems with irregular data

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Abstract. In this paper, we prove the existence of weak solutions for a specific class of nonlinear double phase problems involving variable exponents and irregular data.

Keywords: Nonlinear elliptic equations; weak solutions; double phase problems; variable exponents.

Introduction

This work deals with the existence of solutions of the following boundary value problem:

$$\begin{cases} -\operatorname{div} \left(A(x) |Du|^{p(x)-2} Du \right) - \operatorname{div} \left(F(u) |Du|^{q(x)-2} Du \right) = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (0.150)$$



where Ω is a bounded open domain in $\mathbb{R}^N$ ($N > 2$) with Lipschitz boundary $\partial\Omega$ and $A(\cdot)$ is a measurable function such that:

$$\alpha \leq A(x) \leq \beta, \quad \text{a.e. } x \in \Omega, \quad \text{with } \alpha, \beta > 0. \quad (0.151)$$

Suppose that $F : \mathbb{R} \rightarrow \mathbb{R}$ such that:

$$F(u) = |u|^{(r(\cdot)-1)q(\cdot)+1}, \quad (0.152)$$

and f is a function belongs to $L^{m(\cdot)}(\Omega)$.

Here, the variable exponents $p : \overline{\Omega} \rightarrow (1, \infty)$, $q : \overline{\Omega} \rightarrow (1, \infty)$, $r : \overline{\Omega} \rightarrow (1, \infty)$, and $m : \overline{\Omega} \rightarrow (1, \infty)$ are continuous functions for all $x \in \overline{\Omega}$ such that:

$$r(x)q'(x) > 1, \quad \text{and} \quad 1 < q^- \leq q^+ < p^- \leq p^+ < N \quad (0.153)$$

where $p^- := \min_{x \in \overline{\Omega}} p(x)$, $p^+ := \max_{x \in \overline{\Omega}} p(x)$, $q^- := \min_{x \in \overline{\Omega}} q(x)$, $q^+ := \max_{x \in \overline{\Omega}} q(x)$ and $q'(\cdot) = \frac{q(\cdot)}{q(\cdot)-1}$. Moreover, $p(\cdot) > 1$ is logarithm Hölder continuous function, that is

$$\exists C > 0 : |p(x) - p(y)| \leq -\frac{C}{\ln(|x - y|)}, \quad \forall x \neq y \in \Omega : |x - y| \leq 1/2. \quad (0.154)$$

Our results depend on the presence of the additional term $-\operatorname{div} \left(F(u) |Du|^{q(\cdot)-2} Du \right)$, which has a regularizing effect on the summability of the solution of the problem 0.150. In the particular case $f \in L^1(\Omega)$, the regularizing effect of this term has been recently investigated in [4]. In point of fact, we shall demonstrate that the solutions of problem 0.150 are more regular than the ones of the problems (see [3] and [9]) under an appropriate balance of the parameters $p(\cdot)$, $q(\cdot)$, $r(\cdot)$, and the summability of f . Inspired by [1, 2, 8], we will provide additional regularity results based on the summability of the source term f . To achieve this, we need to define the notion of weak solution we will use in accordance with the summability of the right-hand side. For simplify our investigation, we will only discuss the case when f belongs to $L^{m(\cdot)}(\Omega)$ with $m(\cdot) > \frac{N}{p(\cdot)}$.

Main result:

Definition 0.50. If $f \in L^m(\Omega)$ with $m > 1$, by a weak solution of the problem 0.150 we mean a function $u \in W_0^{1,p(\cdot)}(\Omega)$ such that

$$F(u) |Du|^{q(\cdot)-1} \in L^{p'(\cdot)}(\Omega) \quad (0.155)$$

and the integral identity

$$\int_{\Omega} A(x) |Du|^{p(x)-2} Du \cdot D\varphi \, dx + \int_{\Omega} F(u) |Du|^{q(x)-2} Du \cdot D\varphi \, dx = \int_{\Omega} f \varphi \, dx \quad (0.156)$$

holds for every $\varphi \in W_0^{1,p(\cdot)}(\Omega)$.



We have the following theorem:

Theorem 0.85. *Under the assumptions (0.151)-(0.154). If $f \in L^{m(\cdot)}(\Omega)$ with $m(\cdot) > \frac{N}{p(\cdot)}$ then the problem 0.150 has at least one solution $u \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$.*

The idea of the proof of Theorem 0.85: The proof is based on three steps: First, we introduce a suitable sequence of problems approximating. In addition, we establish some priori estimates for the solutions of the problem 0.150. Finally, we pass to the limit in the approximate problems.

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Limit cycles generated from a zero-Hopf bifurcation in a continuous three-dimensional differential system via averaging theory

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Abstract. In this work, we use the fourth-order averaging theory to investigate the existence of limit cycles bifurcating from a zero–Hopf equilibrium for general polynomial differential systems in $\mathbb{R}^3$ with cubic homogeneous non-linearities. The result indicates there are at most 10 limit cycles that can bifurcate from this zero–Hopf equilibrium.

Keywords: Periodic orbits, differential system, averaging theory.



Introduction

We shall study the maximum number of limit cycles which can bifurcate from a zero-Hopf equilibrium point of a polynomial differential system in $\mathbb{R}^3$ with cubic homogeneous non-linearities using the averaging theory of the fourth-order. More precisely, we consider the differential systems of the form

$$\begin{aligned}\dot{x} &= (a_1\varepsilon + a_2\varepsilon^2 + a_3\varepsilon^3 + a_4\varepsilon^4)x - (b + b_1\varepsilon + b_2\varepsilon^2 + b_3\varepsilon^3 + b_4\varepsilon^4)y + \sum_{j=0}^2 \varepsilon^j X_j(x, y, z), \\ \dot{y} &= (b + b_1\varepsilon + b_2\varepsilon^2 + b_3\varepsilon^3 + b_4\varepsilon^4)x + (a_1\varepsilon + a_2\varepsilon^2 + a_3\varepsilon^3 + a_4\varepsilon^4)y + \sum_{j=0}^2 \varepsilon^j Y_j(x, y, z), \\ \dot{z} &= (c_1\varepsilon + c_2\varepsilon^2 + c_3\varepsilon^3 + c_4\varepsilon^4)z + \sum_{j=0}^2 \varepsilon^j Z_j(x, y, z),\end{aligned}\tag{0.157}$$

where

$$\begin{aligned}X_j(x, y, z) &= a_{j0}x^3 + a_{j1}x^2y + a_{j2}x^2z + a_{j3}xy^2 + a_{j4}xyz + \\ &\quad a_{j5}xz^2 + a_{j6}y^3 + a_{j7}y^2z + a_{j8}yz^2 + a_{j9}z^3,\end{aligned}$$

$Y_j(x, y, z)$ and $Z_j(x, y, z)$ have the same expression as $X_j(x, y, z)$ by replacing a_{ji} respectively by b_{ji} and c_{ji} for $j = \overline{0, 2}$ and $i = \overline{0, 9}$. The coefficients a_{ji} , b_{ji} , c_{ji} , a_1 , a_2 , a_3 , a_4 , b , b_1 , b_2 , b_3 , b_4 , c_1 , c_2 , c_3 , c_4 are real parameters with $b \neq 0$. When $\varepsilon = 0$ the system has eigenvalues $\pm bi$ and 0 at the origin, i.e. the origin is a zero-Hopf equilibrium point.

Main results

Our main result is the following one.

Theorem 0.86. *The following statement are valid*

- (a) *At most 3 limit cycles bifurcate from the origin of system (0.530) when $\varepsilon = 0$ by applying the averaging theory of second order.*
- (b) *At most 3 limit cycles bifurcate from the origin of system (0.530) when $\varepsilon = 0$ by applying the averaging theory of third order.*
- (c) *At most 10 limit cycles bifurcate from the origin of system (0.530) when $\varepsilon = 0$ by applying the averaging theory of the fourth order.*

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Decay Behavior of Solutions to Coupled Biharmonic Schrödinger Equations with Internal Fractional Damping

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Abstract. This study investigates stabilization techniques for coupled Biharmonic Schrödinger equations with fractional boundary damping. First, we establish the well-posedness of our system and demonstrate the existence and uniqueness of the solution using the semi-group theory for linear operators. Then, we demonstrate strong stability of our system. Finally, employing multiplier techniques combined with frequency domain methods, we establish a polynomial decay rate for the solutions. These findings enhance our understanding of the dynamics of coupled Biharmonic Schrödinger equations under fractional dissipation and highlight the effectiveness of our proposed methodology.

Keywords: Biharmonic Schrödinger equations, C_0 semi-group, Strong stability, general decay.



Introduction

In this study, we investigate the existence and decay properties of solutions to the biharmonic Schrödinger equation. The system under examination is expressed as follows:

$$\begin{cases} iu_t(x, t) + \Delta u(x, t) - \Delta^2 u(x, t) + (-1)^j i \gamma \partial^{\alpha, \eta} \Delta^j u(x, t) + il(u(x, t) - v(x, t)) = 0, & \text{in } \Omega \times \mathbb{R}_+, \\ iv_t(x, t) + \Delta v(x, t) - \Delta^2 v(x, t) + (-1)^j i \gamma \partial^{\alpha, \eta} \Delta^j v(x, t) + il(v(x, t) - u(x, t)) = 0, & \text{in } \Omega \times \mathbb{R}_+, \end{cases} \quad (0.158)$$

where Ω is an open bounded domain in $\mathbb{R}^n$, l , ρ_1 , ρ_2 are positive constants, and $j \in \{0, 1, 2\}$.

The term $\partial^{\alpha, \eta}$ represents the generalized Caputo fractional derivative of order α with respect to the time variable (see [1]), defined by

$$\partial^{\alpha, \eta} w(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} w(s) ds, \quad 0 < \alpha < 1, \quad \eta \geq 0, \quad (0.159)$$

where Γ denotes the Gamma function. The system is constrained by the following boundary conditions

$$u(x, t) = v(x, t) = \Delta u(x, t) = \Delta v(x, t) = 0, \quad \text{on } \partial\Omega \times \mathbb{R}_+^*. \quad (0.160)$$

The initial conditions are defined as follows

$$u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad (0.161)$$

here, the initial data u_0 and v_0 belong to a suitable function space.

Main results

*

Preliminary Results In this section*, we reformulate (0.158), (0.160), and (0.161) into an augmented system. This requires the following auxiliary results.

Theorem 0.87 (see [2]). *Let μ be the function:*

$$\mu(\xi) = |\xi|^{(2\alpha-1)/2}, \quad -\infty < \xi < +\infty, \quad 0 < \alpha < 1. \quad (0.162)$$

Then the relationship between the 'input' U and the 'output' O of the system

$$\partial_t \theta(x, \xi, t) + \xi^2 \theta(x, \xi, t) + \eta \theta(x, \xi, t) - U(t) \mu(\xi) = 0, \quad -\infty < \xi < +\infty, \quad \eta \geq 0, \quad t > 0, \quad (0.163)$$

$$\theta(x, \xi, 0) = 0, \quad (0.164)$$

$$O(t) = (\pi)^{-1} \sin(\alpha\pi) \int_{-\infty}^{+\infty} \mu(\xi) \theta(x, \xi, t) d\xi, \quad (0.165)$$

is given by

$$O(t) = I^{1-\alpha, \eta} U(t), \quad (0.166)$$



where

$$[I^{\alpha,\eta}f](t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} e^{-\eta(t-\tau)} f(\tau) d\tau.$$

Taking the input $U(x, t) = y(x, t)$ and combining (0.159) with (0.166), we obtain

$$O(t) = I^{1-\alpha,\eta}y(x, t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} y(x, s) ds = \partial^{\alpha,\eta}y(x, t).$$

By applying the previous theorem, the system (0.158), (0.160), (0.161) can be reformulated as the following augmented model:

$$\begin{cases} u_t(x, t) - i\Delta u(x, t) + i\Delta^2 u(x, t) + (-1)^j \tilde{\gamma} \int_{-\infty}^{+\infty} \mu(\xi) \Delta^j \theta(x, \xi, t) d\xi + l(u(x, t) - v(x, t)) = 0, & (x, \xi, t) \in \Omega \times \mathbb{R} \\ v_t(x, t) - i\Delta v(x, t) + i\Delta^2 v(x, t) + (-1)^j \tilde{\gamma} \int_{-\infty}^{+\infty} \mu(\xi) \Delta^j \vartheta(x, \xi, t) d\xi + l(v(x, t) - u(x, t)) = 0, & (x, \xi, t) \in \Omega \times \mathbb{R} \\ \partial_t \theta(x, \xi, t) + (\xi^2 + \eta) \theta(x, \xi, t) - \mu(\xi) u = 0, & (x, \xi, t) \in \Omega \times \mathbb{R} \\ \partial_t \vartheta(x, \xi, t) + (\xi^2 + \eta) \vartheta(x, \xi, t) - \mu(\xi) v = 0, & (x, \xi, t) \in \Omega \times \mathbb{R} \end{cases} \quad (0.167)$$

with the following initial conditions:

$$\begin{aligned} u(x, t) = v(x, t) = \Delta u(x, t) = \Delta v(x, t) &= 0, & (x, t) \in \partial\Omega \times (0, +\infty), \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), & & x \in \Omega, \\ \theta(x, \xi, 0) = \theta_0(x, \xi), \quad \vartheta(x, \xi, 0) = \vartheta_0(x, \xi), & & (x, \xi) \in \Omega \times \mathbb{R}, \end{aligned} \quad (0.168)$$

where $\tilde{\gamma} = \gamma\pi^{-1} \sin(\alpha\pi)$.

We define the energy related to the solution of the problem (0.167) as follows

$$E(t) = \frac{1}{2} \left(\|u\|_{L^2(\Omega)}^2 + \|v\|_{L^2(\Omega)}^2 \right) + \frac{\tilde{\gamma}}{2} \left(\|\Delta^{\frac{j}{2}} \theta\|_{L^2(\Omega \times \mathbb{R})}^2 + \|\Delta^{\frac{j}{2}} \vartheta\|_{L^2(\Omega \times \mathbb{R})}^2 \right). \quad (0.169)$$

Lemma 0.33. *Let $(u, v, \theta, \vartheta)$ be a regular solution of the problem (0.167). Then the functional energy defined in equation (0.169) satisfies*

$$\begin{aligned} \frac{d}{dt} E(t) &= -\tilde{\gamma} \int_{\Omega} \int_{-\infty}^{+\infty} (\xi^2 + \eta) \left(|\Delta^{\frac{j}{2}} \theta(x, \xi, t)|^2 + |\Delta^{\frac{j}{2}} \vartheta(x, \xi, t)|^2 \right) d\xi dx - l \|u - v\|_{L^2(\Omega)}^2 \\ &\leq 0. \end{aligned} \quad (0.170)$$

The following lemma will be necessary for the sequel.

Lemma 0.34 (see [3]). *If $\lambda \in D_\eta = \mathbb{C} \setminus]-\infty, -\eta]$, then*

$$\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\lambda + \eta + \xi^2} d\xi = \frac{\pi}{\sin \alpha\pi} (\lambda + \eta)^{\alpha-1}.$$

Lemma 0.35.

$$\int_{-\infty}^{+\infty} \frac{|\xi| \mu(\xi)}{(i\lambda + \xi^2 + \eta)^2} d\xi = \frac{1-2\alpha}{4} \cdot \frac{\pi}{\sin\left(\frac{(2\alpha+3)}{4}\pi\right)} (i\lambda + \eta)^{\frac{(2\alpha-5)}{4}} \quad (0.171)$$



$$\int_{-\infty}^{+\infty} \frac{1}{|i\lambda + \xi^2 + \eta|^2} d\xi \leq \frac{\pi}{2} |\lambda + \eta|^{-\frac{3}{2}} \quad (0.172)$$

$$\int_{-\infty}^{+\infty} \frac{|\xi|^2}{|i\lambda + \xi^2 + \eta|^4} d\xi \leq \frac{\pi}{16} |\lambda + \eta|^{-\frac{5}{2}} \quad (0.173)$$

*

Well Posedness In this section*, we aim to establish that system (0.167) is well-posed in the framework of semigroups. By defining the vector function $U = (u, v, \theta, \vartheta)^T$, we can express system (0.167) in an equivalent form

$$\begin{cases} \partial_t U = \mathcal{A}U, \\ U(0) = U_0 = (u_0, v_0, \theta_0, \vartheta_0)^T, \end{cases} \quad (0.174)$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A} \begin{pmatrix} u \\ v \\ \theta \\ \vartheta \end{pmatrix} = \begin{pmatrix} i\Delta u - i\Delta^2 u - (-1)^j \tilde{\gamma} \int_{-\infty}^{+\infty} \mu(\xi) \Delta^j \theta(x, \xi) d\xi - l(u(x, t) - v(x, t)) \\ i\Delta v - i\Delta^2 v - (-1)^j \tilde{\gamma} \int_{-\infty}^{+\infty} \mu(\xi) \Delta^j \vartheta(x, \xi) d\xi - l(v(x, t) - u(x, t)) \\ -(\xi^2 + \eta)\theta(x, \xi) + \mu(\xi)u \\ -(\xi^2 + \eta)\vartheta(x, \xi) + \mu(\xi)v \end{pmatrix} \quad (0.175)$$

with domain

$$D(\mathcal{A}) = \left\{ \begin{array}{l} (u, v, \theta, \vartheta) \text{ in } \mathcal{H} : u, v \in H^4(\Omega) \cap H_0^j(\Omega), \quad j = \overline{0, 2}, \quad |\xi|\theta, |\xi|\vartheta \in L^2(\Omega \times \mathbb{R}), \\ i\Delta u - i\Delta^2 u - (-1)^j \tilde{\gamma} \int_{-\infty}^{+\infty} \mu(\xi) \Delta^j \theta(x, \xi) d\xi - l(u(x, t) - v(x, t)) \in L^2(\Omega) \\ i\Delta v - i\Delta^2 v - (-1)^j \tilde{\gamma} \int_{-\infty}^{+\infty} \mu(\xi) \Delta^j \vartheta(x, \xi) d\xi - l(v(x, t) - u(x, t)) \in L^2(\Omega) \\ -(\xi^2 + \eta)\theta + u(x)\mu(\xi) \in L^2(\Omega \times \mathbb{R}), \\ -(\xi^2 + \eta)\vartheta + v(x)\mu(\xi) \in L^2(\Omega \times \mathbb{R}), \end{array} \right\} \quad (0.176)$$

where

$$\mathcal{H} = (L^2(\Omega))^2 \times \left(H_0^j(\Omega \times (-\infty, +\infty)) \right)^2, \quad j = \overline{0, 2},$$

equipped with the inner product

$$\langle U, U_1 \rangle_{\mathcal{H}} = \int_{\Omega} (u\overline{u_1} + v\overline{v_1}) dx + \tilde{\gamma} \int_{\Omega} \int_{-\infty}^{+\infty} \left(\Delta^{\frac{j}{2}} \theta \Delta^{\frac{j}{2}} \overline{\theta_1} + \Delta^{\frac{j}{2}} \vartheta \Delta^{\frac{j}{2}} \overline{\vartheta_1} \right) d\xi dx,$$

where $U = (u, v, \theta, \vartheta), U_1 = (u_1, v_1, \theta_1, \vartheta_1) \in \mathcal{H}$.

We present the following result regarding existence and uniqueness.

Theorem 0.88.

(1) If $U_0 \in D(\mathcal{A})$, then system (0.174) has a unique strong solution

$$U \in C^0(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

(2) If $U_0 \in \mathcal{H}$, then system (0.174) has a unique weak solution

$$U \in C^0(\mathbb{R}_+, \mathcal{H}).$$



Proof The proof of the theorem involves the following key steps:

- Dissipativity of $\mathcal{A}$: For any $U \in D(\mathcal{A})$, we evaluate

$$\operatorname{Re} \langle \mathcal{A}U, U \rangle_{\mathcal{H}} = 0. \quad (0.177)$$

- Surjectivity of $\lambda I - \mathcal{A}$: For $\lambda > 0$, given $F = (f_1, f_2, f_3, f_4, f_5, f_6) \in \mathcal{H}$, we show that there exists $U \in D(\mathcal{A})$ such that

$$(\lambda I - \mathcal{A})U = F. \quad (0.178)$$

- Application of the Lax-Milgram Theorem.

*

Strong stability of the system In this section*, we will investigate the strong stability of the solution related to problem (0.174) by applying a version of the Arendt-Batty and Lyubich-Vu for Hilbert spaces.

Theorem 0.89 ([1], [2]). *Let $\mathcal{A}$ be the generator of a uniformly bounded C_0 -semigroup $\{S(t)\}_{t \geq 0}$ on a Hilbert space $\mathcal{H}$. If:*

- (i) $\mathcal{A}$ does not have eigenvalues on $i\mathbb{R}$.
- (ii) The intersection* of the spectrum $\sigma(\mathcal{A})$ with $i\mathbb{R}$ is at most a countable set.

then the semigroup $\{S(t)\}_{t \geq 0}$ is asymptotically stable, i.e., $\|S(t)z\|_{\mathcal{H}} \rightarrow 0$ as $t \rightarrow \infty$ for any $z \in \mathcal{H}$.

The following theorem presents our main result

Theorem 0.90. *The C_0 -semigroup $\{S(t)\}_{t \geq 0}$ is strongly stable in $\mathcal{H}$, i.e., for all $U_0 \in \mathcal{H}$, the solution of (0.174) satisfies*

$$\lim_{t \rightarrow \infty} \|S(t)U_0\|_{\mathcal{H}} = 0. \quad (0.179)$$

To prove Theorem 0.90, we require the following lemmas.

Lemma 0.36. *For all $\lambda \in \mathbb{R}$, the operator $i\lambda I - \mathcal{A}$ is injective*

$$\ker(i\lambda I - \mathcal{A}) = \{0\}.$$

Lemma 0.37. *If $\eta > 0$ and $\lambda \in \mathbb{R}$ or $\eta = 0$ and $\lambda \in \mathbb{R}^*$, then $i\lambda I - \mathcal{A}$ is surjective.*

Lemma 0.38. *Assume that $\eta = 0$. Then, the operator $-\mathcal{A}$ is not invertible and consequently $0 \in \sigma(\mathcal{A})$.*



*

Polynomial stability In this section*, we will derive the polynomial decay rate.

Lemma 0.39. *Assume that $\mathcal{A}$ is the generator of a strongly continuous semi-group of contractions $\{S(t)\}_{t \geq 0}$ on a Hilbert space $\mathcal{H}$. If*

$$i\mathbb{R} \subset \rho(\mathcal{A}), \quad (0.180)$$

then for a fixed $\delta > 0$, the following conditions are equivalent:

$$\lim_{|s| \rightarrow \infty} \sup_{s \in \mathbb{R}} \frac{1}{|s|^\delta} \|(isI - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty, \quad (0.181)$$

$$\|S(t)U_0\|_{\mathcal{H}}^2 \leq \frac{c}{t^{\frac{\delta}{2}}} \|U_0\|_{D(\mathcal{A})}^2, \quad U_0 \in D(\mathcal{A}), \quad \text{for some } c > 0. \quad (0.182)$$

Our main result is the following theorem

Theorem 0.91. *The semigroup $\{S(t)\}_{t \geq 0}$ is polynomially stable and*

$$E(t) = \|S(t)U_0\|_{\mathcal{H}}^2 \leq \frac{1}{t^{\frac{\delta}{2}}} \|U_0\|_{D(\mathcal{A})}^2. \quad (0.183)$$

Furthermore, the energy decay rate of $t^{2/(1-\alpha)}$ is optimal for general initial data in $D(\mathcal{A})$.

For the proof, we need the following two lemmas

Lemma 0.40. *We have*

$$\int_{\Omega} \int_{-\infty}^{+\infty} (\xi^2 + \eta) \left(|\Delta^{\frac{j}{2}} \theta_n(x, \xi)|^2 + |\Delta^{\frac{j}{2}} \vartheta_n(x, \xi)|^2 \right) d\xi dx + l \|u_n - v_n\|_{L^2(\Omega)}^2 = \frac{o(1)}{\lambda_n^\delta}. \quad (0.184)$$

and

$$\int_{\Omega} \int_{-\infty}^{+\infty} \left(|\Delta^{\frac{j}{2}} \theta_n(x, \xi)|^2 + |\Delta^{\frac{j}{2}} \vartheta_n(x, \xi)|^2 \right) d\xi dx = \frac{o(1)}{\lambda_n^\delta}. \quad (0.185)$$

Lemma 0.41. *We have*

$$\int_{\Omega} |u_n(x)|^2 dx = \frac{o(1)}{\lambda_n^{\delta+\alpha-1}}, \quad (0.186)$$

and

$$\int_{\Omega} |v_n(x)|^2 dx = \frac{o(1)}{\lambda_n^{\delta+\alpha-1}}. \quad (0.187)$$



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Existence Results for First-Order Boundary Value Problems

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Abstract. This study investigates the existence of solutions for a class of first-order boundary value problems (BVPs). We establish new sufficient conditions that guarantee the existence of at least one solution, one nonnegative solution, and two distinct nontrivial nonnegative solutions. Our approach is built on recent theoretical advancements and introduces a novel methodology that extends and refines existing results. By applying these new conditions, we provide rigorous proofs for the existence of various types of solutions and address gaps left by prior research. This work not only enhances the theoretical understanding of boundary value problems but also offers practical insights for researchers and practitioners. Our results complement and expand upon recent findings, offering a broader perspective on the conditions necessary for solution existence in first-order BVPs.



Keywords: nonnegative solution, fixed point index, first order BVPs, sum of two operators.

Introduction

In this paper we investigate for the existence of solutions of the following first order differential equation

$$x' = f(t, x), \quad t \in [a, b], \quad (0.188)$$

subject to the boundary conditions

$$Mx(a) + Rx(b) = 0, \quad (0.189)$$

where $M, R \in \mathbb{R}$, $M + R \neq 0$.

$$\textbf{(H1)} \quad f \in \mathcal{C}([a, b] \times \mathbb{R}), |f(t, x)| \leq \sum_{j=1}^k a_j(t) |x|^{p_j}, \quad (t, x) \in [a, b] \times \mathbb{R}, \quad a_j \in \mathcal{C}([a, b]), \\ 0 \leq a_j \leq A \text{ on } [a, b], \quad p_j > 0, \quad j \in \{1, \dots, k\},$$

We will investigate the problem (0.470)-(0.351) for existence of solutions. As for the methods used to solve a first degree differential equation, they use on the following four entrances:seperable equations, homoeogenous method, integrating factor, exact differential equations,over years,much effort has been devoltd to these ways for studies of many problems. Motivated by the ideas in the recent works [3], we come up with a new approach to ensure the existence of solutions for certain family of first-order nonlinear differential equations boundary value conditions. Our methods involve a new fixed-point theorems for the sum of two operators.

Main results

In [3], it is proved the the BVP (0.470)-(0.351) is equivalent to the following integral equation

$$x(t) = \int_a^t f(s, x(s))ds - \frac{R}{M+R} \int_a^b f(s, x(s))ds, \quad t \in [a, b]. \quad (0.190)$$



Let $E = \mathcal{C}([a, b])$ be endowed with the maximum norm

$$\|x\| = \max_{t \in [a, b]} |x(t)|.$$

For $x \in E$, define the operator

$$S_1 x(t) = \int_a^t f(s, x(s)) ds - \frac{R}{M+R} \int_a^b f(s, x(s)) ds - x(t), \quad t \in [a, b].$$

By (0.457), it follows that if $x \in E$ satisfies the equation $S_1 x = 0$, then it is a solution to the BVP (0.470)-(0.351). Fix $B > 0$ arbitrarily.

Lemma 0.42. *Suppose that (H1) and (H2) hold. For any $x \in E$ with $\|x\| \leq B$, we have*

$$\|S_1 x\| \leq A \left(1 + \left| \frac{R}{M+R} \right| \right) (b-a) \sum_{j=1}^k B^{p_j} + B.$$

Let $g \in \mathcal{C}([a, b])$ be positive almost everywhere on $[a, b]$ and

$$C = \int_a^b g(t) dt$$

For $x \in E$, define the operator

$$S_2 x(t) = \int_a^t g(\tau) S_1 x(\tau) d\tau, \quad t \in [a, b].$$

Lemma 0.43. *Suppose (H1). If $x \in E$ satisfies the integral equation*

$$S_2 x(t) = 0, \quad t \in [a, b], \quad (0.191)$$

then x is a solution to the BVP (0.470)-(0.351).

Lemma 0.44. *Suppose that (H1) hold. Let $x \in E$ be such that $\|x\| \leq B$. Then*

$$\|S_2 x\| \leq C \left(A \left(1 + \left| \frac{R}{M+R} \right| \right) (b-a) \sum_{j=1}^k B^{p_j} + B \right).$$

*

Existence of at least one solution

Suppose that $B > 0, \epsilon > 0$ be such that

$$(H2) \quad C \left(A \left(1 + \left| \frac{R}{M+R} \right| \right) (b-a) \sum_{j=1}^k B^{p_j} + B \right) < B.$$

$$(H3) \quad \epsilon \left(B + C \left(A \left(1 + \left| \frac{R}{M+R} \right| \right) (b-a) \sum_{j=1}^k B^{p_j} + B \right) \right) \leq B.$$



In the last section*, we will give an example for the constants ϵ , a , b , M , R , A , B , C and functions f , g that satisfy (H1)-(H3).

Let $\tilde{X}$ be the set of all equi-continuous families in E . Let also,

$$X = \{x \in \tilde{X} : \|x\| \leq B\}.$$

For $x \in E$, define the operators

$$Tx(t) = -\epsilon x(t),$$

$$Sx(t) = (1 + \epsilon)x(t) + \epsilon S_2x(t), \quad t \in [a, b].$$

By Lemma 0.43, it follows that any fixed point of the operator $T+S$ is a solution to the BVP (0.470), (0.351).

Lemma 0.45. *Suppose that (H1)-(H3) hold. For $x \in X$, we have*

$$\|(I - S)x\| \leq B \quad \text{and} \quad \|((1 + \epsilon)I - S)x\| < \epsilon B.$$

Our main result in this section* is as follows.

Theorem 0.92. *Suppose that (H1)-(H3) hold. Then the BVP (0.470)-(0.351) has at least one solution in $C^1([a, b])$.*

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Pietsch factorization theorem for Lipschitz ϕ -summing operators

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Abstract. Let $\phi \in [0, \infty[\rightarrow [0, \infty[$ be a modulus function. We introduce the notion of Lipschitz ϕ -summing operator between metric spaces. We give a nonlinear version of Pietsch factorization theorem for such operators and study their basic properties.

Keywords: Lipschitz operators; Lipschitz ϕ -summing operators; Pietsch factorization theorem.



Introduction

A modulus function is a function $\phi: [0, \infty[\rightarrow [0, \infty[$ which is continuous, strictly increasing and subadditive with $\phi(0) = 0$. For Banach spaces X and Y , a bounded linear operator $T: X \rightarrow Y$ is called ϕ -summing if there exists a constant $C \geq 0$ such that

$$\sum_{i=1}^n \phi(\|T(x_i)\|) \leq C \sup_{x^* \in B_{X^*}} \sum_{i=1}^n \phi(|x^*(x_i)|)$$

for all finite set $(x_i)_{i=1}^n$ in X . This concept was first considered with the above definition by B. Rosenberger [6] and R. Khalil and W. Deeb [9], and more generally by N. Tita [12].

Our purpose is to extend this concept to a more general setting introducing the notion of Lipschitz ϕ -summing operator between pointed metric spaces. This paper contains a detailed study of the basic properties of such operators. Moreover, we establish a nonlinear version of Pietsch domination theorem for Lipschitz ϕ -summing operators, and show that the set of Lipschitz 2-summing operators and the set of Lipschitz ϕ -summing operators coincide on the class of Hilbert spaces.

Preliminaries and notation

A pointed metric space X is a metric space with a basepoint that we always will represent by 0. If X is a normed space, its basepoint will be its origin. If there is no place to confusion, we will denote by d the distance in any metric space.

Let X and Y be pointed metric spaces. Let us recall that a map $f: X \rightarrow Y$ is Lipschitz if there exists a real constant $C \geq 0$ such that $d(f(x), f(y)) \leq Cd(x, y)$ for all $x, y \in X$. The infimum of such constants C is denoted by (f) . We denote by $_0(X, Y)$ the set of all Lipschitz maps f from X into Y such that $f(0) = 0$. The elements of (X, Y) are also referred to as Lipschitz operators. If E is a real Banach space, $_0(X, E)$ is a Banach space with the Lipschitz norm. The space $_0(X, \mathbb{R})$ is known as the Lipschitz dual of X and denoted frequently by $X^\#$. It is known that $X^\#$ is a dual Banach space. We will consider the closed unit ball $B_{X^\#}$ of $X^\#$ equipped with the weak* topology. Let $C(B_{X^\#})$ be the Banach space of real-valued continuous functions on $B_{X^\#}$ with the supremum norm, and $(B_{X^\#})$ be the space of all regular Borel measures on $B_{X^\#}$.

A function $\phi: [0, \infty[\rightarrow [0, \infty[$ is said to be a modulus function if it is continuous, strictly increasing and subadditive with $\phi(0) = 0$. The most natural examples of modulus functions are $\phi(t) = t^p$ and $\phi(t) = \max\{t, t^p\}$ when $0 < p \leq 1$, but there are many others like $\phi(t) = \ln(1 + t)$ or $\phi(t) = t/(1 + t)$. Through the paper, we will use the inequality $\phi(at) \leq (1 + a)\phi(t)$ for all $t \geq 0$ when $a > 0$, which can be deduced of the properties of ϕ .

Let E and F be Banach spaces. We denote by (E, F) the Banach space of all bounded linear operators from E into F with the usual norm. As usual, E^* stands for the dual space of E and B_E for the closed unit ball of E .



Lipschitz ϕ -summing operators in metric spaces

Definition 0.51. Let ϕ be a modulus function and let X, Y be pointed metric spaces. A Lipschitz map $T \in (X, Y)$ is called Lipschitz ϕ -summing if there exists a constant $C \geq 0$ such that regardless of the choice of points $(x_i)_{i=1}^n, (x'_i)_{i=1}^n$ in X and the choice of numbers $(a_i)_{i=1}^n$ in $\mathbb{R}^+$, we have the inequality

$$\sum_{i=1}^n a_i \phi(d(T(x_i), T(x'_i))) \leq C \sup_{f \in B_{X^\#}} \sum_{i=1}^n a_i \phi(|f(x_i) - f(x'_i)|).$$

The infimum of such constants C is denoted by $\pi_\phi^L(T)$, and the set of such maps T by $\Pi_\phi^L(X, Y)$.

Remark 0.6. Definition 0.51 is the same if we restrict to $a_i = 1$ (see [4]).

Taking the modulus function $\phi(t) = t^p$ ($0 < p \leq 1$) for all $t \geq 0$, Definition 0.51 covers the notions of Lipschitz p -summing operators with $p \in (0, 1)$ introduced by M. A. S. Saleh [11], and of Lipschitz 1-summing operators given by J. D. Farmer and W. B. Johnson [4].

We now study the relation between a Lipschitz ϕ -summing operator and its linear version. Let us recall that for Banach spaces E and F , an operator $T \in (E, F)$ is ϕ -summing if there is a $C \geq 0$ such that

$$\sum_{i=1}^n \phi(\|T(x_i)\|) \leq C \sup_{x^* \in B_{E^*}} \sum_{i=1}^n \phi(|x^*(x_i)|)$$

for all $(x_i)_{i=1}^n \subset E$. The infimum of such constants C is denoted by $\pi_\phi(T)$, and the set of all such mappings T by $\Pi_\phi(E, F)$.

Proposition 0.10. Let ϕ be a modulus function, let E, F be Banach spaces and let $T: E \rightarrow F$ be a ϕ -summing bounded linear operator. Then T is Lipschitz ϕ -summing and $\pi_\phi^L(T) \leq \pi_\phi(T)$.

Proof Let $(x_i)_{i=1}^n, (x'_i)_{i=1}^n \subset E$. We have

$$\begin{aligned} \sum_{i=1}^n \phi(\|T(x_i) - T(x'_i)\|) &= \sum_{i=1}^n \phi(\|T(x_i - x'_i)\|) \leq \pi_\phi(T) \sup_{x^* \in B_{E^*}} \sum_{i=1}^n \phi(|x^*(x_i - x'_i)|) \\ &= \pi_\phi(T) \sup_{x^* \in B_{E^*}} \sum_{i=1}^n \phi(|x^*(x_i) - x^*(x'_i)|) \leq \pi_\phi(T) \sup_{f \in B_{E^\#}} \sum_{i=1}^n \phi(|f(x_i) - f(x'_i)|), \end{aligned}$$

and the proposition follows. ■

We now show that $\pi_\phi^L(X, Y)$ has the ideal property.

Proposition 0.11. Let ϕ be a modulus function, let X, Y, Z, W be pointed metric spaces, let $B \in_\phi (X, Y)$, $T \in \Pi_\phi^L(Y, Z)$ and $A \in_\phi (Z, W)$. Then $ATB \in \Pi_\phi^L(X, W)$ and $\pi_\phi^L(ATB) \leq_\phi (A)\pi_\phi^L(T)\phi(B)$.



We next state a version of Pietsch domination theorem for Lipschitz ϕ -summing operators.

Theorem 0.93. *Let ϕ be a modulus function, let X, Y be pointed metric spaces and let $T \in (X, Y)$. Then T is Lipschitz ϕ -summing if and only if there exist a constant C and a regular Borel probability measure μ on $B_{X^\#}$ such that*

$$\phi(d(T(x), T(y))) \leq C \int_{B_{X^\#}} \phi(|f(x) - f(y)|) d\mu(f)$$

for all $x, y \in X$. In such a case, $\Pi_\phi^L(T)$ is the minimum of all the constants C for which such a measure exists.

Pietsch factorization theorem for Lipschitz ϕ -summing operators

Our next objective is to formulate a version of Pietsch factorization theorem for Lipschitz ϕ -summing operators. To this end, let us recall that for any pointed metric space X , we have the isometric embeddings

$$\begin{aligned} i_X: X &\rightarrow C(B_{X^\#}), \\ J_X: X &\rightarrow \ell_\infty(X), \end{aligned}$$

defined by

$$\begin{aligned} i_X(x)(f) &= f(x) \quad (f \in B_{X^\#}), \\ J_X(x)(y) &= d(x, y) - d(0, y) \quad (y \in X), \end{aligned}$$

and the canonical injection $j_\infty: C(B_{X^\#}) \rightarrow L_\infty(\mu)$ where $\mu \in (B_{X^\#})$. We also need to introduce the following spaces of functions which is closely related to Orlicz spaces.

Definition 0.52. Let $\mu \in (B_{X^\#})$ with $\mu(B_{X^\#}) = 1$ and let ϕ be a modulus function such that $\phi(1) \leq 1$. We denote by $L_\phi(\mu)$ the space of all (μ -equivalence classes of) measurable functions $f: B_{X^\#} \rightarrow \mathbb{R}$ such that

$$\int_{B_{X^\#}} \phi\left(\frac{|f|}{\rho}\right) d\mu < \infty$$

for some $\rho > 0$. For each $f \in L_\phi(\mu)$, define the norm:

$$\|f\|_\phi = \inf \left\{ \rho > 0: \int_{B_{X^\#}} \phi\left(\frac{|f|}{\rho}\right) d\mu \leq 1 \right\}.$$

It is easy to prove that $L_\infty(\mu) \subset L_\phi(\mu)$ and $\|f\|_\phi \leq \|f\|$ for all $f \in L_\infty(\mu)$.



Proposition 0.12. *Let $\mu \in (B_{X^\#})$ with $\mu(B_{X^\#}) = 1$ and let ϕ be a modulus function such that $\phi(1) \leq 1$. Then the canonical injection $I_{\infty, \phi}: L_\infty(\mu) \rightarrow L_\phi(\mu)$ is Lipschitz ϕ -summing and $\pi_\phi^L(I_{\infty, \phi}) \leq 1$.*

Proof For any $(f_i)_{i=1}^n, (f'_i)_{i=1}^n$ in $L_\infty(\mu)$, we have

$$\begin{aligned} \sum_{i=1}^n \phi(\|f_i - f'_i\|_\phi) &= \sum_{i=1}^n \int_{B_{X^\#}} \phi(\|f_i - f'_i\|_\phi) d\mu \\ &\leq \sum_{i=1}^n \int_{B_{X^\#}} \phi(\|f_i - f'_i\|) d\mu \\ &= \int_{B_{X^\#}} \left(\sum_{i=1}^n \phi(\|f_i - f'_i\|) \right) d\mu \\ &\leq \sup_{g \in B_{L_\infty(\mu)^\#}} \sum_{i=1}^n \phi(|g(f_i) - g(f'_i)|). \end{aligned}$$

Hence $I_{\infty, \phi} \in \Pi_\phi^L(X, Y)$ and $\pi_\phi^L(I_{\infty, \phi}) \leq 1$. ■

Theorem 0.94. *Let ϕ be a modulus function such that $\phi(1) \leq 1$, let X, Y be pointed metric spaces and let $T \in (X, Y)$. Then T is Lipschitz ϕ -summing if and only if there exist a regular Borel probability measure μ on $B_{X^\#}$ together with operators $A \in_\phi (X, L_\infty(\mu))$ and $B \in_\phi (L_\phi(\mu), \ell_\infty(Y))$ such that the following diagram commutes:*

(Linf)

$L_\infty(\mu); (X)$ [above of=Linf] $X; (Y)$ [right of=X] $Y; (li)$ [right of=Y] $\ell_\infty(Y);$
(Lphi) [below of=li] $L_\phi(\mu); [->]$ (X) to node T $(Y); [->]$ (Y) to node J_Y $(li);$
 $[->]$ (X) to node [swap] A (Linf); $[->]$ (Linf) to node $I_{\infty, \phi}$ (Lphi); $[->]$ (Lphi)
to node [swap] B (li);

In this case,

$\Pi_\phi^L(T) = \inf_\phi(A)_\phi(B)$, where this infimum is taken over all such factorizations, and this infimum is attained.

Proof If $T \in \pi_\phi^L(X, Y)$, then Theorem 0.93 gives a probability measure $\mu \in (B_{X^\#})$ such that

$$\phi(d(T(x), T(y))) \leq \Pi_\phi^L(T) \int_{B_{X^\#}} \phi(|f(x) - f(y)|) d\mu(f)$$

for all $x, y \in X$. Consider the composition of canonical isometric embeddings:

$$A = j_\infty i_X: X \xrightarrow{i_X} C(B_{X^\#}) \xrightarrow{j_\infty} L_\infty(\mu).$$

Hence $A \in_\phi (X, L_\infty(\mu))$ with $\phi(A) = 1$. Now denote $M = I_{\infty, \phi} A(X) \subset L_\phi(\mu)$ and define a map $b: M \rightarrow \ell_\infty(Y)$ by $b(I_{\infty, \phi} A(x)) = J_Y T(x)$ for every $x \in X$. Then $b \in_\phi (M, \ell_\infty(Y))$ and $\phi(b) \leq \pi_\phi^L(T)$. By [3, Lemma 1.1. (ii)], b can be



extended to a function $B \in_\phi (L_\phi(\mu), \ell_\infty(Y))$ with $\phi(B) = \phi(b)$. Then $J_Y T = BI_{\infty, \phi} A$ and $\phi(A)_\phi(B) \leq \pi_\phi^L(T)$.

Conversely, suppose that there are a probability measure $\mu \in (B_{X^\#})$ and operators $A \in_\phi (X, L_\infty(\mu))$ and $B \in_\phi (L_\phi(\mu), \ell_\infty(Y))$ such that $J_Y T = BI_{\infty, \phi} A$. Since $I_{\infty, \phi}$ is Lipschitz ϕ -summing by Proposition 0.12, it follows from Proposition 0.11 that $J_Y T$ is Lipschitz ϕ -summing. Hence T is Lipschitz ϕ -summing by Proposition ???. Also,

$$\pi_\phi^L(T) = \pi_\phi^L(J_Y T) = \pi_\phi^L(BI_{\infty, \phi} A) \leq_\phi (A) \pi_\phi^L(I_{\infty, \phi})_\phi(B) \leq_\phi (A)_\phi(B),$$

and this completes the proof. ■

Remark 0.7. Let us recall a metric space Y is Lipschitz 1-injective if for every metric space Z , every subset X of Z and every mapping $f \in (X, Y)$, there is a mapping $\tilde{f} \in (Z, Y)$ extending f with $(\tilde{f}) = (f)$. The real Banach spaces $\ell_\infty(S)$ for a set S and $L_\infty(\mu)$ for a finite measure μ are Lipschitz 1-injective (see Lemma 1.1 and 5. Notes and remarks (p. 31-32) in [3]).

In the statement of Theorem 0.94, one can replace $J_Y: Y \rightarrow \ell_\infty(Y)$ by any isometric embedding of Y into any Lipschitz 1-injective pointed metric space.

The Lipschitz-free Banach space (X) over a pointed metric space X is the closed linear span in $(X^\#)^*$ of the evaluation functionals $\delta_x: X^\# \rightarrow \mathbb{R}$ with $x \in X$, where $\delta_x(f) = f(x)$ for all $f \in X^\#$. This space was called and denoted so by G. Godefroy and N. J. Kalton in [5]. It is known that the Dirac map $\delta_X: X \rightarrow (X)$, given by $\delta_X(x) = \delta_x$, is a (nonlinear) isometry, and $(X)^*$ is isometrically isomorphic to $X^\#$ via the evaluation map $Q_X: X^\# \rightarrow (X)^*$ given by $Q_X(f)(\gamma) = \gamma(f)$ for all $f \in X^\#$ and $\gamma \in (X)$.

Let us recall (see [7, Lemma 3.1]) that if X and Y are pointed metric spaces and $T \in (X, Y)$, then there exists a unique operator $\hat{T} \in ((X), (Y))$ such that $\hat{T}\delta_X = \delta_Y T$. Furthermore, $\|\hat{T}\| = (T)$.

We now study the relationship of a Lipschitz ϕ -summing operator T in (X, Y) and its linearization $\hat{T}$ in $((X), (Y))$.

Proposition 0.13. *Let ϕ be a modulus function, let X, Y be pointed metric spaces and let $T \in (X, Y)$. If $\hat{T}$ is ϕ -summing, then T is Lipschitz ϕ -summing and $\pi_\phi^L(T) \leq \pi_\phi(\hat{T})$.*

Proof Notice first that

$$d(T(x), T(y)) = \|\delta_{T(x)} - \delta_{T(y)}\| = \|\hat{T}(\delta_x) - \hat{T}(\delta_y)\| = \|\hat{T}(\delta_x - \delta_y)\|$$

for all $x, y \in X$. If $\hat{T}$ is ϕ -summing, we have

$$\sum_{i=1}^n \phi(\|\hat{T}(\gamma_i)\|) \leq \pi_p(\hat{T}) \sup_{F \in B_{(X)^*}} \sum_{i=1}^n \phi(|F(\gamma_i)|)$$



for any set $(\gamma_i)_{i=1}^n \subset (X)$. Then

$$\begin{aligned} \sum_{i=1}^n \phi(d(T(x_i), T(y_i))) &\leq \pi_p(\widehat{T}) \sup_{F \in B_{(X)}^*} \sum_{i=1}^n \phi(|F(\delta_{x_i} - \delta_{y_i})|) \\ &= \pi_p(\widehat{T}) \sup_{f \in B_{X^\#}} \sum_{i=1}^n \phi(|Q_X(f)(\delta_{x_i} - \delta_{y_i})|) \\ &= \pi_p(\widehat{T}) \sup_{f \in B_{X^\#}} \sum_{i=1}^n \phi(|f(x_i) - f(y_i)|). \end{aligned}$$

Consequently, T is Lipschitz ϕ -summing and $\pi_\phi^L(T) \leq \pi_\phi(\widehat{T})$. ■

According [1, Definition 2.1], given a pointed metric space X and a Banach space E , it is said that a set $(X, E) \subset (X, E)$ is a Lipschitz operator ideal if it satisfies the following conditions:

1. (X, E) is a linear subspace of (X, E) .
2. $g \cdot e \in (X, E)$ for all $g \in X^\#$ and $e \in E$, where $g \cdot e: X \rightarrow E$ is the mapping defined by $(g \cdot e) = g(x)e$ for all $x \in X$.
3. Ideal property: If Y is a pointed metric space, F is a Banach space, $S \in (Y, X)$, $T \in (X, E)$ and $w \in (E, F)$, then $wTS \in (Y, F)$.

Proposition 0.14. *Let ϕ be a modulus function, let X be a pointed metric space and let E be a Banach space. Then $\Pi_\phi^L(X, E)$ is a Lipschitz operator ideal.*

Proof Let $T, T_1, T_2 \in \Pi_\phi^L(X, E)$ and $\alpha \in \mathbb{R}$. Given $(x_i)_{i=1}^n, (x'_i)_{i=1}^n$ in X , we obtain

$$\begin{aligned} \sum_{i=1}^n \phi(\|\alpha T(x_i) - \alpha T(x'_i)\|) &= \sum_{i=1}^n \phi(|\alpha| \|T(x_i) - T(x'_i)\|) \\ &\leq (1 + |\alpha|) \sum_{i=1}^n \phi(\|T(x_i) - T(x'_i)\|) \\ &\leq (1 + |\alpha|) \pi_\phi^L(T) \sup_{f \in B_{X^\#}} \sum_{i=1}^n \phi(|f(x_i) - f(x'_i)|), \end{aligned}$$



and so $\alpha T \in \Pi_\phi^L(X, E)$ and $\pi_\phi^L(\alpha T) \leq (1 + |\alpha|)\pi_\phi^L(T)$. Moreover, we have

$$\begin{aligned} & \sum_{i=1}^n \phi(\|(T_1 + T_2)(x_i) - (T_1 + T_2)(x'_i)\|) \\ & \leq \sum_{i=1}^n \phi(\|T_1(x_i) - T_1(x'_i)\|) + \sum_{i=1}^n \phi(\|T_2(x_i) - T_2(x'_i)\|) \\ & \leq \pi_\phi^L(T_1) \sup_{f \in B_{X^\#}} \sum_{i=1}^n \phi(|f(x_i) - f(x'_i)|) + \pi_\phi^L(T_2) \sup_{f \in B_{X^\#}} \sum_{i=1}^n \phi(|f(x_i) - f(x'_i)|) \\ & = (\pi_\phi^L(T_1) + \pi_\phi^L(T_2)) \sup_{f \in B_{X^\#}} \sum_{i=1}^n \phi(|f(x_i) - f(x'_i)|), \end{aligned}$$

and thus $T_1 + T_2 \in \Pi_\phi^L(X, E)$ and $\pi_\phi^L(T_1 + T_2) \leq \pi_\phi^L(T_1) + \pi_\phi^L(T_2)$. Therefore $\Pi_\phi^L(X, E)$ is a linear subspace of (X, E) .

Now, let $g \in X^\#$ and $e \in E$. For all $(x_i)_{i=1}^n, (x'_i)_{i=1}^n \subset X$, we have

$$\sum_{i=1}^n \phi(\|g(x_i)e - g(x'_i)e\|) = \sum_{i=1}^n \phi(|g(x_i) - g(x'_i)| \|e\|) \leq (1 + \|e\|)(1 + (g)) \sup_{f \in B_{X^\#}} \sum_{i=1}^n \phi(|f(x_i) - f(x'_i)|),$$

and therefore $g \cdot e \in \Pi_\phi^L(X, E)$ and $\pi_\phi^L(g \cdot e) \leq (1 + (g))(1 + \|e\|)$.

Finally, the ideal property of $\Pi_\phi^L(X, E)$ follows from Proposition 0.11. ■

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Positive solutions results for some nonlocal elliptic problem

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Abstract. We prove the existence of a positive solution to a one-dimensional nonlocal elliptic problem under weak conditions on the reaction terms and the diffusion coefficients. We use Krasnoselskii's fixed point theorem.

Keywords: Nonlocal elliptic problem. Krasnoselskii's fixed point theorem. Reaction term. Diffusion coefficient.

Introduction

Nonlocal elliptic problems are involved in the modeling of several physical and biological systems. Such problems have attracted many researchers over



the years. The authors of [1], [2] have used Krasnoselskii's fixed point theorem (see [[3], Theorem 2.3.4]) to prove the existence of a positive solution to

$$\begin{cases} -\left(\int_0^1 g(u)d\tau\right)^\beta u'' = cg(u)^\alpha & t \in (0,1), \\ u'(0) = u(1) = 0, \end{cases}$$

and

$$\begin{cases} -A\left(\int_0^1 |u|^p d\tau\right)u'' = h(t)g(u), & t \in (0,1), \\ u'(0) = u(1) = 0. \end{cases}$$

we extend results obtained in [1] and [2] by imposing weaker conditions on the reaction term and the diffusion coefficient. Then, we study the existence of a positive solution to the problem

$$\begin{cases} -M\left(t, \int_0^1 f(u)d\tau\right)u''(t) = g(t, u), & t \in (0,1); \\ u'(0) = u(1) = 0. \end{cases} \quad (0.192)$$

Main results

To establish the existence result, we make some assumptions on f, g and M .

- $f : [0, \infty) \rightarrow [0, \infty)$ is continuous function satisfying for some two nondecreasing continuous functions $\alpha, \beta : [0, \infty) \rightarrow [0, \infty)$,

$$\alpha(s) \leq f(s) \leq \beta(s). \quad (0.193)$$

- $g : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$ is continuous function,

$$\xi(t)h(s) \leq g(t, s) \leq \eta(t)k(s), \quad (0.194)$$

while $h, k : [0, \infty) \rightarrow [0, \infty)$ are increasing continuous functions, and the measurable functions $\xi, \eta : [0, 1] \rightarrow [0, \infty)$. Also $g(., 0) \neq 0$.

- $M : [0, 1] \times [0, \infty) \rightarrow (0, \infty)$ is continuous function,

$$\theta(t)v(s) \leq M(t, s) \leq \mu(t)w(s), \quad (0.195)$$

where $v, w : [0, \infty) \rightarrow (0, \infty)$ are increasing continuous functions, and $\theta, \mu : [0, 1] \rightarrow (0, \infty)$ are measurable functions. In addition

$$0 < \int_0^1 \frac{\eta(s)}{\theta(s)} ds < \infty, \quad (0.196)$$

and

$$0 < \int_0^d G(0, s) \frac{\xi(s)}{\mu(s)} ds < \infty, \quad (0.197)$$



with $d \in (0, 1)$ and G denoting the Green function,

$$G(t, s) = \begin{cases} 1 - t, & 0 \leq s \leq t \leq 1; \\ 1 - s, & 0 \leq t \leq s \leq 1. \end{cases} \quad (0.198)$$

Moreover, suppose that

$$\exists R_1 > 0 : \quad \inf_{0 < s \leq R_1} \frac{h((1-d)s)}{sw(\beta(s))} \geq \frac{1}{\int_0^d G(0, s) \frac{\xi(s)}{\mu(s)} ds}. \quad (0.199)$$

$$\exists R_2 > R_1 : \quad \sup_{s \geq R_2} \frac{k(s)}{sv(d\alpha(1-d)s)} \leq \frac{1}{\int_0^1 \frac{\eta(s)}{\theta(s)} ds}. \quad (0.200)$$

Theorem 0.95.

Assume that the conditions (0.241)-(0.200) are verified. Then, the problem (0.240) has a positive solution.

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Nonlinear $p(x)$ -Laplacian Elliptic Equations with Two Singularities and L^1 - Data

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Abstract.

The present paper is concerned with the study of a nonlinear elliptic equation which contains a lower order term, a singular term and a superlinear growth in the gradient. Our approach is based on approximating the initial problem with a non-singular one that is well posed, then we establish the necessary estimates to pass to the limit.

Keywords: Variable exponents, Lower order terms, Gradient term with superlinear growth, Singular convection term.

Introduction

Let Ω be an open, bounded subset of $\mathbb{R}^N$ ($N > 2$), with Lipschitz boundary $\partial\Omega$, and $p \in \mathcal{C}(\overline{\Omega})$, such that $p^+ = \max_{x \in \overline{\Omega}} p(x)$, $p^- = \min_{x \in \overline{\Omega}} p(x)$, and

$$1 \leq p^- \leq p(x) \leq p^+ < N, \quad \forall x \in \overline{\Omega}. \quad (0.201)$$



We will consider the following nonlinear elliptic problem

$$\begin{cases} -\operatorname{div} \left(a(x) |\nabla u|^{p(x)-2} \nabla u \right) + b(x) |u|^{p(x)-1} u = k(x) \frac{|\nabla u|^{q(x)}}{u^{\theta(x)}} + \frac{f}{u^{\gamma(x)}} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (0.202)$$

where f is a nonnegative function in $L^1(\Omega)$, and the functions $q, \theta, \gamma \in \mathcal{C}(\overline{\Omega})$ satisfy the following conditions:

If $\theta(x) = \theta$ (a constant)

$$\begin{cases} 0 < \gamma(x) \leq \theta < 1 \quad \forall x \in \overline{\Omega}, \quad |\nabla \gamma| \in L^\infty(\Omega) & , \\ p(x) - 1 \leq q(x) < \frac{p(x)(p(x) + \gamma(x))}{p(x) + 1}, \quad \forall x \in \overline{\Omega} & . \end{cases} \quad (0.203)$$

If $\gamma(x) = \gamma$ (a constant)

$$\begin{cases} 0 < \theta(x) \leq \gamma < 1 \quad \forall x \in \overline{\Omega}, \quad |\nabla \theta| \in L^\infty(\Omega) & , \\ p(x) - 1 \leq q(x) < \frac{p(x)(p(x) + \theta(x))}{p(x) + 1}, \quad \forall x \in \overline{\Omega} & , \end{cases} \quad (0.204)$$

as well as for some positive numbers α, β, μ and ν , the measurable functions a, b and k satisfy:

$$0 < \alpha \leq a(x) \leq \beta, \quad 0 < \mu \leq b(x) \leq \nu, \quad 0 < k(x) \in L^\infty(\Omega), \quad (0.205)$$

In the last years, problem (0.470) has been widely applied in various scientific fields such as electro-rheological fluids, image processing ... ect. For this reason this type of equations has received a significant amount of attention. To illustrate the originality of this paper, let us discuss some recent research that address problem (0.470). Firstly, we have the constant case, where $p(x) = p$, $\gamma(x) = \gamma$ and $\theta(x) = \theta$ which corresponds to a constant values of the exponent, the existence of weak solutions for the nonlinear elliptic equation with two singularities

$$\begin{cases} -\operatorname{div}(a(x, \nabla u)) + \mu |u|^{p-1} u = b(x) \frac{|\nabla u|^q}{u^\theta} + \frac{f}{u^\gamma} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (0.206)$$

where $f \in L^1(\Omega)$, $0 < \theta \leq 1$ and $0 \leq \gamma < 1$ is obtained in [11]. Then for $\theta, \gamma \geq 0$, $\mu = 0$, $b(x) = 1$, $q = p$, and the nonnegative function f belongs to $L^1(\Omega)$, the problem as (0.206) has been studied in [6] in which they established the existence and uniqueness of nonnegative solutions.

Finally, for the variable exponents case, the authors in [2, 3] have proved the existence and regularity of weak solutions to problem (0.470) with degenerate coercivity, where $b(x) = 0$, $k(x) = B < 0$, $\gamma(x) = 0$ and f is a nonnegative function belonging to $L^{m(\cdot)}(\Omega)$, $m(x) \geq 1$. In [1], the existence and regularity results for the problem (0.470) with $k(x) = 0$ and $0 \leq f \in L^1(\Omega)$ has been proved.



Main results

Our main results are as follows

*

In the case that $\theta(x) = \theta$

Theorem 0.96. *Let $f \in L^1(\Omega)$ and let $\gamma, p, q \in C(\bar{\Omega})$ satisfy conditions (0.201), (0.203) and (0.458). Then there exists a solution u to (0.470), in the sense that $u \in W_0^{1,\eta(\cdot)}(\Omega) \cap L^{p(\cdot)+\gamma(\cdot)}(\Omega)$, satisfies*

$$\forall K \subset\subset \Omega, \exists C_K > 0 : u \geq C_K \text{ a.e. in } K, \quad (0.207)$$

$$\frac{|\nabla u|^{q(x)}}{u^\theta} \in L_{loc}^1(\Omega), \quad (0.208)$$

and that

$$\begin{aligned} & \int_{\Omega} a(x) |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \varphi dx + \int_{\Omega} b(x) u^{p(x)} \varphi dx \\ &= \int_{\Omega} k(x) \frac{|\nabla u|^{q(x)}}{u^\theta} \varphi dx + \int_{\Omega} \frac{f}{u^{\gamma(x)}} \varphi dx, \quad \forall \varphi \in C_c^1(\Omega), \end{aligned} \quad (0.209)$$

where $\eta : \bar{\Omega} \rightarrow]1, +\infty)$ is measurable function, such that

$$1 < \eta(x) < \frac{p(x)(p(x) + \gamma(x))}{p(x) + 1}, \quad \forall x \in \bar{\Omega}. \quad (0.210)$$

*

In the case that $\gamma(x) = \gamma$

Theorem 0.97. *Let $f \in L^1(\Omega)$ and let $\theta, p, q \in C(\bar{\Omega})$ satisfy conditions (0.201), (0.204) and (0.458). Then there exists a solution u to (0.470), in the sense that $u \in W_0^{1,\eta(\cdot)}(\Omega) \cap L^{p(\cdot)+\gamma(\cdot)}(\Omega)$ and satisfies*

$$\forall K \subset\subset \Omega, \exists C_K > 0 : u \geq C_K \text{ in } K, \quad (0.211)$$

$$\frac{|\nabla u|^{q(x)}}{u^{\theta(x)}} \in L_{loc}^1(\Omega), \quad (0.212)$$

and that

$$\begin{aligned} & \int_{\Omega} a(x) |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \varphi dx + \int_{\Omega} b(x) u^{p(x)} \varphi dx \\ &= \int_{\Omega} k(x) \frac{|\nabla u|^{q(x)}}{u^{\theta(x)}} \varphi dx + \int_{\Omega} \frac{f}{u^\gamma} \varphi dx, \quad \forall \varphi \in C_c^1(\Omega), \end{aligned} \quad (0.213)$$

such that

$$1 < r(x) < \frac{p(x)(p(x) + \theta(x))}{p(x) + 1}, \quad \forall x \in \bar{\Omega}. \quad (0.214)$$



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Orthogonal polynomials corresponding to polynomial Szegő measure

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Abstract. The asymptotics behavior orthogonal polynomials have been in the spotlight since the result of G. Szegő in 1921. In this paper we study the pointwise asymptotics inside the unit disk for orthogonal polynomials with respect to a polynomial Szegő measure with an infinite masses points.

Keywords: Orthogonal Polynomials, Asymptotics, Polynomial Szegő class, Szegő condition, infinite masses points, Hardy space.



Introduction

Consider the measure v on $\mathbb{T} \cup \{z_k\}_{k=1}^{\infty}$, z_k are fixed points outside $\mathbb{D}$

$$v = \mu + \sum_{k=1}^{\infty} A_k \delta(z - z_k),$$

where the masses A_k satisfy

$$A_k > 0, \sum_{k=1}^{\infty} A_k < \infty \text{ and } \sum_{k=1}^{\infty} (|z_k| - 1) < +\infty, \quad (0.215)$$

for $k = 1, \dots$ and $\delta(z - z_k)$ is the Dirac measure supported at the point z_k .

Note that the system $\{\Phi_k\}_{k \in \mathbb{N}}$ satisfy the following orthonormality relations:

$$\Phi_n(z) = \gamma_n z^n + \dots (\gamma_n > 0),$$

$$\frac{1}{2\pi} \int_0^{2\pi} \Phi_n(z) \overline{\Phi_m(z)} d\mu(\theta) + \sum_{k=1}^{\infty} A_k \Phi_n(z_k) \overline{\Phi_m(z_k)} = \delta_{mn}. \quad (0.216)$$

$$m, n, \dots, z = e^{i\theta}.$$

The purpose of this paper is to study the pointwise asymptotics inside the unit disk for orthogonal polynomials with respect to a measure from polynomial Szegő class and perturbed by an infinite masses points outside the unit circle. A similar study has been done by Khaldi and R. Benzine in ([3]) where μ is a positive measure on the unit circle satisfying the Szegő condition with an infinite discrete part. To get the asymptotic formula of $\Phi_n(z)$ they have proved two intermediate results.

Main results

We say that μ is a Szegő measure (notation: $\mu \in (S)$) if its singular part μ_s is arbitrary and

$$\int_{\mathbb{T}} \log \mu'_{ac} dm > -\infty$$

where μ'_{ac} is the density of the absolutely continuous part of μ and $dm = dm(t) = d\theta/(2\pi)$, $t = e^{i\theta} \in \mathbb{T}$ is the normalized Lebesgue measure on $\mathbb{T}$.

Now we define Szegő polynomial class (Sp) as follow: Let p be a trigonometric polynomial such that $p(t) \geq 0$, $t \in \mathbb{T}$. Without loss of generality we can assume that

$$p(t) = \prod_{k=1}^N (t - \xi_k)^{2m_k}$$



where ξ_k are points on $\mathbb{T}$ and $m_k > 0$ are their multiplicities. We say that a measure μ belongs to the polynomial Szegő class (denoted by $\mu \in (pS)$) if

$$\int_0^{2\pi} p(e^{i\theta}) \log \mu'_{ac}(e^{i\theta}) d\theta > -\infty,$$

We say that μ belongs to the Erdős class ($\mu \in (E)$) if $\mu'_{ac} > 0$ a.e. on $\mathbb{T}$.

$$S \subset pS \subset E$$

.

*

The space $H^2(G, \tilde{p})$ In what follows, we suppose that the absolutely continuous part of μ with respect to the Lebesgue measure m satisfies the generalized Szegő condition:

Let $\mu \in (pS)$ and the function $\tilde{D}$ be defined in (3), then

Lemma 0.46. *Let $\mu \in (pS)$ and the function $\tilde{D}$ be defined in (3), then*

1. $\tilde{D}(z) \in H^2(D)$,
2. $\tilde{D}(z) \neq 0$ for $|z| < 1$,
3. $|\tilde{D}(t)|^2 = \mu'_{ac}(t)$ a.e. on T ,
4. $\tilde{D}(0) > 0$.

These are the main results of this paper.

Theorem 0.98. *let $v = \mu + \sum_{k=1}^{\infty} A_k \delta(z - z_k)$ such that μ is polynomial Szegő on the unit circle,*

$$\sum_{k=1}^{\infty} A_k < +\infty, \quad A_k > 0,$$

$$\sum_{k=1}^{\infty} (|z_k| - 1) < +\infty, \quad |z_k| > 1.$$

Then

$$\lim_{l \rightarrow \infty} m_n(v_l) = m_n(v), \quad (0.217)$$

$$\lim_{l \rightarrow \infty} \gamma_n^l = \gamma_n. \quad (0.218)$$



Proof The extremal property of $\frac{1}{\gamma_n}\psi_n$ and the fact that $A_k > 0$ implies that

$$m_n(v_l) \leq \frac{1}{2\pi} \int_0^{2\pi} |\Phi_n(e^{i\theta})|^2 d\mu(\theta) + \sum_{k=1}^l A_k |\Phi_n(z_k)|^2 \leq m_n(v), \text{ and}$$

thus

$$m_n(v_l) \leq m_n(v). \quad (0.219)$$

On the other hand, the extremal property of $\frac{1}{\gamma_n}\Phi_n$ implies that

$$m_n(v) \leq \frac{1}{2\pi} \int_0^{2\pi} |\psi_n(e^{i\theta})|^2 d\mu(\theta) + \sum_{k=1}^{\infty} A_k |\psi_n(z_k)|^2 \quad (0.220)$$

$$= m_n(v_l) + \sum_{k=l+1}^{\infty} A_k |\psi_n(z_k)|^2. \quad (0.221)$$

According to the reproducing property of the kernel function $K_n(\xi, z)$ (see [13]), and $\psi_n(z) \in P_n$, we have

$$\psi_n(z_k) = \frac{1}{2\pi} \int_0^{2\pi} \psi_n(e^{i\theta}) \overline{K_n(\xi, z_k)} d\mu(\theta) \quad (0.222)$$

The Scharwz inequality implies that

$$\psi_n(z_k) = \frac{1}{2\pi} \int_0^{2\pi} \psi_n(e^{i\theta}) \overline{K_n(\xi, z_k)} d\mu(\theta) \quad (0.223)$$

$$\leq m_n(v_l) \cdot \frac{1}{2\pi} \int_0^{2\pi} |K_n(\xi, z_k)|^2 d\mu(\theta) \quad (0.224)$$

and the fact that $K_n(\xi, z_k) \in P_n$

$$|\psi_n(z_k)|^2 \leq m_n(v_l) \cdot K_n(z_k, z_k) \quad (0.225)$$

The inequalities (0.215), (0.221), and (0.225) imply

$$m_n(v) \leq m_n(v_l) + \sum_{k=l+1}^{\infty} A_k m_n(v_l) \cdot K_n(z_k, z_k) \quad (0.226)$$

$$= m_n(v_l) \left[1 + \sup_{k \geq l+1} K_n(z_k, z_k) \sum_{k=l+1}^{\infty} A_k \right]. \quad (0.227)$$

so we get

$$\frac{m_n(v)}{m_n(v_l)} \leq 1 + \delta_l \text{ where } \delta_l \rightarrow 0, l \rightarrow \infty. \quad (0.228)$$

Using (0.219) and (0.228), we obtain (0.217). Thus, we obtain (0.218) and the theorem is proved. ■



Theorem 0.99. *let $v = \mu + \sum_{k=1}^{\infty} A_k \delta(z - z_k)$ such that μ is polynomial Szegő measure on the unit circle and;*

label=()

$$\sum_{k=1}^{\infty} A_k < +\infty, \quad A_k > 0, \quad (0.229)$$

lbbel=()

$$\sum_{k=1}^{\infty} (|z_k| - 1) < +\infty, \quad |z_k| > 1, \quad (0.230)$$

lcbel=()

$$\frac{m_n(v_l)}{m_n(\mu)} \leq \left(\prod_{k=1}^l |z_k|^2 \right) \quad (0.231)$$

Then

$$\lim_{n \rightarrow \infty} m_n(v) = \hat{\mu}(v). \quad (0.232)$$

Proof By passing to the limit when l tends to infinity and using theorem (0.98) and the inequality (0.231), we obtain

$$m_n(v) \leq \left(\prod_{k=1}^{\infty} |z_k|^2 \right) m_n(\mu).$$

This implies, by lemma(??) , that

$$\limsup_{n \rightarrow \infty} m_n(v) \leq \left(\prod_{k=1}^{\infty} |z_k|^2 \right) \mu(\tilde{p}) = \hat{\mu}(v).$$

On the other hand, we can find function L in $H^2(G, \tilde{p})$, (see [3],[10]), were

$$\hat{\mu}(v) \leq \|L(z)\|_{H^2(G, \tilde{p})}^2 \leq \liminf_{n \rightarrow \infty} m_n(v),$$

$$\hat{\mu}(v) \leq \liminf_{n \rightarrow \infty} m_n(v).$$

The theorem is then proved. ■

Theorem 0.100. *Let $v = \mu + \sum_{k=1}^{\infty} A_k \delta(z - z_k)$ with μ as in the preceding theorem and such that*

$$\sum_{k=1}^{\infty} A_k < +\infty, \quad A_k > 0, \quad \sum_{k=1}^{\infty} (|z_k| - 1) < +\infty \text{ for } |z_k| > 1,$$



$$\frac{k_n}{\gamma_n^l} \leq \prod_{k=1}^l |z_k|.$$

Let $\{\Phi_n\}_{n=1}^\infty$ be the system of orthonormal polynomials associated to v satisfying relations (0.216). Then

$$\lim_{n \rightarrow \infty} \left\| \frac{1}{\gamma_n} \Phi_n - \varphi^\infty \right\|_{H^2(G, \tilde{p})}^2 = 0 \quad (0.233)$$

Proof Put

$$H_n = \frac{1}{2} \left(\frac{1}{\gamma_n} \Phi_n + \varphi^\infty \right).$$

Then

$$\lim_{n \rightarrow \infty} H_n(\infty) = 1, \quad H_n(z_k) = 0, \quad k = 1, \dots$$

This implies that

$$\liminf_{n \rightarrow \infty} \|H_n\|_{H^2(G, \tilde{p})}^2 \leq \hat{\mu}(v)$$

Using the parallelogram law in $H^2(G, \tilde{p})$, we obtain

$$\left\| \frac{1}{\gamma_n} \Phi_n - \varphi^\infty \right\|_{H^2(G, \tilde{p})}^2 = 2 \left(\left\| \frac{1}{\gamma_n} \Phi_n \right\|_{H^2(G, \tilde{p})}^2 + \|\varphi^\infty\|_{H^2(G, \tilde{p})}^2 \right) - 4 \|H_n\|_{H^2(G, \tilde{p})}^2$$

Hence

$$\limsup_{n \rightarrow \infty} \|H_n\|_{H^2(G, \tilde{p})}^2 \leq 2(\hat{\mu}(v) + \hat{\mu}(v)) - 4\hat{\mu}(v) = 0$$

so that

$$\lim_{n \rightarrow \infty} \left\| \frac{1}{\gamma_n} \Phi_n - \varphi^\infty \right\|_{H^2(G, \tilde{p})}^2 = 0$$

■

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Existence and non-existence results with an Hardy-Leray potential

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Abstract. We will present the existence and non-existence results of the following cooperative Lane -Emden system with singular Hardy potential

$$(GS) \begin{cases} (-\Delta)u = \lambda_1 \frac{v}{|x|^2} + f(x, v, \nabla v) & \text{in } \Omega, \\ (-\Delta)v = \lambda_2 \frac{u}{|x|^2} + g(x, u, \nabla u) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain containing the origin and $N \geq 3$.

Since we are looking for positive solutions, it is natural to require that f and g are non-negative functions. In particular we will focus our analysis in the cases

$$f = v^p, g = u^q, \quad f = |\nabla v|^p, g = |\nabla u|^q, \quad (0.234)$$

Keywords: Singular elliptic system, Hardy inequality, LANE-EMDEN system , Cooperative system



Introduction

The main goal of this paper is to analyze the question of existence and non existence for the cooperative singular system given by

$$(GS) \quad \begin{cases} -\Delta u &= \lambda_1 \frac{v}{|x|^2} + f(x, v, \nabla v) & \text{in } \Omega, \\ -\Delta v &= \lambda_2 \frac{u}{|x|^2} + g(x, u, \nabla u) & \text{in } \Omega, \\ u = v &= 0 & \text{on } \partial\Omega \end{cases}$$

where $\lambda_1, \lambda_2 > 0$.

We will consider two main cases:

1. the case where $f(x, v) = v^p$ and $g(x, u) = u^q$, using suitable process argument, we show the existence of a critical curves in $H(p, q)$ such that the system (S) has a solution according to the sign of H .
2. the case $f(x, v) = |\nabla v|^p$ and $g(x, u) = |\nabla u|^q$. Here the situation is more complicated and we need to use a suitable weighted Sobolev inequality in order to get the critical curves of existence.

Main results

Our main non-existence results for the *potential case* using an algebraic technique are given by :

Theorem 0.101. Suppose that $\min\{p, q\} \geq \frac{N-\mu(\sqrt{\lambda_1\lambda_2})}{\mu(\sqrt{\lambda_1\lambda_2})}$, then System (PS) has non weak supersolution.

Theorem 0.102. Let $\lambda_1, \lambda_2 > 0$ be such that $\sqrt{\lambda_1\lambda_2} \leq \Lambda_N$.

1. $\frac{\mu(\sqrt{\lambda_1\lambda_2})+2}{\mu(\sqrt{\lambda_1\lambda_2})} < p < \frac{N-\mu(\sqrt{\lambda_1\lambda_2})}{\mu(\sqrt{\lambda_1\lambda_2})}$, $q \leq p$ and $\mu(\sqrt{\lambda_1\lambda_2}) \frac{pq-1}{p+1} > 2$,
Or
2. $\frac{\mu(\sqrt{\lambda_1\lambda_2})+2}{\mu(\sqrt{\lambda_1\lambda_2})} < q < \frac{N-\mu(\sqrt{\lambda_1\lambda_2})}{\mu(\sqrt{\lambda_1\lambda_2})}$, $p \leq q$ and $\mu(\sqrt{\lambda_1\lambda_2}) \frac{pq-1}{q+1} > 2$,

then System (PS) has non weak positive supersolution.

And the existence is the following :

Theorem 0.103. Let Ω be a bounded domain containing the origin and that $\sqrt{\lambda_1\lambda_2} \leq \Lambda_N$.

1. Suppose that $p, q < \frac{\mu(\sqrt{\lambda_1\lambda_2})+2}{\mu(\sqrt{\lambda_1\lambda_2})}$, namely $(p, q) \in A$. Then System (PS) has a positive supersolution.



For the gradient case, the results using an iteration technique are the following :

Theorem 0.104. Suppose that $\min\{p, q\} \geq \frac{N-\mu(\sqrt{\lambda_1\lambda_2})}{\mu(\sqrt{\lambda_1\lambda_2})+1}$, then System (PS) has non weak supersolution.

Theorem 0.105. Let $\Omega \subset (R)^N$ be a bounded domain such that $0 \in \Omega$ and $N \geq 3$. Assume that $0 < \lambda_1, \lambda_2 \leq N$ and suppose that $p, q > 0$ satisfying the following assumptions:

1. $q < \frac{\mu(\sqrt{\lambda_1\lambda_2})+2}{\mu(\sqrt{\lambda_1\lambda_2})+1} < p < \frac{N-\mu(\sqrt{\lambda_1\lambda_2})}{\mu(\sqrt{\lambda_1\lambda_2})+1}, q \leq p$ and $H(p, q) < 0$,
2. $p < \frac{\mu(\sqrt{\lambda_1\lambda_2})+2}{\mu(\sqrt{\lambda_1\lambda_2})+1} < q < \frac{N-\mu(\sqrt{\lambda_1\lambda_2})}{\mu(\sqrt{\lambda_1\lambda_2})+1}, p \leq q$ and $H(p, q) < 0$,

Then the System (QS) has no nonnegative weak solution

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Analysis of a bilateral contact problem for piezoelectric material

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Abstract. We consider a mathematical model for the process of frictionless and bilateral contact problem between an electro-elasto-viscoplastic body and a conductive foundation. We derive a variational formulation of the mechanical problem and prove the existence of its unique weak solution. The proof is based on the nonlinear equations involving monotone operators and fixed-point arguments. Finally, we study the behavior of the solution with respect to data and obtain a convergence result.

Keywords: Bilateral contact, Quasistatic process, electro-elasto-viscoplastic material, weak solution, fixed-point.



Introduction

We study a frictionless contact problem for elasto-viscoplastic material with piezoelectric effects, the contact problems with or without friction between bodies are very common in industry and in daily life, whether this contact is between deformable bodies or between a rigid and deformable body. Frictional and frictionless contact problems involving electro-viscoelastic materials were studied in [1]; frictional contact problems for electro-elasto-viscoplastic materials were studied in [2], contact problems involving piezoelectric materials when the foundation is perfectly insulated were studied in [3]. We consider the contact is bilateral between an electro-elasto-viscoplastic body and a conductive foundation it means that the contact between it is maintained at all times. We derive a variational formulation of the problem and the existence of a unique weak solution and prove a convergence result.

Main results

Our main existence and uniqueness result, of the suggested model, is the mechanical problem has a unique weak solution satisfying

$$u \in C^1(0, T; V).$$

$$\varphi \in C(0, T; W).$$

$$\sigma \in C(0, T; \mathcal{H}_1).$$

$$D \in C(0, T; W).$$

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Existence of solutions and numerical simulation for a hyperbolic Kirchhoff-type problem with a free boundary

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Abstract. In this work, we are interested in studying the existence of weak solutions for a hyperbolic Kirchhoff-type problem with a free boundary using an approach often called hyperbolic discrete Morse flow, which is well established in hyperbolic equations. This method replaces the original problem with a sequence of minimization problems at discrete time intervals and proves the existence of a minimizer to the discretized functional. The minimizer serves as a solution to the discretized problem, providing a weak solution to the main problem. We point out that the presence of non-local terms generated by the Kirchhoff term equipped with the free boundary introduces significant challenges that require unconventional treatment to overcome, representing our study's core. Furthermore, we provide a numerical simulation of the main problem to further elucidate our findings' physical implications.

Keywords: Hyperbolic Kirchhoff type problem, Free boundary, Weak solution, Discrete Morse flow, Numerical simulation.



Introduction

In this paper, we address the existence of weak solutions to the hyperbolic free boundary Kirchhoff-type problem by studying the following model:

$$\begin{cases} \chi_{\{v>0\}} v_{tt}(x, t) = \left(a + b \|\nabla v(t)\|_{L^2(\Omega)}^2 \right) \Delta v(x, t) - f(x, t, v(x, t)), & \text{in } \Omega_T, \\ v(x, t) = 0, & \text{on } (0, T) \times \partial\Omega, \\ v(x, 0) = v_0(x), & \text{in } \Omega, \\ v_t(x, 0) = u_0(x), & \text{in } \Omega, \end{cases} \quad (0.235)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a Lipschitz boundary, and $\Omega_T = \Omega \times (0, T)$ with time $T > 0$, $v : \Omega_T \rightarrow \mathbb{R}$ is a scalar function, $\chi_{\{v>0\}}$ refers to the characteristic function to the set

$$\{v > 0\} := \{(x, t) \in \Omega_T; v(x, t) > 0\}. \quad (0.236)$$

$f : \Omega_T \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function with respect v and satisfies the following condition

- $f(x, t, v(x, t)) = 0$ if and only if $v = 0$, for a.e. $(x, t) \in \Omega_T$.
- There exists a constant $0 < C_f < \frac{a}{4}^{\frac{1}{2}}$, and a function $\gamma \in L^2(\Omega_T)$, such that:

$$|f(x, t, v(x, t))| \leq C_f |v(x, t)| + \gamma(x, t), \quad \text{for a.e } (x, t) \in \Omega_T. \quad (0.237)$$

Problem (0.235) presents a model for the free vibration of an elastic string on a flat surface. The displacement of the string is described by the function v , and the external force is represented by $f(x, t, v(x, t))$. The parameters a and b denote the intrinsic properties of the string and the initial tension, respectively. The displacement function $v(x, t)$ is restricted to the surface, where $v(x, t) = 0$ is presented as the surface.

The objective of this paper is a way of proving the existence of a weak solution by the use of a variational method. The variational approach that we used to achieve this goal is known as the Discrete Morse Flow (DMF) of hyperbolic type. The discrete Morse flow is described as a variational method employed for time- dependent problems. It involves replacing the original problem with a sequence of minimization problems at discrete time intervals. The discretized functional is stated to be non-negative, which allows for the demonstration of the existence of a minimizer, offering a crucial advantage over the functional itself. The minimizer of the functional serves as a solution to the discretized problem, providing a weak solution to the main problem.

Main results

In this paper, inspired by this previous work, we will provide a way of proving the existence of a weak solution for the hyperbolic Kirchhoff-type problem by using



the discrete morse flow method. We begin by defining the weak solution and providing the rationale for it. Because we cannot manage the regularity of the free boundary, note that we must limit the test functions to the space $C_0^\infty((\Omega \times [0, T]) \cap (\{v > 0\}))$. We see that examining this kind of set of test functions makes sense if and only if v is sufficiently regular. In this case, continuity is sufficient to ensure that $\{v > 0\}$ is an open set. Therefore, we define the weak solution as similar to the one in [4].

Definition 0.53. The function $v \in H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; H_0^1(\Omega))$ is called to be a weak solution to (0.235) if $v(x, 0) = v_0$, and if $v = 0$ outgoing $\{v > 0\}$ and for all $\varphi(x, t) \in C_0^\infty((\Omega \times [0, T]) \cap \{v > 0\})$ if what follows holds

$$\begin{aligned} \int_0^T \int_\Omega \left[-v_t(x, t) \varphi_t(x, t) + \left(a + b \|\nabla v(t)\|_{L^2(\Omega)}^2 \right) \nabla v(x, t) \nabla \varphi(x, t) + f(x, t, v(x, t)) \varphi(x, t) \right] dx dt \\ - \int_\Omega u_0(x) \varphi(x, 0) dx = 0. \end{aligned} \quad (0.238)$$

The regularity of the free boundary will be required to get this weak solution, and it is unclear whether a solution of this kind exists.

The main result of this work is presented in the following theorem

Theorem 0.106. For u_0, v_0 belong to $\mathbb{H}_0^1(\Omega)$. Then there exists a weak solution to the problem (0.235).

In the remainder of this work, our objective is to establish the existence of the weak solution defined in Definition 1. We begin by constructing the approximate weak solution in section* 1. This involves first introducing the functional $\mathcal{F}$ and subsequently demonstrating the existence of a minimizer, which leads to the derivation of the approximate weak solution. In section* 3, we refine the time step as k approaches zero and investigate the convergence. Through this process, we employ the passage to the limit to establish the existence of the weak solution. To accomplish this, an energy estimate is essential, and we provide it in section* 2. Finally, in section* 4, we present a numerical example of our main problem, showcasing the values of all the unknown terms to present a simulation of our problem.

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**Local existence of solutions in anisotropic Sobolev spaces
for a coupled system of generalized
Kadomtsev-Petviashvili I (KP –I) equations**

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Abstract. The main objective in this work is the study of the local existence and the uniqueness for a coupled system of generalized Kadomtsev-Petviashvili I (KP-I) equations in the anisotropic Sobolev spaces $H^{s_1, s_2}(\mathbb{R}^2)$ with $s_1 > -\frac{\alpha-1}{4}$ and $s_2 \geq 0$, $\alpha \geq 4$. The proof of this results relies on the linear and bilinear estimates.

Keywords: generalized Kadomtsev-Petviashvili I, anisotropic Sobolev spaces, well-posedness.



Introduction

In this paper, we consider a coupled system of generalized KP I equations

$$\begin{cases} \partial_t u + |D_x|^\alpha \partial_x u + \partial_x^{-1} \partial_{yy} u + \frac{1}{2} u \partial_x u + \frac{1}{2} v \partial_x v = 0, \\ \partial_t v - |D_x|^\alpha \partial_x v + \partial_x^{-1} \partial_{yy} v + \frac{1}{2} u \partial_x v + \frac{1}{2} v \partial_x u = 0, \\ u(x, y, 0) = u_0(x, y) \quad , \quad v(x, y, 0) = v_0(x, y). \end{cases} \quad (0.239)$$

where $u = u(x, y, t)$ and $v = v(x, y, t)$ are the unknown function, $(x, y) \in \mathbb{R}^2$ and $t \in \mathbb{R}$, while u_0 and v_0 are a given functions. Here $|D_x|^\alpha$ is the Fourier multiplier give by $\widehat{|D_x|^\alpha f(\xi)} = |\xi|^\alpha \widehat{f}(\xi)$, and ∂_x^{-1} is defined by its Fourier multiplier $-i\xi^{-1}$.

Main results

For the study of the local well-posedness for (0.250) in the anisotropic Sobolev spaces, we have the following main result

Theorem 0.107. *Let $s_1 > -\frac{\alpha-1}{4}$ and $s_2 \geq 0$, $\alpha \geq 4$ then for $(u_0, v_0) \in H^{s_1, s_2}(\mathbb{R}^2) \times H^{s_1, s_2}(\mathbb{R}^2)$, there exist $T > 0$ which depends on (u_0, v_0) such the system (0.250) has a unique solution (u, v) on the interval $[0, T]$ such that*

$$(u, v) \in C([0, T], H^{s_1, s_2}(\mathbb{R}^2)) \times C([0, T], H^{s_1, s_2}(\mathbb{R}^2)).$$

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Nonlinear diffusion model featuring the Caputo fractional derivative and its application in image processing

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Abstract. Partial differential equations (PDEs) have diverse applications, including image processing, where they provide powerful tools for task such as removal, edges detection, and image enhancement. This work introduces a denoising model that combines the nonlinear diffusion equation with Caputo derivative in the diffusivity function. The study explores the existence and uniqueness of weak solution for this model through the application of Schauder's fixed-point theorem. Furthermore, it introduces a comprehensive numerical scheme for addressing the model, utilizing a semi-implicit method that is unconditionally stable and grounded in the Additive Operator Splitting (AOS) technique.

Keywords: Nonlinear diffusion, Fractional derivative, Semi-implicit schemes, Image processing.



Introduction

Image denoising presents a significant challenge in the field of image processing, serving as a fundamental task within the community due to its considerable influence on subsequent stages of analysis. Currently, numerous denoising models are widely utilized across diverse domains, such as medical imaging, multimedia, remote sensing, etc.

Over the past decades, several partial differential equations (PDEs) have been proposed to address the challenge of image denoising, and these approaches have achieved considerable success.

In this work, we propose a model that combines the nonlinear diffusion equation with a fractional gradient in the diffusion coefficient, where the fractional gradient is defined using the Caputo fractional derivative. This model plays a significant role in preserving edges and structural details while reducing noise. Moreover, incorporating the Gaussian Kernel function into the diffusion coefficient enhances the well-posedness of the suggested model. The use of explicit finite difference schemes is constrained by stability requirements, which necessitate very small time steps and consequently lead to decreased efficiency and limits their practical use. Therefore, to numerically solve the model, we employ semi-implicit schemes utilize additive operator splitting (AOS), as originally introduced by Weickert [4].

Main results

*

*The proposed model and its theoretical analysis

$$\frac{\partial u}{\partial t} = \operatorname{div} \left(\frac{1}{1 + |\nabla^\alpha u_\sigma|^2} \nabla u \right), \quad \text{in } \Omega \times (0, T), \quad (0.240)$$

$$u(x, y, 0) = u_0(x, y), \quad \text{in } \Omega, \quad (0.241)$$

$$\nabla u \cdot \vec{n} = 0, \quad \text{on } \partial\Omega \times (0, T), \quad (0.242)$$

where

- Ω is a bounded domain of $\mathbb{R}^2$.
- u_0 is the noisy image.
- $0 < \alpha < 1$, and $u_\sigma = G_\sigma * u$ (G_σ is a Gaussian filter), $\vec{n}$ denote the unit outward normal to $\partial\Omega$.
- ∇ (respectively. $\operatorname{div}(\cdot)$) denotes the gradient (respectively divergence) operator, and
$$|\nabla^\alpha u_\sigma| = \sqrt{|{}^C D_x^\alpha u_\sigma|^2 + |{}^C D_y^\alpha u_\sigma|^2}.$$



${}^C D^\alpha$ is the fractional Caputo drivative of order α , As stated in the following definition

Definition 0.54. [2] The left Caputo fractional derivative of order $\alpha > 0$, of a function $u : [0, \mu] \rightarrow \mathbb{R}$ with $\mu > 0$ is defined by:

$$D_x^\alpha u(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-\eta)^{n-\alpha-1} \frac{d^n u(\eta)}{d\eta^n} d\eta, \quad n-1 < \alpha < n \in \mathbb{N}^*. \quad (0.243)$$

We introduce the space:

$$W(0, T) = \left\{ w \in L^2(0, T; H^1(\Omega)), \frac{dw}{dt} \in L^2(0, T; (H^1(\Omega))') \right\}.$$

Evidently, $W(0, T)$ is a Hilbert space with the norm:

$$\|w\|_W = \|w\|_{L^2(0, T; H^1(\Omega))} + \left\| \frac{\partial w}{\partial t} \right\|_{L^2(0, T; (H^1(\Omega))')}.$$

The following theorem is proven using Schauder's fixed-point theorem and Gronwall's inequality

Theorem 0.108. Assuming that $u_0 \in H^1(\Omega)$, the problem defined by (0.240)-(0.242) admits a unique weak solution $u \in W(0, T)$.

*

*Numerical schemes Nonlinear diffusion equations are typically solved numerically using explicit schemes. However, these schemes are only stable for very small time steps, which results in poor efficiency and limits their practical applicability. In this part, we develop a semi-implicit discrete scheme to numerically solve the problem defined by (0.240)-(0.242). To achieve this, we first introduce the additive operator splitting and then discretize the Caputo fractional derivative. The equation (0.240) can be expressed as follows:

$$\frac{\partial u}{\partial t} = (A_x + A_y) u, \quad (0.244)$$

where, A_x and A_y are differential sub-operators, as mentioned below:

$$\begin{cases} A_x = \frac{\partial}{\partial x} \left(\hbar (|\nabla^\alpha u_\sigma|) \frac{\partial}{\partial x} \right), \\ A_y = \frac{\partial}{\partial y} \left(\hbar (|\nabla^\alpha u_\sigma|) \frac{\partial}{\partial y} \right). \end{cases} \quad (0.245)$$

Applying the backward difference formula to (0.244), we get:

$$\frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t} = (A_x + A_y) u_{i,j}^{n+1}. \quad (0.246)$$



We can write (0.246) in matrix-vector notation as:

$$\frac{U^{n+1} - U^n}{\Delta t} = (A_x + A_y) U^{n+1}. \quad (0.247)$$

Where U^n as the vector with entries $u_{i,j}^n$.

Incorporating U^{n+1} on the right-hand side of equation (0.247) yields the following:

$$[I - \Delta t (A_x + A_y)] U^{n+1} = U^n, \quad (0.248)$$

where I is the identity matrix.

By applying the Additive Operator Splitting (AOS) Schemes, we get:

$$U^{n+1} = \sum_{l=1}^2 [2(I - 2\Delta t A_l)]^{-1} U^n,$$

Based on the definition of the Caputo fractional derivative (0.468), the discretization of this derivative over the interval $[0, x]$ is performed using central finite difference scheme as follows:

$${}^C D_x^\alpha u_\sigma(x_i, y_j, t_n) = \frac{1}{\Gamma(1-\alpha)} \int_0^{x_i} (x_i - s)^{-\alpha} \frac{\partial}{\partial s} u_\rho(s, y_j, t_n) ds. \quad (0.249)$$

The equation (0.249) can be modified to:

$${}^C D_x^\alpha u_\sigma(x_i, y_j, t_n) = \frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^{i-1} \left(\frac{u_{k+1,j}^n - u_{k,j}^n}{\Delta x} \right) \int_{x_k}^{x_{k+1}} (x_i - s)^{-\alpha} ds.$$

Using the space grid point $x_i = i \Delta x$ for $i = 0, 1, \dots, N$, and by integrating, we obtain:

$${}^C D_x^\alpha u_\sigma(x_i, y_j, t_n) = \frac{(\Delta x)^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{i-1} (u_{k+1,j}^n - u_{k,j}^n) \left[(i - (k+1))^{1-\alpha} - (i - k)^{1-\alpha} \right].$$

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Critical curve for existence of non-linear elliptic system involving Hardy potential and potential-gradient terms

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Abstract. The main goal of this work [3] is to study the existence and the nonexistence of a non-negative super-solution of the gradient-potential system with Hardy term. More precisely, we consider the system

$$\begin{cases} -\Delta u &= \lambda \frac{u}{|x|^2} + v^p & \text{in } \Omega, \\ -\Delta v &= \lambda \frac{v}{|x|^2} + |\nabla u|^q & \text{in } \Omega, \\ u, v &> 0 & \text{in } \Omega, \\ u = v &= 0 & \text{on } \partial\Omega, \end{cases} \quad (\text{S})$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded regular domain such that $0 \in \Omega$. Here, $0 < \lambda \leq \Lambda_N$, with $\Lambda_N := \frac{(N-2)^2}{4}$ being the Hardy constant. For $p, q > 1$, we prove the existence of an optimal critical curve in the (p, q) -plane, that separates the existence and nonexistence regions.

Keywords: Hardy's inequality, nonlinear elliptic system, weighted Sobolev spaces, Existence and nonexistence results, gradient/potential terms.



Introduction

In this work, we are interested in understanding the effect of the Hardy potential on the existence and nonexistence of solutions to the non-linear elliptic system where the non-linearity may depends on a first order term and/or a super-linear zero order term. Recently, the authors in [5] studied a class of Lane-Emden system with inverse-square potentials given by:

$$\begin{cases} -\Delta u &= \lambda_1 \frac{u}{|x|^2} + v^p & \text{in } \Omega, \\ -\Delta v &= \lambda_2 \frac{v}{|x|^2} + u^q & \text{in } \Omega, \\ u, v &> 0 & \text{in } \Omega. \end{cases} \quad (0.250)$$

In the case where $\lambda_1 = \lambda_2$, using an iterative scheme, they were able to show existence of a non-negative super-solution (u, v) under the assumptions:

$$\begin{cases} q < \frac{N-\alpha_1}{\alpha_1} & \text{and} & \alpha_1(pq-1) < 2p+2, \\ & \text{or} & \\ p < \frac{N-\alpha_1}{\alpha_1} & \text{and} & \alpha_1(pq-1) < 2q+2, \end{cases}$$

where $\alpha_1 = \frac{N-2}{2} - \sqrt{\Lambda_N - \lambda}$. In [3], we prove new results and introduce new ideas that allow to prove the existence of a solution for the elliptic system with gradient and potential dependent terms.

Main results

Our goal is to show the existence of a critical curve $H(p, q)$ such that existence and nonexistence of solution to the system (S), hold according to the position of (p, q) related to this critical curve.

Let us begin by defining the critical constants.

$$p^+ := \frac{2 + \alpha_1}{\alpha_1}, \quad q^+ := \frac{2 + \alpha_1}{1 + \alpha_1} \quad (0.251)$$

and

$$\bar{p} := \frac{N - \alpha_1}{\alpha_1}, \quad \bar{q} := \frac{N - \alpha_1}{1 + \alpha_1}, \quad (0.252)$$

where $\alpha_1 = \frac{N-2}{2} - \sqrt{\Lambda_N - \lambda}$, $\alpha_2 = \frac{N-2}{2} + \sqrt{\Lambda_N - \lambda}$. Notice that $N - \alpha_1 = \alpha_2 + 2$. The constants in (0.251) were first introduced in [4] and [2].

*

Nonexistence results The nonexistence region in the (p, q) - plane will be presented in the following Theorem.



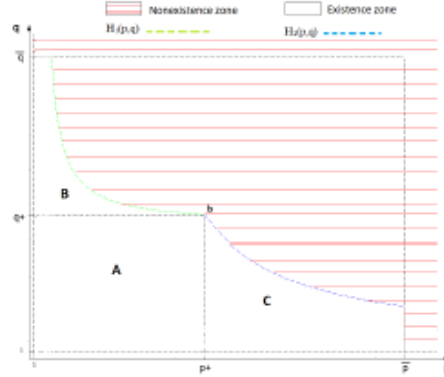


Figure 21: Existence-nonexistence zones for System (S)

Theorem 0.109. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain such that $0 \in \Omega$ and $N \geq 3$. Assume that $0 < \lambda < \Lambda_N$ and suppose that $p, q > 0$ satisfying one of the following assumptions:

- I) $p \geq \bar{p}$ or $q \geq \bar{q}$,
- II) $p^+ < p < \bar{p}$ and $q > \frac{\alpha_1 + 2}{p\alpha_1 - 1}$,
- III) $q^+ < q < \bar{q}$ and $p > \frac{\alpha_1 + 2}{q(\alpha_1 + 1) - 2}$,

then, the system (S) does not have any non-negative weak solution.

In the critical case $\lambda = \Lambda_N$, we have that $\alpha_1 = \alpha_2$. Hence in this case the region B and C are empty. More precisely, we have the following result.

Theorem 0.110. If $\lambda = \Lambda_N$ and suppose $p, q > 1$ are such that: either $p \geq \frac{N+2}{N-2}$ or $q \geq \frac{N+2}{N-2}$. Then, System (S) does not have any non-negative weak solution.

*

Existence result The aim of this subsection** is to show that the nonexistence results obtained in the Theorem 0.109 are optimal. More precisely, we will construct super-solution to system (S) in the complementary region of the (p, q) -plan. To this end, we first consider the radial case $\Omega = B_1(0)$.

Theorem 0.111. Assume that $\Omega = B_1(0)$, $0 < \lambda < \Lambda_N$ and $p, q > 1$ satisfy one of the following assumptions:

- I) $1 < p < p^+$ and $1 < q < q^+$, or



$$II) \quad q^+ \leq q < \bar{q} \text{ and } p < \frac{\alpha_1 + 2}{q(\alpha_1 + 1) - 2}, \text{ or}$$

$$III) \quad p^+ \leq p < \bar{p} \text{ and } q < \frac{\alpha_1 + 2}{p\alpha_1 - 1}.$$

Then, the system (S) has a non-negative weak super-solution.

We deal now with the case where $\lambda = \Lambda_N$, then $p^+ = \bar{p} = \frac{N+2}{N-2}$ and $q^+ = \bar{q} = \frac{N+2}{N}$. In this case the regions B, C are empty. Hence we have the next existence result.

Theorem 0.112. Suppose that $\lambda = \Lambda_N$ and $\Omega = B_\eta(0)$ with $\eta < 1$. Assume that $p < p^+ = \frac{2+\alpha_1}{\alpha_1}$ and $q < q^+ = \frac{2+\alpha_1}{1+\alpha_1}$. Define $w(x) = |x|^{-\alpha_1}(\ln(\frac{1}{|x|}))^\beta$ where $0 < \beta < 1$. Then there exist $c > 0$ such that, setting $(u(x), v(x)) = (cw(x), cw(x))$, we obtain that (u, v) is a weak super-solution of (S).

Remark 0.8. In the case of general bounded domains, using a re-scaling argument, we can prove the existence of super-solution to System (S) under the same conditions of Theorem 0.111. More precisely, if (u, v) is a radial weak super-solution then (u_R, v_R) , where

$$u_R(x) = u\left(\frac{x}{R}\right) \text{ and } v_R(x) = v\left(\frac{x}{R}\right),$$

and R is chosen such that $\Omega \subset B(0, R)$, is weak super-solution on Ω .

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Growth of Analytic Solutions of Linear Differential Equations with Analytic Coefficients Near a Finite Singular Point

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Abstract. In this paper, we investigate the growth of analytic solutions of the linear differential equation

$$f^{(k)} + A_{k-1}(z) \exp \left\{ \frac{a_{k-1}}{(z_0 - z)^n} \right\} f^{(k-1)} + \dots + A_0(z) \exp \left\{ \frac{a_0}{(z_0 - z)^n} \right\} f = 0,$$

where $n \in \mathbb{N} - \{0\}$, $k \geq 2$ is an integer and $A_j(z)$ ($j = 0, \dots, k-1$) are analytic functions in the closed complex plane except a singular point z_0 and a_j ($j = 0, \dots, k-1$) are complex numbers. Under some conditions, we prove that these solutions are of infinite order and their hyper-order is equal to n . We also consider the nonhomogeneous linear differential equations.

Keywords: Linear differential equations, Nevanlinna theory, growth of solutions, a finite singular point.



Introduction

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic function in the complex plane $\mathbb{C}$ and in the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ (see [2, 3, 11]). We denote respectively the order and the hyper-order of growth of a meromorphic function f , by $\sigma(f)$ and $\sigma_2(f)$.

Many authors as in ([2, 5, 3, 9]) have studied the growth of solutions of linear differential equations near a finite singular point. They have investigated different forms of linear differential equations with analytic coefficients near a finite singular point by using adapted notions and properties of Nevanlinna theory. In this paper, we continue this investigation near a finite singular point.

First, we recall the appropriate definitions. Set $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and suppose that f is meromorphic in $\overline{\mathbb{C}} \setminus \{z_0\}$, where $z_0 \in \mathbb{C}$. Define the counting function near z_0 by

$$N_{z_0}(r, f) = - \int_{\infty}^r \frac{n(t, f) - n(\infty, f)}{t} dt - n(\infty, f) \log r,$$

where $n(t, f)$ counts the number of poles of f in the region $\{z \in \mathbb{C} : t \leq |z_0 - z| \} \cup \{\infty\}$, each pole according to its multiplicity, and the proximity function by

$$m_{z_0}(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(z_0 - re^{i\phi})| d\phi.$$

The characteristic function of f is defined in the usual manner by

$$T_{z_0}(r, f) = m_{z_0}(r, f) + N_{z_0}(r, f).$$

In addition, the order of a meromorphic function f near z_0 is defined by

$$\sigma_T(f, z_0) = \limsup_{r \rightarrow 0} \frac{\log^+ T_{z_0}(r, f)}{-\log r}.$$

For an analytic function f in $\overline{\mathbb{C}} \setminus \{z_0\}$, we have also the definition

$$\sigma_M(f, z_0) = \limsup_{r \rightarrow 0} \frac{\log^+ \log^+ M_{z_0}(r, f)}{-\log r},$$

where $M_{z_0}(r, f) = \max_{|z_0 - z| = r} |f(z)|$.

When the order is infinite, we introduce the notion of hyper-order near z_0 that is defined as follows :

$$\sigma_{2,T}(f, z_0) = \limsup_{r \rightarrow 0} \frac{\log^+ \log^+ T_{z_0}(r, f)}{-\log r},$$

$$\sigma_{2,M}(f, z_0) = \limsup_{r \rightarrow 0} \frac{\log^+ \log^+ \log^+ M_{z_0}(r, f)}{-\log r}.$$



Remark 0.9 ([2]). It is shown in [3] that if f is a non-constant meromorphic function in $\mathbb{C} \setminus \{z_0\}$ and $g(\omega) = f(z_0 - \frac{1}{\omega})$, then $g(\omega)$ is meromorphic in $\mathbb{C}$ and we have :

$$T(R, g) = T_{z_0}(\frac{1}{R}, f),$$

where $R > 0$ and so $\sigma_T(f, z_0) = \sigma(g)$. Also, if f is analytic in $\mathbb{C} \setminus \{z_0\}$, then $g(\omega)$ is entire, and thus $\sigma_T(f, z_0) = \sigma_M(f, z_0)$ and $\sigma_{2,T}(f, z_0) = \sigma_{2,M}(f, z_0)$. Then we can use the notations $\sigma(f, z_0)$ and $\sigma_2(f, z_0)$ without any ambiguity.

The linear differential equation

$$f'' + A(z)e^{az}f' + B(z)e^{bz}f = 0, \quad (0.253)$$

where $A(z)$ and $B(z)$ are entire functions is investigated by many authors; see for example [3, 12, 13].

Kwon [12] proved that if $ab \neq 0$ and $\arg a \neq \arg b$ or $a = cb$ with $0 < c < 1$, then every solution $f \not\equiv 0$ of equation (0.253) is of infinite order. Fan and Chen in [5] considered the linear differential equation

$$f^{(k)} + H_{k-1}(z)f^{(k-1)} + \dots + H_1(z)f' + H_0(z)f = 0, \quad (0.254)$$

where $k \geq 2$ is an integer and $H_j(z) (j = 0, 1, \dots, k-1)$ are entire functions. They proved the following result :

Theorem 0.113. *Let $k \geq 2$ be an integer and $a_j (j = 0, \dots, k-1)$ be complex numbers. Suppose that there exist a_s and a_l such that $s < l$, $a_s = d_s e^{i\phi}$, $a_l = -d_l e^{i\phi}$, $d_s > 0$, $d_l > 0$ and for $j \neq s, l$, $a_j = d_j e^{i\phi}$ or $a_j = -d_j e^{i\phi} (d_j \geq 0)$, and $\max\{d_j : j \neq s, l\} = d < \min\{d_s, d_l\}$. If $H_j(z) = h_j(z)e^{a_j z^n}$, where $n \in \mathbb{N} \setminus \{0\}$ and $h_j(z)$ are entire functions with $\sigma(h_j) < n$, $h_s h_l \neq 0$, then every transcendental solution f of equation (0.254) is of infinite order and satisfies $\sigma_2(f) = n$.*

To investigate the counterpart of these results near a finite singular point, Fetouch and Hamouda proved the following result :

Theorem 0.114 ([3]). *Let z_0, a, b be complex constants, such that $\arg a \neq \arg b$ or $a = cb$ ($0 < c < 1$). Let $A(z), B(z) \not\equiv 0$ be analytic functions in $\mathbb{C} \setminus \{z_0\}$ with $\max\{\sigma(A, z_0), \sigma(B, z_0)\} < n$. Then every solution $f \not\equiv 0$ of the differential equation*

$$f'' + A(z) \exp \left\{ \frac{a}{(z_0 - z)^n} \right\} f' + B(z) \exp \left\{ \frac{b}{(z_0 - z)^n} \right\} f = 0$$

satisfies $\sigma(f, z_0) = +\infty$ with $\sigma_2(f, z_0) = n$.

In [2], Cherief and Hamouda have generalized the above result to higher order differential equations and proved the following result :



Theorem 0.115 ([2]). Let $n \in \mathbb{N} \setminus \{0\}$, $k \geq 2$ be an integer and $A_j(z) (j = 0, \dots, k-1)$ be analytic functions in $\mathbb{C} \setminus \{z_0\}$, such that $\sigma(A_j, z_0) < n$ and let $a_j (j = 0, \dots, k-1)$ be complex numbers, such that $a_0 = |a_s|e^{i\theta_0}, a_s = |a_s|e^{i\theta_s}, a_0 a_s \neq 0$ ($0 < s < l \leq k-1$), $\theta_0, \theta_s \in [0, 2\pi)$, $\theta_0 \neq \theta_s$, $A_0 A_s \neq 0$ and for $j \neq 0, s$, a_j satisfies either $a_j = d_j a_0$ ($d_j < 1$) or $\arg a_j = \arg a_s$. Then every solution $f \neq 0$ of the differential equation

$$f^{(k)} + A_{k-1}(z) \exp \left\{ \frac{a_{k-1}}{(z_0 - z)^n} \right\} f^{(k-1)} + \dots + A_0(z) \exp \left\{ \frac{a_0}{(z_0 - z)^n} \right\} f = 0 \quad (0.255)$$

that is analytic in $\overline{\mathbb{C}} \setminus \{z_0\}$ satisfies $\sigma(f, z_0) = +\infty$ and $\sigma_2(f, z_0) = n$.

Main results

In this paper, we give new conditions on $a_j (j = 0, \dots, k-1)$ and investigate the order and the hyper-order of analytic solutions of equation (0.255). We also consider the nonhomogeneous equation. We will prove the following results :

Theorem 0.116. Let $n \in \mathbb{N} \setminus \{0\}$, $k \geq 2$ be an integer and $A_j(z) (j = 0, \dots, k-1)$ be analytic functions in $\mathbb{C} \setminus \{z_0\}$, such that $\sigma(A_j, z_0) < n$. Suppose that there exist $s, l \in \{1, \dots, k-1\}$ such that $A_s A_l \neq 0$, $a_s = d_s e^{i\phi}$, $a_l = -d_l e^{i\phi}$, $\phi \in [0, 2\pi)$, $d_s > 0$, $d_l > 0$ and for $j \neq s, l$, $a_j = d_j e^{i\phi}$ or $a_j = -d_j e^{i\phi}$ ($d_j \geq 0$) and $\max\{d_j : j \neq s, l\} = d < \min\{d_s, d_l\}$. Then every solution f of equation (0.255) that is analytic in $\mathbb{C} \setminus \{z_0\}$ is of infinite order and satisfies $\sigma_2(f, z_0) = n$, where z_0 is an essential singular point for f .

Theorem 0.117. Let $n \in \mathbb{N} \setminus \{0\}$, $k \geq 2$ be an integer and $A_j(z) (j = 0, \dots, k-1)$ be analytic functions in $\mathbb{C} \setminus \{z_0\}$, such that $\sigma(A_j, z_0) < n$. Suppose that there exists $\{a_{i_0}, a_{i_1}, \dots, a_{i_m}\} \subset \{a_1, \dots, a_{k-1}\}$ such that $\arg a_{i_j} (j = 1, \dots, m)$ are distinct and for every nonzero $a_l \in \{a_1, \dots, a_{k-1}\} \setminus \{a_{i_1}, \dots, a_{i_m}\}$, there exists some $a_{i_s} \in \{a_{i_1}, \dots, a_{i_m}\}$ such that $a_l = c_l^{(i_s)} a_{i_s}$ ($0 < c_l^{(i_s)} < 1$). Then every solution f of equation (0.255) that is analytic in $\mathbb{C} \setminus \{z_0\}$ is of infinite order and satisfies $\sigma_2(f, z_0) = n$, where z_0 is an essential singular point for f .

Theorem 0.118. Let $n \in \mathbb{N} \setminus \{0\}$, $k \geq 2$ be an integer, $A_j(z)$ and $a_j (j = 0, \dots, k-1)$ satisfy hypotheses of Theorem 1.1 or those of Theorem 1.2. Let $F \neq 0$ be an analytic function in $\mathbb{C} \setminus \{z_0\}$ of order $\sigma = \sigma(F, z_0) < n$. Then every solution f of equation

$$f^{(k)} + A_{k-1}(z) \exp \left\{ \frac{a_{k-1}}{(z_0 - z)^n} \right\} f^{(k-1)} + \dots + A_0(z) \exp \left\{ \frac{a_0}{(z_0 - z)^n} \right\} f = F \quad (0.256)$$

that is analytic in $\mathbb{C} \setminus \{z_0\}$ is of infinite order and satisfies $\sigma_2(f, z_0) = n$, where z_0 is an essential singular point for f , with at most one exceptional analytic solution f_0 of finite order in $\mathbb{C} \setminus \{z_0\}$.



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Application of a Regularization Method for Reconstructing a Stable Solution to an Inverse Backward Problem in Fractional Diffusion Equations

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Abstract. This study addresses the identification of an unknown backward term within a time-space fractional heat equation. The problem is shown to be ill-posed in the sense of Hadamard when an additional condition at the final time is given. A new fractional Tikhonov regularization method is used to tackle this problem. Under the a-priori parameter choice rules, the error estimates between the exact and its regularized solutions are obtained.

Keywords: Fractional diffusion equation, Inverse backward problem, Ill-posed problem, Tikhonov Regularisation method.



Introduction

We are concerned with the following general one-dimensional fractional diffusion problem:

$$D_{a+}^t u(x, t) = L_q u(x, t) + f(x), \quad (x, t) \in [a, b] \times [0, T], \quad 0 < \alpha < 1 \quad (0.257)$$

$$u(a, t) = u(b, t) = 0, \quad t \in [0, T] \quad (0.258)$$

$$u(x, 0) = g(x), \quad x \in [a, b] \quad (0.259)$$

Here, D_{a+}^t stands for the left-sided Caputo time-fractional derivative, and L_q is a generalization of the Sturm-Liouville operator. Our problem addresses the recovery of the initial condition $g(x)$ along with the solution $u(x, t)$ given the extra condition

$$u(x, T) = h(x), \quad x \in [a, b]. \quad (0.260)$$

In this case, the function $h(x)$ is based on the observation, so it is approximated by the noisy observation data $h^\delta(x)$ satisfying

$$\|h(x) - h^\delta(x)\|_2 \leq \delta, \delta > 0 \quad (1.5)$$

Main results

First we derive the solution set for the problem (1.1)-(1.4) by using Fourier method and the extra condition (1.4). Next, we prove that the problem is ill-posed and we use the fractional Tikhonov regularization method for the reconstruction of the stable solution.

We impose the following bound condition for $g(x) \in H_\omega^p(\Omega)$:

$$\|g\|_{H_\omega^p} \leq E_2 \quad \text{for some } E_2 > 0. \quad (0.261)$$

We put

$$G(x) = \sum_{n=1}^{\infty} g_n E_\alpha(-\lambda_n T^\alpha) X_n(x) \quad (0.262)$$

Lemma 1 If $f \in L^2(\Omega)$, then

$$\|h(x) - \sum_{n=1}^{\infty} \{f_n T^\alpha E_{\alpha; \alpha+1}(-\lambda_n T^\alpha)\} X_n(x)\|_2^2 \leq \|h\|_2^2 + N \|f\|_2^2, \quad (0.263)$$

where $N = \frac{1}{\lambda_1^2}$. Moreover,

$$\|G - G^\delta\|_2^2 \leq 2\delta^2(1 + N). \quad (0.264)$$

Lemma 2 Assume that (1.5) holds. Then we have the following estimate:

$$\|g_v - g_v^\delta\|_2 \leq C_3^{(2\gamma-1)} N_1 \sqrt{2(1+N)} \delta v^{-1/(2\gamma)}, \quad (0.265)$$



where $N_1 = N_1(\gamma, C_2)$.

Lemma 3 Suppose that the a-priori bound condition (2.1) holds. Then the following estimate yields

$$\|g_v - g\|_2 \leq \begin{cases} N_2 v^{p/(2\gamma)} E_2, & 0 < p \leq 2\gamma, \\ N_3 v E_2, & p \geq 2\gamma, \end{cases} \quad (0.266)$$

where $N_2 = N_2(\gamma, p, C_2)$ and $N_3 = N_3(\gamma, p, \lambda_1)$.

Theorem 3 Suppose the a-priori bound condition (2.1) and the noise assumption (1.5) hold. Then:

If $0 < p \leq 2\gamma$, and choose $v = \left(\frac{\delta}{E_2}\right)^{\frac{2\gamma}{p+1}}$, then

$$\|g - g_v^\delta\|_2 \leq \left[N_2 + C_3^{(2\gamma-1)} N_1 \sqrt{2(1+N)} \right] \delta^{\frac{p}{p+1}} E_2^{\frac{1}{p+1}}. \quad (0.267)$$

If $0 < p \leq 2\gamma$, and choose $v = \left(\frac{\delta}{E_2}\right)^{\frac{2\gamma}{p+1}}$, then

$$\|g - g_v^\delta\|_2 \leq \left[N_3 + C_3^{(2\gamma-1)} N_1 \sqrt{2(1+N)} \right] \delta^{\frac{2\gamma}{2\gamma+1}} E_2^{\frac{1}{2\gamma+1}}. \quad (0.268)$$

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Hamilton Bellman Jacobi System in Financial Market

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Abstract. In this work, we study an approach to solving optimal control problems, namely, the method of dynamic programming. Dynamic programming, originated by R. Bellman in the early 1950's, is a mathematical method for making a sequence of inter-related decisions, which can be applied to numerous optimization problems (including optimal control problems). The basic idea of this technique applied to optimal controls is to consider a family of optimal control problems with different initial times and states, to establish relationships between these problems via the so-called Hamilton-Jacobi-Bellman equation (HJB, for short), which is a non-linear first-order (in the deterministic case) or second-order (in the stochastic case) partial differential equation. If the HJB equation is solvable (either analytically or numerically), then one can get an optimal feedback control by taking the maximize/minimize of the Hamiltonian or generalized Hamiltonian involved in the HJB equation. This is the so-called verification technique. Mentioning that this approach actually solves a whole family of problems (with different initial times and states).



Key words: Stochastic Differential equations, value function, Dynamic programming, optimization problems.

Introduction

In this work, we study an approach to solving optimal control problems, namely, the method of dynamic programming.

Main results

$$G(t, x, u, p, P) = \frac{1}{2} \text{tr} (\sigma(t, x, u) \sigma(t, x, u)^\top P + b(t, x, u)^\top p + f(t, x, u)), \quad (1.11)$$

We also need to introduce the second-order infinitesimal generator $\mathcal{L}^u$ associated to the diffusion x with control u

$$\mathcal{L}^u \varphi(t, x) = b(t, x, u) \cdot D_x \varphi(t, x) + \frac{1}{2} \text{tr} (\sigma(t, x, u) \sigma(t, x, u)^\top D_x^2 \varphi(t, x)). \quad (1.12)$$

The classical HJB equation associated to the stochastic control problem (1.9) is

$$-V_t(t, x) - \inf_{u \in A} [\mathcal{L}^u V(t, x) + f(t, x, u)] = 0, \text{ on } [0, T] \times \mathbb{R}^n. \quad (1.13)$$

We give sufficient conditions which enable to conclude that the smooth solution of the HJB equation coincides with the value function this is the so-called verification result.

Theorem 0.119. *Let W be a $C^{1,2}([0, T], \mathbb{R}^n) \cap C([0, T], \mathbb{R}^n)$ function. Assume that f and h are quadratic growth, i.e. there is a constant C such that*

$$|f(t, x, u)| + |h(x)| \leq C [1 + |x|^2], \text{ for all } (t, x, u) \in [0, T] \times \mathbb{R}^n \times U.$$

1. Suppose that $W(T, \cdot) \leq h$, and

$$W_t(t, x) + G(t, x, W(t, x), D_x W(t, x), D_x^2 W(t, x)) \geq 0, \quad (1.14)$$

on $[0, T] \times \mathbb{R}^n$, then $W \leq V$ on $[0, T] \times \mathbb{R}^n$.



2. Assume further that $W(T, \cdot) = h$, and there exists a minimizer $\hat{u}(t, x)$ of $\mathcal{L}^u V(t, x) + f(t, x, u)$, such that

$$\begin{aligned} 0 &= W_t(t, x) + G(t, x, W(t, x), D_x W(t, x), D_x^2 W(t, x)) \\ &= W_t(t, x) + \mathcal{L}^{\hat{u}(t, x)} W(t, x) + f(t, x, u), \end{aligned} \quad (1.15)$$

the stochastic differential equation

$$dX(s) = b(s, X(s), \hat{u}(s, x))ds + \sigma(s, X(s), \hat{u}(s, x))dW(s),$$

defines a unique solution $X(t)$ for each given initial data $X(t) = x$, and the process $\hat{u}(s, x)$ is a well-defined control process in U . Then $W = V$, and $\hat{u}$ is an optimal Markov control process.

Proof The proof of this verification theorem is found in the book by Yong and Zhou [?]. ■

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A new characterization of some classes of operators

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Abstract. Let $A \in \mathcal{B}(\mathcal{H})$ be a bounded linear operator on a Hilbert space $\mathcal{H}$, with its polar decomposition give by $A = U|A|$. The generalized Aluthge transformation of A for any $\alpha, \beta > 0$ is defined as $\Delta_{\alpha, \beta}(A) = |A|^\alpha U |A|^\beta$. In this work, we obtain new characterizations of some classes of operators via the generalized Aluthge transformation

Keywords: Polar decomposition, generalized Aluthge transformation, bi-normal operators.



Introduction

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. For an arbitrary operator $A \in \mathcal{B}(\mathcal{H})$, we denote by $\mathcal{N}(A)$, $\mathcal{R}(A)$ and A^* for the null subspace, the range and the adjoint operator of A , respectively. For any closed subspace M of $\mathcal{H}$, let P_M denote the orthogonal projection onto M .

Recall that for $A \in \mathcal{B}(\mathcal{H})$, there is a unique factorization $A = U|A|$, where $\mathcal{N}(U) = \mathcal{N}(A) = \mathcal{N}(|A|)$, U is a partial isometry, i.e. $UU^*U = U$ and $|A| = (A^*A)^{\frac{1}{2}}$ is modulus of A . This factorization is called the polar decomposition of A . From such a decomposition, the Aluthge transform of A is defined in [1] by

$$\Delta(A) = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}, \quad A \in \mathcal{B}(\mathcal{H}).$$

In [3], Furuta introduced a more general notion called generalized Aluthge transformation which has also been studied in detail. This is defined for $\alpha, \beta > 0$ by

$$\Delta_{\alpha,\beta}(A) = |A|^\alpha U |A|^\beta, \quad A \in \mathcal{B}(\mathcal{H}).$$

Clearly, for $\alpha = \beta = \frac{1}{2}$ we obtain the usual Aluthge transform.

Main results

Theorem 0.120. *Let $A = U|A|$ be the polar decomposition $A \in \mathcal{B}(\mathcal{H})$. Then the following assertions are equivalent:*

1. A is positive.
2. For every $\alpha > 0, \beta > 0$, $\Delta_{\alpha,\beta}(A)$ is positive.
3. There exists $\alpha > 0, \beta > 0$, $\Delta_{\alpha,\beta}(A)$ is positive.

Corollary 0.3. *Let $A = U|A|$ be paranormal and $\Delta_{\alpha,\beta}(A) = U|\Delta_{\alpha,\beta}(A)|$ is the polar decomposition. Then A is binormal and hyponormal.*

Theorem 0.121. *Let $A = U|A| \in \mathcal{B}(\mathcal{H})$ be the polar decomposition and let $\alpha, \beta > 0$ with $\alpha + \beta = 1$. If U is unitary and $A = \lambda \Delta_{\alpha,\beta}(A)$ for $\lambda \in \mathbb{C}$, then A is normal.*

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Solving the resulting differential equations for the Bianchi I cosmological model

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Abstract. In this paper, differential equations were studied in the field of physics that were obtained through Einstein's equations. The field equations of the first spatially homogeneous, anisotropic with the perfect fluid Bianchi model were solved within the framework of the Hubble parameter change law proposed by Berman (1983). He found that the solutions are compatible with modern observations of Type I supernovae.

Keywords: Bianchi space-time, Hubble's parameter, Deceleration parameter, Cosmological models.



Introduction

As we know, differential equations are used in many fields such as applied mathematics, theoretical physics, and others...So that my specialization includes a different kind of differential equations. Physics is based on the search for solutions and their justification using both pure and Applied Mathematics. The differential equations in our physical model facilitate the study and solution of our system of equations.

In this research, we are interested in finding the statement solutions of the field equations and studying the behavior of physical parameters in spacetime of the first type spatially homogeneous, anisotropic and filled with an ideal liquid, which is investigated by the differential field equations of our model.

Main results

The spatially homogeneous and anisotropic Bianchi-I space-time is described by the line element

$$ds^2 = -dt^2 + A^2(t)dx^2 + B^2(t)dy^2 + C^2(t)dz^2, \quad (0.269)$$

where $A(t)$, $B(t)$ and $C(t)$ are potential functions of cosmic time t .

Definition 0.55. The Einstein field equations can be written as follows,

$$G_{\mu\nu} - (6\lambda^2 u_\mu u_\nu + (3 - 6\alpha)\lambda^2 g_{\mu\nu}) = T_{\mu\nu}, \quad (0.270)$$

where $G_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu} = \rho_m u_\mu u_\nu$ is the energy-momentum tensor, λ is real, α is a real and $g^{\mu\nu} u_\mu u_\nu = -1$.

The Hubble parameter in anisotropic models can be defined by the average scale factor $V = (ABC)^{\frac{1}{3}}$ where we write,

$$H = \frac{1}{3} \frac{\dot{V}}{V} = \frac{1}{3} (H_1 + H_2 + H_3), \quad (0.271)$$

where $H_1 = \dot{A}/A$, $H_2 = \dot{B}/B$, and $H_3 = \dot{C}/C$ are the directional Hubble's parameters in the directions of x , y and z respectively. We have assumed $8\pi G = c = 1$ in proper unit. The field equation (0.458) for the line element (0.457)



takes the form

$$\left(\frac{\dot{A}}{A} \frac{\dot{B}}{B} + \frac{\dot{A}}{A} \frac{\dot{C}}{C} + \frac{\dot{B}}{B} \frac{\dot{C}}{C} \right) = \rho_m + 3(1 + 2\alpha)\lambda^2, \quad (0.272)$$

$$\left(\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}}{B} \frac{\dot{C}}{C} \right) = -3(1 - 2\alpha)\lambda^2, \quad (0.273)$$

$$\left(\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}}{A} \frac{\dot{C}}{C} \right) = -3(1 - 2\alpha)\lambda^2, \quad (0.274)$$

$$\left(\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}}{A} \frac{\dot{B}}{B} \right) = -3(1 - 2\alpha)\lambda^2. \quad (0.275)$$

The energy conservation equation

$$T_{\nu;\mu}^\mu = 0, \quad (0.276)$$

leads to

$$\dot{\rho}_m + 3\rho_m H = 0, \quad \dot{\lambda} + \frac{3\lambda}{(1 + 2\alpha)} H = 0. \quad (0.277)$$

where an overhead prime denote derivative with respect to time t only, ρ_m is the energy density of the matter. Subtracting (0.274) from (0.321), (0.275) from (0.321), (0.275) from (0.274) and then integrating twice, we get the relation

$$\frac{A}{B} = \beta_1 \exp \left(\gamma_1 \int \frac{dt}{V} \right), \quad (0.278)$$

$$\frac{A}{C} = \beta_2 \exp \left(\gamma_2 \int \frac{dt}{V} \right), \quad (0.279)$$

$$\frac{B}{C} = \beta_3 \exp \left(\gamma_3 \int \frac{dt}{V} \right), \quad (0.280)$$

where $\beta_1, \beta_2, \beta_3$ and $\gamma_1, \gamma_2, \gamma_3$ are integration constants.
so we get

$$A(t) = \Delta_1 a \exp \left(X_1 \int a^{-3} dt \right), \quad (0.281)$$

$$B(t) = \Delta_2 a \exp \left(X_2 \int a^{-3} dt \right), \quad (0.282)$$

$$C(t) = \Delta_3 a \exp \left(X_3 \int a^{-3} dt \right), \quad (0.283)$$

where $\Delta_1 = (\beta_1 \beta_2)^{\frac{1}{3}}$, $\Delta_2 = (\beta_1^{-1} \beta_3)^{\frac{1}{3}}$, $\Delta_3 = (\beta_2 \beta_3)^{-\frac{1}{3}}$, $X_1 = (\gamma_1 + \gamma_2)/3$, $X_2 = (\gamma_3 - \gamma_2)/3$ and $X_3 = -(\gamma_2 + \gamma_3)/3$ are constants that satisfy the following two relations

$$\Delta_1 \Delta_2 \Delta_3 = 1, \quad X_1 + X_2 + X_3 = 0. \quad (0.284)$$



Theorem 0.122. *To solve the field equations in Bianchi models we assume that $H = Da^{-n}$ where $D > 0$ and $n (\geq 0)$ are constant. By the statement of the Hubble coefficient H we obtain two expressions for the average scale factor $a(t)$*

$$a = (nDt + C_1)^{\frac{1}{n}} \quad \text{for } n \neq 0, \quad (0.285)$$

$$a = C_2 e^{Dt} \quad \text{for } n = 0, \quad (0.286)$$

where C_1 is constant of integration.

We are interested in taking the paper more precisely by solving the field equations by (0.286). The geometry of the space-time (0.457) takes the form

$$ds^2 = -dt^2 + T^{\frac{2}{n}} \left(\Delta_1^2 \exp \left\{ kX_1 T^{\frac{n-3}{n}} \right\} dx^2 + \Delta_2^2 \exp \left\{ kX_2 T^{\frac{n-3}{n}} \right\} dy^2 + \Delta_3^2 \exp \left\{ kX_3 T^{\frac{n-3}{n}} \right\} dz^2 \right), \quad (0.287)$$

where $T = nDt + C_1$ and $k = \frac{2}{D(n-3)}$. The other cosmological parameters of the model have the following expressions

$$H = D(nDt + C_1)^{-1}. \quad (0.288)$$

$$\theta = 3D(nDt + C_1)^{-1}. \quad (0.289)$$

$$A_p = \frac{X_1^2 + X_2^2 + X_3^2}{3D^2} (nDt + C_1)^{\frac{2(n-3)}{n}}, \quad (0.290)$$

$$\sigma^2 = \frac{1}{6} \{ (X_1 - X_2)^2 + (X_2 - X_3)^2 + (X_3 - X_1)^2 \} (nDt + C_1)^{-\frac{6}{n}}. \quad (0.291)$$

where H is the generalized mean Hubble parameter, θ is the expansion scalar, (σ^2) is the shear scalar, and A_p is the anisotropy parameter. The energy density is read as

$$\rho_m = \rho_0 (C_2 \exp \{Dt\})^{-3}, \quad (0.292)$$

where ρ_0 is constant of integration.



- we find Lambda terme in general relativity as follows

$$\Lambda = -6\lambda_0^2(C_2 e^{Dt})^{-\frac{6}{1+2\alpha}}. \quad (0.293)$$

where λ_0 is constant of integration. The dark energy density ρ_{de} and pressure p_{de} in this case as

$$\rho_{de} = 3\lambda_0^2(1+2\alpha)(C_2 \exp\{Dt\})^{-\frac{6}{1+2\alpha}}, \quad (0.294)$$

$$p_{de} = 3\lambda_0^2(1-2\alpha)(C_2 \exp\{Dt\})^{-\frac{6}{1+2\alpha}}. \quad (0.295)$$

In this paper, we have studied the Bianchi type-I universe with special law

of Hubble parameter. We discussed the model for what $n \neq 0$. The feedback obtained is presented below: The $\Lambda(t)$ is negative for all α values until zero at $t \rightarrow \infty$. The anisotropic parameter A_p tends to zero for a long time. It is also noted that $\lim_{t \rightarrow \infty} (\sigma^2/\theta) = 0$.

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Dynamic Properties of a Kirchhoff-Type Plate Equation with Damping Terms And Acoustic Control Boundary Conditions

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Abstract. In this paper, we study the energy of the solution decays exponentially or polynomially, for a Kirchhoff-type plate equation with damping terms and with Dirichlet boundary conditions at one part and acoustic boundary conditions at another part. We show that, by using integral inequalities and multiplier techniques. Instead of using a Lyapunov-type technique for some perturbed energy, where we concentrate on the original energy, showing that it satisfies integral inequality which, in turn, yields the final decay estimate

Keywords: Kirchhoff-type plate equation, decays exponentially, decays polynomially, Acoustic Control Boundary Conditions.



Introduction

In this section*, we consider the following initial boundary-value problem for the Kirchhoff-type plate equation with initial conditions and acoustic boundary conditions:

$$|u_t|^\rho u_{tt} + \Delta^2 u - M(\|\nabla u\|_2^2) \Delta u - \Delta u_{tt} + \int_0^t g(t-s) \Delta u(s) ds + u_t = 0, \quad x \in \Omega, t > 0, \quad (1.1)$$

$$u(x, t) = \frac{\partial u}{\partial \nu}(x, t) = 0, \quad \Delta u(x, t) = 0, \quad x \in \Gamma_1, t > 0, \quad (1.2)$$

$$M(\|\nabla u\|_2^2) \frac{\partial u}{\partial \nu} - \int_0^t g(t-s) \frac{\partial u(s)}{\partial \nu} ds + \frac{\partial u_{tt}}{\partial \nu} + \frac{\partial(\Delta u)}{\partial \nu} = y_t, \quad x \in \Gamma_0, t > 0, \quad (1.3)$$

$$u_t + \alpha(x)y_t + \beta(x)y = 0, \quad x \in \Gamma_0, t > 0, \quad (1.4)$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega. \quad (1.5)$$

Where Ω is a bounded domain in $\mathbb{R}^n (n \geq 1)$, with a sufficiently smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$, $mes(\Gamma_0) > 0$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, $\frac{\partial u}{\partial \nu}$ denotes the unit outer normal derivative, M is a positive C^1 -function, g represents the kernel of the memory term, y is the normal displacement to the boundary at time t at the boundary point x , $\rho > 0$ is a number and the functions $\alpha, \beta : \Gamma_1 \rightarrow \mathbb{R}^+$.

In this part, we will present certain notations, hypotheses, and lemmas that will be utilized in the proof of the results presented in this work. Additionally, we will use the standard $L^p(\Omega)$ norm and $L^p(\Gamma_1)$ norm, which will be denoted as $\|u\|_p$ and $\|u\|_{p, \Gamma_1}$, respectively.

We use the space

$$V = \{u \in H^1(\Omega) | u = 0 \text{ on } \Gamma_0\}, \quad (1.6)$$

the scalar products

$$(u, v) = \int_{\Omega} u(x)v(x)dx, \quad (u, v)_{\Gamma_0} = \int_{\Gamma_0} u(x)v(x)d\Gamma, \quad (1.7)$$

and the norms

$$\|u\|_{L^p(\Omega)} = \left(\int_{\Omega} |u|^p dx \right)^{\frac{1}{p}}, \quad \|u\|_{L^p(\Gamma_0)} = \left(\int_{\Gamma_0} |u|^p d\Gamma \right)^{\frac{1}{p}}. \quad (1.8)$$

The symbol $g * u$ stands for convolution, that is

$$g * u = \int_0^t g(t-s)u(s)ds,$$

and by $\circ$ we denote :

$$g \circ \nabla u(t) = \int_0^t g(t-s) \int_{\Omega} |\nabla u(s) - \nabla u(t)|^2 dx ds = \int_0^t g(t-s) \|\nabla u(s) - \nabla u(t)\|_2^2 ds.$$

We make the following general assumptions on M, α, β , and g .
 (A_1) . functions $\alpha(x)$ and $\beta(x)$ are assumed to satisfy the condition $\alpha(x), \beta(x) \in$



$C(\Gamma_0)$ and $\alpha(x), \beta(x) > 0$ for all $x \in \Gamma_0$. This assumption implies that there exist positive constants α_i and β_i , $i = 0, 1$, such that

$$\alpha_0 \leq \alpha(x) \leq \alpha_1, \quad \beta_0 \leq \beta(x) \leq \beta_1 \quad \text{for all } x \in \Gamma_0.$$

(A₂). The notation $M(s)$, $s \geq 0$ is used for a positive C^1 function satisfying

$$M(s) \geq m_0 > 0 \quad \text{and} \quad \bar{M}(s) \leq sM(s)$$

where $\bar{M}(s) = \int_0^s M(t)dt$.

(A₃). The notation $g : [0, \infty) \rightarrow [0, \infty)$ is used for a nonincreasing C^1 function satisfying

$$m_0 - \int_0^s g(s)ds = l > 0,$$

furthermore, there exists a $\rho \in (2, \infty]$, and a $\xi > 0$ such that

$$g'(t) \leq -\xi g^{1+\frac{1}{\rho}} \quad \text{for all } t \geq 0,$$

(for $\rho = \infty$, $\frac{1}{\rho} = 0$ is set to 0).

Remark 2.1. Assumption (A₂) implies that, for some $t_0 > 0$,

$$\int_0^t g(s)ds \geq \int_0^{t_0} g(s)ds := g_0 \quad \text{for all } t \geq t_0.$$

Remark 2.2. Assumption (A₂) implies that, for $\rho \in (2, \infty)$,

$$g(t) \leq \frac{k}{(1+t)^\rho} \quad \text{for all } t \geq 0.$$

Therefore,

$$g^\sigma(t) \in L^1(0, \infty) \quad \text{for all } t \geq 0 \quad \text{for any } \sigma > \frac{1}{\rho}.$$

In what follows, we shall use the following lemmas

Lemma 0.47. Let $E(t)$ be a nonnegative decreasing function defined on $[0, \infty)$. If

$$\int_s^{+\infty} E(t)dt \leq CE(s) \quad \text{for all } s \geq s_0,$$

for some constants s_0 and $C > 0$, then

$$E(t) \leq E(0)e^{1-\frac{t}{s_0+C}} \quad \text{for all } s \geq t \geq 0.$$

Lemma 0.48. Let $E(t)$ be a nonnegative decreasing function defined on $[0, \infty)$. If

$$\int_s^{+\infty} E(t)dt \leq CE(s) \quad \text{for all } s \geq s_0,$$

for some constants s_0 and $C > 0$, then

$$E(t) \leq E(0)e^{1-t/(s_0+C)} \quad \text{for all } t \geq 0.$$



Lemma 0.49. Let $E(t)$ be a nonnegative decreasing function defined on $[0, \infty)$. If

$$\int_s^{+\infty} E^{1+\theta}(t) dt \leq C E^\theta(0) E(s) \quad \text{for all } s \geq s_0,$$

for some constants s_0 and $C > 0$, then

$$E(t) \leq E(0) \left[\frac{(s_0 + C)(1 + \theta)}{\theta t + s_0 + C} \right]^{\frac{1}{\theta}} \quad \text{for all } t \geq 0.$$

Lemma 0.50. If $g \in C^1(\mathbb{R})$ and u is the solution to (1.1) – (1.6), then

$$\int_0^t g(t-s)(u(t), u_t(t)) ds = -\frac{1}{2} \frac{d}{dt} \left(g \circ u - \int_0^t g(t) ds \|u\|_2^2 \right) + \frac{1}{2} g' \circ u - g(t) \|u\|_2^2.$$

Main results

This work is devoted to provide decays exponentially or polynomially as $t \rightarrow +\infty$ of solution by using the multiplier techniques and some integral inequalities for the problem of the Kirchhoff-type plate equation with Acoustic Control Boundary Conditions by using the multiplier techniques and some integral inequalities computation inspires us to define energy functional as

$$\begin{aligned} E(t) = & \frac{1}{\rho+2} \|u_t\|_{\rho+2}^{\rho+2} + \frac{1}{2} \|\Delta u\|_2^2 + \frac{1}{2} \bar{M} (\|\nabla u\|_2^2) + \frac{1}{2} \left(g \circ \nabla u - \int_0^t g(t) ds \|\nabla u\|_2^2 \right) \\ & + \frac{1}{2} \|\nabla u_t\|_2^2 + \frac{1}{2} \int_{\Gamma_0} \beta(x) y^2(t) d\Gamma. \end{aligned} \quad (2.1)$$

Lemma 0.51. The energy functional $E(t)$ satisfies

$$E'(t) = -\|u_t\|_2^2 + \frac{1}{2} g' \circ \nabla u - g(t) \|\nabla u\|_2^2 - \int_{\Gamma_0} \alpha(x) y_t^2 d\Gamma \leq 0. \quad (2.2)$$

Lemma 0.52. From A_1 , A_2 and (2.1), we have the following inequality holds

$$\begin{aligned} E(t) \geq & \frac{1}{\rho+2} \|u_t\|_{\rho+2}^{\rho+2} + \frac{1}{2} \|\Delta u\|_2^2 + \frac{1}{2} g \circ \nabla u + \frac{l}{2} \|\nabla u\|_2^2 \\ & + \frac{1}{2} \|\nabla u_t\|_2^2 + \frac{1}{2} \int_{\Gamma_0} \beta(x) y^2(t) d\Gamma. \end{aligned}$$

Theorem 0.123. Let u be the global solution of problem (1.1) – (1.6) with conditions (A_1) – (A_3) . Then the following decay estimates are valid:

$$E(t) \leq E(0) \exp \left(1 - \frac{t}{C + t_0} \right), \quad \text{for all } t > 0.$$

$$E(t) \leq \left[\frac{(t_0 + C)(2 + \rho)}{2t + \rho(t_0 + C)} \right]^{\frac{\rho}{2}}, \quad \text{for all } t > 0$$



Proof By multiplying (1.1) by $\phi(t)u(t)$ (where $\phi(t) \rightarrow [0, \infty)$ is nonincreasing, function), integrating the result over $\Omega \times [t_1, t_2](t_0 \leq t_1 \leq t_2)$, and After using the given theories, lemmas and assumptions in the introduction of this researc, and also relying on (2.1) and the given lemmas in this section*, and by following several computational techniques, we obtain the results of our theory. ■

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Exploring Solutions to the Cauchy Problem for the Generalized Double Dispersion Equation

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Abstract. The objective of this work is to examine the global existence and finite-time blow-up of solutions for the Cauchy problem related to the one-dimensional fifth-order Boussinesq equation, which exhibits logarithmic nonlinear character. Initially, we demonstrate the existence and uniqueness of local mild solutions in the energy space through the application of the contraction mapping principle. Additionally, under certain constraints on the initial data, we present the findings on the existence and uniqueness of global solutions and finite-time blow-up of solutions through the utilisation of the potential well method.

Keywords: Cauchy problem, stable set and unstable set, existence of global solution, nonexistence



Introduction

The ability of nonlinear Boussinesq equations to model a wide range of physical phenomena makes them a valuable tool in the fields of mathematical physics and applied engineering. In particular, they are useful for modelling the motion of water waves with small-amplitude, long waves in nonlinear dispersive media. Furthermore, given the extensive literature on polynomial nonlinearity, logarithmic nonlinearity has also been the subject of considerable interest among physicists and mathematicians. It has a wide range of potential applications in various fields of physics. A significant body of research has been conducted into the local, global, existence, nonexistence, and limited time blow-up of solutions to the generalized Boussinesq equations, including those involving logarithmic nonlinearity.

This study investigates the Cauchy problem for the generalized double dispersion equations

$$u_{tt} - u_{xx} - u_{xxtt} + (u_{xx} + f(u))_{xx} - \alpha u_{xxt} + u_{xxxxt} = 0, \quad x \in \mathbb{R}, \quad t > 0, \quad (0.296)$$

subject to initial conditions values

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \mathbb{R}, \quad (0.297)$$

where $u = u(x, t)$ is the unknown real valued function of $x \in \mathbb{R}$ and $t > 0$, $f(u) = |u|^{p-2} u \ln |u|$, ($p > 2$) represents the nonlinearity, $\alpha > 0$ is a constant, and $u_0(x)$ and $u_1(x)$ are given initial data.

Main results

Motivated by prior research, our primary objective is to initially investigate the existence and uniqueness of local mild solutions to the Cauchy problem (0.296)-(0.297) within the energy space $H^1 \times H^{-1}$ with the initial data $(u_0; u_1) \in H^1 \times H^{-1}$. We employ the contraction mapping principle. Using the potential well method and the concavity method, we investigate the global existence and non-existence of mild solutions with initial energy levels $E(0) < d$ and $E(0) = d$. Furthermore, the investigation into the sufficiency and necessary conditions for the blowup of solutions presents an intriguing aspect. with $E(t)$ is the energy functional we will mention it.

Notably, to the best of our knowledge, there are no existing results on the global existence of a solution to the Cauchy problem for a one-dimensional Boussinesq-type equation of the fifth order.

Theorem 0.124. (Local Existence) Suppose $u_0 \in H^1$, $u_1 \in L^2$ and $\Lambda^{-1}u_1 \in L^2$, then there exists a maximal time $T_{\max}$ which depends only on u_0, u_1 such that the Cauchy problem (0.296)-(0.297) admits a unique local mild solution $u \in$



$C([0, T_{\max}); H^1)$, $\Lambda^{-1}u_t \in C([0, T_{\max}); L^2)$ with $\Lambda^{-k}u_{tt} \in C([\varepsilon, T_{\max}); L^2)$, $k = 0, 1$ satisfying

$$E(t) + \alpha \int_0^t \|u_\tau\|^2 d\tau + \int_0^t \|u_{x\tau}\|^2 d\tau = E(0).$$

Moreover if $T_{\max} < \infty$, then

$$\lim_{t \rightarrow T_{\max}} \sup (\|\Lambda^{-1}u_t\|^2 + \|u_t\|^2 + \|u\|_{H^1}^2) = +\infty.$$

Consequently

$$\lim_{t \rightarrow T_{\max}} \|u\|_{L^q}^2 = +\infty, \text{ for all } 1 < q < p + 1. \quad (0.298)$$

Theorem 0.125. Suppose $u_0 \in H^1$, $u_1 \in L^2$, $\Lambda^{-1}u_0 \in L^2$ and $\Lambda^{-1}u_1 \in L^2$ satisfying $(\Lambda^{-1}u_1, \Lambda^{-1}u_0) + (u_1, u_0) > 0$. Assume that $E(0) < d$ and $I(u_0) < 0$, then the solution of the problem (0.296)-(0.297) blows up in finite time, i.e the maximal existence time $T_{\max}$ of the local solution u for problem (0.296)-(0.297) is finite and

$$\lim_{t \rightarrow T_{\max}} (\|\Lambda^{-1}u_t\|^2 + \|u\|_{H^1}^2 + \|u_t\|^2) = +\infty.$$

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On the fourth-order Leray-Lions problem with indefinite weight and nonstandard growth conditions

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Abstract. We prove the existence of at least three weak solutions for the fourth-order problem with indefinite weight involving the Leray-Lions operator with nonstandard growth conditions. The proof of our main result uses variational methods and the critical theorem of Bonanno and Marano

Keywords: Leray-Lions type operator, critical theorem, generalized Sobolev space, variable exponent.



Introduction

In this paper, we shall show the existence of three weak solutions for the following interesting problem

$$\begin{cases} \Delta(a(x, \Delta u)) = \lambda V(x)|u|^{q(x)-2}u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \quad (0.299)$$

where Ω is a bounded domain in N ($N \geq 2$) with a smooth boundary $\partial\Omega$, $\lambda > 0$ is a parameter, V is a function in a generalized Lebesgue space $L^{s(x)}(\Omega)$, functions $p, q, s \in C(\overline{\Omega})$ satisfy the inequalities

$$1 < \min_{x \in \Omega} q(x) \leq \max_{x \in \Omega} q(x) < \min_{x \in \Omega} p(x) \leq \max_{x \in \Omega} p(x) \leq \frac{N}{2} < s(x) \text{ for all } x \in \Omega,$$

and $\Delta(a(x, \Delta u))$ is the Leray-Lions operator of the fourth-order, where a is a Carathéodory function satisfying some suitable supplementary conditions. For more details about this kind of operators the reader is referred to BOUREANU [?] and LERAY-LIONS [4] (and the references therein).

Note that the study of this type of operators is very active in several fields, e.g. in electrorheological fluids, elasticity, stationary thermorheological viscous flows of non-Newtonian fluids, image processing, and mathematical description of the processes filtration of barotropic gas through a porous medium.

Similar problems have been studied before by various authors. In particular, (1) studied the following problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}u) = \lambda V(x)|u|^{q(x)-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (0.300)$$

Under the condition in problem (0.299), he has shown that problem (0.488) has a continuous spectrum and his main argument was the Ekeland variational principal.

Before introducing our main result, we define

$$C_+(\overline{\Omega}) := \{h \mid h \in C(\overline{\Omega}), h(x) > 1, \text{ for all } x \in \overline{\Omega}\},$$

and for $\eta > 0$, $h \in C_+(\overline{\Omega})$, we set

$$h^- := \inf_{x \in \Omega} h(x), \quad h^+ := \sup_{x \in \Omega} h(x)$$

and

$$[\eta]^h := \sup\{\eta^{h^-}, \eta^{h^+}\}, \quad [\eta]_h := \inf\{\eta^{h^-}, \eta^{h^+}\}.$$

Remark 0.10. It is easy to verify that the following holds

$$[\eta]^{\frac{1}{h}} = \sup\{\eta^{\frac{1}{h^-}}, \eta^{\frac{1}{h^+}}\}, \quad [\eta]_{\frac{1}{h}} = \inf\{\eta^{\frac{1}{h^-}}, \eta^{\frac{1}{h^+}}\}.$$

We denote

$$\delta(x) := \sup\{\delta > 0 \mid B(x, \delta) \subseteq \Omega, \text{ for all } x \in \Omega\},$$

where B is the ball of radius δ centered at x . One can prove that there exists $x_0 \in \Omega$ such that $B(x_0, D) \subseteq \Omega$, where $D := \sup_{x \in \Omega} \delta(x)$.



Main results

Theorem 0.126. Assume that hypotheses $(H_1) - (H_5)$ are fulfilled and that there exist $r > 0$ and $d > 0$ such that

$$r < \frac{1}{p^+} \left[\frac{2d(N-1)}{D^2} \right]_p L, \quad (0.301)$$

and

$$\begin{aligned} \bar{w}_r &:= \frac{1}{r} \left\{ \frac{p^+ \frac{q^+}{p^-}}{q^-} [k]^q |V|_{s(x)} \left[[r]^{\frac{1}{p}} \right]^p \right\} \\ &< \gamma_d := \frac{v_0[d]_q}{c_3(2^N - 1) \left(|\alpha|_{\frac{p(x)}{p(x)-1}} \frac{4d(N-1)}{D^2} L^{\frac{1}{p^+}-1} + \left[\frac{4d(N-1)}{D^2} \right]^p \right)}. \end{aligned} \quad (0.302)$$

Then for every $\lambda \in \bar{\Lambda}_r := \left(\frac{1}{\gamma_d}, \frac{1}{\bar{w}_r} \right)$, problem (0.299) admits at least three weak solutions.

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Existence of solutions for a second order problem on the real half-line via Ekeland's variational principle

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Abstract. This work is concerned with the following problem:

$$\begin{cases} -u''(x) + u(x) = \lambda q(x)f(x, u(x)) & x \in (0, +\infty), \\ u(0) = u(+\infty) = 0, \end{cases}$$

where λ is a positive parameter, $q \in L^1(0, +\infty) \cap L^\infty(0, +\infty)$ and $f \in C(\mathbb{R})$. First of all, we establish the functional energy associated to the above problem and we prove that it is of class C^1 on the working space $H_0^1(0, +\infty)$. Next, we determine a compactness result which serves as an essential ingredient in our approach. Moreover, we prove some technical lemmas that we will need to prove the main result of this work. Finally, by using the variational principle of Ekeland, we show the existence of solutions to the above problem.

Keywords: Ordinary differential equation, Variational methods, Critical point, Ekeland's variational principle.



Introduction

Boundary value problems with Dirichlet conditions are an area of the fast developing differential equations theory. Problems of this type arise in various fields of physics, biology, biotechnology etc. Thus, existence and multiplicity of solutions to boundary value problems on the half-line were studied by many authors.

In this work, we are interested with the existence of solutions for the following problem

$$\begin{cases} -u''(x) + u(x) = \lambda q(x)f(x, u(x)) & x \in (0, +\infty), \\ u(0) = u(+\infty) = 0, \end{cases} \quad (0.303)$$

where $\lambda > 0$, f and q satisfy the following assumptions:

(H_0) there exist constants $a, b > 0$ and $\theta \in (0, 1)$ such that $|f(x, u)| \leq a|u|^\theta + b$, $x \in [0, +\infty), \forall u \in \mathbb{R}$,

(H_1) $p : [0, +\infty) \rightarrow (0, +\infty)$ is continuously differentiable and bounded, $q : [0, +\infty) \rightarrow (0, +\infty)$, with $q \in L^1(0, +\infty) \cap L^\infty(0, +\infty)$, $\frac{q}{p^\theta}, \frac{q}{p^2} \in L^1(0, +\infty)$,

$M_0 = \int_0^{+\infty} q(x) \left(\int_x^{+\infty} \frac{ds}{p(s)} \right) dx < +\infty$, and $M = \max\{\|p\|_2, \|p'\|_2\} < +\infty$,

(H_2) $f(x, 0) = 0$, $\lim_{u \rightarrow 0} \frac{f(x, u)}{u} = +\infty$ and $\lim_{u \rightarrow +\infty} \frac{f(x, u)}{u} = 0$, uniformly for $x \in [0, +\infty)$.

We consider our functional space $H_0^1(0, +\infty)$ and the functional energy defined on it by

$$I(u) = \frac{1}{2}\|u\|^2 - \lambda \int_0^{+\infty} q(x)F(x, u)dx.$$

Main result

The main result of this work is the following.

Theorem 0.127. *Suppose (H_0) – (H_2) hold. Then problem (0.303) possesses at least one solution u_λ for every $\lambda \in \left(0, \frac{1}{\|p\|_{L^\infty} M_0}\right)$.*

In order to prove Theorem 1.1, we will need the following technical lemmas.

Lemma 0.53. *The functional I is bounded from below on $\overline{B}_\rho$ where $\rho > 0$.*

Lemma 0.54. *We have $c_b := \inf_{u \in \overline{B}_\rho} I(u) < 0$.*

Our abstract tool is the variational principle of Ekeland.



Proposition 0.15. *Let M be a complete metric space with metric d and let $J : M \longrightarrow \mathbb{R}$ be a lower semicontinuous functional, and bounded from below. Then, for each $\varepsilon > 0$, there exists $u_\varepsilon \in M$ such that*

$$J(u_\varepsilon) \leq \inf_M J + \varepsilon,$$

and whenever $w \in M$

$$J(u_\varepsilon) \leq J(w) + \varepsilon d(u_\varepsilon, w).$$

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Mathematical Modeling of Pest Resistance to Insecticides in Heterogeneous Environments.

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Abstract. We consider a more complex model that incorporates the movement of the insect between patches. By taking advantage of the fact that the dynamical processes in the model evolve on different time scales a fast one for the movement between patches and a slow one, corresponding to the natural time, for the demography we establish global stability results. Numerical simulations provide interesting insights into the dynamics of the pest population. Building on this foundation, we next formulate a model describing the growth of a pest population with seasonal diapause at the larval stage.

Keywords: Aggregated model, Cooperative systems, Insects, Seasonal diapause, Global attractivity, Slow fast dynamics, Numerical simulation.

Introduction

With global warming, the risk of contamination of vineyards by the European Grapevine Moth is increasing globally, particularly in the Mediterranean



basin. This pest should be considered a potential danger on a world- wide scale, as the damages are caused during the larval stage. Research has been conducted to study the behavior and life cycle of the insect *Lobesia botrana*. The acquisition of this knowledge has gradually led to the construction of mathematical models that describe the evolution of insect densities over time and space with varying degrees of precision. Integrated pest management programs typically include chemical and biological controls that act at different stages of the pest's life cycle. The most commonly used method is the dispersion of pheromones to disrupt mating. While chemical control has been extensively used to reduce pest populations, in some areas, the insect has developed resistance to various classes of chemical insecticides. This resistance represents a significant obstacle to the sustainability of pest control. Understanding insect resistance is particularly important when treatments require larger quantities or new and more expensive pesticides to effectively control pest populations. The insect develops resistance through a genetic mutation that neutralizes the ingested pesticides. This mutation disrupts the insect's hormonal function, causing individuals to enter an early diapause or delay their emergence into the adult phase. However, only a few studies have investigated the phenomenon of *Lobesia botrana* diapause, which is still not well understood. Diapause is a strategy adopted by the insect to survive harsh environmental conditions, influenced by factors such as photoperiod (short days and long nights) and the gradual drop in temperature.

Main results

*

*Global Stability

Theorem

- If $R_0 \leq 1$, then $(0, 0, 0)$ is globally asymptotically stable in $\mathbb{R}_+^3$.
- If $R_0 > 1$, then (L^*, V^*, R^*) is globally asymptotically stable in $\mathbb{R}_+^3 \setminus \{(L, V, R) : L = 0\}$.

*

*Asymptotic Behavior and Stability

Proposition

- The trivial equilibrium point of system (5) is globally asymptotically stable if $\max(R_0^1, R_0^2, R_{\text{cri}}) < 1$. (11)
- In case (I), the positive interior equilibrium of (5) is globally asymptotically stable in $\mathbb{R}_+^4 \setminus \{(L_1, L_2, V, R) : L_1 = L_2 = 0\}$.



*

*Existence and Uniqueness of a Positive Periodic Solution

Theorem

For each $u_0 \in \mathbb{R}_+^3$, there exists a unique positive 1-periodic solution $u^*(t)$ of the system.

*

Numerical Simulations

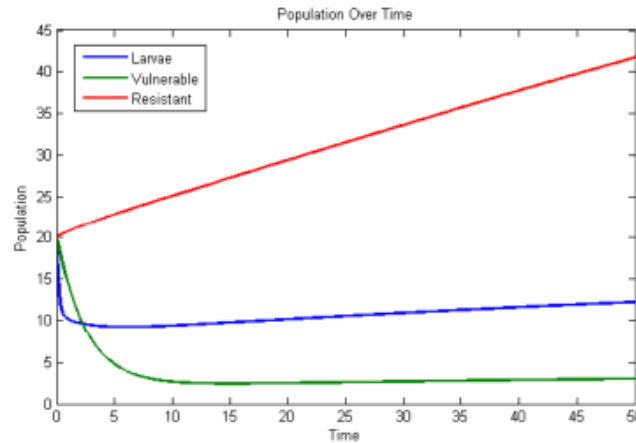


Figure 22: Trajectories of the solutions of the first system for $R_0 = 10.31 > 1$.



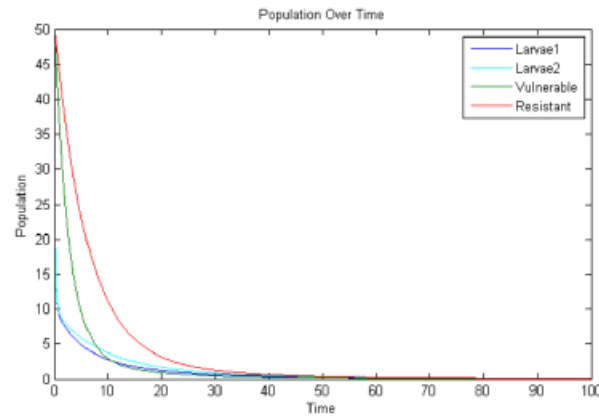


Figure 23: Behavior of the trajectories of the second system for a basic reproduction rate $R_0^i < 1$, $i = 1, 2$, and $R_{cri} < 1$.

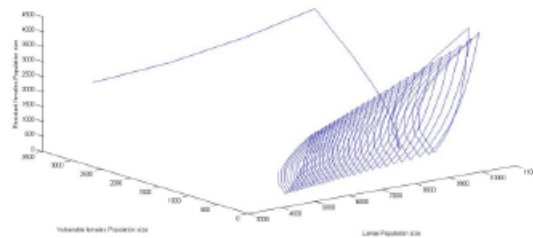


Figure 24: Phase portrait corresponding to the periodic orbit.



Conclusion

The model we have studied in this paper can be applied to other pests with the same characteristics. We have studied the nonlinear dynamics of a three dimensional differential system describing the evolution of a vineyard pest population called *L. botrana*. We have established the well-posedness of the system. The model has two equilibriums, and we have determined a sharp threshold R_0 whose value determines whether the pest population goes to extinction or persists. Under suitable conditions, the proposed model undergoes a transcritical bifurcation. Although some parameters are difficult to obtain, like the proportion of new resistant insects or the insect mortality induced by insecticides, numerical simulations are provided and agree with the theoretical results.

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$\frac{n}{2}$ -normal operators on Hilbert spaces

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Abstract. In this paper, we extend the concept of $\frac{n}{2}$ -normal operators, which is weaker than the case of normal operators. Furthermore, we study this class of operators and give some of their basic properties.

Keywords: $\frac{n}{2}$ -normal, unitary equivalent, tensor sum, tensor product.

Introduction

Let $\mathcal{H}$ be an infinite dimensional separable complex Hilbert space. Let $B(\mathcal{H})^+$ be the algebra of all bounded linear operators defined in $\mathcal{H}$. Let T be an operator



in $B(\mathcal{H})^+$. The operator T is called normal if it satisfies the following condition $T^*T = TT^*$, i.e., (T commutes with T^*); self adjoint if $T^* = T$; skew adjoint if $T^* = -T$; and projection if $T^* = T = T^2$; invertible with inverse S if there exists $S \in B(\mathcal{H})^+$ such that $ST = TS = I$, where $I \in B(\mathcal{H})^+$ is the identity operator.

For an operator $T \in B(\mathcal{H})^+$, if $T^*T = TT^* = I$ (or equivalent or equivalently $T^* = T^{-1}$), then T is called an unitary.

Main results

Definition 0.56. Let $T \in B(\mathcal{H})^+$

$$T^{\frac{n}{2}} = \{S \in B(\mathcal{H})^+; S^{\frac{2}{n}} = T\}$$

we have

$$S^{\frac{2}{n}} = (S^{\frac{1}{n}})^2$$

so

$$S^{\frac{1}{n}} = \{A \in B(\mathcal{H})^+; A^n = S\}$$

thus

$$A^2 = T.$$

Proposition 0.16. If $T \in B(\mathcal{H})^+$ is $\frac{n}{2}$ -normal operator, then $T^{\frac{n}{2}}$ is normal operator.

Theorem 0.128. If $T \in \frac{n}{2}$ -normal then are

1. kT for any real number k .
2. Any $S \in B(\mathcal{H})^+$ that is unitary equivalent to T .
3. The restriction $T|_M$ of T to any closed subspace M of $\mathcal{H}$ that reduces T .

Proposition 0.17. Every self-adjoint is $\frac{n}{2}$ -normal operator.

Lemma 0.55. Let T be $\frac{k}{2}$ -normal and $(\frac{k}{2}+1)$ -normal. If either T or T^* is injective, then T is normal.

Theorem 0.129. Let $T_1, T_2, \dots, T_m$ be $\frac{n}{2}$ -normal operators on $B(\mathcal{H})^+$, then $T_1 \oplus T_2 \oplus \dots \oplus T_m$ is a $\frac{n}{2}$ -normal operator.

Theorem 0.130. Let $T_1, T_2, \dots, T_m$ be $\frac{n}{2}$ -normal operators on $B(\mathcal{H})^+$, then $T_1 \otimes T_2 \otimes \dots \otimes T_m$ is a $\frac{n}{2}$ -normal operator.

Proposition 0.18. Let $T \in B(\mathcal{H})^+$, $F = T^{\frac{n}{2}} + T^*$, and $G = T^{\frac{n}{2}} - T^*$. Then T is an $\frac{n}{2}$ -normal operator if and only if G commutes with F .



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Growth of Solutions to Linear Differential Equations with [p, q]-Order

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Abstract. In the present paper, we investigate the growth and fixed points of solutions to complex linear differential equations of the form

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0,$$

where the coefficients $A_{k-1}(z), \dots, A_0(z)$ are entire functions of $[p, q]$ -order.

Keywords: Entire function, fixed points, differential equation, $[p, q]$ -order.



Introduction

Throughout this paper, we assume the reader is familiar with the basic results and standard notations from Nevanlinna's value distribution theory of meromorphic functions (see [2], [3]). For all $r \in \mathbb{R}$, we define $\exp_1 r := e^r$ and, recursively, $\exp_{p+1} r := \exp(\exp_p r)$ for $p \in \mathbb{N}$. Likewise, for sufficiently large $r \in (0, +\infty)$, we define $\log_1 r := \log r$ and, recursively, $\log_{p+1} r := \log(\log_p r)$ for $p \in \mathbb{N}$. Additionally, we set $\exp_0 r := r = \log_0 r$, $\log_{-1} r := \exp_1 r$, and $\exp_{-1} r := \log_1 r$. We now introduce the concept of $[p, q]$ -order for meromorphic and entire functions.

Definition 0.57. ([1])

Let $p \geq q \geq 1$ be integers and f be a meromorphic function. Then, the $[p, q]$ -order of f is defined by

$$\sigma_{[p,q]}(f) = \limsup_{r \rightarrow +\infty} \frac{\log_p^+ T(r, f)}{\log_q r}.$$

For an entire function f we also define

$$\sigma_{[p,q]}(f) = \limsup_{r \rightarrow +\infty} \frac{\log_{p+1}^+ M(r, f)}{\log_q r} = \limsup_{r \rightarrow +\infty} \frac{\log_p^+ T(r, f)}{\log_q r}.$$

Main results

In this paper, we extend the recent results of Belaïdi and Belmiloud ([1]) on the oscillation of solutions to linear differential equations with analytic coefficients of $[p, q]$ -order in the unit disc to the entire complex plane, and obtain the following findings.

Theorem 0.131. *Let $p \geq q \geq 1$ be integers. Let H be a set of complex numbers satisfying $\text{dens} \{|z| : z \in H\} > 0$, and let $A_0, \dots, A_{k-1}$ be entire functions such that*

$$\max \{ \sigma_{[p,q]}(A_i) : i = 1, 2, \dots, k-1 \} \leq \sigma_{[p,q]}(A_0) = \mu \quad (0 < \mu < +\infty)$$

and for a constant $\alpha \geq 0$, we have

$$\liminf_{|z| \rightarrow +\infty, z \in H} \frac{\log_p |A_0(z)|}{(\log_{q-1} |z|)^\mu} > \alpha$$

and

$$|A_i(z)| \leq \exp_p \{ \alpha (\log_{q-1} |z|)^\mu \} \quad (i = 1, \dots, k-1)$$

as $|z| \rightarrow +\infty$ for $z \in H$. Then every solution $f \not\equiv 0$ of equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \dots + A_1(z) f' + A_0(z) f = 0, \quad (0.304)$$

satisfies $\sigma_{[p,q]}(f) = \infty$ and $\sigma_{[p+1,q]}(f) = \mu$.



Corollary 0.4. *Let $p \geq q \geq 1$ be integers. Let H be a set of complex numbers satisfying $\overline{\text{dens}} \{|z| : z \in H\} > 0$, and let $A_0, \dots, A_{k-1}$ be entire functions such that*

$$\max \{ \sigma_{[p,q]}(A_i) : i = 1, 2, \dots, k-1 \} \leq \sigma_{[p,q]}(A_0) = \mu \quad (0 < \mu < +\infty)$$

and for some real constants α, β where $0 \leq \beta < \alpha$, we have

$$|A_0(z)| \geq \exp_p \{ \alpha (\log_{q-1} |z|)^\mu \}$$

and

$$|A_i(z)| \leq \exp_p \{ \beta (\log_{q-1} |z|)^\mu \}, \quad i = 1, \dots, k-1$$

as $|z| \rightarrow +\infty$ for $z \in H$. Then every solution $f \not\equiv 0$ of equation (0.468) satisfies $\sigma_{[p,q]}(f) = \infty$ and $\sigma_{[p+1,q]}(f) = \mu$.

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On A -normal in semi Hilbertian space

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Abstract. Let $\mathcal{H}$ be a Hilbert space and let A be a positive bounded operator on $\mathcal{H}$. The semi-inner product $\langle u | v \rangle_A := \langle Au | v \rangle$, $u, v \in \mathcal{H}$ induces a semi-norm $\|\cdot\|_A$ on $\mathcal{H}$. This makes $\mathcal{H}$ into a semi-Hilbertian space. In this paper, we introduce a class of operators called A -normal operator in semi-Hilbertian spaces. Some structural properties of this class of operators are established. An operator $T \in \mathcal{B}_A(\mathcal{H})$ is said to be A -normal if

$$T^\sharp T = TT^\sharp$$

equivalently

$$\langle (T^\sharp T - TT^\sharp)x | x \rangle_A = 0 \quad \forall x \in \mathcal{H}$$

Keywords: semi-inner product; normal operator ; A -positive operator ; A -normal operator.



Introduction

The theory of operators classes on Hilbert space has developed quite rapidly during the last decades. In particular the classes of normal operators on Hilbert spaces have been the object of some intensive studies. The classes of normal on Hilbert spaces has been presented by several authors. In very recent years, these classes of operators have been generalized in semi-Hilbertian spaces in the papers and other papers.

In this framework, we are interested to introducing a class in semi-Hilbertian spaces called A -normal operators.

We start by introducing some notations. The symbol $\mathcal{H}$ stands for a complex Hilbert space with inner product $\langle \cdot | \cdot \rangle$ and norm $\|\cdot\|$. We denote by $\mathcal{B}(\mathcal{H})$ the Banach algebra of all bounded linear operators on $\mathcal{H}$, $I = I_{\mathcal{H}}$ being the identity operator. $\mathcal{B}(\mathcal{H})^+$ is the cone of positive (semi-definite) operators, i.e.,

$$\mathcal{B}(\mathcal{H})^+ = \{A \in \mathcal{B}(\mathcal{H}) : \langle Au | u \rangle \geq 0, \forall u \in \mathcal{H}\}$$

For every $T \in \mathcal{B}(\mathcal{H})$ its range is denoted by $\mathcal{R}(T)$, its null space by $\mathcal{N}(T)$ and its adjoint by T^* . If $\mathcal{M} \subset \mathcal{H}$ is a closed subspace, $P_{\mathcal{M}}$ is the orthogonal projection onto $\mathcal{M}$. the closure of $\mathcal{R}(T)$ will be denoted by $\overline{\mathcal{R}(T)}$.

In addition, if $T, S \in \mathcal{B}(\mathcal{H})$, then $T \geq S$ means that $T - S \geq 0$. Any $A \in \mathcal{B}(\mathcal{H})^+$ define a positive semi-definite sesquilinear form, denoted by

$$\begin{aligned} \langle \cdot | \cdot \rangle_A : \mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) &\rightarrow \mathbb{C} \\ (u; v) &\mapsto \langle u | v \rangle_A = \langle Au | v \rangle \end{aligned}$$

We remark that $\langle u | v \rangle_A = \left\langle A^{\frac{1}{2}}u \middle| A^{\frac{1}{2}}v \right\rangle$. The semi-norm induced by $\langle \cdot | \cdot \rangle_A$, which is denoted by $\|\cdot\|_A$, is given by $\|u\|_A = \langle u | u \rangle_A^{\frac{1}{2}} = \left\| A^{\frac{1}{2}}u \right\|$. This makes $\mathcal{H}$ into a semi-Hilbertian space.

Observe that $\|u\|_A = 0$ if and only if $u \in \mathcal{N}(A)$. then $\|\cdot\|_A$ is a norm if and only if A is an injective operator, and the semi-normed space $(\mathcal{B}(\mathcal{H}), \|\cdot\|_A)$ is complete if and only if $\mathcal{R}(A)$ is closed. Moreover $\langle \cdot | \cdot \rangle_A$ induced a seminorm on a certain subspace of $\mathcal{B}(\mathcal{H})$, namely, on the subset of all $T \in \mathcal{B}(\mathcal{H})$ for witch there exists a constant $c > 0$ such that $\|Tu\|_A \leq c\|u\|_A$ for every $u \in \mathcal{H}$ (T is called A -bounded).

Main results

Definition 0.58. Let $T \in \mathcal{B}(\mathcal{H})$, an operator $S \in \mathcal{B}(\mathcal{H})$ is called an A -adjoint of T if for every $u, v \in \mathcal{H}$

$$\langle Tu | v \rangle_A = \langle u | Sv \rangle_A$$

i.e., $AS = T^*A$.



It was observed in that it is possible for an operator T does not have an A -adjoint, and if S is an A -adjoint of T we may find many A -adjoints; In fact, in $AR = 0$ for some $R \in \mathcal{B}(\mathcal{H})$, then $S + R$ is an A -adjoint of T . The set of all A -bounded operators which admit an A -adjoint is denoted by $\mathcal{B}_A(\mathcal{H})$. By Douglas Theorem we have that

$$\mathcal{B}_A(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) / \mathcal{R}(T^*A) \subset \mathcal{R}(A)\}$$

If $T \in \mathcal{B}_A(\mathcal{H})$, then there exists a distinguished A -adjoint operator of T . namely, the reduced solution of equation $AX = T^*A$, i.e., $A^\dagger T^*A$. This operator is denoted by $T^\sharp$. Therefore, $T^\sharp = A^\dagger T^*A$ and

$$AT^\sharp = T^*A, \mathcal{R}(T^\sharp) \subset \overline{\mathcal{R}(A)} \text{ and } \mathcal{N}(T^\sharp) = \mathcal{N}(T^*A).$$

Note that in which $A^\dagger$ is the Moore-Penrose inverse of A . We recapture the following proposition for the convenience of the readers.

Proposition 0.19. *Let $T \in \mathcal{B}_A(\mathcal{H})$. Then the following statements hold.*

- (1) $T^\sharp \in \mathcal{B}_A(\mathcal{H})$, $(T^\sharp)^\sharp = P_{\overline{\mathcal{R}(A)}} T P_{\overline{\mathcal{R}(A)}}$ and $(T^\sharp)^\sharp = T^\sharp$.
- (2) If $S \in \mathcal{B}_A(\mathcal{H})$ then $TS \in \mathcal{B}_A(\mathcal{H})$ and $(TS)^\sharp = S^\sharp T^\sharp$.
- (3) $\|T\|_A = \|T^\sharp\|_A = \|T^\sharp T\|_A^{\frac{1}{2}} = \|TT^\sharp\|_A^{\frac{1}{2}}$.
- (4) If $S \in \mathcal{B}_A(\mathcal{H})$ then $\|TS\|_A = \|ST\|_A$.

In the following we introduce the notion of A -normal operators.

Definition 0.59. [1] An operator $T \in \mathcal{B}_A(\mathcal{H})$ is called an A -normal if $TT^\sharp = T^\sharp T$.

A -normal operators may be regarded as a generalization of normal operators in which $T^\sharp = T^*$. This last property is realized in particular if $A = I$ or if T and A commute and A has a dense range

Theorem 0.132. [2, Theorem 3.1] Let $T \in \mathcal{B}_A(\mathcal{H})$. The following conditions are equivalent :

1. $T \in \mathcal{B}(\mathcal{H})$ is A -normal operator.
2. $\mathcal{R}(TT^\sharp) \subset \overline{\mathcal{R}(A)}$ and $\|Tu\|_A = \|T^\sharp u\|_A$ for all $u \in \mathcal{H}$.

Proof $1 \Rightarrow 2$: Assume that T is a A -normal. This implies that $\mathcal{R}(TT^\sharp) \subset \mathcal{R}(T^\sharp T) \subset \overline{\mathcal{R}(A)}$ since $\mathcal{R}(T^\sharp) \subset \overline{\mathcal{R}(A)}$. Furthermore, for all $u \in \mathcal{H}$ we see that

$$\begin{aligned} T^\sharp T &= TT^\sharp \Rightarrow \langle AT^\sharp Tu/u \rangle = \langle ATT^\sharp u/u \rangle \\ &\Rightarrow \langle T^* ATu/u \rangle = \langle (T^\sharp)^* AT^\sharp u/u \rangle \\ &\Rightarrow \langle ATu/Tu \rangle = \langle AT^\sharp u/T^\sharp u \rangle \\ &\Rightarrow \|Tu\|_A = \|T^\sharp u\|_A \end{aligned}$$

(2) $\Rightarrow$ (1) In a similar way it can be shown that the fact $\|Tu\|_A = \|T^\sharp u\|_A$ for every $u \in \mathcal{H}$ gives

$$\langle A(T^\sharp T - TT^\sharp)u/u \rangle = 0$$



for any $u \in \mathcal{H}$ that $A(T^\sharp T - TT^\sharp) = 0$, which leads to

$$\mathcal{R}(T^\sharp T - TT^\sharp) \subset \mathcal{N}(A)$$

Moreover, since $\mathcal{R}(TT^\sharp) \subset \overline{\mathcal{R}(A)}$, we obtain

$$\mathcal{R}(T^\sharp T - TT^\sharp) \subset \overline{\mathcal{R}(A)} = \mathcal{N}(A)^\perp$$

hence $T^\sharp T - TT^\sharp = 0$. ■

Proposition 0.20. *Let $T, S \in \mathcal{B}_A(\mathcal{H})$ are A -normal operator such that $ST^\sharp = T^\sharp S$, then $T + S$ is Class A -normal operator.*

Proof Let T and S are A -normal

$$\begin{aligned} (T + S)^\sharp(T + S) &= (T^\sharp + S^\sharp)(T + S) \\ &= T^\sharp T + T^\sharp S + S^\sharp T + S^\sharp S \\ &= TT^\sharp + T^\sharp S + S^\sharp T + SS^\sharp \quad (T \text{ and } S \text{ are } A\text{-normal}) \\ &= TT^\sharp + ST^\sharp + TS^\sharp + SS^\sharp \\ &= (T + S)(T^\sharp + S^\sharp) \end{aligned}$$

then $T + S$ is A -normal operator. ■

Proposition 0.21. *Let $T, S \in \mathcal{B}_A(\mathcal{H})$ are an A -normal operator such that $ST^\sharp = T^\sharp S$, then TS is Class A -normal operator.*

Proof Let T and S are an A -normal

$$\begin{aligned} (TS)^\sharp(TS) &= S^\sharp T^\sharp TS \\ &= S^\sharp TT^\sharp S \quad (T \text{ is } A\text{-normal}) \\ &= TS^\sharp ST^\sharp \\ &= TSS^\sharp T^\sharp \quad (S \text{ is } A\text{-normal}) \\ &= (TS)(TS)^\sharp \end{aligned}$$

then TS is A -normal operator. ■

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generalized mean transforms of closed range operators

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Abstract. Let $T \in \mathcal{B}(\mathcal{H})$ be a bounded linear operator on a Hilbert space $\mathcal{H}$. In this talk, we investigate when an operator T and its generalized mean transform both have closed ranges, and show that this transform preserves the class of EP operators.

Keywords: Hilbert space , Generalized mean transform , Moore-Penrose inverse, EP operators.

Introduction

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. For an arbitrary operator $T \in \mathcal{B}(\mathcal{H})$, we shall denote



by $\mathcal{R}(T)$, $\mathcal{N}(T)$ and T^* the range, the null subspace and the adjoint operator of T , respectively. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be positive if $\langle Tx, x \rangle \geq 0$, for all $x \in \mathcal{H}$, normal if $TT^* = T^*T$, quasinormal if T and T^*T commute and hyponormal if $T^*T \geq TT^*$. The Moore-Penrose inverse of $T \in \mathcal{B}(\mathcal{H})$ is the unique operator $T^+ \in \mathcal{B}(\mathcal{H})$ which satisfies:

$$TT^+T = T, \quad T^+TT^+ = T^+, \quad (TT^+)^* = TT^+, \quad (T^+T)^* = T^+T.$$

It is well known that the Moore-Penrose inverse of T exists if and only if $\mathcal{R}(T)$ is closed. It is easy to see that $R(T^+) = R(T^*)$, TT^+ is the orthogonal projection of $\mathcal{H}$ onto $R(T)$ and that T^+T is the orthogonal projection of $\mathcal{H}$ onto $R(T^*)$. The operator T is said to be EP operator, if $R(T)$ is closed and $TT^+ = T^+T$. Clearly

$$T \text{ EP} \iff \mathcal{R}(T) = \mathcal{R}(T^*) \iff \mathcal{N}(T) = \mathcal{N}(T^*).$$

Recall that for $T \in \mathcal{B}(\mathcal{H})$, there is a unique polar decomposition $T = U|T|$, where $\mathcal{N}(U) = \mathcal{N}(T) = \mathcal{N}(|T|)$, U is a partial isometry, i.e. $UU^*U = U$ and $|T| = (T^*T)^{\frac{1}{2}}$ is the modulus of T . This factorization is called the canonical polar decomposition of T . From the polar decomposition, the generalized mean transform of T is defined by

$$\hat{T}_\lambda = \frac{1}{2}(|T|^\lambda U |T|^{1-\lambda} + |T|^{1-\lambda} U |T|^\lambda),$$

where $\lambda \in [0, 1]$. Especially, $\hat{T}_0 = \hat{T}$ and $\hat{T}_{\frac{1}{2}} = \Delta(T)$.

In recent years, the relationship between operators on a Hilbert space and their generalized mean transforms have been investigated by many authors for example, S.H. Lee et al [2] showed that quasinormal operators are exactly the fixed points of the mean transform. The aim of this paper is to investigate the generalized mean transform of closed range operators.

Main results

First, we prove that under certain conditions the operator T and its generalized mean transform both have closed ranges.

Theorem 0.133. *Let $\lambda \in]0, \frac{1}{2}]$ and let $T \in \mathcal{B}(\mathcal{H})$, such that $\mathcal{N}(T) \subset \mathcal{N}(T^*)$. Then*

$$\mathcal{R}(T) \text{ is closed} \iff \mathcal{R}(\hat{T}_\lambda) \text{ is also closed.}$$

As a consequence of the previous Theorem, we get the following result

Corollary 0.5. *Let $T \in \mathcal{B}(\mathcal{H})$ be hyponormal. Then*

$$\mathcal{R}(T) \text{ is closed} \iff \mathcal{R}(\hat{T}_\lambda) \text{ is also closed.}$$

In the following theorem, we prove that the generalized mean transform preserves the class of EP operators.

Theorem 0.134. *Let $T \in \mathcal{B}(\mathcal{H})$. Then for $\lambda \in [0, \frac{1}{2}]$, we have*

$$T \text{ is an EP operator} \iff \hat{T}_\lambda \text{ is EP and } R(T) = R(\hat{T}_\lambda).$$



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Stability and boundedness of solutions for some second-order non-autonomous vector delay differential equations of neutral type

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Abstract.

In the present work, sufficient conditions for the boundedness and the asymptotic stability of zero solution are obtained for a given class of neutral nonlinear differential equations of second order with delay. In addition the square integrability of derivatives of solutions for this class of differential equations is proven using the Lyapunov-Krasovskii functional approach.

Keywords: Stability, Lyapunov functional, boundedness, neutral differential equation of second order.



Introduction

The equation of Hill is one of the equations that are very important in the applications of differential equations, this equation is given by

$$\ddot{x} + a(t)x = 0 \quad (0.305)$$

Some qualitative properties of this equation are explored by some physicists. Many authors are attracted by the qualitative behaviours of the equation of Hill. For instance, Yang [3] has investigated the Hill equation system

$$\ddot{X} + A(t)X = 0 \quad (0.306)$$

where $X \in \mathbb{R}^n$, $A(t) = (a_{ij}(t))$ is a symmetric $n \times n$ -matrix, and has obtained results on the boundedness of its solutions. In an other article Tunc [2] has discussed the uniform boundedness and the boundedness of solutions for the equations

$$\ddot{X} + A(t)F(X) = P(t, X, \dot{X}) \quad (0.307)$$

The purpose of the present paper is to investigate the second-order nonlinear vector differential equations of neutral type of the form :

$$(X(t) + \tau X(t-r))'' + A(t)\dot{X} + B(t)H(X(t-\rho)) = Q(t, X, \dot{X}) \quad (0.308)$$

in which $t \in \mathbb{R}^+$, $\mathbb{R}^+ = [0, \infty)$ and $X \in \mathbb{R}^n$; $A(t) = (a_{ij}(t))$ and $B(t) = (b_{ij}(t))$ are symmetric $n \times n$ -matrix and $H : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $Q : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$. Obviously, Eq. (0.306) and (0.307) are special cases of Eq. (0.308). It is supposed that A, B, H and Q are continuous.

The focus in the work of this paper is on two results. On one hand the uniform asymptotic stability for the first result in the case of $Q = 0$. On the other hand the boundedness of the solution and its derivatives in the case $Q \neq 0$.

Main results

First, some assumptions are stated on the functions that appeared in (0.308). Suppose that there are positive constants $a_0, a_1, a_2, a_3, b_0, b_1$ and α such that the following conditions are satisfied

- (i) The matrices A, B and $\dot{B}$ are symmetric, and $a_1 \geq \lambda_i(A(t)) \geq a_0, b_1 \geq \lambda_i(B(t)) \geq b_0$ and $\lambda_i(\dot{B}(t)) \leq -\alpha < 0$ for all $t \in [0, \infty)$, $(i = 1, 2, \dots, n)$.
- (ii) $H(0) = 0$, $J_H(X)$ is symmetric and $a_3 \geq \lambda_i(J_H(X)) \geq a_2$ for all $X \in \mathbb{R}^n, (i = 1, 2, \dots, n)$.

In the case $Q \equiv 0$ in Eq. (0.308) the following result is established.



Theorem 0.135. Suppose all the assumptions above on the equation (0.308) are satisfied. Then the trivial solution $X = 0$ of the equation (0.308) is uniformly asymptotically stable, provided that

$$\alpha > \frac{b_1 a_3^2 \tau}{a_2}, \quad a_0 > \frac{1}{2} b_1, \quad \tau < \frac{a_0 - \frac{1}{2} b_1}{a_1 + b_1} \quad \text{and} \quad \rho < \frac{a_0 - \frac{1}{2} b_1 - a_1 \tau - b_1 \tau}{\frac{1}{2} b_1 a_3^2 (\tau + 1)}. \quad (0.309)$$

Next, in the case $Q \neq 0$ in Eq. (0.308) the following result is established.

Theorem 0.136. Suppose that the assumptions of Theorem 1 hold and assume also that Q satisfies

$$\|Q(t, X, Y)\| \leq (\mu + \|\bar{Y}\|)\theta(t), \quad \text{for all } (t, X, Y) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \quad (0.310)$$

where $\theta(t)$ is a nonnegative and continuous function of t , and satisfies $\int_0^t \theta(s) ds \leq \gamma < \infty$ for all $t \geq 0$, μ and γ are positive constants. Then there exists a positive constant K such that any solution $(X(t), Y(t))$ of the equation (0.308) determined by

$$X(0) = X_0, \quad Y(0) = Y_0$$

satisfies for all $t \geq 0$,

$$\|X(t)\| \leq K, \quad \|Y(t)\| \leq K.$$

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Exponential Decay for Nonlinear Wave Equations on a Star-Shaped-Network Using Kelvin-Voigt Mechanisms

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Abstract. In this work we study the stabilization of a wave equation with Kelvin-Voigt damping on a star-shaped network composed of a finite string and several infinite edges. We establish the well-posedness of the system and we prove that the energy of the wave equation decays exponentially to zero for all initial data of finite energy.

Keywords: Nonlinear wave equation, Star-shaped network, Kelvin-Voigt damping, exponential energy decay.



Introduction

we have a finite edge I_1 to which we attach $N - 1$ infinite edges $\{I_j\}_{j=2}^N$, all connected at a single vertex. We denote by ℓ the length of the finite edge. We may identify the edge I_1 with the interval $(0, \ell)$, the vertex with 0, and the infinite edges with the semi-infinite interval $(0, +\infty)$. Thus, the problem can be described as follows

$$\begin{cases} \partial_t^2 u_1 - \partial_x^2 u_1 + |u_1|^{p-1} u_1 = 0, & (x, t) \in I_1 \times \mathbb{R}_+, \\ \partial_t^2 u_j - \partial_x^2 u_j - \partial_x (\alpha_j \partial_{xt}^2 u_j) = 0, & (x, t) \in I_j \times \mathbb{R}_+, \quad j = 2, \dots, N, \\ u_i(0, t) = u_j(0, t), \quad t > 0, \quad i, j = 1, \dots, N, \\ \sum_{j=1}^N \partial_x u_j(0, t) = 0, \quad t > 0, \\ u_1(\ell, t) = 0, \quad t > 0, \\ u_j(x, 0) = u_j^0(x), \quad \partial_t u_j(x, 0) = u_j^1(x), \quad x \in I_j, \quad j = 1, \dots, N, \end{cases} \quad (0.311)$$

where $p > 1$ and $\alpha := (\alpha_2, \dots, \alpha_N)$ is the damping coefficient function satisfying

$$\begin{cases} \alpha_j \in L^\infty(I_j), \quad 2 \leq j \leq N, \\ \alpha_j(x) > 0 \text{ on } (0, \ell_j) \subset I_j, \text{ and } \alpha_j(0) = 0, \quad 2 \leq j \leq N. \end{cases} \quad (0.312)$$

The natural energy of system (0.311) is given by,

$$E_u(t) = \sum_{j=1}^N \int_{I_j} \left(\frac{1}{2} |\partial_t u_j(x, t)|^2 + \frac{1}{2} |\partial_x u_j(x, t)|^2 \right) dx + \frac{1}{p+1} \int_{I_1} |u_1(x, t)|^{p+1} dx$$

where $u := (u_1, \dots, u_N)$.

We introduce the Hilbert space

$$\mathcal{H} = V \times X,$$

where

$$V = \left\{ (u_1, \dots, u_N) \in H^1(0, \ell) \times \prod_{j=2}^N H^1(0, \infty) : u_1(\ell) = 0, \quad u_i(0) = u_j(0), \forall i, j = 1, \dots, N \right\},$$

and

$$X = L^2(0, \ell) \times \prod_{j=2}^N L^2(0, +\infty),$$

Main results

We prove that the system (0.311) is well-posed.



Theorem 0.137. *Let $p > 1$. Then for any $U^0 = (u^0, u^1) \in \mathcal{H}$ there exists $T > 0$ such that problem (1) has a unique solution*

$$U(t) = (u(t), v(t)) \in \mathcal{C}([0, T], \mathcal{H}).$$

The main result of this work is to prove that this dissipation is of exponential type. More precisely, we have the following theorem

Theorem 0.138. *Under assumption (0.312), there exist $C > 0$ and $\gamma > 0$ such that inequality*

$$E_u(t) \leq C e^{-\gamma t} E_u(0), \quad \text{for all } t > 0,$$

holds for every solution to problem (0.311) in $\mathcal{H}$.

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**Infinitely many solutions for a perturbation of the
prescribed Q-curvature problem on closed Riemannian
manifolds**

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Abstract. Given a compact Riemannian manifold (M, g) of dimension $n \geq 3$ without boundary, using variational methods, we study the existence of solutions for the elliptic equation

$$P_g^k u = f|u|^{N-2}u + \lambda h|u|^{q-2}u, \quad (0.313)$$

where P_g^k is the GJMS operator of order $2k < n$, $h, f \in C^\infty(M)$, $1 < q < 2$, $\lambda > 0$ and N is the critical Sobolev exponent for the space $H_k^2(M)$. We apply Ljusternik-Schnirelmann theory on C^1 -manifolds to prove that under some conditions, the equation (0.530) admits infinitely many solutions.

Keywords: Geometric PDEs, Variational Methods, Q-curvature.



Introduction

Let (M, g) be a closed Riemannian manifold of dimension $n \geq 3$. Given a function $f \in C^\infty(M)$, finding a metric $\tilde{g} = u^{\frac{4}{n-2}}g$ on M , conformal to g for which the scalar curvature equals f , is the same as solving the elliptic PDE.

$$\Delta_g u + \frac{n-2}{4(n-1)} \text{Scal}_g u = f u^{\frac{n+2}{n-2}}. \quad (0.314)$$

It is remarkable that a problem that appears to be purely geometric in nature can be reframed as an analysis problem, a nonlinear elliptic PDE with critical exponent.

The operator

$$P_g^1 = \Delta_g + \frac{n-2}{4(n-1)} \text{Scal}_g$$

that is found in equation (0.314) is known as the conformal Laplacian. This operator is conformally covariant in the sense that, if $\tilde{g}$ is a metric given by $\tilde{g} = u^{\frac{4}{n-2}}g$, where u is a smooth positive function on M then

$$P_g^1(\phi u) = u^{(n+2)/(n-2)} P_{\tilde{g}}^1 \phi \quad \text{for all } \phi \in C^\infty(M),$$

However, the conformal Laplacian is not the only operator with such a conformal covariance property.

In fact, in their celebrated paper [10], Graham-Jenne-Mason-Sparling, introduced a systematic way to construct for all integer $k < \frac{n}{2}$ a conformally covariant operator P_g^k of the form

$$P_g^k = \Delta_g^k + \text{lower.order.terms}$$

with the following properties:

- The operator P_g^k is formally self-adjoint with respect to the L^2 -scalar product.
- The operator P_g^k is natural, in the sense that, if $\phi : M \rightarrow M$ is a smooth diffeomorphism, then $\phi^* P_g^k = P_{\phi^* g}^k$.
- The operator P_g^k is conformally covariant in the sense that, if $\tilde{g}$ is a metric given by $\tilde{g} = u^{\frac{N-2}{k}}g$, where u is a smooth positive function on M then

$$P_g^k(\phi u) = u^{N-1} P_{\tilde{g}}^k \phi \quad \text{for all } \phi \in C^\infty(M),$$

$$\text{where } N = \frac{2n}{n-2k}.$$

A smooth function $Q_g^k = \frac{2}{n-2k} P_g^k(1)$ called Q-curvature is associated to the GJMS operator P_g^k . The Q-curvature maybe viewed as a generalization of the notion of scalar curvature. In fact, for $k = 1$, the Q-curvature is just the scalar curvature multiplied by a constant.



The notion of Q-curvature is a key concept in conformal geometry and is also connected to the AdS/CFT duality in theoretical physics.

Finding a conformal metric $\tilde{g} = u^{\frac{N-2}{k}} g$ with a prescribed Q-curvature $f \in C^\infty(M)$ is equivalent to finding a positive solution for the equation

$$P_g^k u = \frac{n-2k}{2} f |u|^{N-2} u. \quad (0.315)$$

This problem is particularly hard, as N is the critical Sobolev exponent for the space $H_k^2(M)$. Moreover, the conformal covariance of the operator P_g^k itself creates a geometric obstruction that prevents us from prescribing any arbitrary function $f \in C^\infty(M)$ as a Q-curvature. Certain functions are indeed *forbidden*.

In [13], we were interested in studying the existence of solutions for the elliptic equation

$$P_g^k u = f |u|^{N-2} u + \lambda h |u|^{q-2} u, \quad (0.316)$$

where

- h and f are positive functions in $C^\infty(M)$.
- $q \in (1, 2)$, $\lambda > 0$.

Moreover we assume that the operator P_g^k is coercive i.e. that there exists $\Lambda > 0$ such that

$$\int_M u P_g^k u \, dv_g \geq \Lambda \|u\|_{H_k^2(M)}^2, \quad \forall u \in H_k^2(M), \quad (0.317)$$

This is for example the case if (M, g) is an Einstein manifold with positive scalar curvature.

Main results

Our main result is the following

Theorem 0.139 (O., K. Tahri). [13] *There exists $\lambda_\star > 0$ such that if $\lambda \in (0, \lambda_\star)$, then Equation (0.530) admits infinitely many pairs of weak solutions in $H_k^2(M)$.*

To prove this result, we used the method of Nehari manifolds combined with Lusternik-Schnirelmann theory, more specifically, we used the notion of Kranoselski genus.

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WELL-POSEDNESS AND STABILITY RESULT FOR A TIMOSHENKO SYSTEM WITH THERMODIFFUSION EFFECTS AND TIME-VARYING DELAY TERM

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Abstract. In this work, we consider a new Timoshenko beam model with thermal and mass diffusion effects. Heat and mass exchange with the environment during thermodiffusion in Timoshenko beam with a time-varying delay term. Firstly, by the C_0 -semigroup theory, we prove the well posedness of the considered problem with Dirichlet boundary conditions. Then We illustrate, without assuming the well-known equal wave speeds condition, the general decay result of the problem's solution using a suitable Liapunov function.

Keywords: Timoshenko beam, Time varying delay, Existence and uniqueness, Exponential stability.



Introduction

In this work, we are intersted with the following problem

$$\begin{cases} \rho_1 \varphi_{tt}(x, t) - K(\varphi_x + \psi)_x(x, t) + \mu_1 \varphi_t(x, t) + \mu_2 \varphi(x, t - \tau(t)) = 0, \\ \rho_2 \psi_{tt} - \alpha \psi_{xx} + k(\varphi_x + \psi) - \gamma_1 \theta_x - \gamma_2 P_x = 0, \\ c\theta_t + dP_t - \kappa \theta_{xx} - \gamma_1 \psi_{tx} = 0, \\ d\theta_t + rP_t - \hbar P_{xx} - \gamma_2 \psi_{tx} = 0, \end{cases} \quad (0.318)$$

with the following initial conditions

$$\begin{cases} \varphi(x, 0) = \varphi^0(x), \quad \psi(x, 0) = \psi^0(x), \quad \theta(x, 0) = \theta^0(x) \text{ for } x \in (0, L), \\ P(x, 0) = P^0(x), \quad \varphi_t(x, 0) = \varphi^1(x), \quad \psi_t(x, 0) = \psi^1(x) \text{ for } x \in (0, L), \end{cases} \quad (0.319)$$

and a Dirichlet boundary conditions:

$$\begin{cases} \varphi(0, t) = \psi(0, t) = \theta(0, t) = P(0, t) = 0, & t > 0, \\ \varphi(L, t) = \psi(L, t) = \theta(L, t) = P(L, t) = 0, & t > 0. \end{cases} \quad (0.320)$$

We assume that the symmetric matrix $\Lambda = \begin{pmatrix} c & d \\ d & r \end{pmatrix}$ is positive definite, i.e.

$$\delta = cr - d^2 > 0. \quad (0.321)$$

ote that (0.321) implies that for $\theta, P \neq 0$

$$c\theta^2 + 2d\theta P + rP^2 > 0. \quad (0.322)$$

Main results

We have proven the existence and uniqueness of the solution, and the exponential stability of system (0.318)-(0.319) subject to the boundary conditions (0.351) under the assumption $\sqrt{1 - d}\mu_1 > \mu_2$ and the condition of nonequal wave speeds of propagation

$$\frac{k}{\rho_1} \neq \frac{\alpha}{\rho_2}. \quad (0.323)$$

Theorem 0.140. Assume that $\mu_2 < \sqrt{1 - d}\mu_1$, then for any $U_0 \in \mathcal{H}$, there exists a unique solution $U \in C([0, +\infty), \mathcal{H})$ of system (0.318)-(0.319) subject to the boundary conditions (0.351). Moreover if $U_0 \in D(\mathcal{A})$, then

$$U \in C([0, +\infty), D(\mathcal{A})) \cap C^1([0, +\infty), \mathcal{H}).$$

Theorem 0.141. Assume that (0.323), (??), (??) and $\mu_2 < \sqrt{1 - d}\mu_1$. Then, given any solution to problems (0.318)-(0.319) that is subject to the boundary conditions (0.351) there are two positive constants C and γ that are independent of t , such that we have

$$E(t) \leq Ce^{-\gamma t}, \quad \forall t \geq 0. \quad (0.324)$$



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DESCRETIZATION FOR A 2D-VISCOELASTIC WAVE EQUATION WITH ACOUSTIC DYNAMIC BOUNDARY CONDITIONS

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Abstract. This article presents and analyses a finite element approach for the 2D-viscoelastic wave equation with dynamic boundary conditions and strong damping. We use the Faedo-Galerkin method to prove the global existence of solutions and the multiplier approach to determine the asymptotic behavior in a bounded domain. We show and analyze typical semi-discrete systems as well as an implicit fully discrete scheme. For both semi discrete and fully discrete methods, optimal a priori error estimates are demonstrated. Finally, some numerical findings and a priori error estimate are derived.

Keywords: finite element approach, viscoelastic, Faedo-Galerkin method, dynamic boundary conditions, fully discrete approximation.

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Introduction

In this paper, we consider the following viscoelastic wave equations with strong damping and dynamic boundary conditions:

$$\begin{cases} u_{tt}(x, t) - \Delta u(x, t) - \Delta u_t(x, t) - \Delta u_{tt}(x, t) = f(x, t), & \text{in } Q = \Omega \times [0, T], \\ u(x, t) = 0, & \text{on } \Sigma_0 = \Gamma_0 \times [0, T], \\ \frac{\partial u}{\partial n}(x, t) + \xi(x, t) = 0, & \text{on } \Sigma_1 = \Gamma_1 \times [0, T], \\ \xi_{tt}(x, t) + \xi(x, t) = u_t(x, t) + g(x, t), & \text{on } \Sigma_1 = \Gamma_1 \times [0, T]. \end{cases} \quad (0.325)$$

With the following initial conditions

$$\begin{cases} u(x, 0) = u_0(x), & u_t(x, 0) = u_1(x), & x \in \Omega, \\ \xi(x, 0) = \xi_0(x), & \xi_t(x, 0) = \xi_1(x), & x \in \Gamma_1. \end{cases} \quad (0.326)$$

Where $u = u(x, t)$, $t \geq 0$, $x \in \Omega$, and $\xi(x, t)$ designates the dynamical control. Ω is a bounded open set in $\mathbb{R}^2$ with boundary Γ of class C^2 and $\Gamma = \Gamma_0 \cup \Gamma_1$ with $\Gamma_0 \neq \emptyset$ and $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$.

$\frac{\partial}{\partial \nu}$ denotes the unit outer normal derivative, f and g are functions depend on x, t .

The viscoelastic equation explain how waves travel through materials that have both elastic and viscous characteristics. Understanding the evolution of waves in materials that can simultaneously dissipate energy and deform is made possible by the use of viscoelastic wave equations, for the physical aspects, see Love [6]. The non-ordinary boundary conditions defined on (0.492) are the primary source of difficulty for problem (0.492). Boundary conditions of this kind are referred to as dynamic boundary conditions, moreover these conditions have numerous practical uses in addition to being significant from a theoretical one.

Main results

In this section*, we present some material used in the proof of our main result.

Lemma 0.56. [7] *Let $E : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a non increasing function and $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ a increasing function of class C^1 such that*

$$\phi(0) = 0, \quad \text{and} \quad \phi(t) \rightarrow +\infty \quad \text{as} \quad t \rightarrow +\infty.$$

Assume that there exist $p > -1$ and $\omega > 0$ such that

$$\int_S^{+\infty} E^{p+1}(t) \phi'(t) dt \leq \frac{1}{\omega} E^p(0) E(S), \quad 0 \leq S \leq \infty,$$

then E has the following decay properties



$$E(t) = 0, \quad \forall t \geq \frac{E^p(0)}{\omega|p|}, \quad \text{if } -1 < p < 0$$

$$E(t) \leq E(0)e^{1-\omega\phi(t)}, \quad \forall t \geq 0, \quad \text{if } p = 0$$

$$E(t) \leq E(0) \left(\frac{1+p}{1+\omega p\phi(t)} \right), \quad \forall t \geq 0, \quad \text{if } p > 0$$

Denote by $|\cdot|$, $|\cdot|_1$, $(\cdot, \cdot)$ and $(\cdot, \cdot)_1$ the norms and scalar products in $L^2(\Omega)$ and $L^2(\Gamma_1)$ respectively, and introduce the space

$$H_{\Gamma_0}^1(\Omega) = \{u \in H^1(\Omega)/u|_{\Gamma_0} = 0\}, \quad \text{and } V = \{u \in H_{\Gamma_0}^1(\Omega)/u|_{\Gamma_1} \in L^2(\Gamma_1)\}$$

Multiply the first equation of (0.492) by a test function $v \in V$ then use Green's formula with the initial and boundary conditions to obtain the weak form of (0.492).

Find $u(\cdot, t) \in V$ and $\xi(\cdot, t) \in L^2(\Gamma_1)$ such that, for all $v, w \in V$

$$\begin{cases} (u_{tt}, v) + (\nabla u, \nabla v) + (\nabla u_t, \nabla v) + (\nabla u_{tt}, \nabla v) + (\xi + \xi_t + \xi_{tt}, v)_1 = (f, v), & \text{in } Q, \\ (\xi_{tt}(x, t), w)_1 + (\xi(x, t), w)_1 = (u_t(x, t), w)_1 + (g(x, t), w)_1, & \text{on } \Sigma_1. \end{cases} \quad (0.327)$$

In the next theorem we show the existence and uniqueness of the solution of the problem (0.492) using the Faedo-Galerkin method.

Theorem 0.142. Assume that $u_0, u_1 \in H_{\Gamma_0}^1(\Omega) \cap H^2(\Omega)$, $\xi_0, \xi_1 \in L^2(\Gamma_1)$, $f, \frac{\partial f}{\partial t} \in L^2(Q)$, $g, \frac{\partial g}{\partial t} \in L^2(\Sigma_1)$ and $\frac{\partial u_0}{\partial n}(x) = -\xi_0(x)$, $\frac{\partial u_1}{\partial n}(x) = -\xi_1(x)$ for all $x \in \Gamma_1$.

Then for each $T > 0$, problem (0.492) admits a unique solution (u, ξ) satisfying the following conditions:

$$\begin{aligned} u &\in L^\infty(0, T, H_{\Gamma_0}^1(\Omega)), \\ u_t &\in L^\infty(0, T, L^2(\Omega)) \cap L^\infty(0, T, H_{\Gamma_0}^1(\Omega)), \\ u_{tt} &\in L^\infty(0, T, L^2(\Omega)) \cap L^\infty(0, T, H_{\Gamma_0}^1(\Omega)), \\ \xi, \xi_{tt} &\in L^\infty(0, T, L^2(\Gamma_1)). \end{aligned}$$

Theorem 0.143. Assume that $u_0, u_1 \in H_{\Gamma_0}^1(\Omega) \cap H^2(\Omega)$, $\xi_0, \xi_1 \in L^2(\Gamma_1)$ and $\frac{\partial u_0}{\partial n}(x) = -\xi_0(x)$, $\frac{\partial u_1}{\partial n}(x) = -\xi_1(x)$ for all $x \in \Gamma_1$.

Then for some positive constant k , we obtain the following decay property

$$E(t) \leq E(0)e^{1-k\phi(t)}, \quad \forall t \geq 0,$$

where ϕ is function satisfies certain hypotheses, which we will specify in the proof.



Theorem 0.144. Assume that the assumption of Theorem 0.194 holds with $\xi_0, \xi_1 \in H_{\Gamma_0}^2(\Omega)$ and $\xi \in H^1(0, T, H_{\Gamma_0}^2(\Omega))$, then we have the following estimate

$$|u_t - u_t^h|^2 + |\nabla u - \nabla u^h|^2 + |\nabla u_t - \nabla u_t^h|^2 + |\xi - \xi^h|_1^2 + |\xi_t - \xi_t^h|_1^2 \leq Ch^2,$$

holds for $0 \leq t \leq T$.

*

Error Estimate Let us denote $\hat{u}^{n+1} = \frac{u^{n+1} - u^n}{\Delta t}$, $\hat{\xi}^{n+1} = \frac{\xi^{n+1} - \xi^n}{\Delta t}$, $\hat{u}(t_n) = u_t(t_n)$ and $\hat{\xi}(t_n) = \xi_t(t_n)$.

Theorem 0.145. Assume that the assumption of theorem 0.194 holds, then if $u \in H^3(0, T, L^2(\Omega)) \cap W^{2,\infty}(0, T, H^3(\Omega))$ and $\xi \in H^2(0, T, H^2(\Omega)) \cap W^{1,\infty}(0, T, H^2(\Omega))$ we have

$$|u_t(t_n) - \hat{u}^n|^2 + |\nabla u(t_n) - \nabla u^n|^2 + |\nabla u_t(t_n) - \nabla \hat{u}^n|^2 + |\xi(t_n) - \xi^n|_1^2 + |\xi_t(t_n) - \hat{\xi}^n|_1^2 \leq C(h^2 + (\Delta t)^2).$$

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A class of Kolmogorov differential system with three hyperbolic algebraic limit cycles

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Abstract. For a given algebraic curve, we exhibit Kolmogorov differential systems, and we show that these systems admit precisely the bounded components of the curve as hyperbolic limit cycles if certain conditions on the parameters of the system are satisfied.

Keywords: Planar differential system, invariant curve, periodic solution, hyperbolic limit cycle.



Introduction

Many mathematical models in biology and population dynamics are modeled by systems of ordinary differential equations having the form:

$$\begin{cases} x' = \frac{dx}{dt} = xF(x, y) \\ y' = \frac{dy}{dt} = yG(x, y) \end{cases}$$

Where $x(t)$ and $y(t)$ represent the population density of two species at time t , and $F(x, y)$, $G(x, y)$ are the capita growth rate of each specie, usually, such systems are called Kolmogorov systems.

Generally, Kolmogorov system is introduced as the structure of many natural phenomena models. Their applications can be appear in several fields such as, physics, biology, chemical reactions, hydrodynamics, fluid dynamics, economics, etc. for more detail see ([1]).

We consider the following polynomial differential system

$$\begin{cases} x' = x(ax + by + c) \\ y' = y(mx + ny + l) \end{cases}$$

This system known as Lotka–Volterra predator–prey model, if $a = n = 0$ the system becomes

$$\begin{cases} x' = x(by + c) \\ y' = y(mx + l) \end{cases}$$

where

- The variable x is the population density of prey (for example, the number of rabbits per square kilometre).
- The variable y is the population density of some predator (for example, the number of foxes per square kilometre).
- $\frac{dx}{dt}$ and $\frac{dy}{dt}$ represent the instantaneous growth rates of the two populations.
- t represents time
- The prey's parameters, a and b , describe, respectively, the maximum prey per capita growth rate, and the effect of the presence of predators on the prey growth rate.
- The predator's parameters, m , n , respectively describe the predator's per capita death rate, and the effect of the presence of prey on the predator's growth rate.
- All parameters are positive and real.

The Lotka–Volterra system of equations is a phenomenological model of competing population dynamics, it can model the dynamics of ecological systems with predator-prey interactions, competition, disease and mutualism. It is known



that the model of Lotka-volterra have no limit cycle, but it remains the starting point of several models currently in applied science.

In [6] Kolmogorov developed a more general system than that of Lotka–Volterra predator–prey model, this is the system

$$\begin{cases} x' = xF(x, y) \\ y' = yG(x, y) \end{cases}$$

The functions F and G are respectively the growth rates of the two populations x and y , they are of class $C^1[0, +\infty[$ and satisfy the following conditions :

1. For a fixed prey density, the population growth rate of prey x decreases with the growt of the population of predators y . This leads to the condition

$$\frac{\partial F}{\partial y}(x, y) < 0$$

2. For a fixed density of predators, the growth rate of the predator population y increases by the growth of the prey population x . This leads to the condition

$$\frac{\partial G}{\partial x}(x, y) > 0$$

3. For low densities of both populations, the prey population grows and therefore we have

$$F(0, 0) > 0$$

4. There is a sufficient number of predators for which, a small number of prey cannot grow any longer. This leads to the condition

$$\exists A > 0, \text{ such that } F(0, A) = 0$$

5. The medium admits a limited carrying capacity. This leads to the condition

$$\exists B > 0, \text{ such that } F(B, 0) = 0$$

6. If there is a sufficient number of prey, the number of predators increases, otherwise it decreases. This leads to the condition

$$\exists C > 0, \text{ such that } G(C, 0) = 0$$

7. According to Kolmogorov [6] and Rascigno and Richardson, if $B \leq C$, the predators are in extinction and the prey saturates the environment (this is also true for the model of Lotka–Volterra predator–prey). However, to have a coexistence between the populations of prey and predators, it is necessary to ensure that $C < B$.



To our knowledge, the first result which gives an system differential of kolmogorov type with an algebraic limit cycle whose exact expression is known is found in the article [8], by A. Bendjeddou et al. And then we find the work of A. Bendjeddou, S. Benyoucef ([18]). Who showed the existence of a finite class cycle of the Kolmogorov system, and gives its explicit form.

This work is a contribution to the study of a class of differential systems of the kolmogorov type. It is interesting to see if the differential system recognizes a limit cycle or not and if we know the exact expression for that limit cycle it is even better. Motivated by some papers presenting planar polynomial systems with one or more algebraic term cycles given analytically, and mainly based on the papers of C. Christopher [5], S. Benyoucef [17], we will extend the same concept to Kolmogorov systems, where once we have selected components of the Kolmogorov system that satisfy certain conditions, we can directly infer the number and explicit form of limit cycles.

More precisely, in the following we introduce class of differential system of order n of the Kolmogorov type, in which we will study hyperbolic algebraic limit cycles.

$$\begin{cases} x' = x(R(x, y)U - y\Phi(x, y)U_y) \\ y' = y(S(x, y)U + x\Phi(x, y)U_x) \end{cases} \quad (0.328)$$

Where $R(x, y)$, $S(x, y)$ and $\Phi(x, y)$ are polynomial functions. Our contribution consist to show that the system (0.470) admits all the bounded components of Γ as hyperbolic limit cycles if certain conditions on the parameters are satisfied.

Main results

In this section* we introduce the following theorem that is of particular importance in the study of some class differential kolmogorov system .

Theorem 0.146. *Let U, C^1 function in open subset $V = \{(x, y) \in \mathbb{R}^2, x > 0, y > 0\}$*

We consider a polynomial differential system

$$\begin{cases} x' = x(aU - \alpha xy^2 U_y) \\ y' = y(bU + \alpha yx^2 U_x) \end{cases} \quad (0.329)$$

Where $\alpha \neq 0$, $a + b \neq 0$, then the system (0.351) admits all the bounded components of $U = 0$ as hyperbolic limit cycles.

Proof of theorem 0.146 Let $\Gamma = \{(x, y) \in \mathbb{R}^2, U(x, y) = 0\}$ is a trajectory of (0.351).

To show that all the bounded components of Γ are hyperbolic limit cycles of system (0.351), we will prove that Γ is an invariant curve of the system (0.351), and

$$\int_0^T \text{div}(\Gamma) dt \neq 0$$



1) Calculate the differential of the function $U(x, y)$ with respect to the differential system (0.351):

$$\begin{aligned}\frac{dU}{dt} &= U_x x' + U_y y' \\ &= U_x (x(aU - \alpha xy^2 U_y)) + U_y (y(bU + \alpha yx^2 U_x)) \\ &= U(axU_x + byU_y).\end{aligned}$$

which shows that $U(x, y) = 0$ an invariant curve of the system (0.351) with a cofactor $K(x, y) = (axU_x + byU_y)$.

2) We show that $\int_0^T \text{div}(\Gamma) dt$ is nonzero.

To see this, note that

$$\int_0^T \text{div}(\Gamma) dt = \int_0^T K(x(t), y(t)) dt$$

$$\begin{aligned}\int_0^T K(x(t), y(t)) dt &= \int_0^T (axU_x + byU_y) dt \\ &= \int_0^T (axU_x) dt + \int_0^T (byU_y) dt.\end{aligned}$$

Since

$$dt = \frac{dx}{x(aU - \alpha xy^2 U_y)} = \frac{dy}{y(bU + \alpha yx^2 U_x)}$$

then

$$\int_0^T \text{div}(\Gamma) dt = \oint_{\Gamma} \frac{axU_x}{-\alpha x^2 y^2 U_x} dy + \oint_{\Gamma} \frac{byU_y}{-\alpha y^2 x^2 U_y} dx$$

Since $U(x, y) = 0$ is algebraic curve, then we have $U_x(x, y)dx + U_y(x, y)dy = 0$, thus $U_x(x, y)dx = -U_y(x, y)dy$, and we get

$$\begin{aligned}\int_0^T \text{div}(\Gamma) dt &= \oint_{\Gamma} \frac{a}{\alpha xy^2} dy - \oint_{\Gamma} \frac{b}{\alpha yx^2} dx \\ &= \frac{1}{\alpha} \left(\oint_{\Gamma} \left(\frac{a}{xy^2} \right) dy - \oint_{\Gamma} \left(\frac{b}{x^2 y} \right) dx \right).\end{aligned}$$

By applying the GREEN formula we obtain

$$\begin{aligned}&\left(\oint_{\Gamma} \left(\frac{a}{xy^2} \right) dy - \oint_{\Gamma} \left(\frac{b}{x^2 y} \right) dx \right) \\ &= \iint_{\text{int}(\Gamma)} \left(\frac{\partial \left(\frac{a}{xy^2} \right)}{\partial x} + \frac{\partial \left(\frac{b}{x^2 y} \right)}{\partial y} \right) dx dy \\ &= -\frac{a+b}{\alpha} \iint_{\text{int}(\Gamma)} \frac{1}{x^2 y^2} dx dy,\end{aligned}$$

where $\text{int}(\Gamma)$ denotes the interior of Γ .

As $\alpha \neq 0$, and $a + b \neq 0$, then $\int_0^T \text{div}(\Gamma) dt$ is nonzero. This complete the proof of Theorem 0.146 ■



Application

The following example are given to illustrate our result.

Example 0.5. Let $a = b = 1$, and

$U(x, y) = ((x-2)^2 + (y-2)^2 - 1)^2 - (x-2)^2(y-2) - (x-2)(y-2)^2$,
 then the differentiel system (0.351) becomes

$$\begin{cases} x' = x(((x-2)^2 + (y-2)^2 - 1)^2 - (x-2)^2(y-2) - (x-2)(y-2)^2 \\ \quad - xy^2(4x^2y - 9x^2 - 18xy + 40x + 4y^3 - 24y^2 + 64y - 68)) \\ y' = y(((x-2)^2 + (y-2)^2 - 1)^2 - (x-2)^2(y-2) - (x-2)(y-2)^2 \\ \quad + yx^2(4x^3 - 24x^2 + 4xy^2 - 18xy + 64x - 9y^2 + 40y - 68)) \end{cases} \quad (0.330)$$

It is easy to verify that all conditions of Theorem 0.146 are satisfied. Then system (0.457) has three limit cycles represented by the curve

$U(x, y) = ((x-2)^2 + (y-2)^2 - 1)^2 - (x-2)^2(y-2) - (x-2)(y-2)^2 = 0$. See figure 1

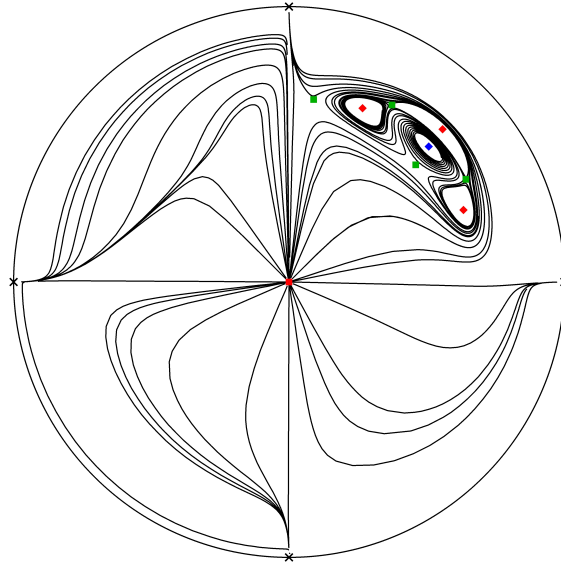


Figure 25: Phase potrait of system (3) on the Poincare disc.



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Nontrivial solutions for nonlinear problem with non-compact resolvent operators

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Abstract. In this work, we study the existence of nontrivial solution to a nonlinear problem with non compact resolvent operator. By means of the Leray-Schauder degree theory, with suitable assumptions on the nonlinearities, we establish the existence of a distributional solution to our problem.

Keywords: Topological degree, non-compact resolvent operators, homotopy.



Introduction

In this work, we concentrate our efforts to extend the results obtained in [4, 5] to the system for the non-compact case. This type of problems has been extensively studied by many authors. In 2021, Zhang and Liu [9] obtained the existence of nontrivial solutions for a quasilinear system. In [7], the authors studied the existence of weak solutions of a quasilinear system of partial differential equations which are a combination of the Perona-Malik equation and the heat equation. Their study is mainly based on the use of the compactness method and the monotonicity arguments. Far from being complete, we refer the interested readers to [7, 10, 11] and the references therein.

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Solutions for a nonlocal elliptic equation with critical Sobolev exponent and singular term

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Abstract. In this paper, we consider a class of nonhomogeneous fractional elliptic equations involving critical Hardy Sobolev exponents as follows

$$\begin{cases} (-\Delta)^s u - \mu \frac{u}{|x|^{2s}} = |u|^{2^*_s-2} u + \lambda \frac{u}{|x|^{2s-\alpha}} + f(x), & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $0 < s < 1$, $\lambda > 0$ is a parameter. We prove the existence of multiple solutions using the variational methods and the Nehari manifold decomposition.

Keywords: Fractional Laplacian, critical Hardy-Sobolev exponent, variational methods.



Introduction

The paper deals with the following fractional Hardy-Sobolev equation with nonhomogeneous term

$$\begin{cases} (-\Delta)^s u - \mu \frac{u}{|x|^{2s}} = |u|^{2^*_s-2} u + \lambda \frac{u}{|x|^{2s-\alpha}} + f(x), & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases} \quad (0.331)$$

being $0 < s < 1$, where Ω is a bounded domain in $\mathbb{R}^N$, ($N > 2s$) containing the origin 0 in its interior, $0 \leq \mu < \bar{\mu}_s := 2^{2s} \frac{\Gamma^2(\frac{N+2s}{4})}{\Gamma^2(\frac{N-2s}{4})}$, λ is a positive parameter, $0 < \alpha < 2s$, $2^*_s = \frac{2N}{N-2s}$ is the fractional critical Hardy-Sobolev exponent. The fractional Laplacian $(-\Delta)^s$ is defined by

$$-2(-\Delta)^s u(x) = C_{N,s} \int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|x-y|^{N+2s}} dy$$

where

$$C_{N,s} = \frac{4^s \Gamma(N \setminus 2 + s)}{\pi^{N/2} |\Gamma(-s)|}.$$

Γ is the Gamma function, f is a given bounded measurable function.

There have been by now a large number of papers concerning the existence, nonexistence as well as qualitative properties of nontrivial solutions to critical elliptic problems of Hardy potential and fractional Laplace operator. For instance, Bennour and all in [1] handled the following singular equation

$$\begin{cases} \Delta^2 u - \mu \left(\frac{u}{|x|^4} \right) = \left(\frac{|u|^{2^*(s)-2} u}{|x|^s} \right) + \lambda \left(\frac{u}{|x|^{4-\alpha}} \right) + f(x), & x \in \Omega, \\ u = \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ and $N \geq 5$, Under sufficient conditions on the data, the existence and multiplicity of solutions was proven, via Eklund's variation principle and the Mountain Pass Lemma principle.

In the local setting case ($s = 1$) the problem (0.530) is reduced to the semilinear problem with Sobolev-Hardy exponents

$$\begin{cases} -\Delta u - \mu \frac{u}{|x|^2} = |u|^{2^*_s-2} u + \lambda \frac{u}{|x|^{2-\alpha}} + f(x), & x \in \Omega, \\ u = 0 & x \in \partial\Omega. \end{cases}$$

This problem was further studied by Chen and Rocha [2], who based on variational methods obtained the existence of four non-trivial solutions.

Recently, the existence of nontrivial solutions for nonlinear fractional elliptic equations with Hardy's potential type

$$(-\Delta)^s u - \mu \frac{u}{|x|^{2s}} = g(u) \quad (0.332)$$



have been studied by several authors. Wang and all [5] studied (0.489) with $g(u) = |u|^{2^*-2}u + au$, $a > 0$ and discussed the infinitely many solutions. Daoues and all [3] studied (0.489) with $g(u) = \lambda|u|^{q-2}u + \frac{|u|^{2^*(t)-2}u}{|x|^t}$ and obtained the existence and nonexistence of nonnegative distributional solutions. In what follows, we state the main result for which we consider the following hypothesis

$$\inf \left\{ \gamma_{N,s}(T(u))^{\frac{N+2s}{4s}} - \int_{\Omega} f u dx : u \in X, \int_{\Omega} |u|^{2^*} dx = 1 \right\} > 0. \quad (\mathcal{F})$$

Where X is a Hilbert space defined as

$$X = \{u \in H^{2s}(\mathbb{R}^N) : u = 0 \text{ in } \mathbb{R}^N \setminus \Omega\},$$

where $H^{2s}(\mathbb{R}^N)$ the usual fractional Sobolev space,

$$\gamma_{N,s} = \frac{4s}{N-2s} \left(\frac{N-2s}{N+2s} \right)^{\frac{N+2s}{4s}}$$

and

$$T(u) = C_{N,s} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy - \mu \int_{\Omega} \frac{u^2}{|x|^{2s}} dx - \lambda \int_{\Omega} \frac{u^2}{|x|^{2s-\alpha}} dx.$$

Main results

we can define the best constant of fractional Sobolev-Hardy

$$A_{\mu,s}(\Omega) := \inf_{u \in X \setminus \{0\}} \frac{C_{N,s} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy - \mu \int_{\Omega} \frac{u^2}{|x|^{2s}} dx}{\left(\int_{\Omega} |u|^{2^*} dx \right)^{\frac{2}{2^*}}}.$$

Ghoussoub and Shakerian [4] proved that there exists radial solutions $U_{\mu}(x) \in H^s(\mathbb{R}^N)$ positive, symmetric, decreasing and solves

$$\begin{cases} (-\Delta)^s u - \mu \frac{u}{|x|^{2s}} = |u|^{2^*-2}u & \text{in } \mathbb{R}^N, \\ u \geq 0 & \text{in } \mathbb{R}^N, \end{cases}$$

satisfying $\lim_{|x| \rightarrow \infty} U_{\mu}(x) = 0$ and $U_{\mu} \in C^1(\mathbb{R}^N \setminus \{0\})$. Furthermore, U_{μ} has the following properties

$$\begin{aligned} \lim_{|x| \rightarrow 0} |x|^{\beta^-(\mu)} U_{\mu}(x) &= \delta_0, \\ \lim_{|x| \rightarrow \infty} |x|^{\beta^+(\mu)} U_{\mu}(x) &= \delta_{\infty}, \end{aligned}$$



where δ_0 and δ_∞ are positive constants and $\beta^-(\mu)$, $\beta^+(\mu)$ are zeros of the function

$$\Psi_{N,s}(\beta) = 4^s \frac{\Gamma(\frac{N-\beta}{2})\Gamma(\frac{2s+\beta}{2})}{\Gamma(\frac{N-2s-\beta}{2})\Gamma(\frac{\beta}{2})} - \mu, \quad \beta > 0, \quad 0 \leq \mu < \bar{\mu}_s,$$

and satisfying

$$0 \leq \beta^-(\mu) < \frac{N-2s}{2} < \beta^+(\mu) \leq N-2s.$$

The author in [4] proved that $A_{\mu,s}$ is attained in $\mathbb{R}^N$ by the function

$$y_\varepsilon(x) = \varepsilon^{\frac{2s-N}{2}} U_\mu\left(\frac{x}{\varepsilon}\right), \quad \forall \varepsilon > 0,$$

and achieved

$$C_{N,s} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|y_\varepsilon(x) - y_\varepsilon(y)|^2}{|x-y|^{N+2s}} dx dy - \mu \int_{\Omega} \frac{y_\varepsilon^2}{|x|^{2s}} dx = \int_{\Omega} |y_\varepsilon|^{2^*} dx = A_{\mu,s}^{\frac{N}{2s}}.$$

Let the cut-off function $\varphi(x) = \varphi(|x|) \in C_0^\infty(\Omega)$ such that $0 \leq \varphi(x) \leq 1$ in $B(0, R)$ and $\varphi(x) = 1$ in $B(0, \frac{R}{2})$. Set $u_\varepsilon = \varphi(x)y_\varepsilon(x)$, the following asymptotic properties hold.

Proposition 0.22. *Suppose that $N > 2s$, $\mu \in [0; \bar{\mu}_s[$. Then*

- (1) $\|u_\varepsilon\|^2 = A_{\mu,s}^{\frac{N}{2s}} + \mathcal{O}(\varepsilon^{2\beta^+(\mu)+2s-N})$.
- (2) $\int_{\Omega} |u_\varepsilon|^{2^*} dx = A_{\mu,s}^{\frac{N}{2s}} + \mathcal{O}(\varepsilon^{2\beta^+(\mu)-N})$.
- (3) $\int_{\Omega} \frac{|u_\varepsilon|^2}{|x|^{2s-\alpha}} dx = \mathcal{O}(\varepsilon^\alpha)$, where $0 < \alpha < 2\beta^+(\mu) + 2s - N$.
- (4) $\int_{\Omega} u_\varepsilon dx = \mathcal{O}(\varepsilon^{\frac{2\beta^+(\mu)+2s-N}{2}})$.

Due to the fractional Hardy inequality, the operator $\mathcal{L} = (-\Delta)^s - \mu \frac{1}{|x|^{2s}}$ is defined on X . Moreover, the following eigenvalue problem with Hardy potentials and singular coefficient

$$\begin{cases} (-\Delta)^s u - \mu \frac{u}{|x|^{2s}} = \lambda \frac{u}{|x|^{2s-\alpha}} & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases}$$

where $0 < \alpha < 2s$, $\lambda \in \mathbb{R}$, has the first eigenvalue λ_1 given by:



$$\lambda_1 = \inf_{u \in X \setminus \{0\}} \frac{C_{N,s} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy - \mu \int_{\Omega} \frac{u^2}{|x|^{2s}} dx}{\int_{\Omega} \frac{u^2}{|x|^{2s-\alpha}} dx}.$$

The energy functional associated to (0.530) is given by the following expression:

$$I(u) = \frac{1}{2} \|u\|^2 - \frac{\lambda}{2} \int_{\Omega} \frac{u^2}{|x|^{2s-\alpha}} dx - \frac{1}{2^*_s} \int_{\Omega} |u|^{2^*_s} dx - \int_{\Omega} f u dx, \quad \forall u \in X.$$

We see that I is well defined in X and belongs to $C^1(X, \mathbb{R})$ and is not bounded. Moreover, all the critical points of I are precisely the solutions of (0.530). We define the Nehari manifold $\mathcal{N}$ associated with the functional by

$$\mathcal{N} := \{u \in X, \langle I'(u), u \rangle = 0\}.$$

Thus, we write our results in the following theorem:

Theorem 0.147. *Let $0 < \mu < \overline{\mu}_s$, $0 < \lambda < \lambda_1$ and f is a bounded measurable function satisfying the condition $(\mathcal{F})$, then (0.530) has at least two nontrivial solutions, if $0 < \alpha < 2\beta^+(\mu) + 2s - N$.*

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Energy decay of solutions for a linear wave equation with dynamical boundary control

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Abstract. In this paper, we study the 2D linear wave equation with dynamical control on the boundary. We use the Faedo-Galerkin method to prove the global existence of solutions and the multiplier approach to determine the asymptotic behavior in a bounded domain.

Keywords: wave, equation , dynamical, control, asymptotic behavior, energy decay.



Introduction

The study of different distributed systems with dynamical boundary controls, from a theoretical point of view has been done by several authors see for example ([1], [3], and [4]). Studying the 2D linear equation with dynamical boundary controls presents new theoretical and numerical difficulties.

The numerical study of the wave equation with dynamical boundary controls is a fairly recent study. Recently, some problems of this type have been studied, for example [2].

In our work, we relied on the problem studied in [2] and we added some terms in the 2D wave equation, these terms that we added to the equation created a new problem which led us to another change in the dynamical conditions. After these modifications, we obtain a 2D linear equation with boundary controls, then we proved the existence and uniqueness of the solution by using the Faedo-Galerkin method. Next we prove the energy decay and we obtain exponential stability using Martinez lemma (see [5]).

Let Ω be a bounded open set in $\mathbb{R}^2$ with boundary Γ of class C^2 and $\Gamma = \Gamma_0 \cup \Gamma_1$ with $\Gamma_0 \neq \emptyset$ and $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$. The position $u(x, t)$ of the point x of the beam, at the instant t , is governed by the following wave equation:

$$\begin{cases} u_{tt}(x, t) - \Delta u(x, t) - \Delta u_t(x, t) - \Delta u_{tt}(x, t) = 0, & \text{in } Q = \Omega \times [0, T], \\ u(x, t) = 0, & \text{on } \Sigma_0 = \Gamma_0 \times [0, T], \\ \frac{\partial u}{\partial n}(x, t) + \xi(x, t) = 0, & \text{on } \Sigma_1 = \Gamma_1 \times [0, T], \\ \xi_{tt}(x, t) + \xi(x, t) = 0, & \text{on } \Sigma_1 = \Gamma_1 \times [0, T]. \end{cases} \quad (0.333)$$

With the following initial conditions

$$\begin{cases} u(x, 0) = u_0(x), & u_t(x, 0) = u_1(x), & x \in \Omega, \\ \xi(x, 0) = \xi_0(x), & \xi_t(x, 0) = \xi_1(x), & x \in \Gamma_1. \end{cases} \quad (0.334)$$

Where $\xi(x, t)$ designates the dynamical control, and $\frac{\partial}{\partial n}$ is the unit outward normal to the boundary.

Main results

First, we define the energy function of problem (0.492)-(0.493) as

$$E(t) = \frac{1}{2} \int_{\Omega} (|u_t|^2 + |\nabla u|^2 + |\nabla u_t|^2) dx + \frac{1}{2} \int_{\Gamma_1} (|\xi_t|^2 + |\xi|^2) d\sigma \quad (0.335)$$

Lemma 0.57. *Let (u, ξ) be a solution of problem (0.492)-(0.493). Then the energy function (0.442) is a non-increasing function on $[0, T]$ and satisfies*

$$E'(t) = - \int_{\Omega} |\nabla u_t|^2 dx - \int_{\Gamma_1} |\xi_{tt} + \xi|^2 d\sigma \leq 0, \quad \forall 0 \leq t \leq T. \quad (0.336)$$



Lemma 0.58. [5] Let $E : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a non increasing function and $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ a increasing function of class C^1 such that

$$\phi(0) = 0, \quad \text{and} \quad \phi(t) \rightarrow +\infty \quad \text{as} \quad t \rightarrow +\infty.$$

Assume that there exist $p > -1$ and $\omega > 0$ such that

$$\int_S^{+\infty} E^{p+1}(t) \phi'(t) dt \leq \frac{1}{\omega} E^p(0) E(S), \quad 0 \leq S \leq \infty,$$

then E has the following decay properties

$$\begin{aligned} E(t) &= 0, \quad \forall t \geq \frac{E^p(0)}{\omega|p|}, \quad \text{if } -1 < p < 0 \\ E(t) &\leq E(0)e^{1-\omega\phi(t)}, \quad \forall t \geq 0, \quad \text{if } p = 0 \\ E(t) &\leq E(0) \left(\frac{1+p}{1+\omega p\phi(t)} \right), \quad \forall t \geq 0, \quad \text{if } p > 0 \end{aligned}$$

Theorem 0.148. Assume that $u_0, u_1 \in H_{\Gamma_0}^1(\Omega) \cap H^2(\Omega)$, $\xi_0, \xi_1 \in L^2(\Gamma_1)$ and $\frac{\partial u_0}{\partial n}(x) = -\xi_0(x)$, $\frac{\partial u_1}{\partial n}(x) = -\xi_1(x)$ for all $x \in \Gamma_1$. Then for each $T > 0$, problem (0.492)-(0.493) admits a unique solution (u, ξ) satisfying the following conditions:

$$\begin{aligned} u &\in L^\infty(0, T, H_{\Gamma_0}^1(\Omega)), \quad u_t, u_{tt} \in L^\infty(0, T, L^2(\Omega)) \cap L^\infty(0, T, H_{\Gamma_0}^1(\Omega)), \\ \xi, \xi_{tt} &\in L^\infty(0, T, L^2(\Gamma_1)). \end{aligned}$$

Moreover, for some positive constant k , we obtain the following decay property

$$E(t) \leq E(0)e^{1-k\phi(t)}, \quad \forall t \geq 0,$$

where ϕ is function satisfies certain hypotheses, which we will specify in the proof.

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Optimal Control of The Problems Evolution Equations

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Abstract. In this work we study the optimal control of a nonlinear model described by Marguerre von Karman equations with long memory with variable change to simplify this problem, first we prove the uniqueness and existence of optimal control for the quadratic cost. Then, by proving the strong Gâteaux differentiability of nonlinear solution mapping $u \rightarrow y(u)$ we establish necessary optimality condition for the optimal control corresponding to the quadratic cost

Keywords: Marguerre von Karman system with long memory, optimal control, Gâteaux differentiability, necessary optimality condition.



Introduction

We consider the following Marguerre von Kármán system with long memory and the clamped boundary condition in the variables χ , representing the position of the plate and the Airy's stress function ψ and θ represents the geometric shape of this shell.

$$\begin{cases} \chi_{tt} - \Delta \chi_{tt} + \Delta^2 \chi + K * \Delta^2 \chi = [\psi, \chi + \theta] + f & \text{in } (0, T) \times \Omega, \\ \Delta^2 \psi = -[\chi, \chi + 2\theta] & \text{in } (0, T) \times \Omega, \\ \chi = \partial_\nu \chi = \psi = \partial_\nu \psi = 0 & \text{on } (0, T) \times \partial\Omega, \\ \chi(0, x) = \chi_0(x) \quad \chi_t(0, x) = \chi_1(x) & \text{in } \Omega, \end{cases} \quad (0.337)$$

such that :

- Ω bounded open of R^2 with smooth boundary $\partial\Omega$.
- $(K * \Delta^2)(\varpi)\chi = \int_0^t K(t - \varpi) \Delta^2 \chi(\varpi) d\varpi$, and $K \in C^1([0, T])$.
- The vector ν an external normal, $f \in L^2$.
- $[\chi, \psi] = \frac{\partial^2 \chi}{\partial x_1^2} \frac{\partial^2 \psi}{\partial x_2^2} + \frac{\partial^2 \psi}{\partial x_1^2} \frac{\partial^2 \chi}{\partial x_2^2} - 2 \frac{\partial^2 \chi}{\partial x_1 \partial x_2} \frac{\partial^2 \psi}{\partial x_1 \partial x_2}$ and call the non-linear, symmetrical Monge-Ampère bracket

By variable change according to ([1]) we have:

$$\begin{cases} y_{tt} - \Delta y_{tt} + \Delta^2 y + K * \Delta^2 y = [y, \varphi] + \Delta^2 \theta + f & \text{in } (0, T) \times \Omega \\ \Delta^2 \varphi = -[y, y] + [\theta, \theta] & \text{in } (0, T) \times \Omega \\ y = \frac{\partial y}{\partial \nu} = \varphi = \frac{\partial \varphi}{\partial \nu} = 0 & \text{on } (0, T) \times \partial\Omega \\ y(0, x) = y_0(x), \quad y_t(0, x) = y_1(x) & \text{in } \Omega, \end{cases} \quad (0.338)$$

Firstly, we survey the well-posedness of Eq. (0.459) with respect to y in the Hadamard sense relying on some previous results. Especially, in order to prove the local Lipschitz continuity of the solution mapping, we employ the energy equality of Volterra-type integro-differential equation which is proved in [2].

Secondly, based on this result, we study the following optimal control problem

$$\text{Minimize} \quad J(u), \quad (0.339)$$

we propose the quadratic cost functional J as studied in Lions [5], which is to be minimized within U_{ad} , an admissible set of control variables in U . The problem 2 consists of two problems:

- The existence of optimal control.
- The necessary condition for the optimal control.



We need to show the existence of $u^* \in U_{ad}$ that minimizes the quadratic cost functional J . However, differently from the linear equation case, we are faced with difficulty that the weak convergence of the controlled state $y(u_n)$ is insufficient to cover the convergence of the nonlinear part of Eq (0.459). Also, this method is quite useful in proving the strong Gâteaux differentiability of the nonlinear solution mapping $u \rightarrow y(u)$, which is used to define the associate adjoint system. Then, we establish a necessary condition of optimality of the optimal control u^* .

Main results

*

The uniqueness of the solution:

Theorem 0.149. Assume that $(y_0, y_1) \in H_0^2 \times H_0^1$, $K \in C^1([0, T])$, $f \in L^2(0, T; L^2)$, and $\theta \in H_0^2$. Then Eq. (0.459) has a unique weak solution y . Moreover, the solution mapping $p = (y_0^1, y_1^1, f_1) \rightarrow (y(p), y_t(p), \varphi(p))$ of $\wp = H_0^2 \times H_0^1 \times L^2(0, T; L^2)$ into $C([0, T]; H_0^2) \times C([0, T]; H_0^1) \times C([0, T]; W_\infty^2)$ is locally Lipschitz continuous.

Indeed, let $p_1 = (y_0^1, y_1^1, f_1) \in \wp$ and $p_2 = (y_0^2, y_1^2, f_2) \in \wp$. We prove Theorem ?? by showing the inequality:

$$\|\nabla(y'(p_1; t) - y'(p_2; t))\| + \|\Delta(y(p_1; t) - y(p_2; t))\| + \|\varphi(p_1; t) - \varphi(p_2; t)\|_{W_\infty^2} \leq C \|p_1 - p_2\|_\wp, \quad (0.340)$$

where $C > 0$ is a constant depending on data, and

$$\|p_1 - p_2\|_\wp = (\|y_0^1 - y_0^2\|_{H_0^2} + \|y_1^1 - y_1^2\|_{H_0^1} + \|f_1 - f_2\|_{L^2(0, T; L^2)})^{\frac{1}{2}}.$$

*

The existence of the solution:

Theorem 0.150. Assume that the hypotheses of Theorem (2.1) and [4] are satisfied. Then there exists at least one optimal control u for the control this problem with the cost :

$$J(u) = \|Cy(u) - Y_d\|_M^2 + (Ru, u)_U, \quad \text{for } u \in U \quad (0.341)$$

where C is an operator called the observer, M is a Hilbert space of observation variables, $Y_d \in M$ is a desired value of $y(u)$, and $R \in L(U, U)$ is symmetric and positive.

*

Necessary condition of an optimal control:



Theorem 0.151. *The optimal control u^* is characterized by the following system of equations and inequality:*

$$\begin{cases} y_{tt}(u^*) - \Delta y_{tt}(u^*) + \Delta^2 y + K * \Delta^2 y(u^*) = [y(u^*), \varphi(u^*)] + \Delta^2 \theta + Bu^* & \text{in } (0, T) \times \Omega \\ \Delta^2 \varphi(u^*) = -[y(u^*), y(u^*)] + [\theta, \theta] & \text{in } (0, T) \times \Omega \\ y(u^*) = \frac{\partial y(u^*)}{\partial \nu} = \varphi(u^*) = \frac{\partial \varphi(u^*)}{\partial \nu} = 0 & \text{on } (0, T) \times \partial \Omega \\ y(0, u^*) = y_0(u^*), \quad y_t(0, u^*) = y_1(u^*) & \text{in } \Omega, \end{cases} \quad (0.342)$$

using the Gâteaux differentiable derivative of $y(u)$, with the direction $u - u^*$, we find:

$$\begin{cases} p_{tt}(u^*) - \Delta p_{tt}(u^*) + \Delta^2 p + \int_t^T K(\sigma - t) \Delta^2 p(u^*, \sigma) d\sigma = [p(u^*), G(-[y(u^*), y(u^*)] + [\theta, \theta]) \\ + 2[y(u^*), -G[p(u^*), y(u^*)]] + y(u^*) - Y_d & \text{in } (0, T) \times \Omega \\ p(u^*) = \frac{\partial p(u^*)}{\partial \nu} = 0 & \text{on } (0, T) \times \partial \Omega \\ p(u^*, T) = 0, \quad p_t(u^*, T) = 0 & \text{in } \Omega, \end{cases} \quad (0.343)$$

Then

$$\int_0^T (p(u^*), B(u - u^*))_2 dt + (Ru^*, u - u^*)_2 \geq 0 \quad \forall u \in U_{ad}.$$

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Free-boundary problem and image contour enhancement with self-similar solution

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Abstract. Many models that use non-linear PDEs have been extensively used for different tasks of edge enhancement in image processing. In this work, we introduce a new evolution model that generalizes mean curvature motion and provides a comprehensive study of edge enhancement by formulating a free boundary problem. This model is particularly effective in analyzing the process of edge enhancement in images, allowing for more precise control over the diffusion process around image edges. This represents a novel approach, providing a new perspective on image contour enhancement. We present numerical simulations for the proposed model on 2-D gray-level images to validate the theoretical findings. The numerical experiments demonstrate the effectiveness of the model in preserving and enhancing edges, confirming the practical applicability of the theoretical results.

Keywords: Nonlinear diffusion equations ; Mean curvature motion ; Image enhancement ; Self ; similar solutions.



Introduction

For the past decades, methods based on partial differential equations (PDEs) have been extensively used to model processes in numerous real-world applications, ranging from physics over life sciences to economy. Thus, it is not surprising that many partial differential equations (PDEs) and variational approaches have been contributed substantially to the mathematical foundations of signal and image analysis. For instance, they appear as Euler-Lagrange equations when solving continuous optimization problems that result from variation models or regularizations of ill-posed problems. As a result of a certain degeneracy of the asymptotic forms of equations mentioned in [5], Barenblatt [1] noted the possibility to construct a more general class of equations which generalized mean curvature motion and Beltrami flow. An asymptotic treatment in one dimensional case of this class of equations has been investigated theoretically in [1, 2] and [4]. The authors report that the generalized mean curvature flow equation forms a sharp step in the vicinity of edges.

Free boundary problem to contour enhancement

We introduce the following free boundary problem for determining the image intensity evolution in the boundary layer: Given an increasing function $u_0(x)$ defined in an interval $I = (a_1, a_2)$ with end values $u_0(a_1) = 0$, $u_0(a_2) = 1$, to find a continuous function $u(x, t)$ and continuous curves $x = l(t)$ and $x = r(t)$ such that:

1. $l(0) = a_1$, $r(0) = a_2$, and $l(t) < r(t)$ for some time interval $t \in (0, T)$.
2. u solves the following problem in $\Omega = \{(x, t) : 0 < t < T, l(t) < x < r(t)\}$:

$$\begin{cases} \frac{\partial u}{\partial t} = (1-p)u_x^{q-p}u_{xx}, & \text{in } \Omega \\ u(x, 0) = u_0(x), & \text{for } a_1 \leq x \leq a_2 \\ u(l(t), t) = 0, u_x(l(t), t) = +\infty & \text{for } 0 < t < T \\ u(r(t), t) = 1, u_x(r(t), t) = +\infty & \text{for } 0 < t < T \end{cases}, \quad (0.344)$$

for $q < p < 1$ and $q > p > 1$. T can be finite or infinite time.

This is a free boundary problem because the unknowns are the function $u(x, t)$ and the moving boundaries, $l(t)$ and $r(t)$, which must be determined from the problem by the over-specifical conditions. For this reason the interfaces $x = l(t)$ and $x = r(t)$ are also called "free-boundaries".

The mathematical theory for the free-boundary problem (0.344) was treated in [2]. The solution $u(x, t)$ sharpens as t grows; actually the two free boundaries $x = l(t)$ and $x = r(t)$ shrink. If the two moving boundaries meet, a vertical front is formed, representing completed enhancement. We want to find a solution written in the form:

$$u(x, t) = \phi(\xi), \quad \xi = \frac{x - x_0}{a(t)}, \quad (0.345)$$



Main results

We try now to solve problem (0.344) by using the self-similar solution under the form (0.345). Taking account this form, Substituting (0.345) into problem (0.344), we obtain, the equation:

$$-\frac{\dot{a}}{a}\xi\frac{d\phi}{d\xi} = (1-p)\frac{1}{a^{-(q+p)+2}}\frac{d^2\phi}{d\xi^2}\left(\frac{d\phi}{d\xi}\right)^{-(q+p)}, \quad \text{for } q > 1-p. \quad (0.346)$$

A separation of variables argument implies that the following conditions must hold :

$$\frac{\dot{a}}{a} = -\alpha\frac{1}{a^{-(q+p)+2}}, \quad (0.347)$$

where α is arbitrary positive constant.

The equation for the profile ϕ , becomes:

$$\alpha\xi\frac{d\phi}{d\xi} = (1-p)\frac{d^2\phi}{d\xi^2}\left(\frac{d\phi}{d\xi}\right)^{-(q+p)}.$$

After easy integration, we get:

$$\frac{d\phi}{d\xi} = \left[\frac{\alpha(q+p)}{2(1-p)}\right]^{\frac{-1}{q+p}} C^{\frac{-2}{q+p}} \left[1 - \left(\frac{\xi}{C}\right)^2\right]^{\frac{-1}{q+p}}, \quad (0.348)$$

The self-similar solution assumes the form

$$u(x, t) = \left[\frac{\alpha(q+p)}{2(1-p)}\right]^{\frac{-1}{q+p}} C^{\frac{q+p-2}{q+p}} \int_{-1}^{\frac{x-x_0}{C\alpha(t)}} [1 - \eta^2]^{\frac{-1}{q+p}} d\eta, \quad (0.349)$$

We formulate a free boundary problem that describes the evolution of image intensity in boundary layers around the edges. This formulation is a significant innovation, as it offers a more dynamic and adaptable method for handling image edges, unlike many fixed-boundary methods used in previous research. The free-boundary approach allows for better modeling of the natural variations and complexities present in real-world images. Numerical Simulations and Experimental Validation: we also presents some numerical simulations for the proposed model on 2-D gray-level images to validate the theoretical findings. The numerical experiments demonstrate the effectiveness of the model in preserving and enhancing edges.

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A Numerical strategy for trapped waves in the β – plane

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Abstract. In this work a simplified barotropic-baroclinic model will be used to study both monsoon oscillation and the ITCZ breakdown, for various locations, strengths, and migration events of the ITCZ. The mathematical study will be carried both analytically using linear wave theory and numerically, and both the dry and moist cases, with and without moisture/latent heat effect, will be considered in order to pin down the key physical mechanisms responsible for these phenomena and compare to both existing studies and observations.

Keywords: Inter-tropical convergence zone (ITCZ),Non-linear Primitive equation,Galerkin projection method , parabolic cylindrical functions, spectral method.



Introduction

The Inter-tropical convergence zone (ITCZ) is a strip of persistent precipitation and cloudiness, located in the tropical region that runs parallel to the equator. Its migration from the equatorial region during the northern/southern hemisphere winter to the northern/southern tropics during the northern/southern hemisphere summer leads to the onset of summer monsoons. Monsoons occur in tropical regions where the warm ocean water is located on the equatorial side of a major continental land mass. Monsoon seasons are vital to many local economics and ecosystems, including regions such as South Asia, the Sahel, Northern Australia, and South-Western North America. With the advent of global warming their stability and predictability becomes more and more uncertain. Moreover, the dynamical instability of the ITCZ is known to induce wave disturbances with oscillations varying from days to several weeks that are believed to be responsible for weathervariability on these scales and to control the amounts of rain that fall during the monsoon season. In oceanic regions of the tropics, away from the land masses, such as the Eastern Pacific and the tropical Atlantic, the ITCZ is known to undulate and breakdown to form vortices that potentially amplify and spin up into tropical cyclones that travel long distances and become devastating hurricanes and typhoons. There is evidence that the ITCZ breakdown may occur both as a manifestation of the instability of the associated barotropic shear or due to interactions with easterly waves synoptic scale (1000-5000 kmbwavelength) atmospheric waves that are abundant in the tropics.

Primitive equations

We consider the following non-linear primitive equations:

$$\begin{aligned}\frac{\partial V}{\partial t} + V \cdot \nabla V + W \frac{\partial V}{\partial z} + f(y)V^\perp &= -\nabla \Phi, \\ \frac{\partial \Phi}{\partial z} &= \frac{g\Theta}{\theta_0}, \\ \frac{\partial \Theta}{\partial t} + V \cdot \nabla \Theta + W \frac{\partial \Theta}{\partial z} + \frac{N^2 \theta_0}{g} W &= 0, \\ \nabla \cdot V + \frac{\partial W}{\partial z} &= 0.\end{aligned}\tag{0.350}$$

Here $V = (u, v)$ is the horizontal velocity field, where u is the zonal component and v is the meridional (North-South) component, and W is the vertical velocity. The scalars Φ and Θ are the pressure and potential temperature perturbations, respectively. the total potential temperature, including a background that depends only on height, is given by

$$\Theta^{\text{total}} = \theta_0 + \bar{\theta}(z) + \Theta(x, y, z, t)$$



where $\theta_0 \approx 300\text{K}$ is a reference constant temperature and $\bar{\theta}$ defines the vertical profile background stratification and satisfies $N^2 = (g/\theta_0)(\partial\bar{\theta}/\partial z) > 0$ where N is known as the buoyancy frequency. Here x, y, z are, respectively, the zonal, meridional, and vertical coordinates and $t > 0$ is time. β is the gradient of Coriolis parameter.

The primitive equations are supplemented with the rigid boundary conditions

$$W|_{z=0}^{z=H} = 0.$$

Using the β -plane approximation the Coriolis force $f(y) = 2\Omega \sin(y)$ is linearized about a latitude of reference $y = y_0$: $f(y) \approx f_0 + \beta(y - y_0)$. (we can write $y - y_0$ as a new variable like y) Then, we convert the dependent and independent variables into non-dimensional form as

$$\begin{aligned} \hat{V} &= \frac{V}{c}, & \hat{W} &= \frac{W}{\nu}, & \hat{\Phi} &= \frac{\Phi}{c^2}, & \hat{\theta} &= \frac{\theta}{\bar{\alpha}}, \\ (\hat{x}, \hat{y}) &= \frac{(x, y)}{L}, & \hat{z} &= \frac{\pi}{H}z, & \hat{t} &= \frac{t}{T}, \end{aligned} \quad (0.351)$$

Where $c = NH/\pi$, $L = \sqrt{c/\beta}$, $T = L/c$ and the temperature scale is $\bar{\alpha} = HN^2\theta_0/\pi g$. Thus, using scalings (0.351), the equations (0.470) are non-dimensionalized to give

$$\begin{aligned} \frac{\partial V}{\partial t} + V \cdot \nabla V + W \frac{\partial V}{\partial z} + \left(\frac{f_0 L}{c} + y\right) V^\perp &= -\nabla \Phi, \\ \frac{\partial \Phi}{\partial z} &= \Theta, \\ \frac{\partial \Theta}{\partial t} + V \cdot \nabla \Theta + W \frac{\partial \Theta}{\partial z} + W &= 0, \\ \nabla \cdot V + \frac{\partial W}{\partial z} &= 0. \end{aligned} \quad (0.352)$$

We decompose the non-dimensional equations (0.457) into their barotropic and baroclinic parts by expansion as

$$\begin{aligned} \begin{pmatrix} V \\ \Phi \end{pmatrix}(x, y, z, t) &= \begin{pmatrix} \bar{V} \\ \bar{p} \end{pmatrix}(x, y, t) + \begin{pmatrix} V_1 \\ p_1 \end{pmatrix}(x, y, t) \sqrt{2} \cos(\pi z/H), \\ \begin{pmatrix} W \\ \Theta \end{pmatrix}(x, y, z, t) &= \begin{pmatrix} w_1 \\ \theta_1 \end{pmatrix}(x, y, t) \sqrt{2} \sin(\pi z/H). \end{aligned} \quad (0.353)$$

When the primitive equations are Galerkin projected into the barotropic and first baroclinic modes according to (0.458), we obtain the following non-linearly



coupled systems, for barotropic and first baroclinic modes, respectively. We have

$$\begin{aligned} \frac{\partial \bar{V}}{\partial t} + \bar{V} \cdot \nabla \bar{V} + V_1 \cdot \nabla V_1 + (\nabla \cdot V_1) V_1 + \left(\frac{f_0 L}{c} + y\right) \bar{V}^\perp &= -\nabla \bar{p}, \\ \nabla \cdot \bar{V} &= 0, \\ \frac{\partial V_1}{\partial t} + \bar{V} \cdot \nabla V_1 + V_1 \cdot \nabla \bar{V} + \left(\frac{f_0 L}{c} + y\right) V_1^\perp - \nabla \theta_1 &= 0, \\ \frac{\partial \theta_1}{\partial t} + \bar{V} \cdot \nabla \theta_1 - \nabla \cdot V_1 &= 0. \end{aligned} \quad (0.354)$$

In order to apply the linear theory, first we consider shear background solution for the equation as $\bar{V} = \bar{V}_0 = (\bar{u}(y), 0)$, $\bar{p} = \bar{p}_0 = \bar{p}(y)$ for barotropic and $V_1 = \bar{V}_1 = (\bar{u}_1(y), 0)$, $\theta_1 = \bar{\theta}_1 = \bar{\theta}_1(y)$ for baroclinic. As a result, $\bar{V}_0, \bar{p}_0, \bar{V}_1, \bar{\theta}_1$ are solutions to the equations (0.525) provided that

$$\begin{aligned} \left(\frac{f_0 L}{c} + y\right) \bar{u}(y) &= -\frac{\partial \bar{p}(y)}{\partial y}, \\ \left(\frac{f_0 L}{c} + y\right) \bar{u}_1(y) &= \frac{\partial \bar{\theta}_1(y)}{\partial y}. \end{aligned}$$

Next, We consider the solution to equations (0.525) as $\bar{V} = \bar{V}_0 + \varepsilon V'_0$, $\bar{p} = \bar{p}_0 + \varepsilon p'_0$ for barotropic part and $V = \bar{V}_1 + \varepsilon V'_1$, $\theta = \bar{\theta}_1 + \varepsilon \theta'_1$ for baroclinic part, where $\bar{V}_0 = (\bar{u}(y), 0)$, $\bar{V}_1 = (\bar{u}_1(y), 0)$, $\bar{p}_0 = \bar{p}(y)$, $\bar{\theta}_1 = \bar{\theta}_1(y)$ as defined above. Here, V'_0, V'_1 are vectors and p'_0, θ'_1 are scalars depending on x, y, t . We plug them in equations (0.525), we obtain the result:

$$\begin{aligned} \frac{\partial u'_0}{\partial t} + \bar{u}(y) \frac{\partial u'_0}{\partial x} + v'_0 \frac{\partial \bar{u}(y)}{\partial y} + \bar{u}_1(y) \frac{\partial u'_1}{\partial x} + v'_1 \frac{\partial \bar{u}_1(y)}{\partial y} + \bar{u}_1(y) \left(\frac{\partial u'_1}{\partial x} + \frac{\partial v'_1}{\partial y}\right) - \left(\frac{f_0 L}{c} + y\right) v'_0 &= -\frac{\partial p'_0}{\partial x}, \\ \frac{\partial v'_0}{\partial t} + \bar{u}(y) \frac{\partial v'_0}{\partial x} + \bar{u}_1(y) \frac{\partial v'_1}{\partial x} + \left(\frac{f_0 L}{c} + y\right) u'_0 &= -\frac{\partial p'_0}{\partial y}, \\ \frac{\partial u'_0}{\partial x} + \frac{\partial v'_0}{\partial y} &= 0, \\ \frac{\partial u'_1}{\partial t} + \bar{u}(y) \frac{\partial u'_1}{\partial x} + v'_0 \frac{\partial \bar{u}_1(y)}{\partial y} + \bar{u}_1(y) \frac{\partial u'_0}{\partial x} + v'_1 \frac{\partial \bar{u}(y)}{\partial y} - \left(\frac{f_0 L}{c} + y\right) v'_1 &= \frac{\partial \theta'_1}{\partial x}, \\ \frac{\partial v'_1}{\partial t} + \bar{u}(y) \frac{\partial v'_1}{\partial x} + \bar{u}_1(y) \frac{\partial v'_0}{\partial x} + \left(\frac{f_0 L}{c} + y\right) u'_1 &= \frac{\partial \theta'_1}{\partial y}, \\ \frac{\partial \theta'_1}{\partial t} + \bar{u}(y) \frac{\partial \theta'_1}{\partial x} + v'_0 \frac{\partial \bar{\theta}_1(y)}{\partial y} - \left(\frac{\partial u'_1}{\partial x} + \frac{\partial v'_1}{\partial y}\right) &= 0. \end{aligned} \quad (0.355)$$

Because of the incompressibility constraint, the barotropic equations can be written in vorticity-stream function form. Let $\xi'_0 = \frac{\partial v'_0}{\partial x} - \frac{\partial u'_0}{\partial y}$ be the vertical vorticity of the flow and ψ'_0 be the associated stream function. we do cross-differentiation of the first and second equations in (0.531), we obtain:



$$\begin{aligned}
& \frac{\partial}{\partial t} \left\{ (\psi'_0)_{xx} + (\psi'_0)_{yy} \right\} + \bar{u}(y) \frac{\partial}{\partial x} \left\{ (\psi'_0)_{xx} + (\psi'_0)_{yy} \right\} + \bar{u}_1(y) \frac{\partial}{\partial x} \left\{ \frac{\partial v'_1}{\partial x} - \frac{\partial u'_1}{\partial y} \right\} - \frac{\partial \psi'_0}{\partial x} \frac{\partial^2 \bar{u}(y)}{\partial y^2} \\
& - \frac{\partial \bar{u}_1(y)}{\partial y} \left\{ \frac{\partial u'_1}{\partial x} + \frac{\partial v'_1}{\partial y} \right\} - v'_1 \frac{\partial^2 \bar{u}_1(y)}{\partial y^2} - \frac{\partial \bar{u}_1(y)}{\partial y} \left\{ \frac{\partial u'_1}{\partial x} + \frac{\partial v'_1}{\partial y} \right\} \\
& - \bar{u}_1(y) \frac{\partial}{\partial y} \left\{ \frac{\partial u'_1}{\partial x} + \frac{\partial v'_1}{\partial y} \right\} + \frac{\partial \psi'_0}{\partial x} = 0, \\
& \frac{\partial u'_1}{\partial t} + \bar{u}(y) \frac{\partial u'_1}{\partial x} + \frac{\partial \psi'_0}{\partial x} \frac{\partial \bar{u}_1(y)}{\partial y} + \bar{u}_1(y) \frac{\partial}{\partial x} \left(-\frac{\partial \psi'_0}{\partial y} \right) + v'_1 \frac{\partial \bar{u}(y)}{\partial y} - \left(\frac{f_0 L}{c} + y \right) v'_1 = \frac{\partial \theta'_1}{\partial x}, \\
& \frac{\partial v'_1}{\partial t} + \bar{u}(y) \frac{\partial v'_1}{\partial x} + \bar{u}_1(y) \frac{\partial^2 \psi'_0}{\partial x^2} + \left(\frac{f_0 L}{c} + y \right) u'_1 = \frac{\partial \theta'_1}{\partial y}, \\
& \frac{\partial \theta'_1}{\partial t} + \bar{u}(y) \frac{\partial \theta'_1}{\partial x} + \frac{\partial \psi'_0}{\partial x} \frac{\partial \bar{\theta}_1(y)}{\partial y} - \left(\frac{\partial u'_1}{\partial x} + \frac{\partial v'_1}{\partial y} \right) = 0, \quad (0.356)
\end{aligned}$$

Next we solve the equations (0.356) using Galerkin method which consists of projecting the equations in the y-direction onto the first few parabolic cylinder functions.

By projecting the equations onto the parabolic cylindrical functions we obtain one-dimensional system



$$\begin{aligned}
& \begin{pmatrix} I_x + \frac{1}{2}D_1 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \frac{\partial \tilde{W}}{\partial t} + \begin{pmatrix} I + \frac{\alpha_0}{2\sqrt{2}}E_1 & -2\alpha_1 I - \alpha_1 F_1 & 0 & 0 \\ -\frac{\alpha_1}{2}F_1 + \alpha_1 I & \frac{\alpha_0}{2\sqrt{2}}B_1 & 0 & -I \\ 0 & 0 & \frac{\alpha_0}{2\sqrt{2}}B_1 & 0 \\ \frac{\alpha_1}{\sqrt{2}}\frac{f_0 L}{c}B_1 + \frac{\alpha_1}{2}D_2 & -I & 0 & \frac{\alpha_0}{2\sqrt{2}}B_1 \end{pmatrix} \frac{\partial \tilde{W}}{\partial x} \\
& + \begin{pmatrix} 0 & 0 & -\frac{2\alpha_1}{\sqrt{2}}B_2 - \frac{\alpha_1}{2\sqrt{2}}E_1 & 0 \\ 0 & 0 & \alpha_0 I - \frac{f_0 L}{c}I - \frac{1}{\sqrt{2}}B_1 & 0 \\ 0 & \frac{f_0 L}{c}I + \frac{1}{\sqrt{2}}B_1 & 0 & -\frac{1}{\sqrt{2}}B_2 \\ 0 & 0 & -\frac{1}{\sqrt{2}}B_2 & 0 \end{pmatrix} \tilde{W} \\
& + \begin{pmatrix} \frac{\alpha_0}{\sqrt{2}}B_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \frac{\partial^3 \tilde{W}}{\partial x^3} + \begin{pmatrix} 0 & 0 & \frac{\alpha_0}{\sqrt{2}}B_1 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\alpha_1}{\sqrt{2}}B_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \frac{\partial^2 \tilde{W}}{\partial x^2} \\
& \quad (0.357)
\end{aligned}$$

where,

$\tilde{W} = ((\tilde{\psi}'_0)_0, \dots, (\tilde{\psi}'_0)_{N-1}, (\tilde{u}'_1)_0, \dots, (\tilde{u}'_1)_{N-1}, (\tilde{v}'_1)_0, \dots, (\tilde{v}'_1)_{N-1}, (\tilde{\theta}'_1)_0, \dots, (\tilde{\theta}'_1)_{N-1})$
and the matrices are given by

$$\begin{aligned}
I_x &= \begin{pmatrix} \partial_x^2 & 0 & \dots & 0 \\ 0 & \partial_x^2 & \dots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \dots & \partial_x^2 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & \sqrt{2} & \\ & \sqrt{2} & \ddots & \ddots \\ & & \ddots & 0 & \sqrt{N-1} \\ 0 & & \sqrt{N-1} & 0 \end{pmatrix}, \\
B_2 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & \sqrt{2} & \\ & -\sqrt{2} & \ddots & \ddots \\ & & \ddots & 0 & \sqrt{N-1} \\ 0 & & -\sqrt{N-1} & 0 \end{pmatrix},
\end{aligned}$$



$$E_1 = \begin{pmatrix} 0 & -3 & 0 & \sqrt{6} & 0 & \cdots & & & & 0 \\ 1 & 0 & -4\sqrt{2} & 0 & \sqrt{24} & 0 & \cdots & & & 0 \\ 0 & 0 & 0 & -5\sqrt{3} & 0 & \sqrt{60} & 0 & \cdots & & 0 \\ \sqrt{6} & 0 & -\sqrt{3} & 0 & -6\sqrt{4} & 0 & \sqrt{120} & 0 & \cdots & 0 \\ & \sqrt{24} & 0 & -2\sqrt{4} & 0 & -7\sqrt{5} & 0 & \sqrt{210} & \cdots & \\ \vdots & & \ddots & & \ddots & & \ddots & & \ddots & \\ 0 & \cdots & & & & \cdots & 0 & -(N-3)\sqrt{N-1} & 0 \end{pmatrix}$$

$$D_1 = \begin{pmatrix} -1 & 0 & \sqrt{2} & 0 & \cdots & & & 0 \\ 0 & -3 & 0 & \sqrt{6} & 0 & \cdots & & 0 \\ \sqrt{2} & 0 & -5 & 0 & \sqrt{12} & \ddots & & \\ \vdots & & & \ddots & & \ddots & & \\ & & & & \sqrt{(N-1)(N-2)} & -(2N-3) & 0 \\ & & & & & 0 & -(2N-1) \end{pmatrix},$$

$$D_2 = \begin{pmatrix} 1 & 0 & \sqrt{2} & 0 & \cdots & & & 0 \\ 0 & 3 & 0 & \sqrt{6} & 0 & \cdots & & 0 \\ \sqrt{2} & 0 & 5 & 0 & \sqrt{12} & \ddots & & \\ \vdots & & & \ddots & & \ddots & & \\ & & & & \sqrt{(N-1)(N-2)} & (2N-3) & 0 \\ & & & & & 0 & (2N-1) \end{pmatrix},$$

$$F_1 = \begin{pmatrix} -1 & 0 & \sqrt{2} & 0 & \cdots & & & 0 \\ 0 & -1 & 0 & \sqrt{6} & 0 & \cdots & & 0 \\ \sqrt{2} & 0 & -1 & 0 & \sqrt{12} & \ddots & & \\ \vdots & & & \ddots & & \ddots & & \\ & & & & \sqrt{(N-1)(N-2)} & -(2N-3) & 0 \\ & & & & & 0 & -1 \end{pmatrix},$$

Now, we do a linear analysis of this system by assuming wavelike solutions as

$$\tilde{W} = \hat{W} \exp(i(kx - \omega t)), \quad (0.358)$$

where $i^2 = -1$, k is zonal wave number and ω is the frequency. We plug (0.358)



in ():

$$\begin{aligned}
 & \exp(i(kx - \omega t))\hat{W} \{B(-i\omega) + A(ik) + C + D(ik)^3 + E(ik)^2\} = 0, \\
 & \implies [-i\omega B + Aki + C - ik^3 D - k^2 E]\hat{W} = 0, \\
 & \implies [-\omega B + Ak - Ci - Dk^3 + k^2 iE]\hat{W} = 0, \\
 & \implies [-\omega I + B^{-1}(Ak - Dk^3 + i(Ek^2 - C))]\hat{W} = 0, \tag{0.359}
 \end{aligned}$$

which requires us to find the eigenvalues $\omega(k)$ and eigenvectors $\hat{W}$ for the matrix $M = B^{-1}(Ak - Dk^3 + i(Ek^2 - C))$. The formula for wave number is $k = \frac{2\pi}{\lambda}$, where λ is the wavelength. Set $\lambda = \frac{\tilde{p}}{l}$, $l = 1, 2, \dots$ in non-dimensional unit, where $\tilde{p} = \frac{40000}{1500}$. Thus, the wave number in non-dimensional unit is $\tilde{k} = \tilde{k}_0 l$, where $\tilde{k}_0 = \frac{2\pi}{\tilde{p}}$ for $l = 1, 2, 3, 4$.

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Analysis of Bifurcation, Chaos, and Entropy in the Fractional Chaotic Burgers Map

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Abstract. This work presents the fractional Burgers map. The dynamic behavior of this map is analyzed using bifurcation diagrams, Lyapunov exponents, and spectral entropy. The findings conclude that the proposed map exhibits chaotic behavior, which is related to the value of the fractional order.

Keywords: Bifurcation diagram, Spectral entropy, Fractional, Caputo-like.



Introduction

Fractional systems have gained increasing attention in recent years, as fractional calculus can effectively model many real-world problems. Significant findings, such as chaos and bifurcation, have been discovered in fractional systems. However, research on discrete fractional systems remains in its early stages, particularly in the area of chaos dynamics. Moreover, researchers have obtained intriguing results in the study of discrete fractional chaotic systems. Despite the progress, identifying chaos in fractional chaotic maps remains an open challenge. Our primary objective is to apply discrete fractional calculus to the Burgers map in order to gain a deeper understanding of the underlying discrete dynamic behaviors. Using the theory of delta difference equations, we seek to uncover the discrete chaotic behaviors present in the fractional Burgers map.

Main results

Let $a \in \mathbb{R}$ fixed and let $\mathbb{N}_a = \{a, a+1, a+2, \dots\}$ denotes the isolated time scale. For the function $u(n)$, the delta difference operator Δ is defined as

$$\Delta u(n) = u(n+1) - u(n). \quad (0.360)$$

Definition 0.60. Let $u : \mathbb{N}_a \rightarrow \mathbb{R}$ and $v > 0$. Then the fractional sum of v order is defined by

$$\Delta_a^{-v} u(t) = \frac{1}{\Gamma(v)} \sum_{s=a}^{t-v} (t - \sigma(s))^{(v-1)} u(s), \quad t \in \mathbb{N}_{a+v}, \quad (0.361)$$

where a is the starting point, $\sigma(s) = s+1$ and t^v is the falling function defined in terms of the Gamma function as

$$t^{(v)} = \frac{\Gamma(t+1)}{\Gamma(t+1-v)}. \quad (0.362)$$

Definition 0.61. For $v > 0$, $v \notin \mathbb{N}$ and $u(t)$ defined on $\mathbb{N}_a$, the Caputo-like delta difference is defined by

$${}^c \Delta_a^v u(t) = \Delta_a^{-(m-v)} \Delta^m u(t) \quad (0.363)$$

$$= \frac{1}{\Gamma(m-v)} \sum_{s=a}^{t-(m-v)} (t - \sigma(s))^{(m-v-1)} \Delta_s^m u(s), \quad (0.364)$$

where $t \in \mathbb{N}_{a+m-v}$, $m = [v] + 1$.

Theorem 0.152. For the delta fractional difference equation

$$\begin{cases} {}^c \Delta_a^v u(t) = f(t+v-1, u(t+v-1)) \\ \Delta^k u(a) = u_k, m = [v] + 1, k = 0, \dots, m-1, \end{cases}$$



the equivalent discrete integral equation can be obtained as

$$u(t) = u_0(t) + \frac{1}{\Gamma(v)} \sum_{s=a+m-v}^{t-v} (t-\sigma(s))^{(v-1)} \times f(s+v-1, u(s+v-1)), \quad t \in \mathbb{N}_{a+m}, \quad (0.365)$$

where

$$u_0(t) = \sum_{k=0}^{m-1} \frac{(t-a)^{(a)}}{k!} \Delta^k u(a). \quad (0.366)$$

The fractional-order version of the Burgers map:

The first-order difference of can be easily formulated as

$$\begin{cases} x_{n+1} = (1-a)x_n - y_n^2, \\ y_{n+1} = (1+b)y_n + x_n y_n. \end{cases} \quad (0.367)$$

The fractional Burger map, can be given by :

$$\begin{cases} \Delta_a^v x(t) = (1-a)x(t+1-v) - y(t+1-v)^2 - x(t+1-v), \\ \Delta_b^v y(t) = (1+b)y(t-1+v) + y(t-1+v)x(t-1+v) - y(t-1+v). \end{cases} \quad (0.368)$$

According to Theorem 1, we can express the discrete fractional-order Burgers map in the following form:

$$\begin{cases} x(t) = x(0) + \frac{1}{\Gamma(v)} \sum_{s=1-v}^{t-v} (t-s-1)^{(v-1)} ((1-a)x(s+1-v) - y(s+1-v)^2 - x(s+1-v)), \\ y(t) = y(0) + \frac{1}{\Gamma(v)} \sum_{s=1-v}^{t-v} (t-s-1)^{(v-1)} ((1+b)y(s-1+v) + y(s-1+v)x(s-1+v) - y(s-1+v)). \end{cases} \quad (0.369)$$

In conclusion, the numerical solution of the discrete fractional-order Burgers map is given by:

$$\begin{cases} x(n) = x(0) + \frac{1}{\Gamma(v)} \sum_{j=1}^n \frac{\Gamma(n-j+v)}{\Gamma(n-j+1)} ((1-a)x(j-1) - y(j-1)^2 - x(j-1)), \\ y(n) = y(0) + \frac{1}{\Gamma(v)} \sum_{j=1}^n \frac{\Gamma(n-j+v)}{\Gamma(n-j+1)} ((1+b)y(j-1) + x(j-1)y(j-1) - y(j-1)). \end{cases} \quad (0.370)$$

where $x(0)$, $y(0)$ is the initial condition, v is the fractional order such as $0 < v \leq 1$.

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Modeling Infectious Disease Incidence in Algeria: Comparative Efficacy of SARIMA, NNAR, and Hybrid Models

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Abstract. In Algeria, infectious diseases present a persistent public health challenge characterized by both endemicity and sporadic outbreaks. This study provides a comprehensive epidemiological overview, evaluating diverse prediction models using surveillance data from Tébessa province. Utilizing Seasonal Autoregressive Integrated Moving Average (SARIMA) models, Autoregressive Neural Network (NNAR) models, and SARIMA-NNAR hybrid models, the research aims to forecast monthly disease notifications. Predictive accuracy is systematically evaluated using metrics such as Root Mean Squared Error (RMSE), Mean Absolute Error (MAE), and Mean Absolute Percentage Error (MAPE). Results reveal that $SARIMA(2, 1, 3)(1, 1, 1)_{12}$, $NNAR(12, 1, 6)_{12}$ and $SARIMA(2, 1, 3)(1, 1, 1)_{12} - NNAR(5, 1, 4)_{12}$ exhibit optimal fit to the data. Significantly, SARIMA and SARIMA-NNAR models outperform the NNAR model, evidenced by lower RMSE, MAE, and MAPE values. These findings underscore the suitability of SARIMA and SARIMA-NNAR models for accurately predicting disease patterns, providing valuable insights into epidemiological dynamics. The precision of forecasts and incidence mapping aids policymakers in crafting informed, targeted strategies and timely interventions. In conclusion, the development of robust prediction models for infectious disease outbreaks in Algeria, utilizing SARIMA, NNAR, and hybrid approaches, contributes to our understanding of epidemiological dynamics. These insights



empower policymakers to make informed decisions and implement proactive measures to mitigate the impact of outbreaks in Algerian regions.

Keywords: Predictive modeling, Time series analysis, Neural networks, Disease outbreaks.

Introduction

Human brucellosis is still a problem for public health. This disease currently affects every province in the country and is spread enzootically across various animal species (bovine, ovine, caprine, and camelina), with sporadic epidemic breakouts in some regions. Tebessa, an endemo-epidemic province, leads the eastern part of the country in terms of notification cases, making it a suitable choice to provide an overview, perform statistical analysis, and develop predictive models.

Modeling Methods

Among popular statistical models used to model recorded cases over time and make predictions, we can refer to Autoregressive (AR) Moving Average (MA) models which are used as a reference point for the evaluation of new prediction methods. Artificial neural networks (ANNs) and combined models have been developed to improve predictions.

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Seasonal Autoregressive Integrated Moving Average Model The monthly data are not normally distributed (Jarque-Bera 188.9388, p-value 0.0000). The training data set of human brucellosis original time series, named bruc, was used in the model identification. The time series bruc exhibits non-stationary variance, a non-stationary mean, and seasonal components. To stabilise the variance, the logarithm was applied.



To stabilise the mean, a first-order differencing was applied. Peaks occur at lags of 12 months ; therefore, a seasonal differencing was required ($D = 1$). ADF test confirmed the stationary of the transformed data time series

$$dsbructr = \nabla \nabla_{12} \log(bruc) = (1 - B)(1 - B^{12}) \log(bruc)$$

It is worth noting that $dsbructr$ is normally distributed (Jarque-Bera = 2.6734 and p-value = 0.2627), and the points of Q-Q plot seems approximately over the straight line. It should be mentioned that we lost the first 13 values in the training set; therefore, 179 data points were used in the building process. The autocorrelation and partial autocorrelation functions are displayed in allowing us to identify possible values for p, q, P, and Q orders.

Several possibilities have been considered and on the basis of the best-fitted model criteria (the smallest AIC and RMSE, the invertibility condition, the significance of AR and MA roots, and the white noise condition for residuals), $SARIMA(2, 1, 3)(1, 1, 1)_{12}$ model which fulfills all the features of the best-fit model were selected as appropriate (Tables 6 and 5).

Table 5: Best SARIMA model

[0.5mm] Model	AIC	RMSE
[0.5mm] $SARIMA(3, 1, 3)(1, 1, 1)_{12}$	393.34	0.291
$SARIMA(2, 1, 3)(1, 1, 0)_{12}$	312.82	0.291
$SARIMA(2, 1, 3)(1, 1, 1)_{12}$	288.43	0.286
$SARIMA(2, 1, 3)(0, 1, 1)_{12}$	288.86	0.328
$SARIMA(3, 1, 4)(0, 1, 1)_{12}$	292.29	0.289
[0.5mm]		

The shows the residuals for the fitted model. Both the ACF and PACF show significant spikes, which indicates that some additional non-seasonal terms should be included in the model, but none gave a smaller AIC value. As a result, we select the $SARIMA(2, 1, 3)(1, 1, 1)_{12}$ model. The BOX test also shows that the residuals have no remaining autocorrelations. The results Q^* are not significant (the p-value=0.57 is comparatively large). As a result, we can conclude that the residuals cannot be distinguished from a white noise process.

The final model equation is given by:

$$Z_t = -1.719Z_{t-1} - 0.903Z_{t-2} - 0.164Z_{t-12} + \epsilon_t + 1.124\epsilon_{t-1} - 0.109\epsilon_{t-2} - 0.615\epsilon_{t-3} - 0.721\epsilon_{t-12}$$

where $Z_t = dsbructr$.

The $SARIMA(2, 1, 3)(1, 1, 1)_{12}$ was then used to forecast the monthly cases making use of the test set. The simulated values and reported cases matched reasonably well as displayed with a strong correlation (Pearson product-moment



Table 6: Parameter estimates of seasonal autoregressive integrated moving average model

[0.5mm]	ar1	ar2	ma1	ma2	ma3	sar1	sma1
[0.5mm] coefficient	-1.719	-0.903	1.124	-0.109	-0.615	-0.164	-0.721
t-stat	-25.732	-15.677	13.896	-1.049	-8.855	-1.602	-8.620
p-value	0.001	0.001	0.001	0.293	0.001	0.100	0.001
[0.5mm]							

correlation coefficient; $r = 0.866$). Moreover, the monthly predicted values for the year 2020 demonstrated a considerable agreement with the actual data with a very strong correlation ($r = 0.915$). All these conclusions attest to the adequacy of $SARIMA(2, 1, 3)(1, 1, 1)_{12}$.

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NNAR And Hybrid SARIMA-NNAR Models

A number of 12 time-lagged variables and one hidden layer with 6 neurons as input features were created while developing the NNAR. Therefore, 180 values were compared in the training set. The $NNAR(12, 1, 6)_{12}$ with 12 lagged inputs was selected as the best-fitting model. The fitted values were highly correlated to actual values ($r = 0.99$); nonetheless, the correlation between forecasted and actual values was moderate ($r = 0.539$). In the development of hybrid SARIMA-NNAR, the obtained residuals from the SARIMA model were modeled using the NNAR model. $SARIMA(2, 1, 3)(1, 1, 1)_{12} - NNAR(5, 1, 4)_{12}$ was selected as best fitting model. The forecast values were obtained by adding the forecast values of SARIMA with NNAR residual forecasts. The forecasted and actual values matched very well with a very strong correlation ($r = 0.87$). Moreover, the predicted and actual values for the year 2020 were strongly correlated ($r = 0.912$).

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Comparison Of The Three Models

Table 7: Comparaison of the fitting and forecasting accuracy of the three models

[0.5mm] Model (r)2-4 (r)5-7	Training set			Test set		
	RMSE	MAE	MAPE	RMSE	MAE	MAPE
[0.5mm] SARIMA	0.476	0.361	12.488	0.286	0.247	5.962
NNAR	0.131	0.089	3.047	0.808	0.637	19.737
SARIMA-NNAR	0.131	0.280	8.365	0.314	0.272	6.644
[0.5mm]						

For the comparison of the fitting accuracy of the three models, the training set was used taking into account lost values for each built model. In fact, for the



SARIMA model, the first 13 values in the training set were lost since a first-order difference and a seasonal difference were applied to the original time series to achieve the stationary, therefore, the remaining 179 values were used to build the model and therefore they were used to compare the fitting accuracy. In a similar vein, for the NNAR model 180 values were compared for fitting accuracy since 12 time-lagged variables were created as input features, and 179 values were compared for the fitting accuracy of the SARIMA-NNAR model. The test data set was used to compare the forecast performance for the three models. All error measurements of the NNAR model were far lower than those of SARIMA and SARIMA-NNAR models in the training set, and there were slight differences between the metric measures for SARIMA and SARIMA-NNAR models.

The lowest metric measure values using the test set were achieved with the SARIMA model followed by the hybrid SARIMA-NNAR model (Table 7). It was concluded that both models SARIMA and SARIMA-NNAR can successfully predict human brucellosis cases.

The predictions of human brucellosis in Tebessa province from January to December 2020 (Figure 26 and Table 8) indicate that both SARIMA-NNAR and SARIMA models are much better in performance accuracy than NNAR model and that SARIMA-NNAR model outperforms SARIMA model. The RMSE, MAE and MAPE measures for SARIMA model were far lower than those of NNAR model, and they were slightly different from those of the SARIMA-NNAR model. The model NNAR gives worse prediction than either SARIMA or SARIMA-NNAR models.

Table 8: Actual, SARIMA, NNAR, and Hybrid SARIMA-NNAR 2020 predictions of human brucellosis with performance metrics

[0.5mm]	Performance metric	SARIMA	NNAR	SARIMA-NNAR
[0.5mm]	MAE	25.3	42.7	24.9
	RMSE	29.3	48.8	28.7
	MAPE	32.6	43.3	30.9
[0.5mm]				

conclusions

the SARIMA and hybrid SARIMA-NNAR models have demonstrated their effectiveness in accurately predicting human brucellosis cases. The application of



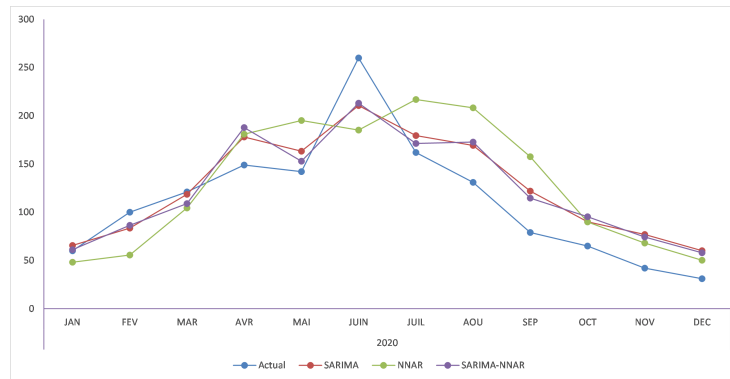


Figure 26: Actual, SARIMA, NNAR, and Hybrid SARIMA-NNAR 2020 predictions of human brucellosis.

these models, can provide valuable support to veterinary and health policymakers.

By utilizing predictive modeling techniques, policymakers can gain insights into the future trends and patterns of brucellosis outbreaks. This knowledge enables the development of informed, effective, and targeted policies that address the specific needs and challenges associated with brucellosis control and prevention.

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**ON THE CAUCHY PROBLEM FOR A WEAKLY
COUPLED SYSTEM OF SEMI-LINEAR FRACTIONAL
 σ -EVOLUTION EQUATION WITH DIFFERENT
NONLINEAR MEMORY**

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Abstract. In this presentation, we would like to consider global existence of small data solution to the Cauchy problem for a multi-component weakly coupled system of semi-linear Fractional σ -evolution with different nonlinear memory. We use $L^q - L^r$ estimates for solutions to the corresponding linear Cauchy problems to prove the global (in time) existence of small data Sobolev solutions to the weakly coupled systems of semi-linear models from suitable function spaces for any $\sigma \geq 1$

Keywords: Fractional, coupled systems, small data, σ -evolution.



Introduction

In this talk, we consider the following the Cauchy problems for weakly coupled systems of semi-linear Fractional σ -evolution equations with non linear memory:

$$\begin{cases} \partial_t^{1+\alpha_1} u_1 + (-\Delta)^{\sigma_1} u_1 + M_1^2 u_1 = F_{\mu_1, p_1}(u_k), \\ \partial_t^{1+\alpha_2} u_2 + (-\Delta)^{\sigma_2} u_2 + M_2^2 u_2 = F_{\mu_2, p_2}(u_1), \\ \vdots \\ \partial_t^{1+\alpha_k} u_k + (-\Delta)^{\sigma_k} u_k + M_k^2 u_k = F_{\mu_k, p_k}(u_{k-1}), \\ u_\ell(0, x) = u_{0\ell}(x) \quad \partial_t u_\ell(0, x) = 0, \quad \ell = 1, 2, \dots, k \end{cases} \quad (0.371)$$

where for $k \geq 2$, for $\ell = 1, 2, \dots, k$, $\alpha_\ell \in (0, 1)$, $\sigma_\ell \geq 1$, $M_\ell > 0$, $p_\ell > 1$, $(t, x) \in [0, \infty) \times \mathbb{R}^n$, with

$$F_{\mu, p}(u)(t, x) = \int_0^t (t-s)^\mu |u(s, x)|^p$$

Main results

Theorem 0.153. (*Loss of decay*) Let us assume $0 < \alpha_\ell < 1$, $\alpha_\ell < \mu_\ell < 1$, $\sigma_\ell \geq 1$, $m_l \geq 1$ and $M_\ell > 0$ for all $\ell = 1, \dots, k$. Assume that for all $\delta > 0$

$$\max \left\{ 1, \frac{\alpha_1 - \mu_1 + 1}{\mu_k - \alpha_k}, \frac{m_k}{m_1} - \delta \right\} < p_1 < \frac{1}{\mu_k - \alpha_k},$$

,

$$\max \left\{ 1, \frac{\alpha_2 - \mu_2 + 1}{\mu_1 - \alpha_1 - \gamma_{(\alpha_k)}^{(\mu_k)}(p_1)}, \frac{m_1}{m_2} - \delta \right\} < p_2 < \frac{1}{\mu_1 - \alpha_1 - \gamma_{(\alpha_k)}^{(\mu_k)}(p_1)}$$

and for $l = 3, \dots, k-1$

$$p_\ell < \frac{1}{\mu_{\ell-1} - \alpha_{\ell-1} - \gamma_{(\alpha_k, \dots, \alpha_{\ell-2})}^{(\mu_k, \dots, \mu_{\ell-2})}(p_1, \dots, p_{\ell-1})},$$

$$p_l > \max \left\{ 1, \frac{\alpha_\ell - \mu_\ell + 1}{\mu_{\ell-1} - \alpha_{\ell-1} - \gamma_{(\alpha_k, \dots, \alpha_{\ell-2})}^{(\mu_k, \dots, \mu_{\ell-2})}(p_1, \dots, p_{\ell-1})}, \frac{m_{l-1}}{m_l} - \delta \right\}$$

and

$$p_k > \max \left\{ \frac{m_k}{m_{k-1}} - \delta, \frac{1}{\mu_{k-1} - \alpha_{k-1} - \gamma_{(\alpha_k, \dots, \alpha_{k-2})}^{(\mu_k, \dots, \mu_{k-2})}(p_1, \dots, p_{k-1})} \right\},$$

where, for $l = 3, \dots, k-1$

$$\begin{cases} \gamma_{(\alpha_k)}^{(\mu_k)}(p_1) = 1 - p_1(\mu_k - \alpha_k) \\ \gamma_{(\alpha_k, \alpha_1)}^{(\mu_k, \mu_1)}(p_1, p_2) = 1 - p_2(\mu_1 - \alpha_1) + p_2 \gamma_{(\alpha_k)}^{(\mu_k)}(p_1) \\ \gamma_{(\alpha_k, \dots, \alpha_{\ell-1})}^{(\mu_k, \dots, \mu_{\ell-1})}(p_1, \dots, p_\ell) = 1 - p_\ell(\mu_\ell - \alpha_\ell) + p_\ell \gamma_{(\alpha_k, \dots, \alpha_{\ell-2})}^{(\mu_k, \dots, \mu_{\ell-2})}(p_1, \dots, p_{\ell-1}). \end{cases} \quad (0.372)$$



Then there exists a positive constant ε such that for any data

$$(u_{01}, \dots, u_{0k}) \in \mathcal{A}_k := \prod_{\ell=1}^k (L^{m_\ell}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)) \quad \text{with } \|(u_{01}, \dots, u_{0k})\|_{\mathcal{A}_k} \leq \varepsilon$$

, we have a uniquely determined global (in time) Sobolev solution

$$u \in \prod_{\ell=1}^k \mathcal{C}([0, \infty), L^{m_\ell}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$$

to the Cauchy problem (0.371). Moreover, the solution satisfies for all $s \geq 0$ and $l = 2, \dots, k-1$, the decay estimate

$$\begin{aligned} \|u_1(s, \cdot)\|_{L^q} &\lesssim (1+s)^{\alpha_1 - \mu_1 + \gamma_{\alpha_k}^{\mu_k}(p_1)} \|u_{01}\|_{L^{m_1} \cap L^\infty} \quad \text{for all } q \in [m_1, \infty], \\ \|u_\ell(s, \cdot)\|_{L^q} &\lesssim (1+s)^{\alpha_\ell - \mu_\ell + \gamma_{(\alpha_k, \dots, \alpha_{\ell-1})}^{(\mu_k, \dots, \mu_{\ell-1})}(p_1, \dots, p_\ell)} \|u_{0\ell}\|_{L^{m_\ell} \cap L^\infty} \quad \text{for all } q \in [m_\ell, \infty], \\ \|u_k(s, \cdot)\|_{L^q} &\lesssim (1+s)^{\alpha_k - \mu_k} \|u_{0k}\|_{L^{m_k} \cap L^\infty} \quad \text{for all } q \in [m_k, \infty]. \end{aligned}$$

Lemma 0.59. Suppose that $\theta \in [0, 1)$, $a \geq 0$ and $b \geq 0$. Then there exists a constant $C = C(a, b, \theta) > 0$ such that for all $t > 0$ the following estimate holds:

$$\begin{aligned} &\int_0^t (t-\tau)^{-\theta} (1+t-\tau)^{-a} (1+\tau)^{-b} d\tau \\ &\leq \begin{cases} C(1+t)^{-\min\{a+\theta, b\}} & \text{if } \max\{a+\theta, b\} > 1, \\ C(1+t)^{-\min\{a+\theta, b\}} \ln(2+t) & \text{if } \max\{a+\theta, b\} = 1, \\ C(1+t)^{1-a-\theta-b} & \text{if } \max\{a+\theta, b\} < 1. \end{cases} \end{aligned} \quad (0.373)$$



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Proof of Theorem 0.153 We introduce for all $T > 0$ the space $X^k(T)$ as follows:

$$X^k(T) := \prod_{\ell=1}^k C([0, T], L^{r_\ell}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$$

with the norm

$$\|u\|_{X^k(T)} := \sup_{0 \leq t \leq T} \left\{ (1+t)^{-\gamma_{(\alpha_k)}^{(\mu_k)}(p_1)} M_1(t, u_1) + \sum_{l=2}^{k-1} (1+t)^{-\gamma_{(\alpha_k, \dots, \alpha_{\ell-1})}^{(\mu_k, \dots, \mu_{\ell-1})}(p_1, \dots, p_\ell)} M_\ell(t, u_l) + M_k(t, u_k) \right\},$$

where

$$M_k(t, u_k) := (1+t)^{\mu_k - \alpha_k} (\|u_k(t, \cdot)\|_{L^{r_k}} + \|u_k(t, \cdot)\|_{L^\infty}),$$

$$M_\ell(t, u_l) = (1+t)^{\mu_\ell - \alpha_\ell} (\|u_\ell(t, \cdot)\|_{L^{r_\ell}} + \|u_\ell(t, \cdot)\|_{L^\infty}),$$

$$\cdot \left\{ \begin{array}{l} \gamma_{(\alpha_k)}^{(\mu_k)}(p_1) = 1 - p_1(\mu_k - \alpha_k) \\ \gamma_{(\alpha_k, \alpha_1)}^{(\mu_k, \mu_1)}(p_1, p_2) = 1 - p_2(\mu_1 - \alpha_1) + p_2 \gamma_{(\alpha_k)}^{(\mu_k)}(p_1) \\ \gamma_{(\alpha_k, \alpha_1, \alpha_2)}^{(\mu_k, \mu_1, \mu_2)}(p_1, p_2, p_3) = 1 - p_3(\mu_2 - \alpha_2) + p_3 \gamma_{(\alpha_k, \alpha_1)}^{(\mu_k, \mu_1)}(p_1, p_2) \\ \vdots \\ \gamma_{(\alpha_k, \dots, \alpha_{\ell-1})}^{(\mu_k, \dots, \mu_{\ell-1})}(p_1, \dots, p_\ell) = 1 - p_\ell(\mu_{l-1} - \alpha_{l-1}) + p_\ell \gamma_{(\alpha_k, \dots, \alpha_{\ell-2})}^{(\mu_k, \dots, \mu_{\ell-2})}(p_1, \dots, p_{\ell-1}), \end{array} \right.$$

for $l = 3, \dots, k-1$. and the operator N by

$$N : u = (u_1, u_2, \dots, u_k) \in X^k(T) \rightarrow N(u) = N(u)(t, x) := u^{ln}(t, x) + u^{nl}(t, x).$$

To get the global (in time) existence and uniqueness of Sobolev solutions in $X(T)$, we can consider a global (in time) Sobolev solution to (0.371) as a fixed point of the operator N . We will prove that the operator N satisfies for $u = (u_1, u_2, \dots, u_k)$; $\bar{u} = (\bar{u}_1, \bar{u}_2, \dots, \bar{u}_k)$ in $X^k(T)$, the following two inequalities:

$$\|N(u)\|_{X^k(T)} \lesssim \|(u_{01}, u_{02}, \dots, u_{0k})\|_{\mathcal{A}_k} + \sum_{\ell=1}^{\ell=k} \|u\|_{X^k(T)}^{p_\ell}, \quad (0.374)$$

$$\|N(u) - N(\bar{u})\|_{X^k(T)} \lesssim \|u - \bar{u}\|_{X^k(T)} \sum_{\ell=1}^{\ell=k} \left(\|u\|_{X^k(T)}^{p_\ell - 1} + \|\bar{u}\|_{X^k(T)}^{p_\ell - 1} \right) \quad (0.375)$$

Using the definition of the norm in $X^k(T)$ and the Propositions ??, we may conclude:

$$\|u^{ln}\|_{X^k(T)} \lesssim \|(u_{01}, u_{02}, \dots, u_{0k})\|_{\mathcal{A}_k}.$$



Hence, in order to complete the proof of (??) it is reasonable to we shall show the following inequality

$$\|u^{nl}\|_{X^k(T)} \lesssim \sum_{\ell=1}^{\ell=k} \|u\|_{X^k(T)}^{p_\ell}.$$

If $u := (u_1, u_2, \dots, u_k) \in X^k(T)$, then by interpolation we derive for $l = 2, \dots, k-1$

$$\begin{aligned} \|u_1(t, \cdot)\|_{L^q} &\lesssim (1+t)^{(\alpha_1-\mu_1)+\gamma_{(\alpha_k)}^{(\mu_k)}(p_1)} \|u\|_{X^k(T)} \quad \text{for all } q \in [r_1, \infty] \\ \|u_\ell(t, \cdot)\|_{L^q} &\lesssim (1+t)^{(\alpha_\ell-\mu_\ell)+\gamma_{(\alpha_k, \dots, \alpha_{\ell-1})}^{(\mu_k, \dots, \mu_{\ell-1})}(p_1, \dots, p_\ell)} \|u\|_{X^k(T)} \quad \text{for all } q \in [r_\ell, \infty], \\ \|u_k(t, \cdot)\|_{L^q} &\lesssim (1+t)^{\alpha_k-\mu_k} \|u\|_{X^k(T)} \quad \text{for all } q \in [r_k, \infty]. \end{aligned}$$

On the other hand, for $q \in [r_1, \infty]$, we have

$$\|u_1^{nl}(t, \cdot)\|_{L^q} \lesssim \|u\|_{X(T)}^{p_1} I_1(t) \quad \text{for all } t \in [0, T] \quad \text{and } p_1 q \in [r_k, \infty],$$

where

$$I_1(t) = \int_0^t (1+t-\tau)^{-(1+\alpha_1)} \int_0^\tau (\tau-s)^{\alpha_1-1} \int_0^s (s-\rho)^{-\mu_1} (1+\rho)^{-p_1(\mu_k-\alpha_k)} d\rho ds d\tau. \quad (0.376)$$

We are interested to estimate the right-hand side of (0.376). For this we need the Lemma 0.60. We put

$$\omega(s) = \int_0^s (s-\rho)^{-\mu_1} (1+\rho)^{-p_1(\mu_k-\alpha_k)} d\rho.$$

Thanks to Lemma 0.60 we obtain $\omega(s) \lesssim (1+s)^{1-\mu_1-p_1(\mu_k-\alpha_k)}$, if we assume that $p_1 < \frac{1}{\mu_k-\alpha_k}$.

Once more we apply Lemma 0.60 to obtain

$$\begin{aligned} I_1(t) &\lesssim \int_0^t (1+t-\tau)^{-(1+\alpha_1)} \int_0^\tau (\tau-s)^{\alpha_1-1} (1+s)^{1-\mu_1-p_1(\mu_k-\alpha_k)} ds d\tau \\ &\lesssim \int_0^t (1+t-\tau)^{-(1+\alpha_1)} (1+\tau)^{1+\alpha_1-\mu_1-p_1(\mu_k-\alpha_k)} d\tau \\ &\lesssim (1+t)^{1+\alpha_1-\mu_1-p_1(\mu_k-\alpha_k)} \\ &\lesssim (1+t)^{\alpha_1-\mu_1+\gamma_{\alpha_k}^{\mu_k}(p_1)}. \end{aligned}$$

On other hands, the conditions $q \in [r_1, \infty]$ and $p_1 q \in [r_k, \infty]$ implies that $p_1 \geq \frac{r_k}{r_1}$.



For $l = 2$ and $q \in [r_2, \infty]$, we have

$$\begin{aligned} \|u_2^{nl}(t, \cdot)\|_{L^q} &\lesssim \int_0^t (1+t-\tau)^{-(1+\alpha_2)} \int_0^\tau (\tau-s)^{\alpha_2-1} \|u_1(s, \cdot)\|_{L^q}^{p_2} ds d\tau \\ &\lesssim \int_0^t (1+t-\tau)^{-(1+\alpha_2)} \int_0^\tau (\tau-s)^{\alpha_2-1} \int_0^s (s-\rho)^{-\mu_2} \|u_1(\rho, \cdot)\|_{L^{p_2q}}^{p_2} d\rho ds d\tau \\ &\lesssim \|u\|_{X(T)}^{p_2} I_2(t) \quad \text{for all } t \in [0, T] \quad \text{and } p_2q \in [r_1, \infty], \end{aligned}$$

where

$$\begin{aligned} I_2(t) &= \int_0^t (1+t-\tau)^{-(1+\alpha_2)} \int_0^\tau (\tau-s)^{\alpha_2-1} \\ &\quad \times \int_0^s (s-\rho)^{-\mu_2} (1+\rho)^{-p_2(\mu_1-\alpha_1)+p_2\gamma_{(\alpha_k)}^{(\mu_k)}(p_1)} d\rho ds d\tau. \end{aligned} \quad (0.377)$$

On the other hand, we have We are interested to estimate the right-hand side of (0.377). For this we need the Lemma 0.60. We put

$$\omega(s) = \int_0^s (s-\rho)^{-\mu_2} (1+\rho)^{-p_2(\mu_1-\alpha_1)+p_2\gamma_{(\alpha_k)}^{(\mu_k)}(p_1)} d\rho.$$

Thanks to Lemma 0.60 we obtain

$$\omega(s) \lesssim (1+s)^{1-\mu_2-p_2(\mu_1-\alpha_1)+p_2\gamma_{(\alpha_k)}^{(\mu_k)}(p_1)},$$

if we assume that $p_2 < \frac{1}{\mu_1-\alpha_1-\gamma_{(\alpha_k)}^{(\mu_k)}(p_1)}$ and $p_1 > \frac{1+\alpha_1-\mu_1}{\mu_k-\alpha_k}$.

Once more we apply Lemma 0.60 to obtain

$$\begin{aligned} I_2(t) &\lesssim \int_0^t (1+t-\tau)^{-(1+\alpha_2)} \int_0^\tau (\tau-s)^{\alpha_2-1} (1+s)^{1-\mu_2-p_2(\mu_1-\alpha_1)+p_2\gamma_{(\alpha_k)}^{(\mu_k)}(p_1)} ds d\tau \\ &\lesssim \int_0^t (1+t-\tau)^{-(1+\alpha_2)} (1+\tau)^{1+\alpha_2-\mu_2-p_2(\mu_1-\alpha_1)+p_2\gamma_{(\alpha_k)}^{(\mu_k)}(p_1)} d\tau \\ &\lesssim (1+t)^{1+\alpha_2-\mu_2-p_2(\mu_1-\alpha_1)+p_2\gamma_{(\alpha_k)}^{(\mu_k)}(p_1)} \\ &\lesssim (1+t)^{1+\alpha_2-\mu_2-p_2(\mu_1-\alpha_1)+p_2\gamma_{(\alpha_k)}^{(\mu_k)}(p_1)} \\ &\lesssim (1+t)^{\alpha_2-\mu_2+\gamma_{(\alpha_k, \alpha_1)}^{(\mu_k, \mu_1)}(p_1, p_2)}. \end{aligned}$$

On other hands, the conditions $q \in [r_2, \infty]$ and $p_2q \in [r_1, \infty]$ implies that

$$p_2 \geq \frac{r_1}{r_2}.$$

For $l = 3, \dots, k-1$ and $q \in [r_l, \infty]$, we have

$$\begin{aligned} \|u_l^{nl}(t, \cdot)\|_{L^q} &\lesssim \int_0^t (1+t-\tau)^{-(1+\alpha_\ell)} \int_0^\tau (\tau-s)^{\alpha_\ell-1} \|u_{\ell-1}(s, \cdot)\|_{L^q}^{p_\ell} ds d\tau \\ &\lesssim \int_0^t (1+t-\tau)^{-(1+\alpha_\ell)} \int_0^\tau (\tau-s)^{\alpha_\ell-1} \int_0^s (s-\rho)^{-\mu_\ell} \|u_{\ell-1}(\rho, \cdot)\|_{L^{p_\ell q}}^{p_\ell} d\rho ds d\tau \\ &\lesssim \|u\|_{X(T)}^{p_\ell} I_\ell(t) \quad \text{for all } t \in [0, T] \quad \text{and } p_\ell q \in [r_{\ell-1}, \infty], \end{aligned}$$



where

$$I_\ell(t) = \int_0^t (1+t-\tau)^{-(1+\alpha_\ell)} \int_0^\tau (\tau-s)^{\alpha_\ell-1} \times \int_0^s (s-\rho)^{-\mu_\ell} (1+\rho)^{-p_\ell(\mu_{\ell-1}-\alpha_{\ell-1})+p_\ell\gamma_{(\alpha_k,\dots,\alpha_{\ell-2})}^{(\mu_k,\dots,\mu_{\ell-2})}(p_1,\dots,p_{\ell-1})} d\rho ds d\tau. \quad (0.378)$$

On the other hand, we have We are interested to estimate the right-hand side of (0.378). For this we need the Lemma 0.60. We put

$$\omega(s) = \int_0^s (s-\rho)^{-\mu_\ell} (1+\rho)^{-p_\ell(\mu_{\ell-1}-\alpha_{\ell-1})+p_\ell\gamma_{(\alpha_k,\dots,\alpha_{\ell-2})}^{(\mu_k,\dots,\mu_{\ell-2})}(p_1,\dots,p_{\ell-1})} d\rho.$$

Thanks to Lemma 0.60 we obtain

$$\omega(s) \lesssim (1+s)^{1-\mu_\ell-p_\ell(\mu_{\ell-1}-\alpha_{\ell-1})+p_\ell\gamma_{(\alpha_k,\dots,\alpha_{\ell-2})}^{(\mu_k,\dots,\mu_{\ell-2})}(p_1,\dots,p_{\ell-1})},$$

if we assume that

$$p_\ell < \frac{1}{(\mu_{\ell-1}-\alpha_{\ell-1})-\gamma_{(\alpha_k,\dots,\alpha_{\ell-2})}^{(\mu_k,\dots,\mu_{\ell-2})}(p_1,\dots,p_{\ell-1})}$$

and

$$p_{l-1} > \frac{1+\alpha_{l-1}-\mu_{l-1}}{\mu_{\ell-2}-\alpha_{\ell-2}-\gamma_{(\alpha_k,\dots,\alpha_{\ell-3})}^{(\mu_k,\dots,\mu_{\ell-3})}(p_1,\dots,p_{\ell-2})}$$

On other hands, the conditions $q \in [r_\ell, \infty]$ and $p_\ell q \in [r_{\ell-1}, \infty]$ implies that $p_\ell \geq \frac{r_{\ell-1}}{r_\ell}$.

Once more we apply Lemma 0.60 to obtain

$$\begin{aligned} I_\ell(t) &\lesssim \int_0^t (1+t-\tau)^{-(1+\alpha_\ell)} \int_0^\tau (\tau-s)^{\alpha_\ell-1} (1+s)^{1-\mu_\ell-p_\ell(\mu_{\ell-1}-\alpha_{\ell-1})+p_\ell\gamma_{(\alpha_k,\dots,\alpha_{\ell-2})}^{(\mu_k,\dots,\mu_{\ell-2})}(p_1,\dots,p_{\ell-1})} ds d\tau \\ &\lesssim \int_0^t (1+t-\tau)^{-(1+\alpha_\ell)} (1+s)^{\alpha_\ell-\mu_\ell+1-p_\ell(\mu_{\ell-1}-\alpha_{\ell-1})+p_\ell\gamma_{(\alpha_k,\dots,\alpha_{\ell-2})}^{(\mu_k,\dots,\mu_{\ell-2})}(p_1,\dots,p_{\ell-1})} d\tau \\ &\lesssim (1+t)^{\alpha_\ell-\mu_\ell+1-p_\ell(\mu_{\ell-1}-\alpha_{\ell-1})+p_\ell\gamma_{(\alpha_k,\dots,\alpha_{\ell-2})}^{(\mu_k,\dots,\mu_{\ell-2})}(p_1,\dots,p_{\ell-1})} \\ &\lesssim (1+t)^{\alpha_\ell-\mu_\ell+\gamma_{(\alpha_k,\dots,\alpha_{\ell-1})}^{(\mu_k,\dots,\mu_{\ell-1})}(p_1,\dots,p_\ell)} \end{aligned}$$

Finally, for $q \in [r_k, \infty]$, we have

$$\begin{aligned} \|u_k^{nl}(t, \cdot)\|_{L^q} &\lesssim \int_0^t (1+t-\tau)^{-(1+\alpha_k)} \int_0^\tau (\tau-s)^{\alpha_k-1} \|u_{k-1}(s, \cdot)\|_{L^q}^{p_k} ds d\tau \\ &\lesssim \int_0^t (1+t-\tau)^{-(1+\alpha_k)} \int_0^\tau (\tau-s)^{\alpha_k-1} \int_0^s (s-\rho)^{-\mu_k} \|u_{k-1}(\rho, \cdot)\|_{L^{p_k q}}^{p_k} d\rho ds d\tau \\ &\lesssim \|u\|_{X(T)}^{p_k} I_k(t) \quad \text{for all } t \in [0, T] \quad \text{and } p_k q \in [r_{k-1}, \infty], \end{aligned}$$



where

$$I_k(t) = \int_0^t (1+t-\tau)^{-(1+\alpha_k)} \int_0^\tau (\tau-s)^{\alpha_k-1} \times \int_0^s (s-\rho)^{-\mu_k} (1+\rho)^{-p_k} (\mu_{k-1}-\alpha_{k-1}-\gamma_{(\alpha_1,\dots,\alpha_{k-2})}^{(\mu_1,\dots,\mu_{k-2})}(p_1,\dots,p_{k-1})) d\rho ds d\tau. \quad (0.379)$$

We are interested to estimate the right-hand side of (0.379). For this we need the Lemma 0.60. We put

$$\omega(s) = \int_0^s (s-\rho)^{-\mu_k} (1+\rho)^{-p_k} (\mu_{k-1}-\alpha_{k-1}-\gamma_{(\alpha_1,\dots,\alpha_{k-2})}^{(\mu_1,\dots,\mu_{k-2})}(p_1,\dots,p_{k-1})) d\rho.$$

Thanks to Lemma 0.60 we obtain $\omega(s) \lesssim (1+s)^{-\mu_k}$, if we assume that

$$p_k > \frac{1}{\mu_{k-1}-\alpha_{k-1}-\gamma_{(\alpha_k,\dots,\alpha_{k-2})}^{(\mu_k,\dots,\mu_{k-2})}(p_1,\dots,p_{k-1})}$$

and

$$\mu_{k-1}-\alpha_{k-1}-\gamma_{(\alpha_k,\dots,\alpha_{k-2})}^{(\mu_k,\dots,\mu_{k-2})}(p_1,\dots,p_{k-1}) > 0$$

wich equivalent to

$$p_{k-1} > \frac{\alpha_{k-1}-\mu_{k-1}+1}{(\mu_{k-2}-\alpha_{k-2})-\gamma_{(\alpha_k,\dots,\alpha_{k-3})}^{(\mu_k,\dots,\mu_{k-3})}(p_1,\dots,p_{k-2})}.$$

Once more we apply Lemma 0.60 to obtain

$$\begin{aligned} I_k(t) &\lesssim \int_0^t (1+t-\tau)^{-(1+\alpha_k)} \int_0^\tau (\tau-s)^{\alpha_k-1} (1+s)^{-\mu_k} d\rho ds d\tau \\ &\lesssim \int_0^t (1+t-\tau)^{-(1+\alpha_k)} (1+\tau)^{\alpha_k-\mu_k} d\tau \\ &\lesssim (1+t)^{\alpha_k-\mu_k} \end{aligned}$$

Since the number of pages is limited , if I am accepted into this global event , I will end this proof in a private meeting. This completes the proof.

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Prime de crédibilité dans le cas pondéré équilibré

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Abstract.

Cette étude se concentre sur une technique largement utilisée en théorie de la crédibilité, à savoir l'évaluation de la prime. Nous tenterons d'établir des primes de crédibilité sans recourir à la spécification de la distribution des sinistres Y . On applique la méthode de l'entropie maximale sous la fonction de perte pondérée équilibrée pour déterminer la prime.

Keywords: Prime de crédibilité, Estimation, Fonction de perte quadratique équilibrée, La méthode de l'entropie maximale.

Introduction

Dans la pratique, les actuaires ont constaté que les compagnies d'assurance risquent la faillite si elles n'obligent les assurés qu'à payer la prime nette. Pour résoudre ce problème et améliorer la qualité de l'estimation des risques futurs, certains auteurs ont remplacé la forme de la fonction de perte quadratique classique par une fonction ayant une forme équilibrée appelée la fonction de perte équilibrée pondérée : $L(\delta, \theta) = \omega h(x)(\delta_0(x) - P)^2 + (1 - \omega)h(x)(\theta - P)^2$,

où $\delta_0(x)$ est une fonction des données observées choisie par l'actuaire, $0 \leq \omega \leq 1$ est un facteur de pondération déterminé par l'actuaire. En utilisant la fonction de perte équilibrée pondérée, on peut obtenir des primes avec plus de flexibilité de sorte que différentes compagnies d'assurance peuvent prendre des fonctions de pondération positives particulières $h(x)$ pour répondre à leurs besoins et faire en sorte que la prime soit compétitive sur les marchés.



La méthode de l'entropie maximale repose sur l'hypothèse que lorsqu'on estime la distribution de probabilité. On doit sélectionner une distribution à partir d'une famille donnée de distributions de probabilité, ce qui lui laisse la plus grande incertitude restante (c'est-à-dire l'entropie maximale) en fonction de contraintes données.

Main results

Notre théorème ci-dessous montre comment obtenir une nouvelle prime de crédibilité dans le cadre de la fonction de perte équilibrée en appliquant la méthode de l'entropie maximale et en résolvant un problème de minimisation.

Theorem 0.154. *Supposons que les sinistres d'un contrat au cours des n dernières années sont $X_1, X_2, \dots, X_n$ alors un estimateur linéaire non homogène optimal du prochain sinistre X_{n+1} est donné par*

$$P_{n+1}^{MEM} = \alpha_0^* + \sum_{k=1}^n \alpha_k^* X_k$$

qui satisfait aux conditions suivantes :

(i) minimiser la fonction de perte carrée équilibrée espérée de

$$E \left[\omega \left(\delta_0(X) - \alpha_0 - \sum_{i=1}^n \alpha_i X_i \right)^2 + (1 - \omega) \left(X_{n+1} - \alpha_0 - \sum_{i=1}^n \alpha_i X_i \right)^2 \right],$$

(ii) maximiser l'entropie
avec

$$\alpha_0^* = \omega E[\delta_0(X)] + \frac{(1 - \omega) E(X_{n+1})}{\left\{ 1 + \sum_{i=1}^n \exp \left[\lambda \sum_{j=1}^n \text{Cov}(X_i, X_j) \frac{(1 - \omega) E(X_{n+1})}{E(X_i)} \right] \right\}},$$

$$\alpha_k^* = \frac{(\alpha_0^* - \omega E[\delta_0(X)]) \exp \left[\lambda \sum_{j=1}^n \text{Cov}(X_k, X_j) \frac{(1 - \omega) E(X_{n+1})}{E(X_k)} \right]}{E(X_k)},$$

où λ est la solution de l'équation suivante,

$$\begin{aligned} & \sum_{i=1}^n \exp \left[\lambda \sum_{j=1}^n \text{Cov}(X_i, X_j) \frac{(1 - \omega) E(X_{n+1})}{E(X_i)} \right] \sum_{j=1}^n \text{Cov}(X_i, X_j) \frac{(1 - \omega) E(X_{n+1})}{E(X_i)} \\ &= \left[1 + \sum_{i=1}^n \exp \left[\lambda \sum_{j=1}^n \text{Cov}(X_i, X_j) \frac{(1 - \omega) E(X_{n+1})}{E(X_i)} \right] \right] \omega \sum_{j=1}^n \text{Cov}(\delta_0(X), X_j). \end{aligned}$$



Conclusion

Dans ce travail, nous avons utilisé une technique très simple pour obtenir des primes de crédibilité dans le cadre d'une fonction de perte d'équilibre. Cette méthodologie qui n'est pas très connue dans la littérature actuarielle nous donne une prime de crédibilité linéaire après un calcul simple.

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Non-parametric estimation of stochastic differential equations driven by mixed bi-fractional Brownian motion

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Abstract. The main objective of this paper is to investigate the problem of estimating the trend function $S_t = S(x_t)$ for a process satisfying stochastic differential equations of the type:

$$dX_t = S(X_t) dt + \varepsilon d\widetilde{W}_t^{H,K}, \quad X_0 = x_0, \quad 0 \leq t \leq T,$$

where $(\widetilde{W}_t^{H,K})_{t \in [0, T]}$ is a mixed bi-fractional Brownian motion with known parameters $H \in]1/2, 1[$, $K \in]1/2, 1]$ and $HK \in]1/2, 1[$. We estimate the unknown trend function $S(\cdot)$ by a Kernel type estimator $\widehat{S}_t$ and obtain the asymptotic properties as $\varepsilon \rightarrow 0$.

Keywords: Mixed bi-fractional Brownian motion, bi-fractional Brownian motion, mixed fractional Brownian motion, trend function, Kernel estimator.



Introduction

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ is filtered probability space satisfying the usual conditions, and the processes discussed in the following are $(\mathcal{F}_t)_{t \in [0, T]}$ -adapted i.e., $((\mathcal{F}_t)_{t \in [0, T]})$ -mesurable). Further, the natural filtration of a process is understood as the $\mathbb{P}$ -completion of the filtration generated by this process. Let $\widetilde{W}^H := (\widetilde{W}_t^H)_{t \in [0, T]}$ be a mixed fractional Brownian motion defined by

$$\widetilde{W}_t^H = W_t + W_t^H, \quad 0 \leq t \leq T, \quad (0.380)$$

where $W := (W_t)_{t \in [0, T]}$ is a centered Brownian motion and $W^H := (W_t^H)_{t \in [0, T]}$ is a fractional Brownian motion. For more details see, for instance [3].

Let $W^{H,K} := (W_t^{H,K})_{t \in [0, T]}$ be a bi-fractional Brownian motion is a centered Gaussian process, starting from zero, with covariance function

$$R_{H,K}(t, s) := \frac{1}{2^K} \left((t^{2H} + s^{2H})^K - |t - s|^{2HK} \right), \quad 0 \leq t, s \leq T, \quad (0.381)$$

where $H \in]1/2, 1[$, $K \in]1/2, 1[$ and $HK \in]1/2, 1[$. For more details see, for instance [2] and [1].

The mixed bi-fractional Brownian motion (mbfBm in short) is a generalization of the bi-fractional Brownian motion. It is characterized by long-range dependence and self-similarity, combining properties of Brownian motion, fractional Brownian motion and bi-fractional Brownian motion. These characteristics have been validated in various studies, including the work by Sghir and Hadiri (2017) [2]. Let

$$\widetilde{W}_t^{H,K} = \widetilde{W}_t^H + W_t^{H,K}, \quad 0 \leq t \leq T, \quad (0.382)$$

the process $\widetilde{W}^{H,K} = (\widetilde{W}_t^{H,K})_{t \in [0, T]}$ is called the mixed bi-fractional Brownian motion with Hurst parameters H and K , it has the following properties:

1. The expectation of mixed bi-fractional Brownian motion is

$$\mathbb{E} \left[\widetilde{W}_t^{H,K} \right] = 0.$$

2. For all $t \in [0, T]$,

$$\mathbb{E} \left[\left(\widetilde{W}_t^{H,K} \right)^2 \right] = t + t^{2H} + t^{2HK}.$$

3. For all $s, t \in [0, T]$,

$$\text{Cov} \left(\widetilde{W}_t^{H,K}, \widetilde{W}_s^{H,K} \right) = (t \wedge s) + \frac{1}{2} \left(t^{2H} + s^{2H} - |t - s|^{2H} \right) + \frac{1}{2^K} \left((t^{2H} + s^{2H})^K - |t - s|^{2HK} \right).$$



4. $\widetilde{W}^{H,K}$ is said to be mixed self-similar

$$\left(\widetilde{W}_{at}^{H,K}\right)_{t \in [0, T]} \stackrel{d}{=} \left(\widetilde{W}_t^{H,K} \left(a^{\frac{1}{2}}, a^H, a^{HK}\right)\right)_{t \in [0, T]}, \quad \forall a > 0.$$

Non-parametric estimation of the trend function for stochastic differential equations (SDEs in short) has attracted the attention of many researchers. The topic was first investigated by Kutoyants (1994) [3] for SDEs driven by the standard Wiener process (Brownian motion). Subsequently, the problem was generalized by Mishra and Prakasa Rao (2011) [1] for SDEs driven by fractional Brownian motion. Later, Prakasa Rao (2019) [3] examined nonparametric estimation of SDEs driven by mixed fractional Brownian motion. Recently, Keddi et al (2020) [1] studied the non-parametric estimation of the trend function for SDEs driven by bi-fractional Brownian motion. In this context, we use the method developed by Kutoyants (1994) [3] to construct an estimator for the trend function $S(\cdot)$ in a model described by SDEs driven by mixed bi-fractional Brownian motion

$$dX_t = S(X_t) dt + \varepsilon d\widetilde{W}_t^{H,K}, \quad X_0 = x_0, \quad 0 \leq t \leq T, \quad (0.383)$$

where $\left(\widetilde{W}_t^{H,K}\right)_{t \in [0, T]}$ is a mixed bi-fractional Brownian motion with parameters $H \in]1/2, 1[$, $K \in]1/2, 1[$ and $HK \in]1/2, 1[$, and $S(\cdot)$ is an unknown trend function. In this study, we adopt a similar approach to Kutoyants (1994) [3] to estimate the trend function $S(\cdot)$ in the stochastic process. We consider the estimation as follows:

$$\widehat{S}_t = \frac{1}{\varphi_\varepsilon} \int_0^T G\left(\frac{\tau - t}{\varphi_\varepsilon}\right) dX_\tau, \quad (0.384)$$

where G is a bounded kernel with finite support with $\varphi_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$.

The paper is organized as follows: In section* , we assume some assumptions about the trend function. section* contains the main theoretical results of the paper, i.e., the uniform consistency, the rate of convergence and the asymptotic normality for the non-parametric estimation procedure.

Model assumptions

Let $(X_t)_{t \in [0, T]}$ be a process governed by the following stochastic differential equation

$$dX_t = S(X_t) dt + \varepsilon d\widetilde{W}_t^{H,K}, \quad X_0 = x_0, \quad 0 \leq t \leq T, \quad (0.385)$$



where the function $S(\cdot)$ is unknown and the Hurst parameters H and K are known ($H \in]1/2, 1[$ and $H \in]1/2, 1]$). Suppose x_t is the solution of the differential equation when $\varepsilon = 0$. Then, we can write

$$\frac{dx_t}{dt} = S(x_t), \quad x_0, \quad 0 \leq t \leq T. \quad (0.386)$$

Also, we suppose that the function $S : \mathbb{R} \rightarrow \mathbb{R}$ satisfies the following assumptions

(H1) There exists $L > 0$ such that, $\Theta_0(L)$ denote the class of all functions $S(x)$ satisfying the condition

$$|S(x) - S(y)| \leq L|x - y|, \quad x, y \in \mathbb{R}.$$

(H2) There exists a constant $M > 0$ such that

$$|S(x)| \leq M(1 + |x|), \quad x \in \mathbb{R}.$$

(H3) Since the function x_t satisfies Equation (0.386), it follows that

$$|S(x_t) - S(x_s)| \leq L|x_t - x_s| = L \left| \int_s^t S(x_u) du \right| \leq L_1|t - s|, \quad t, s \in \mathbb{R},$$

for some constant $L_1 > 0$.

(H4) Let $\Theta_k(L)$ denote the class of all functions $S(x)$ which are uniformly bounded by the same constant L and which are k -times differentiable with respect to t satisfying the condition

$$|S^{(k)}(x) - S^{(k)}(y)| \leq L|x - y|, \quad x, y \in \mathbb{R}.$$

(H5) Let $\Theta_{k+1}(L)$ denote the class of all functions $S(x)$ which are uniformly bounded by the same constant L and which are $(k+1)$ -times differentiable with respect to t satisfying the condition

$$|S^{(k+1)}(x) - S^{(k+1)}(y)| \leq L|x - y|, \quad x, y \in \mathbb{R}.$$

Lemma 0.60. *Let the function $S(\cdot)$ satisfy condition (H1). Let X_t and x_t be the solutions of equations (0.385) and (0.386), respectively. Then, with probability one,*

$$(i). \quad |X_t - x_t| < \varepsilon \left| \widetilde{W}_t^{H,K} \right| e^{Lt},$$

$$(ii). \quad \sup_{0 \leq t \leq T} \mathbb{E} [|X_t - x_t|^2] \leq e^{2LT} \varepsilon^2 (T + T^{2H} + T^{2HK}).$$



Main results

In this section*, we establish the uniform consistency, the rate of convergence and the asymptotic normality of non-parametric estimation for a stochastic differential equation driven by a mixed bi-fractional Brownian motion. We define a kernel estimator of the trend $S_t = S(x_t)$ using the method developed by Kutoyants (1994) [3]

$$\hat{S}_t = \frac{1}{\varphi_\varepsilon} \int_0^T G\left(\frac{\tau - t}{\varphi_\varepsilon}\right) dX_\tau, \quad (0.387)$$

where $G(u)$ is a bounded function with finite support $[A, B]$ with $A < 0 < B$ satisfying the following assumptions

- (R1) $G(u) = 0$ for $u \notin [A, B]$, and $\int_A^B G(u) du = 1$,
- (R2) $\int_{\mathbb{R}} G^2(u) du < \infty$,
- (R3) $\int_{\mathbb{R}} u^{2(k+1)} G^2(u) du < \infty$ for $k \in \mathbb{N}$,
- (R4) $\int_{\mathbb{R}} \{G(u)\}^{\frac{1}{H-K}} du < \infty$,
- (R5) $\int_{\mathbb{R}} u^j G(u) du = 0$ for $j = 1, 2, \dots, k$,
- (R6) $\int_{\mathbb{R}} |G(u) u^{k+1}| du < \infty$ for $k \in \mathbb{N}^*$,
- (R7) we suppose that the normalizing function φ_ε satisfies

$$\varphi_\varepsilon \longrightarrow 0 \text{ and } \varepsilon^2 \varphi_\varepsilon^{-1} \longrightarrow 0 \text{ as } \varepsilon \longrightarrow 0.$$

In the following theorem, we prove that the estimator $\hat{S}_t$ is uniformly consistent.

Theorem 0.155. *Suppose that the trend function $S(x) \in \Theta_0(L)$ and suppose that conditions (H1) and (H3). Further, suppose that conditions (R1)-(R4) holds. Then, for any $0 < c \leq d < T$, the estimator $\hat{S}_t$ is uniformly consistent, that is,*

$$\lim_{\varepsilon \rightarrow 0} \sup_{S(x) \in \Theta_0(L)} \sup_{c \leq t \leq d} \mathbb{E} \left[\left| \hat{S}_t - S(x_t) \right|^2 \right] = 0.$$



The rate of convergence of the estimator $\hat{S}_t$ is established in the following theorem.

Theorem 0.156. *Suppose that the function $S(x) \in \Theta_k(L)$ and suppose that condition (H4). Further, suppose that conditions (R1), (R3) and (R5) holds. We have,*

$$\limsup_{\varepsilon \rightarrow 0} \sup_{S(x) \in \Theta_k(L)} \sup_{c \leq t \leq d} \mathbb{E} \left[\left| \hat{S}_t - S(x_t) \right|^2 \right] < \infty.$$

Finally, the following theorem presents the asymptotic normality of the estimator $\hat{S}_t$.

Theorem 0.157. *Suppose that the function $S(x) \in \Theta_{k+1}(L)$ and suppose that condition (H5). Further, suppose that conditions (R1), (R3) and (R5) holds. We have*

$$\left(\hat{S}_t - S(x_t) \right) \xrightarrow{d} \mathcal{N}(\mu, \sigma_{H,K}^2), \text{ as } \varepsilon \rightarrow 0$$

where,

$$\mu = \frac{S^{(k+1)}(x_t)}{(k+1)!} \int_{\mathbb{R}} G(u) u^{k+1} du$$

and

$$\begin{aligned} \sigma_{H,K}^2 = & \int_{\mathbb{R}} \int_{\mathbb{R}} G(u) G(v) \left(\alpha_{H,K} (uv)^{2H-1} (u^{2H} + v^{2H})^{K-2} + \beta_{H,K} |u-v|^{2HK-2} \right) du dv \\ & + H(2H-1) \int_{\mathbb{R}} \int_{\mathbb{R}} G(u) G(v) |u-v|^{2H-2} du dv + \int_{\mathbb{R}} G^2(u) du, \end{aligned}$$

with

$$\alpha_{H,K} = 2^{-K+2} H^2 K(K-1) \quad \text{and} \quad \beta_{H,K} = 2^{-K+1} 2H(2HK-1).$$

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**The existence and stability of coincidence point for
mappings acting from b -metric space into generalized
metric space**

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Abstract. In this work, we will present a new results of coincidence point for mappings acting from b -metric space to a space with distance satisfies only the axiom of identity.

Keywords: Metric space, Covering mapping, b -metric space, Lipschitz mapping.

Mathematics Subject Classification: 54E40, 54H25, 46A30



Introduction

In this work, we will state a new theorem for the existence of coincidence point. The mappings in this theorem are defined from a b -metric space to a nonempty space equipped with a generalized distance. One of the mapping is α -covering and closed, and the other is β -Lipschitz. Also, we obtain the stability of the coincidence points for applications ψ_n and φ_n with some convergence towards ψ and φ , respectively.

Main results

Firstly, we give the definition of the space X (b -metric space).

Definition 0.62. Let X be a nonempty set, we define the application $\rho : X \times X \rightarrow \mathbb{R}_+$, and let $s \geq 1$. If the following relations are satisfied for all $x, y, z \in X$:

- (b_1) $\rho(x, y) = 0$ iff $x = y$,
- (b_2) $\rho(x, y) = \rho(y, x)$,
- (b_3) $\rho(x, z) \leq s(\rho(x, y) + \rho(y, z))$,

so ρ is called a b -metric on X and (X, ρ) is a b -metric space.

Secondly, the space Y is a generalized metric space (that's mean distance satisfied only the axiom of identity). (Y, d) is a generalized metric space where Y is a nonempty set and d is a distance satisfied only the axiom of identity i.e.:

$$\forall y_1, y_2 \in Y \quad d(y_1, y_2) = 0 \quad \Leftrightarrow \quad y_1 = y_2. \quad (0.388)$$

Now, we give the definitions of covering mapping and Lipschitz mapping, respectively.

Definition 0.63. Let $\alpha > 0$, the mapping $\psi : X \rightarrow Y$ is called α -covering, if satisfying the relation :

$$\forall x_0 \in X \quad \forall y \in Y \quad \exists x \in X : \psi(x) = y \quad \text{and} \quad \rho(x, x_0) \leq \frac{1}{\alpha} d(\psi(x), \psi(x_0)).$$

Definition 0.64. : Let $\beta > 0$, the mapping $\varphi : X \rightarrow Y$ is called β -Lipschitz, if satisfying the relation:

$$d(\varphi(x), \varphi(x_0)) \leq \beta \rho(x, x_0) \quad \forall x, x_0 \in X$$

We remained the definition of coincidence point for two mappings $\psi, \varphi : X \rightarrow Y$.

Definition 0.65. A point $x \in X$ is called a coincidence point for the mappings $\psi : X \rightarrow Y$ and $\varphi : X \rightarrow Y$ if the following relation is holds:

$$\psi(x) = \varphi(x).$$



Now, we have the following assertion.

Theorem 0.158. *Let a b -metric space (X, ρ) be complete and (Y, d) is space with a generalized distance d . If the following conditions are satisfied:
 An application $\psi : X \rightarrow Y$ is α -covering and closed, and an application $\varphi : X \rightarrow Y$ is β -Lipschitz with $\alpha > \beta$.
 Then, for all $x_0 \in X$ there exists a coincidence point ξ for ψ and φ such that:*

$$\rho(x_0, \xi) \leq \frac{s^2 \alpha^{t_0-1} R(s\beta/\alpha, t_0 - 1) + s(s\beta)^{t_0-1}}{\alpha^{t_0} - s\beta^{t_0}} d(\psi(x_0), \varphi(x_0)), \quad (0.389)$$

where, $t_0 = \min\{j \in \mathbb{N} / s\beta^j < \alpha^j\}$, and for $\theta \in [0, 1)$ and $n \in \mathbb{N}$, we have $R(\theta, n) = \frac{1-\theta^n}{1-\theta}$.

Using Theorem 0.158, we obtain the stability conditions for the coincidence points of the mappings $\psi_n, \varphi_n : X \rightarrow Y$, $n \in \mathbb{N}$ under small changes. For every $n \in \mathbb{N}$, define the mappings $\psi_n, \varphi_n : X \rightarrow Y$. The stability of the set of coincidence points is interpreted here as the convergence, when $n \rightarrow \infty$, of the sequence of coincidence points of the mappings ψ_n, φ_n towards the coincidence point of the mappings ψ and φ , in the case where there is some convergence of ψ_n to ψ and φ_n to φ .

Recall that a coincidence point of the mappings ψ_n, φ_n is the solution of the equation:

$$\psi_n(x) = \varphi_n(x). \quad (0.390)$$

Theorem 0.159. *Let (X, d_b) be a complete b -metric space, and let $\xi \in X$ be the coincidence point of the mappings $\psi, \varphi : X \rightarrow Y$. For each $n \in \mathbb{N}$, given $0 \leq \beta_n < \alpha_n$, we define:*

$$B_n = \frac{s^2 \alpha_n^{t_0-1} R(s\beta_n/\alpha_n, t_0 - 1) + s(s\beta_n)^{t_0-1}}{\alpha_n^{t_0} - s\beta_n^{t_0}} d(\psi_n(\xi), \varphi_n(\xi)).$$

We assume that for each $n \in \mathbb{N}$, $x \in X$, the mapping ψ_n is α_n -covering and closed, the mapping φ_n is β_n -Lipschitz. If $B_n \rightarrow 0$ as $n \rightarrow \infty$, then for all $n \in \mathbb{N}$, equation (0.390) has at least one solution ξ_n such that $\xi_n \rightarrow \xi$ as $n \rightarrow \infty$.

This result is a generalization of the results published in [1] and [2].

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Étude d'un modèle de pêche avec stock variable

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Abstract. Dans ce travail, nous étudions un modèle de pêche qui décrit la dynamique d'une ressource marine exploitée avec stock variable.

Notre objectif est de maximiser les captures et les revenus de la pêche sans risquer l'extinction du stock exploité. Pour cela, nous avons développé un modèle mathématique et nous avons étudié les points d'équilibres, la stabilité local et global en utilisant le critère de Routh-Hurwitz et la fonction de Lyapunov, la bifurcation de Hopf et nous avons fini l'analyse du modèle par les simulations numériques.

Keywords: Systèmes dynamiques, modélisation mathématique, modèle de pêche, simulations numériques, stabilité.



Introduction

La pêche joue un rôle crucial en fournissant des moyens de subsistance durables à des millions de personnes à travers le monde. Les activités de pêche englobent un large éventail de rôles, allant des pêcheurs qui capturent directement le poisson, aux personnes impliquées dans la transformation, la commercialisation et la vente du produits de la mer.

Dans la modélisation bioéconomique, l'intégration des perspectives écologiques et économiques est essentielle pour relever les défis complexes du monde réel. En considérant ces deux aspects, les chercheurs peuvent développer des modèles plus complets qui représentent avec précision la dynamique des systèmes naturels tout en tenant compte des implications économiques de diverses stratégies de gestion.

Dans les modèles de pêche, la densité de poissons et l'effort de la pêche sont pris toujours comme variable. Dans cet article, nous allons inclure une nouvelle variable qui est le stock du poissons.

Modèle mathématique

Nous considérons un modèle mathématique de pêche avec stock variable. Pour ce modèle nous avons étudié les points d'équilibres et leurs stabilité local et global en utilisant le critère de Routh-Hurwitz et la fonction de Lyapunov ainsi que la bifurcation de Hopf qui fait référence à la naissance d'une solution périodique.

Tous les résultats précédents sont illustrés par les simulations numériques.

Conclusion

Dans ce travail, un modèle de pêche a été discuter en introduisant une variable stock pour contrôler la variation du stock de poissons avec le temps. Dans l'analyse du modèle, nous avons cherché des conditions de stabilité local et global pour les points d'équilibres dans le but d'assurer une pêche durable.

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DNA codes from skew cyclic codes over $\mathbb{F}_4[v]/\langle v^2 - v \rangle$

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Abstract. In this work, we investigate the DNA codes over the finite ring $R = \mathbb{F}_4 + v\mathbb{F}_4$, $v^2 = v$. We construct DNA codes from reversible skew cyclic codes over R by generalizing the method of their construction from reversible cyclic codes in [2].

Keywords: Skew polynomial ring, skew cyclic codes, reversible DNA codes,

Introduction

Deoxyribonucleic acid (DNA) is a nucleic acid containing genetic information for the biological evolution of life. DNA strands is a sequences of four nucleotides: adenine (A), guanine (G), thymine (T), and cytosine (C). Two DNA strands are linked with a rule named as Watson–Crick complement (WCC). According



to this rule, every adenine (A) has a link with a thymine (T), and we write $A^c = \bar{A} = T$ and every guanine (G) has a link with cytosine (C), and we write $C^c = \bar{C} = G$, and vice versa.

DNA computing was first introduced by Leonhard Adleman in 1994, in [1]. Since then, DNA cyclic codes have been discussed by several authors over finite fields and finite rings of 4^n elements by mapping the DNA nucleotides to the elements of finite fields and finite rings.

In [2], the authors presented an approach to solve the reversibility problem for DNA double pairs, and they obtained DNA codes over $\mathbb{F}_4 + v\mathbb{F}_4$ by using a special automorphism and properties of cyclic codes over $\mathbb{F}_4 + v\mathbb{F}_4$. In this talk, We extend the approach introduced in [?] for obtaining DNA codes using skew cyclic codes over $\mathbb{F}_4 + v\mathbb{F}_4$.

Skew cyclic codes over R

Let $\mathbb{F}_4 = \mathbb{F}_2[w] = \{a_0 + a_1w | a_0, a_1, a_2 \in \mathbb{F}_2\}$, where $w^2 + w + 1 = 0$, and let σ be the Frobenius automorphism of $\mathbb{F}_4$ defined by $\sigma(a) = a^2$, for any $a \in \mathbb{F}_4$. The skew polynomial ring $\mathbb{F}_4[x; \sigma]$ is formed by the set of polynomials in the indeterminate x

$$\mathbb{F}_4[x; \sigma] = \{a_0 + a_1x + a_2x^2 + \dots + a_nx^n \text{ with } a_i \in \mathbb{F}_4\}, \quad (0.391)$$

and two operations, namely addition and multiplication, on the set. Addition of two polynomials, denoted by $+$, is defined in the usual way whereas multiplication is governed by the distributive law and the rule

$$(ax^i) \cdot (bx^j) = a\sigma^i(b)x^{i+j}, a, b \in \mathbb{F}_4. \quad (0.392)$$

Consider the finite ring $R = \mathbb{F}_4 + v\mathbb{F}_4 = \{a + vb | a, b \in \mathbb{F}_4\}$ of 16 elements. The ring R is a semi-local ring, it contains two maximal ideals $\langle v \rangle$ and $\langle 1 - v \rangle$. So any element of R can be written as follows: $r = (1 - v)r_1 + vr_2$, where $r_1, r_2 \in \mathbb{F}_4$. The automorphism σ of $\mathbb{F}_4$ is extended to $\mathbb{F}_4 + v\mathbb{F}_4$ as follows: For $a + vb \in \mathbb{F}_4 + v\mathbb{F}_4$, $\sigma(a + vb) = \sigma(a) + v\sigma(b)$, where $a, b \in \mathbb{F}_4$. The skew polynomial ring $R[x; \sigma]$ is defined as follow

$$R[x; \sigma] = \{a_0 + a_1x + a_2x^2 + \dots + a_nx^n \text{ with } a_i \in R\}, \quad (0.393)$$

where the addition is the usual addition of polynomials, whereas the multiplication is given by the distributive law and the rule

$$(ax^i) \cdot (bx^j) = a\sigma^i(b)x^{i+j}, a, b \in R. \quad (0.394)$$

Consider the map Θ defined as

$$\begin{aligned} \Theta : R[x; \sigma] &\rightarrow R[x; \sigma] \\ \sum_{i=0}^r a_i x^i &\mapsto \sum_{i=0}^r \sigma(a_i) x^i. \end{aligned}$$



It is a semi-linear isomorphism.

Consider also the non-trivial automorphism over $\mathbb{F}_4 + v\mathbb{F}_4$ defined in [?] by:

$$\begin{aligned}\psi : R &\rightarrow R \\ a + vb &\mapsto a + (1 + v)b.\end{aligned}$$

A linear code C of length n over R is an R -submodule of R^n . Let C be a linear code of length n over R and let

$$\begin{aligned}C_1 &= \{x + y \in \mathbb{F}_4^n | v(x + y) + (1 + v)x \in C, \text{ for some } x, y \in \mathbb{F}_4^n\} \\ C_2 &= \{x \in \mathbb{F}_4^n | v(x + y) + (1 + v)x \in C, \text{ for some } y \in \mathbb{F}_4^n\}.\end{aligned}$$

The sets C_1 and C_2 are linear codes of length n over $\mathbb{F}_4$, and $C = vC_1 \oplus (1 + v)C_2$ with $|C| = |C_1||C_2|$.

Definition 0.66. A linear code C of length n over R is a skew cyclic code if

$$(c_0, c_1, \dots, c_{n-1}) \in C \implies (c_{n-1}, c_0, \dots, c_{n-2}) \in C.$$

Let A and B are two linear codes. The sets $A \oplus B$ and $A \otimes B$ are given by $A \oplus B = \{a + b | a \in A \text{ and } b \in B\}$ and $A \otimes B = \{a \cdot b | a \in A \text{ and } b \in B\}$.

Theorem 0.160. ([2]) Let C be a linear code over R and $C = vC_1 \oplus (1 + v)C_2$. Then, C is a skew cyclic code if and only if C_1 and C_2 are skew cyclic codes over $\mathbb{F}_4$.

Definition 0.67. Let C be a linear code of length n over R .

The code C is said to be reversible if for any $(c_0, c_1, \dots, c_{n-1}) \in C$, we have $(c_{n-1}, c_{n-2}, \dots, c_0) \in C$.

Definition 0.68. A skew polynomial $g \in \mathbb{F}_4[x; \sigma]$, with $\deg(g) = r$, is said to be σ -palindromic if $g(x) = \alpha(\Theta(g_r + g_{r-1}x + \dots + g_0x^r))$, where i is a positive integer and $\alpha \in \mathbb{F}_4^*$.

Theorem 0.161. Let $C = \langle g(x) \rangle$ be a skew cyclic code of dimension k and of even length n over $\mathbb{F}_4$ and g is a monic divisor of $x^n - 1$ in $\mathbb{F}_4[x, \sigma]$. Then the code C is reversible if and only if σ^{k+1} -palindromic.

Theorem 0.162. Let $C = \langle vg_1 + (1 + v)g_2 \rangle$ be a skew cyclic code of even length n over $\mathbb{F}_4 + v\mathbb{F}_4$, where g_i is a monic divisors of $x^n - 1$ in $\mathbb{F}_4[x, \sigma]$ and generates the skew cyclic code C_i of dimension k_i , $i = 1, 2$. Then C is reversible over $\mathbb{F}_4 + v\mathbb{F}_4$ if and only if g_i is σ^{k_i+1} -palindromic, $i = 1, 2$.



Reversible DNA codes over R

Definition 0.69. Let C be an additive code of length n over R . Define the map

$$\begin{aligned} \xi : C &\rightarrow S_{D_4}^{2n} \\ (c_0, c_1, \dots, c_{n-1}) &\mapsto (\xi(c_0), \xi(c_1), \dots, \xi(c_{n-1})). \end{aligned}$$

Let $x = (x_1, \dots, x_n) \in R^n$. The reverse of x is defined as $x^r = (x_n, x_{n-1}, \dots, x_1)$, the complement of x is defined as $x^c = (\bar{x}_1, \dots, \bar{x}_n)$ and the reverse-complement of x is defined as $x^{rc} = (\bar{x}_{n-1}, \dots, \bar{x}_1)$, x^{rc} is also called the WCC of x .

A code is called DNA code if it satisfies some or all of the following constraints:

1. The Hamming constraint: For any $c_1, c_2 \in C$ with $c_1 \neq c_2$, $H(c_1, c_2) \geq d$.
2. The reverse constraint: For any $c_1, c_2 \in C$ with $c_1 \neq c_2$, $H(c_1, c_2^r) \geq d$.
3. The reverse complement constraint: For any $c_1, c_2 \in C$ with $c_1 \neq c_2$, $H(c_1, c_2^c) \geq d$.
4. The fixed GC content constraint: For any $c \in C$ contains the same number of G and C element.

Definition 0.70. A linear code C of length n over R is reversible if for any $c \in C$, $c^r \in C$, complement if for any $c \in C$, $c^c \in C$, and reversible-complement if for any $c \in C$, $c^{rc} \in C$.

Definition 0.71. Let n be an even integer, g_1 and g_2 be polynomials in $\mathbb{F}_4[x, \sigma]$ of degree m_1 and m_2 , respectively, and divide $x^n - 1$ in $\mathbb{F}_4[x, \sigma]$. Let $g = vg_1 + (1 + v)g_2 \in (\mathbb{F}_4 + v\mathbb{F}_4)[x, \sigma]$ and $\bar{g} = \Theta^\epsilon(\psi(g))$, $\epsilon \in \{0, 1\}$. Define the set

$$E_{\sigma, \psi}(g) = \{E_0, E_1, \dots, E_{m-1}, \bar{E}_0, \bar{E}_1, \dots, \bar{E}_{m-1}\},$$

where $E_i = x^i g$ and $\bar{E}_i = x^i \bar{g}$, $i = 0, 1, \dots, m-1$, $m = \min(n - m_1, n - m_2)$. The set $E_{\sigma, \psi}(g)$ is a linear code over R .

We consider the DNA-correspondence defined in [2, Table 3] by

$$\begin{aligned} \xi : C &\rightarrow S_{D_4}^{2n} \\ (c_0, c_1, \dots, c_{n-1}) &\mapsto (\xi(c_0), \xi(c_1), \dots, \xi(c_{n-1})). \end{aligned} \quad (0.395)$$

Definition 0.72. An linear code of length n over R is called a skew cyclic DNA code if



- C is skew cyclic code.
- For any $c \in C$, $c \neq c^{rc}$ and $c^{rc} \in C$.

The following theorem shows how to construct reversible DNA codes of even length over R using reversible or conjugate reversible skew cyclic codes over R .

Theorem 0.163. *Let n be an even integer and let g_1 and g_2 be skew polynomials in $\mathbb{F}_4[x, \sigma]$ such that g_i divides $x^n - 1$ in $\mathbb{F}_4[x, \sigma]$ and g_i is $\Theta^{m_i+\epsilon}$ -palindromic, $\epsilon \in \{0, 1\}$, with $\deg(g_i) = m_i$, $i = 1, 2$, and $m_1 = m_2 \pmod{2}$.*

- 1- *If $m_1 = m_2$, then $g = vg_1 + (1+v)g_2$ and $\bar{g} = \Theta^{\epsilon+1}(\psi(g_0g))$.*
- 2- *If $m_1 \neq m_2$, then let $l = |m_1 - m_2|$ and*

$$g = \begin{cases} vx^{l/2}g_1 + (1+v)g_2 & \text{if } m_1 < m_2, \\ vg_1 + (1+v)x^{l/2}g_2 & \text{if } m_1 > m_2 \end{cases}$$

$$\text{and } \bar{g} = \Theta^{\epsilon+1}(\psi(g_0g)).$$

Then, the code $\mathcal{C} = \langle E_{\sigma, \psi}(g) \rangle$ is a linear code and $\xi(\mathcal{C})$ is a reversible DNA code over R .

Example 0.6. *Let $n = 18$ and let $g_1 = x^8 + x^6 + w * x^5 + w * x^3 + x^2 + 1$ and $g_2 = x^6 + w * x^5 + w * x^4 + w * x^3 + w * x^2 + w * x + 1$ be divisors of $x^{18} - 1$ in $\mathbb{F}_4[x, \sigma]$ such that g_i is σ^2 -palindromic, $i = 1, 2$, $\epsilon = 0$, with $m_1 = 8$, $m_2 = 6$, $l = 2$, and $m = 10$. Note that g_i generates a conjugate reversible skew cyclic code C_i of length 18 over $\mathbb{F}_4$.*

Let $g = vg_1 + (1+v)g_2 = v * x^8 + (v+1) * x^7 + (w * v + w^2) * x^6 + (v + w^2) * x^5 + (w^2 * v + w^2) * x^4 + (v + w^2) * x^3 + (w * v + w^2) * x^2 + (v+1) * x + v$. We have $\tau_8(g) = g$. and $\bar{g} = \Theta(\psi(g)) = (1+v) * x^8 + (v) * x^7 + (w^2 * v + 1) * x^6 + (v + w^2) * x^5 + (w * v) * x^4 + (v + w^2) * x^3 + (w^2 * v + 1) * x^2 + (v) * x + (1+v)$;.

Then, the code $\mathcal{C} = \langle E_{\sigma, \psi}(g) \rangle$ is a linear code and $\xi(\mathcal{C})$ is a reversible DNA code over R .

Let $\alpha_1 = w$, $\alpha_5 = v$, $\alpha_9 = w + v$, $\beta_1 = 1$, $\beta_5 = w * v$, $\beta_9 = w^2$ and $\alpha_i = \beta_i = 0$, $i = 0, 2, 3, 4, 6, 7, 8$. Let $c(x) = \alpha_1 E_1 + \alpha_5 E_5 + \alpha_9 E_9 + \beta_1 \bar{E}_1 + \beta_5 \bar{E}_5 + \beta_9 \bar{E}_9 \in \mathcal{C}$. It corresponds to $c = (0, w^2 * v + 1, w^2 * v + w, w^2 * v + w, w^2 * v + 1, v + w^2, v + 1, w^2 * v + w, v + w, w * v + w, w^2 * v + w, v, w, w^2, w * v + w, v, v + w, w^2)$, and $\xi(c) = AACTTCTCCTCGATTCGCACTCTACCGGACTAGCGG$.

By Theorem 0.163, $c'(x) = \alpha_1 \bar{E}_8 + \alpha_5 \bar{E}_4 + \alpha_9 \bar{E}_0 + \beta_1 E_8 + \beta_5 E_4 + \beta_9 E_0 \in \mathcal{C}$ and it corresponds to $c' = (w^2, v + w^2, 1 + v, w * v, w^2, w, 1 + v, w^2 * v + 1, w * v, v + w^2, w^2 * v + 1, v, v + w, w^2 * v + w, w^2 * v + 1, w^2 * v + 1, w^2 * v + w, 0)$, and we have $\xi(c') = GGCGATCAGGCCATCTCACGCTTAGCTCCTCTTCAA$ is in $\xi(\mathcal{C})$, hence we obtain $\xi(c') = \xi(c)^r$.



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On the Diophantine equation: $d(kn) = \varphi(\varphi(n))$

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Abstract. Let $d(n)$ and $\varphi(n)$ denote the number of positive integers dividing the positive integer n and the Euler's phi function representing the numbers less than and prime to n , respectively. In this paper, we determine all solutions of the equation $d(n) = \varphi(\varphi(n))$ and we prove that the equation $d(kn) = \varphi(\varphi(n))$ has a finite number of solutions for any $k \geq 1$. Further, we characterize all solutions of the last equation when k is prime.

Keywords: Arithmetic functions; diophantine equations; primes; factorization.



Introduction

Let $d(n)$ be the divisor function, which counts the number of positive divisors of n , i.e., if n has the prime factorization $n = q_1^{a_1} q_2^{a_2} \dots q_k^{a_k}$ with distinct primes $q_1, q_2, \dots, q_k$ and positive integers $a_1, a_2, \dots, a_k$, then

$$d(n) = (a_1 + 1)(a_2 + 1) \dots (a_k + 1).$$

In our main results, we will use the following inequalities (see; eg. [2],[4]):

1. For any positive integer n , we have

$$d(n) \leq 2\sqrt{n}. \quad (0.396)$$

2. For all positive integers m and n , we have

$$d(mn) \leq d(m) d(n). \quad (0.397)$$

Moreover, if $(x, y) > 1$, then

$$d(x) d(y) > d(xy) \quad (0.398)$$

3. If x divides y , then

$$d(y) \geq d(x). \quad (0.399)$$

Let $\varphi(n)$ be the Euler function, which counts the number of positive integers $m \leq n$ with $(m, n) = 1$. It is well-known that

$$\varphi(n) = q_1^{a_1-1}(q_1-1)q_2^{a_2-1}(q_2-1)\dots q_k^{a_k-1}(q_k-1).$$

And we will use the following inequalities (see also; eg. [2],[4]):

1. for $n \neq 2$ and $n \neq 6$

$$\varphi(n) \geq \sqrt{n} \quad (0.400)$$

- 2.

$$\varphi(n)\varphi(m) \leq \varphi(nm); \quad m, n = 1, 2, 3... \quad (0.401)$$

- 3.

$$\varphi(nm) = \varphi(n)\varphi(m) \frac{d}{\varphi(d)}; \quad m, n = 1, 2, 3... \quad (0.402)$$

4. If $(n, m) > 1$ then (0.401) and (0.402) give us this inequality:

$$\varphi(n)\varphi(m) < \varphi(nm); \quad (0.403)$$



Recall that various diophantine equations involving the divisor function and Euler's phi function were investigated by many authors. For example, see [1] and [3].

In [2, Problem 705, page 78], it is shown that $\varphi(d(n)) = d(\varphi(n))$ has infinitely many solutions; while in [4, pages 110-111], it is shown that $d(n) = \varphi(n)$ has the only solutions 1, 3, 8, 10, 24 and 30, where $d(n) < \varphi(n)$ for $n \geq 31$. Using these multiplicative functions, we are interested here in problems involving the number of positive divisors of $\varphi(n)$. For this purpose, define for any positive integer k the following sets:

$$\mathbb{E}_k := \{d(k \cdot n) = \varphi(\varphi(n))\} \quad (0.404)$$

$$\mathbb{L}_k := \{d(k \cdot n) < \varphi(\varphi(n))\} \quad (0.405)$$

$$\mathbb{G}_k := \{d(k \cdot n) > \varphi(\varphi(n))\} \quad (0.406)$$

In this presentation, we characterize all the elements of $\mathbb{E}_1$, $\mathbb{L}_1$ and $\mathbb{G}_1$, respectively. And we prove that the sets $\mathbb{G}_k$ and $\mathbb{E}_k$ are finite, while $\mathbb{L}_k$ is infinite.

Main results

*

The equation $d(n) = \varphi(\varphi(n))$

Theorem 0.164. *The numbers 1, 5, 7, 15, 22, 26, 40, 56, 66, 70, 78, 108, 120, 126, 168, 210 are the only solutions of the equation $d(n) = \varphi(\varphi(n))$.*

Moreover, for every $n \neq 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 18, 20, 22, 24, 26, 28, 30, 36, 40, 42, 48, 54, 56, 60, 66, 70, 72, 78, 84, 90, 108, 120, 126, 168, 180$ and 210 we have $d(n) < \varphi(\varphi(n))$.

*

On the equation $d(k \cdot n) = \varphi(\varphi(n)), k \geq 2$

Theorem 0.165. *Let $k \geq 2$. The sets $\mathbb{G}_k$ and $\mathbb{E}_k$ are finite, while $\mathbb{L}_k$ is infinite.*

Proposition 0.23. *We have $\bigcap_{k \geq 1} \mathbb{G}_k = \mathbb{G}_1$.*

Proposition 0.24. *Let $r, s \geq 2$. If r divides s , then $\mathbb{G}_r \subset \mathbb{G}_s$.*

Theorem 0.166. *Let p be a prime number with $p \geq 23$ and $p \neq 31$. Then $E_p = \{11, 13, 33, 34, 35, 39, 62, 63, 76, 88, 98, 102, 104, 105, 110, 130, 154, 186, 228, 234, 264, 280, 294, 312, 330, 390, 462, 504, 540, 630, 840\}$.*



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Modeling Chagas Disease Transmission: A Semigroup approach

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Abstract. Using operator theory, we develop an abstract perturbed Cauchy problem to model Chagas disease transmission. Our model accounts for both domestic and sylvatic habitats and employs skew Brownian motion to describe movement between them.

Keywords: Chagas disease, reaction-diffusion equations, skew Brownian motion, sylvatic transmission,

Introduction

Chagas disease, a vector-borne illness caused by the *Trypanosoma cruzi* parasite, is endemic in Latin America but has spread globally. Despite efforts to control the disease through insecticide spraying, re-infestation remains a challenge [1].

This study proposes a new mathematical model to understand Chagas disease transmission. The model considers two interconnected habitats: one for susceptible individuals and one for infected individuals. The movement of individuals between these habitats is modeled using skew Brownian motion. The model also includes reaction-diffusion equations to describe population dynamics and spatial dispersal within each habitat.



The goal of the study is to analyze the behavior of the model and determine the key factors influencing the spread of Chagas disease. This information can be used to inform public health interventions and control strategies.

The model

The complete reaction-diffusion model considered in this paper is the following

$$\begin{cases} \frac{\partial J_I}{\partial t} = d_J^- \Delta J_I + ((1 - \tau^-) \sigma_J^- - 1) J_I + f_A^- \sigma_A^- A_I + T_5 \Lambda_J, & t > 0, (x, y) \in \Omega_I \\ \frac{\partial A_I}{\partial t} = d_A^- \Delta A_I + \tau^- \sigma_J^- J_I + (\sigma_A^- - 1) A_I + T_4 \Lambda_A, & t > 0, (x, y) \in \Omega_I \\ \frac{\partial H_I}{\partial t} = d_H^- \Delta H_I + ((1 + \nu f_H^-) \sigma_H^- - 1) H_I + T_3 \Lambda_H, & t > 0, (x, y) \in \Omega_I \\ \frac{\partial J_S}{\partial t} = d_J^+ \Delta J_S + ((1 - \tau^+) \sigma_J^+ - 1) J_S + f_A^+ \sigma_A^+ A_S + T_1, & t > 0, (x, y) \in \Omega_S \\ \frac{\partial A_S}{\partial t} = d_A^+ \Delta A_S + \tau^+ \sigma_J^+ J_S + (\sigma_A^+ - 1) A_S, & t > 0, (x, y) \in \Omega_S \\ \frac{\partial H_S}{\partial t} = d_H^+ \Delta H_S + ((1 + f_H^+) \sigma_H^+ - 1) H_S + T_2, & t > 0, (x, y) \in \Omega_S, \end{cases} \quad (0.407)$$

with the initial conditions

$$\begin{cases} J_I(0, x, y) = J_I^0(x, y), \\ A_I(0, x, y) = A_I^0(x, y), \\ H_I(0, x, y) = H_I^0(x, y), \end{cases} \quad (x, y) \in \Omega_I \quad \begin{cases} J_S(0, x, y) = J_S^0(x, y), \\ A_S(0, x, y) = A_S^0(x, y), \\ H_S(0, x, y) = H_S^0(x, y), \end{cases} \quad (x, y) \in \Omega_S, \quad (0.408)$$

the boundary conditions

$$\begin{cases} (J_I(t, \sigma_1, \sigma_2), A_I(t, \sigma_1, \sigma_2), H_I(t, \sigma_1, \sigma_2)) = (0, 0, 0), & t > 0, (\sigma_1, \sigma_2) \in \partial\Omega_I \setminus \Gamma \\ (J_S(t, \sigma_1, \sigma_2), A_S(t, \sigma_1, \sigma_2), H_S(t, \sigma_1, \sigma_2)) = (0, 0, 0), & t > 0, (\sigma_1, \sigma_2) \in \partial\Omega_S \setminus \Gamma, \end{cases} \quad (0.409)$$

and the transmission conditions [2]

$$\begin{cases} J_I(t, 0, y) = J_S(t, 0, y), & t > 0, y \in (0, 1) \\ A_I(t, 0, y) = A_S(t, 0, y), & t > 0, y \in (0, 1) \\ H_I(t, 0, y) = H_S(t, 0, y), & t > 0, y \in (0, 1) \\ p_J d_J^- \frac{\partial J_I}{\partial x}(t, 0, y) = (1 - p_J) d_J^+ \frac{\partial J_S}{\partial x}(t, 0, y), & t > 0, y \in (0, 1) \\ p_A d_A^- \frac{\partial A_I}{\partial x}(t, 0, y) = (1 - p_A) d_A^+ \frac{\partial A_S}{\partial x}(t, 0, y), & t > 0, y \in (0, 1) \\ p_H d_H^- \frac{\partial H_I}{\partial x}(t, 0, y) = (1 - p_H) d_H^+ \frac{\partial H_S}{\partial x}(t, 0, y), & t > 0, y \in (0, 1). \end{cases} \quad (0.410)$$

Conclusion

Our objective is twofold

- to reformulate the previous model as an abstract Cauchy problem.
- to solve the Dirichlet problem by applying functional analysis tools.



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Analysis and Dynamics of Tuberculosis (TB) with Vaccination and Treatment.

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Abstract. Tuberculosis remains a major global health challenge, especially in developing countries such as Pakistan. Despite being a preventable and curable disease, TB remains the world's leading infectious killer and a significant cause of morbidity and mortality. In this paper, we formulate a deterministic model to study the dynamics of TB. We develop the model without control variables and present its basic mathematical analysis. The stability of the model is analyzed at the disease-free equilibrium when $\mathcal{R}_0 < 1$, and it is further proven that the infection will persist uniformly when $\mathcal{R}_0 > 1$

Keywords: Tuberculosis epidemic, basic reproduction number, stability.



Model formulation

In the present section*, we formulate and discuss the proposed TB transmission model with vaccination and treatment. The model is formulated by dividing the total population denoted by $M(t)$ at any time t into five different epidemiological subclasses. The susceptible class is characterized as $S(t)$ and $V(t)$ stands for vaccinated class, while the latent is symbolized as $L(t)$. The individuals with active TB and under treatment are denoted by $I(t)$ and $R(t)$, respectively. Finally, $R(t)$ shows the people who recovered after proper treatment, so that

$$\begin{cases} \dot{S} = \Lambda - \frac{\beta(I+\rho R)S}{M} - (\mu + \nu)S + \nu_1 V, \\ \dot{V} = \nu S - \frac{\beta\rho_1(I+\rho R)S}{M} - (\mu + \nu_1)V, \\ \dot{L} = \frac{\beta(I+\rho R)S}{M} + \frac{\beta\rho_1(I+\rho R)S}{M} - (\mu + \varepsilon)L + (1 - \eta)\delta R, \\ \dot{I} = \varepsilon L + \eta\delta R - (\mu + \gamma + \sigma_1)I, \\ \dot{R} = \gamma I - (\mu + \delta + \sigma_2)R, \end{cases} \quad (1.1)$$

The associated initial conditions are

$$S(0) = S_0 \geq 0, V(0) = V_0 \geq 0, L(0) = L_0 \geq 0, I(0) = I_0 \geq 0, R(0) = R_0 \geq 0.$$

In system (1.1), Λ is the constant recruitment rate. ν represents the vaccination rate of susceptible individuals, and they move to the vaccination class. The parameter ν_1 denotes the loss of immunity, and the individuals become susceptible again. The susceptible people gain TB infection at the rate $\frac{\beta(I+\rho R)}{M}$ after a successful interaction with individuals capable of transmitting the infection. Parameter β is the disease transmission coefficient, and $\rho < 1$ is used for the reduction in infection among individuals with active TB who are under treatment. The disease-induced mortality rates for individuals in the compartments I and R are σ_1 and σ_2 , respectively. The latent individuals enter to TB-active class at the rate of ε , whereas the TB-active individuals move to treatment class at the rate of γ . It is assumed that the individuals in the exposed stage are not capable of transmitting the infection. The vaccinated individuals gain infection at the rate of $\frac{\beta\rho_1(I+\rho R)}{M}$, where $\rho_1 < 1$ represents a reduction in the risk of infection owing to vaccination. The parameter δ denotes incomplete treatment (or treatment failure) where the term $\eta\delta R$ denotes a fraction of treated individuals who leave treatment at early stage and they join the infected class I , whereas the term $(1 - \eta)\delta R$ represents the treated people who leave the treatment at a later stage and they enter the latent class.

Analysis of the model

*

Positivity and boundedness



Theorem 0.167. Let $T(0) \geq 0$ denote the initial condition, where $T := (S, V, L, I, R)$, then there exists a nonnegative solution of the TB model (1.1) for all $t > 0$. Moreover,

$$\lim_{t \rightarrow \infty} \sup M(t) \leq \frac{\Lambda}{\mu}$$

Theorem 0.168. For any non-negative initial conditions in $\mathbb{R}_+^6$, the feasible region given by $\Omega \subset \mathbb{R}_+^6$ is positively invariant for the TB transmission model (1.1).

The basic reproduction number

The mode (1.1) has a **DFE**, obtained by setting the right-hand sides of the equations in the model to zero, given by

$$E_0 = (S^0, V^0, 0, 0, 0),$$

where $S^0 = \frac{\Lambda(\mu+\nu)}{(\mu+\nu)(\mu+\nu_1)-\mu\nu}$, $V^0 = \frac{\Lambda\nu}{(\mu+\nu)(\mu+\nu_1)-\mu\nu}$. The linear stability of E_0 can be obtained using the next-generation operator method on

system (1.1). Using the notation in [1], the matrices F and V are, respectively, given by

$$F = \begin{pmatrix} 0 & \frac{\beta(S^0+\rho_1 V^0)}{S^0+V^0} & \frac{\beta\rho(S^0+\rho_1 V^0)}{S^0+V^0} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } V = \begin{pmatrix} T_1 & 0 & (\eta-1)\delta \\ -\varepsilon & T_2 & -\eta\delta \\ 0 & -\gamma & T_3 \end{pmatrix}$$

$$\text{where } T_1 = (\mu + \varepsilon), T_2 = (\mu + \sigma_1 + \gamma), T_3 = (\mu + \delta + \sigma_1),$$

Hence, using $\mathcal{R}_0 = r(FV^{-1})$, the expression for the basic reproduction number is obtained as

$$\mathcal{R}_0 = \frac{\varepsilon (\gamma (\rho_1 \rho \beta V^0 + \rho \beta S^0) + T_3 (\beta S^0 + \beta V^0 \rho_1))}{M_0 (T_1 (T_2 T_3 - \gamma \eta \delta) - \gamma \delta (1 - \eta) \varepsilon)}$$

Stability of disease-free equilibrium

Theorem 0.169. System (1.1) at E_0 is global asymptotic stability (GAS) for $\mathcal{R}_0 < 1$ and unstable for $\mathcal{R}_0 > 1$.

Proof Consider the following Lyapunov function:

$$H = \alpha_1 L + \alpha_2 I + \alpha_3 R,$$

■

Conclusion

A deterministic model for the transmission dynamics of TB is developed and analyzed. We demonstrate that the basic reproduction number characterizes



the disease transmission dynamics: if $\mathcal{R}_0 \leq 1$, only the disease-free equilibrium exists and is globally asymptotically stable. However, if $\mathcal{R}_0 > 1$, a disease endemic equilibrium arises.

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Exploring confidence bands for nonparametric relative error regression in survival analysis with functional covariables

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Abstract. Technological advancements have enabled the collection of high-resolution data that can be represented as functions (functional data). This numerical study investigates the asymptotic distribution of a nonparametric relative regression estimator for data subject to truncation and censoring, with predictors of functional type. We focus on building confidence intervals for predictions using some simulated data, showcasing the reliability and the practical aspect of the estimator.

Keywords: Functional regression, Functional data, Truncated data, Censored data, Relative error regression.



Introduction

The task in scalar on function regression, is to model the link between a real random variable denoted by Y and a functional explanatory variable $\mathcal{X}$ taking its values in $(\mathcal{F}, d)$, where $\mathcal{F}$ is a semi metric space and d its corresponding semi metric, such that $Y = r(\mathcal{X}) + \epsilon$, $r(\cdot)$ is a regression operator mapping $\mathcal{F}$ to $\mathbb{R}$ and ϵ some random noise. It is worth noting that in some practical cases, the variable of interest Y , can't be fully observed as in survival analysis, where the lifetime may be left truncated and/or right censored. Right censoring occurs when for some individuals, the event of interest is observed only if it happens prior to some specified times and these times vary from an individual to another. We have left truncation when only the individuals who experience the event of interest after a certain delayed entry time are included in the study. Usually r is estimated by minimizing the least squares criterion. The estimations obtained then are sensitive to outliers since it assumes that all the variables have an equal weight, which makes them inadequate in some situations with large outliers. A possible solution is the use of the following mean quadratic relative error as a loss function:

$$\mathbb{E} \left[\left(\frac{Y - r(\mathcal{X})}{Y} \right)^2 \mid \mathcal{X} \right], \text{ for } Y > 0. \quad (0.411)$$

It is to show that the minimizer of (0.411) is given by:

$$r(\chi) = \frac{\mathbb{E} [Y^{-1} \mid \mathcal{X} = \chi]}{\mathbb{E} [Y^{-2} \mid \mathcal{X} = \chi]} =: \frac{r_1(\chi)}{r_2(\chi)}, \quad (0.412)$$

where $r_1(\chi)$ and $r_2(\chi)$ are assumed finite almost surely.

Construction of the estimator and main result

Under this particular setting, one has to account for both left truncation and right censoring effect and by combining the ideas of [2] and [3], we propose the following mean relative error regression estimator:

$$\hat{r}_{RER}(\chi) = \frac{\sum_{i=1}^n \frac{\delta_i Z_i^{-1}}{L_n(Z_i)G_n(Z_i)} K\left(\frac{d(\chi, \mathcal{X}_i)}{h}\right)}{\sum_{i=1}^n \frac{\delta_i Z_i^{-2}}{L_n(Z_i)G_n(Z_i)} K\left(\frac{d(\chi, \mathcal{X}_i)}{h}\right)}.$$

More details on the construction of the estimator can be found in our published work (see [1]). In order to build confidence intervals for the estimates obtained using the proposed estimator we need to know its distribution.

Under some hypotheses, it can be shown that:

$$\sqrt{\frac{n\phi_\chi(h)}{\sigma(\chi)^2}} (\hat{r}_n(\chi) - r(\chi)) \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1) \text{ as } n \rightarrow +\infty,$$



where

$$\phi_{\chi}(h) = \mathbb{P}(d(\mathcal{X}, \chi) < h) \text{ and } \sigma^2(\chi) = \frac{(\mu_2(\chi) - 2r(\chi)\mu_3(\chi) + \mu_4(\chi)r^2(\chi))}{r_2^2(\chi)} \times \frac{\alpha\mathcal{M}_2}{\mathcal{M}_1^2},$$

with

$$\mathcal{M}_1 = K(1) - \int_0^1 (K(t))' \Gamma_{\chi}(t) dt, \mathcal{M}_2 = K^2(1) - \int_0^1 (K^2(t))' \Gamma_{\chi}(t) dt,$$

$$\mu_m(\chi) = \mathbb{E} \left[\frac{Y^{-m}}{L(Y)\overline{G}(Y)} \mid \mathcal{X} = \chi \right] \text{ for } m = 2, 3, 4.$$

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Estimation non paramétrique par la méthode des Sieves

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Abstract. We consider the estimation of the operator of a functional autoregressive process by the sieves method of Grenander in the case of dependent random variables framework. We give the sieve estimator of this parameter when the operator is an Hilbertian operator. In such case, one has to perform maximization over an approximating spaces (sieves) of the original parameter space and allows the dimension of the approximating space to grow as the sample size increases.

Keywords: Processus autoregressif, Estimation non paramétrique, Estimateur Sieves, Enigalité exponentielle.

Probabilités et Statistiques

Introduction

Quand le paramètre à estimer prend ses valeurs dans un espace de paramètre Θ de dimension infinie, la méthode du maximum de vraisemblance ne s'applique pas directement sans des modifications substantielles.

GRENANDER (1981) a introduit la méthode des Sieves pour remédier à l'estimation dans ce type de problèmes. L'approche de GRENANDER est basée



sur la maximisation de la vraisemblance sur les sousensembles Θ_m qui sont de dimension finie de telle sorte que le maximum de vraisemblance local existe. D'autre part, quand la dimension de Θ_m croît vers la taille de l'échantillon le but étant d'obtenir la convergence de l'estimateur de maximum de vraisemblance local sur Θ_m vers la vraie valeur du paramètre.

Soient $(\Omega, \mathcal{A}, \mathcal{P})$ un espace probabilisé et $(H, \mathcal{H}, \|\cdot\|)$ un espace de Hilbert séparable, $(\varepsilon_n, n \in \mathbb{Z})$ un H-bruit blanc et ρ une application mesurable dans $\mathcal{L}(H)$ avec ρ un opérateur linéaire dans l'espace H .

Considérons une suite de v.a. $(X_n, n \in \mathbb{Z})$ définie sur $(\Omega, \mathcal{A}, \mathcal{P})$ et à valeur dans H vérifiant la relation suivante:

$$X_n = \rho X_{n-1} + \varepsilon_n \quad (0.413)$$

Le processus $(X_n, n \in \mathbb{Z})$ est appelé processus auto-régressif hilbertien d'ordre 1 noté ARHA(1)

Plus précisément, Soit (Θ, d) un espace paramétrique muni d'une distance d . Soit $\{\Theta_m\}_m$ une famille de sous-ensembles de Θ . La famille $\{\Theta_m\}_m$ est appelée sieve si Θ_m est compact, $\{\Theta_m\}_m$ est une suite croissante et $\bigcup \Theta_m$ dense dans Θ . Et on appelle une suite d'estimateurs $\{\hat{\theta}_m\}$, estimateurs sieves correspondants à la sieve $\{\Theta_m\}$ si elle satisfait: $L_n(X^n, \hat{\theta}_m) = \sup_{\theta \in \Theta_m} L_n(X^n, \theta)$ avec L_n est la fonction de vraisemblance.

Dans cette étude on considère un processus autoregressif Hilbertien d'ordre 1 donné par l'équation (0.470) avec un opérateur ρ particulier.

Nous considérons un opérateur ρ admettant une décomposition spectrale et on montre que le maximum de vraisemblance est atteint pour une sieve particulière et nous donnons une forme explicite de l'estimateur sieve. Notre but est d'obtenir une convergence presque sûre de l'estimateur sieve de cet opérateur.

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Vertex–distinguishing threshold of graph

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Abstract. The vertex-distinguishing number of a graph G is the smallest integer d such that G has a vertex coloring with d colors that is preserved solely by the identity automorphism. The vertex distinguishing threshold is defined as the minimum number r such that every r -vertex coloring of G is distinguishing. In this paper, we investigate the vertex-distinguishing threshold for certain classes of graphs.

Keywords: Distinguishing coloring, Automorphism group of graph, Distinguishing number, Distinguishing threshold.

Introduction

Breaking symmetry in graphs was introduced by *Frank Rubin* [16] in the "Journal of Recreational Mathematics" in 1979 as "the blind man's keys problem," defined as follows:



"A blind professor wants to distinguish the keys on his key ring by using different handle shapes. How many shapes does the professor need to identify each key?"

Motivated by this problem, Albertson and Collins [3] proposed to model it as a graph: the keys are represented by vertices, and two vertices are adjacent if and only if the corresponding keys are adjacent on the key ring. The handle shapes are represented by the cycle $C_n = \{v_0, v_1, v_2, \dots, v_i, v_{i+1}, \dots, v_{n-1}, v_0\}$, where n is the number of keys.

To distinguish each key, Albertson et al. [3] proposed to assign to the vertices of C_n a vertex coloring called distinguishing coloring.

A vertex coloring $c : V \rightarrow \{1, 2, \dots, d\}$ is said to be d -distinguishing if non-trivial automorphisms of G preserve all of the vertex colors. Note that the coloring is not necessarily proper.

The distinguishing number of a graph G , denoted by $D(G)$ is defined as the smallest integer d such that there exists a vertex coloring of G using d colors that is distinguishing. It is shown in [3] that for cycle C_n , we have $D(C_n) = 3$ if $3 \leq n \leq 5$, and $D(C_n) = 2$ for all $n \geq 6$. Other results are presented in [3] for different classes of graphs, such as the path P_n of length n , where $D(P_n) = 2$ for $n \geq 2$. It is clear that $D(K_n) = n$. If $m > n$, we find that $D(K_{m,n}) = m$, and it is straightforward to see that $D(K_{n,n}) = n + 1$. Since $\text{Aut}(\overline{G}) = \text{Aut}(G)$, we can conclude that $D(\overline{G}) = D(G)$. Thus, we have $D(J_n) = D(K_n)$ where $J_n = \overline{K_n}$.

The distinguishing number has been extensively studied since its introduction, and many articles have focused on this invariant: trees [3], hypercubes [4], product graphs [13], and certain circulant graphs [7].

Several generalizations of distinguishing coloring have already been made. Among them is the concept of distinguishing graphs through edge coloring, which was first studied by Kalinowski and Pilsniak [12]. They defined the distinguishing index $D'(G)$ as the minimum number of colors required for an asymmetric edge coloring of a connected graph G that is not isomorphic to K_2 . Furthermore, they demonstrated that for a finite connected graph G with $|G| \geq 6$, it holds that $D'(G) \leq \Delta(G)$, and $D'(G) \leq D(G)$ when $|G| \geq 3$.

To precisely calculate the distinguishing number of the lexicographic product and X -join of graphs, Ahmadi, Alinaghpour, and Shekarriz [2] defined certain indices related to the concept of distinguishing graphs. They introduced $\phi_k(G)$ as the number of non-equivalent distinguishing vertex colorings of the graph G using at most k colors, and $\varphi_k(G)$ as the number of non-equivalent distinguishing vertex colorings of G using exactly k colors. They demonstrated that, in general, calculating $\phi_k(G)$ and $\varphi_k(G)$ is not straightforward. This led them to define the vertex distinguishing threshold [2], denoted $\theta(G)$, as the smallest number r of colors such that any arbitrary coloring of G with t colors is distinguishing for any $t \geq \theta(G)$.

The vertex distinguishing threshold seems to be a noteworthy index that has been investigated by Shekarriz et al. [17]. They established a connection



between the cycle structure of the graph's automorphisms and $\theta(G)$. Moreover, they demonstrated that if $|G| = 2$, then $\theta(G) = 2$. They also examined graphs for which $D(G) = \theta(G)$.

In what follow, we study the vertex–distinguishing threshold of some classes of graphs. We start our study by giving some definitions and fundamentals results then we give the main results.

Definitions and Preliminaries Results

Definition 0.73. [2] Let c_1 and c_2 be two vertex colorings of a graph G . The colorings c_1 and c_2 are considered equivalent if there exists an automorphism $\alpha \in \text{Aut}(G)$ such that $c_1(v) = c_2(\alpha(v))$ for every vertex $v \in V(G)$.

Let $\phi_k(G)$ the number of non–equivalent distinguishing coloring of the graph G with at most k colors, and let $\varphi_k(G)$ the number of non equivalent distinguishing coloring of the graph G with $\{1, 2, \dots, k\}$ the set of colors. The relationship between these indices is given in [2]:

$$\phi_k(G) = \sum_{i=D(G)}^k \binom{i}{k} \varphi_k(G)$$

They demonstrated that for $n \geq 1$ and $k \geq 1$, $\phi_k(P_n) = \frac{1}{2}(k^n - k^{\lceil \frac{n}{2} \rceil})$ and $\phi_k(K_n) = \binom{n}{k}$ when $n \geq 2$ and $k \geq n$.

In general, calculating and determining the values of these indices is not straightforward. To address this, Ahmadi et al. [2] introduced the concept of the distinguishing threshold $\theta(G)$, defined as the smallest integer r such that for any $t \geq \theta(G)$, any arbitrary coloring of G with t colors is distinguishing.

Furthermore, for every graph G , we have $\varphi_k(G) = \frac{k! \cdot nk}{|\text{Aut}(G)|}$ for any $k \geq \theta(G)$, where nk represents the Stirling number of the second kind.

As previously discussed in the introduction, an alternative definition is provided by Shekarriz et al. [17], which utilizes the cycle structure of the elements of the automorphism group of a graph G . Following the notation established by Shekarriz et al. [17], we denote the number of cycles of a nontrivial automorphism α by $c(\alpha)$, while $c(id) = 0$. To distinguish α , at least one nontrivial cycle must be colored with at least two different colors. By applying the pigeonhole principle, we obtain:

Definition 0.74. [17] For any graph G : $\theta(G) = \max\{c(\alpha) : \alpha \in \text{Aut}(G)\} + 1$.



Main results

For any graph G , the distinguishing threshold $\theta(G)$ is the minimum number t such that for any $k \geq t$, any arbitrary coloring of G with k colors is distinguishing. For example $\theta(K_n) = \theta(\overline{K}_n) = n$ and $\theta(K_{n,m}) = n + m$. Note, moreover, that for an asymmetric graph G , we have $\theta(G) = D(G) = 1$. Also, we always have $\theta(G) \geq D(G)$.

The distinguishing threshold can be determined for paths and cycles as follow:

Proposition 0.25. *For any $n \geq 2$ we have: $\theta(P_n) = \lceil \frac{n}{2} \rceil + 1$ and for any $n \geq 3$ we have: $\theta(C_n) = \lfloor \frac{n}{2} \rfloor + 2$.*

Theorem 0.170. $\theta(G) = 2$ if and only if $G \cong K_2$ or $G \cong \overline{K}_2$.

We note that for any graph G , the distinguishing threshold is constrained by $D(G) \leq \theta(G) \leq |V(G)|$. The upper bound is due to the fact that any coloring that utilizes $|V(G)|$ colors is trivially distinguishing. In the following discussion, we will investigate the graphs for which $\theta(G) = D(G)$.

Theorem 0.171. *If for a graph G , we have $\theta(G) = D(G)$, then either $\theta(G) = 1$ or $\theta(G) = |V(G)|$*

We can extend this result as follows:

Theorem 0.172. *For a graph G on n vertices, we have $\theta(G) = D(G)$ if and only if $|Aut(G)| = 1$, $G \cong K_n$ or $G \cong \overline{K}_n$.*

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Application de l'algorithme MCEM sur la dynamique du VIH via le modèle espace d'état à effets mixtes.

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Abstract. Dans ce travail, nous avons appliqué l'algorithme MCEM que nous avons proposé précédemment sur un jeu de données longitudinales de la charge virale chez un nombre de patients atteints du VIH afin d'estimer le modèle Espace d'états à effets mixtes proposé par Liu et al.(2011). Les résultats obtenus montrent une bonne qualité d'ajustement et confirment la performance du modèle. Notre méthode a beaucoup d'avantages par rapport aux méthodes proposés étant donné qu'elle ne requiert aucune restriction sur le modèle à condition qu'il existe d'une loi à postériori de l'effet aléatoire calculée de manière récursive.

Mots clés: Modèle Espace D'états, effets mixtes, Filtre particulaire, VIH.

Classification: 65C05; 65C35; 62L12



Introduction

L'analyse des données longitudinales, aussi appelées données répétées, constitue un domaine important avec des spécificités propres. Ces données proviennent d'un grand nombre d'individus et incluent plusieurs observations par individu sur une période donnée. Elles sont largement utilisées dans les sciences biomédicales, comme illustré par l'exemple de l'essai clinique de Lavielle (2014). Deux types de modèles sont couramment employés pour leur analyse : les modèles à effets fixes et à effets aléatoires. Les premiers capturent la dynamique globale de la population en considérant tous les individus, tandis que les seconds intègrent les variations individuelles en ajoutant une composante aléatoire au modèle. La combinaison de ces deux approches donne lieu aux modèles à effets mixtes. Liu et al. (2011) ont proposé de combiner les approches des modèles espace d'état et des effets mixtes pour étudier la dynamique du VIH. Leur modèle, appelé Modèle Espace d'État à Effets Mixtes (MESS). Ce modèle peut être adapté à différents contextes en incorporant des covariables explicatives supplémentaires. Diverses méthodes d'estimation des paramètres du modèle MESS ont été proposées. Cependant, leur efficacité dépend souvent de l'application de contraintes spécifiques au modèle. Motivée par des études sur la dynamique du VIH,

Définition du modèle

Le modèle MESS est défini selon les expressions suivantes :

$$\begin{cases} x_{i,t+1} = F(\theta_i)x_{i,t} + v_{i,t}, & v_{i,t} \sim \mathcal{N}(0, Q), \\ y_{i,t} = G(\theta_i)x_{i,t} + w_{i,t}, & w_{i,t} \sim \mathcal{N}(0, R), \\ \theta_i = \theta + b_i, & b_i \sim \mathcal{N}(0, D), \end{cases}$$

où $y_{i,t}$ est un vecteur de dimension $q \times 1$ des variables observées et i et t ($i = 1, \dots, m$ et $t = 1, \dots, n_i$) représentent les dimensions individuelles et temporelles respectivement. $x_{i,t}$ est un vecteur de dimension $p \times 1$ des variables d'état. $v_{i,t}$ et $w_{i,t}$ sont les termes d'erreurs associés à l'équation d'état et à l'équation d'observation, respectivement, tels que pour tout $t, h \in \mathbb{Z}$, $\mathbb{E}(v_{i,t}v'_{i,t+h}) = \delta_{h,0}Q$, $\mathbb{E}(w_{i,t}w'_{i,t+h}) = \delta_{h,0}R$. Les erreurs $v_{i,t}$ et $w_{i,t}$ ne sont pas corrélés entre elles et les individus sont supposés indépendants. θ_i est le vecteur des paramètres des effets mixtes de taille $r \times 1$. Il contient à la fois des effets fixes, θ , que nous voulons estimer et supprimer, et des effets aléatoires, b_i , qui contribuent à la variabilité. $F(\theta_i)$ est la matrice de transition de taille $p \times p$ et $G(\theta_i)$ est la matrice de l'équation observable de taille $q \times p$. Ces matrices sont en fonction du vecteur des paramètres θ_i . Enfin, $x_{i,0}$ les états initiaux pour chaque individu sont supposés gaussiens de moyenne τ et de matrice de variance-covariance A .



Résultats

Nous avons appliqué le modèle espace d'état à effets mixtes à un ensemble de données provenant d'un essai clinique sur le VIH afin d'évaluer la performance de la méthode d'estimation que nous avons proposée Hamdi et Lellou(2024). La méthode MCEM est une combinaison de l'algorithme EM et le filtrage et le lissage particulière(Boussaha et Hamdi, 2018, Storvik, 2002 Doucet et Johansen, 2009).L'ensemble de données provient du protocole 315 du Groupe d'essais cliniques sur (ACTG) (voir Wu et Ding, 1999). Dans ce protocole, 53 patients infectés par le VIH ont été traités avec des médicaments antiviraux puissants (ritonavir, 3TC et AZT). Parmi les 53 patients traités, 5 ont abandonné le traitement. Le traitement a commencé le jour 0 et la concentration plasmatique de l'ARN du VIH-1 a été mesurée à plusieurs reprises les jours 0, 2, 7, 10, 14, 21, 28 et les semaines 8, 12, 24, 48 après le début du traitement. L'application numérique confirme la performance de la méthode et sa capacité de gérer le cas de présence des données manquantes. Les représentations graphiques montrent une bonne qualité d'ajustement. Les observations se trouvent en majorité à l'intérieur des intervalles de prédiction.

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Autoencoders-Based Feature Extraction for Random Forest Classifier

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Abstract. In the realm of predictive modeling, the challenge of high-dimensional data can significantly hinder the performance of classification algorithms. This study explores the integration of Autoencoders and Random Forest classifiers to enhance classification tasks. Autoencoders serve as a powerful tool for dimensionality reduction, extracting essential features while mitigating noise from complex datasets. Subsequently, a Random Forest model is trained on these transformed features to improve classification accuracy. The results are compared against several baseline models, including Naive Bayes, K-Nearest Neighbors (KNN), Support Vector Machine (SVM), and Logistic Regression. This comparative analysis demonstrates the effectiveness of the Autoencoder-Random Forest framework, emphasizing its potential to improve predictive modeling performance.

Keywords: Classification, Autoencoders ,Random Forest,Performance .



Introduction

In recent years, the fields of machine learning and deep learning have experienced unprecedented growth and transformation, becoming pivotal in solving complex problems across various domains. Machine learning, a subset of artificial intelligence, enables systems to learn from data, identify patterns, and make decisions without explicit programming. Techniques such as supervised learning, unsupervised learning, and reinforcement learning have empowered machines to tackle diverse tasks, ranging from image recognition and natural language processing to fraud detection and medical diagnosis.

Deep learning, a specialized area within machine learning, utilizes multi-layered neural networks to model intricate relationships in data. With its ability to process large amounts of unstructured data, deep learning has revolutionized fields like computer vision and speech recognition. The use of deep learning algorithms, such as convolutional neural networks (CNNs) and recurrent neural networks (RNNs), allows for automatic feature extraction and enhanced performance in tasks that were once deemed challenging for traditional machine learning approaches.

The integration of machine learning and deep learning has led to the development of sophisticated frameworks that harness the strengths of both methodologies. By combining the interpretability and efficiency of machine learning models with the representational power of deep learning architectures, researchers and practitioners can create more robust solutions. This synergy is particularly evident in handling high-dimensional data, where traditional models often struggle with the curse of dimensionality.

One promising approach to improving predictive modeling is the use of Autoencoders for dimensionality reduction followed by classification with Random Forests. Autoencoders, a type of unsupervised neural network, compress input data into a lower-dimensional latent space, enabling the extraction of essential features while reducing noise. This process enhances the quality of the input data for subsequent classification tasks, effectively addressing the challenges posed by high-dimensional datasets.

Random Forests, an ensemble learning technique, excel in classification tasks by aggregating the predictions of multiple decision trees. This robustness and resilience against overfitting make Random Forests an ideal choice for classification, especially when paired with the rich features derived from Autoencoders. The combination of these two powerful methodologies creates a framework that leverages the strengths of both, leading to improved classification performance and the potential to advance predictive modeling across various applications.



In this study, we investigate the effectiveness of integrating Autoencoders for dimensionality reduction with Random Forest classifiers for classification tasks. By comparing the performance of this combined approach against baseline models, including Naive Bayes, K-Nearest Neighbors (KNN), Support Vector Machine (SVM), and Logistic Regression, we aim to highlight the advantages of the Autoencoder-Random Forest framework and its potential for enhancing predictive modeling in Classification task.

Main results

*

Autoencoders

Autoencoders are a type of artificial neural network designed for unsupervised learning. Their main goal is to efficiently encode input data into a lower-dimensional latent representation and then reconstruct the original input from that representation. The architecture consists of two key components: the encoder and the decoder.

*

*1. Encoder The encoder function f maps the input data $\mathbf{x}$ to a latent representation $\mathbf{z}$:

$$\mathbf{z} = f(\mathbf{x}) = \sigma(\mathbf{W}\mathbf{x} + \mathbf{b})$$

where:

- $\mathbf{W}$: weight matrix of the encoder
- $\mathbf{b}$: bias vector of the encoder
- σ : activation function

*

*2. Decoder The decoder function g reconstructs the input from the latent representation:

$$\hat{\mathbf{x}} = g(\mathbf{z}) = \sigma'(\mathbf{W}'\mathbf{z} + \mathbf{b}')$$

where:

- $\mathbf{W}'$: weight matrix of the decoder
- $\mathbf{b}'$: bias vector of the decoder
- σ' : activation function of the decoder



*

*3. **Training Objective** The autoencoder is trained to minimize the reconstruction error between the original input $\mathbf{x}$ and the reconstructed output $\hat{\mathbf{x}}$. This is typically achieved using the mean squared error (MSE) loss function:

$$\mathcal{L}(\mathbf{x}, \hat{\mathbf{x}}) = \frac{1}{n} \sum_{i=1}^n (\mathbf{x}_i - \hat{\mathbf{x}}_i)^2$$

where n is the number of input samples.

The dimensionality reduction achieved by autoencoders can help mitigate the curse of dimensionality and enhance the performance of subsequent classification tasks.

*

Random Forests

Random Forests are an ensemble learning technique commonly used for classification and regression tasks. They build multiple decision trees during training and combine their outputs to improve model accuracy and robustness against overfitting.

*

*1. **Bootstrap Sampling** Randomly sample n instances from the training dataset with replacement to create multiple subsets known as bootstrap samples.

*

*2. **Tree Construction** For each bootstrap sample, a decision tree is constructed. At each node, a random subset of features is selected, and the best feature is chosen to split the data based on a specific criterion (e.g., Gini impurity G or entropy H):

- **Gini Impurity:**

$$G = 1 - \sum_{k=1}^K p_k^2$$

where p_k is the probability of class k .

- **Entropy:**

$$H = - \sum_{k=1}^K p_k \log(p_k)$$



*

*3. Prediction For a given input instance $\mathbf{x}$, each decision tree f_i in the forest makes a prediction. The final output $\hat{y}$ of the Random Forest model is obtained by aggregating predictions from all the trees:

$$\hat{y} = \text{mode}(f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_m(\mathbf{x}))$$

where m is the total number of trees in the forest. For classification, the mode (most common class) is selected, while for regression, the average of the predictions is computed.

This ensemble method enhances generalization and model performance on unseen data, making Random Forests a powerful tool for various predictive tasks.

Methodology

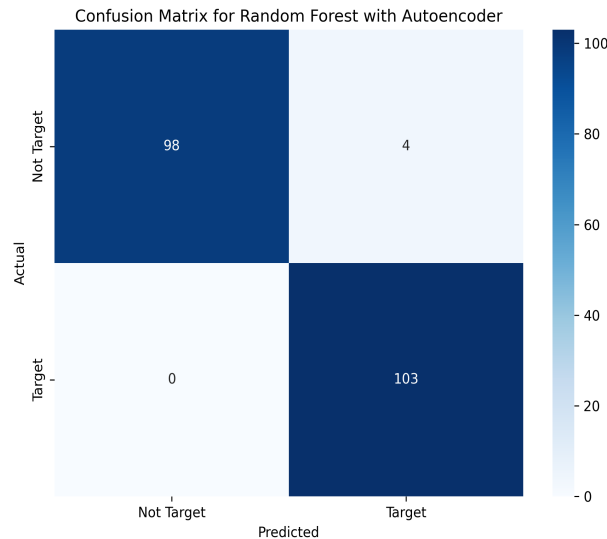
The dataset used in this study is the Heart Disease dataset, downloaded from Kaggle, which comprises various features that contribute to predicting the presence of heart disease. The dataset is preprocessed as follows:

- Encoding Categorical Variables: Categorical variables are encoded using the Label Encoder to convert them into numerical format, facilitating their use in machine learning models.
- Data Splitting: The dataset is split into training and testing sets using an 80-20 ratio, ensuring the model can be evaluated on unseen data.
- Feature Scaling: The features are scaled using the StandardScaler to normalize the data, which helps improve the convergence of models and ensures that all features contribute equally to the model.
- Training Models : Various classification models, including Naive Bayes, K-Nearest Neighbors (KNN), Support Vector Machine (SVM), and Logistic Regression, are trained on the scaled training data. Their performance is evaluated using accuracy, precision, recall, and F1 score.
- Autoencoder Model Building: An Autoencoder model is constructed with an input layer and two main components: the encoder and the decoder. The model is trained to minimize the reconstruction loss.
- Data Transformation: After training, the encoder is used to transform the original features into a lower-dimensional representation.
- Random Forest Training: Finally, a Random Forest classifier is trained using the transformed features from the Autoencoder. The model's performance is evaluated using the same metrics as before, allowing for a comparative analysis against the baseline models.



Results

- The integration of Autoencoders with Random Forest resulted in outstanding performance, achieving an accuracy of 98.05% alongside precision, recall, and F1 scores of 0.9812 and 0.9805. To illustrate this model's classification effectiveness, we present the confusion matrix, which details the model's predictions and highlights its accuracy in classifying the target classes.



- The results of the comparative analysis of various machine learning models, including the integration of Autoencoder and Random Forest, are presented in Table 1. The table highlights the performance metrics for each model, including accuracy, precision, recall, and F1 score.

The Random Forest model combined with autoencoders demonstrated exceptional performance, achieving an accuracy of 0.9805, precision of 0.9812, recall of 0.9805, and an F1 score of 0.9805. The high metrics indicate that this model effectively balances precision and recall, minimizing both false positives and false negatives.

In contrast, Naive Bayes, while effective given its simplicity and the independence assumption between features, yielded lower performance with an accuracy of 0.8000, precision of 0.8105, recall of 0.8000, and an F1 score of 0.7982. Its simplistic approach limited its ability to capture the complex patterns present in the data, placing it behind the Random Forest with Autoencoder.



K-Nearest Neighbors (KNN) achieved an accuracy of 0.8341, with precision and recall both at 0.8387 and an F1 score of 0.8335. Although KNN performed better than Naive Bayes due to its lazy learning approach, it struggled with higher-dimensional data and was outperformed by the enhanced Random Forest model.

Support Vector Machines (SVM) showed strong performance with an accuracy of 0.8878, precision of 0.8923, recall of 0.8878, and an F1 score of 0.8875. While SVM effectively created complex decision boundaries, it could not match the performance of the Random Forest with Autoencoder, likely due to its sensitivity to feature scaling and the intricacies of the kernel used.

Logistic Regression, despite its straightforward nature, recorded an accuracy of 0.7951, with precision and recall at 0.8023 and an F1 score of 0.7938. This model is effective for linear relationships but falls short in capturing more complex patterns, leading to lower performance compared to the Random Forest with Autoencoder.

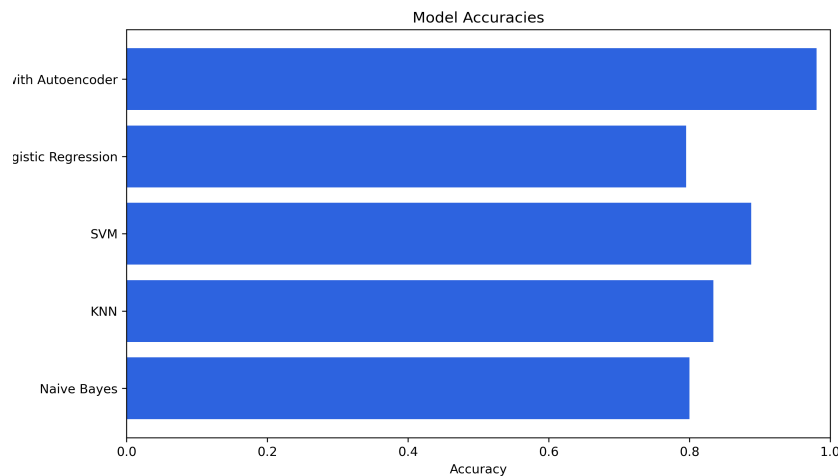
Decision Trees achieved an accuracy of 0.8200, with precision, recall, and F1 scores of 0.8250, 0.8200, and 0.8195, respectively. While Decision Trees provide decent performance, they are prone to overfitting. The Random Forest model, as an ensemble of multiple decision trees, inherently offers better generalization capabilities.

Model	Accuracy	Precision	Recall	F1 Score
Naive Bayes	0.8000	0.8105	0.8000	0.7982
KNN	0.8341	0.8387	0.8341	0.8335
SVM	0.8878	0.8923	0.8878	0.8875
Logistic Regression	0.7951	0.8023	0.7951	0.7938
Random Forest with Autoencoder	0.9805	0.9812	0.9805	0.9805

Table 9: Model Evaluation Metrics

To provide a visual comparison of the model performances, we present a plot illustrating the accuracy of each model evaluated in this study.





Conclusion

This study effectively demonstrates the integration of Autoencoders for dimensionality reduction and Random Forest classifiers for predictive modeling. By employing Autoencoders, we can extract essential features from high-dimensional datasets, thereby reducing noise and improving the quality of the input data for classification tasks. The results indicate a significant performance enhancement with the Autoencoder-Random Forest framework, highlighting the power of combining deep learning techniques with traditional machine learning algorithms.

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QUADRINOMIAL-LIKE VERSIONS FOR WOLSTENHOLME AND MORLEY CONGRUENCES

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Abstract. The quadrinomial coefficient is defined as the coefficient of x^k in the polynomial expansion of $(1 + x + x^2 + x^3)^n$, where n and k are nonnegative integers. In the present paper, we derive some congruences involving the quadrinomial coefficients. For instance, we establish two congruences that are analogous to those of Morley and Wolstenholme.

Keywords: Wolstenholme's congruence; quadrinomial coefficient; Euler numbers; Fermat quotient; Pell sequence.

Introduction

Let $s \geq 1, n \geq 0$ and k be integers. The bi^snomial coefficient denoted by $\binom{n}{k}_s$ is defined as

$$\binom{n}{k}_s := \begin{cases} [t^k] (1 + t + \cdots + t^s)^n, & \text{for } 0 \leq k \leq sn \\ 0, & \text{for } k < 0 \text{ or } k > sn \end{cases} \quad (0.414)$$

where $[t^k] f(t)$ denotes the coefficient of t^k in the formal power series $f(t)$; see Belbachir et al. [1]. The study of bisnomial coefficients dates back to de Moivre [2] and Euler [3]. Combinatorially, the bi^snomial coefficient $\binom{n}{k}_s$ counts the number of different ways of distributing k objects among n cells where each cell contains at most s objects [1].

For $s = 1$, one obtains the binomial coefficient $\binom{n}{k}_1 = \binom{n}{k}$.

Some known properties of the bis nomial coefficients are as follows:



- Symmetry relation

$$\binom{n}{k}_s = \binom{n}{sn-k}_s,$$

- De Moivre alternating summation

$$\binom{n}{k}_s = \sum_{i=0}^s \binom{n-1}{k-i}_s,$$

- The explicit form of the bis nomial coefficient in term of the binomial coefficients

$$\binom{n}{k}_s = \sum_{j_1+j_2+\dots+j_s=k} \binom{n}{j_1} \binom{j_1}{j_2} \dots \binom{j_{s-1}}{j_s}.$$

Throughout the paper we consider p as an odd prime number.
In 1819, Babbage [2] established that

$$\frac{1}{2} \binom{2p}{p} = \binom{2p-1}{p-1} \equiv 1 \pmod{p^2}, \quad (0.415)$$

and in 1862 Wolstenholme [3] showed the same result modulo p^3 , for $p \geq 5$.
In 1895, Morley [4] showed that, for $p \geq 5$, it holds that

$$\binom{p-1}{(p-1)/2} \equiv (-1)^{(p-1)/2} 4^{p-1} \pmod{p^3}. \quad (0.416)$$

For $s = 3$ in (0.414), we get the quadrinomial coefficients. For a non negative integer n , the central quadrinomial coefficient κ_n is the coefficient of x^{3n} in the expansion of $(1+x+x^2+x^3)^{2n}$.

Main results

Motivated by the previous results, we study some congruence properties of quadrinomial coefficients. Our focus lies on those involving central quadrinomial coefficients.

We introduce our first congruence which is similar to the Wolstenholme congruence.

Theorem 0.173. *Let $p \geq 5$. Then*

$$\kappa_p \equiv 4 + 8p^2 \left(\frac{-1}{p} \right) E_{p-3} \pmod{p^3}, \quad (0.417)$$

where E_n is the n -th Euler number.



The second congruence is an analogue to the Morley congruence (2) modulo squares of primes.

Theorem 0.174. *Let $p \geq 5$. Then*

$$\kappa_{(p-1)/2} \equiv \left(\frac{-2}{p}\right) + p \left(q_p(2) \left(\frac{7}{2} \left(\frac{-2}{p} \right) - 3 \left(\frac{-1}{p} \right) \right) - 2 \left(\frac{2}{p} \right) A_p \right) \pmod{p^2} \quad (0.418)$$

where

$$A_p := \frac{(-1)^{(p-1)/2} P_p - (-8)^{(p-1)/2}}{p}$$

in which $(P_n)_n$ is the Pell sequence.

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Regional Controllability of Nonlinear Cellular Automata using a Preimage generation Algorithm

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Abstract.

Controllability is a fundamental concept in control theory, describing the ability of an external input, known as control inputs, in order to steer a dynamical system from any initial state to any desired final state within a finite time. The focus of this paper lies in exploring the regional controllability issue within a numerical framework for a specific class of spatially-distributed dynamic systems, which are characterized using cellular automata (CA) models. In CA, control is defined as a forcing function denoted as $u(t)$, where t represents discrete time intervals within a set of all possible inputs U . This control function operates at each time step based on a predefined objective. This research presents an innovative approach to examining the problem of regional controllability in one-dimensional nonlinear elementary CA, in which their global dynamics is unpredictable and irregular. To address this challenge, a preimages algorithm has been devised to decide whether a configuration is reachable from another configuration by applying boundary controls. The study aims to introduce a novel method for generating preimages utilizing a new characterization tool called a Tree.

Keywords: Elementary Cellular Automata, Regional Controllability, Tree.



Introduction

Control theory focuses on the behavior of dynamical systems, with one of its key challenges being controllability, a fundamental property for understanding a system's behavior. This paper centers on the concept of regional controllability, a specific form of controllability. Regional controllability refers to the ability to guide a system to a desired state within a particular part of its domain, using controls applied either at the boundary or within the region of interest. This problem has been extensively studied in the context of distributed parameter systems, which are modeled by partial differential equations involving both time and space variables. Additionally, regional controllability has been explored through cellular automata (CA), which are discrete dynamical systems [1][2][3]. CA can be categorized based on their neighborhood configuration, update rules, and behavior. In this work, We present the algorithm used to check whether the system is regionally controllable using a tool called a Tree.

Main results

Definition 0.75. An elementary cellular automaton is a collection of ones and zeros arranged on an one-dimensional strip. Each cell interacts with its immediate neighbors on its left and right.

Definition 0.76. A cellular automaton is said to be regionally controllable for ω at time T if there exists a control sequence $u = (u_i(c_0), u_i(c_{n+1})), i = 0, \dots, T-1$ such as

$$s(T) = s_d \quad \text{on} \quad \omega$$

Where s_T is the final configuration at time T and s_d is the desired configuration.

Definition 0.77. The Hamming distance (H) between two sequences of bits having the same length is the number of positions where the bits differ.

Example 0.7. The rules 150 and 22 correspond to the output tables 10010110 and 00010110, resp. In this case, their Hamming distance is equal to 1; in this case this distance represents the degree of non-linearity.

*

Principle of the algorithm for generating the preimages: The idea to check whether a desired configuration is reachable starting from an initial configuration consists of generating all preimages of the controlled region of the nonlinear rule that have a hammming distance equal to 1 with respect to the nearset linear rule, see Figure (27).



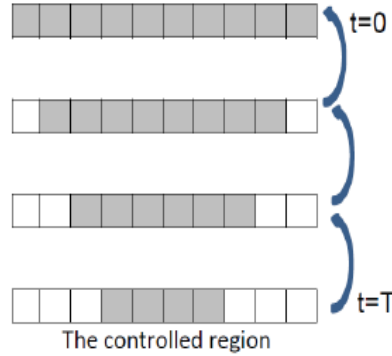


Figure 27: Preimages

The state of the controlled region $\omega^T = \{c_{posmin}, \dots, c_{posmax}\}$ at time T depends of the state of the region $\omega^{T-1} = \{c_{posmin-1}, \dots, c_{posmax+1}\}$ at time $T-1$ and so on. For that we give an algorithm that computes the preimages of the controlled region and construct the tree.

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Generalized tempered ultradistributions

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Abstract. The aim of this communication is to introduce and to study a new classes of generalized functions of Colombeau type containing the space of tempered ultradistributions for this we are going to recall the definition of Simplified Colombeau algebra and the embedding of space of distributions in this algebra, then give the definition of the space of ultradistribution, finely we give the definition of the algebra of tempered ultradistribution.

Keywords: colombeau generalized functions, generalized ultradistribution, Roumieu ultradistributions, tempered ultradistribution.



COLOMBEAU'S THEORY OF GENERALIZED FUNCTIONS

*

Introduction

Multiplication problem example

$$\delta = \delta \left(x V_p \frac{1}{x} \right) \neq (\delta x) V_p \frac{1}{x} = 0$$

*

Construction of such algebra

We want to construct a commutative, associative algebra $(\mathcal{A}(\Omega), +, \circ)$ satisfy:

- (i) $\mathcal{D}'(\Omega)$ is linearly embedded into $\mathcal{A}(\Omega)$ and $f(x) \equiv 1$ is the unity in $\mathcal{A}(\Omega)$.
- (ii) There exist derivation operators $\partial_i : \mathcal{A}(\Omega) \rightarrow \mathcal{A}(\Omega)$, that are linear and satisfy the Leibniz rule.
- (iii) $\partial_i / \mathcal{D}'(\Omega)$ is the usual partial derivative
- (iv) $\circ /_{C(\Omega) \times C(\Omega)}$ coincides with the pointwise product of functions.

Theorem 0.175. *There is no associative, commutative algebra $(\mathcal{A}(\Omega), +, \circ)$ satisfying (i) – (iii) and (iv) .*

(iv)' $\circ /_{C^\infty(\Omega) \times C^\infty(\Omega)}$ coincides with the pointwise product of functions.

Colombeau algebras are constructed to satisfy the conditions (i) – (iii). and condition (iv) , replaced by (iv)' they keep only the product of smooth functions (infinitely differentiable).

Definition 0.7(8) we define the space of moderate elements , denoted $\mathcal{E}_m(\Omega)$, by:

$$\mathcal{E}_m(\Omega) = \left\{ \begin{array}{l} (u_\varepsilon)_\varepsilon \in C^\infty(\Omega)^I; \forall K \Subset \Omega; \forall \alpha \in \mathbb{Z}_+^n; \exists m > 0, \\ \exists c > 0, \exists \varepsilon_0 \in I; \forall \varepsilon \leq \varepsilon_0; \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| \leq c \varepsilon^{-m} \end{array} \right\} \quad (0.419)$$

(ii) we define The space of null elements, denoted $\mathcal{N}(\Omega)$ by:

$$\mathcal{N}(\Omega) = \left\{ \begin{array}{l} (u_\varepsilon)_\varepsilon \in C^\infty(\Omega)^I; \forall K \Subset \Omega; \forall \alpha \in \mathbb{Z}_+^n; \forall q > 0, \exists c > 0, \\ \exists \varepsilon_0 \in I; \forall \varepsilon \leq \varepsilon_0; \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| \leq c \varepsilon^q \end{array} \right\} \quad (0.420)$$

Proposition 0.2(6) *The space of moderate elements $\mathcal{E}_m$ is an algebra stable by derivation.*



(2) The space $\mathcal{N}$ is an ideal of $\mathcal{E}_m$

Definition 0.79. The simplified Colombeau algebra of generalized functions on Ω is defined as

$$\mathcal{G}(\Omega) = \mathcal{E}_m(\Omega) / \mathcal{N}(\Omega)$$

The space $C^\infty(\Omega)$ is canonically embedded into $\mathcal{G}(\Omega)$ by the map

$$\begin{aligned} i : C^\infty(\Omega) &\rightarrow \mathcal{G}(\Omega) \\ f &\rightarrow (f_\varepsilon)_\varepsilon + \mathcal{N}(\Omega) \end{aligned} \quad (0.421)$$

Where $f_\varepsilon = f$, $\forall \varepsilon \in I$.

Theorem 0.176. The map

$$\begin{aligned} i_A : \mathcal{D}'(\Omega) &\rightarrow \mathcal{G}(\Omega) \\ T &\rightarrow [T] = cl((T * \phi_\varepsilon)_\varepsilon / \Omega) \end{aligned} \quad (0.422)$$

is an embedding.

the space $C^\infty(\Omega)$ can be embedded into $\mathcal{G}(\Omega)$ by two ways, shown by following commutative diagram

$$\begin{array}{ccc} C^\infty(\Omega) & \rightarrow & \mathcal{D}'(\Omega) \\ & \searrow i & \downarrow i_A \\ & & \mathcal{G}(\Omega) \end{array}$$

Generalized Roumieu ultradistributions

*

Roumieu ultradistributions

let $(Mp)_{p \in \mathbb{Z}_+}$ be sequences of positive numbers, recall the following properties

(H1) Logarithmic convexity :

$$M_p^2 \leq M_{p-1} M_{p+1}$$

(H2) Stability under ultradifferentiation :

$$\exists A > 0; \exists H > 0; M_{p+q} \leq A H^{p+q} M_p M_q; \forall p \geq 0; \forall q \geq 0$$

(H2)' Stability under differentiation :

$$\exists A > 0; \exists H > 0; M_{p+1} \leq A H^{p+1} M_p; \forall p \geq 0;$$

(H3)' Non-quasi-analyticity :

$$\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < +\infty$$



Definition 0.80. The associated function of the sequence $(M_p)_{p \in \mathbb{Z}_+}$ is the function defined by

$$M(t) = \sup_P \log \frac{t^P}{M_P}; t \in \mathbb{R}_+^*$$

Definition 0.81. The class of ultradifferentiable functions of class M , denoted $E^M(\Omega)$, is the space of all $f \in C^\infty(\Omega)$ satisfying for every compact subset K of Ω , $\exists c > 0$, $\forall \alpha \in \mathbb{Z}_+^n$,

$$\sup_{x \in K} |\partial^\alpha f(x)| \leq c^{|\alpha|+1} M_{|\alpha|} \quad (0.423)$$

This space is also called the space of Donjoy-Carleman.

A differential operator of infinite order $P(D) = \sum_{\gamma \in \mathbb{Z}_+^n} a_\gamma D^\gamma$ is called an ultradifferential operator of class $(M_p)_{p \in \mathbb{Z}_+}$, if for every $h > 0$ there exist $c > 0$ such that $\forall \gamma \in \mathbb{Z}_+^n$,

$$|a_\gamma| \leq c \frac{k^{|\gamma|}}{M_{|\gamma|}} \quad (0.424)$$

The basic properties of the space $E^M(\Omega)$ are summarized in the following proposition.

Let the sequence $(M_p)_{p \in \mathbb{Z}_+}$ satisfy condition $(H1)$, then the space $E^M(\Omega)$ is an algebra

moreover, if $(M_p)_{p \in \mathbb{Z}_+}$ satisfies $(H2)'$, then $E^M(\Omega)$ is stable by differential operators of finite order with coefficients in $E^M(\Omega)$.

The space $\mathcal{D}^M(\Omega) = E^M(\Omega) \cap \mathcal{D}(\Omega)$ is not trivial if and only if the sequence $(M_p)_{p \in \mathbb{Z}_+}$ satisfies $(H3)'$.

Definition 0.82. The strong dual of $\mathcal{D}^M(\Omega)$, denoted $\mathcal{D}'^M(\Omega)$, is called the space of Roumieu ultradistributions.

The space of moderate elements, denoted $\mathcal{E}_m^M(\Omega)$, is the space of $(f_\varepsilon)_\varepsilon \in C^\infty(\Omega)^I$ satisfying for every compact K of Ω , $\forall \alpha \in \mathbb{Z}_+^n$, $\exists k > 0$, $\exists c > 0$, $\exists \varepsilon_0 \in I$, $\forall \varepsilon \leq \varepsilon_0$,

$$\sup_{x \in K} |\partial^\alpha f_\varepsilon(x)| \leq c \exp(M(\frac{k}{\varepsilon})) \quad (0.425)$$

The space of null elements, denoted $\mathcal{N}^M(\Omega)$, is the space of $(f_\varepsilon)_\varepsilon \in C^\infty(\Omega)^I$ satisfying for every compact K of Ω , $\forall \alpha \in \mathbb{Z}_+^n$, $\forall k > 0$, $\exists c > 0$, $\exists \varepsilon_0 \in I$, $\forall \varepsilon \leq \varepsilon_0$,

$$\sup_{x \in K} |\partial^\alpha f_\varepsilon(x)| \leq c \exp(-M(\frac{k}{\varepsilon})) \quad (0.426)$$

- 1) The space of moderate elements $\mathcal{E}_m^M(\Omega)$ is an algebra stable by derivation.
- 2) The space $\mathcal{N}^M(\Omega)$ is an ideal of $\mathcal{E}_m^M(\Omega)$.



Definition 0.83. The algebra of generalized Roumieu ultradistributions of class $(M_p)_{p \in \mathbb{Z}_+}$, denoted $\mathcal{G}^M(\Omega)$, is the quotient algebra:

$$\mathcal{G}^M(\Omega) = \frac{\mathcal{E}_m^M(\Omega)}{\mathcal{N}^M(\Omega)}.$$

Theorem 0.177. Let $T \in \mathcal{E}'_{MN}(\Omega)$ with $\text{supp} T \subset K$, then there exists $P(D) = \sum_{\gamma \in \mathbb{Z}_+^n} a_\gamma D^\gamma$ an ultradifferential operator of class $(M_p N_p)_{p \in \mathbb{Z}_+}$, $C > 0$, and continuous functions f_γ with $\text{supp} f_\gamma \subset K$, $\forall \gamma \in \mathbb{Z}_+^n$, and $\sup_{\gamma \in \mathbb{Z}_+^n, x \in K} |f_\gamma(x)| \leq C$, such that:

$$T = \sum_{\gamma \in \mathbb{Z}_+^n} a_\gamma D^\gamma f_\gamma.$$

*

Embedding of Roumieu ultradistributions

$$\begin{array}{ccc} \mathcal{D}^{\frac{M}{N}p^l}(\Omega) & \rightarrow & \mathcal{G}^M(\Omega) \\ & \searrow \uparrow & \\ & & \mathcal{E}'_{MN}(\Omega) \end{array}$$

where

$$\begin{array}{ccc} \mathcal{E}'_{MN}(\Omega) & \rightarrow & \mathcal{G}^M(\Omega) \\ f & \rightarrow & [f] = cl((f * \phi_\varepsilon)_{\varepsilon})_{/\Omega} \end{array}$$

and $\mathcal{D}^{\frac{M}{N}p^l}(\Omega) \rightarrow \mathcal{G}^M(\Omega)$ by the canonical embedding.

Mains Results

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tempered Roumieu ultradistribution

Definition 0.84. The space of smooth functions ϕ on R^n such that, for all $h > 0$,

$$\sigma_h(\phi) = \left(\sum_{\alpha, \beta} \int \left| \frac{h^{\alpha+\beta}}{M_{|\alpha|} M_{|\beta|}} (1+|x|^2)^{\frac{\beta}{2}} \partial^\alpha \phi(x) \right|^2 \right)^{\frac{1}{2}} < \infty, \text{ equipped with the topology induced by the norm } \sigma_h(\phi) \text{ is denoted by } S_2^{M_P, h}.$$

The strong duals of

$$S^{\{M_P\}} = \text{ind}_{h \rightarrow \infty} S_2^{M_P, h}.$$

is called the space of tempered Roumieu ultradistributions. denoted $S'^{\{M_P\}}$



*

Generalized Roumieu tempered ultradistributions

the space of tempered moderate elements, denoted $\mathcal{E}_{m,\tau}^{\{M\}}(\Omega)$, is the space of $(f_\varepsilon)_\varepsilon \in C^\infty(\Omega)^I$ satisfying $\forall \alpha \in \mathbb{Z}_+^n, \exists k > 0, \exists p > 0, \exists c > 0, \exists \varepsilon_0 \in I, \forall \varepsilon \leq \varepsilon_0$,

$$\sup_{x \in \mathbb{R}^n} (1 + |x|)^{-p} |\partial^\alpha f_\varepsilon(x)| \leq c \exp(M(\frac{k}{\varepsilon})) \quad (0.427)$$

The space of tempered null elements, denoted $\mathcal{N}_\tau^{\{M\}}(\Omega)$, is the space of $(f_\varepsilon)_\varepsilon \in C^\infty(\Omega)^I$ satisfying $\forall \alpha \in \mathbb{Z}_+^n, \forall K > 0, \exists p > 0, \exists c > 0, \exists \varepsilon_0 \in I, \forall \varepsilon \leq \varepsilon_0$,

$$\sup_{x \in \mathbb{R}^n} (1 + |x|)^{-p} |\partial^\alpha f_\varepsilon(x)| \leq c \exp(-M(\frac{k}{\varepsilon})) \quad (0.428)$$

- 1) The space of tempered moderate elements $\mathcal{E}_{m,\tau}^{\{M\}}(\Omega)$ is an algebra stable by derivation.
- 2) The space $\mathcal{N}_\tau^{\{M\}}(\Omega)$ is an ideal of $\mathcal{E}_{m,\tau}^{\{M\}}(\Omega)$.

The algebra of generalized tempered Roumieu ultradistributions of class $(M_p)_{p \in \mathbb{Z}_+}$ denoted $\mathcal{G}_\tau^{\{M\}}(\Omega)$, is the quotient algebra:

$$\mathcal{G}_\tau^{\{M\}}(\Omega) = \frac{\mathcal{E}_{m,\tau}^{\{M\}}(\Omega)}{\mathcal{N}_\tau^{\{M\}}(\Omega)}$$

*

Embedding of Roumieu tempered ultradistributions

Let $N = (N_p)_{p \in \mathbb{Z}_+}$ be a sequence satisfying the conditions (H1), (H2), (H3)' and $N_0 = 1$, the space $S^N(R^n)$ is the space of functions $\varphi \in C^\infty(R^n)$ such that $\forall m > 0$, we have

$$\|\varphi\|_m = \sup_{\alpha, x} \left\{ \frac{m^{|\alpha|}}{N_{|\alpha|}} |\partial^\alpha \varphi(x)| \exp N(m|x|) \right\} < \infty.$$

Define Σ^N as the set of functions $\phi \in S^N(R^n)$ satisfying

$$\int \phi(x) dx = 1 \text{ and } \int x^\alpha \phi(x) dx = 0, \quad \forall \alpha \in \mathbb{Z}_+^n \setminus \{0\}.$$

Definition 0.85. The net $\phi_\varepsilon = \varepsilon^{-n} \phi(\cdot/\varepsilon)$, $\varepsilon \in I$, where $\phi \in \Sigma^N$ is called a N -mollifier net.

Theorem 0.178. The following commutative diagram is an embeddings

$$\begin{array}{ccc} O_{\mathcal{M}}^{\frac{M}{N}p!}(\Omega) & \rightarrow & \mathcal{G}_\tau^M(\Omega) \\ & \searrow & \uparrow \\ & & S'^{MN}(\Omega) \end{array}$$



where

$$\begin{aligned} S'^{MN}(\Omega) &\rightarrow \mathcal{G}_\tau^M(\Omega) \\ f &\rightarrow [f] = cl((f * \phi_\varepsilon)_{/\Omega}) \end{aligned}$$

and $O_{\mathcal{M}}^{\frac{M}{N}p!}(\Omega) \rightarrow \mathcal{G}_\tau^M(\Omega)$ by the canonical embedding.

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Modeling Risk in Finance: The One-Default Mathematical Model

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Abstract. In this paper, we focus on an important financial model called the one-default model, which is represented by a stochastic differential equation known as the natural equation. The main objective is to establish the existence and uniqueness of the solution to this equation under the non-Lipschitz condition, which is less restrictive than the traditional Lipschitz condition.

Keywords: Financial Risk Modeling, One-Default Model, Default Risk, SDEs.

Introduction

Firstly, we will give a description of the one-default model. Precisely, we define a filtered probability space $(\Omega, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, for any continuous increasing process Λ null at the origin, for any continuous non-negative local martingale N such that $Z_t = N_t e^{-\Lambda_t}$ satisfies $0 < Z_t \leq 1, t \geq 0$, for any continuous local martingale Y , for any Lipschitz function f on $\mathbb{R}$ null at the origin, there



exists a random variable τ such that the family of conditional expectations $dX_t^u = \mathbb{Q}[\tau \leq u | \mathcal{F}_t]$, $u > 0$, $t < \infty$, satisfies the following one-dimensional stochastic differential equation:

$$\begin{cases} dX_t = X_t \left(-\frac{e^{-\Lambda_t}}{1-Z_t} dN_t + f(X_t - (1-Z_t)) dY_t \right), & t \in [u, \infty) \\ X_u = x, \end{cases} \quad (0.429)$$

where the initial condition x can be any $\mathcal{F}_u$ -mesurable random variable. This equation, referred to as the natural equation in [3], describes the evolution of the defaultable market. The integral form of equation (0.429) is expressed as follows:

$$X_t = x + \int_u^t X_s \left(-\frac{e^{-\Lambda_s}}{1-Z_s} \right) dN_s + \int_u^t X_s f(X_s - (1-Z_s)) dY_s, \quad u \leq s \leq t, \quad (0.430)$$

Main results

Now we are in a position to give the heart of our work. Our focus is to introduce modifications to equation (0.429). We have no idea whether the quantity $\left(-\frac{e^{-\Lambda_s}}{1-Z_s} \right)$ is finite or not. So, we introduce the following stopping time: $\tau_n = \inf \{ t : 1 - Z_t < \frac{1}{n} \}$. Hence, we assume the process $\tilde{X}$ instead of X .

The modified equation can be expressed as follows:

$$d\tilde{X}_t = \tilde{X}_t \left(-\frac{e^{-\Lambda_t}}{1-Z_{t \wedge \tau_n}} \right) dN_t + \tilde{X}_t f(\tilde{X}_t - (1-Z_t)) dY_t, \quad u \leq t \leq T, \quad (0.431)$$

such that $\tilde{X}_t = X_t$, $\forall t \leq \tau_n$, and $n \in \mathbb{N}$.

Equation (0.431) can be expressed in its integral form as follows:

$$\tilde{X}_t = x + \int_u^t \tilde{X}_s \left(-\frac{e^{-\Lambda_s}}{1-Z_{s \wedge \tau_n}} \right) dN_s + \int_u^t \tilde{X}_s f(\tilde{X}_s - (1-Z_s)) dY_s, \quad s \in [u, t], \quad (0.432)$$

Throughout our work, we establish and depend on a defined set of assumptions, fundamental to our mathematical framework and analytical discussions.

(A1) We assume that the stochastic integral $\int_u^t \frac{e^{-\Lambda_s}}{1-Z_s} dN_s$, $u \leq t < \infty$, exists and defines a continuous local martingale.

(A2) We suppose that f is continuous, bounded and satisfying

$$|f(x) - f(y)|^2 \leq \lambda(|x - y|^2), \quad \forall x, y \in \mathbb{R}, \quad (0.433)$$

where $\lambda(\cdot)$ is a concave non-decreasing continuous function from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that $\lambda(0) = 0$, $\lambda(v) > 0$ for $v > 0$ and $\int_{0+} \frac{dv}{\lambda(v)} = \infty$.

(A3) Assuming the existence of a positive constant K such that f satisfies the linear growth condition

$$|f(x)|^2 \leq K(1 + |x|^2), \quad \forall x \in \mathbb{R}, \quad (0.434)$$



Now, the theorem regarding the existence and uniqueness of solutions for the natural equation (0.431) with the initial value $\tilde{X}_u = x$ is introduced under the previously mentioned non-Lipschitz condition and linear growth condition.

Theorem 0.179. *Suppose that (A1), (A2) and (A3) hold, then there exists a unique solution to the natural equation (0.431).*

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Successive Approximations for Caputo-Fabrizio Fractional Differential Equations

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Abstract. In this work we deal with a uniqueness result of solutions for a class of fractional differential equations involving the Caputo-Fabrizio derivative. We provide a result on the global convergence of successive approximations. In this article, we begin the study of the global convergence of successive approximations for Caputo-Fabrizio fractional differential equation(CFFDE)

$$\begin{cases} ({}^{CF}D_0^s \rho)(t) = \varphi(t, \rho(t)); & t \in \Upsilon := [0, \lambda] \\ \rho(0) = \rho_0, \end{cases} \quad (0.435)$$

Here ${}^{CF}D_t^s$ is for the CFFDE, $0 < s < 1$, $\varphi : [0, \lambda] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and $\rho_0 \in \mathbb{R}$.

Keywords: Caputo-Fabrizio fractional derivative; generalized solution; global convergence; successive approximations.



Introduction

Fractional differential equations was of great interest to physics, mathematics, engineering, biology, image processing, electricity, control theory, economics, biophysics, and mechanics, etc. For a few fundamental results in the theory of fractional differential equations.

Main results

Definition 0.86. The solution of the problem (0.530) is a continuous function $\rho \in \mathcal{C}$ which satisfies the equation $({}^{CF}D_0^s \rho)(t) = \varphi(t, \rho(t))$ on Υ and initial condition $\rho(0) = \rho_0$.

Theorem 0.180. Suppose the hypotheses $(H_1) - (H_3)$ hold. Then the successive approximations $\rho_n; n \in \mathbb{N}$ are defined and we obtain a convergence towards the unique solution of problem (0.530) uniformly on Υ .

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ETUDE QUALITATIVE DE QUELQUES SYSTEMES DIFFERENTIELS POLYNOMIAUX.

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Résumé La solution périodique d'une classe de systèmes différentiels polynomiaux d'ordre cinq et leur forme globale dans le plan sont l'objet de cette recherche. Plus précisément, en employant la méthode de moyennisation du second ordre, nous analysons les cycles limites et leur structure globale, qui bifurquent pour un système différentiel polynomial de la forme : $\dot{x} = P_1(x, y) + P_5(x, y)$, $\dot{y} = Q_1(x, y) + Q_5(x, y)$, où P_n et Q_n sont des polynômes homogènes de degré i .

Mots clés: Cycle limite, systeme differentiel, methode de moyennisation, solution périodique.

Introduction

Dans ce travail, nous allons utilisé la méthode de la moyennisation du second ordre pour étudier la forme globale des cycles limites pour une classe des



systèmes différentiels polynomiaux de la forme

$$\begin{cases} \dot{x} = P_1(x, y) + P_5(x, y), \\ \dot{y} = Q_1(x, y) + Q_5(x, y), \end{cases} \quad (1)$$

où P_n et Q_n représentent des polynômes homogènes de degré n .
Il est connu que le système (1) est équivalent à la forme

$$\begin{cases} \dot{x} = \lambda x - y + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)x^5 + (-5\beta_1 - 3\beta_2 - \beta_3 + \beta_4 \\ \quad + 3\beta_5 + 5\beta_6)x^4y + (-10\alpha_1 - 2\alpha_2 + 2\alpha_3 + 2\alpha_4 - 2\alpha_5 - 10\alpha_6)x^3y^2 \\ \quad + (-10\beta_1 - 2\beta_2 - 2\beta_3 + 2\beta_4 + 2\beta_5 - 10\beta_6)x^2y^3 + (5\alpha_1 - 3\alpha_2 + \alpha_3 \\ \quad + \alpha_4 - 3\alpha_5 + 5\alpha_6)xy^4 + (-\beta_1 + \beta_2 - \beta_3 + \beta_4 - \beta_5 + \beta_6)y^5, \\ \dot{y} = x + \lambda y + (\beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6)x^5 + (5\alpha_1 + 3\alpha_2 + \alpha_3 - \alpha_4 \\ \quad - 3\alpha_5 - 5\alpha_6)x^4y + (-10\beta_1 - 2\beta_2 + 2\beta_3 + 2\beta_4 - 2\beta_5 - 10\beta_6)x^3y^2 \\ \quad + (10\alpha_1 + 2\alpha_2 + 2\alpha_3 - 2\alpha_4 - 2\alpha_5 + 10\alpha_6)x^2y^3 + (5\beta_1 - 3\beta_2 + \beta_3 \\ \quad + \beta_4 - 3\beta_5 + 5\beta_6)xy^4 + (\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4 + \alpha_5 - \alpha_6)y^5, \end{cases} \quad (2)$$

où l'origine est un foyer. Plus précisément, nous étudierons la forme globale des cycles limites de la bifurcation de Hopf à l'origine du système (2) et aussi la forme globale des cycles limites du système (2) qui bifurquent des orbites périodiques du centre linéaire

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = x. \end{cases}$$

La méthode de la moyennisation donne une relation quantitative entre les solutions d'un système différentiel non-autonome et les solutions de son système différentiel moyenné, qui est autonome. Afin d'utiliser cette méthode à l'ordre 2, on transforme le système (2) en l'équation différentielle d'Abel à l'aide de la transformation de Cherkas.

Résultats principaux

Nous utilisons l'approximation du second ordre de la méthode de la moyennisation pour chercher les solutions périodiques qui bifurquent de l'origine du



système d'ordre cinq

$$\begin{cases} \dot{x} = -y + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)x^5 + (-5\beta_1 - 3\beta_2 - \beta_3 + \beta_4 \\ \quad + 3\beta_5 + 5\beta_6)x^4y + (-10\alpha_1 - 2\alpha_2 + 2\alpha_3 + 2\alpha_4 - 2\alpha_5 - 10\alpha_6)x^3y^2 \\ \quad + (-10\beta_1 - 2\beta_2 - 2\beta_3 + 2\beta_4 + 2\beta_5 - 10\beta_6)x^2y^3 + (5\alpha_1 - 3\alpha_2 + \alpha_3 \\ \quad + \alpha_4 - 3\alpha_5 + 5\alpha_6)xy^4 + (-\beta_1 + \beta_2 - \beta_3 + \beta_4 - \beta_5 + \beta_6)y^5 + \varepsilon^2\alpha_0x, \\ \dot{y} = x + (\beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6)x^5 + (5\alpha_1 + 3\alpha_2 + \alpha_3 - \alpha_4 \\ \quad - 3\alpha_5 - 5\alpha_6)x^4y + (-10\beta_1 - 2\beta_2 + 2\beta_3 + 2\beta_4 - 2\beta_5 - 10\beta_6)x^3y^2 \\ \quad + (10\alpha_1 + 2\alpha_2 + 2\alpha_3 - 2\alpha_4 - 2\alpha_5 + 10\alpha_6)x^2y^3 + (5\beta_1 - 3\beta_2 + \beta_3 \\ \quad + \beta_4 - 3\beta_5 + 5\beta_6)xy^4 + (\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4 + \alpha_5 - \alpha_6)y^5 + \varepsilon^2\alpha_0y, \end{cases} \quad (3)$$

en $\varepsilon = 0$. Par la même méthode, nous étudions les cycles limites qui bifurquent des orbites périodiques du centre linéaire $\dot{x} = -y, \dot{y} = x$, quand nous le perturbons par des polynômes de degré cinq

$$\begin{cases} \dot{x} = -y + \varepsilon[(a_1 + a_2 + a_3 + a_4 + a_5 + a_6)x^5 + (-5b_1 - 3b_2 - b_3 + b_4 \\ \quad + 3b_5 + 5b_6)x^4y + (-10a_1 - 2a_2 + 2a_3 + 2a_4 - 2a_5 - 10a_6)x^3y^2 \\ \quad + (-10b_1 - 2b_2 - 2b_3 + 2b_4 + 2b_5 - 10b_6)x^2y^3 + (5a_1 - 3a_2 + a_3 \\ \quad + a_4 - 3a_5 + 5a_6)xy^4 + (-b_1 + b_2 - b_3 + b_4 - b_5 + b_6)y^5] + \varepsilon^2\alpha_0x, \\ \dot{y} = x + \varepsilon[(b_1 + b_2 + b_3 + b_4 + b_5 + b_6)x^5 + (5a_1 + 3a_2 + a_3 - a_4 \\ \quad - 3a_5 - 5a_6)x^4y + (-10b_1 - 2b_2 + 2b_3 + 2b_4 - 2b_5 - 10b_6)x^3y^2 \\ \quad + (10a_1 + 2a_2 + 2a_3 - 2a_4 - 2a_5 + 10a_6)x^2y^3 + (5b_1 - 3b_2 + b_3 \\ \quad + b_4 - 3b_5 + 5b_6)xy^4 + (a_1 - a_2 + a_3 - a_4 + a_5 - a_6)y^5] + \varepsilon^2\alpha_0y. \end{cases} \quad (4)$$

Les résultats principaux sont les théorèmes suivants.

Théorème 1 Considérons le système différentiel (3) avec $\alpha_1 = -\frac{8}{5}\alpha_3$ et $\alpha_0M > 0$, où

$$\begin{aligned} M = & -105\alpha_1\beta_1 + 40(\alpha_1\beta_3 + \alpha_3\beta_1 + \alpha_4\beta_1 + \alpha_1\beta_4) + 64(\alpha_3\beta_3 \\ & + \alpha_2\beta_4 + \alpha_4\beta_2) + 24(\alpha_1\beta_5 + \alpha_5\beta_1) - 80(\alpha_2\beta_1 + \alpha_1\beta_2), \end{aligned}$$

alors, il existe un cycle limite qui bifurque de l'origine pour $\varepsilon = 0$. De plus, pour $\varepsilon > 0$ suffisamment petit, l'expression de ce cycle limite en coordonnées polaires (r, θ) est donnée par

$$\begin{aligned} r(\theta, \varepsilon) = & \varepsilon^{\frac{1}{4}}\sigma_0^{\frac{1}{4}} + \frac{1}{4}\varepsilon^{\frac{5}{4}}\sigma_0^{\frac{5}{4}}b(\theta) + \frac{1}{4}\varepsilon^{\frac{5}{4}}\sigma_0^{\frac{5}{4}}c(\theta) + \frac{5}{16}\varepsilon^{\frac{9}{4}}\sigma_0^{\frac{9}{4}}b(\theta)c(\theta) \\ & + \frac{5}{32}\varepsilon^{\frac{9}{4}}\sigma_0^{\frac{9}{4}}b^2(\theta) - \frac{3}{32}\varepsilon^{\frac{9}{4}}\sigma_0^{\frac{9}{4}}c^2(\theta) + O(\varepsilon)^{\frac{13}{4}}, \end{aligned}$$



où

$$\sigma_0 = 8\sqrt{\frac{\alpha_0}{M}},$$

et

$$\begin{aligned} b(\theta) &= \left(\frac{15}{8}\alpha_1 + \alpha_2 - \alpha_4\right)\sin(2\theta) + \left(-\frac{5}{8}\beta_1 + \beta_2 + \beta_4\right)\cos(2\theta) \\ &\quad + \left(\alpha_1 - \alpha_5\right)\sin(4\theta) + \left(-\frac{1}{4}\beta_1 + \beta_5\right)\cos(4\theta) + \left(-\frac{5}{8}\alpha_1 \right. \\ &\quad \left. - \alpha_6\right)\sin(6\theta) + \left(\frac{5}{8}\beta_1 + \beta_6\right)\cos(6\theta) + \left(\frac{5}{8}\beta_1 + \beta_3\right), \\ c(\theta) &= \left(-\frac{25}{8}\alpha_1 + \alpha_2 + 3\alpha_4\right)\sin(2\theta) + \left(\frac{35}{8}\beta_1 + \beta_2 - 3\beta_4\right)\cos(2\theta) \\ &\quad + \left(-\frac{5}{4}\alpha_1 + 2\alpha_5\right)\sin(4\theta) + \left(\frac{5}{4}\beta_1 - 2\beta_5\right)\cos(4\theta) + \left(\frac{25}{24}\alpha_1 \right. \\ &\quad \left. + \frac{5}{3}\alpha_6\right)\sin(6\theta) + \left(-\frac{25}{24}\beta_1 - \frac{5}{3}\beta_6\right)\cos(6\theta). \end{aligned} \quad (5)$$

Théorème 2 Considérons le système différentiel (4) avec $a_1 = -\frac{8}{5}a_3$ et $\alpha_0 N > 0$, où

$$\begin{aligned} N &= -105a_1b_1 + 40(a_1b_3 + a_3b_1 + a_4b_1 + a_1b_4) + 64(a_3b_3 + a_2b_4 + a_4b_2) \\ &\quad + 24(a_1b_5 + a_5b_1) - 80(a_2b_1 + a_1b_2), \end{aligned}$$

alors, il existe un cycle limite, qui bifurque des solutions périodiques du centre linéaire pour $\varepsilon = 0$ avec un rayon

$$\rho_0 = 8\sqrt{\frac{\alpha_0}{N}}.$$

De plus, pour $\varepsilon > 0$ suffisamment petit, l'expression de ce cycle limite en coordonnées polaires (r, θ) est donnée par

$$\begin{aligned} r(\theta, \varepsilon) &= \rho_0^{\frac{1}{4}} + \frac{1}{4}\varepsilon\rho_0^{\frac{5}{4}}c(\theta) + \frac{1}{4}\varepsilon\rho_0^{\frac{5}{4}}b(\theta) + \frac{5}{16}\rho_0^{\frac{9}{4}}\varepsilon^2b(\theta)c(\theta) \\ &\quad + \frac{5}{32}\varepsilon^2\rho_0^{\frac{9}{4}}b^2(\theta) - \frac{3}{32}\varepsilon^2\rho_0^{\frac{9}{4}}c^2(\theta) + O(\varepsilon)^3, \end{aligned}$$

où $b(\theta)$ et $c(\theta)$ sont définis comme dans le théorème 1 en changeant seulement α_i et β_i par a_i et b_i , respectivement.

Applications

Exemple Soit le système différentiel de degré 5 donné par

$$\begin{cases} \dot{x} = 5\varepsilon^2x - y + 6x^5 - 15x^4y - 16x^3y^2 - 82x^2y^3 + 10xy^4 - 3y^5, \\ \dot{y} = x + 5\varepsilon^2y + 17x^5 - 32x^4y - 58x^3y^2 - 68x^2y^3 + 37xy^4 - 4y^5. \end{cases} \quad (6)$$



Ce système est équivalent au système (3). Les calculs de M et σ_0 sont données par

$$\begin{aligned} M &= -105\alpha_1\beta_1 + 40(\alpha_1\beta_3 + \alpha_3\beta_1 + \alpha_4\beta_1 + \alpha_1\beta_4) + 64(\alpha_3\beta_3 \\ &\quad + \alpha_2\beta_4 + \alpha_4\beta_2) + 24(\alpha_1\beta_5 + \alpha_5\beta_1) - 80(\alpha_2\beta_1 + \alpha_1\beta_2), \\ M &= 4608, \end{aligned}$$

et

$$\sigma_0 = \sqrt{\frac{64\alpha_0}{M}} = 2\sqrt{\frac{5}{288}}.$$

D'après les expressions de (5), on obtient

$$\begin{aligned} b(\theta) &= -14\sin(2\theta) + \frac{3}{2}\cos(2\theta) - 8\sin(4\theta) \\ &\quad - \cos(4\theta) + 5\sin(6\theta) + \frac{11}{2}\cos(6\theta) + \frac{17}{2}, \end{aligned}$$

et

$$\begin{aligned} c(\theta) &= 26\sin(2\theta) + \frac{27}{2}\cos(2\theta) + 10\sin(4\theta) \\ &\quad + 5\cos(4\theta) - \frac{25}{3}\sin(6\theta) - \frac{55}{6}\cos(6\theta). \end{aligned}$$

La forme du cycle limite au voisinage de l'origine du système (6) est donné par

$$\begin{aligned} r(\theta, \varepsilon) &= 0.7164810171\varepsilon^{\frac{1}{4}} + (0.4012198179 - 0.1730752155\cos(6\theta) \\ &\quad + 0.1888093261\cos(4\theta) + 0.7080349728\cos(2\theta) \\ &\quad + 0.5664279787\sin(2\theta) - 0.1573411051\sin(6\theta) \\ &\quad + 0.9440466300\sin(4\theta))\varepsilon^{\frac{5}{4}}. \end{aligned}$$

* Pour $\varepsilon = 10^{-1}$, la forme de cycle limite s'écrit

$$\begin{aligned} r(\theta, 10^{-1}) &= 0.4254691330 - 0.9732734604\cos(6\theta) \\ &\quad + 0.1061752866\cos(4\theta) + 0.3981573249\cos(2\theta) \\ &\quad + 0.3185258602\sin(2\theta) - 0.8847940555\sin(6\theta) \\ &\quad + 0.5308764330\sin(4\theta). \end{aligned}$$

* La forme de cycle limite pour $\varepsilon = 10^{-2}$ est

$$\begin{aligned} r(\theta, 10^{-2}) &= 0.2278399599 + 10^{-3}(-0.5473118875\cos(6\theta) \\ &\quad + 0.5970675139\cos(4\theta) + 0.2239003177\cos(2\theta) \\ &\quad + 0.1791202543\sin(2\theta) - 0.4975562617\sin(6\theta) \\ &\quad + 0.2985337568\sin(4\theta)). \end{aligned}$$



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A NEW QUADRATIC EXPONENTIAL DISTRIBUTION WITH TWO-PARAMETER

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Abstract. This paper presents a novel probability distribution, namely the new XLindley distribution, derived from a unique combination of exponential and gamma distributions through a special mixture formulation. The study extensively investigates the mathematical properties of the proposed distribution, including but not limited to the moment generation function, moments of different orders, mode identification, and the quantile function. Furthermore, the research employs a Monte Carlo simulation to assess and compare the performance of various estimators in estimating the unknown parameter of the new XLindley distribution. These estimators are carefully evaluated and analyzed in terms of their efficiency and accuracy, providing valuable insights into the practical application of the new distribution in statistical modeling and data analysis contexts. The voltage and failure time data in the field of engineering are used to model the proposed distribution. The new model is compared with many current distributions such as XLindley, gamma,



Weibull, exponential, Lindley, Shanker, Akash, Zeghdoudi, Chris-Jerry, and Xgamma. Among all models, it is concluded that the new one-parameter distribution performed the best in modeling based on criteria such as the Akaike information criterion, Bayesian information criterion, and others. The real data results show that the proposed distribution exhibits greater flexibility and improved goodness of fit compared to alternative distributions. The new XLindley distribution could be useful in modeling real-life data and may warrant further exploration in future research. Overall, this study contributes to the field of probability distributions and provides new insights for statistical modeling.

Keywords: Exponential distribution, Xlindley distribution, quantile function, estimation methods.

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Intermittency in fractional Duffing equation

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Abstract. In this paper, we characterize the chaos in the Duffing equation with negative linear stiffness and a fractional damping term given by a Caputo fractional derivative of order α ranging from 0 to 2. We use two different numerical methods to compute the solutions, one of them new. The system is very rich with distinct behaviours as intermittency. We classify some cases as intermittency type I.

Keywords: fractional calculus, Nonlinear oscillations and coupled oscillators for ordinary differential equations, Chaos

Introduction

The fractional Duffing equation is a nonlinear differential equation with fractional-order:

$$\begin{cases} \ddot{x} + \gamma D_t^\alpha x(t) - x + x^3 = f_0 \cos(\omega t), \\ x(0) = x_0, x'(0) = v_0. \end{cases} \quad (0.436)$$

$D_t^\alpha x(t)$ is the Caputo fractional derivative, α is the order of fractional derivative in the equation. In this paper, we present our results concerning intermittency in the fractional Duffing equation. To analyse the behaviour in the fractional system, we start with the theory of intermittency, and especially, with results for the classical Duffing equation. We use this to validate the tools that we apply to analyse the fractional model.



Main results

*

Phase space of fractional Duffing equation In the fractional equation, when we start the solution of the fractional equation at time t_1 different from the initial time t_0 , we end with a fractional system started at time $t = 0$ but with a phase shift.

$$\begin{cases} \ddot{x} + \gamma(t_1 D_t^\alpha)x(t) - x + x^3 = f_0 \cos(\omega t), \\ x(t_1) = a, \quad \dot{x}(t_1) = b \end{cases} \quad (0.437)$$

This (0.437) equivalent to:

$$\begin{cases} y'' + \gamma_0 D_s^\alpha y(s) - y + y^3 = f_0 \cos(\omega s + \phi_0), \\ \phi_0 = \omega t_1, \quad y(0) = a, \quad y'(0) = b \end{cases} \quad (0.438)$$

with $y(s) = x(s + t_1)$. We conclude that our phase space has three dimensions corresponding to the number of parameters determining that state. We prove also that there are no T -periodic solutions to the initial value problem.

*

Intermittency in fractional Diffing equation

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Intermittency with two reinjection channels

Now, we study an intermittent behaviour with two reinjection channels for $\alpha = 1.1$, $x_0 = v_0 = 0$.

- When the control parameter f_0 is below the critical value $f_{0c} = 0.250007$, the attractor takes the form of a regular curve with period T .
- At the critical value $f_{0c} = 0.250007$, we observe a loss of stability in the solution. A chaotic burst interrupts the regular phase which oscillates between the two fixed points of the underlying Hamiltonian system.
- As the parameter deviates significantly from the critical value associated with intermittency, the laminar phases disappear entirely, giving way to a state of complete chaos in the motion.

$$[f_0 = 0.250013] [f_0 = 0.250027] [f_0 = 0.253]$$

Figure 28: Stroboscopic map and solution $x(t)$ showing the intermittency behaviour as f_0 increases with two re-injection channels.



*

Intermittency type I Classification of intermittency as type I: return map We present the return map x_n, x_{n+1} , derived from the data corresponding to Figure ???. Using the values x_n of the stroboscopic map, we plot x_{n+1} versus x_n to see the two channels obtained near the points constituting the laminar phases.

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The operating diagram of the competition model between plasmid-bearing and plasmid-free organisms in the chemostat

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Abstract. A mathematical model of competition between plasmid-bearing and plasmid-free organisms for a single limiting resource in a chemostat is analyzed. This model was previously introduced without operating diagrams. Through the two operating parameters of the model, represented by the dilution rate and the input concentrations of the substrate, we give necessary and sufficient conditions for existence and stability of all equilibria. By means of operating diagrams we describe the asymptotic behavior of the model with respect to those operating parameters. Numerical simulations are carried out to illustrate the mathematical results.

Keywords: Chemostat, Competition, Plasmid, Stability, Operating diagram.



Introduction

Genetic alteration generally takes place by the introduction of DNA into the cell in the form of a plasmid. The plasmid-bearing organism is likely to be a lesser competitor than the plasmid-free organism because of the added load on its metabolic machinery. During replication, the plasmid can be lost by not being passed to the daughter cells, producing a plasmid-free organism.

In this work, we consider a competition model between plasmid-bearing and plasmid-free organisms in a chemostat that incorporates both general response functions and distinct removal rates. The model with identical removal rates was proposed in a paper by Stephanopoulos and Lapidus [1], and the analysis there was local. The complete global analysis of the model in the case of identical removal rates was given in a paper by Hsu, Waltman and Wolkowicz [2]. However, it is important to consider the more general situation that removal rates are distinct. This part is based similarly on the work of B. Li, Y. Kuang and H. L. Smith [3]. The model takes the form:

$$\begin{aligned} S' &= (S^0 - S)D - f_1(S)\frac{x}{\beta} - f_2(S)\frac{y}{\beta} \\ x' &= [f_1(S) - D_1]x + qf_2(S)y \\ y' &= [(1 - q)f_2(S) - D_2]y \end{aligned} \quad (0.439)$$

with $S(0) \geq 0$, $x(0) > 0$ and $y(0) > 0$. Where $S(t)$ denote the concentration of nutrient in the vessel, $x(t)$ and $y(t)$ denote the concentration of the plasmid-free and plasmid-bearing organisms at time t , respectively. The constant q is the probability that plasmid-bearing organisms lose the plasmid in reproduction and so convert into plasmid-free organisms ($0 < q < 1$). The parameters e_1 and e_2 are death rates of x and y respectively.

Main results

We assume that the response functions f_i , $i = 1, 2$ are monotone increasing on a suitable interval, those functions are $\mathcal{C}^1$ -functions satisfying the following conditions:

C: For $i = 1, 2$, $f_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $f_i(0) = 0$ and $f'_i(S) > 0$ for all $S \geq 0$.

The following result proves that the model (0.632) preserves the biological significance where all solutions of the system are non-negative and bounded for any non-negative initial condition.

Proposition 0.27. *Assume that assumption (C) hold. For non-negative initial conditions, all solutions of system (0.632) are bounded and remain non-negative for all $t > 0$. In addition, the set*

$$\Omega = \left\{ (S, x, y) \in \mathbb{R}_+^3 : \frac{\beta DS^0}{D_{\max}} \leq \beta S + x + y \leq \frac{\beta DS^0}{D_{\min}} \right\}$$

is positively invariant and globally attracting for (??), where $D_{\max} = \max\{D, D_1, D_2\}$ and $D_{\min} = \min\{D, D_1, D_2\}$.



Hereafter we use the following conditions and notations: for functions f_i , $i = 1, 2$, satisfying the condition (C_1) . If the equations $f_1(S) = D_1$ and $(1 - q)f_2(S) = D_2$ admit solutions, they are unique and we define the break-even concentrations

$$\lambda_1(D) = f_1^{-1}(D_1) \quad \text{and} \quad \lambda_2(D) = f_2^{-1}\left(\frac{D_2}{1-q}\right). \quad (0.440)$$

Otherwise, we pose: $\lambda_i = +\infty$, $i = 1, 2$.

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Existence of equilibria The existence of equilibria of system (0.632) is stated by the following result:

Proposition 0.28. *Assume that (C) is satisfied. System (0.632) has at most three equilibria:*

- The washout equilibrium $E_0 = (S^0, 0, 0)$, that always exists.
- The equilibrium $E_1 = (\lambda_1, \frac{D_1}{\beta}(S^0 - \lambda_1), 0)$, of extinction of plasmid-bearing organism y . This equilibrium exists if and only if $\lambda_1 < S^0$.
- The coexistence equilibrium, $E_c = (\lambda_2, x_c, y_c)$, where x_c and y_c are given by

$$x_c = \frac{-q\beta D f_2(\lambda_2)(S^0 - \lambda_2)}{D_2 f_1(\lambda_2) - D_1 f_2(\lambda_2)} \quad \text{and} \quad y_c = \frac{\beta D (f_1(\lambda_2) - D_1)(S^0 - \lambda_2)}{D_2 f_1(\lambda_2) - D_1 f_2(\lambda_2)}. \quad (0.441)$$

This equilibrium exists if and only if $\lambda_2 < \min(S^0, \lambda_1)$.

*

Local stability of equilibria In the following we investigate the local stability of equilibrium points of (0.632).

Proposition 0.29. *Assume that (C) is satisfied. The stability of equilibria of (0.632) is as follows:*

- The equilibrium E_0 is LES if and only if $S^0 < \min(\lambda_2, \lambda_1)$.
- The equilibrium E_1 , if it exists, has at least three dimensional stable manifolds and is LES if and only if $\lambda_1 < \lambda_2$.
- The equilibrium E_c , is LES if and only if

$$H = B_2 B_1 - B_0 > 0 \quad (0.442)$$

where B_n , $n = 0, 1, 2$ are defined by:

$$\begin{cases} B_2 = m_{11} + m_{22}, \\ B_1 = m_{11}m_{22} + m_{12}m_{21} + \frac{D_2}{\beta(1-q)}m_{31}, \\ B_0 = \frac{qD_2}{1-q}m_{12}m_{31} + \frac{D_2}{\beta(1-q)}m_{22}m_{31}. \end{cases} \quad (0.443)$$



with

$$\begin{cases} m_{11} = D + f'_1(\lambda_2)x_c/\beta + f'_2(\lambda_2)y_c/\beta, & m_{21} = f'_1(\lambda_2)x_c + qf'_2(\lambda_2)y_c, \\ m_{12} = f_1(\lambda_2)/\beta, & m_{22} = D_1 - f_1(\lambda_2), & m_{31} = (1 - q)f'_2(\lambda_2)y_c. \end{cases} \quad (0.444)$$

Operating diagrams

The effect of the operating parameters on the asymptotic behavior of the system can be summarised using the operating diagram. The operating diagram has the operating parameters as its coordinates, and the different regions defined in it correspond to qualitatively different dynamics. Our objective is now to describe the operating diagram of the system (0.632).

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Homogenization of quadratic forms

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Abstract.

In this paper, we examine the process of homogenization of energy within a quadratic form in 3 dimension, which is represented by the following integral functional: $F_\epsilon : L^2(0, 1)^3 \rightarrow \mathbb{R}^+ \cup \{+\infty\}$

$$F_\epsilon(u) = \begin{cases} \int_0^1 \langle A_\epsilon(t). \nabla u, \nabla u \rangle dx & \text{if } u \in H_0^3(0, 1)^3, \\ +\infty & \text{else.} \end{cases} \quad (0.445)$$

where: $A_\epsilon : \mathbb{R} \rightarrow \mathbb{R}^{3 \times 3}$ invertible, measurable and

$$0 < b \leq A_\epsilon^{i,j} \leq c < +\infty \quad i, j \in \{1, 2, 3\} \quad (0.446)$$

Keywords: Homogenization, Γ -Limit, Elliptic equations, Quadratic forms.



Introduction

Homogenization is a mathematical method used when a material with a heterogeneous component and a periodic structure is being studied physically. A small, positive parameter is defined, representing the size of the domain before the size of the parameter. When ε tends towards 0, this corresponds, from a practical point of view, to a transition from a non-homogeneous medium to a so-called 'homogenized' (i.e. a problem posed in a homogeneous medium). Homogenization problems are often addressed by studying minimization problems and associated derivative equations. The minimization problems and the partial differential equations that govern the physical properties being studied are key considerations. Γ -convergence is used to homogenize the energetic, ensuring that the minimizers converge to the minimum value of the energy associated with the problem. This concept was first introduced in 1975 by E. De Giorgi and T. Franzoni [7]. The objective of homogenization is to provide the macroscopic properties of the composite by considering the properties of the microscopic structure. In the first part of this document, we will review some fundamental definitions and theorems, including the classical theorems of integration theory, the application of functional analysis tools, the concept of the sub-differential, and the notion of Γ -convergence. In this section*, we will present some of the fundamental tools that will be used throughout the remainder of this document. The second part of this work will focus on functional homogenization results under the convexity condition. After proposing a condition that is more straightforward than the convexity condition, we utilize De Giorgi's slicing technique. The final objective of our work is to homogenize an elliptic equation of Dirichlet type associated with the minimization problem using the tools of convexity and Γ -convergence.

Main results

Definition 0.87. Let E be a complete metric space. The functional $F^{hom} : E \rightarrow \mathbb{R} \cup +\infty$ is called the (sequential) Γ -limit of the functionals $F_\varepsilon : E \rightarrow \mathbb{R} \cup +\infty$, $\varepsilon > 0$ if the following two conditions are satisfied:

- i) (lim inf-inequality)
 For all family $(u_\varepsilon) \subset X$ such that $u_\varepsilon \rightarrow u$ (resp $u_\varepsilon \rightharpoonup u$), we have

$$F_{hom}(u) \leq \liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon),$$



ii) (lim sup-inequality)

For all $u \in E$ there exists a recovery sequence $(u_\varepsilon) \in E$, that is, $u_\varepsilon \rightarrow u$ ($u_\varepsilon \rightharpoonup u$) in E and

$$F_{hom}(u) \geq \limsup_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon)$$

Theorem 0.181. Let F_{hom} a functional defined by

$$F_{hom}(u) = \begin{cases} \int_0^1 \langle A \nabla u, \nabla u \rangle dx & \text{if } u \in H_0^3(0,1)^3, \\ +\infty & \text{else.} \end{cases} \quad (0.447)$$

Assume that A fulfills condition (0.2) and the following assertions hold: $A_\varepsilon^{-1} \rightharpoonup A^{-1}$ in the weak sense of $L^1 - L^\infty$, then F_ε Γ -converge to the integral functional F defined on $H_0^3(0,1)^3$ by

$$F_{hom}(u) = \begin{cases} \int_{(0,1)^3} \langle A(x) \nabla u, \nabla u \rangle dx & \text{if } u \in H_0^3(0,1)^3, \\ +\infty & \text{sinon.} \end{cases} \quad (0.448)$$

Sketch of proof

i) inner bound:

Let $u_\varepsilon \rightarrow u$ in $L^2(0,1)^3$, without loss of generality, we assume that $\liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) < +\infty$, by the equi-coercivity of F_ε , by an extraction family $\{u_{g(\varepsilon)}\}_{\varepsilon > 0}$ bonded in $H_0^3(0,1)$ in one hand, and by other hand $u_{g(\varepsilon)} \rightarrow u$, it follows that:

$$\begin{aligned} F_{g(\varepsilon)}(u_{g(\varepsilon)}) &= \int_{(0,1)^3} \langle A_\varepsilon(t) \cdot \nabla u, \nabla u \rangle = \int_{(0,1)^3} \langle A_{g(\varepsilon)}(\nabla u_{g(\varepsilon)} - A \cdot A_{g(\varepsilon)}^{-1} \nabla u), (\nabla u_{g(\varepsilon)} - A \cdot A_{g(\varepsilon)}^{-1} \nabla u) \rangle dx \\ &+ 2 \int_{(0,1)^3} \langle A \nabla u, \nabla u_{g(\varepsilon)} \rangle dx - \int_{(0,1)^3} \langle A \nabla u, A \cdot A_{g(\varepsilon)}^{-1} \nabla u \rangle dx \\ &\geq 2 \int_{(0,1)^3} \langle A \nabla u_{g(\varepsilon)}, \nabla u_{g(\varepsilon)} \rangle dx - \int_{(0,1)^3} \langle A \nabla u, A \cdot A_{g(\varepsilon)}^{-1} \nabla u \rangle \end{aligned}$$



Hence,

$$F_{hom}(u) \leq \lim_{\epsilon \rightarrow 0} F_{g(\epsilon)}(u_{g(\epsilon)}) = \liminf_{\epsilon \rightarrow 0} F_{\epsilon}(u_{\epsilon})$$

ii) *The upper bound:*

Let $u \in L^2(0,1)^3$, without loss of generality assume that $F_{hom}(u) < +\infty$, then $u \in H_0^3(0,1)^3$, thanks to Sobolev immersion, there exist $\tilde{u} \in C([0,1])$:

$$\tilde{u} = u \quad a.e. x \in (0,1)^3$$

We pose:

$$\begin{aligned} \tilde{u}_{\epsilon}(x) &= \int A_{\epsilon}(A \nabla u) dx. \\ |\tilde{u}_{\epsilon}(x)| &\leq \frac{c}{b} \int_{(0,1)^3} |\nabla u dx| \\ &\leq \frac{c}{b} \|\nabla u\|_2 = Cte \in L^2(0,1) \end{aligned} \quad (0.449)$$

Hence, by the dominant convergence theorem $\tilde{u}_{\epsilon} \xrightarrow{L^2} u$.

We take the limit and we find:

$$\lim_{\epsilon \rightarrow 0} F_{\epsilon}(\tilde{u}_{\epsilon}) = \limsup_{\epsilon \rightarrow 0} F_{\epsilon}(\tilde{u}_{\epsilon}) = F_{hom}(u).$$

Application to the Dirichlet Problem

The basic property of Γ -convergence, see e.g.[2] imply that for every $g \in L^2(\Omega)$, the solutions u_{ϵ} of the minimizer of the problem

$$\min \left\{ \int_{\Omega} \frac{1}{2} \left(\left\langle A \left(\frac{x}{\epsilon} \right) \nabla u, \nabla u \right\rangle - g(x)u \right) dx : u \in H_0^1(\Omega) \right\} \quad (0.450)$$

converge in $L^2(\Omega)$, as $\epsilon \rightarrow 0$, to the solution u_{hom} of the minimization problem

$$\min \left\{ \int_{\Omega} \frac{1}{2} (\langle A_{hom} \nabla u, \nabla u \rangle - g(x)u) dx : u \in H_0^1(\Omega) \right\} \quad (0.451)$$

By considering the Euler-Lagrange equation's, see e.g.[3] of the problems (1) and (2) in this special case, from the previous result we obtain the homogenization theorem for symmetric elliptic operators in divergence form, which asserts that for every $g \in L^2(\Omega)$ the solutions u_{ϵ} of the Dirichlet problems

$$\begin{cases} -div \left(A \left(\frac{x}{\epsilon} \right) \nabla u_{\epsilon} \right) = g & \text{on } \Omega \\ u_{\epsilon} = 0 & \text{on } \partial\Omega. \end{cases}$$

converge in $L^2(\Omega)$ to the solution u_{hom} of the Dirichlet problem

$$\begin{cases} -div (A_{hom} \nabla u) = g & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$



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Estimation force de blocage de fluide Herschel Bulkley en régime stationnaire

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Abstract. L'objectif principal est d'estimer la force de blocage du fluide de Herschel-Bulkley. Pour cela, nous démontrons l'existence d'une force de blocage maximale pour un écoulement stationnaire dans un domaine borné, en nous appuyant sur un modèle mathématique décrivant cet écoulement ainsi que sur l'analyse du comportement asymptotique de la solution en fonction de la force.

Keywords: Blocking force, Herschel-Bulkley fluid, variational inequality, fluide incompressible.



Introduction

L'estimation de la force de blocage constitue une caractéristique physique importante pour les fluides de Herschel- Bulkley. Les expériences montrent que, lorsque la limite d'élasticité augmente, les zones rigides s'élargissent, pouvant entraîner un blocage complet du flux. Ce phénomène de blocage peut avoir des effets bénéfiques, tels que glissements de terrain, mais également des conséquences négatives, comme forages pétroliers concernant le transport du pétrole dans les pipelines.

Main results

On considère un problème mécanique qui décrit l'écoulement stationnaire du fluide incompressible et viscoplastique de Herschel-Bulkley dans un domaine borné $\Omega \subset \mathbb{R}^n$ ($n = 2, 3$), de frontière régulière $\partial\Omega = \Gamma$ de classe C^1 .

Problème (P1): Trouver le champ des vitesses $\mathbf{u} = (u_i) : \Omega \longrightarrow \mathbb{R}^n$ et le champ des contraintes $\sigma = (\sigma_{ij}) : \Omega \longrightarrow S_n$ tels que

$$\mathbf{u} \cdot \nabla \mathbf{u} = \text{div} \sigma + \mathbf{f} \text{ sur } \Omega. \quad (2.1)$$

$$\left. \begin{aligned} \sigma^D &= \mu |\mathbf{D}(\mathbf{u})|^{p-2} \mathbf{D}(\mathbf{u}) + g \frac{\mathbf{D}(\mathbf{u})}{|\mathbf{D}(\mathbf{u})|} \quad \text{si } |\mathbf{D}(\mathbf{u})| \neq 0 \\ |\sigma^D| &\leq g \quad \text{si } |\mathbf{D}(\mathbf{u})| = 0 \end{aligned} \right\} \text{ sur } \Omega. \quad (2.2)$$

$$\text{div} \mathbf{u} = 0 \text{ sur } \Omega. \quad (2.3)$$

$$\mathbf{u} = 0 \text{ sur } \Gamma. \quad (2.4)$$

Rappelons ici que (2.1) représente l'équation de l'équilibre stationnaire du fluide, telle que $\text{div} \sigma = (\sigma_{ij,j})$ et f est la densité des forces volumiques agissant sur le fluide. La relation (2.2) représente la loi de comportement du fluide incompressible de Herschel-Bulkley dont le coefficient de consistence μ et le seuil de plasticité g . Le paramètre p tel que $1 < p \leq 2$; est l'exposant de la loi de puissance du fluide. L'équation (2.3), où $\text{div} \mathbf{u} = u_{i,i}$, décrit la condition d'incompressibilité du fluide. (2.4) est une condition sur la vitesse sur la frontière Γ .

Comportement asymptotique de la solution

Soit $\mathbf{f}$ une force de blocage. On note C l'ensemble

$$C = \left\{ \mathbf{v} \in W_{p,\text{div}} \mid g \int_{\Omega} |\mathbf{D}(\mathbf{v})| dx = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} dx \right\}.$$

l'ensemble C est un cône convexe dans $W_{p,\text{div}}$.

Introduisons pour $\varepsilon > 0$ la force de blocage perturbée

$$\mathbf{f}_{\varepsilon} = (1 + \varepsilon^{p-1}) \mathbf{f}, \quad (2.5)$$

et notons par $\mathbf{u}_{\varepsilon}$ la solution du problème correspondant, c'est-à-dire



$$B(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{v} - \mathbf{u}_\varepsilon) + \mu \int_{\Omega} |\mathbf{D}(\mathbf{u}_\varepsilon)|^{p-2} \mathbf{D}(\mathbf{u}_\varepsilon) \cdot \mathbf{D}(\mathbf{v} - \mathbf{u}_\varepsilon) dx + \\ g \int_{\Omega} |\mathbf{D}(\mathbf{v})| dx - g \int_{\Omega} |\mathbf{D}(\mathbf{u}_\varepsilon)| dx \geq \int_{\Omega} \mathbf{f}_\varepsilon \cdot (\mathbf{v} - \mathbf{u}_\varepsilon) dx \quad \forall \mathbf{v} \in W_{p,div}. \quad (2.6)$$

On pose maintenant

$$\mathbf{w}_\varepsilon = \frac{\mathbf{u}_\varepsilon}{\varepsilon}, \quad \forall \varepsilon > 0. \quad (2.7)$$

Dans ce qui suit nous établissons un résultat de convergence pour $(\mathbf{w}_\varepsilon)_{\varepsilon>0}$ quand ε tend vers 0.

Théoreme

Supposons que $\mathbf{f}$ une force de blocage. Alors, $(\mathbf{w}_\varepsilon)_{\varepsilon>0}$ converge fortement vers $\mathbf{w}$ solution de l'inégalité variationnelle suivante quand ε tend vers 0 dans $W_{p,div}$

$$\mathbf{w} \in C \mid \mu \int_{\Omega} |\mathbf{D}(\mathbf{w})|^{p-2} \mathbf{D}(\mathbf{w}) \cdot \mathbf{D}(\mathbf{v} - \mathbf{w}) dx \geq \int_{\Omega} \mathbf{f} \cdot (\mathbf{v} - \mathbf{w}) dx \quad \forall \mathbf{v} \in C. \quad (2.8)$$

Corollaire

En notant $\mathbf{w}_0$ l'unique solution de l'équation variationnelle donnée par

$$\mu \int_{\Omega} |\mathbf{D}(\mathbf{w}_0)|^{p-2} \mathbf{D}(\mathbf{w}_0) \cdot \mathbf{D}(\mathbf{v}) dx = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} dx \quad \forall \mathbf{v} \in W_{p,div}. \quad (2.9)$$

Alors, les estimations suivantes tiennent

$$\|\mathbf{D}(\mathbf{w})\|_{L^p(\Omega)^{n \times n}} \leq \|\mathbf{D}(\mathbf{w}_0)\|_{L^p(\Omega)^{n \times n}} \quad \text{et} \quad \int_{\Omega} \mathbf{f} \cdot \mathbf{w} dx \leq \int_{\Omega} \mathbf{f} \cdot \mathbf{w}_0 dx \quad (2.10)$$

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Non Markovian Queue with Two Types service Optional Re-service and General Vacation Distribution

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Abstract. We consider a single server batch arrival queueing system, where the server provides two types of heterogeneous service. A customer has the option of choosing either type 1 service with probability p_1 or type 2 service with probability p_2 with the service times follow general distribution. After the completion of either type 1 or type 2 service a customer has the option to repeat or not to repeat the type 1 or type 2 service. As soon as the customer service is completed, the server will take a vacation with probability θ or may continue staying in the system with probability $1 - \theta$. The re-service periods and vacation periods are assumed to be general. Using supplementary variable technique, the Laplace transforms of time dependent probabilities of system state are derived and thus we deduce the steady state results. We obtain the average queue size and average waiting time. Some system performance measures and numerical illustrations are discussed.

Keywords: Two type of service; Time dependent solution; Optional re-service; Average queue size; Average waiting time.



Introduction

Queueing models are essential for designing and monitoring of several communication systems. In queueing theory, customers arrive randomly according to a Poisson process and form a queue. In some practical situations like stored and forward communication networks the arrivals cannot be characterized by a Poisson process the arriving messages are converted into a random number of packets depending upon the size of the message. Similarly, in railway yards and ports the cargo handling is done in batches of random size. For analyzing these sort of situations the queueing models with bulk arrivals are developed. The bulk arrivals can be well characterised by a compound Poisson process. In a typical manufacturing situation, the work-pieces arrive at a machine center in batches and they leave in batches. A batch consists of identical work-pieces that are processed and then transported in batches for further processing. Such a situation can be modeled as queues with bulk arrivals. There is a discipline within the mathematical theory of probability, called a bulk queue (also called batch queue) where customers are served in groups of random size. The purpose of present paper is to provide overview of queueing models with phase service and its applications in real life queueing problems. The motivation for studying the queueing systems with phase service comes from numerous versatile applications in the performance evaluation and dimensioning of production and manufacturing systems, computer and communication networks, inventory and distribution systems, and so forth. During the last few decades, attention has been paid increasingly by many researchers in studying the phase service queueing models. In recent years, computer networks and data communication systems are the fastest growing technologies, which have led to significant development in applications such as swift advance in internet, audio data traffic, video data traffic, etc.

Main results

8. Numerical results

The most common distributions which are frequently used in the real time systems are the exponential and Poisson distributions. In-fact, the exponential distribution is widely accepted because the queueing models having exponential distribution are of practical utility and very easy to be handled. The main target of selecting a proper distribution and estimating their parameters is to provide a tractable analytical model giving a close approximation to the real life system under consideration.

To numerically illustrate the results obtained in this work, we consider that the service times and vacation times are exponentially distributed with rates μ_1, μ_2 and γ . In order to see the effect of various parameters on server's idle time Q , utilization factor ρ and various other queue characteristics such as L_q, L, W_q, W . We base our numerical example on the result found in case 1. For this purpose in Table I, we choose the following arbitrary values: $E(I) = 0.3$, $E(I(I-1)) =$



0.04, $r_1 = 0.4$, $r_2 = 0.5$, $\mu_1 = 6$, $\mu_2 = 4$ and $p_1 = 0.4$, $p_2 = 0.6$ while λ varies from 0.1 to 1.0 such that the stability condition is satisfied.

It clearly shows as long as increasing the arrival rate, the server's idle time decreases while the utilization factor, the average queue size, system size of our queueing model are all increases.

We base our numerical example on the result found in case 2 . For this purpose in Table II we choose the following arbitrary values: $E(I) = 0.3$, $E(I(I - 1)) = 0.04$, $r_1 = 0.3$, $\mu_1 = 4$, $\theta = 0.6$, $\lambda = 2$ while γ varies from 1 to 10 such that the stability condition is satisfied. It clearly shows as long as increasing the vacation rate, the server's idle time increases while the utilization factor, average queue size, system size of our queueing model are all decreases. For the effect of the parameters λ, γ, W_q, W on the system performance measures, two dimensional graphs are drawn.

Table I: Computed values of various queue characteristics

λ	Q	ρ	L_q	L
0.1	0.990450	0.009550	0.000729	0.010279
0.2	0.980900	0.019100	0.001647	0.020747
0.3	0.971350	0.028650	0.002759	0.031409
0.4	0.961800	0.038200	0.004070	0.042270
0.5	0.952250	0.047750	0.005588	0.053338
0.6	0.942700	0.057300	0.007317	0.064617
0.7	0.933150	0.066850	0.009266	0.076116
0.8	0.923600	0.076400	0.011439	0.087839
0.9	0.914050	0.085950	0.013846	0.099796
1.0	0.904500	0.095500	0.016492	0.111992

Table II: Computed values of various queue characteristics

γ	Q	ρ	L_q	L
1	0.445000	0.555000	0.537798	1.092798
2	0.625000	0.375000	0.207760	0.582760
3	0.685000	0.315000	0.141372	0.450372
4	0.715000	0.285000	0.114969	0.399969
5	0.733000	0.267000	0.101191	0.368191
6	0.745000	0.255000	0.092846	0.347846
7	0.753570	0.246430	0.087290	0.333718
8	0.760000	0.240000	0.083342	0.323342
9	0.765000	0.235000	0.080401	0.315401
10	0.769000	0.231000	0.078129	0.309129

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Elliptic Curves and their applications in Cryptography

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Abstract. In this talk, we present a method of cryptography based on elliptic curves. In the first time, we study the elliptic curve over finite fields. Next, we give a numerical example of encryption over the elliptic curve group $E(\mathbb{F}_q)$ with a secret key.

Keywords: elliptic curve, torsion group, finite field, cryptography.

Introduction

Elliptic curve cryptography (ECC) has been an active area of research since 1985 when Koblitz [6] and Miller [8] independently suggested using elliptic curves



for public-key cryptography. ECC is a form of public-key cryptography based on the algebraic structure of elliptic curves over finite fields. ECC is recognized for its ability to provide the same level of security as traditional public-key cryptosystems (like RSA) but with much smaller key sizes. However, their security is based on the hardness of a different problem, namely the Elliptic Curve Discrete Logarithmic Problem (ECDLP).

In this talk we will present some properties of elliptic curves over any field K , then we define the group law geometrically and explicitly on these curves. We will mention briefly some of their properties over finite fields. Next, we will provide an overview of cryptography and its classifications; after that, we will discuss the elliptic curve discrete Logarithm problem, Diffie Hellman key exchange, and digital signatures based on elliptic curves with an illustrated example. Finally, we will conclude our presentation with some notes and future work on ECC.

Auxiliary results

An elliptic curve is defined as a smooth(projective) algebraic curve of genus 1 with a specified point at infinity $\mathcal{O}$. In affine coordinates, an elliptic curve E can be represented by the equation:

$$E : y^2 = x^3 + ax + b, \quad (0.452)$$

where a and b are constants that define the curve's shape.

The discriminant $\Delta = -16(4a^3 + 27b^2)$ determines the non-singularity of the curve.

Elliptic curves have several important properties and invariants:

- **Torsion Points:** Elliptic curves have a finite set of rational points, known as torsion points, which form a cyclic group.
- **Rank:** The rank of an elliptic curve is an integer representing the number of independent rational points that form an infinite subgroup. The rank can be 0 or any positive integer.
- **J-Invariant:** The j -invariant is an invariant that distinguishes non-isomorphic elliptic curves. It is a complex number calculated from the coefficients of the elliptic curve equation.

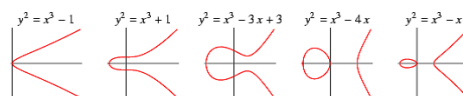


Figure 29: Elliptic Curves over $\mathbb{R}$



In the realm of mathematics and cryptography, we define an elliptic curve over $\mathbb{F}_p$ to be an equation of the form

$$E : y^2 = x^3 + ax + b \pmod{p}$$

and then we look at the points on E with coordinates in $\mathbb{F}_p$, which we denote by $E(\mathbb{F}_p)$.

The next theorem show us that the group of points $E(\mathbb{F}_p)$ is always either a cyclic group or the direct product of two cyclic groups.

Theorem 0.182. *Let E be an elliptic curve over the finite field $\mathbb{F}_p$. Then*

$$E(\mathbb{F}_p) \cong \mathbb{Z}_n \quad \text{or} \quad \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2}$$

for some integer $n \geq 1$, or for some integers $n_1, n_2 \geq 1$ with $n_1 | n_2$.

To set ECC it is important to know the order of the group. Hasse's theorem gives bounds for the group of points on an elliptic curve over a finite field.

Theorem 0.183. (Hasse) *Let E be an elliptic curve over a finite field $\mathbb{F}_p$, then the order of $E(\mathbb{F}_p)$ satisfies*

$$|p + 1 - \#E(\mathbb{F}_p)| \leq 2\sqrt{p}$$

Elliptic Curve Cryptography (ECC)

In this section*, we will discuss some cryptosystems based on elliptic curves especially on the discrete logarithm problem (DLP) for elliptic curves.

Elliptic Curve Discrete Logarithm Problem (ECDLP): is the problem of finding an integer n such that $Q = nP$. In other words, we need to find out how many times P must be added to itself in order to get Q .

Elliptic Curve Diffie –Hellman key exchange (ECDH): is a method for securely exchanging cryptographic keys over a public channel. Let $E(\mathbb{F}_p)$ be an elliptic curve over a finite field and let $P \in E(\mathbb{F}_p)$. The Elliptic Curve Diffie–Hellman Problem (ECDHP) is the problem of computing the value of $n_1 n_2 P$ from the known values of $n_1 P$ and $n_2 P$.

Elliptic curve digital signatures: offer a reliable method to ensure the authenticity and integrity of messages. By using their private key, a sender can sign a message, while the recipient can verify the signature's legitimacy with the corresponding public key. Digital signatures based on ECC are extensively employed in secure communication protocols.

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Prime Bayésienne du système Bonus-malus basée sur la fréquence et le montant du sinistre

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Abstract. Dans ce travail, nous étudions un modèle de pêche qui décrit la dynamique d'une ressource marine exploitée avec stock variable. Notre objectif est de maximiser les captures et les revenus de la pêche sans risquer l'extinction du stock exploité. Pour cela, nous avons développé un modèle mathématique et nous avons étudié les points d'équilibre, la stabilité local et global en utilisant le critère de RouthHurwitz et la fonction de Lyapunov, la bifurcation de Hopf et nous avons fini l'analyse du modèle par les simulations numériques.

Keywords: Système Bonus-Malus, Méthode Bayésienne, Poisson-Akash, Inverse-Gamma Lindley.

Introduction

La majorité des systèmes bonus-malus fixent les primes pour chaque assuré en se basant uniquement sur le nombre de sinistres. Opter pour cette approche serait inéquitable. Un conducteur qui fait une réclamation pour des dommages légers ne peut être soumis aux mêmes sanctions qu'un conducteur responsable d'une réclamation impliquant des dommages importants. Pour rétablir l'équilibre du portefeuille entre les bons et les mauvais conducteurs, une partie supplémentaire de la prime est déterminée comme un ajustement. Pour ce faire, le montant des sinistres est introduit afin de fournir une prime équitable. Cette approche repose sur l'hypothèse que la fréquence des sinistres suit une distribution de



Poisson-Akash, mais que la gravité des sinistres suit une distribution nouvellement proposée, Inverse-Gamma Lindley. On a déterminé les primes en utilisant la méthodologie Bayésienne, où la prime est évaluée en se basant sur la fonction de perte Linex. Une comparaison des primes en utilisant le logiciel R sera présenté.

La fréquence de sinistres

La distribution Poisson-Akash utilisée est le mélange des deux distributions comme suit :

$$\begin{aligned} f(l) &= \int_0^\infty P(l|\theta)\pi(\theta)d\theta \\ &= \int_0^\infty \frac{e^{-\theta}\theta^l}{l!} \left(\frac{\gamma^3}{\gamma^2+2} \right) (1+\theta^2) e^{-\gamma\theta} d\theta \\ f(l) &= \frac{\gamma^3}{\gamma^2+2} \frac{\gamma^2+2\gamma+l^2+3l+3}{(1+\gamma)^{l+3}} \end{aligned}$$

$l_1, l_2, \dots, l_t$ est un échantillon de taille t . $N = \sum_{i=1}^t l_i$ est le nombre total de réclamations qu'un assuré a faites en t ans, où l_i est le nombre de réclamations faites par un assuré sur i ans, $i = 1, 2, \dots, t$. La fonction de vraisemblance est :

$$L(\theta; l_1, l_2, \dots, l_n) = \prod_{i=1}^t \frac{e^{-\theta}\theta^{l_i}}{l_i!} = \frac{1}{\prod l_i!} e^{-t\theta} \theta^{\sum l_i} \propto e^{-t\theta} \theta^N.$$

La distribution a priori est : $\pi(\theta) = \frac{\gamma^3}{\gamma^2+2} (1+\theta^2) e^{-\gamma\theta} \propto (1+\theta^2) e^{-\gamma\theta}$.

Ainsi, la fonction de distribution a posteriori pour la fréquence des sinistres utilisant la méthode bayésienne est exprimée par

$$\pi^*(\theta|l_1, \dots, l_n) = \frac{(t+\gamma)^{n+3}}{\Gamma(n+1)[(n+2)(n+1) + (t+\gamma)^2]} * \theta^N (1+\theta^2) e^{-\theta(t+\gamma)}.$$

Le montant du sinistre

La distribution Inverse-Gamma Lindley est présentée comme suit :

$$\begin{aligned} f(x) &= \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{-\alpha-1} e^{-\frac{\lambda}{x}} \frac{\beta^2}{\beta+1} (\lambda+1) e^{-\beta\lambda} d\lambda \\ &= \frac{x^{-\alpha-1} \beta^2}{\Gamma(\alpha)(\beta+1)} \int_0^\infty \lambda^\alpha (\lambda+1) e^{-\lambda(\beta+\frac{1}{x})} d\lambda \\ f(x) &= \frac{x^{-\alpha-1} \beta^2 (\alpha+1)}{(\beta+1)(\beta+\frac{1}{x})^{\alpha+2}} \left[\alpha+1+\beta+\frac{1}{x} \right]. \end{aligned}$$

$N = \sum_{i=1}^t l_i$ est le nombre total de sinistres effectués par un assureur sur t ans. Soit X la taille du sinistre l pour $l = 1, 2, \dots, N$. La fonction de vraisemblance



est :

$$L(\lambda|x_1, \dots, x_n) = f(x_1, \dots, x_n|\lambda) = \prod_{i=1}^n \frac{\lambda^\alpha x_i^{-\alpha-1}}{\Gamma(\alpha)} e^{-\lambda \frac{1}{x_i}} \propto \lambda^{n\alpha} e^{-\lambda \sum_{i=1}^n \frac{1}{x_i}}.$$

La distribution a priori est : $\pi(\lambda) \propto (\lambda + 1)e^{-\beta\lambda}$. Ainsi, la fonction de distribution a posteriori pour le montant est donnée par :

$$\pi^*(\lambda|x_1, \dots, x_n) = \frac{\left(\beta + \sum_{i=1}^n \frac{1}{x_i}\right)^{\alpha n + 2}}{\Gamma(\alpha n + 1) \left(\alpha n + 2 + \beta + \sum_{i=1}^n \frac{1}{x_i}\right)} * \lambda^{n\alpha} (1 + \lambda) e^{-\lambda \left(\beta + \sum_{i=1}^n \frac{1}{x_i}\right)}.$$

Prime Finale basé sur la fréquence et le montant du sinistre

La prime finale basée sur la distribution Poisson-Akash pour la fréquence du sinistre et l'Inverse-Gamma Lindley pour le montant du sinistre que l'assuré doit payer est :

$$\begin{aligned} Premium_{t+1} &= \frac{(n+1)(n+2)(n+3) + (n+1)(t+\gamma)^2}{(t+\gamma)^3 + (t+\gamma)(n+2)(n+1)} \\ &\quad * \frac{(\alpha n + 1)(\alpha n + 2 + \beta + \sum_{i=1}^n \frac{1}{x_i})}{(\alpha - 1) \left(\beta + \sum_{i=1}^n \frac{1}{x_i}\right) (\alpha n + 2 + \beta + \sum_{i=1}^n \frac{1}{x_i})}. \end{aligned}$$

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Impact of Missing Data Percentage on the Consistency of a non parametric Gamma Estimators

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Abstract. In this paper, we examine how the missing data percentage p affects the consistency performance of a non parametric Gamma estimator and modified version of this estimator. We assess their consistency by computing the empirical mean squared error (MSE), via simulation studies.

Keywords: Gamma estimator, Missing Data, Regression function, Hille's lemma.



Introduction

Nonparametric regression estimation methods are well-developed for completely observed data, but ignoring missing data components may lead to inconsistent estimation. In this paper, we propose a generalized nonparametric estimator of the regression function with non-negative support based on an incomplete sample $(X_i, Y_i, \zeta_i)_{1 \leq i \leq n}$ drawn from a discrete time stationary and ergodic process (X, Y, ζ) , where ζ is a Bernoulli random variable (r.v.). Our approach uses asymmetric gamma probability densities as kernels. Unlike symmetric kernels, gamma kernels are flexible and have non-negative support. They also match the supports of the design variables, ensuring that the estimator assigns positive mass only to the relevant regions. We will conduct simulation studies to evaluate the finite sample performance of two proposed estimators: gamma estimator and the modified gamma estimator (referred to them as CLS, CLSM).

Gamma estimator and the modified Gamma estimator

Following the perturbed approach of Chaubey et al. (2010)[1] for complete data, we consider the smooth regression estimator for incomplete data $(X_i, Y_i, \zeta_i)_{1 \leq i \leq n}$, defined by

$$\hat{m}_{\psi_{y,n}}(x^+) = \frac{n^{-1} \sum_{i=1}^n \zeta_i \psi_y(Y_i) Q_{x^+, v_n}(X_i)}{n^{-1} \sum_{i=1}^n \zeta_i Q_{x^+, v_n}(X_i)} = \frac{\hat{m}_{\psi_{y,2}}(x^+)}{\hat{m}_{\psi_{y,1}}(x^+)}, \quad x^+ \neq 0 \quad (0.453)$$

whenever $\sum_{i=1}^n \zeta_i Q_{x^+, v_n}(X_i) \neq 0$. Here, $x^+ =: x + \epsilon_n$, $x \geq 0$, and $(\epsilon_n =: \epsilon)$ is a sequence of non-negative real numbers tending to zero at an appropriate (sufficiently slow) rate as $n \rightarrow +\infty$. Q_{x^+, v_n} is a Gamma kernel density defined, for any $t > 0$, by $Q_{x^+, v_n}(t) = \frac{1}{\beta_{x^+}^\alpha \Gamma(\alpha)} t^{\alpha-1} e^{-\frac{\alpha t}{x^+}}$ with $\alpha =: \alpha_n = 1/v_n^2$ and $\beta_{x^+} =: \beta_{x^+, n} = v_n^2 x^+$.

The estimator (0.453) is based on an extension of Hill's Lemma, which avoids possible inconsistency at 0 (for more details, refer to section* of Chaubey et al. (2010))[1].

The proposed estimators, denoted by $\hat{m}_{\psi_{y,n}}(x^+)$, and the modified Gamma estimator $\hat{m}_{mod}(x^+)$ defined below (referred to as CLS and CLSM, respectively):

$$\hat{m}_{\psi_{y,n}}(x^+) = \frac{\sum_{i=1}^n \zeta_i \psi_y(Y_i) Q_{x+\epsilon, v_n}(X_i)}{\sum_{i=1}^n \zeta_i Q_{x+\epsilon, v_n}(X_i)} \quad \text{and} \quad \hat{m}_{mod}(x^+) = \frac{\sum_{i=1}^n \zeta_i \psi_y(Y_i) Q_{\rho_\epsilon(x), v_n}(X_i)}{\sum_{i=1}^n \zeta_i Q_{\rho_\epsilon(x), v_n}(X_i)} \quad x \geq 0, \quad (0.454)$$

where $\rho_x = \max(0.5x + \epsilon, x)$, $x^+ = x + \epsilon$ and $\epsilon = \epsilon_n$.

Remark 0.11. In the first estimator, the perturbation involves replacing x with $x + \epsilon$ for some $\epsilon := \epsilon_n$ that tends to 0. This perturbation is applied to all values of x , even x is very far from 0. However, in the second estimator, we replace x by



a perturbation function $\rho_\epsilon(x)$ that behaves similar to $x + \epsilon$. We take $\rho_\epsilon(x) = x$ if $x \geq 2\epsilon$ and $\rho_\epsilon(x) = \frac{x+2\epsilon}{2}$ if $x < 2\epsilon$. This choice allows to applies perturbation only near the boundary.

The properties of the perturbed estimator have been discussed in a recent study by [2], in the context of density estimation.

Selection of Smoothing Parameters

To select the values of smoothing parameters (v_n, ϵ_n) , we adopt the Cross Validation (CV) approach. The CV estimate is computed as the mean square error obtained by leaving one observation out:

$$CV(v_n, \epsilon_n) = \frac{1}{n} \sum_{j=1}^n (Y_j - M_j(X_j))^2,$$

where M_j stands for either $\hat{m}_{\psi_{y,n}}$ or $\hat{m}_{mod}$, evaluated at X_j but based on the X_j -deleted sample $\{X_1, \dots, X_{j-1}, X_{j+1}, \dots, X_n\}$, $1 \leq j \leq n$. The optimal values $(\hat{v}_n, \hat{\epsilon})$ are determined by minimizing the CV function: $(\hat{v}_n, \hat{\epsilon}) = \arg \min_{v_n, \epsilon_n} CV(v_n, \epsilon_n)$. We repeat this process 200 times to obtain the optimal parameter estimates.

Numerical study

*

Illustration through a Simulation Study We will illustrate our method through a simulation study using a simulated dataset from the additive model:

$$Y = m(X) + \varepsilon, \text{ with } \varepsilon \hookrightarrow \mathcal{N}(0; 0.5^2), \quad (0.455)$$

where $m(X)$ takes the following forms: $m_1(X) = \frac{10}{1+X^2}$ and $m_2(X) = \frac{3}{4} \cos(5\pi X)$.

The r.v. X is generated from two different dependent models:

Model 1 (M1). $X_i = \frac{1}{2}X_{i-1} + \eta_i$, where the η_i 's are i.i.d symmetric Bernoulli r.v.'s taking values -1 and 1 . This process is ergodic but not α -mixing.

Model 2 (M2). $X_i = \frac{1}{2}X_{i-1} + \eta_i$, where the η_i 's are i.i.d. and follow a normal distribution $\mathcal{N}(0, 3.5^2)$.

In the following, we assume that the response variable Y is subject to a MAR with a logistic function $p(x) = \mathbb{P}(\zeta = 1 \mid X = x) = \text{expit}(2\alpha x^2) = \frac{e^{2\alpha x^2}}{1+e^{2\alpha x^2}}$, where $\alpha \geq 0$ and ζ is a Bernoulli r.v. The degree of dependency between X and the missing variable ζ is controlled by the parameter α with missing proportion $\bar{\zeta} = 1 - \frac{1}{n} \sum_{j=1}^n \zeta_j$. We take $\alpha \in \{0, 1\}$, which leads to $\bar{\zeta} = 50\%$ and $\bar{\zeta} = 20\%$, respectively.



*

Simulation: Impact of Missing Data Percentage on the Consistency of Estimators $\hat{m}_{\psi_{y,n}}$ and $\hat{m}_{mod}$ In the subsequent simulations, we examine how the missing data percentage p affects the consistency performance of the estimators $\hat{m}_{\psi_{y,n}}$ and $\hat{m}_{mod}$. We assess their consistency by computing the empirical mean squared error (MSE), which quantifies, in this case, the average squared difference between the estimator and the true function. We generate a sample of size $n = 100$ based on Models (M1) and (M2). The estimation procedures are replicated 500 times. Tables 10 and 11 present the optimal MSE values for different MAR rates in Models (M1) and (M2).

Table 10: Optimal parameters and MSE with different MAR rates for Model (M1).

2 * $n = 100$	CLS			CLSM		
	(ϵ_n^*, v_n^*)	$\bar{\zeta}$	MSE	(ϵ_n^*, v_n^*)	$\bar{\zeta}$	MSE
3 * m_1	(0.012,0.101)	50%	0.3250	(0.089,0.088)	50%	0.3280
	(0.012,0.101)	20%	0.3131	(0.089,0.088)	20%	0.3086
3 * m_2	(0.036,0.030)	50%	0.5056	(0.088,0.035)	50%	0.4469
	(0.036,0.030)	20%	0.4517	(0.088,0.035)	20%	0.3987

Comments. The simulation results reveal that both estimators $\hat{m}_{\psi_{y,n}}$ and $\hat{m}_{mod}$ are sensitive to the percentage of missing data. As the percentage increases, the performance of the estimators deteriorates, leading to higher values of mean squared error (MSE). Conversely, as the percentage of missing data decreases, the performance of the estimators improves. We observe that the CLSM method tends to outperform the CLS method, particularly in Model 2.

Graphical Illustrations. Figure 30 presents the estimated regression curves for Model 1 with $m = m_i$; $i = 1, 2$. The true regression curve is illustrated by a black line. These figures reveal that both methods perform similarly well for both models, with a slight advantage for the Gamma modified (CLSM).

Figure 31 presents the simulated Average Squared Error for Model (M1) with $m(x) = m_1(x)$ and MAR rates of 50% and 20% at different values of x . The results indicate that the (CLSM) estimator outperforms the (CLS) estimator when the MAR rate is close to zero. However, the two methods have similar performances when the MAR rate is far from zero.



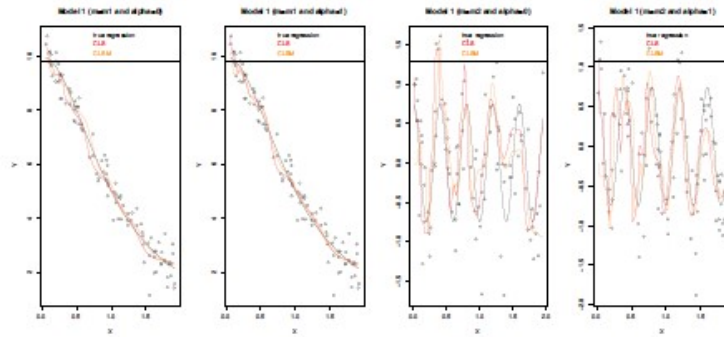


Figure 30: Scatterplot and regression estimators for Model (M1) with $\bar{\zeta} = 50\%$ and $\bar{\zeta} = 20\%$.

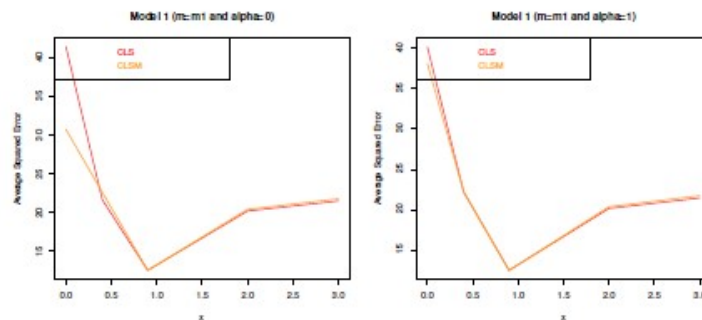


Figure 31: Simulated MSE for (M1) with $m(x) = m_1(x)$ and $\bar{\zeta} = 50\%$ and 20% .



Table 11: Optimal parameters and MSE with different MAR rates for Model (M2).

3 * $n = 100$	CLS			CLSM		
	(ϵ_n^*, v_n^*)	$\bar{\zeta}$	MSE	(ϵ_n^*, v_n^*)	$\bar{\zeta}$	MSE
3 * m_1	(0.017,0.150)	50%	0.3485	(0.124,0.137)	50%	0.3440
	(0.017,0.150)	20%	0.3113	(0.124,0.137)	20%	0.3120
3 * m_2	(0.359,0.104)	50%	0.6377	(0.280,0.229)	50%	0.5961
	(0.359,0.104)	20%	0.5662	(0.280,0.229)	20%	0.5706

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New Iterative Modification of the Conjugate Gradient Method for Unconstrained Problems

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Abstract. Conjugate gradient (CG) methods are a fundamental tool for solving large-scale unconstrained optimization problems. Recently, much attention has been given to these methods due to their efficiency. In this work, we introduce a new modification of the MDY (Modification for Dai-Yuan) conjugate gradient method by adding a third term to the update formula. This additional term plays a crucial role in determining the descent direction, enhancing the convergence properties and performance of the method. Our modified three-term CG approach demonstrates superior solution efficiency and improved convergence characteristics compared to the traditional MDY method. Numerical experiments validate the effectiveness of our proposed method, showing its capability to address a wide range of optimization problems across various domains.

Keywords: Conjugate gradient direction, descent condition, global convergence, numerical tests.



Introduction

Main results

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and the following unconstrained optimization problem

$$\min \{f(x), x \in \mathbb{R}^n\}.$$

We previously saw that the various conjugate gradient methods generate a sequence $\{x_k\}_{k \in \mathbb{N}}$ in the following manner:

$$\begin{cases} x_0 \text{ chosen in } \mathbb{R}^n, \\ x_{k+1} = x_k + \alpha_k d_k, \quad k \geq 0, \end{cases} \quad (0.456)$$

The step size $\alpha \in]0, +\infty[$ is determined by a line search, and it is very important for the global convergence of conjugate gradient methods. It is often required that the line search satisfies the Wolfe conditions [1, 2]:

$$f(x_k + \alpha_k d_k) - f(x_k) \leq \delta \alpha_k g_k^t d_k, \quad (0.457)$$

$$g_k^t d_{k-1} \geq \sigma g_{k-1}^t d_{k-1}, \quad (0.458)$$

Where $0 < \delta < \sigma < 1$. $d_k \in \mathbb{R}^n$ is the search direction defined by the following formula:

$$\begin{cases} d_0 = -g_0, \\ d_k = -g_k + \beta_k d_{k-1}, \quad k \geq 1. \end{cases} \quad (0.459)$$

Where $g_k = \nabla f(x_k)$ and $\beta_k \in \mathbb{R}^*$ is known as the conjugate gradient coefficient. There are several classical formulas for β_k [3], including Hestenes-Stiefel (1952), Fletcher-Reeves (1964), Polak-Ribière-Polyak (1969), Conjugate Descent (1987), Liu-Storey (1991), and Dai-Yuan (1999).

$$\begin{aligned} \beta_k^{HS} &= \frac{g_k^t y_{k-1}}{d_{k-1}^t y_{k-1}} & \beta_k^{FR} &= \frac{\|g_k\|^2}{\|g_{k-1}\|^2} & \beta_k^{PRP} &= \frac{g_k^t y_{k-1}}{\|g_{k-1}\|^2} \\ \beta_k^{CD} &= \frac{\|g_k\|^2}{-g_{k-1}^t d_{k-1}} & \beta_k^{LS} &= \frac{g_k^t y_{k-1}}{-g_{k-1}^t d_{k-1}} & \beta_k^{DY} &= \frac{\|g_k\|^2}{d_{k-1}^t y_{k-1}} \end{aligned}$$

Where $y_k = g_k - g_{k-1}$ and $\|\cdot\|$ represents the Euclidean norm. Hailin Liu, Sui Sun, and Xiaoyong Li [4] proposed a modification of the classic DY (Dai-Yuan) method with the following adjustment:

$$\beta_k^{MDY} = \frac{\|g_k\|^2}{\mu \|d_{k-1}^t g_k\| + d_{k-1}^t y_{k-1}}, \quad \mu > 1. \quad (0.460)$$



Our method consists of modifying the classic MDY conjugate gradient direction into a three-term conjugate gradient direction by adding the term $\theta_k g_k$ is given in the following form

$$d_k = -g_k + \beta_k d_{k-1} - \theta_k g_k.$$

Where β_{MDY}^k is defined in equation (0.460) and θ_k is defined that our method verify the descent direction, so it given by the following formula:

$$\theta_k = \frac{g_k^t d_{k-1}}{\mu \|d_{k-1}^t g_k\| + d_{k-1}^t y_{k-1}}, \quad \mu > 1 \quad (0.461)$$

Finally, numerical results are presented to compare our new method with other conjugate gradient methods.

Preliminaries

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The descent direction

Theorem 0.184. *If $\mu > 1$, then the direction is a sufficient descent direction for all $k \in \mathbb{N}$, i.e*

$$g_k^t d_k \leq -C \|g_k\|^2, \forall k \geq 0, C > 0.$$

Proof For $k = 0$, we have the directions $d_0 = -g_0$, then

$$g_0^t d_0 = -\|g_0\|^2, \text{ for } C = 1$$

For $k > 0$, considering the condition (0.461), we can establish that the direction satisfy the sufficient descent condition with $C = 1$. ■

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Global convergence In this section*, we present the global convergence analysis for our direction. To establish the global convergence, certain basic assumptions are required. Consider the level set $\Delta = \{x \in \mathbb{R}^n : f(x) \leq f(x_0)\}$ to be bounded, where x_0 is the initial point of the iteration. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuously differentiable function in a neighborhood ν of Δ . In other words, there exists a constant $D > 0$ such that

$$\|x\| \leq D, \quad \forall x \in \nu. \quad (0.462)$$

The gradient of the function f is Lipschitz continuous in the neighborhood $\mathbb{N}$. In other words, there exists a positive constant $L > 0$ such that the following inequality holds true for all x_1, x_2 in $\mathbb{N}$

$$\|g(x_1) - g(x_2)\| \leq L \|x_1 - x_2\|, \quad \forall x_1, x_2 \in \mathbb{N}. \quad (0.463)$$

We determine the result of the global convergence of our new method based on the two lemmas and the theorem.



Lemma 0.61. Suppose that assumptions 3.1 and 3.2 are satisfied. Let the sequence $\{x_k\}_{k \in \mathbb{N}}$ be generated by our method, and let $d_k \in \mathbb{R}^n$ satisfy the descent condition. α_k is determined by the Wolfe line search conditions (0.457), (0.458). If

$$\sum_{k=0}^{\infty} \frac{1}{\|d_k\|^2} = \infty, \quad (0.464)$$

Then

$$\lim_{k \rightarrow \infty} \inf \|g_k\| = 0.$$

Lemma 0.62. Suppose that assumptions 3.1 and 3.2 are satisfied and that the sequence $\{x_k\}_{k \in \mathbb{N}}$ generated by our method, with $d_k \in \mathbb{R}^n$, satisfies the descent condition. α_k is determined by the Wolfe line search conditions (0.457), (0.458). Then

$$\alpha_{k-1} \geq \frac{(1 - \sigma) |g_{k-1}^t d_{k-1}|}{L \|d_{k-1}\|^2}. \quad (0.465)$$

Theorem 0.185. Suppose that assumptions 3.1 and 3.2 are satisfied and that the sequence of vectors $\{x_k\}_{k \in \mathbb{N}}$ is generated by our method. Then, α_k is determined by the Wolfe line search conditions (0.457), (0.458), and condition (0.464) is satisfied. Then

$$\lim_{k \rightarrow \infty} \inf \|g_k\| = 0. \quad (0.466)$$

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Numerical results In this section*, we provide numerical test results that compare the performance of our direction, which implements the Wolfe line search conditions (0.457), (0.458) with $\delta = 0.0001$ and $\sigma = 0.1$, using the parameter $\mu = 1.1$. We compare it with other conjugate gradient methods. The new method provides significantly better performance than the other methods.

Conclusion

We have presented a new modified conjugate gradient direction. This modification shows that we have achieved the descent direction and global convergence using the Wolfe line search conditions. Numerical results demonstrate that our method is effective compared to well-known conjugate gradient methods.

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Optimal Control in Partially Observed McKean-Vlasov FBSDEs with Random Jumps: A Stochastic Maximum Principle Approach

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Abstract. This paper investigates the stochastic maximum principle for a partially observable optimal control problem involving forward backward stochastic differential equations (FBSDEs) of the McKean-Vlasov type, which are driven by both a Poisson random measure and an independent Brownian motion. The system's coefficients and the cost functional depend on the state of the solution, its probability distribution, and the control variable. Necessary optimality conditions are established under the assumption of a convex control domain, with the results relying on Girsanov's theorem and the derivatives with respect to probability law.



Keywords: FBSDEs with jump processes, Control systems with partial observation, Stochastic maximum principle, McKean-Vlasov differential equations.

Introduction

This study investigate the partially observed optimal control problem for forward-backward stochastic differential equations (FBSDEs) of the McKean-Vlasov type, driven by both Poisson random measures and independent Brownian motion. These problems have significant applications in areas like engineering, economics, and finance. Previous research on the stochastic maximum principle for McKean-Vlasov systems without partial observation includes Buckdahn et al.[13], who used second-order derivatives with respect to probability measures, and Carmona and Delarue [9], who provided sufficient conditions for the existence of optimal control. Additionally, Meherrem and Hafayed extended the principle to McKean-Vlasov FBSDEs with Lévy processes [14].

The partially observed stochastic optimal control problem for forward-backward stochastic differential equations (FBSDEs) has been widely studied. Wu [1] proved the maximum principle for such systems, while Shi and Wu [2] addressed fully coupled FBSDEs. Nie and Yan [1] extended these problems to cases where both the state and observation depend on joint distributions. Xiao and Wang [3] investigated systems with random jumps, and Zhou and Zhang applied Malliavin calculus and Lévy processes with regime-switching. This paper aims to prove the necessary conditions for partially observed optimal control of McKean-Vlasov FBSDEs with jumps, where system coefficients depend on both the state and its probability measures. The control domain is assumed to be convex, and the proof uses variational techniques and stochastic differential equation estimates. The aim of this paper is to establish both necessary conditions for the partially observed optimal control problem involving McKean-Vlasov forward-backward stochastic differential equations (FBSDEs) with jumps. The system's coefficients and the cost functional depend on the state variables $(x^v(t), y^v(t), z^v(t), \bar{z}^v(t), r^v(t))$, as well as their associated probability measures $(P_{x^v(t)}, P_{y^v(t)}, P_{z^v(t)}, P_{\bar{z}^v(t)}, P_{r^v(t)})$. The control domain is assumed to be convex. The proof of the main result relies on variational techniques and precise estimates of the corresponding stochastic differential equations.

Main results

In this paper, we present the necessary conditions for optimality in a McKean-Vlasov type system, focusing on the partially observed optimal control problem, assuming a solution exists. The proof is constructed using convex





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perturbation techniques and involves key estimates of the system's state processes and the observation process.



We establish the following variational inequality.

Lemma 0.63.

$$\begin{aligned}
0 \leq & \mathbb{E} \left[Z_T M_x(x_T, P_{x_T}) x_T^1 + Z_T \widehat{\mathbb{E}} \left[\partial_\mu^{P_x} M(x_T, P_{x_T}, \widehat{x}_T) \widehat{x}_T^1 \right] \right] \\
& + \mathbb{E} \left[Z_T^1 M(y_0, P_{y_0}) y_0^1 + \widehat{\mathbb{E}} \left[\partial_\mu^{P_y} h(y_0, P_{y_0}, \widehat{y}_0) \widehat{y}_0^1 \right] \right] \\
& + \mathbb{E} \int_0^T \left[Z_t^1 l(t) + Z_t \left(l_x(t) x_t^1 + \widehat{\mathbb{E}} \left[\partial_\mu^{P_x} l(t, \widehat{x}_t) \widehat{x}_t^1 \right] \right) + Z_t \left(l_y(t) y_t^1 + \widehat{\mathbb{E}} \left[\partial_\mu^{P_y} l(t, \widehat{y}_t) \widehat{y}_t^1 \right] \right) \right. \\
& + Z_t \left(l_z(t) z_t^1 + \widehat{\mathbb{E}} \left[\partial_\mu^{P_z} l(t, \widehat{z}_t) \widehat{z}_t^1 \right] \right) + Z_t \left(l_{\bar{z}}(t) \bar{z}_t^1 + \widehat{\mathbb{E}} \left[\partial_\mu^{P_{\bar{z}}} l(t, \widehat{\bar{z}}_t) \widehat{\bar{z}}_t^1 \right] \right) \\
& \left. + Z_t \left(l_r(t) r_t^1 + \widehat{\mathbb{E}} \left[\partial_\mu^{P_r} l(t, \widehat{r}_t) \widehat{r}_t^1 \right] \right) + Z_t l_v(t) v_t \right] dt.
\end{aligned}$$

The main result of this paper is presented in the following theorem.

Theorem 0.186 (Partial necessary conditions of optimality). *Let $(x, y, z, \bar{z}, r, u)$ be an optimal solution of our partially observed optimal control problem. Then, there are $(p, q, k, \bar{k}, n(\cdot))$ and $(P, \bar{Q}, Q)$ of $\mathbb{F}$ -adapted processes and that for all $v \in \mathcal{U}$, we have*

$$\mathbb{E}^u [H_v(t)(v_t - u_t) / \mathcal{F}_t^Y] \geq 0, a.e., a.s., \quad (0.467)$$

where the Hamiltonian function

$$H(t) = H(t, x_t, P_{x_t}, y_t, P_{y_t}, z_t, P_{z_t}, \bar{z}_t, P_{\bar{z}_t}, r_t, P_{r_t}, u_t, p_t, q_t, k_t, \bar{k}_t, n_t(\cdot), Q_t),$$

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Stochastic Modelling of Epidemic Systems with Fractional Brownian Motion

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Abstract. In this paper, we studied the dynamics of a stochastic *SIRDS* epidemic model and formulate the model as a fractional stochastic differential equation (FSDE, in short) with fractional Brownian Motion with Hurst index $H > \frac{1}{6}$. We prove the existence and uniqueness of the global positive solution using random Lyapunov function theory. The sufficient conditions for disease extinction are obtained.

Keywords: COVID-19; *SIRDS* model; Fractional Brownian motion; Existence and Uniqueness; Extinction.



Introduction

Epidemiology study the spread of diseases to identify their causes or risk factors. The theory and applications are very advanced, thanks to mathematical researches. When modelling epidemic diseases, it makes more sense to classify individual hosts into groups, taking into account different interaction patterns, different numbers of sexual partners, or different geographic considerations. Kermack Mckendrick's classic *SIR* model is the basis of almost all models [1]. The *SIR* model is divided into three compartments: *S*, *I*, and *R* represent the proportions of susceptible, infected, and recovered individuals, respectively. After the *SIR* model, many other models were created according to the nature of the epidemic or pandemic.

In general terms, the mathematical models applied to the epidemiology of infectious diseases can be classified into two types : The deterministic and stochastic models.

A limitation of deterministic modelling is that it is very challenging to anticipate a system's dynamics with any degree of accuracy. Deterministic models fail to account for the impact of a changing environment. As a result, the stochastic model has become essential for comprehending infectious dynamics.

We employ models based on a fractional Brownian motion (fBm in short) with a Hurst parameter $H \in (0, 1)$. Leonardo Lima [4] studied the transmission of infections (COVID-19) using fractional Brownian motion with $\frac{1}{2} < H < 1$. Elisa Als et al [5] studied a fractional model for the COVID-19 pandemic with application to Italian data with $H > \frac{1}{2}$. In our case we take $H > \frac{1}{6}$ (short memory), due to the high rate of infection transmission, individuals are implementing automatic quarantine measures and practising social distancing through isolation. This precautionary action leads to fluctuations in the transmission rate characterised by short-range dependence. The noise in the transmission rate arises as a consequence of these preventive measures and the resulting changes in social interactions.

In this paper, we defied stochastic *SIRDS* model this model takes into account a total population size of N_t and divides the population into four compartments: S_t , I_t , and R_t represent the proportions of susceptible, infected, and recovered individuals at time t , for $t \geq 0$, respectively, and D_t is the proportion of people who died as a result of COVID-19 at time t .

The model is shown in Figure 32: The deterministic model (0.468) below is

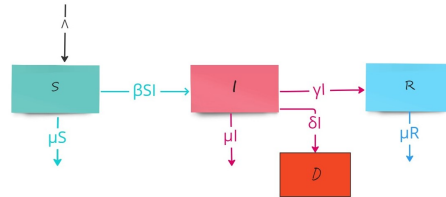


Figure 32: The flowchart of an *SIRDS* model.



derived from the description and flow diagram (see Figure 32) .

$$\begin{cases} \frac{dS_t}{dt} = \Lambda - \beta_t I_t S_t - \mu S_t + \theta R_t, \\ \frac{dI_t}{dt} = \beta_t I_t S_t - (\gamma + \delta + \mu) I_t, \\ \frac{dR_t}{dt} = \gamma I_t - (\theta + \mu) R_t, \\ \frac{dD_t}{dt} = \delta I_t. \end{cases} \quad (0.468)$$

The parameters in the model are summarised as following : Λ is the constant migration rate of susceptible populations. β_t represents the transmission rate between S and I at time t . γ represents the constant recovery rate for the disease infected individual. μ represents the natural death rate of S , I , and R compartments. δ is the death rate due to disease (COVID-19). θ is the rate at which the recovered individual loses immunity and returns to the susceptible individuals,

where parameters Λ , μ , are positive constants, and γ , δ , θ are non-negative constants.

Main results

In this paper, we take $H > \frac{1}{6}$ (there is short-range dependence) because the rate of infection transmission is high, then people apply automatic quarantine rules and maintain a distance through isolation; this is what causes the noise in the transmission rate with short-range dependence.

The stochastic version corresponding to the deterministic model (0.468) takes the following form:

$$\begin{cases} dS_t = [\Lambda - \beta_t I_t S_t - \mu S_t + \theta R_t]dt - \sigma I_t S_t dB_t^H, \\ dI_t = [\beta_t I_t S_t - (\gamma + \delta + \mu) I_t]dt + \sigma I_t S_t dB_t^H, \\ dR_t = [\gamma I_t - (\theta + \mu) R_t]dt, \\ dD_t = [\delta I_t]dt. \end{cases} \quad (0.469)$$

This system of FSDEs is called an FSDEs *SIRDS* model.

The following theorem provides the result that analyses the problem's existence analysis:

Theorem 0.187. *Let $X_t = (S_t, I_t, R_t, D_t)$. For any given initial Value $(S_0, I_0, R_0, D_0) \in \mathbb{R}_+^4$, the system of FSDEs (0.469) has a unique positive solution $\{S_t, I_t, R_t, D_t\}$ for all $t \geq 0$, and this solution stay in $\mathbb{R}_+^4$, with probability one. Namely,*

$$\mathbb{P}\{(S_t, I_t, R_t, D_t) \in \mathbb{R}_+^4, \quad \forall t \geq 0\} = 1.$$

In this section*, we investigate the conditions for the extinction of the disease.

Theorem 0.188. *For any initial value $(S_0, I_0, R_0, D_0) \in \mathbb{R}_+^4$ in model (0.469), if $R_0^S < 1$, then the disease will die out exponentially with probability one.*

$$\lim_{t \rightarrow \infty} \sup \frac{\log I_t}{t} \leq (\gamma + \mu + \delta)(R_0^S - 1) < 0 \quad a.s.$$

In addition

$$\lim_{t \rightarrow \infty} S_t = \frac{\Lambda}{\mu},$$

and

$$\lim_{t \rightarrow \infty} R_t = \lim_{t \rightarrow \infty} I_t = 0 \quad a.s.$$



Conclusion

In this paper, we studied a stochastic mathematical model *SIRDS* for the spread of COVID-19 driven by fractional Brownian motion for $H > \frac{1}{6}$. We showed that the model has a global positive solution, and obtained conditions for the extinction of the disease.

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Nonlinear Problem with Neumann Condition

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Abstract. Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain with $0 \in \partial\Omega$, $N \geq 3$, $0 \leq s < 2$, and define the critical Hardy-Sobolev exponent by $2^*(s) \triangleq \frac{2(N-s)}{N-2}$. In this paper, We establish the existence of positive weak solutions of the singular critical problem

$$-\Delta u = \frac{|u|^{2^*(s)-2}u}{|x|^s} + \lambda u$$

with Neumann conditions on Ω , where λ are positive parameter.

Keywords: Positive solutions, Critical Hardy-Sobolev exponent, elliptic equations, Neumann Condition.

Introduction

We study the existence of positive solutions to the following Neumann problem:

$$\begin{cases} -\Delta u &= \frac{|u|^{2^*(s)-2}u}{|x|^s} + \lambda u, & \text{if } x \in \Omega \\ \frac{\partial u}{\partial \nu} &= 0, & \text{if } x \in \partial\Omega, \end{cases} \quad (0.470)$$

where Ω is a smooth open bounded domain of $\mathbb{R}^N$ ($N \geq 3$) with $0 \in \partial\Omega$, λ are positive parametres, $0 \leq s < 2$ and $2^*(s) = 2(N-s)/(N-2)$ is the critical Hardy-Sobolev exponent. In the case where $s = 0$, the problem was addressed by Wang [2]. He demonstrated the existence of positive solutions under certain conditions on the boundary $\partial\Omega$. The main result is the following theorem [1]:

Theorem 0.189. *Let Ω be a bounded regular domain in $\mathbb{R}^n$ ($n \geq 3$) with a boundary of class C^2 . Suppose that $0 < s < 2$, $0 \in \partial\Omega$, and that the mean curvature of $\partial\Omega$ in the vicinity of 0 is positive. Then, the Neumann problem (0.470) admits a positive solution for all $\lambda < 0$.*





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On Controllability Of Linear Constant Coefficient Differential-Algebraic Equations

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Abstract. We investigate different concepts related to the controllability of linear constant coefficient differential-algebraic equations. Concepts like impulse controllability, controllability at infinity, strong and complete controllability are described and defined by using an equivalent form for linear time-invariant singular systems. Some real-life problems are given to illustrate the presented theory.

Keywords: Controllability, Differential-Algebraic Equations.

Introduction

Many physical phenomena are modeled by differential equations, but sometimes this mathematical model contains algebraic equations. Systems containing two types of differential and algebraic equations are called differential-algebraic equations (DAEs) [also known as singular systems, implicit systems, descriptor systems]. These systems have become a model of a wide



variety of problems in economics, demography, mechanical systems, the simulation of the dynamics of multi-body systems, electrical networks, fluid mechanics and chemistry, which often cannot be modeled by standard ODE systems. In this work, we consider a singular system of the form

$$E\dot{x}(t) = Ax(t) + Bu(t) \quad (0.471)$$

where $E, A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{m \times r}$ and $\text{rank} E = r_0 \leq \min\{m, n\}$. The unknown $x(t) \in \mathbb{R}^n$ is the dynamic state and $u(t) \in \mathbb{R}^r$ is the control. Contrary to the theory of controllability of ordinary systems, there are various concepts of controllability for singular system: C-controllability, R-controllability, I-controllability, infinite controllability, ..., etc.

Preliminary results

A trajectory $(x, u) : \mathbb{R} \rightarrow \mathbb{R}^n \times \mathbb{R}^r$ is said to be a solution of (0.471) if, and only if, it belongs to the behavior:

$$\mathfrak{B}_{[E, A, B]} := \{(x, u) \in W_{loc}^{1,1} \times L_{loc}^1(\mathbb{R}, \mathbb{R}^r) : (x, u) \text{ satisfies (0.471) for almost all } t \in \mathbb{R}\}. \quad (0.472)$$

Now, we will present some findings regarding controllability for descriptor systems which we will need in the next section*s. All these results are extracted from [4].

Definition 0.88. The system (0.471) is called controllable at infinity if and only if

$$\forall x_0 \in \mathbb{R}^n, \exists (x, u) \in \mathfrak{B}_{[E, A, B]} : x(0) = x_0.$$

Definition 0.89. The system (0.471) is called impulse controllable (I-controllable) if and only if

$$\forall x_0 \in \mathbb{R}^n, \exists (x, u) \in \mathfrak{B}_{[E, A, B]} : Ex(0) = Ex_0.$$

Definition 0.90. The system (0.471) is called completely controllable (C-controllable) if and only if

$$\exists T > 0, \forall x_0, x_f \in \mathbb{R}^n, \exists (x, u) \in \mathfrak{B}_{[E, A, B]} : x(0) = x_0 \text{ and } x(T) = x_f.$$

Definition 0.91. The system (0.471) is called strongly controllable (S-controllable) if and only if

$$\exists T > 0, \forall x_0, x_f \in \mathbb{R}^n, \exists (x, u) \in \mathfrak{B}_{[E, A, B]} : Ex(0) = Ex_0 \text{ and } Ex(T) = Ex_f.$$

Proposition 0.30. The system (0.471) is controllable at infinity if and only if

$$\text{rank}[E, B] = \text{rank}[E, A, B] \quad (0.473)$$

Proposition 0.31. The system (0.471) is C-controllable if and only if condition (0.544) is satisfied together with

$$\text{rank}[\lambda E - A, B] = \text{rank}[E, A, B], \quad \forall \lambda \in \mathbb{C}. \quad (0.474)$$

Proposition 0.32. The system (0.471) is I-controllable if and only if

$$\text{rank} \begin{bmatrix} E & 0 & 0 \\ A & E & B \end{bmatrix} = \text{rank}[E, A, B] + \text{rank} E. \quad (0.475)$$

Proposition 0.33. The system (0.471) is S-controllable if and only if both the conditions (0.545) and (0.546) are satisfied.



Main results

Now, we're going to give an equivalent form of singular system (0.471). For that, we assume that $\text{rank}(E) = r_0 \leq \min\{m, n\}$. From matrix theory, we know that there exist nonsingular matrices $P \in \mathbb{R}^{m \times m}$ and $Q \in \mathbb{R}^{n \times n}$ such that

$$PEQ = \begin{bmatrix} I_{r_0} & 0 \\ 0 & 0 \end{bmatrix}. \quad (0.476)$$

By taking the coordinate transformation

$$x = Q \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, x_1 \in \mathbb{R}^{r_0}, x_2 \in \mathbb{R}^{n-r_0},$$

system (0.471) is equivalent to

$$(EF) \begin{cases} \dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2 + B_1u(t) \\ 0 = A_{21}x_1(t) + A_{22}x_2 + B_2u(t) \end{cases} \quad (0.477)$$

where

$$PAQ = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, PB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}. \quad (0.478)$$

where $A_{11} \in \mathbb{R}^{r_0 \times r_0}$, $A_{12} \in \mathbb{R}^{r_0 \times (n-r_0)}$, $A_{21} \in \mathbb{R}^{(m-r_0) \times r_0}$, $A_{22} \in \mathbb{R}^{(m-r_0) \times (n-r_0)}$, $B_1 \in \mathbb{R}^{r_0 \times r}$ and $B_2 \in \mathbb{R}^{(m-r_0) \times r}$. The system (0.477) is an equivalent form (EF) of (0.471), and it clearly reflects the physical meaning of the singular systems.

Theorem 0.190. *The system (0.471) is controllable at infinity if and only if the system (0.477) satisfies*

$$\text{rank } B_2 = \text{rank} \begin{bmatrix} A_{21} & A_{22} & B_2 \end{bmatrix}. \quad (0.479)$$

Now, for further discussion on I-controllability for descriptor system (0.471), consider another descriptor system as

$$\mathcal{E}\dot{y}(t) = \mathcal{A}y(t) + \mathcal{B}u(t), \quad (0.480)$$

where $\mathcal{E} = B_2$, $\mathcal{A} = A_{21}$ and $\mathcal{B} = A_{22}$.

Theorem 0.191. *The system (0.471) is I-controllable if and only if the system (0.547) is controllable at infinity, i.e.,*

$$\text{rank} \begin{bmatrix} A_{22} & B_2 \end{bmatrix} = \text{rank} \begin{bmatrix} A_{21} & A_{22} & B_2 \end{bmatrix}. \quad (0.481)$$

Theorem 0.192. *The system (0.471) is C-controllable if and only if condition (0.552) is satisfied together with*

$$\text{rank} [\lambda I_{r_0} - A_{11} \quad -A_{12} \quad B_1] = r_0, \quad \forall \lambda \in \mathbb{C}. \quad (0.482)$$

Theorem 0.193. *The system (0.471) is S-controllable if and only if both the conditions (0.493) and (0.492) are satisfied.*

Illustrating Examples

In this section*, we present some numerical examples which illustrate the effectiveness of the presented theory.

Example 0.8. *Consider the circuit system shown in Figure 33. in which the voltage source u_e is the control input. Choose the state variable*

$$x = [u_{C_1} \quad u_{C_2} \quad I_1 \quad I_2]$$

where u_{C_1} and u_{C_2} are the voltages of C_1 , C_2 and the amperage of the currents flowing over them. According to Kirchoff's second law (Smith, 1966) we may establish the following state equation

$$\begin{cases} C_1 u_{C_1} = I_2, \\ C_2 u_{C_2} = I_1, \\ -L\dot{I}_1 = -u_{C_1} + u_{C_2}, \\ 0 = u_{C_1} + RI_2 - u_e. \end{cases} \quad (0.483)$$



9.png

Figure 33: Circuit électrique LCR.

The above system of equations can be written as

$$\begin{bmatrix} C_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \\ 0 & 0 & -L & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{u}_{C_1} \\ \dot{u}_{C_2} \\ \dot{I}_1 \\ \dot{I}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & R \end{bmatrix} \begin{bmatrix} u_{C_1} \\ u_{C_2} \\ I_1 \\ I_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} u_e(t). \quad (0.484)$$

In this description equation, we take $C_1 = C_2 = 1$, $L = 1$, and $R = 1$. We have $\text{rank} E = 3$. By using singular value decomposition of matrix E , we get $E = USV^T$ with

$$U = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } V = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

By taking $P = U$ and $Q = (V^T)^{-1}$, we obtain

$$PEQ = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}, \quad PAQ = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad \text{and} \quad PB = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}.$$

In this system, we have

$$\text{rank} B_2 = \text{rank} \begin{bmatrix} A_{21} & A_{22} & B_1 \end{bmatrix} = 1, \quad (0.485)$$

Therefore, by Theorem (0.190), the circuit network (0.484) is controllable at infinity. We have

$$\text{rank} \begin{bmatrix} A_{22} & B_2 \end{bmatrix} = \text{rank} \begin{bmatrix} A_{21} & A_{22} & B_2 \end{bmatrix} = 1. \quad (0.486)$$

Hence, from theorem (0.191), the system (0.484) is I-controllable. We have

$$\text{rank} \begin{bmatrix} \lambda I_r - A_{11} & -A_{12} & B_1 \end{bmatrix} = 3, \quad \forall \lambda \in \mathbb{C}. \quad (0.487)$$

Hence, from theorem (0.192), the system (0.484) is C-controllable. Moreover, from (0.486), (0.487) and theorem (0.193), the circuit network (0.484) is S-controllable.

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Numerical study of fractional boundary value problem with a conformable fractional derivative of caputo-fabrizio and its fractional integral

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Abstract. In the text, we investigate the existence and uniqueness of the solution of a fractional boundary value problem with conformable fractional derivative of the Caputo-Fabrizio type. In order to study this problem we used a newdefinition of fractional integral as an inverse of the conformable fractional derivative of Caputo-Fabrizio, therefore, so we transformed the problem to a equivalent linear Volterra-Fredholm integral equations of the second kind.

Keywords: Caputo-Fabrizio fractional derivative; Fractional boundary value problem; Fractional integral; Volterra-Fredholm integral equation



Introduction

The usual integral and derivative are (to say the least) a staple for the new technology, essential as a means of understanding and working with natural and artificial. Recently, papers have been published that deal with the existence and multiplicity of the solution of nonlinear initial fractional differential equation by the use of techniques of nonlinear analysis. However, Our aim is to study the existence and the uniqueness of the solution for a class of fractional boundary value problems.

Main results

*

Problem

$$\begin{cases} \mathcal{D}^{(\rho)}v(t) + q(t)v(t) = f(t), & 0 \leq t \leq 1 \\ v(0) = a, \quad v(1) = b \end{cases} \quad (0.488)$$

where $1 < \rho < 2$ is a real number, q is the potential function, and $f : [0, 1] \rightarrow \mathbb{R}$ is continuous. and $\mathcal{D}^{(\rho)}$ is the new fractional derivative, using this fractional integral upon problem (0.488) to obtain an equivalent linear Volterra-Fredholm integral equations of second kind.

Definition 0.92. Let $\Omega = [\alpha, \beta]$ ($-\infty < \alpha < \beta < +\infty$) be a finite interval on the real axis . The left-sided CF-FD ${}^{CF}\mathcal{D}_{\alpha+}^{\gamma}v$ of order $\gamma \in [0, 1[$ of a function v is defined by:

$$v(t) = \frac{M(\gamma)}{1-\gamma} \int_{\alpha}^t v'(\tau) \exp \left[-\frac{\gamma(t-\tau)}{1-\gamma} \right] \tau \quad (0.489)$$

where $\gamma \in [0, 1]$ and $\alpha \in]-\infty, t)$, $v \in H^1(\alpha, \beta)$, $\beta > \alpha$, and $M(\gamma)$ is a normalization function such that $M(0) = M(1) = 1$ for example $M(\gamma) = 1 - 0.2\sin(2\pi\gamma)$.

Definition 0.93. [1] (A new fractional integral)
Let $n \geq 1$, $\gamma \in [0, 1]$, and $f \in {}^1[\alpha, \beta]$. The formula :

$$I_{\alpha}^{n+\gamma}f(t) = \frac{1}{M(\gamma) \cdot n!} \int_{\alpha}^t (t-\tau)^{n-1} [\gamma(t-\tau) + n(1-\gamma)] f(\tau) \tau \quad (0.490)$$

is a new fractional integral of order $(n + \gamma)$.

Lemma 0.64. [1] Let $\rho \in (n, n+1)$, $n = [\rho] \geq 0$. Assume that $v \in {}^n[\alpha, \beta]$, then those statements hold:

1. if $v(\alpha) = 0$, then $(I_{\alpha}^{\rho}v(t)) = v(t)$.
2. $I_{\alpha}^{\rho}(v(t)) = v(t) + \sum_{i=0}^n \alpha_i t^i$, $\alpha_i \in \mathbb{R}$, $i = 0, 1, \dots, n$.

Analytic Study

Lemma 0.65. The solution of problem (0.488) is solution of linear integral equation Volterra-Fredholm of the second kind

$$v(t) + \int_0^t L(t, \tau)v(\tau) \tau + \int_0^1 F(t, \tau)v(\tau) \tau = g(t) \quad (0.491)$$

where $g(t) = (b-a)t + a + \int_0^t [\gamma(t-\tau) + (1-\gamma)] f(\tau) d\tau + \int_0^1 t(\gamma\tau - 1) f(\tau) d\tau$, and $L(t, \tau) = [\gamma(t-\tau) + (1-\gamma)]q(\tau)$, $F(t, \tau) = t(\gamma\tau - 1)q(\tau)$.



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DESCRETIZATION FOR A 2D-VISCOELASTIC WAVE EQUATION WITH ACOUSTIC DYNAMIC BOUNDARY CONDITIONS

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Abstract. This article presents and analyses a finite element approach for the 2D-viscoelastic wave equation with dynamic boundary conditions and strong damping. We use the Faedo-Galerkin method to prove the global existence of solutions and the multiplier approach to determine the asymptotic behavior in a bounded domain. We show and analyze typical semi-discrete systems as well as an implicit fully discrete scheme. For both semi discrete and fully discrete methods, optimal a priori error estimates are demonstrated. Finally, some numerical findings and a priori error estimate are derived.

Keywords: finite element approach, viscoelastic, Faedo-Galerkin method, dynamic boundary conditions, fully discrete approximation.

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Introduction

In this paper, we consider the following viscoelastic wave equations with strong damping and dynamic boundary conditions:

$$\begin{cases} u_{tt}(x, t) - \Delta u(x, t) - \Delta u_t(x, t) - \Delta u_{tt}(x, t) = f(x, t), & \text{in } Q = \Omega \times [0, T], \\ u(x, t) = 0, & \text{on } \Sigma_0 = \Gamma_0 \times [0, T], \\ \frac{\partial u}{\partial n}(x, t) + \xi(x, t) = 0, & \text{on } \Sigma_1 = \Gamma_1 \times [0, T], \\ \xi_{tt}(x, t) + \xi(x, t) = u_t(x, t) + g(x, t), & \text{on } \Sigma_1 = \Gamma_1 \times [0, T]. \end{cases} \quad (0.492)$$

With the following initial conditions

$$\begin{cases} u(x, 0) = u_0(x), & u_t(x, 0) = u_1(x), & x \in \Omega, \\ \xi(x, 0) = \xi_0(x), & \xi_t(x, 0) = \xi_1(x), & x \in \Gamma_1. \end{cases} \quad (0.493)$$

Where $u = u(x, t)$, $t \geq 0$, $x \in \Omega$, and $\xi(x, t)$ designates the dynamical control.

Ω is a bounded open set in $\mathbb{R}^2$ with boundary Γ of class C^2 and $\Gamma = \Gamma_0 \cup \Gamma_1$ with $\Gamma_0 \neq \emptyset$ and $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$.

$\frac{\partial}{\partial \nu}$ denotes the unit outer normal derivative, f and g are functions depend on x, t .

The viscoelastic equation explain how waves travel through materials that have both elastic and viscous characteristics. Understanding the evolution of waves in materials that can simultaneously dissipate energy and deform is made possible by the use of viscoelastic wave equations, for the physical aspects, see Love [6].

The non-ordinary boundary conditions defined on (0.492) are the primary source of difficulty for problem (0.492). Boundary conditions of this kind are referred to as dynamic boundary conditions, moreover these conditions have numerous practical uses in addition to being significant from a theoretical one.

Main results

In this section*, we present some material used in the proof of our main result.

Lemma 0.66. [7] *Let $E : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a non increasing function and $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ a increasing function of class C^1 such that*

$$\phi(0) = 0, \quad \text{and} \quad \phi(t) \rightarrow +\infty \quad \text{as} \quad t \rightarrow +\infty.$$

Assume that there exist $p > -1$ and $\omega > 0$ such that

$$\int_S^{+\infty} E^{p+1}(t) \phi'(t) dt \leq \frac{1}{\omega} E^p(0) E(S), \quad 0 \leq S \leq \infty,$$

then E has the following decay properties

$$E(t) = 0, \quad \forall t \geq \frac{E^p(0)}{\omega|p|}, \quad \text{if } -1 < p < 0$$

$$E(t) \leq E(0) e^{1-\omega\phi(t)}, \quad \forall t \geq 0, \quad \text{if } p = 0$$

$$E(t) \leq E(0) \left(\frac{1+p}{1+\omega p \phi(t)} \right), \quad \forall t \geq 0, \quad \text{if } p > 0$$

Denote by $\|\cdot\|$, $\|\cdot\|_1$, $(\cdot, \cdot)$ and $(\cdot, \cdot)_1$ the norms and scalar products in $L^2(\Omega)$ and $L^2(\Gamma_1)$ respectively, and introduce the space

$$H_{\Gamma_0}^1(\Omega) = \{u \in H^1(\Omega) / u|_{\Gamma_0} = 0\}, \quad \text{and} \quad V = \left\{u \in H_{\Gamma_0}^1(\Omega) / u|_{\Gamma_1} \in L^2(\Gamma_1)\right\}$$



Multiply the first equation of (0.492) by a test function $v \in V$ then use Green's formula with the initial and boundary conditions to obtain the weak form of (0.492).

Find $u(., t) \in V$ and $\xi(., t) \in L^2(\Gamma_1)$ such that, for all $v, w \in V$

$$\begin{cases} (u_{tt}, v) + (\nabla u, \nabla v) + (\nabla u_t, \nabla v) + (\nabla u_{tt}, \nabla v) + (\xi + \xi_t + \xi_{tt}, v)_1 = (f, v), & \text{in } Q, \\ (\xi_{tt}(x, t), w)_1 + (\xi(x, t), w)_1 = (u_t(x, t), w)_1 + (g(x, t), w)_1, & \text{on } \Sigma_1. \end{cases} \quad (0.494)$$

In the next theorem we show the existence and uniqueness of the solution of the problem (0.492) using the Faedo-Galerkin method.

Theorem 0.194. Assume that $u_0, u_1 \in H_{\Gamma_0}^1(\Omega) \cap H^2(\Omega)$, $\xi_0, \xi_1 \in L^2(\Gamma_1)$, $f, \frac{\partial f}{\partial t} \in L^2(Q)$, $g, \frac{\partial g}{\partial t} \in L^2(\Sigma_1)$ and $\frac{\partial u_0}{\partial n}(x) = -\xi_0(x)$, $\frac{\partial u_1}{\partial n}(x) = -\xi_1(x)$ for all $x \in \Gamma_1$.

Then for each $T > 0$, problem (0.492) admits a unique solution (u, ξ) satisfying the following conditions:

$$\begin{aligned} u &\in L^\infty(0, T, H_{\Gamma_0}^1(\Omega)), \\ u_t &\in L^\infty(0, T, L^2(\Omega)) \cap L^\infty(0, T, H_{\Gamma_0}^1(\Omega)), \\ u_{tt} &\in L^\infty(0, T, L^2(\Omega)) \cap L^\infty(0, T, H_{\Gamma_0}^1(\Omega)), \\ \xi, \xi_{tt} &\in L^\infty(0, T, L^2(\Gamma_1)). \end{aligned}$$

Theorem 0.195. Assume that $u_0, u_1 \in H_{\Gamma_0}^1(\Omega) \cap H^2(\Omega)$, $\xi_0, \xi_1 \in L^2(\Gamma_1)$ and $\frac{\partial u_0}{\partial n}(x) = -\xi_0(x)$, $\frac{\partial u_1}{\partial n}(x) = -\xi_1(x)$ for all $x \in \Gamma_1$.

Then for some positive constant k , we obtain the following decay property

$$E(t) \leq E(0)e^{1-k\phi(t)}, \quad \forall t \geq 0,$$

where ϕ is function satisfies certain hypotheses, which we will specify in the proof.

Theorem 0.196. Assume that the assumption of Theorem 0.194 holds with $\xi_0, \xi_1 \in H_{\Gamma_0}^2(\Omega)$ and $\xi \in H^1(0, T, H_{\Gamma_0}^2(\Omega))$, then we have the following estimate

$$|u_t - u_t^h|^2 + |\nabla u - \nabla u^h|^2 + |\nabla u_t - \nabla u_t^h|^2 + |\xi - \xi^h|_1^2 + |\xi_t - \xi_t^h|_1^2 \leq Ch^2,$$

holds for $0 \leq t \leq T$.

*

Error Estimate Let us denote $\hat{u}^{n+1} = \frac{u^{n+1} - u^n}{\Delta t}$, $\hat{\xi}^{n+1} = \frac{\xi^{n+1} - \xi^n}{\Delta t}$, $\hat{u}(t_n) = u_t(t_n)$ and $\hat{\xi}(t_n) = \xi_t(t_n)$.

Theorem 0.197. Assume that the assumption of theorem 0.194 holds, then if $u \in H^3(0, T, L^2(\Omega)) \cap W^{2,\infty}(0, T, H^3(\Omega))$ and $\xi \in H^2(0, T, H^2(\Omega)) \cap W^{1,\infty}(0, T, H^2(\Omega))$ we have

$$|u_t(t_n) - \hat{u}^n|^2 + |\nabla u(t_n) - \nabla u^n|^2 + |\nabla u_t(t_n) - \nabla \hat{u}^n|^2 + |\xi(t_n) - \xi^n|_1^2 + |\xi_t(t_n) - \hat{\xi}^n|_1^2 \leq C(h^2 + (\Delta t)^2).$$



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On Classification of Supermodules and Supermodule superalgebras

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Abstract. In this work, we study supermodule and supermodule superalgebras structures of finite dimensional Hopf superalgebras. In particular, we determine low dimensional supermodule and supermodule superalgebra structures for all Hopf superalgebras obtained in the classification established in [1].

Keywords: superalgebra, superbialgebra, Hopf superalgebras, supermodule, supermodule superalgebra.

Introduction

Supermodules over superalgebras have applications in various mathematical and physical disciplines. They provide a framework for studying supersymmetry in theoretical physics and can be regarded as a generalization of superspaces over a field. In category theory, the class of all supermodules over a superalgebra forms a category, with supermodule homomorphisms as the morphisms. This category is symmetric monoidal and closed under the super tensor product, with the internal Hom functor given by the Hom construction.



Main results

Theorem 0.198. *Every non-trivial 2-dimensional Hopf superalgebra is isomorphic to the 2-dimensional Hopf superalgebra $\mathcal{H} = [x]/(x^2)$ with $\deg(x) = 1$ and such that*

$$\Delta(x) = 1 \otimes x + x \otimes 1, \quad \epsilon(x) = 0, \quad S(x) = -x.$$

Theorem 0.199. *Every non-trivial 4-dimensional Hopf superalgebra, where $\dim A_0 = 3$, is isomorphic to the 4-dimensional Hopf superalgebra $\mathcal{H}_0 = [x, y]/(x^2 + y^2 - 1, xy)$ with $\deg(x) = 0$, $\deg(y) = 1$ and such that*

$$\begin{aligned} \Delta(x) &= x \otimes x - \alpha y \otimes y, & \epsilon(x) &= 1, & S(x) &= x, \\ \Delta(y) &= x \otimes y + y \otimes x, & \epsilon(y) &= 0, & S(y) &= \alpha y, \end{aligned}$$

where α is a primitive 4th root of unity.

Every non-trivial 4-dimensional Hopf superalgebra, where $\dim A_0 = 2$, is isomorphic to one of the following pairwise non-isomorphic Hopf superalgebras:

1. $\mathcal{H}_1 = [x, y]/(x^2 - x, y^2)$ with $\deg(x) = 0$, $\deg(y) = 1$ and such that

$$\begin{aligned} \Delta(x) &= 1 \otimes x + x \otimes 1 - 2x \otimes x, & \Delta(y) &= 1 \otimes y + y \otimes 1, \\ \epsilon(x) &= \epsilon(y) = 0, & S(x) &= x, \quad S(y) = -y. \end{aligned}$$

2. $\mathcal{H}_2 = \langle x, y \rangle / (x^2 - x, y^2, xy + yx - y)$ with $\deg(x) = 0$, $\deg(y) = 1$ and such that

$$\begin{aligned} \Delta(x) &= 1 \otimes x + x \otimes 1 - 2x \otimes x, & \Delta(y) &= 1 \otimes y + y \otimes 1 - 2x \otimes y - 2y \otimes x, \\ \epsilon(x) &= \epsilon(y) = 0, & S(x) &= x, \quad S(y) = y. \end{aligned}$$

3. $\mathcal{H}_3 = \langle x, y \rangle / (x^2, y^2, xy + yx)$ with $\deg(x) = \deg(y) = 1$ and such that

$$\begin{aligned} \Delta(x) &= 1 \otimes x + x \otimes 1, & \Delta(y) &= 1 \otimes y + y \otimes 1, \\ \epsilon(x) &= \epsilon(y) = 0, & S(x) &= -x, \quad S(y) = -y. \end{aligned}$$

4. $\mathcal{H}_4 = [x, y]/(x^2 - x, y^2)$ with $\deg(x) = 0$, $\deg(y) = 1$ and such that

$$\begin{aligned} \Delta(x) &= 1 \otimes x + x \otimes 1 - 2x \otimes x, & \Delta(y) &= 1 \otimes y + y \otimes 1 - 2x \otimes y, \\ \epsilon(x) &= \epsilon(y) = 0, & S(x) &= x, \quad S(y) = -y + 2xy, \end{aligned}$$

where $\langle x, y \rangle$ stands for noncommutative polynomials.

Definition 0.94. A supermodule over a superalgebra (A, μ_A, η_A) is a pair $(V, \triangleright)$ consisting of a superspace V over $\mathbb{K}$ and a morphism $\triangleright : A \otimes V \rightarrow V$ such that the following identities hold for all $a, b \in A$ and $v \in V$

$$(ab) \triangleright v = a \triangleright (b \triangleright v). \quad (0.495)$$

$$1_A \triangleright v = v. \quad (0.496)$$

Definition 0.95. Let H be a superbialgebra or a Hopf superalgebra. A supermodule superalgebra over H is a superalgebra (A, μ_A, η_A) together with an H -supermodule structure $\triangleright : H \otimes A \rightarrow A$ such that the following identities hold for all $h \in H$ and $a, a' \in A$

$$h \triangleright (aa') = \sum_{(h)} (-1)^{|h_{(2)}||a|} (h_{(1)} \triangleright a) (h_{(2)} \triangleright a'), \quad (0.497)$$

$$h \triangleright 1_A = \epsilon(h) 1_A. \quad (0.498)$$

A supermodule (resp. supermodule superalgebra) is identified by its structure constants with respect to a fixed basis. It turns out that the axioms of a supermodule structure (resp. supermodule superalgebra structure) can be translated into a system of polynomial equations. The resolution of the polynomial equations system lead to a description of these structures.



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Almost complete convergence and asymptotic normality of nonparametric estimator.

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Abstract. The objective of this work is the relative error prediction of the nonparametric regression in multivariate spatial data. We have focused our study on the case where the variables are spatially dependent. More precisely, we have obtained almost complete convergence and asymptotic normality.

Keywords: Kernel methos, Relative error, Nonparametric estimation, Associated variable.

Introduction

Let $Z_i = (X_i, Y_i), i \in \mathbb{N}^N$ be a $\mathbb{R}^d \times \mathbb{R}$ valued measurable strictly stationary spatial process. We consider the problem of estimating the regression function of Y_i give nX_i . We construct an alternative kernel estimate of the regression function based on the minimization of the mean squared relative error. Under some general mixing assumptions, the almost complete consistency and the asymptotic normality of this estimator are obtained. Its finite sample performance is compared with a standard kernel regression estimator via Monte Carlo study and real data example.



Main results

Theorem 0.200. Under assumptions if $\inf_{x \in S} g_2(x) > 0$, we have:

$$\sup_x |\tilde{\theta}(x) - \theta(x)| = O(h^2) + O_{a.co.} \left(\sqrt{\frac{\log \hat{n}}{\hat{n}h^d}} \right) \quad a.co. \quad (0.499)$$

Lemma 0.67. Under hypotheses (H1), (H2), and (H5), we have, for $l = 1, 2$, that:

$$|E\tilde{g}_l(x) - g_l(x)| = O(h^2).$$

Theorem 0.201. we have, for any $x \in \mathcal{A}$,

$$\left(\frac{\hat{n}h^d}{\sigma^2(x)} \right)^{1/2} (\tilde{\theta}(x) - \theta(x)) \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1) \quad \text{as } n \rightarrow \infty. \quad (0.500)$$

where

$$\sigma^2(x) = \frac{(g_2(x) - 2\theta(x)r_3(x) + \theta^2(x)r_4(x))}{g_2^2(x)} \int_{R^d} K^2(z)dz,$$

$$\mathcal{A} = \left\{ x \in S, (g_2(x) - 2\theta(x)r_3(x) + \theta^2(x)r_4(x)) g_2^2(x) \neq 0 \right\}$$

and $\xrightarrow{\mathcal{D}}$ means the convergence in distribution.

Lemma 0.68. Under the hypotheses of Theorem ??, we obtain:

$$\left(\frac{\hat{n}h^d}{g_2^2(x)\sigma^2(x)} \right)^{1/2} (B_n - \mathbb{E}[B_n]) \rightarrow N(0, 1).$$

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Nonparametric estimation of the relative regression function for left truncated and right censored data: Methodology and Applications

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Abstract. This paper introduces a nonparametric estimator for the regression function in the presence of left truncation and right censoring, derived through the minimization of the mean squared relative error. The utility and practical relevance of the proposed estimator are illustrated via two real world case studies.

Keywords: kernel estimator, nonparametric relative error regression, truncated-censored data.

Introduction

Let $\{(\mathbf{X}_i, Y_i); i = 1, \dots, N\}$ be a sequence of random vectors from the pair $(\mathbf{X}, Y)$, where $\mathbf{X}$ takes values in $\mathbb{R}^d$ and Y is a real variable. In the field of nonparametric statistics, it is common to investigate the relationship between the response variable Y and the explanatory variable $\mathbf{X}$. A frequently adopted nonparametric approach for modeling this relationship is based on the following model:

$$Y = m(\mathbf{X}) + \epsilon,$$

where $m(\cdot)$ represents an unknown function and ϵ is a random error variable. The function $m(\cdot)$ can be obtained by minimizing the mean squared relative error (MSRE), given by:

$$\mathbb{E} \left[\left(\frac{Y - m(\mathbf{X})}{Y} \right)^2 | \mathbf{X} \right], \text{ for } Y > 0. \quad (0.501)$$



Previous work [3] shown that the solution of the minimization problem in equation (0.501) can be explicitly expressed as:

$$m(\mathbf{X}) = \frac{\mathbb{E}[Y^{-1}|\mathbf{X}]}{\mathbb{E}[Y^{-2}|\mathbf{X}]} \quad (0.502)$$

In this study, we focus on estimating $m(\cdot)$ using the kernel method when the variable of interest Y is subject to left truncation by a random variable (rv) T and right censoring by a rv W . Let $\{T_i; i = 1, \dots, N\}$ and $\{W_i; i = 1, \dots, N\}$ be sequences of N samples from T and W , respectively. Under the random left truncated and right censored (LTRC) model, we observe only $\{(X_i, Z_i, T_i, \delta_i); i = 1, \dots, N\}$ for which $Z_i \geq T_i$, where $Z_i = \min(Y_i, W_i)$ and $\delta_i = 1_{\{Y_i \leq W_i\}}$. Hence, the observed sample is $\{(X_i, Z_i, T_i, \delta_i); i = 1, \dots, n\}$, with $n \leq N$. Following the work of [2], we propose a new estimator, $\hat{m}_n(\cdot)$, for the relative regression function defined in equation (0.502), accounting for truncation and censoring. The estimator for any $\mathbf{x} \in \mathbb{R}^d$ is given by:

$$\hat{m}_n(\mathbf{x}) =: \frac{\hat{\psi}_1(\mathbf{x})}{\hat{\psi}_2(\mathbf{x})}, \quad (0.503)$$

where

$$\hat{\psi}_\ell(\mathbf{x}) = \frac{\alpha_n}{nh_n^d} \sum_{i=1}^n \frac{\delta_i Z_i^{-\ell}}{L_n(Z_i) \bar{G}_n(Z_i)} K_d \left(\frac{\mathbf{x} - \mathbf{X}_i}{h_n} \right) \text{ for } \ell = 1, 2.$$

Here, $K_d: \mathbb{R}^d \rightarrow \mathbb{R}$ represents the kernel function, h_n is the bandwidth that approaches zero $n \rightarrow \infty$, $L_n(\cdot)$ denotes the Lynden-Bell estimator of the distribution function (df) L for the truncation rv T , $\bar{G}_n(\cdot) = 1 - G_n(\cdot)$, where G_n is the TJW type estimator of the df G for the censoring rv W , and α_n is the estimator of $\alpha := \mathbb{P}(Z \geq T)$.

Real data application 1

In this section*, we evaluate the performance of the relative error regression estimator using a real world dataset that involves left truncation and right censoring. The dataset, reported by [1], includes the lifespans of 97 male residents at the Channing House retirement community in Palo Alto, California, observed between January 1964 and July 1975. It provides information on the age at entry, age at death or departure, and survival status at the study's conclusion.

This dataset exhibits left truncation, as residents were required to be at least 55 years old to enter the community, and right censoring, due to the study's end date or loss to follow-up. The primary objective of this analysis is to investigate the relationship between the age of entry and the age of death.

Real data application 2

We further assess the performance of the relative error regression estimator using a dataset related to AIDS patients in Australia diagnosed before July 1, 1991, obtained from Dr. Patty Solomon [4]. The dataset contains 1276 patients who survived at least one year after diagnosis. It includes detailed information such as the date of diagnosis, date of death, gender, age at diagnosis, survival status at the end of the study, as well as state and transmission category.

In this analysis, we are particularly interested in examining the relationship between the time of diagnosis and the time of death, taking into account the left truncation and right censoring present in the data.

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SZL Distribution: Properties and Simulation

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Abstract. This paper introduces the new one-parameter SZLindley distribution (SZLD). The shape of the SZLD will be discussed, there are also quantile functions, moment generation functions, Rényi entropy, order statistics, and survival and hazard functions and quantile functions can help to adequately depict the utility of the proposed distribution. The greatest likelihood estimation method was utilized to calculate parameters. A complete simulation analysis is carried out to assess the performance of these suggested estimators, which use MLE for a range of parameters.

Keywords: ZLindley distribution, moments, reliability analysis, simulation.

Introduction

Beghriche et al. and Saaidia et al. [1,2] recently introduced a novel statistical family known as the new one-parameter polynomial exponential distribution (NPED). This distribution's probability density function (p.d.f) is

$$f(t, \theta) = \frac{\exp(-\theta t)P(t, \theta)}{\sum_{k=0}^n \frac{t^k}{\theta^{k+1}} a_{k, \theta}}; \quad t > 0, \theta > 0.$$



Where,

$$P(t, \theta) = \sum_{k=0}^n t^k a_{k,\theta},$$

and $a_{k,\theta}$ depends on k and θ .

The following are the main objectives of our study: The objective is to identify its qualities and provide a new enlargement of the SZL distribution, to use the maximum likelihood estimation technique to determine the parameters of the new distribution. Where the SZL distribution (SZLD) is obtained by applying a transformation $Y = T^{\frac{1}{2}}$, where T follows the Z-Lindley distribution. The random variable Y is distributed according to the SZLD, which is symbolically represented as $Y \sim SZLD(\theta)$. The corresponding probability density function (PDF) is

$$f(y) = \frac{\theta y(1 + 2\theta + \theta y^2)}{(1 + \theta)} e^{-\theta y^2}, \quad y > 0, \theta > 0.$$

Statistical Properties of the SZL Distribution

$$\mu = E(X) = \frac{\theta^{-\frac{1}{2}}(5 + 4\theta)\Gamma\left(\frac{3}{2}\right)}{4(1 + \theta)}$$

$$\text{Var}(X) = \frac{4(6 + 4\theta)(1 + \theta) + (5 + 4\theta)^2\Gamma\left(\frac{3}{2}\right)^2}{16\theta(1 + \theta)^2}$$

Survival function and failure rate

The cumulative distribution function of the SZLD is

$$F(y) = 1 - \left(1 + \frac{\theta y^2}{2(1 + \theta)}\right) e^{-\theta y^2}, \quad y > 0, \theta > 0.$$

The survival function of the new distribution is given by

$$S(y) = 1 - F(y) = \left(1 + \frac{\theta y^2}{2(1 + \theta)}\right) e^{-\theta y^2}.$$

The hazard function associated with LZD is

$$h(x) = \frac{f(x)}{1 - F(x)} = \frac{2\theta y(1 + 2\theta + \theta y^2)}{2 + 2\theta + \theta y^2}, \quad x > 0, \theta, \gamma > 0.$$

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A remark on the Hájek-Rényi inequality and some limit theorems for a class of dependent random variables

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Abstract. In this research, we determine a Hájek-Rényi type inequality and establish the strong law of large numbers for a sequence of widely orthant dependent random variables.

Keywords: Widely orthant dependence, The Hájek-Rényi inequality, the strong law of large numbers.

Introduction

In probability theory, the study of limit theorems plays a role in statistics to establish the estimator's convergence. One of the important tools in proving those theorems and also getting the rate of convergence are the probability inequalities (usual inequalities, exponential inequalities, and so forth).

Especially when we deal with dependent random variables, in our study, we treat our results under the widely orthant dependency concept, which is defined as follows:



Definition 0.96. Let $\{Y_n, n \geq 1\}$ be a sequence of random variables and $(y_i \in \mathbb{R}, 1 \leq i \leq n)$ be a real realisations, then if

$$\mathbb{P}\left(\bigcap_{i=1}^n Y_i > y_i\right) \leq g_u(n) \prod_{i=1}^n \mathbb{P}(Y_i > y_i)$$

And

$$\mathbb{P}\left(\bigcap_{i=1}^n Y_i \leq y_i\right) \leq g_l(n) \prod_{i=1}^n \mathbb{P}(Y_i \leq y_i)$$

Then $\{Y_n, n \geq 1\}$ are widely orthant dependent random variables, where the finite real series $\{g_u(n), n \geq 1\}$ and $\{g_l(n), n \geq 1\}$ are known as dominant coefficients.
 $g(n) = \max\{g_u(n), g_l(n)\}$ in all of the paper.

Lemmas

Lemma 0.69. Let $\{W_n, n \geq 1\}$ be an increasing (or decreasing) real function and Let $\{Y_n, n \geq 1\}$ be a sequence of widely orthant dependent random variables, then the random variables $W_n(Y_n)$ are widely orthant dependent for each $n \geq 1$.

Lemma 0.70. Let $\{Y_n, n \geq 1\}$ be a sequence of widely orthant dependent random variables with $\mathbb{E}(Y_n) = 0$ and $\mathbb{E}(Y_n^2) < \infty$, for all $n \geq 1$, where C represents different positive constants in all this paper, then we have

$$\mathbb{E}\left(\max_{1 \leq i \leq n} \left|\sum_{i=1}^n Y_i\right|\right)^2 \leq C \log^2(n)(1 + g(n)) \sum_{i=1}^n \mathbb{E}(|Y_i|^2) \quad (0.504)$$

Main results

Theorem 0.202. Let $\{Y_n, n \geq 1\}$ be a sequence of widely orthant dependent random variables with $\mathbb{E}(Y_n) = 0$ and $\mathbb{E}(Y_n^2) < \infty$, for all $n \geq 1$. Assume that $\{r_n, n \geq 1\}$ be a positive increasing sequence of real values, then for every $\epsilon > 0$, we have

$$\mathbb{P}\left(\max_{1 \leq l \leq n} \left|\sum_{i=1}^l Y_i\right|/r_l \geq \epsilon\right) \leq C 4\epsilon^{-2} \log^2(n)[(g(n) + 1) \sum_{i=1}^n \mathbb{E}(Y_i^2)/r_i] \quad (0.505)$$

Corollary 0.6. Let $\{Y_n, n \geq 1\}$ be a sequence of widely orthant dependent random variables with $\mathbb{E}(Y_n) = 0$ and $\mathbb{E}(Y_n^2) < \infty$, for all $n \geq 1$. Assume that $\{r_n, n \geq 1\}$ be a positive increasing sequence of real values, then for every $\epsilon > 0$, and positive integers $m < n$, we have

$$\mathbb{P}\left(\max_{m \leq l \leq n} \left|\sum_{i=1}^l Y_i\right|/r_l \geq \epsilon\right) \leq C 8\epsilon^{-2} \log^2(n)[(g(n) + 1) \sum_{j=m+1}^n \mathbb{E}(Y_j^2)/r_j + \sum_{j=1}^m \mathbb{E}(Y_j^2)/r_m] \quad (0.506)$$

Theorem 0.203. Let $\{Y_n, n \geq 1\}$ be a sequence of widely orthant dependent random variables with $\mathbb{E}(Y_n) = 0$ and $\mathbb{E}(Y_n^2) < \infty$, for all $n \geq 1$. Assume that $\{r_n, n \geq 1\}$ be a positive increasing sequence of real values, where

$$\sum_{i=1}^{\infty} \mathbb{E}(Y_i^2)/r_i^2 < \infty \quad (0.507)$$

Then,

(a) If $\lim_{n \rightarrow \infty} g(n) < \infty$, we get

$$\mathbb{E}(\sup_n \left|\sum_{i=1}^n Y_i\right|/r_n \log(n))^q < \infty \quad (0.508)$$

for every $0 < q < 2$,



(b) If $\lim_{n \rightarrow \infty} g(n) \log^{-d}(n) = 0$ for each $d \geq 1$ and $r_n = \infty$ as $n \rightarrow \infty$, we get

$$\sum_{i=1}^n Y_i / (r_n \log^{\theta/2}(n)) \text{ converge almost surely to } 0 \quad (0.509)$$

for every $\theta \geq 3$.

Corollary 0.7. Let $\{Y_n, n \geq 1\}$ be a sequence of widely orthant dependent random variables with $\mathbb{E}(Y_n) = 0$ and $\mathbb{E}(Y_n^2) < \infty$, for all $n \geq 1$. Assume that $\{r_n, n \geq 1\}$ be a positive increasing sequence of real values, where

$$\sum_{i=1}^{\infty} \mathbb{E}(Y_i^2) / r_i^2 < \infty \quad (0.510)$$

Then, If $\lim_{n \rightarrow \infty} g(n) < \infty$, we get

$$\mathbb{E}(\sup_n |\sum_{i=1}^n Y_i| / (r_n \log^{3/2}(n)))^2 < \infty \quad (0.511)$$

Theorem 0.204. Let $\{Y_n, n \geq 1\}$ be a sequence of widely orthant dependent random variables with $\mathbb{E}(Y_n) = 0$ and $\mathbb{E}(Y_n^2) < \infty$, for all $n \geq 1$, where

$$\sum_{i=1}^{\infty} \mathbb{E}(Y_i^2) < \infty \quad (0.512)$$

Then, If $\lim_{n \rightarrow \infty} g(n) n^{-2/v} = 0$ for each $d \geq 1$ and $0 < v < 2$, we get

$$\sum_{i=1}^n Y_i / (n^{1/v} \log(n)) \text{ converge almost surely to } 0 \quad (0.513)$$

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Matrices de Lucas et une nouvelle théorie du codage.

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Abstract. Dans cette étude, nous nous concentrons sur l'utilisation des suites récurrentes linéaires dans le domaine du codage. Il s'agit d'abord de présenter le théorème de Zeckendorf avec la version Lucas, puis on montre comment coder et décoder un entier en utilisant la suite de Lucas classique. Enfin, on développe la partie codage en utilisant la matrice QL de Lucas (simple et généralisée) et on prouve les hautes capacités de détection et de correction basées sur le déterminant de la matrice code.

Keywords: Nombre de Lucas, Nombre de Fibonacci, Algorithme, Complexité.

Introduction

Depuis la découverte des suites de Fibonacci et de Lucas, les chercheurs ne cessent de découvrir de nouvelles applications de ces suites. On distinguera deux codes basés sur les nombres de Fibonacci. Le premier défini par Apostolico et Frankel en 1987 dans [1], nommé



le code universel de Fibonacci qui repose essentiellement sur le théorème de Zeckendorf, ce dernier propose une représentation des entiers en utilisant les nombres de Fibonacci, suivit par un préfixe. Le deuxième propose par Stankhov en 1999 dans [2], qui utilise la Qp-matrice de Fibonacci. Rachid chergui en 2020 a développé arithmétique de Zeckendorf pour les nombres de Lucas dans [3] Dans ce travail, on va développer la partie du codage en utilisant le théorème de Zeckendorf on montrera comment coder et décoder un entier en utilisant la suite de Lucas classique, la suite Lucas d'ordre supérieur. Enfin, on va développer la partie codage en utilisant la matrice QL de Lucas.

Suite de Lucas

*

Suite de Lucas

Definition 0.97. Les nombres de Lucas, $(L_n)_n$, sont définis par la formule de récurrence :

$$\begin{cases} L_{n+2} = L_{n+1} + L_n, & n \geq 0 \\ L_0 = 2, & L_1 = 1 \end{cases}$$

et pour tout $n \geq 1$, on a l'identité bien connue

$$L_n = F_{n+1} + F_{n-1}.$$

*

Suite de Lucas généralisée

Definition 0.98. On dit que (g_n) est une suite de Lucas généralisée d'ordre k si elle satisfait la récurrence :

$$\begin{cases} g_n^{(k)} = g_{n-1}^{(k)} + g_{n-2}^{(k)} + \dots + g_{n-k}^{(k)}, & \forall n \geq 2 \\ g_0^{(k)} = 2, g_1^{(k)} = 1. \\ g_{-n}^{(k)} = \dots = g_{-2}^{(k)} = g_{-1}^{(k)} = 0 \end{cases}$$

Ci-dessous, nous présentons les valeurs de ces nombres pour les premières valeurs k et n .

k	Nom	Premiers termes non nuls ($n \geq 0$)
2	Lucas	2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, ...
3	3 -Lucas	2, 1, 3, 6, 10, 19, 35, 64, 118, 217, 399, 734, 1350, 2483, 4567, ...
4	4 -Lucas	2, 1, 3, 6, 12, 22, 43, 83, 160, 308, 594, 1145, 2207, 4254, 8200, ...
5	-Lucas	2, 1, 3, 6, 12, 24, 46, 91, 179, 352, 692, 1360, 2674, 5257, 10335, ...
6	6 -Lucas	2, 1, 3, 6, 12, 24, 48, 94, 187, 371, 736, 1460, 2896, 5744, 11394, ...
7	7 -Lucas	2, 1, 3, 6, 12, 24, 48, 96, 190, 379, 755, 1504, 2996, 5968, 11888, ...
8	8 -Lucas	2, 1, 3, 6, 12, 24, 48, 96, 192, 382, 763, 1523, 3040, 6068, 12112, ...
9	9 -Lucas	2, 1, 3, 6, 12, 24, 48, 96, 192, 384, 766, 1531, 3059, 6112, 12212, ...
10	10-Lucas	2, 1, 3, 6, 12, 24, 48, 96, 192, 384, 768, 1534, 3067, 6131, 12256, ...

Table 2.2- Premiers nombres k -Lucas non nuls.

*

La p -Lucas En 2006, les p -nombres de Lucas ont été introduits par Stakhov et Rozin (voir[1]). Pour un entier $p = 0, 1, 2, 3, \dots$, les p -nombres de Lucas sont donnés par la relation de récurrence suivante :

$$\begin{cases} L_p(n) = L_p(n-1) + L_p(n-p-1), & \text{si } n > p+1 \\ L_p(1) = L_p(2) = \dots = L_p(p) = 1, L_p(p+1) = p+2 \end{cases} \quad (0.514)$$

Pour $p = 1, L_1(n) = L_n$ sont connus sous le nom de nombres classiques de Lucas.



*

$Q_{p,m}$ -matrice de Lucas Il a été introduit dans [1] que les matrices de Lucas généralisées sont désignées par $Q_{p,m}$ pour un p donné ($p = 0, 1, 2, 3, \dots$) et $m > 0$.

Definition 0.99. On définit une nouvelle matrice appelée $Q_{p,m}$ matrice de Lucas d'ordre $(p+1)$ sur la m -extension des nombres p de Lucas où $p > 0$ est entier et $m > 0$.

$$Q_{p,m} = \begin{pmatrix} m & 1 & 0 & . & . & . & . & 0 & 0 \\ 0 & 0 & 1 & . & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & . & 0 & 1 \\ 1 & 0 & 0 & . & . & . & . & 0 & 0 \end{pmatrix}$$

Figure 34: Matrice $Q_{p,m}$



voici les $Q_{p,m}$ matrices qui leurs correspond :

$$Q_{1,m} = \begin{pmatrix} m & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} L_{1,m}(2) & L_{1,m}(1) \\ L_{1,m}(1) & L_{1,m}(0) \end{pmatrix}$$

$$Q_{2,m} = \begin{pmatrix} m & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} L_{2,m}(2) & L_{2,m}(1) & L_{2,m}(0) \\ L_{2,m}(0) & L_{2,m}(-1) & L_{2,m}(-2) \\ L_{2,m}(1) & L_{2,m}(0) & L_{2,m}(-1) \end{pmatrix}$$

$$Q_{3,m} = \begin{pmatrix} m & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} L_{3,m}(2) & L_{3,m}(1) & L_{3,m}(0) & L_{3,m}(-1) \\ L_{3,m}(-1) & L_{3,m}(-2) & L_{3,m}(-3) & L_{3,m}(-4) \\ L_{3,m}(0) & L_{3,m}(-1) & L_{3,m}(-2) & L_{3,m}(-3) \\ L_{3,m}(1) & L_{3,m}(0) & L_{3,m}(-1) & L_{3,m}(-2) \end{pmatrix}$$

$$Q_{4,m} = \begin{pmatrix} m & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} L_{4,m}(2) & L_{4,m}(1) & L_{4,m}(0) & L_{4,m}(-1) & L_{4,m}(-2) \\ L_{4,m}(-2) & L_{4,m}(-3) & L_{4,m}(-4) & L_{4,m}(-5) & L_{4,m}(-6) \\ L_{4,m}(-1) & L_{4,m}(-2) & L_{4,m}(-3) & L_{4,m}(-4) & L_{4,m}(-5) \\ L_{4,m}(0) & L_{4,m}(-1) & L_{4,m}(-2) & L_{4,m}(-3) & L_{4,m}(-4) \\ L_{4,m}(1) & L_{4,m}(0) & L_{4,m}(-1) & L_{4,m}(-2) & L_{4,m}(-3) \end{pmatrix}$$

Théorème de Zeckendorf pour les nombres de Lucas

Il est connu que les nombres de Lucas sont complets[2] au sens que chaque entier positif peut être représenté comme somme de nombres distincts de Lucas. En général, sa représentation ne sont pas uniques par exemple $4 = L_3 = L_1 + L_2$, $12 = L_1 + L_3 + L_4 = L_0 + L_2 + L_4$. Avant d'énoncer le Théorème, certains Lemmes sont utiles :

Lemma 0.71. $L_n - 1 = L_{n-1} + L_{n-3} + L_{n-5} + \dots + L_{1,2}(n)$, pour $n \geq 2$ où

$$L_{1,2}(n) = \begin{cases} 2L_1 & \text{si } n \text{ pair} \\ L_2 & \text{si } n \text{ impair} \end{cases}$$

Lemma 0.72. $L_{n+2} = 1 + \sum_{i=0}^n L_i$ pour $n \geq 0$.

Theorem 0.205. [3]

Soit n un entier positif satisfaisant $0 \leq n \leq L_k$ pour $k \geq 1$, alors

$$n = \sum_{i=0}^{k-1} \alpha_i L_i$$

où $\alpha_i \in \{0, 1\}$ tels que

$$\begin{cases} \alpha_i \alpha_{i+1} = 0 & \text{pour } i \geq 0 \\ \alpha_0 \alpha_2 = 0 \end{cases}$$

Cette représentation est unique.



*

Méthode de la décomposition de Zeckendorf : Pour décomposer un entier x de la forme de Zeckendorf $x = \sum_{r=1}^{\infty} e_r F_r$ il suffit de suivre les étapes suivantes :

1. Trouver le plus grand nombre de Fibonacci $F_r \leq x$;
2. Effectuer la soustraction $X = x - F_r$; affecter un '1' à e_r et conserver ce coefficient;
3. Affecter X à x et répéter les étapes 1 et 2 jusqu'à avoir un X nul;
4. Affecter des 0 aux e_i où $0 < i < r$ et $e_i \neq 1$.

Le résultat de cette décomposition est : un vecteur de r éléments qui contient les coefficients e_r de la décomposition.

Exemple 0.9. Exemple : Décomposition de l'entier 50, ce tableau explique la représentation

Suite de Lucas	2	1	3	4	7	11	18	29	47	76
Vecteur des coefficients e_r	0	0	1	0	0	0	0	0	1	0

$50_L = 001000001$

Proposition 0.34. Soit $m > 1$, si $L_n \leq m$ alors $n \leq \frac{\ln(m-1)}{\ln(\varphi)}$, où φ est le nombre d'or.

Main results

*

Applications au Codage

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counter-example of Park's necessary condition of realizability over $SL_2(R[x_1^\pm, \dots, x_k^\pm])$

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Abstract. The objective of this work is to develop a new algorithm that can determine the realization of matrices in SL_2 over the multivariate Laurent polynomial ring $R[x_1^\pm, \dots, x_k^\pm]$; We are interested in Park's algorithm [2] with which one can test whether the matrix A allows a factorization into elementary matrices and if it does give its explicit factorization.

Keywords: Suslin's stability theorem, Laurent polynomial ring, monomial orders, Park's algorithm, S-pairs.

Introduction

the symbol R denote an arbitrary Euclidean ring, which allows a suitable generalization of the Euclidean division algorithm. It's known that every matrix in $SL_n(R[x_1^\pm, \dots, x_k^\pm])$ for $n \geq 3$ is realizable based on Suslin's stability theorem [3], furthermore, it was proved by P. M. Cohn in [1] that this theorem cannot be applied in the case where $n = 2$. Now if we extend



these results over the multivariate Laurent polynomial ring the Cohn matrix is realizable and a factorization is given by Tolhuizen, Hollmann, Kalker in [1];

$$C = \begin{pmatrix} 1 & 0 \\ -\frac{y}{x} & 1 \end{pmatrix} \begin{pmatrix} 1 & x^2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{x}{y} & 1 \end{pmatrix}.$$

In this work we construct a new realization algorithm over $SL_2(R[x_1^\pm, \dots, x_k^\pm])$ using the S-pairs with the appropriate monomial order to determine the realisation on the proposed matrices.

Main results

Definition 0.100. $SL_n(R)$ is the set of $n \times n$ matrices of determinant 1 whose entries are elements of R .

2. An elementary matrix E_{ij} over R is defined as follows : for $i \neq j$

$$E_{ij}(a) = I + ae_{ij}, \text{ where } e_{ij} = \begin{cases} 1 & \text{for } (i, j) \\ 0 & \text{Elsewhere} \end{cases}$$

3. $A \in SL_n(R)$ is called realizable if it can be written as a product of elementary matrices.

Theorem 0.206 (Park's theorem [2]). *Let $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in SL_2(R[x_1^\pm, \dots, x_k^\pm])$, for a fixed monomial order $' > '$. If A is a nonconstant realizable matrix, then either A has a zero entry or one of the row vectors of $LT(A)$ is a monomial multiple of the other row.*

Let $A = \begin{pmatrix} \frac{1}{x^4y^2} + \frac{\frac{1}{xy} + 1}{x^2y^2} & xy \\ \frac{1}{x^3y} + \frac{1}{xy} & \frac{1}{x^2} + 2 \end{pmatrix}$, by fixing the lexicographic order with $y > x$ we have $LT(A) = \begin{pmatrix} \frac{1}{xy} & xy \\ \frac{1}{x^3y} & 2 \end{pmatrix}$, despite that neither of the two row vectors $(1, xy)$ or $(\frac{1}{xy}, 2)$ is a monomial multiple of the other, we notice that the matrix A is realizable and by using the formula (0.515) we obtain

$$A = E_{21} \left(\frac{1}{xy} \left(\frac{1}{x^2} + 1 \right) \right) E_{12}(xy) E_{21} \left(\frac{1}{x^2y^2} \right).$$

Therefore, Theorem 0.206 is not always valid for $SL_2(R[x_1^\pm, \dots, x_k^\pm])$.

Lemma 0.73. *Let $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in SL_2(R[x_1^\pm, \dots, x_k^\pm])$, if one of the entries of A is zero or invertible then A is realizable. The explicit factorization of A in each case is given as follows :*

- If $a_2 = 0$

$$A = E_{12}(-a_4^{-1}) E_{21}(a_4) E_{12}(1 - a_1 - a_3^{-1}a_4) E_{21}(-1) E_{12}(1).$$

- If $a_2^{-1} \in SL_2(R[x_1^\pm, \dots, x_k^\pm])$

$$A = E_{21}(a_2^{-1}(a_4 - 1)) E_{12}(a_2) E_{21}(a_2^{-1}(a_1 - 1)). \quad (0.515)$$



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Nouvelle distribution zéro tronquée

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Abstract. Une nouvelle distribution à troncature zéro, appelée distribution poisson Lindley modifiée à troncature zéro, est introduite. Ses propriétés statistiques, y compris l'expression générale des probabilités, les moments, la fonction cumulative et la fonction quantile, ont été examinées. Différentes propriétés statistiques telles que la méthode des moments, l'estimation par maximum de vraisemblance et la fonction quantile sont identifiées. L'estimation des paramètres de la distribution Lindley modifiée à troncature zéro est expliquée par les méthodes d'estimation.

Keywords: distribution de Poisson-Lindley, distribution de Poisson-modifiée Lindley, distribution Lindley modifiée, estimation des moments.



Introduction

En sciences statistiques, les distributions tronquées sont plus couramment utilisées lorsqu'une variable aléatoire est restreinte à être observée sur une plage donnée. Les distributions à troncature nulle, en théorie des probabilités, sont des distributions discrètes positives qui modélisent des données en excluant les zéros. De nombreux travaux ont étudié les distributions à troncature nulle : Ghitany, Al-Mutairi et Nadarajah [1,2], Shanker et al. [3,4], Shanker et Fesshaye [5].

Récemment, C. Chesneau, L. Tomy et J. Gillariose, ont introduit une nouvelle distribution, appelée distribution Lindley modifiée, et ont étudié ses propriétés mathématiques et statistiques, ainsi que l'estimation des paramètres, dont la fonction de densité est donnée par

$$f(x, \theta) = \frac{\theta}{1 + \theta} e^{-2\theta x} \left[(1 + \theta) e^{\theta x} + 2\theta x - 1 \right] \dots x \succ 0 \quad (1)$$

De plus, Christophe C, Lishamol T, Veena G ont introduit d'autre nouvelle distribution, telle que la distribution de Poisson-Lindley modifiée, qui est un cas particulier. La fonction de masse de probabilité de la distribution de Poisson-Lindley modifiée est donnée par

$$fp_{LM}(x; \theta) = \frac{\theta [(2\theta + 1)^{x+2} + (2\theta x - 1)(\theta + 1)^x]}{(\theta + 1)^{x+1} (2\theta + 1)^{x+2}}, x \in \mathbb{N} \quad (2)$$

Distribution de zéro tronquée poisson Lindley modifiée

En utilisant $P(x, \theta) = \frac{P_0(x, \theta)}{1 - P_0(0, \theta)} \dots x = 1, 2, \dots$ et la fonction de masse de probabilité (f.m.p.) de la distribution de poisson Lindley modifiée (2), nous avons obtenu la f.m.p. de la distribution de Lindley modifiée tronquée en zéro (ZTLM).

$$P(x, \theta) = \frac{\theta [(2\theta + 1)^{x+2} + (2\theta x - 1)(\theta + 1)^x]}{(\theta + 1)^{x+1} (2\theta + 1)^{x+2}} \times \frac{(\theta + 1)(2\theta + 1)}{(\theta + 1)(2\theta + 1)^2 - \theta [(2\theta + 1)^2 - 1]}. \quad x \in \mathbb{N}$$

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Existence of solution for a Fractional Boundary Value Problem with higher fractionnal order.

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Abstract. The goal of this work is to study a boundary value problem dependent on a fractional derivative of Riemann-Liouville type with homogeneous conditions by Critical Point Theory.

Keys words: Existence of solution, fractional differential equation, boundary value problem, critical point theory, minimization principle, Mountain pass theorem.

Introduction

Fractional derivative and integrals are used in the modeling different physical and natural phenomena.

Definition 0.101. [1](R-L fractional integrals)

Let u be a function defined on $[a, b]$. The left and right Riemann-Liouville fractional integrals of order $\alpha > 0$ for a function u denoted by $I_+^\alpha u$ and $I_-^\alpha u$, respectively, are defined by

$$I_+^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} (t-s)^{\alpha-1} u(s) ds, \quad t \in [0, +\infty),$$

and

$$I_-^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_t^{+\infty} (s-t)^{\alpha-1} u(s) ds, \quad t \in [0, +\infty),$$

provided that the right-hand side is pointwise defined on $[0, +\infty)$, where $\Gamma(\alpha)$ is the gamma function.

Definition 0.102. [1](R-L fractional derivatives)

Let u be a function defined on $[0, +\infty)$. For $n-1 \leq \alpha < n$ ($n \in \mathbb{N}^*$), the left and right Riemann-Liouville fractional derivatives of order $\alpha > 0$ for a function u denoted by $D_+^\alpha u$ and $D_-^\alpha u$, respectively, are defined by

$$\begin{aligned} D_+^\alpha u(t) &= \frac{d^n}{dt^n} I_+^{n-\alpha} u(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} u(s) ds, \quad t \in [0, +\infty), \end{aligned}$$



and

$$\begin{aligned} D_-^\alpha u(t) &= (-1)^n \frac{d^n}{dt^n} I_-^{n-\alpha} u(t) \\ &= \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^{+\infty} (s-t)^{n-\alpha-1} u(s) ds, \quad t \in [0, +\infty), \end{aligned}$$

provided that the right-hand side is pointwise defined.

Definition 0.103. Let $\varphi \in C^1(X, \mathbb{R})$. If any sequence $(u_n) \subset X$ for which $(\varphi(u_n))$ is bounded in $\mathbb{R}$ and $\varphi'(u_n) \rightarrow 0$ when $n \rightarrow +\infty$ in X' , the dual space of X , possesses a convergent subsequence, then we say that φ satisfies the Palais-Smale condition (denoted by (P.S) condition).

Theorem 0.207. (mountain pass theorem) Let X be a real Banach space and $\varphi \in C^1(X, \mathbb{R})$ satisfying the (P.S) condition. Suppose that

- (i) $\varphi(0) = 0$;
- (ii) there exist $\rho > 0$ and $\sigma > 0$ such $\varphi(z) \geq \sigma$ for all $z \in X$ with $\|z\| = \rho$;
- (iii) there exists z_1 in X with $\|z_1\| > \rho$ such that $\varphi(z_1) \leq \sigma$.

Then φ possesses a critical value such that $c \geq \sigma$. Moreover, c can be characterized as

$$c = \inf_{g \in \Lambda} \max_{z \in g([0,1])} \varphi(z),$$

where

$$\Lambda = \{g \in C([0,1], X) : g(0) = 0, g(1) = z_1\}.$$

Theorem 0.208. (Minimisation principale) Let X be a real reflexive Banach space. If the functional $\phi : X \rightarrow \mathbb{R}$ is weakly lower semi-continuous and coercive, i.e. $\lim_{\|u\| \rightarrow +\infty} \phi(u) = +\infty$, then there exists $u_0 \in X$ such that $\phi(u_0) = \inf_{u \in X} \phi(u)$. Moreover, if ϕ is also Gateaux differentiable on X , then $\phi'(u_0) = 0$

Main results

We consider the following boundary value problem

$$\begin{cases} D_{T-}^\alpha (D_{0+}^\alpha (D_{T-}^\alpha (D_{0+}^\alpha (D_{T-}^\alpha (D_{0+}^\alpha u(t))))) = f(t, u(t)), & t \in [0, T], \\ u(0) = u(T) = 0, \\ D_{T-}^\alpha (D_{0+}^\alpha u(0)) = D_{T-}^\alpha (D_{0+}^\alpha u(T)) = 0 \end{cases} \quad (0.516)$$

where $0 < \alpha \leq 1$, $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, $D_{0+}^\alpha, D_{T-}^\alpha$ are respectively the left and right fractional Riemann-Liouville derivatives. The main result is obtained by using the critical point theory in suitable space.

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The weak convergence of the smooth estimator of the distribution function under double truncation data

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Abstract. One of the biggest problems in data collection is the loss of observations, since this loss occurs under certain conditions that make it difficult to find the results of the study. This problem is known as truncation. In this paper we dealt with the asymptotic properties of an estimator defined under the condition of double truncation for the distribution function.

Keywords: Double truncation, distribution function, The convergence rate, The cumulative kernel function.

Introduction

The problem of double truncation is considered when the observation is known only if it satisfies certain conditions, for this reason we have dealt with three random variables in this case. First let X be the random variable of interest with the distribution function F and (U, V) are the left and the right truncation limit with the joint truncation limit G , let α be the probability of non truncation.

$$\tilde{F}(t) = P(T^* \leq t | U^* \leq T^* \leq V^*) = \alpha^{-1} \int_0^t \int_{u \leq x \leq v} G(u, v) du dv F(dx). \quad (0.517)$$

We defined the joint distribution function of truncation limit as before

$$\tilde{G}(u, v) = P(U^* \leq u, V^* \leq v | U^* \leq T^* \leq V^*) = \alpha^{-1} \int_{a_{K_1}}^u \int_{\max\{x, a_{K_2}\}}^v \int_x^y f(z) dz G(x, y) dx, dy. \quad (0.518)$$



The density estimators in Moreira2 is integrated for defined the estimator of distribution function

$$\begin{aligned} F_b(x) &= \int_{a_F}^x f_n(t) dt \\ &= \alpha_n \int \left(\int_{l \leq y \leq m} dG_n(l, m) \right)^{-1} H\left(\frac{x-y}{b}\right) d\tilde{F}_n(dy), \end{aligned}$$

and the semiparametric estimator of df by

$$\begin{aligned} F_{b;\hat{\theta}}(x) &:= \int_{a_X}^x f_{b;\hat{\theta}}(z) dz \\ &= \alpha_{\hat{\theta}} \frac{1}{n} \sum_{i=1}^n \left(\int_{l \leq x \leq m} dG_{\hat{\theta}}(l, m) \right)^{-1} H\left(\frac{x - X_i}{b}\right), \end{aligned}$$

where $H(t) = \int_{a_F}^t k(x) dx$ is the cumulative kernel function. First, we defined the asymptotic equivalent of estimator $F_b^*(t) = \frac{\alpha}{n} \sum_{i=1}^n \left(\int_{l \leq X_i \leq m} dG(l, m) \right)^{-1} H\left(\frac{x - X_i}{b}\right)$. Under the same condition of regularity of the kernel and bandwidth and let $F(x)$ and $\left(\int_{l \leq x \leq m} dG(l, m) \right)^{-1} F(x)$ are twice continuously differentiable., and assuming that H is have symmetric support on $[-1, 1]$ we gate

$$\begin{aligned} E(F_b(x)) &= F(x) + \frac{b^2}{2} F''(x) \mu_2(K) + o(b^2). \\ \text{Var}(F_b(x)) &= n^{-1} \alpha \left[\int_{l \leq x \leq m} dG(l, m) \right]^{-1} F(x) \{ F(x) [1 - F(x)] + b f(x) [J(k)] + o((nb)^{-1}) \}, \end{aligned}$$

where $J(k) = \int_{-1}^1 H^2(u) du - 1$.

Main results

Theorem 0.209. *Let K be a kernel function whose satisfied $0 < k(x)$ and bounded with $\int k(x) = 1$, $\lim_{x \rightarrow \infty} |xk(x)| = 0$, and $b \rightarrow 0$ with increasing n . Then our estimators is asymptotic unbiased and consists.*

Proof Based on the theorem of density under double truncation defined in [2] and the inequality instantly which based on fubini theorem lead to the first part of the theorem.

$$\begin{aligned} |EF_b^*(x) - F(x)| &= |E \int f_b^*(x) - f(x) dx| \\ &\leq \int E |f_b^*(x) - f(x)| dx \end{aligned}$$

Hence from [2] we can find that $E |f_b^*(x) - f(x)| \xrightarrow{n \rightarrow \infty} 0$ which leads to result of the part 1. Now for proof the consist of our estimators it results from the the first part of the theorem and the mean square error (MSE)

$$MSE(F_n^*(x)) = \text{Var}(F_n^*(x)) + (EF_n^*(x) - F(x))^2$$

hence as $n \rightarrow \infty$ the proof is completed. ■

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Consistency of a kernel estimator of the relative error regression function for incomplete and dependent data.

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Abstract. Let $(Y_i)_{1 \leq i \leq n}$ be a sequence of dependent random variables (r.v.) of interest distributed as Y and $(X_i)_{1 \leq i \leq n}$ be a corresponding d -dimensional vector of covariates taking values on $\mathbb{R}^d$. The r.v. Y is twice censored and satisfies the α -mixing property. In this paper, we are interested in the relative error regression estimation in case of twice censored data under strong mixing condition. Under suitable assumptions, the estimator's strong uniform almost sure consistency with rate is established. We have highlighted the covariance term, which is relatively uncommon in this context. Furthermore, we give a result about the α -mixing rate in the case for twice censoring. As far as we know there is no analogous result with proof in the context of twice censoring.

Keywords: Kernel estimate; Patilea-Rolin estimator; relative regression error; strong mixing condition; twice-censored data; uniform almost sure consistency.

Introduction

Consider the regression model $Y = r(X) + \epsilon$, where Y is a random variable (rv) of interest, X is a random covariate such that the error ϵ has zero mean and is uncorrelated with X . In the classical regression model, the purpose is to find a function $r^*(x)$ which achieves the minimum of the mean squared error

$$\mathbb{E} \left[(Y - r(x))^2 | X \right]. \quad (0.519)$$

However, this loss function which is considered as a measure of the prediction performance may be unadapted to some situations, especially in the presence of outliers. So, in the following, we circumvent the limitations of this classical regression by estimating the operator of regression with respect to the minimization of the following mean squared relative error (MSRE), for $Y > 0$

$$\min_r \mathbb{E} \left[\left(\frac{Y - r(x)}{Y} \right)^2 | X \right] \quad (0.520)$$



it is easy to see that the solution of the minimization problem of (0.520) is given by

$$\frac{\mathbb{E}(Y^{-1}|X)}{\mathbb{E}(Y^{-2}|X)} =: \frac{r_1(x)}{r_2(x)}$$

where $r_\ell(x) = \int_{\mathbb{R}} y^{-\ell} f_{X,Y}(x, y) dy$, for $\ell = 1, 2$.

[3] showed that

$$\frac{\mathbb{E}(Y^{-1}|X)}{\mathbb{E}(Y^{-2}|X)} \leq \mathbb{E}(Y|X) \quad \text{a.s.}, \quad (0.521)$$

provided the first two conditional inverse moments of Y given X are finite.

Unfortunately, in practice the data are not always independent and identically distributed, they are rather dependent. In this work we are particularly interested in the α -mixing dependent case. In the context of incomplete and dependent data, several works have been carried out such as [2] who studied the asymptotic properties of the kernel estimator of the classical regression function, [1] established the strong uniform consistency. Our goal is to build a kernel estimator of the mean squared relative error prediction for the regression function under twice censored and under α -mixing condition. Then, we give the uniform consistency result over a compact set with rate of our estimator.

In the twice censored case defined in model 1 of [4], let $(Y_i)_{1 \leq i \leq n}$ be a sample of r.v. of interest and let $(X_i)_{1 \leq i \leq n}$ be a corresponding d -dimensional vector of covariates with density $f(\cdot)$. We define $\tilde{r}_n(x)$ as a "pseudo-estimator" of $r(x)$ that is for any $x \in \mathbb{R}^d$ we have

$$\tilde{r}_n(x) = \frac{\sum_{i=1}^n \frac{1_{\{A_i=0\}}}{S_R(Z_i)F_L(Z_i)} Z_i^{-1} K_d\left(\frac{x - X_i}{h_n}\right)}{\sum_{i=1}^n \frac{1_{\{A_i=0\}}}{S_R(Z_i)F_L(Z_i)} Z_i^{-2} K_d\left(\frac{x - X_i}{h_n}\right)} = \frac{\tilde{r}_{1,n}(x)}{\tilde{r}_{2,n}(x)}. \quad (0.522)$$

In practice S_R and F_L are usually unknown, we replace them by their corresponding estimates (see [4]). So a feasible estimator of $r(\cdot)$ is given by

$$r_n(x) = \frac{\sum_{i=1}^n \frac{1_{\{A_i=0\}}}{\hat{S}_R(Z_i)\hat{F}_L(Z_i)} Z_i^{-1} K_d\left(\frac{x - X_i}{h_n}\right)}{\sum_{i=1}^n \frac{1_{\{A_i=0\}}}{\hat{S}_R(Z_i)\hat{F}_L(Z_i)} Z_i^{-2} K_d\left(\frac{x - X_i}{h_n}\right)} \quad (0.523)$$

where

$$r_{\ell,n}(x) = \frac{\sum_{i=1}^n \frac{1_{\{A_i=0\}}}{\hat{S}_R(Z_i)\hat{F}_L(Z_i)} Z_i^{-\ell} K_d\left(\frac{x - X_i}{h_n}\right)}{\sum_{i=1}^n K_d\left(\frac{x - X_i}{h_n}\right)} \quad (0.524)$$

Main results

Theorem 0.210. *Under suitable assumptions, we have for n large enough*

$$\sup_{x \in C} |r_n(x) - r(x)| = O(h_n) + O_{a.s.} \left(\sqrt{\frac{\log n}{nh_n^d}} + \sqrt{h_n^{d(v-2)} \log n} \right).$$

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PRODUCT OF ASYMMETRIC TOEPLITZ MATRICES.

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Abstract. In this paper, we study products of asymmetric Toeplitz matrices, we give necessary and sufficient conditions for the product of two asymmetric Toeplitz matrices compatible sizes is asymmetric Toeplitz matrix. We also give some results related to the isometry.

Keywords: Toeplitz matrix, asymmetric Toeplitz matrix, product of asymmetric Toeplitz matrices, isometry of asymmetric Toeplitz matrices.

Introduction

The product of matrices is an important technique in numerical calculations. Some authors have deduced several properties of asymmetric Toeplitz matrices based on the operators known as asymmetric truncated Toeplitz operators, such as [4, 5, 6]. When talking about asymmetric Toeplitz operators, they are introduced by [1, 2], the characterization of product of asymmetric truncated Toeplitz operators has been discussed in [7]. More interesting results about Toeplitz matrices can be found in [3, 8].



Preliminaries

Let $\mathbb{D} = \{z \in \mathbb{C}, |z| < 1\}$ and $\mathbb{T} = \{z \in \mathbb{C}, |z| = 1\}$ denote the open unit disk, the unit circle, respectively, in $\mathbb{C}$.

The Hardy space H^2 is defined as $H^2 = \{f \in L^2 : \hat{f}(n) = 0, \forall n \leq -1\}$, where $\hat{f}(n)$ denotes the n -th Fourier coefficient of f .

The standard Lebesgue space is denoted by $L^2 = L^2(\mathbb{T}, m)$, and The Lebesgue space is denoted by $L^\infty = L^\infty(\mathbb{T}, m)$, where m be the Lebesgue measure on $\mathbb{T}$.

On H^2 , we define the unilateral shift operator S by $(Sf)(z) = zf(z)$. Or, if $f = \sum_{n=0}^{\infty} a_n z^n \in H^2$ by $S(a_0, a_1, \dots) = (0, a_0, a_1, \dots)$, for $(a_0, a_1, \dots) \in \ell^2(\mathbb{N})$.

A bounded analytic function u is said to be inner if $|u(z)| = 1$ a.e on $\mathbb{T}$.

The model space corresponding to inner function u is the closed subspace K_u of H^2 defined as

$$K_u := (uH^2)^\perp = \{f \in H^2 : \langle f, uh \rangle = 0\} \quad \forall h \in H^2.$$

Let α and β are inner functions. An asymmetric truncated Toeplitz operator $A_\varphi^{\alpha, \beta}$ on K_α with symbol $\varphi \in L^2$ is defined by

$$A_\varphi^{\alpha, \beta} f = P_\beta(\varphi f), \quad f \in K_\alpha,$$

where P_β denotes the orthogonal projection onto K_β .

Let $\alpha = z^m$ and $\beta = z^n$. Then the matrix representation of asymmetric truncated Toeplitz operator $A_\varphi^{\alpha, \beta}$ with respect to the standard orthonormal basis $\{1, z, z^2, \dots, z^{n-1}\}$ is

$$A = \begin{bmatrix} a_0 & \bar{a}_1 & \cdots & \bar{a}_{m-1} \\ a_1 & a_0 & & \vdots \\ \vdots & \ddots & & \\ a_{n-1} & & & \end{bmatrix}.$$

Objective

Our goal is to find the necessary and sufficient conditions on two asymmetric Toeplitz matrices A and B until the product AB is an asymmetric Toeplitz matrix. Finally, we characterize isometric asymmetric Toeplitz matrices.

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Two Parameter Lindley Frailty Model: Properties, Estimation and Application

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Abstract. Standard survival models make the assumption that every individual in the population who could experience the event of interest is identical. Nonetheless, a certain level of unobserved heterogeneity exists in many real-world situations, which can be generated by a variety of factors including genetic or environmental ones. Many issues, such as an inaccurate estimate of the regression coefficients and an overestimation of the hazard rate, could occur if the heterogeneity is underestimated. Typically, individual heterogeneity is modeled using frailty models. Our work presents a new univariate frailty model. It is considered that the frailty variable follows the Two-parameter Lindley distribution. To estimate the model parameters, the maximum likelihood method is applied. A simulation research is carried out under the assumption that the baseline hazard functions have Weibull distribution. We analyze the distribution's properties and evaluate the model's adequacy using a powerful hypothesis test.

Keywords: Frailty, Heterogeneity, Maximum likelihood, Time to event data.



Introduction

A crucial assumption in survival analysis is the independent and identical distribution of the time-to-event data. The data are subject to unobserved heterogeneity or frailty, which could affect a person's chance of survival, hence this assumption is not always accurate. Frailty is usually described by a distribution known as the frailty distribution (Duchateau and Janssen (2007)). Although a non-parametric specification of the frailty distribution can be obtained. Almeida et al. (2020), and Wienke (2010) pointed out that the parametric approach is frequently used due to mathematical simplicity.

The main motivation of this work is to introduce the Two-Parameter Lindley frailty model (TPLF), a new flexible frailty model developed based on the (TPL) model suggested by Shanker and Mishra (2013), to overcome the limitations of the gamma, compound Poisson, log-normal frailty models, and other frailty distributions. The inference is presented using the maximum likelihood estimation method to estimate our model's parameters with the Weibull baseline hazard function. Simulation studies are performed in cases of completed and censored data. We carry out a powerful hypothesis test, in order to show the adequacy of our model.

Main results

*

The Two Parameter Lindley Frailty model

According to Shanker and Mishra (2013), the TPL model's probability density function (PDF) can be written as

$$f_{\alpha, \theta}(y) = \frac{1}{\alpha\theta + 1} \theta^2 (\alpha + y) \exp(-\theta y) | y > 0, \theta > 0, \alpha\theta > -1, \quad (0.525)$$

Let's consider that the frailty variable Z has a TPL distribution (0.525) with a mean of one, or $E[Z] = 1$. This assumption is essential in order to identify the resulting frailty model. As a result, by using the alternative parameterization of the TPL model in terms of mean, the TPLF model's PDF becomes

$$f_{\theta}(z) = \frac{1}{3} \theta^2 (1 - \theta) \left[\frac{\theta + 2}{\theta(1 - \theta)} + z \right] \exp(-\theta z) | z > 0, \quad (0.526)$$

where the unknown shape parameter is denoted by $\theta > 0$. In general, the variance of the frailty distribution is used to quantify the amount of unobserved heterogeneity present in a study's population. Considering that the PDF (0.531) is a frailty distribution. The variance is expressed by:

$$\sigma^2 = \frac{1}{9\theta^2} \left[(1 - \theta) (2\theta + 10) + (\theta + 2)^2 \right], \quad (0.527)$$

Depending on its variance, the frailty PDF (0.531)'s Laplace transform is given by:

$$L_f(s) = \frac{\tau(\sigma^2) - 2}{3[s(\sigma^2) - 2]} \left\{ \frac{D(\sigma^2)}{(1 + 9\sigma^2)} + \frac{[\tau(\sigma^2) - 2] C(\sigma^2)}{(1 + 9\sigma^2)[s(\sigma^2) - 2]} \right\} | s \in \mathbb{R}, \quad (0.528)$$

where $\tau(\sigma^2) = 3\sqrt{2(1 + 7\sigma^2)}$, $D(\sigma^2) = 18\sigma^2 + \tau(\sigma^2)$, $s(\sigma^2) = s(1 + 9\sigma^2) + \tau(\sigma^2)$ and $C(\sigma^2) = 3 + 9\sigma^2 - \tau(\sigma^2)$. For the sake of simplicity, we evaluate equation (0.528) at $s = \Lambda_0(t_i)\xi_i$, where $\xi_i = \exp(x_i^T \beta)$, and determine that the marginal survival function under the TPLF model can be obtained by

$$S(t_i | x_i) = \frac{\tau(\sigma^2) - 2}{3\Lambda_0(\xi_i, \sigma^2)} \left\{ \frac{D(\sigma^2)}{1 + 9\sigma^2} + \frac{[\tau(\sigma^2) - 2] C(\sigma^2)}{(1 + 9\sigma^2) \Lambda_0(\xi_i, \sigma^2)} \right\}, \quad (0.529)$$

where

$$\Lambda_0(\xi_i, \sigma^2) = \Lambda_0(t_i) \xi_i (1 + 9\sigma^2) + \tau(\sigma^2) - 2.$$





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Large deviation principle of the single server queue

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Abstract. The theory of large deviations focuses on rare events with a low probability of occurrence. Let (X_n) be a sequence of random variables with values in a Polish space $\mathbb{E}$. We say that (X_n) verifies a PLD with a good rate function T and of speed $s_n = n$ if $P(X_n \in \mathbb{E}) \simeq e^{-nT(x)}$. The aim of this work is to apply the PLD for a sequence of independent and equidistributed random variables representing the sojourn time of stays in the M/M/1 queuing system

Keywords: Queueing theory, sojourn time, Large deviation, rate function.

Introduction

The theory of large deviations is a branch of probability theory focused on estimating the likelihood of rare events, which are significant deviations from a system's typical or average behavior. In the context of queueing systems, such as the M/M/1 model, this theory helps analyze infrequent occurrences, like exceptionally long waiting times or unusually extended service durations. The theory of large deviation can be applied to a variety of scientific fields, from queue theory to statistics, from finance to engineering. It is also used in statistical physics, telecommunication, reliability, etc...

The problem of large deviations and its importance can be well understood by studying the question which motivated the development of the theory which concerns the law of large numbers and the central limit theorem.

Let (X_n) be a sequence of random variables independent identically distributed (i.i.d.) with expectation μ : $M_n = \sum_{l=1}^n X_l$, $\overline{M}_n = \frac{M_n}{n}$.



- Law of large numbers:
If $\mathbb{E}(|X_1|) < \infty$ then $\overline{M}_n \rightarrow \mu$ p.s when $n \rightarrow \infty$.
- Central limit theorem:
If $\sigma^2 = \text{Var}(X_1) < \infty$ then $\sqrt{n}(\overline{M}_n - \mu)/\sigma$ converges in distribution to a standard normal Gaussian distribution, denoted $\mathcal{N}(0, 1)$.
According to these previous theorems, the events $\{\sqrt{n}(\overline{M}_n - \mu)/\sigma > \tau\}$ and $\{\overline{M}_n > \tau\}$, pour $\tau > \mu$ are rare, and we seek to quantify the speed of convergence to zero of the probabilities of these events.
- Principle of large deviations:
If $\mathbb{E}(e^{\alpha X_1}) < \infty$, $\alpha \in \mathbb{R}$, then $\mathbb{P}(\overline{M}_n > \tau + \rho) \simeq e^{-ng(\tau, \rho)}$ for a suitable function g .
The central limit theorem gives an approximation of M_n when n is large. But its interest is, in a way, complementary to that of LDP. Indeed, it shows how $\frac{M_n}{n}$ concentrates around the mean μ .
Here, we want to estimate probabilities that we know to be low. In this case, the central limit theorem is not a very good tool. On the contrary, the LDP gives the behavior of the tail of the distribution, but says nothing about the typical behavior of the $\frac{M_n}{n}$.

Main results

This paper summarizes the results of theory of large deviations and presents an application in the M/M/1 queueing system by studying the asymptotic behavior of a customer's stay time in the system.

Definition 0.104. The function

$$\psi^*(x) = \sup_{\alpha \in \mathbb{R}} \left\{ \alpha x - \log \mathbb{E} e^{\alpha X} \right\} \quad (0.530)$$

is called The Legendre transform (or convex dual) of ψ .

Theorem 0.211. Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of independent and identically distributed random variables with a rate function ψ^* . The large deviations principle is verified for the sequence $(P_n)_{n \in \mathbb{N}}$ defined by:

$$P_n(F) = \mathbb{P}(\overline{M}_n \in F), \forall F \in \mathbb{R}$$

If $\mu = E(X)$, then

$$P(\overline{M}_n \in [\mu + \delta, +\infty]) \simeq e^{-n\psi^*(\mu + \delta)} \quad (0.531)$$

Lemma 0.74. The function ψ^* is convex and lower semicontinuous.

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Hybrid Taylor and block-pulse functions operational matrix algorithm and its application to obtain the approximate solution of stochastic Itô-Volterra integral equations

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Abstract. The definition of block pulse functions (fb) and the definition of Taylor polynomail have been used. The approximate of exact solution by (fb) and the approximate of exact solution by Hybrid Taylor polynomail have been studied. Relationship matrix between block function and Hybrid Taylor polynomail have been involved,The stochastic operational matrix method for solving stochastic Itô Volterra have been found. The theoremes and the remarks have been used.The algebraic system of a numerical method for the approximation of exact solution have been found.

Keywords: Equation, stochastic operational matrix, approximation,Brownian motion,numerical method of Hybrid Taylor polynomail,algebraic system.



Introduction

There are many equations that no one has found its exact solution, but these equations are very important in scientific problems, we have to find a rough solution to solve these equations. In this work we implement operational matrix method based on Hybrid Taylor and block-pulse functions to obtain the numerical solution of equation.[1]
 Consider the stochastic differential equation for $t \in [0, 1]$:[2]

$$V(t) = U(t) + \int_0^t K_0(s, t)V(s)ds + \sum_{i=1}^n \int_0^t K_i(s, t)V(s)dB_i(s), i = 1, 2, \dots, n, \quad (0.532)$$

where $B(t)$ is standard Brownian motion and $\int_0^t K(s, t)V(s)dB_i(s)$ is Itô integral .
 We use an algebraic system to solve the equation (1), this algebraic system resulted in many matrices.

In this paper, We study numerical solutions by using block pulse functions (fb) and Hybrid Taylor polynomial in order to find the algebraic system and approaching the exact solution of the equation (1) .

Main results

Definition 0.105. block-pulse functions [1]

$$\psi_i(t) = \begin{cases} 1, & \frac{i-1}{m} \leq t < \frac{i}{m}, \\ 0, & \text{sinon,} \end{cases} \quad (0.533)$$

where $t \in [0, 1]$, $m \in \mathbb{N}$, $h = 1/m$ and $i = 1, \dots, m$.

$$f(t) \simeq \sum_{i=1}^m f_i \psi_i(t) = F^T \Psi(t) = \Psi(t)^T F, \quad (0.534)$$

$$F = [f_1, f_2, \dots, f_m]^T$$

$$f_i = \frac{1}{h} \int_0^1 f(t) \psi_i(t) dt. \quad (0.535)$$

$$\Psi^T(t) = [\psi_1(t), \psi_2(t), \dots, \psi_m(t)]$$

Definition 0.106. Taylor polynomials and block-pulse functions [1]

Consider well-known Taylor polynomials $T^j(t) = t^j$ on the interval $[0, 1]$. The hybrid functions constructed by block-pulse functions and Taylor polynomials are defined in the next definition.

$$W_{i,j}(t) = \begin{cases} T_j(Kt - i + 1), & \frac{i-1}{K} \leq t \leq \frac{i}{K}, \\ 0, & \text{sinon,} \end{cases} \quad (0.536)$$

Where $j = 0, 1, 2, \dots, J-1$, $i = 1, 2, \dots, K$ and $k \in \mathbb{N}$, T_j is Taylor polynomial. Approximation of $z(t)$ and $z(s, t)$ via Taylor polynomials and block-pulse functions

$$z(t) \simeq \sum_{i=1}^K \sum_{j=0}^{J-1} c_{i,j} W_{i,j}(t) = C^T W(t). \quad (0.537)$$

and

$$z(s, t) \simeq W^T(s) Z W(t) = W^T(t) Z^T W(s), \quad (0.538)$$

Theorem 0.212. [2]

$$W(t) = Q \Psi(t), \quad (0.539)$$

where



$$Q_{ij} = W_i \left(\frac{2j-1}{2m} \right), i, j = 1, 2, \dots, m. \quad (0.540)$$

By using Lemma 1, 2, 3 , Remark 1, 2 and Theorem 3.1, 3.2, 3.3 in [2] we get

$$V^T W(t) = U^T W(t) + W(t)^T k_0^T \hat{V} R W(t) + \sum_{i=1}^n W(t)^T k_i^T \hat{V} R_s W(t). \quad (0.541)$$

So

$$V^T W(t) - \hat{Y}_0^T W(t) - \sum_{i=1}^n \hat{Y}_i^T W(t) = U^T W(t). \quad (0.542)$$

Therefore

$$V^T - \hat{Y}_0^T - \sum_{i=1}^n \hat{Y}_i^T = U^T. \quad (0.543)$$

Where

$$Y_0 = k_0^T \hat{V} R, Y_i = k_i^T \hat{V} R_s,$$

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Modeling Seismic Activity: A Theoretical and Numerical Study of Nonlinear Volterra Systems

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Abstract. In this research, we introduce a pioneering scientific approach to comprehending seismic activity, incorporating the concept of variable resistance. Our investigation focuses on a comprehensive earthquake model defined by a system of nonlinear integro-differential Volterra equations. This model's solution is intricately tied to both velocity and resistance factors. Under a spectrum of reasonable conditions, we demonstrate the existence of a unique solution. Utilizing Nyström's method, we conduct a series of numerical tests that validate the practicality and effectiveness of our study, offering a comprehensive insight into the complexities of this natural phenomenon.

Keywords: Nonlinear Volterra equation, · Integro-differential, earthquakes, Fix point



Introduction

In their study, Salah et al.[1] presented a comprehensive, deterministic physical model elucidating seismic phenomena. This model reflects the authors' dedication to enhancing our understanding of seismic processes, drawing inspiration from the Incorporated Research Institutions for Seismology (IRIS) demonstration model (www.iris.edu). Their achievement in developing numerical approximations closely aligned with empirical observations is noteworthy.

In this work, we consider a model comprising two coupled mass-spring oscillators. The first oscillator includes two masses, M_1 and m_1 , connected by a spring with a natural length of L_1 when $t = 0$. Similarly, the second oscillator comprises two masses, M_2 and m_2 , connected by a spring with a natural length of L_2 when $t = 0$. Both oscillators are allowed free movement. The objective is to engineer a coherent system that maintains the essential properties of the phenomenon in question.

To refine the model, we will disregard the effects of friction and external forces. The system shall be situated on a level, unblemished surface, thus providing an idealized context for analysis.

We designate the forces acting upon the masses m_1 and m_2 as F_1 and F_2 , respectively. Furthermore, we denote by R the resistance encountered by the larger masses M_1 and M_2 . Since each mass can oscillate, we invoke Newton's second law for each entity. Let $x_1(t)$ and $x_2(t)$ represent the displacements of masses M_1 and M_2 from their equilibrium positions, respectively. Similarly, let $y_1(t)$ and $y_2(t)$ signify the analogous displacements for masses m_1 and m_2 . The mass m_1 is subjected to a force F_1 propelling it rightward, whereas the mass m_2 is acted upon by a force F_2 directing it leftward. Implementing Hooke's law for the spring connecting m_1 and M_1 , we derive:

$$T_1(t) = \varrho_1 l_1(t),$$

where T_1 is the tension in the spring, ϱ_1 is the stiffness constant of the spring, and $l_1(t)$ denotes the temporal variation in length between the two masses m_1 and M_1 , such that:

$$l_1(t) = L_1 + \frac{p}{2} - (x_1(t) - y_1(t)).$$

Where p denotes the width of the plates, upon application of Hooke's law to the spring connecting m_2 and M_2 , we obtain:

$$T_2(t) = \varrho_2 l_2(t),$$

$$l_2(t) = L_2 + \frac{p}{2} - (y_2(t) - x_2(t)),$$

where T_2 is the tension in the spring, $l_2(t)$ is the change in length between the masses m_2 and M_2 at time t , ϱ_2 is the stiffness constant of the spring.

Applying the modified Newton's law to these objects and utilizing a Heaviside function, we obtain a system of equations in the state of rest. Then, by employing the conventional version of this law to these equations, we obtain another system of equations in the state of motion. Using the delta function, the states can be unified into one system, as shown in the following form:

$$\begin{cases} (1 - \delta(x'_1(t)))(\varrho_1 l_1(t) - R(t)) + \delta(x'_1(t))H(\varrho_1 l_1(t) - R(t))(\varrho_1 l_1(t) - R(t)) = M_1 x''_1(t), \\ (1 - \delta(x'_2(t)))(R(t) - \varrho_2 l_2(t)) + \delta(x'_2(t))H(\varrho_2 l_2(t) - R(t))(R(t) - \varrho_2 l_2(t)) = M_2 x''_2(t), \\ F_1 - \varrho_1 l_1(t) = m_1 y''_1(t), \\ -F_2 + \varrho_2 l_2(t) = m_2 y''_2(t), \\ l_1(t) = L_1 + \frac{p}{2} - (x_1(t) - y_1(t)), \\ l_2(t) = L_2 + \frac{p}{2} - (y_2(t) - x_2(t)) \end{cases} \quad (0.544)$$

Where,

$$\delta(\tau) = \begin{cases} 1, & \text{if } \tau = 0, \\ 0, & \text{if } \tau \neq 0. \end{cases}$$

The velocities of the masses M_1 and M_2 , $x'_1(t)$ and $x'_2(t)$, respectively, are also considered.



In the context of modeling seismic activity and to ensure a well-posed problem, we introduce seismic functions u_1 , and u_2 , given by

$$u_1(\tau_1) = \varrho_1 l_1(\tau_1) - R(t),$$

$$u_2(\tau_2) = \varrho_2 l_2(\tau_2) - R(t).$$

Accordingly, the aforementioned system of equations (0.544) can be reformulated as

$$u_1(t) + \frac{1}{2} \delta(x'_1(t))(|u_1(t)| - u_1(t)) = M_1 x''_1(t), \quad (0.545)$$

$$u_2(t) + \frac{1}{2} \delta(x'_2(t))(|u_2(t)| - u_2(t)) = -M_2 x''_2(t), \quad (0.546)$$

$$F_1 - R(t) - u_1(t) = m_1 y''_1(t), \quad (0.547)$$

$$-F_2 + R(t) + u_2(t) = m_2 y''_2(t), \quad (0.548)$$

$$u_1(t) = \varrho_1 \left(L_1 + \frac{p}{2} - (x_1(t) - y_1(t)) \right) - R(t), \quad (0.549)$$

$$u_2(t) = \varrho_2 \left(L_2 + \frac{p}{2} - (y_2(t) - x_2(t)) \right) - R(t). \quad (0.550)$$

Where the relation

$$H(\tau)\tau = \frac{1}{2}(|\tau| + \tau) \quad (0.551)$$

is used.

In our analysis, we perform integrations on equations (0.545) and (0.547), subsequently substituting the results into (0.549). Additionally, we conduct integrations on equations (0.546) and (0.548), replacing the outcomes in (0.550). These operations are carried out while adhering to the following initial conditions:

$$\begin{cases} x_1(0) = L_1 + \frac{p}{2} \\ x_2(0) = L_1 + \frac{3p}{2} \\ y_1(0) = y'_1(0) = x'_1(0) = 0 \\ y_2(0) = L_1 + L_2 + 2p \\ x'_2(0) = y'_2(0) = 0 \end{cases} \quad (0.552)$$

In light of the preceding analysis, we derive the following system of equations:

$$\begin{aligned} u_1(t) - \frac{\varrho_1 C_1}{\varrho_2(\varrho_1 - C_1)} u_2(t) &= \int_0^t (t-s) \left(-\varrho_1 \left(\frac{\varrho_1}{(\varrho_1 - C_1)M_1} + \frac{1}{m_1} \right) u_1(s) + \frac{\varrho_1 C_1}{m_2(\varrho_1 - C_1)} u_2(s) + \right. \\ &\quad \left. + \varrho_1 \left(\frac{C_1}{m_2(\varrho_1 - C_1)} - \frac{1}{m_1} \right) R(s) - \frac{\varrho_1^2}{2M_1(\varrho_1 - C_1)} \delta_\xi \left(\frac{C_1}{\varrho_1 \varrho_2} u'_2(t) + \frac{C_1 - \varrho_1}{\varrho_1^2} u'_1(t) + \right. \right. \\ &\quad \left. \left. + \frac{\varrho_1 - C_1}{m_1 \varrho_1} \int_0^s (F_1 - R(\alpha) - u_1(\alpha)) d\alpha + \frac{C_1}{m_2 \varrho_1} \int_0^s (u_2(\alpha) + R(\alpha) - F_2) d\alpha \right) (|u_1(s)| - u_1(s)) ds \right) + \\ &\quad \left. + \frac{\varrho_1}{2} \left(\frac{F_1}{m_1} - \frac{C_1}{m_2(\varrho_1 - C_1)} F_2 \right) t^2. \right. \quad (0.553) \end{aligned}$$

$$\begin{aligned} u_2(t) - \frac{C_1 \varrho_2}{\varrho_1(\varrho_2 - C_1)} u_1(t) &= \int_0^t (t-s) \left(-\varrho_2 \left(\frac{\varrho_2}{M_2(\varrho_2 - C_1)} + \frac{1}{m_2} \right) u_2(s) + \frac{\varrho_2 C_1}{m_1(\varrho_2 - C_1)} u_1(s) + \right. \\ &\quad \left. + \varrho_2 \left(\frac{C_1}{m_1(\varrho_2 - C_1)} - \frac{1}{m_2} \right) R(s) - \frac{\varrho_2^2}{2M_2(\varrho_2 - C_1)} \delta_\xi \left(\frac{\varrho_2 - C_1}{\varrho_2^2} u'_2(t) - \frac{C_1}{\varrho_1 \varrho_2} u'_1(t) + \right. \right. \\ &\quad \left. \left. + \frac{C_1}{m_1 \varrho_2} \int_0^s (F_1 - R(\alpha) - u_1(\alpha)) d\alpha + \frac{\varrho_2 - C_1}{m_2 \varrho_1} \int_0^s (u_2(\alpha) + R(\alpha) - F_2) d\alpha \right) (|u_2(s)| - u_2(s)) ds \right) + \\ &\quad \left. + \frac{\varrho_2}{2} \left(\frac{F_2}{m_2} - \frac{C_1 F_1}{m_1(\varrho_2 - C_1)} \right) t^2. \right. \quad (0.554) \end{aligned}$$

$$R(t) + C_1 \left(\frac{u_2(t)}{\varrho_2} + \frac{u_1(t)}{\varrho_1} \right) = -C_1 \int_0^t (t-s) \left(\frac{1}{m_2} u_2(s) + \frac{1}{m_1} u_1(s) + \left(\frac{1}{m_1} + \frac{1}{m_2} \right) R(s) \right) ds +$$



$$+ C_1 \left(\frac{F_1}{2m_1} + \frac{F_2}{2m_2} \right) t^2 \quad (0.555)$$

These equations extend to their corresponding differential forms:

$$\begin{aligned} u_1'(t) - \frac{\varrho_1 C_1}{\varrho_2(\varrho_1 - C_1)} u_2'(t) = & \int_0^t \left(-\varrho_1 \left(\frac{\varrho_1}{(\varrho_1 - C_1)M_1} + \frac{1}{m_1} \right) u_1(s) + \frac{\varrho_1 C_1}{m_2(\varrho_1 - C_1)} u_2(s) + \right. \\ & + \varrho_1 \left(\frac{C_1}{m_2(\varrho_1 - C_1)} - \frac{1}{m_1} \right) R(s) - \frac{\varrho_1^2}{2M_1(\varrho_1 - C_1)} \delta_\xi \left(\frac{C_1}{\varrho_1 \varrho_2} u_2'(t) + \frac{C_1 - \varrho_1}{\varrho_1^2} u_1'(t) + \right. \\ & + \left. \frac{\varrho_1 - C_1}{m_1 \varrho_1} \int_0^s (F_1 - R(\alpha) - u_1(\alpha)) d\alpha + \frac{C_1}{m_2 \varrho_1} \int_0^s (u_2(\alpha) + R(\alpha) - F_2) d\alpha \right) (|u_1(s)| - u_1(s)) ds \Big) + \\ & + \varrho_1 \left(\frac{F_1}{m_1} - \frac{C_1}{m_2(\varrho_1 - C_1)} F_2 \right) t. \end{aligned} \quad (0.556)$$

$$\begin{aligned} u_2'(t) - \frac{C_1 \varrho_2}{\varrho_1(\varrho_2 - C_1)} u_1'(t) = & \int_0^t \left(-\varrho_2 \left(\frac{\varrho_2}{M_2(\varrho_2 - C_1)} + \frac{1}{m_2} \right) u_2(s) + \frac{\varrho_2 C_1}{m_1(\varrho_2 - C_1)} u_1(s) + \right. \\ & + \varrho_2 \left(\frac{C_1}{m_1(\varrho_2 - C_1)} - \frac{1}{m_2} \right) R(s) - \frac{\varrho_2^2}{2M_2(\varrho_2 - C_1)} \delta_\xi \left(\frac{\varrho_2 - C_1}{\varrho_2^2} u_2'(t) - \frac{C_1}{\varrho_1 \varrho_2} u_1'(t) + \right. \\ & + \left. \frac{C_1}{m_1 \varrho_2} \int_0^s (F_1 - R(\alpha) - u_1(\alpha)) d\alpha + \frac{\varrho_2 - C_1}{m_2 \varrho_1} \int_0^s (u_2(\alpha) + R(\alpha) - F_2) d\alpha \right) (|u_2(s)| - u_2(s)) ds \Big) + \\ & + \varrho_2 \left(\frac{F_2}{m_2} - \frac{C_1 F_1}{m_1(\varrho_2 - C_1)} \right) t. \end{aligned} \quad (0.557)$$

Numerical results

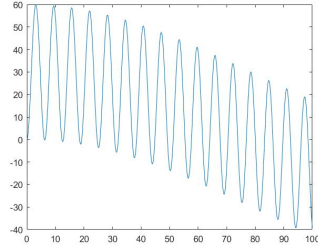


Figure 35: Seismic Function $u_1(t)$

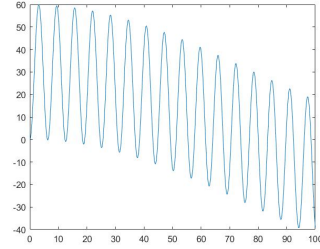


Figure 36: Seismic Function $u_2(t)$

The graphs (Figs. 35 and 36) illustrate the seismic functions u_1 and u_2 . The time axis extends up to 100 seconds, while the vertical axis represents the values of the seismic response in Newtons. Initially, both functions start at 0 and rise to 60 Newtons within the first 5 seconds. This is followed by a decrease back to 0 over the next 7 seconds. Subsequently, another peak reaching 60 Newtons is observed at 10 seconds, after which a gradual decline to 0 occurs over 15 seconds. This periodic behavior continues, with the amplitude gradually decreasing to -40 Newtons by the end of the 100-second period. The negative values represent a reversal in the direction of the seismic force, based on the adopted reference frame. These results for both u_1 and u_2 validate the effectiveness and consistency of our model in simulating seismic behaviour.



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Energy levels of Dirac particle confined on a curved surface

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Introduction

The recent advances in synthesizing low-dimensional nanostructures with curved geometries have renewed interest in the dynamics of particles constrained on a curved two-dimensional surface. Historically, two relevant methods have been formulated to describe the quantum mechanical motion of particles on curved surface: a first by DeWitt [1] based on canonical quantization procedure, which led however to an ambiguity of operator ordering. A second much suitable scheme, the thin-layer quantization approach has been proposed by Jensen and Koppe [2] and later by da Costa [3] showing that the quantum dynamics of a particle confined to a curved surface presents an additional scalar geometric potential even though there is no interaction. The confinement process is achieved by the action of an external potential V_λ , which is assumed to be strong enough to keep the particle permanently on surface. This potential can be described by an ideal model that only depends on the normal coordinate q_3 and has the limitation

$$V_\lambda(q_3) = \begin{cases} 0, & q_3 = 0 \\ \infty, & q_3 \neq 0, \end{cases} \quad (0.558)$$

where λ is a parameter. This model could be used to describe confined particles in nanostructures, more precisely in nano-tubes which are generally constructed by very thin layers of different materials [5]. In this way, the three-dimensional Schrodinger equation for the system is reduced to an effective two-dimensional equation for the spatial components parallel to the curved surface.

To our knowledge, the thin-layer quantization formalism for a non-relativistic particle on the curved surface has been successfully discussed both theoretically and experimentally []. However, for the relativistic case, very few studies have discussed this problem, particularly regarding the ambiguity of non-appearance of the geometric potential in the non-relativistic limit [[6][7]]. It would therefore be interesting to shed some light on this subject.

The objective of this study is twofold. First, we will show by concrete examples that the non-appearance of this potential in the non-relativistic limit is essentially due to the inclusion of spin connection terms in the Dirac equation. These terms give a scalar repulsive terms which contribute to cancel the geometric potential. Second, we deduce the energy spectrum for relativistic particle confined on a cylindrical and a spherical surfaces using the thin-layer approach.

As this geometric potential results from the separation of the Laplace Beltrami operator, it seems natural to consider the quadratic form of the Dirac equation. Moreover, this form will help us to deduce the non-relativistic limit. For simplicity, we adopt here a more realistic model for the confining potential as hard walls with a constant width d [4]

$$V_\lambda(q_3) = \begin{cases} 0, & |q_3| < d/2 \\ \infty, & |q_3| \geq d/2, \end{cases} \quad (0.559)$$

the limit $q_3 \rightarrow 0$ is simply obtained by considering very small values of d . This confining process is then introduced in the Dirac equation as minimal coupling.

The present paper is organized as follows. In Sec II, the Dirac equation in curved space-time is briefly reviewed. In Sec. III, the thin-layer quantization formalism is employed on quadratic Dirac equation for the relativistic particle confined on curved surface. In Sec. IV, a particle on cylinder. In Sec. V, we briefly summarize the obtained results.



Dirac equation in curved space-time

Even in flat Euclidean space it might be useful to use curvilinear coordinates. However, in a curved space we have no other choice, because Cartesian coordinates can only exist locally, that is, in an infinitesimal neighborhood. In this section* we will see how to write the Dirac equation in a curved space. In reality, what we want to write is the covariant Dirac equation for an arbitrary metric. Since this equation is written for an arbitrary metric, we can also use it to write the Dirac equation on a curved surface.

Let's start by writing the Dirac equation in flat space-time for a particle of mass M and spin $1/2$ in cartesian coordinates x^a , $a = 0, 1, 2, 3$, as follows

$$(i\hbar\gamma^a\partial_a - Mc)\psi = 0, \quad (0.560)$$

where γ^a are constant matrices relative to Minkowski metric $\eta_{ab} = \text{diag}(1, -1, -1, -1)$. We can extend the Dirac equation from flat space-time to curved space-time using the tetrad formalism. These tetrads e^a_μ and its inverses e_a^μ allow us to transform between a local inertial frame and a global frame. Indeed, in this formalism the metric $h_{\mu\nu}$ of a curved space-time is linked to the Minkowski metric by the relation

$$h_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab} \quad (0.561)$$

Since η_{ab} is a constant matrix, these tetrads necessarily depend on space-time coordinates. In addition, these tetrads verify the orthogonality relations

$$e_\mu^a e_a^\nu = \delta_\mu^\nu, \quad e^a_\mu e_b^\mu = \delta^a_b$$

and

$$e_{a\mu} = h_{\mu\nu} e_a^\nu, \quad e_a^\mu = \eta^{ab} e_{b\mu}$$

from where we get the relation

$$\eta_{ab} = e_a^\mu e_b^\nu h_{\mu\nu} \quad (0.562)$$

To write the Dirac equation (0.560) in curved space-time, the constant matrices γ^a must be replaced by coordinates dependent matrices γ^a as

$$\gamma^a \rightarrow \tilde{\gamma}^\mu = e_a^\mu \gamma^a,$$

these $\tilde{\gamma}^\mu$ satisfy the Clifford algebra

$$\tilde{\gamma}^\mu \tilde{\gamma}^\nu + \tilde{\gamma}^\nu \tilde{\gamma}^\mu = 2h^{\mu\nu},$$

we also have to put the covariant derivative instead of the partial derivative because the partial derivative of a spinor does not transform like a spinor. That is

$$\partial_a \rightarrow \mathcal{D}_\mu = \partial_\mu + \Gamma_\mu,$$

where the spinorial affine connection Γ_μ satisfies the following relation in terms of Dirac matrices

$$\Gamma_\mu = \frac{1}{4} \omega_{ab\mu} \gamma^a \gamma^b, \quad (0.563)$$

the quantity $\omega_{ab\mu}$ is called the spin connection and can be calculated from the relation

$$\omega_{ab\mu} = \eta_{ac} e^c_\beta \nabla_\mu e_b^\beta, \quad (0.564)$$

with

$$\nabla_\mu e_b^\beta = \partial_\mu e_b^\beta + \Gamma_{\mu\lambda}^\beta e_b^\lambda, \quad (0.565)$$

where $\Gamma_{\mu\lambda}^\beta$ are Christoffel Symbols. We note that $\omega_{ab\mu}$ are antisymmetric with respect to the indices a and b . In this way, Dirac's equation takes the following form in curved space-time

$$(i\hbar\tilde{\gamma}^\mu \mathcal{D}_\mu - Mc)\psi = 0. \quad (0.566)$$

This equation will be used in the following section* to describe the dynamics of a relativistic particle constrained on a curved surface.

multiplions par $(i\hbar\tilde{\gamma}^\nu \mathcal{D}_\nu + Mc)$ on obtient

$$\left(\tilde{\gamma}^\mu \mathcal{D}_\mu \tilde{\gamma}^\nu \mathcal{D}_\nu + \frac{M^2 c^2}{\hbar^2} \right) \psi = 0$$



on a

$$\tilde{\gamma}^\mu D_\mu \tilde{\gamma}^\nu D_\nu = \frac{1}{\sqrt{-h}} D_\mu \left(\sqrt{-h} h^{\mu\nu} D_\nu \right) + \frac{\mathcal{R}}{4}$$

mais (voir appendice) ou $\mathcal{R}$ est le scalaire de Ricci. Ainsi,

$$\left(\frac{1}{\sqrt{-h}} D_\mu \left(\sqrt{-h} h^{\mu\nu} D_\nu \right) + \frac{\mathcal{R}}{4} + \frac{M^2 c^2}{h^2} \right) \psi = 0$$

Dirac equation on curved surface

We consider a two-dimensional surface (S) with parametric equations $\mathbf{r} = \mathbf{r}(q^1, q^2)$. In a three-dimensional Euclidean space, the very close neighborhood of this surface is described by the vector

$$\mathbf{R} = \mathbf{r}(q^1, q^2) + q^3 \mathbf{n}(q^1, q^2), \quad (0.567)$$

where $\mathbf{n}$ is the unit vector normal to (S) and q^3 is the coordinate indicating the distance from (S). The components of the assosiate metric tensor are defined by

$$G_{IJ} = \partial_I \mathbf{R} \cdot \partial_J \mathbf{R}, \quad I, J = 1, 2, 3 \quad (0.568)$$

where ∂_I is the abbreviation of $\partial_I = \partial/\partial q^I$. The metric tensor on the surface $g_{ij} = \partial_i \mathbf{r} \cdot \partial_j \mathbf{r}$, ($i, j = 1, 2$). The relation between the determinant G of the matrix G_{IJ} and the determinant g of the matrix g_{ij} is

$$\sqrt{G} = \sqrt{g} (1 + 2\mathcal{M}q_3 + \kappa q_3^2), \quad (0.569)$$

where $\mathcal{M}$ is the mean curvature and κ is the Gaussian curvature of the surface.

A point in space-time is characterized by the coordinates $(q^0 = ct, q^1, q^2, q^3)$. Thus, the space-time very close to the surface (S) is described by the metric

$$\begin{aligned} ds^2 &= h_{\mu\nu} dq^\mu dq^\nu \\ &= (dq^0)^2 - G_{ij}(q^1, q^2, q^3) dq^i dq^j - (dq^3)^2, \end{aligned} \quad (0.570)$$

that is

$$\begin{aligned} h_{00} &= 1, \quad h_{33} = -1 \\ h_{ij} &= -G_{ij}, \quad i, j = 1, 2 \end{aligned}$$

We now assume that a particle confined near the surface (S) by a confining potential $V_\lambda(q^3)$. As mentionned in the introduction, this potential is introduced in the Dirac equation via a minimal coupling as

$$D_\nu \rightarrow D_\nu + \frac{iq}{c} A_\nu, \quad (0.571)$$

so that

$$A_0 \equiv V_\lambda(q_3), \quad \mathbf{A} = \mathbf{0}. \quad (0.572)$$

Let's now show that the connections $\Gamma_0 = 0$. Indeed,

$$\Gamma_0 = \frac{1}{4} \omega_{ab0} \gamma^a \gamma^b, \quad (0.573)$$

with $a, b = 0, 1, 2, 3$ and

$$\omega_{ab0} = \eta_{ac} e_\beta^c \nabla_0 e_b^\beta \quad (0.574)$$

since the tetrads do not depend on time, we get

$$\omega_{ab0} = \eta_{ac} e_\beta^c \Gamma_{0\lambda}^\beta e_b^\lambda$$

and because

$$\begin{aligned} \Gamma_{0\lambda}^\beta &= \frac{h^{\ell\beta}}{2} \left(\frac{\partial h_{\lambda\ell}}{\partial q^0} + \frac{\partial h_{\ell 0}}{\partial q^\lambda} - \frac{\partial h_{0\lambda}}{\partial q^\ell} \right) \\ &= 0 \end{aligned}$$



we have $\Gamma_0 = 0$.

Then, the Dirac equation (0.566) is written as

$$\left(\frac{1}{\sqrt{-h}} \left(D_i \sqrt{-h} h^{ij} D_j \right) + \frac{\mathcal{R}}{4} + \partial_0^2 + \frac{M^2 c^2}{h^2} \right) \psi = 0$$

since $h = -G$.

$$\left[\frac{1}{\sqrt{G}} \left(\partial_i \sqrt{G} G^{ij} \partial_j \right) + 2h^{ij} \Gamma_i \partial_j + \frac{1}{\sqrt{G}} \partial_i \left(\sqrt{G} h^{ij} \Gamma_j \right) + h^{ij} \Gamma_i \Gamma_j + \frac{\mathcal{R}}{4} + \frac{M^2 c^2}{h^2} \right] \psi = 0$$

$$\left[-\frac{1}{\sqrt{G}} \left(\partial_i \sqrt{G} G^{ij} \partial_j \right) - 2G^{ij} \Gamma_i \partial_j - \frac{1}{\sqrt{G}} \partial_i \left(\sqrt{G} G^{ij} \Gamma_j \right) - G^{ij} \Gamma_i \Gamma_j + \frac{\mathcal{R}}{4} + \left(\partial_0 + \frac{iq}{c} V_\lambda \right)^2 + \frac{M^2 c^2}{h^2} \right] \psi = 0$$

$$\left[-\frac{1}{\sqrt{G}} \left(\partial_i \sqrt{G} G^{ij} \partial_j \right) - 2G^{ij} \Gamma_i \partial_j - \frac{1}{\sqrt{G}} \partial_i \left(\sqrt{G} G^{ij} \Gamma_j \right) - G^{ij} \Gamma_i \Gamma_j + \frac{\mathcal{R}}{4} + \left(\partial_0 + \frac{iq}{c} V_\lambda \right)^2 + \frac{M^2 c^2}{h^2} \right] \psi = 0$$

in the inside region

$$\left[-\frac{1}{\sqrt{G}} \left(\partial_i \sqrt{G} G^{ij} \partial_j \right) - 2G^{ij} \Gamma_i \partial_j - \frac{1}{\sqrt{G}} \partial_i \left(\sqrt{G} G^{ij} \Gamma_j \right) - G^{ij} \Gamma_i \Gamma_j + \frac{\mathcal{R}}{4} + \partial_0^2 + \frac{M^2 c^2}{h^2} \right] \psi = 0$$

$$\left[-\frac{h^2}{\sqrt{G}} \left(\partial_i \sqrt{G} G^{ij} \partial_j \right) - 2h^2 G^{ij} \Gamma_i \partial_j - \frac{h^2}{\sqrt{G}} \partial_i \left(\sqrt{G} G^{ij} \Gamma_j \right) - h^2 G^{ij} \Gamma_i \Gamma_j + \frac{h^2}{4} \mathcal{R} + M^2 c^2 - \frac{E^2}{c^2} \right] \psi = 0$$

$$\frac{E^2}{c^2} - M^2 c^2 \approx 2ME_{NR}$$

$$\left[-\frac{h^2}{2M\sqrt{G}} \partial_i \sqrt{G} G^{ij} \partial_j - \frac{h^2}{M} G^{ij} \Gamma_i \partial_j - \frac{h^2}{2M\sqrt{G}} \partial_i \left(\sqrt{G} G^{ij} \Gamma_j \right) - \frac{h^2}{2M} G^{ij} \Gamma_i \Gamma_j + \frac{h^2}{8M} \mathcal{R} \right] \psi = E_{NR} \psi$$

apply the confining process

$$\psi = f^{-\frac{1}{2}} \chi$$

$$\mathcal{R} = 2\kappa$$

where

$$-\frac{h^2}{2M\sqrt{G}} \left(\partial_i \sqrt{G} G^{ij} \partial_j \right) \rightarrow -\frac{h^2}{2M\sqrt{g}} \left(\partial_a \sqrt{g} g^{ab} \partial_b \right) + V_g$$

$$V_g = -\frac{h^2}{2M} (\mathcal{M}^2 - \kappa)$$

we say that the is no scalar geometric potential in non relativistic limit when the following condition is satisfied

$$V_g + \frac{h^2 \kappa}{4M} - \frac{h^2}{2M} \lim_{q^3 \rightarrow 0} G^{ij} \Gamma_i \Gamma_j = 0 \quad (0.575)$$

*

cylindrical surface

$$\Gamma_\varphi = -\frac{1}{2} \gamma^1 \gamma^3, \quad \Gamma_z = 0, \quad \Gamma_\tau = 0. \quad (0.576)$$

$$\kappa = 0, \quad \mathcal{M} = 1/2a$$

$$V_g = -\frac{h^2}{8Ma^2}$$

$$-\frac{h^2}{2M} \lim_{q^3 \rightarrow 0} G^{ij} \Gamma_i \Gamma_j = \frac{h^2}{8Ma^2}$$



$$\left[-\frac{\hbar^2}{2M} \left(\frac{1}{a^2} \partial_\varphi^2 + \partial_z^2 \right) - \frac{\hbar^2}{M} g^{\varphi\varphi} \Gamma_\varphi \partial_\varphi \right] \psi = E_t \psi$$

$$\left[i\hbar \tilde{\gamma}^I \mathcal{D}_I + i\hbar \tilde{\gamma}^0 \left(\partial_0 + \Gamma_0 + \frac{iq}{c} V_\lambda(q_3) \right) - Mc \right] \psi = 0. \quad (0.577)$$

Thus, the Dirac equation takes the following form

$$\left(i\hbar \tilde{\gamma}^I \mathcal{D}_I + i\hbar \tilde{\gamma}^0 \left(\partial_0 + \frac{iq}{c} V_\lambda(q_3) \right) - Mc \right) \psi = 0. \quad (0.578)$$

The metric $\gamma_{\mu\nu}$ is connected to the metric η_{ab} by means of tetrads as

$$\gamma_{\mu\nu} = e^\alpha_\mu e^b_\nu \eta_{ab}. \quad (0.579)$$

Thus, we can choose e^a_μ in matrix form as

$$e^a_\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & A & B & 0 \\ 0 & C & D & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (0.580)$$

where A , B , C and D are components to be determined. Thus, from (0.579) we obtain

$$G_{11} = A^2 + C^2 \quad (0.581)$$

$$G_{22} = B^2 + D^2 \quad (0.582)$$

$$G_{12} = G_{21} = AB + CD. \quad (0.583)$$

For diagonal metric $G_{12} = G_{21} = 0$, we simply choose

$$A = \sqrt{G_{11}}, \quad B = C = 0, \quad D = \sqrt{G_{22}}, \quad (0.584)$$

in this case, the Dirac matrices for curved space-time $\tilde{\gamma}^\mu = e_a^\mu \gamma^a$ are given in terms of the constant matrices γ^a as

$$\tilde{\gamma}^0 = \gamma^0, \quad \tilde{\gamma}^3 = \gamma^3$$

$$\tilde{\gamma}^1 = \frac{1}{\sqrt{G_{11}}} \gamma^1, \quad \tilde{\gamma}^2 = \frac{1}{\sqrt{G_{22}}} \gamma^2,$$

we adopte

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

Relativistic particle constrained on cylindrical surface

For a particle confined on the surface of a cylinder of radius a and parametric equations

$$\mathbf{r}(\varphi, z) = (a \cos \varphi, a \sin \varphi, z) \quad (0.585)$$

where φ is the polar angle, the metric on the surface is given by

$$g_{ij} = \begin{pmatrix} a^2 & 0 \\ 0 & 1 \end{pmatrix}. \quad (0.586)$$

We introduce the normal Gaussian coordinates (φ, z, τ) defined in the very close space to this surface whose position vector is

$$\mathbf{R}(\varphi, z, q_3) = \mathbf{r}(\varphi, z) + \tau \mathbf{n}, \quad \tau \ll a \quad (0.587)$$



where $\mathbf{n} = (\cos \varphi, \sin \varphi, 0)$ is a vector normal to this surface. Then, from (??) we obtain the metric G_{ab} as

$$G_{ab} = \begin{pmatrix} (a + \tau)^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (0.588)$$

Therefore, the space-time metric $\gamma_{\mu\nu}$ for a point in the neighboring space (ct, φ, z, τ) near the surface is

$$\gamma_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -(a + \tau)^2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (0.589)$$

according to () the vierbeins are

$$e^a_\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a + \tau & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (0.590)$$

The gamma matrix $\tilde{\gamma}^\mu$ are given by

$$\begin{aligned} \tilde{\gamma}^0 &= \gamma^0 \\ \tilde{\gamma}^\varphi &= \frac{1}{a + \tau} \gamma^1 \\ \tilde{\gamma}^z &= \gamma^2 \\ \tilde{\gamma}^3 &= \gamma^3. \end{aligned} \quad (0.591)$$

To write the Dirac equation for this system, we need to find the spin connections Γ_φ, Γ_z and Γ_τ . Indeed, by using (0.563) and (0.564) we obtain

$$\Gamma_\varphi = -\frac{1}{2} \gamma^1 \gamma^3, \quad \Gamma_z = 0, \quad \Gamma_\tau = 0. \quad (0.592)$$

Using the above requirements, we the Dirac equation (0.578) becomes

$$\left(i\hbar \gamma^0 \partial_0 + \frac{i\hbar}{a + \tau} \gamma^1 \partial_\varphi + i\hbar \gamma^2 \partial_z + i\hbar \gamma^3 \partial_\tau + \frac{i\hbar}{2(a + \tau)} \gamma^3 - \frac{q\hbar}{c} \gamma^0 V_\lambda(\tau) - Mc \right) \psi = 0, \quad (0.593)$$

At this stage, we propose a model of the confining potential $V_\lambda(\tau)$. We therefore assume that this particle is confined between two coaxial cylinders of radius $a \pm d/2$, that is ;

$$V_\lambda(\tau) = \begin{cases} 0 & \text{if } |\tau| < d/2 \\ \infty & \text{if } |\tau| \geq d/2. \end{cases} \quad (0.594)$$

We focus our attention on the interior region $|\tau| < d/2$. Thus, we apply to following operator

$$i\hbar \gamma^0 \partial_0 + \frac{i\hbar}{a + \tau} \gamma^1 \partial_\varphi + i\hbar \gamma^2 \partial_z + i\hbar \gamma^3 \partial_\tau + \frac{i\hbar}{2(a + \tau)} \gamma^3 + Mc \quad (0.595)$$

we obtain

$$-\hbar^2 \left(\nabla_{//}^2 + \nabla_\perp^2 \right) \psi + i\hbar^2 \sigma^2 \otimes I_2 \partial_\varphi + (\hbar^2 \partial_0^2 + M^2 c^2) \psi = 0, \quad (0.596)$$

where

$$\begin{aligned} \nabla_{//}^2 &= \frac{1}{(a + \tau)^2} \left(\partial_\varphi^2 - \frac{1}{4} \right) + \partial_z^2, \\ \nabla_\perp^2 &= \partial_\tau^2 + \frac{1}{(a + \tau)} \partial_\tau. \end{aligned}$$

As we hope to obtain a surface wave function which depends only on the coordinates φ and z , we are naturally led to introduce a new wave function for which we can write the separation



$\chi(\varphi, z, \tau) = \chi_{//}(\varphi, z) \chi_{\perp}(\tau)$. Then, we assume that the wave function ψ is normalizable such that the norm is given by

$$N = \int \sqrt{G} |\psi|^2 d\varphi dz d\tau = \int |\chi_{//}|^2 dS \int |\chi_{\perp}|^2 d\tau, \quad (0.597)$$

where $dS = \sqrt{g} d\varphi dz$. The proper transformation in light of (0.569) is

$$\psi = \frac{N}{\sqrt{a + \tau}} \chi_{//}(\varphi, z) \chi_{\perp}(\tau) e^{-\frac{i}{\hbar} E \tau},$$

thus, equation (0.596) gives

$$\left(-\frac{\hbar^2}{a^2} \partial_{\varphi}^2 + \frac{i\hbar^2}{a^2} \sigma^2 \otimes I_2 \partial_{\varphi} - \hbar^2 \partial_z^2 + \hbar^2 k_{\perp}^2 - \frac{E^2}{c^2} + M^2 c^2 \right) \chi_{//}(\varphi, z) = 0 \quad (0.598)$$

$$\partial_{\tau}^2 \chi_{\perp}(\tau) + k_{\perp}^2 \chi_{\perp}(\tau) = 0 \quad (0.599)$$

the limit $\tau \rightarrow 0$ is performed after calculating all derivatives with respect to τ . Rotation $\frac{\pi}{2}$

$$\chi_{//}^{rot} = e^{\frac{i}{\hbar} S_x(\pi/2)} \chi_{//}(\varphi, z)$$

we get

$$\exp\left(\frac{i}{\hbar} S_x(\pi/2)\right) S_y \exp\left(-\frac{i}{\hbar} S_x(\pi/2)\right) = S_z$$

$$\exp\left(-i\sigma^1 \frac{\pi}{4}\right) \sigma^2 \exp\left(i\sigma^1 \frac{\pi}{4}\right) = \sigma^3$$

$$\left(-\frac{\hbar^2}{a^2} \partial_{\varphi}^2 + \frac{2i\hbar}{a^2} S_z \otimes I_2 \partial_{\varphi} - \hbar^2 \partial_z^2 + \hbar^2 k_{\perp}^2 - \frac{E^2}{c^2} + M^2 c^2 \right) \chi_{//}^{rot}(\varphi, z) = 0 \quad (0.600)$$

$$\chi_{//}^{rot} = e^{ik_z z} e^{in\varphi} (\chi_s \otimes I_2), \quad n \in \mathbb{Z}, \quad s = \pm 1$$

$$E^2 - M^2 c^4 = \frac{\hbar^2 c^2}{a^2} (n^2 - 2ns) + \hbar^2 c^2 k_z^2 + \hbar^2 c^2 k_{\perp}^2 \quad (0.601)$$

$$E^2 - M^2 c^4 \simeq 2Mc^2 \mathcal{E} \quad (0.602)$$

$$\mathcal{E}_{n,s} \simeq \frac{\hbar^2 n^2}{2Ma^2} - \frac{\hbar^2 ns}{Ma^2} + \frac{\hbar^2 k_z^2}{2M} + \frac{\hbar^2 k_{\perp}^2}{2M} \quad (0.603)$$

Relativistic particle constrained on spherical surface

Our aim in this section* is to apply the thin layer quantization approach for particle confined on spherical surface, we consider the infinitesimal distance ds^2 for (1+3) dimensional Minkowski space-time is given by

$$ds^2 = (dq^0)^2 - G_{ij}(q, q^3) dq^i dq^j - (dq^3)^2. \quad (0.604)$$

Where $G_{00} = 1$, and the latin indices $(i, j = 1, 2)$, we obtain the metric on the sphere as

$$\mathbf{g} = \begin{pmatrix} r_0^2 & 0 \\ 0 & r_0^2 \sin^2 \theta \end{pmatrix}, \quad (0.605)$$

and the surface potential defined by

$$V_s = -\frac{\hbar^2}{2m} (H^2 - k) = 0. \quad (0.606)$$



Now, with $dq^0 = cdt$. We can write the components space-time metric γ by the matrix forme

$$\gamma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -(r_0 + \tau)^2 & 0 & 0 \\ 0 & 0 & -\sin^2 \theta (r_0 + \tau)^2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (0.607)$$

the element τ represent the coordinat of the neighborhood space with $r_0 - d/2 \leq \tau \leq r_0 + d/2$, the determinant of G given by

$$\sqrt{G} = r_0^2 \sin \theta \left[1 + \frac{2}{r_0} \tau + \frac{1}{r_0^2} \tau^2 \right] \quad (0.608)$$

we can write the space-time metric for a point in the neighborhood space $(ct, \theta, \varphi, \tau)$ very close to the space in this case as

$$ds^2 = c^2 dt^2 - (r_0 + \tau)^2 d\theta^2 - \sin^2 \theta (r_0 + \tau)^2 d\varphi^2 - d\tau^2 \quad (0.609)$$

where the derminant of the metric space is

$$\gamma = \begin{pmatrix} \gamma_{00} = 1 & 0 & 0 & 0 \\ 0 & \gamma_{\theta\theta} = -(r_0 + \tau)^2 & 0 & 0 \\ 0 & 0 & \gamma_{\varphi\varphi} = -\sin^2 \theta (r_0 + \tau)^2 & 0 \\ 0 & 0 & 0 & \gamma_{\tau\tau} = -1 \end{pmatrix}$$

$$\det \gamma = \gamma = -G \quad (0.610)$$

By using the vierbeins are represent by

$$e_a^\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & r_0 + \tau & 0 & 0 \\ 0 & 0 & (r_0 + \tau) \sin \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow e_a^\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{r_0 + \tau} & 0 & 0 \\ 0 & 0 & \frac{1}{(r_0 + \tau) \sin \theta} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (0.611)$$

the gamma matrices $\tilde{\gamma}^\mu$ are

$$\begin{aligned} \tilde{\gamma}^0 &= \gamma^0 \\ \tilde{\gamma}^\theta &= \frac{1}{r_0 + \tau} \gamma^1 \\ \tilde{\gamma}^\varphi &= \frac{1}{(r_0 + \tau) \sin \theta} \gamma^2 \\ \tilde{\gamma}^\tau &= \gamma^3 \end{aligned} \quad (0.612)$$

Thus, the Dirac equation is written as

$$\left(i\hbar \gamma^0 \gamma^3 D_\tau + \frac{i\hbar}{(r_0 + \tau)} \gamma^0 \gamma^1 D_\theta + \frac{i\hbar}{(r_0 + \tau) \sin \theta} \gamma^0 \gamma^2 D_\varphi - \gamma^0 M c \right) \psi = \left(-i\hbar \partial_0 + \frac{q\hbar}{c} V(\tau) \right) \psi \quad (0.613)$$

Now, we will get to calculate the affine-connections $(\Gamma_\tau, \Gamma_\theta, \Gamma_\varphi)$. From (??), we have

$$e_0^{ct} = 1, \quad e_1^\theta = \frac{1}{(r_0 + \tau)}, \quad e_2^\varphi = \frac{1}{(r_0 + \tau) \sin \theta} \text{ and } e_3^\tau = 1$$

by using the following definition with all compnents

$$\begin{aligned} \Gamma_\tau &= 0 \\ \Gamma_\theta &= -\frac{1}{2} \gamma^1 \gamma^3 \\ \Gamma_\varphi &= \frac{1}{2} \cos \theta \gamma^1 \gamma^2 - \frac{1}{2} \sin \theta \gamma^2 \gamma^3 \end{aligned} \quad (0.614)$$



Using the standard representation of constant Dirac matrices, which represents by

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \quad (0.615)$$

Dirac equation becomes

$$\left(i\hbar\gamma^0\partial_0 + i\hbar\frac{\gamma^1}{(r_0 + \tau)} \left(\partial_\theta + \frac{\cot\theta}{2} \right) + \frac{i\hbar}{(r_0 + \tau)\sin\theta} \gamma^2\partial_\varphi + i\hbar\gamma^3 \left(\partial_\tau + \frac{1}{r_0 + \tau} \right) - \gamma^0 \frac{q\hbar}{c} V(\tau) - Mc \right) \psi = 0 \quad (0.616)$$

In order to continue our analysis, we can propose at this stage a model of the potential $V(\tau)$ as in the previous section*.

$$V(\tau) = \begin{cases} 0 & \text{si } |\tau| < d/2 \\ \infty & \text{si } |\tau| \geq d/2. \end{cases} \quad (0.617)$$

we apply the operator

$$i\hbar\gamma^0\partial_0 + i\hbar\frac{\gamma^1}{(r_0 + \tau)} \left(\partial_\theta + \frac{\cot\theta}{2} \right) + \frac{i\hbar}{(r_0 + \tau)\sin\theta} \gamma^2\partial_\varphi + i\hbar\gamma^3 \left(\partial_\tau + \frac{1}{r_0 + \tau} \right) + Mc = 0 \quad (0.618)$$

we obtain

$$\begin{aligned} & -\frac{\hbar^2}{(r_0 + \tau)^2} \nabla_{//}^2 \psi - \hbar^2 \nabla_\perp^2 \psi + \frac{\hbar^2}{(r_0 + \tau)^2} (\Omega \otimes I_2) \psi + (\hbar^2 \partial_0^2 + M^2 c^2) \psi = 0, \quad (0.619) \\ & \left[-\frac{\hbar^2}{2M\sqrt{G}} \partial_i \sqrt{G} G^{ij} \partial_j - \frac{\hbar^2}{M} G^{ij} \Gamma_i \partial_j - \frac{\hbar^2}{2M\sqrt{G}} \partial_i (\sqrt{G} G^{ij} \Gamma_j) - \frac{\hbar^2}{2M} G^{ij} \Gamma_i \Gamma_j + \frac{\hbar^2}{8M} \mathcal{R} \right] \psi = E_{NR} \psi \\ & \frac{1}{(r_0 + \tau)^2} \left(P_\theta^2 + \frac{L_z^2}{\sin^2 \theta} \right) \psi - \hbar^2 \left(\partial_\tau^2 + \frac{2}{r_0 + \tau} \partial_\tau \right) \psi - \frac{2}{(r_0 + \tau)^2} (\Omega \otimes I_2) \psi + (\hbar^2 \partial_0^2 + M^2 c^2) \psi = 0, \end{aligned} \quad (0.620)$$

where

$$P_\theta = \frac{\hbar}{i} \left(\partial_\theta + \frac{\cot\theta}{2} \right), \quad L_z = \frac{\hbar}{i} \partial_\varphi$$

and

$$\Omega = \left(\frac{\cot\theta}{\sin\theta} S_z - \frac{S_x}{\sin\theta} \right) L_z + S_y P_\theta$$

we put

$$\psi = \frac{N}{r_0 + \tau} \chi_{//}(\theta, \varphi) \chi_\perp(\tau) e^{-\frac{i}{\hbar} E t}, \quad (0.621)$$

and $\tau \rightarrow 0$

$$\left(\frac{1}{r_0^2} \left(P_\theta^2 + \frac{L_z^2}{\sin^2 \theta} \right) - \frac{2}{r_0^2} (\Omega \otimes I_2) + \hbar^2 k_\perp^2 - \frac{E^2}{c^2} + M^2 c^2 \right) \chi_{//}(\theta, \varphi) = 0 \quad (0.622)$$

$$\partial_\tau^2 \chi_\perp(\tau) + k_\perp^2 \chi_\perp(\tau) = 0 \quad (0.623)$$

$$\chi_{//} = e^{im\varphi} \Lambda_{(m)}(\theta), \quad m \in \mathbb{Z},$$

we have

$$\left(\frac{1}{r_0^2} \left(P_\theta^2 + \frac{\hbar^2 m^2}{\sin^2 \theta} \right) - \frac{2}{r_0^2} \left[\hbar \left(\frac{\cot\theta}{\sin\theta} S_z - \frac{S_x}{\sin\theta} \right) m + S_y P_\theta \right] \otimes I_2 + \hbar^2 k_\perp^2 - \frac{E^2}{c^2} + M^2 c^2 \right) \Lambda_{(m)}(\theta) = 0 \quad (0.624)$$



for $m = 0$ wave

$$\left(\frac{P_\theta^2}{r_0^2} - \frac{2}{r_0^2} S_y P_\theta \otimes I_2 + \hbar^2 k_\perp^2 - \frac{E^2}{c^2} + M^2 c^2 \right) \Lambda_{(0)}(\theta) = 0 \quad (0.625)$$

$$\Lambda_{(0)}^{rot}(\theta) = \mathcal{R}_x(\pi/2) \Lambda_{(0)}(\theta)$$

$$\exp\left(\frac{i}{\hbar} S_x(\pi/2)\right) S_y \exp\left(-\frac{i}{\hbar} S_x(\pi/2)\right) = S_z$$

$$\left(\frac{P_\theta^2}{r_0^2} - \frac{2}{r_0^2} S_z P_\theta \otimes I_2 + \hbar^2 k_\perp^2 - \frac{E^2}{c^2} + M^2 c^2 \right) \Lambda_{(0)}^{rot}(\theta) = 0 \quad (0.626)$$

$$P_\theta v(\theta) = p_\theta v(\theta)$$

$$\frac{dv(\theta)}{v(\theta)} = \frac{i}{\hbar} p_\theta d\theta - \frac{\cot \theta}{2} d\theta$$

$$v(\theta) = \frac{N}{\sqrt{|\sin \theta|}} \exp\left(\frac{i}{\hbar} p_\theta \theta\right)$$

single valued function

$$v(\theta + \pi) = v(\theta)$$

$$\frac{i}{\hbar} p_\theta \pi = 2\pi i n$$

$$p_\theta = 2\hbar n, \quad n \in \mathbb{Z},$$

$$\Lambda_{(0)}^{rot}(\theta) = v(\theta) (\chi_s \otimes I_2)$$

$$E^2 - M^2 c^2 = \frac{4c^2 \hbar^2}{r_0^2} (n^2 - sn) + c^2 \hbar^2 k_\perp^2, \quad n \in \mathbb{Z}, \quad (0.627)$$

Conclusion

$$\exp\left(-\frac{i}{\hbar} \frac{\hbar \sigma^1}{2} \frac{\pi}{2}\right) \sigma^2 \exp\left(i \sigma^1 \frac{\pi}{4}\right) = \sigma^3$$

$$\left(\partial_\tau + \frac{1}{r_0 + \tau}\right)^2 = \partial_\tau^2 + \frac{2}{r_0 + \tau} \partial_\tau$$

$$\left(\partial_\theta + \frac{\cot \theta}{2}\right)^2 = \partial_\theta^2 + \cot \theta \partial_\theta - \frac{1}{4 \sin^2 \theta} - \frac{1}{4}$$

$$\Gamma_\varphi = \frac{1}{4} \omega_{ab\varphi} \gamma^a \gamma^b, \quad (0.628)$$

with $a, b = 0, 1, 2, 3$ and

$$\omega_{ab\varphi} = \eta_{ac} e_\beta^c \nabla_\varphi e_b^\beta \quad (0.629)$$

we get

$$\omega_{ab\varphi} = \eta_{ac} e_\beta^c \Gamma_{\varphi\lambda}^\beta e_b^\lambda$$

and because

$$\Gamma_{ij}^k = \frac{\gamma^{k\ell}}{2} \left(\frac{\partial \gamma_{\ell i}}{\partial q^j} + \frac{\partial \gamma_{\ell j}}{\partial q^i} - \frac{\partial \gamma_{ij}}{\partial q^\ell} \right)$$



$$\begin{aligned}\Gamma_{\varphi\lambda}^{\beta} &= \frac{\gamma^{\beta\ell}}{2} \left(\frac{\partial\gamma_{\ell\varphi}}{\partial q^{\lambda}} + \frac{\partial\gamma_{\ell\lambda}}{\partial\varphi} - \frac{\partial\gamma_{\varphi\lambda}}{\partial q^{\ell}} \right) \\ &= \frac{\gamma^{\beta\varphi}}{2} \frac{\partial\gamma_{\varphi\varphi}}{\partial q^{\lambda}} - \frac{\gamma^{\beta\tau}}{2} \frac{\partial\gamma_{\varphi\lambda}}{\partial\tau}\end{aligned}$$

$$\Gamma_{\varphi\tau}^{\varphi} = \frac{\gamma^{\varphi\varphi}}{2} \frac{\partial\gamma_{\varphi\varphi}}{\partial\tau} = \frac{1}{(a+\tau)}$$

$$\Gamma_{\varphi\varphi}^{\tau} = \frac{1}{2} \frac{\partial\gamma_{\varphi\varphi}}{\partial\tau} = -(a+\tau)$$

$$\omega_{ab\varphi} = \eta_{ac} e_{\beta}^c \Gamma_{\varphi\tau}^{\varphi} e_b{}^{\tau} + \eta_{ac} e_{\beta}^c \Gamma_{\varphi\varphi}^{\tau} e_b{}^{\varphi}$$

$$\omega_{13\varphi} = -1$$

$$\omega_{31\varphi} = 1$$

$$e^a{}_{\mu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a+\tau & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Gamma_{\varphi} = \frac{1}{4} \omega_{ab\varphi} \gamma^a \gamma^b = -\frac{1}{2} \gamma^1 \gamma^3$$

$$\gamma^1 \gamma^3 = i\sigma^y \otimes I_2$$

$$\gamma^1 \gamma^2 = -i\sigma^z \otimes I_2$$

$$\gamma^2 \gamma^3 = -i\sigma^x \otimes I_2$$

$$\frac{1}{\sqrt{-h}} \left(D_i \sqrt{-h} h^{ij} D_j \right) = \frac{1}{\sqrt{G}} \left(D_i \sqrt{G} h^{ij} D_j \right)$$

ACKNOWLEDGMENTS

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Analysis of the modified SIR model for infectious diseases with fractional derivatives

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Abstract. This study presents a mathematical model known as the fractional SIR model, which represents the dynamics of transmission and spread of infectious diseases. Additionally, the study addresses the existence unique solutions and the stability of the proposed mathematical model, along with analyzing various stability outcomes related to specific conditions to prevent the spread of infectious diseases.

Keywords: SIR model, fractional derivative, system, existence, stability analysis.

Introduction

The infectious diseases have been a persistent challenge for thousands of years, with scientists and researches continuously striving to understand and combat them. These diseases vary in their causes, transmission, and impacts, encompassing a wide range of conditions, from



common illnesses to global pandemics. Studying these diseases and analyzing their effects is of utmost importance for understanding ways to address and prevent their spread.

This model requires the division of the total population N into three epidemiological classes namely Susceptible S , Infected I , and Recovered class R .

Parameters' definition of the SIR model could be presented as follows:

- Λ : indicates the rate of increase in the susceptible individuals.
- $\mu \leq \Lambda$: is the rate of natural death population.
- v represents the death rate due to the disease.
- γ is the vaccine rate of the suspected population.
- β is the rate of contacts between susceptible S and infectious population I .
- δ is the recovery rate from infectious populations I .
- κ is the rate of transfer of recovered population R even those who were vaccinated to the susceptible population S .

In recent decades, studying fractional calculus has been regarded as essential as it provides an effective alternative to integer-order models, and can describe and process various structures' properties.

For $0 \leq t \leq T < \infty$, and $0 < \alpha \leq 1$, we have:

$$\begin{cases} {}^C\mathcal{D}_{0+}^{\alpha,p} S(t) = \Lambda N(t) + \kappa R(t) - \left(\frac{\beta}{N(t)} I(t) + \gamma + \mu \right) S(t) \\ {}^C\mathcal{D}_{0+}^{\alpha,p} I(t) = \frac{\beta}{N(t)} S(t) I(t) - (\delta + v + \mu) I(t) \\ {}^C\mathcal{D}_{0+}^{\alpha,p} R(t) = \gamma S(t) + \delta I(t) - (\kappa + \mu) R(t) \end{cases} \quad (0.630)$$

The alterations in infectious disease transmission within the SIR model (0.630) can be interpreted through reference to the following chart:

Our primary objective is to efficiently find the solution to the system of nonlinear fractional differential equations (0.630). We aim to determine the solution's main properties, including stability, existence, boundedness, and computation of equilibrium points. Using the fixed point theory, we establish the necessary conditions for obtaining at least one solution or its uniqueness.

Main results

The SIR model(0.630) will be investigated within a biologically feasible region in $\mathbb{R}^3$, as defined in the subsequent lemma.

Lemma 0.75. *Let N_0 be the initial total population at $t = 0$, where $0 \leq t \leq T < \infty$, then the solution of the model under consideration is restricted to the feasible region given by*

$$(S, I, R) \in \mathbb{R}_+^3, \quad N(t) \leq N_0 \exp\left(\frac{(\Lambda - \mu) T^\alpha}{\Gamma(\alpha + 1)}\right).$$

This shows the boundedness of the considered population dynamic model.

In the following, we assume the existence of a constant $\mathcal{N} > 0$

$$\mathcal{N} \leq N_0 \exp\left(\frac{(\Lambda - \mu) T^\alpha}{\Gamma(\alpha + 1)}\right),$$

To normalize the model (0.630), we put:

$$S(t) = \frac{S(t)}{\mathcal{N}}, \quad \mathcal{I}(t) = \frac{I(t)}{\mathcal{N}}, \quad \mathcal{R}(t) = \frac{R(t)}{\mathcal{N}}, \quad (0.631)$$

then we obtain

$$\begin{cases} {}^C\mathcal{D}_{0+}^{\alpha} S(t) &= \Lambda + \kappa \mathcal{R}(t) - (\beta \mathcal{I}(t) + \gamma + \mu) S(t), \\ {}^C\mathcal{D}_{0+}^{\alpha} \mathcal{I}(t) &= \beta \mathcal{I}(t) S(t) - (\delta + v + \mu) \mathcal{I}(t), \\ {}^C\mathcal{D}_{0+}^{\alpha} \mathcal{R}(t) &= \gamma S(t) + \delta \mathcal{I}(t) - (\kappa + \mu) \mathcal{R}(t), \end{cases} \quad (0.632)$$

along with the positive initial conditions

$$S(0) = u_1, \quad \mathcal{I}(0) = u_2, \quad \mathcal{R}(0) = u_3. \quad (0.633)$$



SIR.pdf

Figure 37: Flowchart of SIR.

Theorem 0.213. Let $\beta, \delta, \kappa, \gamma, \tau, v, \mu, \alpha, T \in \mathbb{R}_+$, be such that $\alpha \in (0, 1]$ and

$$T < \left(\frac{\Gamma(\alpha + 1)}{\mu + \max\{\beta + \gamma, \beta + \delta + v, \kappa\}} \right)^{\frac{1}{\alpha}}, \quad (0.634)$$

then the problem (0.632)–(0.633) has at least one solution on $[0, T]$.

Theorem 0.214. We give $\alpha \in (0, 1]$, $T > 0$, and

$$\eta = \max\{\beta + \gamma, \beta + \delta + v, \kappa\},$$

for some $\beta, \delta, \kappa, \gamma, v, \mu \in \mathbb{R}_+$. If

$$\frac{(\mu + \eta) T^\alpha}{\Gamma(\alpha + 1)} < 1, \quad (0.635)$$

thus, there is a unique solution to the problem (0.632)–(0.633) on $[0, T]$.

Theorem 0.215. Assuming that (0.635) holds, system (0.632) is Ulam-Hyers stable. Furthermore, it can also be asserted that (0.632) is a generalized Ulam-Hyers stable system.



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**Étude des équations différentielles abstraites complète de
type elliptique provenant de modèles physiques concrets,
sur des domaines cylindriques hétérogènes dans un cadre
holdérien: application au potentiel électrostatique.**

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Abstract. Nous analysons une famille de problèmes de transmission elliptique sur des domaines hétérogènes dans les espaces de Hölder, en présentant des résultats d'existence, d'unicité et de la régularité maximale des solutions strictes à cette famille de problèmes abstraits. Nous utilisons la théorie des semi-groupes généralisés, les résultats de SINISTRARI et la théorie des espaces d'interpolation. Par ailleurs, nous présentons des exemples illustrant l'application de notre théorie abstrait.

Keywords: équations différentielles abstraits, potentiel électrostatique, semi-groupe analytique, solution stricte, régularité maximale.



Introduction

Nous proposons d'étudier, dans le cadre holdérien, une classe d'équations différentielles abstraites de second ordre de la forme

$$u''(x) + 2Bu'(x) + Au(x) = g(x)$$

posé sur un domaine hétérogène $[a, \delta]$, où $a \in [-\infty, 0[$ peut être non infini. Ici A et B sont des opérateurs linéaires fermés à valeurs dans un espace de Banach E , définis sur des domaines non nécessairement denses.

Dans cette analyse, le domaine hétérogène se compose de deux segments bien définis $[a, 0[$ et $]0, \delta]$. Chaque segment est caractérisé par des coefficients différents, μ_- pour la première partie et μ_+ pour la seconde, cela induit une discontinuité de

$$u = \begin{cases} u_+ & \text{sur } [0, \delta] \\ u_- & \text{sur } [a, 0] \end{cases} \quad \text{et } g = \begin{cases} g_+ & \text{sur } [0, \delta] \\ g_- & \text{sur } [a, 0] \end{cases}$$

où g est supposé hœldérien par morceaux i.e.,

$$g_- \in C^{2\alpha}([a, 0], E), \quad \text{et } g_+ \in C^{2\alpha}([0, \delta], E), \quad \text{où } 0 < 2\alpha < 1.$$

Il s'agit alors à l'étude d'un problème soumis à des conditions de transmission sur l'interface $\{0\}$ entre les deux segments,

$$u(0_+) = u(0_-), \quad \mu_- u'(0_-) = \mu_+ u'(0_+)$$

Sur les bords on impose les conditions de type Robin :

$$\mu u'(\delta) = f_+, \quad (u^\delta)'(a) = f_-$$

où f_+ et f_- sont des éléments donnés dans E .

Ce travail se concentre sur un cas particulier où B est un opérateur nul et $a = -1$, avec pour objectif d'étudier l'existence, l'unicité et la régularité maximale de la solution stricte de ce problème abstrait

$$(p) \begin{cases} (e.g) \begin{cases} (u_-)''(x) + Au_-(x) = g_-(x), & x \in [-1, 0] \\ (u_+)''(x) + Au_+(x) = g_+(x), & x \in [0, \delta] \end{cases} \\ (c.l) \begin{cases} \mu (u_+)'(\delta) = f_+ \\ (u_-)'(-1) = f_- \end{cases} \\ (c.t) \begin{cases} u_+(0) = u_-(0) \\ \mu_- u_-(0) = \mu_+ u_+(0) \end{cases} \end{cases}$$

sous l'unique hypothèse d'ellipticité suivante

$$\begin{cases} \rho(A) \supset [0, +\infty[\quad \text{et } \exists C > 0 \text{ tel que} \\ \forall \lambda \in [0, +\infty[: \|(A - \lambda I)^{-1}\| \leq \frac{C}{1 + \lambda}, \end{cases} \quad (H.1)$$

où $\rho(A)$ est l'ensemble résolvant de A .

C'est-à-dire cherchons une fonction $u = (u_-, u_+)$ vérifie

$$\begin{cases} u_+ \in C^2([0, \delta], E) \cap C([0, \delta], D(A)) \\ u_- \in C^2([-1, 0], E) \cap C([-1, 0], D(A)) \end{cases}$$

et

$$Au_+, (u_+)'' \in C^{2\alpha}([0, \delta], E) \quad \text{et} \quad Au_-, (u_-)'' \in C^{2\alpha}([-1, 0], E).$$

Ce type des problèmes issus de certains modèles concrets de la physique posés sur des domaines cylindriques. On peut considérer, par exemple, le problème du potentiel électrostatique, régi par l'équation suivante

$$\frac{1}{\mu} \nabla \cdot (\mu \nabla u^\delta) = g^\delta$$

posé dans un cylindre hétérogène $\Omega^\delta =]-1, \delta[\times G$ constitué par la jonction de $\Omega =]-1, 0[\times G$ et la couche $\Omega_+^\delta =]0, \delta[\times G$ tel que G est une partie de $\mathbb{R}^n$, $n \in \mathbb{N}$.



Ici, la fonction u^δ est le potentiel électrostatique, μ est le coefficient de conductivité, la fonction g^δ est une densité électrique telles que

$$\begin{cases} g^\delta / [-1, 0] := g_- \in C^{2\alpha}([-1, 0] \times G), \\ g^\delta / [0, \delta] := g_+^\delta \in C^{2\alpha}([0, \delta] \times G). \end{cases}$$

On impose sur les bords $]-1, \delta[\times \partial G$, $\{\delta\} \times G$ et $\{-1\} \times G$ les conditions aux limites suivantes

$$(cl) \begin{cases} u^\delta = 0 & \text{sur }]-1, \delta[\times \partial G \\ \mu \frac{\partial}{\partial v} u^\delta = f_+^\delta & \text{sur } \{\delta\} \times G \\ \frac{\partial}{\partial v} u^\delta = f_- & \text{sur } \{-1\} \times G \end{cases}$$

où v est le vecteur normal, f_+^δ et f_- sont des potentiels des deux surfaces $]-1, \delta[\times \partial G$, $\{\delta\} \times G$ respectivement.

Sur l'interface $\{0\} \times G$, on impose les conditions de transmission

$$(ct) \begin{cases} u^\delta(0_+, y) = u^\delta(0_-, y) \\ \mu_- \frac{\partial}{\partial v} u^\delta(0_-, y) = \mu_+ \frac{\partial}{\partial v} u^\delta(0_+, y). \end{cases}$$

En utilisant les propriétés des opérateurs différentielles, on trouve que

$$\frac{1}{\mu} \nabla (\mu \nabla u^\delta)(x, y) = \Delta u^\delta(x, y) = \frac{\partial^2}{\partial x^2} u^\delta(x, y) + \Delta_y u^\delta(x, y)$$

Donc le problème posé devient

$$(P_c) \begin{cases} (eq) \begin{cases} \Delta u_-(x, y) = g_-(x, y), & (x, y) \in [-1, \delta] \times G \\ \Delta u_+^\delta(x, y) = g_+^\delta(x, y), & (x, y) \in [0, \delta] \times G \\ u_+^\delta(x, y) = u_-(x, y) = 0 & \text{sur }]-1, \delta[\times \partial G \end{cases} \\ (cl) \begin{cases} \mu_+ \frac{\partial}{\partial x} u_+^\delta(\delta, y) = f_+^\delta(y) & \text{si } y \in G \\ \frac{\partial}{\partial x} u^\delta(-1, y) = f_-(y) & \text{si } y \in G \end{cases} \\ (ct) \begin{cases} u_+^\delta(0, y) = u_-(0, y) \\ \mu_- \frac{\partial}{\partial x} u_-(0, y) = \mu_+ \frac{\partial}{\partial x} u_+^\delta(0, y) & \text{si } y \in G \end{cases} \end{cases}$$

Formulation abstraite du problème:

Puisque le domaine $\Omega^\delta =]-1, \delta[\times G$ est cylindrique, donc on peut utiliser les notations vectorielles usuelles

$$u^\delta(x)(y) := u^\delta(x, y), \quad g^\delta(x)(y) := g^\delta(x, y)$$

pour cacher la variable y . Le problème précédent peut être formulé, dans l'espace de Banach $E = C(G)$, par des équations différentielles abstraites. En introduisant l'opérateur A défini par

$$\begin{cases} D(A) = \{\phi \in E, \Delta_y \phi \in E : \phi / \partial G = 0\} \\ (A\phi)(y) = \Delta_y \phi(y) \end{cases}$$

l'opérateur A , est l'action par rapport à y . Ainsi le problème (P_c) s'écrit sous la forme abstraite (P) . Par exemple, en dimension 2, (c'est-à-dire $n = 1$) et pour $G = [0, 1] \subset \mathbb{R}$ (est une partie bornée), donc le domaine $\Omega^\delta =]-1, \delta[\times G$ est le rectangle hétérogène suivant. Ainsi l'opérateur A est défini par

$$\begin{cases} D(A) = \{\phi \in C^2([0, 1]) : \phi(0) = \phi(1) = 0\} \\ \quad \quad \quad := C_0([0, 1]) \cap C^2([0, 1]) \\ (A\phi)(y) = \phi''(y) \end{cases}$$

et lorsque $G = \mathbb{R}$, est une partie non bornée, donc le domaine $\Omega^\delta =]-1, \delta[\times G$ est la bande hétérogène suivante ici, l'opérateur A est

$$\begin{cases} D(A) = C^2(\mathbb{R}). \\ (A\phi)(y) = \phi''(y) \end{cases}$$



Capture 1.PNG

Capture n2.PNG

En dimension 3 ($n = 2$) lorsque $G = D(0, 1) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$, donc le domaine $\Omega^\delta =]-1, \delta[\times G$ est le cylindre hétérogène suivant ensuite l'opérateur A est

$$\begin{cases} D(A) = \{\phi \in E, \Delta_y \phi \in E : \phi|_{C(0,1)} = 0\} \\ (A\phi)(y_1, y_2) = \frac{\partial^2}{\partial y_1^2} \phi(y_1, y_2) + \frac{\partial^2}{\partial y_2^2} \phi(y_1, y_2) \end{cases}$$

tel que $\partial G = C(0, 1)$.

Main results

Les résultats obtenus pour le **problème abstrait** sont donnés par les deux théorèmes suivants

Theorem 0.216 (Solution Stricte). *On suppose (H.1). Soient*

$$g_+ \in C^{2\alpha}([0, \delta]; E), \quad g_- \in C^{2\alpha}([-1, 0]; E)$$

et $f_-, f_+ \in D((-A)^{1/2})$. Alors le problème posé admet une unique solution stricte si et seulement si

$$g_+(0) - g_-(0), \quad (-A)^{1/2} f_-, \quad (-A)^{1/2} f_+ \in \overline{D(A)}.$$



$$\begin{pmatrix} L_{p,m}(n+1) & L_{p,m}(n) & \dots & L_{p,m}(n-p+2) & L_{p,m}(n-p+1) \\ L_{p,m}(n-p+1) & L_{p,m}(n-p) & \dots & L_{p,m}(n-2p+2) & L_{p,m}(n-2p+1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ L_{p,m}(n-1) & L_{p,m}(n-2) & \dots & L_{p,m}(n-p) & L_{p,m}(n-p-1) \\ L_{p,m}(n) & L_{p,m}(n-1) & \dots & L_{p,m}(n-p+1) & L_{p,m}(n-p) \end{pmatrix}$$

Theorem 0.217 (Régularité maximale). *On suppose (H.1). Soient*

$$g_+ \in C^{2\alpha}([0, \delta]; E), \quad g_- \in C^{2\alpha}([-1, 0]; E)$$

avec $0 < 2\alpha < 1$ et $f_-, f_+ \in D((-A)^{1/2})$. Alors

$$(u_+)'' , Au_+ \in C^{2\alpha}([0, \delta]; E), \quad u_-'' , Au_- \in C^{2\alpha}([-1, 0]; E)$$

si et seulement si

$$g_+(0) - g_-(0), \quad (-A)^{1/2} f_-, \quad (-A)^{1/2} f_+ \in D_A(\alpha, +\infty).$$

où

$$D_A(\theta, \infty) = \left\{ x \in E : \sup_{t>0} \|t^{1-\theta} A e^{At} x\| < \infty \right\}.$$

Il est difficile de déterminer le domaine de de l'opérateur $(-A)^{1/2}$ pour cela on donne le résultat pour le **problème de potentiel électrostatique** dans le cas où $f_- = f_+ = 0$, et le résultat obtenu est le suivant

Theorem 0.218. *Soient $(g_+, g_-) \in C^{2\alpha}([0, \delta]; C(G)) \times C^{2\alpha}([-1, 0]; C(G))$ avec $0 < 2\alpha < 1$. Alors le problème (P_c) admet une unique solution stricte si et seulement si*

$$g_+(0) - g_-(0) \in C_0([0, 1]).$$

En plus,

$$\begin{cases} \frac{\partial^2}{\partial x^2} u_+, \Delta u_+ \in C^{2\alpha}([0, \delta]; C(G)), \\ \frac{\partial^2}{\partial x^2} u_-, \Delta u_- \in C^{2\alpha}([-1, 0]; C(G)) \end{cases}$$

si et seulement si

$$g_+(0) - g_-(0) \in C_0^{2\alpha}(G).$$





Les techniques employées ici reposent sur une représentation explicite de la solution à l'aide de la racine carrée de l'opérateur positif $-A$ et les semi groupes analytiques généralisé et les résultats de Sinestrari. En donnant des conditions nécessaires et suffisantes sur les données f_- , f_+ et $g_+(0) - g_-(0)$ pour établir l'existence, l'unicité et la régularité maximale de la solution stricte.

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Null controllability of linear heat equation with mixed boundary conditions

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Abstract. In this work, we study the null controllability of linear heat equation with mixed boundary conditions. Where to achieve this result we use one of the applications of Carleman estimates to controllability problems.

Keywords: Null controllability, heat equation, carleman estimates, boundary conditions.

Introduction

Overall, Carleman estimates provide powerful tools for analyzing complex mathematical systems and have numerous applications across areas such as photography.

Control theory is a branch of engineering and mathematics that deals with the analysis and design of controllable systems to achieve the required behavior. This theory is concerned with creating systems capable of maintaining or changing their condition based on feedback, and involves using mathematical models to describe system behaviour and developing control algorithms to handle system input in order to achieve the desired output. The goal of control theory is to develop techniques and tools that can be used to design systems capable of achieving specific performance goals, such as stability, response, and accuracy, control theory uses all disciplines.

An example of steering a vehicle, driving an aircraft or satellite to a geostationary orbit, improving the flow of information into a network, coding and decoding a digital image or SMS, regulating the thermostat, refining oil, controlling the pH of chemical reactions or optimising profits from stock market flows... Control can also reduce pain and prolong life. For example, a blood pressure regulator is designed to maintain this pressure at a constant and appropriate level; we can also control an epidemic such as studying brain tumor treatment or performing laser surgery. Plus robots. It is also used in areas such as economics, biology, and social sciences to model and control complex systems.



Main results

In this work we study the null controllability of heat equation with mixed boundary conditions. Where $\Omega \subset \mathbb{R}^n$, $n \geq 1$ be a bounded spatial domain with smooth boundary Γ of class C^2 and let $T > 0$. We denote $Q = \Omega \times]0, T[$, and $\Sigma = \Gamma \times]0, T[$ where Σ_1 is a piece of the boundary Σ and $\Sigma_2 = \Sigma \setminus \Sigma_1$. Consider the following parabolic equation:

$$\begin{cases} \partial_t u - \Delta u + a_0 u = h_0 \chi_{\mathcal{O}} + w \chi_{\omega} & \text{in } Q, \\ u = 0 & \text{on } \Sigma_1, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Sigma_2, \\ u(x, 0) = 0 & \text{in } \Omega, \end{cases} \quad (0.636)$$

with $a_0 \in L^\infty(Q)$, $\mathcal{O} \subset \Omega$ be some non-empty open subset, is called an observatory set, and ω be an open and non-empty subset of Ω .

Where $h \in L^2(Q)$, and where $\chi_{\mathcal{O}}$ and χ_{ω} denote the characteristic functions of $\mathcal{O}$ and ω respectively. Then problem (2.1) admits a unique solution u that satisfies $u \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$.

Theorem 0.219. *Let $u \in \mathcal{V}$ say defined in (3.3), then there exists a positive constant $C = C(\Omega; \omega; \mathcal{O}; T; a_0)$ such that*

$$\int_Q \frac{1}{\rho^2} |u|^2 dx dt \leq C \left[\int_Q |Lu|^2 dx dt + \int_0^T \int_{\omega} |u|^2 dx dt \right], \quad (0.637)$$

where $\rho \in C^2(Q)$ positive with $\frac{1}{\rho}$ bounded.

This inequality is called observability inequality resulting from Carleman simplification estimates

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Multilinear $\vec{p}$ -th power factorable operators

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Abstract. The generalization of operator ideals to ideals of multilinear operators is a problem recently carried out for some kind of operators. Following this trend we extend the ideal of $\mathcal{F}_p$ -factorable operators to the multilinear case.

Keywords: operator ideal, multilinear operator, p -th power factorable, multilinear summing.

Introduction

The class of p -th power factorable operators were introduced in [5, Chapter 5] and include a very large class of linear operators from σ -order continuous Banach function spaces over a positive finite measure into Banach spaces. In fact, every linear operator between these spaces is 1-th power factorable. They have been applied to the study of several topics and analyzed from several points of view. The extension of this class of operators between Banach function spaces and Banach spaces to general operators between arbitrary Banach spaces was accomplished by means of a factorization diagram that characterize the well known ideal of (p, q) -factorable operators [2, Theorem 4.5]. This work is devoted to generalize the concept of p -th power factorization for linear operators to the multilinear context. Let us give some terminology. A B.f.s. is σ -order continuous (σ -o.c. in brief) if for every sequence of positive functions $(f_i)_i$ such that $f_i \downarrow 0$ we have that $\|f_i\| \downarrow 0$ and it is *Fatou* if $f_i \uparrow f \in L^0(\mu)$ and $\sup_i \|f_i\|_{X(\mu)} < \infty$ implies that $f \in X(\mu)$ and $\|f_i\|_{X(\mu)} \uparrow \|f\|_{X(\mu)}$. Let $Y(\mu)$ be a B.f.s., the space of multiplication operators is $X(\mu)^{Y(\mu)} := \{g \in L^0(\mu) : gX(\mu) \subseteq Y(\mu)\}$. The *Köthe dual space* of $X(\mu)$ is the subspace of integral functionals of the dual space $X(\mu)^*$ and is denoted by $X(\mu)'$. Notice that $X(\mu)' = X(\mu)^{L^1(\mu)}$, also σ -order continuity of $X(\mu)$



implies that $X(\mu)' = X(\mu)^*$ and the Fatou property of $X(\mu)$ implies that $X(\mu)'' = X(\mu)$ (see [4, p.29-30]). A Banach lattice is a Banach space with an order compatible with the linear structure with least upper bound operation $\vee$ and greatest lower bound operation $\wedge$, so that the order of the absolute values implies the order of the norms (see for example [4, Definition 1.a.1]). Notice that a B.f.s. is a Banach lattice with the order $f \leq g$ if and only if $|f| \leq |g|$ and that $\chi_\Omega \in L^\infty(\mu)$ is a weak order unit in any σ -order continuous Banach function space, i.e. $f \wedge n\chi_n \uparrow$ for all $f \in X(\mu)$ (see [5, Proposition 2.22(v)]).

If $0 < p < \infty$, the p -th power space of a B.f.s. $X(\mu)$ is defined as $X(\mu)_{[p]} := \{g \in L^0(\mu) : |g|^{1/p} \in X(\mu)\}$ endowed with the quasi-norm $\|f\|_{X(\mu)}^{1/p}$. One main property of this space is that for $1 < p \leq q < \infty$, $X(\mu) = X(\mu)_{[1]} \subseteq X(\mu)_{[p]} \subseteq X(\mu)_{[q]}$, having in mind that μ is finite (see [5, Lemma 2.21(iv)]). We denote the inclusion $i_{[p]}: X(\mu) \hookrightarrow X(\mu)_{[p]}$, let E be a Banach space and let $X(\mu)$ be a σ -o.c. B.f.s. and let $p \in [1, \infty)$, an operator $T \in \mathcal{L}(X(\mu), E)$ is p -th power factorable if there exists an operator $T_{[p]}: X(\mu)_{[p]} \rightarrow E$ so that $T = T_{[p]} \circ i_{[p]}$ (see [5, Definition 5.1]). This definition is equivalent to the existence of some $K > 0$ such that $\|Tf\|_E \leq K\|f\|_{X(\mu)_{[p]}}$ for every $f \in X(\mu)$ (these concepts and related ones can be seen in [5, Chapters 2 and 5]).

Multilinear $\vec{p}$ -th power factorable operators

The concept of p -th power factorization was extended in [3, Definition 3.1] to the (p, q) -th power factorization for those operators $T \in \mathcal{L}(X(\mu_1), Y(\mu_2))$ such that $T = S \circ R$, where $R: X(\mu_1) \rightarrow E$ is p -th power factorable, $S: E \rightarrow Y(\mu_2)$ is so that S^* is q -th power factorable and E is a Banach space. Notice that we have used two measures while the original definition just uses one. With mild conditions (see [3, Theorem 3.5]) we can obtain a characterization by means of the following inequality

$$|\langle Tf, g \rangle| \leq K\|f\|_{X(\mu_1)_{[p]}}\|g\|_{(Y(\mu_2)^*)_{[q]}}$$

for every $f \in X(\mu_1)$, $g \in Y(\mu_2)^*$ and some $K > 0$. The factorization scheme that is given in this case is $T = i_{[q]}^* \circ T_{[p,q]} \circ i_{[p]}$, for some suitable $T_{[p,q]} \in \mathcal{L}(X(\mu_1)_{[p]}, ((Y(\mu_2)^*)_{[q]})^*)$. Observe that this (p, q) -th power factorability is equivalent to the factorization scheme of the bilinear map $T_b: X(\mu_1) \times Y(\mu_2)^* \rightarrow \mathbb{R}$ defined as $T_b(f, g) := \langle Tf, g \rangle$, given by

$$X(\mu_1) \times Y(\mu_2)^*[dr]^{(i_{[p]}, i_{[q]})}[rr]^{T_b} \mathbb{R} X(\mu_1)_{[p]} \times (Y(\mu_2)^*)_{[q]}[ur]^{T_b[p,q]},$$

where we assume the structural conditions for this diagram makes sense. This is the first example of (bilinear) (p, q) -th power factorable operator. Let us formalize this definition for general multilinear operators.

Definition 0.107. Let F be a Banach space. Let $X_1(\mu_1), \dots, X_n(\mu_n)$ be σ -o.c. Banach function spaces over positive, finite measures, let $T \in \mathcal{L}(X_1(\mu_1), \dots, X_n(\mu_n); F)$ and let $\vec{p} := (p_1, \dots, p_n)$, where $1 \leq p_1, \dots, p_n < \infty$. We say that T is n -linear $\vec{p}$ -th power factorable if there exists $T_{[\vec{p}]} \in \mathcal{L}(X_1(\mu_1)_{[p_1]}, \dots, X_n(\mu_n)_{[p_n]}; F)$ such that

$$X_1(\mu_1) \times \dots \times X_n(\mu_n)[dr]^{(i_{[p_1]}, \dots, i_{[p_n]})}[rr]^{T_b} F X_1(\mu_1)_{[p_1]} \times \dots \times X_n(\mu_n)_{[p_n]}[ur]^{T_b[\vec{p}]}.$$

We will denote by $\mathcal{F}_{\vec{p}}$ the class of all n -linear $\vec{p}$ -th power factorable operators.

Remark 0.12. Notice that diagram above implies the following one

$$X_1(\mu_1) \otimes \dots \otimes X_n(\mu_n)[dr]^{i_{[p_1]} \otimes \dots \otimes i_{[p_n]}}[rr]^{T_b} F X_1(\mu_1)_{[p_1]} \otimes \dots \otimes X_n(\mu_n)_{[p_n]}[ur]^{T_b[\vec{p}]},$$

where $\widehat{T}$ and $\widehat{T}_{[\vec{p}]}$ denotes the linearizations of T and $T_{[\vec{p}]}$, respectively. We will use this scheme in the last result.

By means of the guidelines of [3] and [?, section* 3.1], we obtain the following characterizations.



Proposition 0.35. Let $(\Omega_1, \Sigma_1, \mu_1), \dots, (\Omega_n, \Sigma_n, \mu_n)$ be positive finite measure spaces. Let $X_1(\mu_1), \dots, X_n(\mu_n)$ be σ -o.c. Banach function spaces and let F be a Banach space. For $1 \leq p_1, \dots, p_n < \infty$ and $\vec{p} := (p_1, \dots, p_n)$, the following statements are equivalent for $T \in \mathcal{L}(X_1(\mu_1), \dots, X_n(\mu_n); F)$:

1. T is n -linear $\vec{p}$ -th power factorable.

2. There exists some $K > 0$ such that

$$\|T(f_1, \dots, f_n)\|_F \leq K \|f_1\|_{X_1(\mu_1)_{[p_1]}} \cdots \|f_n\|_{X_n(\mu_n)_{[p_n]}}, \quad (0.638)$$

for every $f_j \in X_j(\mu_j)$, where $j = 1, \dots, n$.

3. There exists some $K > 0$ such that

$$|\langle T(f_1, \dots, f_n), y^* \rangle| \leq K \|f_1\|_{X_1(\mu_1)_{[p_1]}} \cdots \|f_n\|_{X_n(\mu_n)_{[p_n]}} \|y^*\|_{F^*}, \quad (0.639)$$

for every $f_j \in X_j(\mu_j)$ for $j = 1, \dots, n$ and every $y^* \in F^*$.

4. The following factorization diagram holds

$$X_1(\mu_1) \times \cdots \times X_n(\mu_n) [dr]^{([i]_{p_1}, \dots, [i]_{p_n})} [rr]^T F L^{p_1}(m_{T_1}) \times \cdots \times L^{p_n}(m_{T_n}) [ur]^{I_{m_{T_1}, \dots, m_{T_n}}^{(\vec{p})}},$$

where $T_j: X_j(\mu_j) \rightarrow F$ defined by $T_j(f_j) := T(\chi_{\Omega_1}, \dots, \chi_{\Omega_{j-1}}, f_j, \chi_{\Omega_{j+1}}, \dots, \chi_{\Omega_n})$ is continuous for $j = 1, \dots, n$.

5. There exist some $K > 0$ such that

$$\|T(f_1, \dots, f_n)\|_F \leq K \|f_1\|_{L^{p_1}(m_{T_1})} \cdots \|f_n\|_{L^{p_n}(m_{T_n})}, \quad (0.640)$$

for every $f_j \in X_j(\mu_j)$, where $j = 1, \dots, n$ and $T_j: X_j(\mu_j) \rightarrow F$ defined by $T_j(f_j) := T(\chi_{\Omega_1}, \dots, \chi_{\Omega_{j-1}}, f_j, \chi_{\Omega_{j+1}}, \dots, \chi_{\Omega_n})$ is continuous for $j = 1, \dots, n$.

6. There exists some $K > 0$ such that

$$|\langle T(f_1, \dots, f_n), y^* \rangle| \leq K \|f_1\|_{L^{p_1}(m_{T_1})} \cdots \|f_n\|_{L^{p_n}(m_{T_n})} \|y^*\|_{F^*},$$

for every $f_j \in X_j(\mu_j)$, where $j = 1, \dots, n$, $y^* \in F^*$ and $T_j: X_j(\mu_j) \rightarrow F$ defined by $T_j(f_j) := T(\chi_{\Omega_1}, \dots, \chi_{\Omega_{j-1}}, f_j, \chi_{\Omega_{j+1}}, \dots, \chi_{\Omega_n})$ is continuous for $j = 1, \dots, n$.

Proof (1) $\implies$ (2). Let T be a n -linear $\vec{p}$ -th power factorable, so there exists a continuous n -linear operator $T_{[\vec{p}]} \in \mathcal{L}(X_1(\mu_1)_{[p_1]}, \dots, X_n(\mu_n)_{[p_n]}; F)$, such that $T = T_{[\vec{p}]} \circ (i_{[p_1]}, \dots, i_{[p_n]})$. For every $f_j \in X_j(\mu_j)$ ($j = 1, \dots, n$), we have

$$\begin{aligned} \|T(f_1, \dots, f_n)\|_F &= \|T_{[\vec{p}]}(i_{[p_1]}(f_1), \dots, i_{[p_n]}(f_n))\|_F \\ &\leq \|T_{[\vec{p}]}\| \|f_1\|_{X_1(\mu_1)_{[p_1]}} \cdots \|f_n\|_{X_n(\mu_n)_{[p_n]}} \end{aligned}$$

In other words, (0.638) holds with $K = \|T_{[\vec{p}]}\|$.

(2) $\implies$ (1). We assume that T satisfies (0.638), since $X_j(\mu_j) = i_{[p_j]}(X_j(\mu_j))$ is dense in $X_j(\mu_j)_{[p_j]}$ ($j = 1, \dots, n$), T has a unique extension $S: X_1(\mu_1)_{[p_1]} \times \cdots \times X_n(\mu_n)_{[p_n]} \rightarrow F$ such that $T = S \circ (i_{[p_1]}, \dots, i_{[p_n]})$, hence T is $\vec{p}$ -th power factorable.

(2) $\implies$ (3). Let $f_j \in X_j(\mu_j)$ for $j = 1, \dots, n$ and $y^* \in F^*$.

$$\begin{aligned} |\langle T(f_1, \dots, f_n), y^* \rangle| &= \|y^*\| |\langle T(f_1, \dots, f_n), \frac{y^*}{\|y^*\|} \rangle| \\ &\leq \|y^*\| \sup_{x^* \in B_{F^*}} |\langle T(f_1, \dots, f_n), x^* \rangle| \\ &= \|y^*\| \|T(f_1, \dots, f_n)\|_F \\ &\leq K \|f_1\|_{X_1(\mu_1)_{[p_1]}} \cdots \|f_n\|_{X_n(\mu_n)_{[p_n]}} \|y^*\| \end{aligned}$$



(3) $\implies$ (2). Let $K > 0$ such that

$$|\langle T(f_1, \dots, f_n), y^* \rangle| \leq K \|f_1\|_{X_1(\mu_1)_{[p_1]}} \cdots \|f_n\|_{X_n(\mu_n)_{[p_n]}} \|y^*\|_{F^*},$$

for every $y^* \in F^*$ and every n -tuple $(f_1, \dots, f_n) \in X_1(\mu_1) \times \cdots \times X_n(\mu_n)$. Then

$$\|T(f_1, \dots, f_n)\|_F = \sup_{y^* \in B_{F^*}} |\langle T(f_1, \dots, f_n), y^* \rangle| \leq K \|f_1\|_{X_1(\mu_1)_{[p_1]}} \cdots \|f_n\|_{X_n(\mu_n)_{[p_n]}}.$$

(2) $\implies$ (4) It is clear that if T is $\vec{p}$ -th power factorable, then the map T_j is p_j -th power factorable for every $j = 1, \dots, n$. Indeed, for every $f_j \in X_j(\mu_j)$ we have

$$\begin{aligned} \|T_j(f_j)\|_F &= \|T(\chi_{\Omega_1}, \dots, \chi_{\Omega_{j-1}}, f_j, \chi_{\Omega_{j+1}}, \dots, \chi_{\Omega_n})\|_F \\ &\leq K \left(\prod_{\substack{i=1 \\ i \neq j}}^n \|\chi_{\Omega_i}\|_{X_i(\mu_i)_{[p_i]}} \right) \|f_j\|_{X_j(\mu_j)_{[p_j]}}. \end{aligned}$$

So $T_j = I_{\vec{m}_{T_j}}^{(p_j)} \circ [i]_{p_j}$ and $[i]_{p_j}(X_j(\mu_j)) \subseteq L^{p_j}(m_{T_j})$ is dense for each $j = 1, \dots, n$. Also, the map T is $\vec{p}$ -th power factorable and so is continuous, by means of a density argument, T has a unique extension $I_{\vec{m}_T}^{(\vec{p})}$ such that

$$X_1(\mu_1) \times \cdots \times X_n(\mu_n) [dr]^{([i]_{p_1}, \dots, [i]_{p_n})} [rr]^T FLP^1(m_{T_1}) \times \cdots \times L^{p_n}(m_{T_n}) [ur]^{I_{\vec{m}_T}^{(\vec{p})}},$$

where $I_{\vec{m}_T}^{(\vec{p})}$ depends on $m_{T_1}, \dots, m_{T_n}$.

(4) $\implies$ (5). We assume that T factorize as in (4), so there exists a continuous n -linear operator $I_{\vec{m}_T}^{(\vec{p})} \in \mathcal{L}(L^{p_1}(m_{T_1}), \dots, L^{p_n}(m_{T_n}); F)$, such that $T = I_{\vec{m}_T}^{(\vec{p})} \circ ([i]_{p_1}, \dots, [i]_{p_n})$. For every $f_j \in X_j(\mu_j)$ ($j = 1, \dots, n$), we have

$$\begin{aligned} \|T(f_1, \dots, f_n)\|_F &= \|I_{\vec{m}_T}^{(\vec{p})}([i]_{p_1}(f_1), \dots, [i]_{p_n}(f_n))\|_F \\ &\leq \|I_{\vec{m}_T}^{(\vec{p})}\| \|f_1\|_{L^{p_1}(m_{T_1})} \cdots \|f_n\|_{L^{p_n}(m_{T_n})} \end{aligned}$$

In other words, (0.640) holds with $K := \|I_{\vec{m}_T}^{(\vec{p})}\|$.

(5) $\implies$ (2). We assume that (5) is satisfied, that is, there exists some $K > 0$ such that $\|T(f_1, \dots, f_n)\|_F \leq K \|f_1\|_{L^{p_1}(m_{T_1})} \cdots \|f_n\|_{L^{p_n}(m_{T_n})}$, where $T_j: X_j(\mu_j) \rightarrow F$ is continuous, then $[i]_1(X_j(\mu_j)) \subseteq L^1(m_{T_j})$ densely for $j = 1, \dots, n$, which implies that there is a unique extension $I_{\vec{m}_T}$ of T such that $T = I_{\vec{m}_T} \circ ([i]_1, \dots, [i]_1)$. Moreover

$$\begin{aligned} \|T_j(f_j)\|_F &= \|T(\chi_{\Omega_1}, \dots, \chi_{\Omega_{j-1}}, f_j, \chi_{\Omega_{j+1}}, \dots, \chi_{\Omega_n})\|_F \\ &\leq K \left(\prod_{\substack{i=1 \\ i \neq j}}^n \|\chi_{\Omega_i}\|_{L^{p_i}(m_{T_i})} \right) \|f_j\|_{L^{p_j}(m_{T_j})} \\ &= M \|[i]_{p_j}(f_j)\|_{L^{p_j}(m_{T_j})} \leq M \|[i]_{p_j}\| \cdot \|f_j\|_{X_j(\mu_j)}, \end{aligned}$$

where $M = K \left(\prod_{\substack{i=1 \\ i \neq j}}^n \|\chi_{\Omega_i}\|_{L^{p_i}(m_{T_i})} \right)$. Then $[i]_{p_j}(X_j(\mu_j)) \subseteq L^{p_j}(m_{T_j})$ continuously, which implies that $[i]_1(X_j(\mu_j)_{[p_j]}) \subseteq L^1(m_{T_j})$ (see the proof of [3, Lemma 3.2]). Now we consider the restriction of $I_{\vec{m}_T}$ to $[i]_1(X_1(\mu_1)_{[p_1]}) \times \cdots \times [i]_1(X_n(\mu_n)_{[p_n]})$, i.e. $I_{\vec{m}_T} \circ ([i]_1, \dots, [i]_1)$, that is continuous. Notice that here, the inclusion/quotient maps start from $X_j(\mu_j)_{[p_j]}$, instead of $X_j(\mu_j)$ (for $j = 1, \dots, n$), as above. Then for every $(f_1, \dots, f_n) \in X_1(\mu_1) \times \cdots \times X_n(\mu_n)$, we have that

$$\begin{aligned} \|T(f_1, \dots, f_n)\|_F &= \|I_{\vec{m}_T} \circ ([i]_1 \cdot \dots \cdot [i]_1) \circ ([i]_{p_1}, \dots, [i]_{p_n})(f_1, \dots, f_n)\|_F \\ &= \|I_{\vec{m}_T}(f_1, \dots, f_n)\|_F \leq C \prod_{j=1}^n \|f_j\|_{L^1(m_{T_j})} \leq D \prod_{j=1}^n \|f_j\|_{X_j(\mu_j)_{[p_j]}}, \end{aligned}$$



for some $C, D > 0$.

(5) $\iff$ (6). Occurs immediately. ■

Next example is the trivial one of n -linear $\vec{p}$ -th power factorable operator.

Example 0.10. Let $1 \leq p_1, \dots, p_n < \infty$, E be a Banach space and $L^{p_1}(\mu_1), \dots, L^{p_n}(\mu_n)$ be Banach function spaces over positive, finite measures and let a n -linear operator $T: L^{p_1}(\mu_1) \times \dots \times L^{p_n}(\mu_n) \rightarrow E$. Let $q_j \geq p_j$, thus $L^{q_j}(\mu_j) \subseteq L^{p_j}(\mu_j)$ for $j = 1, \dots, n$, hence $L^{q_j/r_j}(\mu_j) = L^{q_j}(\mu_j)_{[r_j]} \subseteq L^{p_j}(\mu_j)$ for every $r_j \in [1, q_j/p_j]$ ($j = 1, \dots, n$). In consequence the restrictions of T :

$$T_{(q_1, \dots, q_n)}: L^{q_1}(\mu_1) \times \dots \times L^{q_n}(\mu_n) \rightarrow E$$

are $(r_1, \dots, r_n)$ -th power factorable for every $r_j \in [1, q_j/p_j]$, where $q_j \geq p_j$, for $j = 1, \dots, n$.

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Numerical solution of fractional hopefield neural networks using reproducing kernel Hilbert space method

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Abstract. Artificial neural networks (ANN) are consisting of a group from the virtual neurons that are designed by computerized programs which use a variety of mathematical fractional equations. In this paper, we introduce the Reproducing Kernel Hilbert Space (RKHS) method to solving some certain fractional differential systems in the artificial neural networks field which is the Hopfield network using the Conformable derivative.

Keywords: Reproducing Kernel Hilbert Space Method (RKHS), Fractional Derivative, Artificial Neural Networks, Differential Systems, Chaotic attractors.

Introduction

Artificial Neural Networks (ANN) is a powerful recently emerging computer-aided design technology (CAD) for modeling devices and circuits. These networks consist of a set of virtual neurons that are generated by computer programs that use a number of fractional



mathematical equations to process the data that comes from the neurons. The Hopfield network is a variety of recurrent artificial neural networks. Its idea arose from the behavior of particles in a magnetic field such that each particle is communicated (completely linked) with another particle by magnetic forces. This is referred to as activation in the case of neurons. As a result, both particles or neurons spin, encouraging one another to continue this rotation. Hopfield neural networks come in two versions: binary and continuous. In the binary form, all neurons are linked to each other there is no connection from a neuron to itself, in the continuous version, all connections, including self-connections, are allowed. The Hopfield network neurons will allow each other to rotate while they are in a spinning state. The movement of the particles is to process information so they will be in the activation situation. For example, if two particles are in a rotating state to process information, this is known as binary activation in Hopfield neural networks.

For, obtaining the solution of the Hopfield neural network equations systems, we propose the reproducing kernel Hilbert space method which was used for the first time at the beginning of the 20th century by S. Zarembo for the harmonic and biharmonic functions to find solutions for boundary value problems (BVPs). The RKHS approach is a valuable framework for creating numerical solutions in applied sciences. This theory has been successfully extended to a variety of important applications in numerical analysis, computational mathematics, image processing, machine learning, probability and statistics, and finance, and it has been shown to be very effective in different fields of integrative equations, integrative differential equations and partial differential equations, and others, especially when the derivative order is fractional. Recently a new definition in the fractional calculus has been introduced which is the conformable fractional derivative. this concept a natural extension of the first-order derivative, and it satisfies some of the properties which are lost in the other fractional definitions, such as the derivative of product and quotient of two functions formulas, and the chain rule.

Main results

Figure 38: Chaotic attractors for $\alpha = 0.95$ for example 1.

(a) (b)

Figure 39: Chaotic attractors of HNN for example 2 such that: (a): $\alpha = 0.95$, $k = 0.95$ / (b): $\alpha = 0.85$, $k = 0.95$

Figure 40: Chaotic attractors for $\alpha = 1$ and $k = -9$ for example 3.

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Analyse de la régularité d'un problème de transmission gouverné par une équation abstraite dans un cadre höldérien.

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Abstract. L'objectif de mon travail est l'étude dans le cadre continue, l'existence, l'unicité et la régularité maximale de la solution stricte d'un problème de transmission (P^δ) , pour tout $\delta > 0$ sous certaines hypothèses sur les données, dans un domaine non borné composé d'une demi-droite $]-\infty, 0]$ et d'une couche mince $[0, \delta]$,

en utilisant essentiellement la théorie des semi-groupes et le le travail de Sinesstrari [?]

Keywords: Problèmes de transmission, les semi-groupes, les espaces d'interpolation, les espaces de Hölder, solution stricte, régularité maximale.



Introduction

On se propose d'étudier le problème abstrait du second ordre suivant

$$\left((P^\delta) \right) \begin{cases} u''(x) + Au(x) = g(x), & x \in]-\infty, 0[\cup]0, \delta[, \\ u'(\delta) = f_+, \text{ et } u(-\infty) = f_-, \\ u(0_-) = u(0_+) \text{ et } \mu_- u'(0_-) = \mu_+ u(0_+). \end{cases}$$

posée dans un espace de Banach E , avec

- A est un opérateurs linéaires fermés à valeur dans E .
- f_- , f_+ sont deux éléments de E .
- μ_- , μ_+ sont deux constantes réelles positives non nulles et distinctes exprimant la conductivité des deux milieux.

L'étude se fait sous l'unique l'hypothèse d'ellipticité sur l'opérateur linéaire fermé A de domaine $D(A) \subset E$ i.e.,

$$(H) \begin{cases} \rho(A) \supset [0, +\infty[, \text{ où } \rho(A) \text{ est l'ensemble résolvant de } A \\ \exists C > 0 \text{ tel que } \forall \lambda \in [0, +\infty[: \|(A - \lambda I)^{-1}\| \leq \frac{1}{1 + \lambda} \end{cases}$$

dans le cadre continue, plus précisément lorsque le second membre $g = (g_-, g_+)$ est holdérienne

$$g_- \in BUC^{2\alpha}(-\infty, 0; E), \quad g_+ \in C^{2\alpha}(0, \delta; E)$$

d'exposant $2\alpha \in]0, 1[$.

L'objectif est de montrer l'existence, l'unicité et la régularité maximale de la solution stricte c'est à dire les restrictions (u_+, u_-) de la solution u sur les deux intervalles $[0, \delta]$ et $]-\infty, 0]$ vérifient

$$\begin{cases} u_+ \in C^2([0, \delta]; E) \cap C([0, \delta]; D(A)), \\ u_- \in BUC^2([-\infty, 0]; E) \cap BUC([-\infty, 0]; D(A)). \end{cases}$$

en plus $u_+, Au_+ \in C^{2\alpha}(0, \delta; E)$ et $u_-, Au_- \in BUC^{2\alpha}(-\infty, 0; E)$.

Main results

Theorem 0.220 (Solution Stricte). *Supposons (H). Soient*

$$g_+ \in C^{2\alpha}([0, \delta]; E), \quad g_- \in BUC^{2\alpha}([-\infty, 0]; E) \text{ avec } 0 < 2\alpha < 1$$

Si $f_+ \in D\left((-A)^{\frac{1}{2}}\right)$ et $f_- \in D(A)$. Alors le problème (P_0) admet une unique solution stricte si et seulement si $g_+(0) - g_-(0) \in \overline{D(A)}$, $Af_-, (-A)^{\frac{1}{2}} f_+ \in \overline{D(A)}$.



Theorem 0.221 (Régularité). *Supposons (H_1) . Soient*

$$g_+ \in C^{2\alpha}([0, \delta]; E), \quad g_- \in BUC^{2\alpha}(-\infty, 0]; E)$$

avec $0 < 2\alpha < 1$ et $f_- \in D(A)$, $f_+ \in D((-A)^{\frac{1}{2}})$. Alors

$$u''_+, Au_+ \in C^{2\alpha}(0, \delta; E); \quad u''_+, Au_+ \in BUC^{2\alpha}(-\infty, 0; E)$$

si seulement si $g_+(0) - g_-(0)$, Af_- , $(-A)^{\frac{1}{2}}f_+ \in D_A(\alpha, +\infty)$ où

$$D_A(\alpha, \infty) = \left\{ x \in E, \|x\|_\alpha = \sup_{t>0} \|t^{1-\alpha} A e^{At} x\| < \infty \right\}.$$

Les techniques employées ici reposant sur la théorie de semi-groupes (voir par exemple [?] pour cette théorie) et le travail de Sinesstrari [?].

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Analyse de la régularité d'un problème de transmission gouverné par une équation abstraite dans un cadre höldérien.

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Abstract. L'objectif de mon travail est l'étude dans le cadre continue, l'existence, l'unicité et la régularité maximale de la solution stricte d'un problème de transmission (P^δ) , pour tout $\delta > 0$ sous certaines hypothèses sur les données, dans un domaine non borné composé d'une demi-droite $]-\infty, 0]$ et d'une couche mince $[0, \delta]$,

en utilisant essentiellement la théorie des semi-groupes et le le travail de Sinesstrari [?]

Keywords: Problèmes de transmission, les semi-groupes, les espaces d'interpolation, les espaces de Hölder, solution stricte, régularité maximale.



Introduction

On se propose d'étudier le problème abstrait du second ordre suivant

$$\left((P^\delta) \right) \begin{cases} u''(x) + Au(x) = g(x), & x \in]-\infty, 0[\cup]0, \delta[, \\ u'(\delta) = f_+, \text{ et } u(-\infty) = f_-, \\ u(0_-) = u(0_+) \text{ et } \mu_- u'(0_-) = \mu_+ u(0_+). \end{cases}$$

posée dans un espace de Banach E , avec

- A est un opérateurs linéaires fermés à valeur dans E .
- f_- , f_+ sont deux éléments de E .
- μ_- , μ_+ sont deux constantes réelles positives non nulles et distinctes exprimant la conductivité des deux milieux.

L'étude se fait sous l'unique l'hypothèse d'ellipticité sur l'opérateur linéaire fermé A de domaine $D(A) \subset E$ i.e.,

$$(H) \begin{cases} \rho(A) \supset [0, +\infty[, \text{ où } \rho(A) \text{ est l'ensemble résolvant de } A \\ \exists C > 0 \text{ tel que } \forall \lambda \in [0, +\infty[: \|(A - \lambda I)^{-1}\| \leq \frac{1}{1 + \lambda} \end{cases}$$

dans le cadre continue, plus précisément lorsque le second membre $g = (g_-, g_+)$ est holdérienne

$$g_- \in BUC^{2\alpha}(-\infty, 0; E), \quad g_+ \in C^{2\alpha}(0, \delta; E)$$

d'exposant $2\alpha \in]0, 1[$.

L'objectif est de montrer l'existence, l'unicité et la régularité maximale de la solution stricte c'est à dire les restrictions (u_+, u_-) de la solution u sur les deux intervalles $[0, \delta]$ et $]-\infty, 0]$ vérifient

$$\begin{cases} u_+ \in C^2([0, \delta]; E) \cap C([0, \delta]; D(A)), \\ u_- \in BUC^2([-\infty, 0]; E) \cap BUC([-\infty, 0]; D(A)). \end{cases}$$

en plus $u_+, Au_+ \in C^{2\alpha}(0, \delta; E)$ et $u_-, Au_- \in BUC^{2\alpha}(-\infty, 0; E)$.

Main results

Theorem 0.222 (Solution Stricte). *Supposons (H). Soient*

$$g_+ \in C^{2\alpha}([0, \delta]; E), \quad g_- \in BUC^{2\alpha}([-\infty, 0]; E) \text{ avec } 0 < 2\alpha < 1$$

Si $f_+ \in D((-A)^{\frac{1}{2}})$ et $f_- \in D(A)$. Alors le problème (P_0) admet une unique solution stricte si et seulement si $g_+(0) - g_-(0) \in \overline{D(A)}$, $Af_-, (-A)^{\frac{1}{2}} f_+ \in \overline{D(A)}$.



Theorem 0.223 (Régularité). *Supposons (H_1) . Soient*

$$g_+ \in C^{2\alpha}([0, \delta]; E), \quad g_- \in BUC^{2\alpha}(-\infty, 0]; E)$$

avec $0 < 2\alpha < 1$ et $f_- \in D(A)$, $f_+ \in D((-A)^{\frac{1}{2}})$. Alors

$$u''_+, Au_+ \in C^{2\alpha}(0, \delta; E); \quad u''_+, Au_+ \in BUC^{2\alpha}(-\infty, 0; E)$$

si seulement si $g_+(0) - g_-(0)$, Af_- , $(-A)^{\frac{1}{2}}f_+ \in D_A(\alpha, +\infty)$ où

$$D_A(\alpha, \infty) = \left\{ x \in E, \|x\|_\alpha = \sup_{t>0} \|t^{1-\alpha} A e^{At} x\| < \infty \right\}.$$

Les techniques employées ici reposant sur la théorie de semi-groupes (voir par exemple [?] pour cette théorie) et le travail de Sinesstrari [?].

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Analysis of a dynamic viscoelastic problem with long-term memory

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Abstract. We consider a dynamic contact problem associated to a non-linear viscoelastic system in a thin domain $\Omega^\varepsilon \subset \mathbb{R}^3$. We discuss the variational analysis of the problem. then, we study the asymptotic behavior when one dimension of the domain tends to zero. A generalized weak equation and the uniqueness results for the limit problem are established.

Keywords: Variational formulation, Asymptotic analysis, Tresca law, A priori estimates.

Introduction

In mathematical literature, there are many research papers regarding the asymptotic analysis of the problems that arise in a thin domain of which we mention: Benseridi and Dilmi studied the asymptotic analysis of a dynamical problem of Tresca type in [1]. The theoretical analysis of a frictionless contact between two general elastic bodies in a stationary regime in a



three-dimensional thin domain with Tresca friction law in [1]. In the present paper, we consider the problem of deformation of a non linear viscoelastic body in the dynamic regime occupying a bounded, homogenous thin domain $\Omega^\varepsilon \subset \mathbb{R}^3$. The boundary Γ^ε of the domain is assumed to be Lipschitz continuous and is partitioned into three disjoint measurable parts, Γ_1^ε , Γ_L^ε and ω with $\text{mes}(\Gamma_1^\varepsilon) > 0$. This paper deals with the asymptotic behavior of a non linear problem in a dynamic regime with Tresca free boundary friction conditions occupying a bounded homogeneous domain.

Main results

Let $0 < \varepsilon < 1$ is a small parameter that will tend to zero. We denote by Ω^ε the domain of the flow $\Omega^\varepsilon = \{(x', x_3) \in \mathbb{R}^3, (x', 0) \in \omega, 0 < x_3 < \varepsilon h(x')\}$. We consider a mathematical problem governed by a nonlinear operator in a three dimensional thin domain:

Problème (P^ε) : Find a displacement field $u^\varepsilon = (u_i^\varepsilon)_{1 \leq i \leq 3} : \Omega^\varepsilon \times]0, T[\rightarrow \mathbb{R}^3$ such that:

$$\frac{\partial^2 u^\varepsilon}{\partial t^2} - \text{Div } \sigma^\varepsilon - f^\varepsilon = 0, \quad \text{in } \Omega^\varepsilon \times]0, T[\quad (0.641)$$

$$\sigma_{ij}^\varepsilon(u^\varepsilon) = 2\mu d_{ij}(u^\varepsilon) + \lambda d_{kk}(u^\varepsilon) \delta_{ij} + \int_0^t b_{ijkh}(t-s) d_{kh}(u^\varepsilon(s)) ds, \quad \text{in } \Omega^\varepsilon \times]0, T[\quad (0.642)$$

$$u^\varepsilon = 0, \quad \text{in } (\Gamma_1^\varepsilon \cup \Gamma_L^\varepsilon) \times]0, T[\quad (0.643)$$

$$\frac{\partial u^\varepsilon}{\partial t} \cdot n = 0, \quad \text{in } \omega \times]0, T[\quad (0.644)$$

$$\left. \begin{aligned} |\sigma_\tau^\varepsilon| < k^\varepsilon &\implies \left(\frac{\partial u^\varepsilon}{\partial t}\right)_\tau = 0 \\ |\sigma_\tau^\varepsilon| = k^\varepsilon &\implies \exists \lambda \geq 0, \left(\frac{\partial u^\varepsilon}{\partial t}\right)_\tau = -\lambda \sigma_\tau^\varepsilon, \end{aligned} \right\} \quad \text{in } \omega \times]0, T[. \quad (0.645)$$

$$u^\varepsilon(x, 0) = u_0(x), \quad \frac{\partial u^\varepsilon}{\partial t}(x, 0) = u_1(x), \quad \text{and} \quad \forall x \in \Omega^\varepsilon. \quad (0.646)$$

Where $f^\varepsilon = (f_1^\varepsilon, f_2^\varepsilon, f_3^\varepsilon)$, is the density of forces.

Expression (2.1) is representing the law of conservation of momentum. The constitutive law for a nonlinear viscoelastic isotropic material with long memory is given by relation (2.2) with $\mathcal{B} = ((b_{ijkh}(x, t) \varepsilon_{kh}))$ is the relaxation tensor. Furthermore, conditions (2.3)-(2.4) is the displacements and the velocity functions respectively. The Tresca friction law is given by (2.6) with friction coefficient k^ε . Finally, the initial conditions of the displacement and the velocity field are given by (2.7).

Next, we derive the variational formulation of the problem. For this purpose, we define the following set: $H^1(\Omega^\varepsilon)^3 = \left\{ \varphi \in (L^2(\Omega))^\times : \frac{\partial \varphi_i}{\partial x_j} \in L^2(\Omega) \forall i, j = 1, 2, 3 \right\}$,

$K^\varepsilon = \left\{ \varphi \in H^1(\Omega^\varepsilon)^3 : \varphi = 0 \text{ on } \Gamma_1^\varepsilon \cup \Gamma_L^\varepsilon, \varphi \cdot n = 0 \text{ on } \omega \right\}$,

and we will use the technique of scaling in Ω^ε on the coordinate x_3 . By introducing the change of the variables $z = \frac{x_3}{\varepsilon}$ we obtain a fixed domain Ω which is independent of ε . We establish some estimate independent on this parametre, for displacement and the velocity. We obtain some convergence results. Finally, we give the main results concerning the limit problem with a specific weak form of the weak generalized equation and the uniqueness of the displacement when the thickness tends to zero.





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Some Results on the Harmonicity of 1-forms

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Abstract. In this paper we show that the harmonicity of a vector field (resp. 1-form) on a Riemannian manifold is equivalent to the harmonicity of the corresponding 1-form (resp. vector field).

Keywords: Cotangent bundle, Sasaki metric, harmonic maps.

Introduction

Harmonic maps are of particular interest in geometric analysis, they are related to many areas of theoretical physics. Given a smooth mapping $\varphi : (M, g) \rightarrow (N, h)$ between Riemannian manifolds, φ is said to be harmonic if it is a critical point of the energy functional $E(\varphi)$

$$E(\varphi) = \frac{1}{2} \int_D |d\varphi|^2 v_g, \quad (0.647)$$

for any compact domain D of M . The Euler-Lagrange equation of (0.647) is [2, 7]

$$\tau(\varphi) = \text{Tr}_g(\nabla d\varphi) = 0,$$

where $\tau(\varphi)$ is the tension field of φ . The energy functional (0.647) have acquired an extensive literature see for instance [2, 5, 6, 7] and references therein.

Denote by $\Omega^1(M)$ the set of all smooth 1-forms on M and by $g_{\tilde{S}}$ the Sasaki metric on the cotangent bundle T^*M . Any $\omega \in \Omega^1(M)$ defines a smooth map from (M, g) to $(T^*M, g_{\tilde{S}})$. In this work, we will examine the harmonicity of $\omega \in \Omega^1(M)$.



Main results

Let (M, g) be an m -dimensional Riemannian manifold and (TM, g_S) (resp. $(T^*M, g_{\tilde{S}})$) be its tangent bundle (resp. its cotangent bundle) equipped with the Sasaki metrics. As the metric g is nondegenerated, the musical isomorphisms $\flat : X \mapsto \flat(X) = g(X, \cdot)$ and $\sharp$ define a bundle isomorphism between the tangent and the cotangent bundle of M . Moreover, $\flat$ and $\sharp$ are isometries between (TM, g_S) and $(T^*M, g_{\tilde{S}})$, from [12] we have

$$d\flat(X^v) = (\flat X)^{\tilde{v}}, \quad d\flat(X^h) = (X)^{\tilde{h}}, \quad (0.648)$$

and

$$d\sharp(\omega^{\tilde{v}}) = (\sharp\omega)^v, \quad d\sharp(X^{\tilde{h}}) = (X)^h. \quad (0.649)$$

Thus if $Y \in \Gamma(TM)$, we get

$$d\flat(d\sharp\omega(Y)) = d\omega(Y), \quad d\sharp(d\omega(Y)) = d\sharp\omega(Y), \quad (0.650)$$

On the other hand, from [5, pp. 50] we have

$$d\sharp\omega(Y) = \{Y^h + (\nabla_Y \sharp\omega)^v\} \circ \sharp\omega. \quad (0.651)$$

Therefore, using (0.648), (0.650) and (0.651), we find

$$d\omega(Y) = d\flat(d\sharp\omega(Y)) = \{Y^{\tilde{h}} + (\nabla_Y \omega)^{\tilde{v}}\} \circ \omega. \quad (0.652)$$

Now, we give the formula of the tension field $\tau(\omega)$ of $\omega \in \Omega^1(M)$.

Theorem 0.224. *Let (M, g) be an m -dimensional Riemannian manifold, a 1-form $\omega : (M, g) \rightarrow (T^*M, g_{\tilde{S}})$ is a harmonic map if and only if the corresponding vector field $\sharp\omega : (M, g) \rightarrow (TM, g_S)$ is a harmonic map, moreover the tension field $\tau(\omega)$ of ω is given by*

$$\tau(\omega) = (-S(\sharp\omega))^{\tilde{h}} + (-\tilde{\Delta}\omega)^{\tilde{v}}, \quad (0.653)$$

where $\tilde{\Delta}\omega$ is the rough Laplacian on 1-forms given by $\tilde{\Delta}\omega = \sum_{i=1}^m (\nabla_{\nabla_{e_i} e_i} \omega - \nabla_{e_i} \nabla_{e_i} \omega)$.

Theorem 0.225. *Let (M, g) be an m -dimensional Riemannian manifold, a 1-form $\omega : (M, g) \rightarrow (T^*M, g_{\tilde{S}})$ is a harmonic map if and only if $S(\sharp\omega) = 0$ and $\tilde{\Delta}\omega = 0$.*

Theorem 0.226. *Let (M, g) be an m -dimensional Riemannian manifold, a 1-form $\omega : (M, g) \rightarrow (T^*M, g_{\tilde{S}})$ is harmonic map if and only if the corresponding vector field $\sharp\omega : (M, g) \rightarrow (TM, g_S)$ is harmonic map.*

Example 0.11. *Let Sol_3 be the Sol space i.e. $\mathbb{R}^3(x, y, z)$ equipped with metric g given by*

$$g = e^{2z}(dx)^2 + e^{-2z}(dy)^2 + (dz)^2.$$

The frame $\{e_1 = e^{-z} \frac{\partial}{\partial x}, e_2 = e^z \frac{\partial}{\partial y}, e_3 = \frac{\partial}{\partial z}\}$ constitute an orthonormal basis of the Lie algebra $\mathfrak{g}$ of Sol_3 . We take the 1-form $\omega = f(z)\theta_3$, where $f(z)$ is a smooth real function and $\theta_3 = e_3$, based on computations from [1], we obtain

$$\begin{aligned} \tilde{\Delta}\omega &= (2f - f'')\theta_3, \\ S(\sharp\omega) &= S(fe_3) = 0, \end{aligned}$$

thus $\omega = f(z)\theta_3$ is a harmonic map if and only if the function f satisfies

$$f'' - 2f = 0. \quad (0.654)$$

The solution of (0.654) is

$$f(z) = c_1 e^{\sqrt{2}z} + c_2 e^{-\sqrt{2}z}, \quad (0.655)$$

where c_1, c_2 are real constants.



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Application du théorème de Tikhonov pour les systèmes lents-rapides à un modèle épidémiologique

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Abstract. Un modèle singulièrement perturbé se figure lorsqu'on combine un modèle épidémiologiques SEIRS dont l'échelle de temps est le jour, avec un modèle démographique dont l'échelle de temps est l'année. A l'aide du Théorème de Tikhonov pour les systèmes lents-rapides, les solutions du modèle combiné de dimension 5 peuvent être approchées dans leurs phases lentes par des solutions d'un modèle planaire linéaire ou affine selon les conditions initiales et ce pour des temps arbitrairement grands.

Keywords: Modèles épidémiologiques, perturbations singulières, réduction de modèles, Théorème de Tikhonov.



Introduction

Généralement, l'élaboration de modèles mathématiques implique la considération de la description des phénomènes à différentes échelles de temps ou d'espace [4]. Nous nous intéressons ici au cas de la hiérarchisation d'une dynamique à plusieurs échelles de temps. Une telle modélisation met en évidence les caractéristiques essentielles de la dynamique en fournissant des modèles de dimensions d'espace réduites, allant du plus rapide au plus lent. La théorie des perturbations singulières, dans sa forme topologique ou géométrique, est généralement utilisée pour étudier de tels modèles, quand ils sont exprimés par des équations différentielles ordinaires, telles que la théorie de Tikhonov [1, 5] ou celle de Fenichel [2].

Dans sa forme standard, un problème singulièrement perturbé s'écrit sous la forme

$$\begin{aligned} \epsilon \frac{dx}{dt} &= f(x, y, \epsilon) & x(0) &= x_0 \\ \frac{dy}{dt} &= g(x, y, \epsilon) & y(0) &= y_0 \end{aligned} \quad (0.656)$$

où x et y représentent respectivement la variable rapide et la variable lente. Lorsque la dynamique rapide, décrite par l'équation $dx/d\tau = f(x, y, 0)$, admet des équilibres asymptotiquement stables paramétrés par y , on pense à utiliser la théorie de Tikhonov pour les systèmes lents-rapides pour approcher la phase lente du problème.

Dans [3] ont élaboré un modèle épidémiologique non linéaire en combinant un modèle démographique à deux stades et un modèle basique SIS. En tenant compte des échelles de temps, leur modèle est de type lent-rapide. À l'aide du théorème de Tikhonov, ils ont montré que le modèle peut être approché avec une bonne précision, quand le paramètre de perturbation est petit, par des modèles linéaires de dimension inférieure. Un résultat similaire est obtenu dans [6] pour un modèle SIRS. Nous proposons de généraliser cette approche au modèle SEIRS moins simpliste.

Application du théorème de Tikhonov

Considérons le modèle démographique qui décrit l'évolution de la population humaine en deux stades, les juvéniles de taille n_1 et les adultes de taille n_2

$$\begin{cases} \dot{n}_1 = -a_1 n_1 - \mu_1 n_1 + b n_2 \\ \dot{n}_2 = a_1 n_1 - \mu_2 n_2 \end{cases} \quad \begin{matrix} n_1(0) = n_{10} \\ n_2(0) = n_{20} \end{matrix} \quad (0.657)$$

On considère aussi le modèle SEIRS non vital avec immunité temporelle suivant

$$\begin{cases} \dot{S} = -\beta SI + \gamma R \\ \dot{E} = \beta SI - \delta E \\ \dot{I} = \delta E - \alpha I \\ \dot{R} = \alpha I - \gamma R \end{cases} \quad \begin{matrix} S(0) = S_0 \\ E(0) = E_0 \\ I(0) = I_0 \\ R(0) = R_0 \end{matrix} \quad (0.658)$$

Nous tenons compte du fait que l'unité de temps dans le modèle démographique est l'année et celle du modèle épidémiologique le jour. Ainsi la démographie est relativement plus lente que le processus épidémiologique. Pour cela, nous ne combinons les deux modèles qu'après avoir multiplié le modèle épidémiologique par le grand nombre 365 ou, plus généralement par $1/\epsilon$, où ϵ est un petit paramètre positif. Nous obtenons alors le modèle de la dynamique de l'épidémie dans la population

$$\begin{cases} \epsilon \dot{s} = \epsilon(a_1 n_1 - \mu_2 s) + -\beta si + \gamma r & s(0) = s_0 \\ \epsilon \dot{i} = -\epsilon \mu_2^* i + \delta(n_2 - s - i - r) - \alpha i & i(0) = i_0 \\ \epsilon \dot{r} = -\epsilon \mu_2 r + \alpha i - \gamma r & r(0) = r_0 \\ \dot{n}_1 = -a_1 n_1 - \mu_1 n_1 + b n_2 & n_1(0) = n_{10} \\ \dot{n}_2 = a_1 n_1 - \mu_2 n_2 - (\mu_2^* - \mu_2) i & n_2(0) = n_{20} \end{cases} \quad (0.659)$$



Remarquons que (0.656) et (0.659) ont la même forme qui convient à l'application du théorème de Tikhonov.

Selon la théorie des perturbations singulières, la phase rapide du modèle est d'écrite par l'équation rapide de (0.659) et qui est définie par

$$\begin{cases} S' = -\beta SI + \gamma R \\ I' = \delta n_2 - \delta S - (\delta + \alpha)I - \delta R \\ R' = \alpha I - \gamma R \end{cases} \quad (0.660)$$

où $(') = \frac{d}{d\tau}$, $\tau = \frac{1}{\epsilon}$ et où n_1, n_2 sont considérés comme des paramètres.

Le lieu des points d'équilibre du système (0.660) définit ce qu'on appelle la variété lente de (0.659). Il en existe ici deux composantes V_1 et V_2 de dimension 2, définies dans $\mathbb{R}_+^2$ respectivement par

$$\begin{aligned} (s, i, r) &= \phi_1(n_1, n_2) := (n_2, 0, 0) \\ (s, i, r) &= \phi_2(n_1, n_2) := (\phi_{21}(n_1, n_2), \phi_{22}(n_1, n_2), \phi_{23}(n_1, n_2)) \end{aligned}$$

avec

$$\phi_{21}(n_1, n_2) = \frac{\alpha}{\beta} = \nu, \quad \phi_{22}(n_1, n_2) = \frac{\gamma\delta(n_2 - \nu)}{\delta\alpha + \delta\gamma + \gamma\alpha}, \quad \phi_{23}(n_1, n_2) = \frac{\alpha\delta(n_2 - \nu)}{\delta\alpha + \delta\gamma + \gamma\alpha}$$

Dans la théorie de Tikhonov, la variété lente doit être isolée, or les deux composantes se coupent quand $n_2 = \nu$. Elles ne sont donc isolées l'une de l'autre. Cela nous mènera à définir deux systèmes lents en fonction des conditions initiales en dehors de la région d'intersection des deux variétés.

Le lemme suivant donne les conditions d'attractivité des variétés lentes, ce qui équivaut à démontrer la stabilité asymptotique (uniforme) des équilibres de (0.660).

Lemma 0.76.

1. La variété V_1 est attractive si $n_2 < \nu$ et répulsive si $n_2 > \nu$.
2. La variété V_2 est attractive si $n_2 > \nu$ et répulsive si $n_2 < \nu$.

Proof La preuve du premier point est consiste à calculer les valeurs propres de la matrice jacobienne associée au système rapide au point $(n_2, 0, 0)$, où la matrice admet trois valeurs propres qui sont strictement négatives ssi $n_2 < \nu$.

Le second point du lemme découle de l'application du critère de Routh-Hurwitz à la matrice jacobienne au point $(\phi_{21}(n_1, n_2), \phi_{22}(n_1, n_2), \phi_{23}(n_1, n_2))$. ■

Les *équations lentes* décrivent la phase lente de (0.659) près des variétés lentes respectives V_1 et V_2 . Elles sont obtenues en remplaçant les variables (s, i, r) dans le sous-système (n_1, n_2) de (0.659) respectivement par $\phi_1(n_1, n_2)$ et $\phi_2(n_1, n_2)$, ce qui donne les système linéaires

$$\begin{cases} \dot{n}_1 = -(a_1 + \mu_1)n_1 + bn_2 \\ \dot{n}_2 = a_1n_1 - \mu_2n_2 \end{cases} \quad (0.661)$$

$$\begin{cases} \dot{n}_1 = -(a_1 + \mu_1)n_1 + bn_2 \\ \dot{n}_2 = a_1n_1 - \rho_1n_2 + \rho_2 \end{cases} \quad (0.662)$$

où

$$\rho_1 = \frac{\alpha\delta\mu_2 + \gamma\mu_2^*(\delta + \alpha)}{\delta\alpha + \delta\gamma + \gamma\alpha}, \quad \rho_2 = \frac{\nu(\mu_2^* - \mu_2)(\delta\gamma + \gamma\alpha)}{\delta\alpha + \delta\gamma + \gamma\alpha}$$

On définit alors l'équation de la couche limite de (0.659) par

$$\begin{cases} s' = -\beta si + \gamma r \\ i' = \delta n_{20} - \delta s - (\delta + \alpha)i - \delta r \\ r' = \alpha i - \gamma r \end{cases} \quad \begin{cases} s(0) = s_0 \\ i(0) = i_0 \\ r(0) = r_0 \end{cases} \quad (0.663)$$

Les *problèmes réduits* de (0.659) sont donnés par les équations lentes (0.661) ou (0.662) auxquelles on adjoint les conditions initiales (n_{10}, n_{20}) . Nous les désignons par (PR_1) et



(PR_2). Nous pouvons dès à présent appliquer le théorème de Tikhonov en affirmant principalement que si (s_0, i_0, r_0) est dans le bassin d'attraction de V_i , $i = 1, 2$, et si $T > 0$ se trouve dans l'intervalle positif de la solution $(\bar{n}_1(t), \bar{n}_2(t))$ de (PR_i), alors, pour ϵ assez petit, la solution $(s_\epsilon, i_\epsilon, r_\epsilon, n_{1\epsilon}, n_{2\epsilon})$ de (0.659) est définie au moins sur $[0, T]$ et converge quand ϵ tend vers zéro, pour $t \in]0, T]$, vers $(\phi_i(\bar{n}_1(t), \bar{n}_2(t)), \bar{n}_1(t), \bar{n}_2(t))$. Nous voudrions cependant obtenir des approximations pour des temps arbitrairement grands. Il suffit de choisir, pour les équations lentes, des compacts du plan qui soient positivement invariants, quitte à faire quelques restrictions sur les conditions initiales. Considérons les sous-ensembles compacts

$$U_1 = \{(n_1, n_2) \in \mathbb{R}_+^2 : n_1 \leq \frac{\mu_2 \bar{\nu}}{a_1} ; n_2 \leq \bar{\nu}\}$$

$$U_2 = \{(n_1, n_2) \in \mathbb{R}_+^2 : k_1 \geq n_1 > \frac{\rho_1 \nu - \rho_2}{a_1} ; k_2 \geq n_2 \geq \bar{\nu}\}$$

où k_1 et k_2 sont deux réels arbitrairement grands et $\bar{\nu} < \nu < \tilde{\nu}$.

Lemma 0.77.

1. Si $(0, 0)$ est asymptotiquement stable pour (0.657), alors U_1 est positivement invariant pour le champs de (0.660).
2. Si $(0, 0)$ est point selle pour (0.657), alors U_2 est positivement invariant pour le champs de (0.661).

Proof La preuve consiste à étudier la direction des champs de vecteurs pour les systèmes lents. ■

Theorem 0.227. 1- Supposons que $(n_{10}, n_{20}) \in U_1$, que $(0, 0)$ est asymptotiquement stable pour le système (0.657) et que (s_0, i_0, r_0) est dans le bassin d'attraction de V_1 . Soit $T > 0$ arbitrairement grand. Alors, pour ϵ assez petit, la solution $(s_\epsilon, i_\epsilon, r_\epsilon, n_{1\epsilon}, n_{2\epsilon})$ de (0.659) est définie au moins sur $[0, T]$ et vérifie :

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} s_\epsilon(t) &= \bar{n}_2(t) & \forall t \in]0, T], \\ \lim_{\epsilon \rightarrow 0} e_\epsilon(t) &= 0 & \forall t \in]0, T], \\ \lim_{\epsilon \rightarrow 0} i_\epsilon(t) &= 0 & \forall t \in]0, T], \\ \lim_{\epsilon \rightarrow 0} r_\epsilon(t) &= 0 & \forall t \in]0, T], \\ \lim_{\epsilon \rightarrow 0} n_{1\epsilon}(t) &= \bar{n}_1(t) & \forall t \in [0, T], \\ \lim_{\epsilon \rightarrow 0} n_{2\epsilon}(t) &= \bar{n}_2(t) & \forall t \in [0, T], \end{aligned}$$

où $(\bar{n}_1(t), \bar{n}_2(t))$ est la solution du problème réduit (PR_1).

2- Supposons que $(n_{10}, n_{20}) \in U_2$, que $(0, 0)$ est instable pour le système (0.657) et que (s_0, i_0, r_0) est dans le bassin d'attraction de V_2 . Supposons que l'équilibre $(\bar{n}_1^*, \bar{n}_2^*)$ de (PR_2) se trouve à l'intérieur de U_2 . Alors, pour ϵ assez petit, la solution $(s_\epsilon, i_\epsilon, r_\epsilon, n_{1\epsilon}, n_{2\epsilon})$ de (0.659) est définie pour tout $t \geq 0$ et vérifie :

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} s_\epsilon(t) &= \nu & \forall t > 0, \\ \lim_{\epsilon \rightarrow 0} e_\epsilon(t) &= \frac{\gamma \alpha (\bar{n}_2(t) - \nu)}{\delta \alpha + \delta \gamma + \gamma \alpha} & \forall t > 0, \\ \lim_{\epsilon \rightarrow 0} i_\epsilon(t) &= \frac{\gamma \delta (\bar{n}_2(t) - \nu)}{\delta \alpha + \delta \gamma + \gamma \alpha} & \forall t > 0, \\ \lim_{\epsilon \rightarrow 0} r_\epsilon(t) &= \frac{\alpha \delta (\bar{n}_2(t) - \nu)}{\delta \alpha + \delta \gamma + \gamma \alpha} & \forall t > 0, \\ \lim_{\epsilon \rightarrow 0} n_{1\epsilon}(t) &= \bar{n}_1(t) & \forall t \geq 0, \\ \lim_{\epsilon \rightarrow 0} n_{2\epsilon}(t) &= \bar{n}_2(t) & \forall t \geq 0, \end{aligned}$$

où $(\bar{n}_1(t), \bar{n}_2(t))$ est la solution du problème réduit (PR_2).



Proof La première partie découle directement de l'application du théorème de Tikhonov sur un intervalle de temps borné, avec cette particularité que T est arbitrairement grand. La seconde partie découle de l'extension du Théorème de Tikhonov aux intervalles de temps non bornés, sachant que le problème réduit admet un équilibre asymptotiquement stable. ■

Conclusion

Nous avons prouvé, à l'aide de la théorie de Tikhonov, que si la population est stable et que les deux conditions initiales ne sont pas trop grandes, alors, pour des maladies de courte durée, la population totale se comporte, sur des intervalles de temps arbitraire, comme s'il n'y avait pas de maladie du tout.

En revanche, dans le cas du second théorème, la maladie persiste dans la population. Ce cas peut aussi faire l'objet d'une approximation pour tout $t > 0$, quand le modèle réduit (0.662) admet un équilibre asymptotiquement stable.

Susceptiblesn2lessnu.png

Figure 41: La précision de l'approximation pour différentes valeurs de ϵ ($(n_2 < \nu)$)

Sous les hypothèses du *théorème 2.1* pour $n_2 < \nu$ la figure 1 montre la convergence de solution "susceptibles" du problème 0.659 vers n_2 la solution du PR_1 pour un temps T arbitrairement



grand.

La figure 2 montre la convergence de solution "susceptibles" du problème 0.659 vers ν la solution du PR_2 pour un temps T arbitrairement grand.

Remarquons aussi que la précision d'approximation augmente lorsque le epsilon proche de 0.



Figure 42: La précision de l'approximation pour différentes valeurs de ϵ ($(n_2 > \nu)$)

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On ternary fuzzy preorder

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Abstract. In this talk, we examine ternary preorder L -relations, where L is a residuated lattice. We provide a characterization of the three types of transitivity found in ternary relations and discuss their characteristics. Additionally, we present necessary and sufficient conditions for a ternary relation T to be considered a preorder, focusing on the key properties of reflexivity and transitivity within this algebraic framework.

Keywords: Ternary relation, Relational composition, Associativity, B-K compositions, traces

Introduction

The study of relations is a cornerstone of mathematics with broad applications in areas such as logic, computer science, artificial intelligence, and algebra. Although relational theory has roots dating back to the 19th century, it wasn't until Augustus De Morgan's work in 1860 that relations were formalized as distinct mathematical objects. Building on this, Charles Sanders Peirce advanced the calculus of binary relations, a foundational tool in set theory and



logic, followed by Schröder's contributions in the late 19th century. Despite these early developments, the study of higher-order relations, particularly ternary and beyond, has remained comparatively underexplored.

Ternary relations first gained attention with the study of the relation of betweenness. In the 20th century, researchers such as Novák and Novotný advanced the theory of ternary relations, revealing their utility across various scientific disciplines, from theoretical physics to computer science and artificial intelligence. Applications include knowledge graphs, data structures, and even social network analysis. Despite this, the formal exploration of the compositions of ternary relations has lagged behind the extensive work done on binary relations.

One of the most crucial areas in relational theory is the composition of relations. While binary relation compositions have been studied extensively, ternary and higher-order compositions have not received the same focus. In the binary case, significant progress was made when Bandler and Kohout introduced the concept of BK-compositions, where logical implications replace the more traditional existential quantifiers and logical conjunctions. This novel approach expanded the use of compositions in practical applications, such as medical diagnosis and decision-making.

However, the composition of ternary relations remains an open problem in relational theory. My work seeks to address this by providing a systematic approach to the compositions of ternary relations. In this thesis, I propose a general definition of ternary compositions that includes four-point and five-point compositions. I will explore their properties, such as associativity, mixed associativity, mixed commutativity, and the existence of neutral elements, which are critical for understanding their algebraic structure.

Additionally, I utilize tools such as the projections of ternary relations and the cylindrical extensions of binary relations to express these new ternary compositions in terms of the well-known binary ones. Interestingly, I found a strong connection between the traces of binary relations and their BK-compositions, which led me to introduce the notion of traces of ternary relations based on BK-compositions. This link between binary and ternary compositions offers a deeper insight into how higher-order relations can be structured and manipulated.

In summary, this research aims to fill a gap in the study of ternary relations, particularly their compositions, by building on the existing knowledge of binary relations and extending it to higher-order cases. By formalizing these concepts and exploring their algebraic properties, we open new pathways for understanding complex relational structures in various scientific fields. I believe this work not only advances the theory of relations but also provides new tools for practical applications in areas like artificial intelligence, data science, and beyond.

Main results

First, we introduce Bandler–Kohout-like compositions of ternary relation, we show their properties and use these compositions to introduce the notion of the traces of ternary relations. First, we introduce the BK-compositions of ternary relations and show their basic properties.

Definition 0.108. Let $p \in \{0, \dots, 215\}$ with $p = (kji)_6$. For any two ternary relations S and T on X , the sub-composition ST , super-composition ST and square-composition ST are defined as:

$$\begin{aligned}
 S_p^0 T &= \{\sigma_k(x, y, z) \in X^3 \mid (\forall t \in X)(\sigma_i(x, y, t) \in S \Rightarrow \sigma_j(x, t, z) \in T)\}; \\
 S_p^0 T &= \{\sigma_k(x, y, z) \in X^3 \mid (\forall t \in X)(\sigma_i(x, y, t) \in S \Leftarrow \sigma_j(x, t, z) \in T)\}; \\
 S_p^0 T &= \{\sigma_k(x, y, z) \in X^3 \mid (\forall t \in X)(\sigma_i(x, y, t) \in S \Leftrightarrow \sigma_j(x, t, z) \in T)\}; \\
 S_p^1 T &= \{\sigma_k(x, y, z) \in X^3 \mid (\forall t \in X)(\sigma_i(x, y, t) \in S \Rightarrow \sigma_j(z, t, z) \in T)\}; \\
 S_p^1 T &= \{\sigma_k(x, y, z) \in X^3 \mid (\forall t \in X)(\sigma_i(x, y, t) \in S \Leftarrow \sigma_j(z, t, z) \in T)\}; \\
 S_p^1 T &= \{\sigma_k(x, y, z) \in X^3 \mid (\forall t \in X)(\sigma_i(x, y, t) \in S \Leftrightarrow \sigma_j(z, t, z) \in T)\}; \\
 S_p^2 T &= \{\sigma_k(x, y, z) \in X^3 \mid (\forall t \in X)(\sigma_i(y, y, t) \in S \Rightarrow \sigma_j(x, t, z) \in T)\}; \\
 S_p^2 T &= \{\sigma_k(x, y, z) \in X^3 \mid (\forall t \in X)(\sigma_i(y, y, t) \in S \Leftarrow \sigma_j(x, t, z) \in T)\}; \\
 S_p^2 T &= \{\sigma_k(x, y, z) \in X^3 \mid (\forall t \in X)(\sigma_i(y, y, t) \in S \Leftrightarrow \sigma_j(x, t, z) \in T)\}.
 \end{aligned}$$



The following example illustrate the composition $S_0^0 T$ for two simple ternary relations S and T .

Example 0.12. Let S and T be the ternary relations on the set $X = \{x_1, x_2, x_3, x_4\}$ given by

$$S = \{(x_3, x_3, x_1), (x_3, x_2, x_3)\},$$

$$T = \{(x_3, x_1, x_3), (x_3, x_1, x_2), (x_1, x_1, x_1)\}.$$

We have $S_0^0 T = \{(x_1, x_1, x_3), (x_2, x_2, x_1), (x_1, x_3, x_1), (x_2, x_1, x_1), (x_1, x_2, x_2), (x_2, x_3, x_2), (x_1, x_3, x_2), (x_1, x_1, x_2), (x_2, x_1, x_3), (x_1, x_2, x_1), (x_2, x_2, x_2), (x_1, x_1, x_1), (x_2, x_1, x_1), (x_2, x_2, x_3)\}$

These compositions can be expressed in terms of three representative ones, just as in the case of the composition of binary relations.

Proposition 0.36. Let $p = (kji)_6 \in \{0, \dots, 215\}$ and $r \in \{0, 1, 2\}$. For any two ternary relations S and T on X , the compositions $S_p^r T$, $S_p^r T$ and $S_p^r T$ can be written in terms of the compositions $S_0^0 T$, $S_0^0 T$ and $S_0^0 T$ as:

1. $S_p^r T = (S^{\sigma_i^{-1}} T^{\sigma_j^{-1}})^{\sigma_k};$
2. $S_p^r T = (S^{\sigma_i^{-1}} T^{\sigma_j^{-1}})^{\sigma_k};$
3. $S_p^r T = (S^{\sigma_i^{-1}} T^{\sigma_j^{-1}})^{\sigma_k}.$

Here, we study the properties of the newly introduced compositions.

The following proposition concerns the refinement of the BK-compositions of ternary relations.

Proposition 0.37. Let $p \in \{0, \dots, 215\}$ and $r \in \{0, 1, 2\}$. For any two ternary relations S and T on X , the following inclusions hold:

1. $S_p^r T \subseteq S_p^r T \subseteq S_p^r T;$
2. $S_p^r T \subseteq S_p^r T \subseteq S_p^r T.$

Although the compositions, and are not commutative in general, interesting mixed-commutativity properties can be established.

Proposition 0.38. Let $p = (kji)_6 \in \{0, \dots, 215\}$. For any two ternary relations S and T on X , the following equalities hold:

1. $S_p^0 T = T_{p'}^0 S;$
2. $S_p^1 T = T_{p'}^2 S;$
3. $S_p^0 T = T_{p'}^0 S;$
4. $S_p^1 T = T_{p'}^2 S;$

with $p' = (\pi(k)\pi(i)\pi(j))_6$ and π the permutation of $\{0, \dots, 5\}$ given by Table 12:

Table 12: The permutation π .

u	0	1	2	3	4	5
$\pi(u)$	1	0	4	5	2	3

and their BK-compositions We show some interactions between the 4-point compositions and the BK-compositions of ternary relations.

The following proposition shows a mixed-associativity property of BK-compositions.



Proposition 0.39. Let $p \in \{0, 43, 86, 129, 172, 215\}$. For any three ternary relations R , S , and T on X , the following equality holds:

$$R_p^0(S_p^0 T) = (R_p^0 S)_p^0 T.$$

The following example illustrates that the above property does not hold for the compositions r_p and r_p with $r \in \{1, 2\}$.

Example 0.13. Consider the ternary relations R , S and T on the set $X = \{x_1, x_2, x_3, x_4, x_5\}$ given as:

$$\begin{aligned} R &= \{(x_1, x_2, x_3)\}, \\ S &= \{(x_4, x_3, x_5)\}, \\ T &= \{(x_4, x_5, x_4)\}. \end{aligned}$$

It is clear that $R_0^1(S_0^1 T) = \{(x_1, x_2, x_4)\} \neq (R_0^1 S)_0^1 T = \emptyset$.

Proposition 0.40. Let $p \in \{0, 43, 86, 129, 172, 215\}$. For any three ternary relations R , S , and T on X , the following equalities hold:

1. $R_p^0(S_p^0 T) = (R_p^0 S)_p^0 T$;
2. $R_p^0(S_p^0 T) = (R_p^0 S)_p^0 T$.

Now, we show that the BK-compositions of ternary relations are a valuable tool to express the greatest solutions of relational equations. The following proposition expresses an adjointness-like relationship [?] between the different types of compositions of ternary relations.

Proposition 0.41. For any three ternary relations R , S , and T on X , the following equivalences hold:

$$R_0^0 S \subseteq T \iff R \subseteq T_0^0 S^{\sigma_1} \iff S \subseteq R^{\sigma_1}_0 T.$$

The following proposition extends the previous result to account for all $p \in \{0, \dots, 215\}$.

Proposition 0.42. Let $p = (kji)_6 \in \{0, \dots, 215\}$. For any three ternary relations R , S , and T on X , the following equivalences hold:

$$R_p^0 S \subseteq T \iff R \subseteq T_{p'}^0 S \iff S \subseteq R_{p''}^0 T,$$

with $p' = (i\pi(j)k)$ and $p'' = (jk\pi(i))$.

The following proposition concerns the greatest solution of the most basic ternary relational equation.

Proposition 0.43. Let S and T be two ternary relations on X . The following statements hold:

1. The relational equation $S_0^0 U = T$ in the unknown relation U is solvable if and only if $S^{\sigma_1}_0 T$ is its greatest solution.
2. The relational equation $U_0^0 S = T$ in the unknown relation U is solvable if and only if $T_0^0 S^{\sigma_1}$ is its greatest solution.

Proposition 0.44. Let $p \in \{0, \dots, 215\}$. For any two ternary relations S and T on X , the following statements hold, where p' and p'' are as in Proposition 0.42:

1. The relational equation $S_p^0 U = T$ in the unknown relation U is solvable if and only if $T_{p'}^0 S^{\sigma_1}$ is its greatest solution.
2. The relational equation $U_p^0 S = T$ in the unknown relation U is solvable if and only if $S^{\sigma_1}_{p''} T$ is its greatest solution.



Now, we introduce the traces of ternary relation based on the BK-compositions of ternary relations. Similarly as for binary relations, we can introduce the traces of a ternary relation T as the BK-compositions of T with itself.

Definition 0.109. Let $p = (kji)_6 \in \{0, \dots, 215\}$ and $r \in \{0, 1, 2\}$. The traces of a ternary relation T on X are the ternary relations defined as:

$$T_p^r := T_p^r T = T_{p'}^{r'} T,$$

where r' and p' are as in Proposition 0.38.

Example 0.14. The following example gives the relation T_p^r where T is given as in Example 0.12.

We have $T_p^r = \{(x_3, x_2, x_3), (x_3, x_3, x_3), (x_1, x_1, x_2), (x_1, x_2, x_1), (x_1, x_3, x_1), (x_3, x_1, x_2), (x_1, x_1, x_3), (x_3, x_2, x_2), (x_3, x_1, x_1), (x_3, x_3, x_2), (x_3, x_1, x_3), (x_1, x_1, x_1)\}$

Obviously, these traces are not all distinct. Among them, we identify those that preserve the properties in Theorems ??-??.

Proposition 0.45. Let T be a ternary relation on X .

1. The following traces are left-reflexive and left-transitive:
 - (a) $T_1^0 = T_6^0 = T_{16}^0 = T_{23}^0 = T_{26}^0 = T_{33}^0$;
 - (b) $T_{37}^0 = T_{42}^0 = T_{52}^0 = T_{59}^0 = T_{62}^0 = T_{69}^0$.
2. The following traces are middle-reflexive and middle-transitive:
 - (a) $T_{73}^0 = T_{78}^0 = T_{88}^0 = T_{95}^0 = T_{98}^0 = T_{105}^0$;
 - (b) $T_{109}^0 = T_{114}^0 = T_{124}^0 = T_{131}^0 = T_{134}^0 = T_{141}^0$.
3. The following traces are right-reflexive and right-transitive:
 - (a) $T_{145}^0 = T_{150}^0 = T_{160}^0 = T_{167}^0 = T_{170}^0 = T_{177}^0$;
 - (b) $T_{181}^0 = T_{186}^0 = T_{196}^0 = T_{203}^0 = T_{206}^0 = T_{213}^0$.

The following proposition expresses the link between the traces in statements (a) and (b) of Proposition 0.45.

Proposition 0.46. Let T be a ternary relation on X .

1. $T_{37}^0 = (T_1^0)^{\sigma_1}$;
2. $T_{109}^0 = (T_{73}^0)^{\sigma_1}$;
3. $T_{181}^0 = (T_{145}^0)^{\sigma_1}$.

Similarly as in the binary setting, we select three of the traces that will play a specific role in what follows, namely the left trace T^ℓ , the middle trace T^m and the right trace T^r :

1. $T^\ell := T_1^0 = \{(x, y, z) \in X^3 \mid (\forall a \in X)((x, a, y) \in T \Rightarrow (x, a, z) \in T)\}$;
2. $T^m := T_{73}^0 = \{(x, y, z) \in X^3 \mid (\forall a \in X)((x, y, a) \in T \Rightarrow (z, y, a) \in T)\}$;
3. $T^r := T_{145}^0 = \{(x, y, z) \in X^3 \mid (\forall a \in X)((y, a, z) \in T \Rightarrow (x, a, z) \in T)\}$.

The following proposition expresses that some of these traces coincide in the face of symmetry.

Proposition 0.47. For any ternary relation T on X , the following statements hold:

1. if T is left-symmetric, then $T^\ell = T^m$;



2. if T is middle-symmetric, then $T^r = T^\ell$;
3. if T is left-symmetric, then $T^m = T^r$.

The following theorem shows that the above-defined traces are solutions to relational equations. It extends Theorem ?? to the ternary setting.

Theorem 0.228. *For any ternary relation T on X , the following equalities hold:*

1. $T = T_0 T^\ell = T_{43}^\ell T$;
2. $T = T_{86} T^m = T_{129}^m T$;
3. $T = T_{172} T^r = T_{215}^r T$.

Now, we characterize some properties of a ternary relation in terms of its traces. The first proposition characterizes different reflexivity properties and extends Theorem ?? to the ternary setting.

Proposition 0.48. *For any ternary relation T on X , the following equivalences hold:*

1. T is left-reflexive $\iff T^\ell \subseteq T$;
2. T is middle-reflexive $\iff T^m \subseteq T$;
3. T is right-reflexive $\iff T^r \subseteq T$.

The following proposition characterizes different transitivity properties of a ternary relation in terms of its traces. This result extends Theorem ?? to the ternary setting.

Proposition 0.49. *For any ternary relation T on X , the following equivalences hold:*

1. T is left-transitive $\iff T \subseteq T^\ell$;
2. T is middle-transitive $\iff T \subseteq T^m$;
3. T is right-transitive $\iff T \subseteq T^r$.

The following corollary follows immediately from Propositions 0.48 and 0.49.

Corollary 0.8. *For any ternary relation T on X , the following equivalences hold:*

1. T is left-reflexive and left-transitive $\iff T^\ell = T$;
2. T is middle-reflexive and middle-transitive $\iff T^m = T$;
3. T is right-reflexive and right-transitive $\iff T^r = T$.

Now, we consider formal ternary counterparts of the concept of a binary equivalence relation, *i.e.*, a binary relation that is reflexive, symmetric, and transitive. As each of these properties comes in many flavors, we aim to avoid a combinatorial explosion and select relevant combinations only.

Definition 0.110. A ternary relation T on X is said to be:

1. a left ternary equivalence relation if it is left-reflexive, left-symmetric and left-transitive;
2. a middle ternary equivalence relation if it is middle-reflexive, middle-symmetric and middle-transitive;
3. a right ternary equivalence relation if it is right-reflexive, right-symmetric and right-transitive.

The following definition and proposition provide a setting in which ternary equivalence relations naturally arise.

Definition 0.111. With any partition $\mathcal{P}$ of X^2 , we associate the ternary relations T_ℓ , T_m and T_r on X defined as:



1. $T_\ell = \{(x, y, z) \in X^3 \mid (\exists C \in \mathcal{P})((x, y) \in C \wedge (x, z) \in C)\};$
2. $T_m = \{(x, y, z) \in X^3 \mid (\exists C \in \mathcal{P})((x, y) \in C \wedge (y, z) \in C)\};$
3. $T_r = \{(x, y, z) \in X^3 \mid (\exists C \in \mathcal{P})((x, z) \in C \wedge (y, z) \in C)\}.$

Proposition 0.50. *For any partition $\mathcal{P}$ of X^2 , the following statements hold:*

1. T_ℓ is a left ternary equivalence relation;
2. T_m is a middle ternary equivalence relation;
3. T_r is a right ternary equivalence relation.

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The asymptotic properties for a conditional class of the kernel estimators

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Abstract. This study examines the asymptotic properties of the Nadaraya-Watson kernel estimator, specifically focusing on the conditional density estimator for independent and identically distributed (i.i.d.) data. The key aim is to demonstrate the almost complete convergence of the conditional density estimator, offering a thorough theoretical framework for its analysis. To reinforce these theoretical insights, an extensive simulation study is conducted, showcasing the estimator's practical performance and utility.

Keywords: Nadaraya-Watson estimator, Conditional density estimation, Almost complete convergence, Kernel estimator.



Introduction

The Nadaraya-Watson kernel estimator is a fundamental nonparametric technique widely used for estimating conditional densities. Introduced by [1] Nadaraya (1964) and Watson (1964), this method allows for flexible modeling of data without assuming a specific parametric form. It estimates the conditional density by weighting observations based on their proximity to a target point, effectively smoothing the data.

Significant contributions to the understanding of this estimator include the works of [2] Rosenblatt (1971), who established the foundational concepts of conditional density estimation, and [3] Fan and Gijbels (1996), who provided insights into its asymptotic properties. Recent studies, such as [1] Ferraty and Vieu (2010), have further refined the approach, addressing issues like consistency for functional data.

Despite extensive research, the asymptotic properties of the Nadaraya-Watson kernel estimator, particularly for independent and identically distributed (i.i.d.) data, remain an important area of investigation. In this paper, we aim to explore these properties, focusing on the almost complete convergence of the estimator and supporting our theoretical findings with a comprehensive simulation study.

Main results

Definition 0.112. The conditional density kernel estimator is defined by:

$$\hat{f}_n^x(y) = \frac{\sum_{i=1}^n K\left(\frac{x - X_i}{h}\right) g^{-1} K_0\left(\frac{y - Y_i}{g}\right)}{\sum_{i=1}^n K\left(\frac{x - X_i}{h}\right)}$$

h and g are sequences of positive decreasing real numbers.
 K and K_0 are kernels.

Assumptions

H1: K is a type 1 kernel and $K_0 = H'$ is of type zero;

H2: $\lim_{n \rightarrow \infty} h = 0$, $\lim_{n \rightarrow \infty} g = 0$, $\lim_{n \rightarrow \infty} \frac{\log n}{n g h} = 0$ and $\exists \xi \geq 0$, $\lim_{n \rightarrow \infty} g n^\xi = \infty$;

H3: $\exists C < \infty$, $\forall (x, x_0) \in \mathbb{R} \times \mathbb{R}$, $|K'(x) - K'(x_0)| \leq C|x - x_0|$;

H4: $\lim_{|x' - x| \rightarrow 0} f^{x'}(y) = f^x(y)$ uniformly on $y \in S$
 $\forall y \in \mathbb{R}$, $\lim_{|y' - y| \rightarrow 0} f^x(y') = f^x(y)$;

H5: $\exists \epsilon > 0$, $\exists ! y_0 \in S$ such that $y \mapsto f^x(y)$ is strictly increasing on $]y_0 - \epsilon, y_0[$ and strictly decreasing on $]y_0, y_0 + \epsilon[$;

Theorem 0.229. Under the assumptions H1 – H5, we have

$$\lim_{n \rightarrow \infty} \sup_{y \in S} |\hat{f}_n^x(y) - f^x(y)| = 0 \quad p.co.$$





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M/M/1 Retrial Queuing System with Negative arrivals and two types Breakdowns

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Abstract. This paper analyzes a single repairable server retrial queuing system with negative arrivals and breakdowns. The server is subject to passive and active Poisson breakdowns. Passive breakdowns can occur when the server is free while active breakdowns can occur when the server is busy. When an active breakdown occurs, the customer in service is lost, after repair, the server becomes free. A negative arrival has the effect of eliminating a customer from the orbit. If the server is free, an arriving positive customer is served immediately and leaves the system after service completion (if no breakdowns had occurred during his service time). Otherwise, the positive customer joins the orbit and becomes a repeated customer. Order to derive the system performance measures the matrix geometric technique is used combined with direct truncation method. From the reliability view point, we analyze the mean time to the first failure of the system by using the Laplace transform method. Finally, we investigate the effects of some parameters, such as the arrival rate, the repair rate, and the breakdown rates, on the system performance measures and the server reliability (the mean time to the first failure). We present numerical results to illustrate these effects.



Keywords: Negative arrivals, Reliability, Active breakdown, Passive breakdown.

Introduction

The theory of queuing systems with retrials is a crucial component of modern teletraffic theory, providing key insights into the behavior of complex service systems. In a retrial queue, when a customer arrives to find the server busy, they join a retrial group (or orbit) and attempt to access the server again after a random interval. These models are highly applicable to real-world scenarios in telecommunications and computing, particularly in situations where service disruptions or delays lead to retry attempts, such as in telephone switching systems and computer networks.

The fundamental aspects of retrial queues are thoroughly discussed in the monographs by Artalejo and Gomez-Corral (2008) and Falin and Templeton (1997), which delve into the theoretical frameworks and analytical techniques for modeling these systems. A more recent survey by Kim and Kim (2016) further explores the latest advancements and practical applications of retrial queueing theory.

This study focuses on a single-server retrial queueing system with repairable components, incorporating negative arrivals and two types of server breakdowns: passive and active. The model extends the work of Gelenbe (1991), who introduced the concept of queueing systems with both positive and negative arrivals. In this context, positive arrivals represent customers seeking service, while negative arrivals remove customers from the system. This framework is particularly relevant for practical applications such as distributed computing systems or databases, where negative arrivals can represent commands to cancel transactions.

Negative arrivals have been extensively studied in the literature, with significant contributions from Anisimov and Artalejo (2001), Artalejo and Economou (2004), Atencia and Moreno (2005), Muthu Canapathi Subramanion et al. (2009), and Ayyappan et al. (2010, 2011).

In this paper, we investigate a single unreliable server retrial queue characterized by the presence of negative arrivals and two types of breakdowns. Positive customers are served by the server as expected, but if the server is busy, they join the orbit and become retrial customers. Negative arrivals, meanwhile, remove customers from the orbit regardless of the server's status. The server can also experience passive or active breakdowns, affecting the system's performance.

The paper is structured as follows: section* 2 outlines the model, section* 3 discusses the stability condition, section* 4 presents the steady-state distribution, section* 5 focuses on reliability analysis, and section* 6 provides numerical results.

Model Description

We consider a single unreliable server retrial queueing system with FCFS orbit and negative arrivals. Customers arrive according to a Poisson process with rate λ for positive arrivals and δ for negative arrivals. We assume that the service times are exponentially distributed with rate μ . If the server is free, the arriving positive customer is served immediately and leaves the system after service completion (if no breakdowns had occurred during his service time). Otherwise, the positive customer joins the orbit and becomes a repeated customer. The time intervals between each repeated attempts is assumed to be independent and exponentially



distributed with rate ν . The server is subject to two types of Poisson breakdowns (passive and active). Active (passive) breakdowns can occur when the server is busy (free) with rate $\theta_1(\theta_0)$. As soon as the server fails it is sent for repair. The repair times have an exponential distribution with rate μ_0 for passive breakdown and μ_1 for active breakdown.

Let $N(t)$ and $C(t)$ be the number of customers in orbit and the server state at time t respectively.

$$C(t) = \begin{cases} 0 & \text{if the server is idle at time } t, \\ 1 & \text{if the server is busy at time } t, \\ 2 & \text{if the server is under repair during an passive breakdown at time } t, \\ 3 & \text{if the server is under repair during an active breakdown at time } t. \end{cases}$$

The system state is described by the Markov process $(N(t), C(t)), t \geq 0$ whose state space is $(n, k)/n = 0, 1, 2, 3; k = 0, 1, 2, 3$.

Stability condition

The inequality

$$\frac{\lambda}{\mu} \left(\frac{(\lambda + \nu)(\mu_1 + \theta_1)}{\mu_1(\mu + \delta + \theta_1)} + \frac{\theta_0}{\mu_0} \right) < 1 \quad (0.664)$$

is the necessary and sufficient condition for the system stability.

Steady state distribution

We denote by P_{nk} the stationary probability of the state (n, k) , and $P_K(z) = \sum_{n \geq 0} z^n P_{nk}$

$$\begin{aligned} P_0(z) &= \frac{(\nu(\mu_1 + \lambda(1 - z))[\gamma(z) - (\mu + \delta)]\nu\mu_1\theta_1)(\mu_0 + \lambda(1 - z))}{\gamma(z)(\mu_1 + \lambda(1 - z))h(z) - (\lambda z + \nu)(\mu_0 + \lambda(1 - z))\beta(z)} P_{00}, \\ P_1(z) &= \frac{\lambda z + \nu}{\gamma(z)P_0(z) + \frac{\nu}{\gamma(z)}} P_{00}, \\ P_2(z) &= \frac{\theta_0}{(\mu_1 + \lambda(1 - z))\gamma(z)} P_0(z), \\ P_3(z) &= \frac{\theta_1(\lambda z + \nu)}{(\mu_1 + \lambda(1 - z))\gamma(z)} P_0(z) + \frac{\nu\theta_1}{(\mu_1 + \lambda(1 - z))\gamma(z)}, \\ P_{00} &= \left(\frac{\alpha\phi}{\mu_0\mu_1(\mu + \delta + \theta_1)\varphi} - \frac{\nu(\mu_1 + \theta_1)}{\mu_1(\mu\delta + \theta_1)} \right)^{-1} \end{aligned}$$

Where

$$h(z) = (\lambda + \nu + \theta_0)(\mu_0 + \lambda(1 - z)) - \mu_0\theta_0$$

$$\beta(z) = (\mu + \delta)(\mu_1 + \lambda(1 - z)) + \mu_1\theta_1$$

$$\alpha = \nu\mu_0(\mu_1(\mu + \delta + \theta_1 - \lambda) - \lambda\theta_1)$$

$$\varphi = \mu_0(\lambda + \nu)(\mu + \delta + \theta_1)(\mu_1 - \lambda) + \lambda\mu_0(\lambda + \nu)(\mu + \delta - \mu_1) - \lambda\mu_1(\mu + \delta + \theta_1)(\mu_0 + \theta_0)$$

From the Theorem 2, we can obtain after long and hard calculationS

(i) the server state probabilities $P_k = (\lim_{z \rightarrow 1} P_k(z)), k = 0, 3$,

(ii) the mean orbit size according to the server state $M_k = (\lim_{z \rightarrow 1} \frac{dP_K(z)}{dz}), k = 0, 3$,



(iii) the generating function of the number of customers in the system $G(z) = P_0(z) + zP_1(z) + P_2(z) + P_3(z)$ Therefore the mean orbit size is given by $M_K = P_3k = 0$ Mk and the mean number of customer in the system is given by $N = \frac{dG(z)}{dz}|_{z=1} = M + P_1(1)$

Reliability analysis

In this section* we focus on the time to the first failure of the server and its reliability. We assume that the system is empty at time $t = 0$. Denote ψ the time to the first failure of the server. The reliability of the server is defined as $R(t) = P(\psi > t)$. The Laplace transform of $R(t)$ is given by

$$R^*(s) = \frac{\nu(s + \theta_1)(s + \lambda + \theta_0 + \nu) + \nu(s + \theta_1)(\mu + \delta) - F(s, 1)}{(\mu + \delta)F(s, 1)} P_{00}^*(s) + \frac{((s + \mu + \delta + \theta_1)(\mu + \delta) + (\lambda + \nu)(s + \delta))}{(\mu + \delta)F(s, 1)}$$

where $f(s)$ is the root of the equation $F(s, z)$ inside $|z| = 1$, $Re(s) > 0$

Corollary 0.9. The mean time to the first failure $MTTF$ of the server is given by

$$MTTF = \frac{\nu(\theta_1)(\lambda + \theta_0 + \nu) + \nu(\theta_1)(\mu + \delta) - F(0, 1)}{(\mu + \delta)F(0, 1)} P_{00}^*(0) + \frac{((\mu + \delta + \theta_1)(\mu + \delta) + (\lambda + \nu)(\delta))}{(\mu + \delta)F(0, 1)} \quad (0.665)$$

Numerical Result

In this section*, we present numerical examples to illustrate how various system parameters affect performance measures. Specifically, we examine the influence of the negative arrival rate (δ) and the retrial rate (ν) on the mean number of customers in the orbit (MNCO) and the probabilities associated with the server's states.

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Model Uncertainty Stochastic Partial Differential Equations (SPDEs) Control and Application

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Abstract. In this paper, we consider the problem of optimal control of stochastic partial differential equations (SPDEs) under model uncertainty. We represent this model uncertainty control problem as a stochastic differential game of stochastic partial differential equations with two players. The control of one of the players, representing the uncertainty, is a measure-valued stochastic process $\Theta(t)$ and the control of the other player is a classical real-valued stochastic process $u(t)$. We prove a sufficient and necessary maximum principle for Nash equilibria for such games in the general nonzero-sum case and for saddle points in zero-sum games. Finally, we apply the results to solve a problem of linear quadratic games problem for stochastic partial differential equations

Keywords: SPDEs; Maximum Principle; zero-sum; nonzero-sum; model uncertainty.



Introduction

The game theory as a branch of mathematics has also received much attention in other areas of applied sciences. For example it has been proven useful in social sciences as an approach to model decision making of interacting individuals in certain social situations. Other applications of this theory pertain e.g. to the description of evolutionary processes in biology, the modelling of interactive computation or the design of fair division in political science. In this paper, We study optimal stochastic control problems under model uncertainty. Our paper is organised as follows: in section* 2, We represent this model uncertainty control problem as a stochastic differential game of stochastic partial differential equations with two players. Then in section* 3, we state general stochastic maximum principles for stochastic differential games, both in the nonzero-sum case and for the zero-sum case .

A SPDE maximum principle for stochastic differential games

We first give a general formulation of the stochastic control problem we consider. Suppose that the state $Y(t, x) = Y^{(u, \theta)}(t, x)$ at (t, x) is described by a stochastic partial differential equation of the form

$$\begin{aligned} dY(t, x) = & \left[L_{(u, \theta)(t, x)} Y(t, x) + b(t, x, Y(t, x), u(t, x), \theta(t, x)) \right] dt \\ & + \sigma(t, x, Y(t, x), u(t, x), \theta(t, x)) dB(t), \quad (t, x) \in (0, T) \times D \end{aligned} \quad (0.666)$$

with boundary conditions

$$Y(0, x) = \xi(x); \quad x \in D \quad (0.667)$$

$$Y(t, x) = \eta(t, x); \quad (t, x) \in [0, T] \times \partial D \quad (0.668)$$

Nonzero-sum game

In this section*, we formulate and prove a sufficient and a necessary maximum principle for stochastic differential games in the framework of SPDE control.

[Nonzero-sum game] We consider the general nonzero-sum stochastic game to find $(u^*, \theta^*) \in \mathcal{A}_1 \times \mathcal{A}_2$, such that

$$J_1(u, \theta^*) \leq J_1(u^*, \theta^*), \quad \text{for all } u \in \mathcal{A}_1 \quad (0.669)$$

and

$$J_2(u^*, \theta) \leq J_2(u^*, \theta^*), \quad \text{for all } \theta \in \mathcal{A}_2 \quad (0.670)$$

The pair (u^*, θ^*) is called a *Nash equilibrium*.

The cost functionals are assumed to be on the form

$$\begin{aligned} J_i(u, \theta) = & E \left[\int_0^T \left(\int_D f_i(t, x, Y(t, x), u(t, x), \theta(t, x)) dx \right) dt \right. \\ & \left. + \int_D g_i(x, Y(T, x)) dx \right]; \text{ for } i = 1, 2. \end{aligned} \quad (0.671)$$

Definition 0.113. (The Hamiltonian) For $i = 1, 2$ we define the Hamiltonian

$$H_i : \mathbb{R} \times \mathbb{R} \times \mathcal{U} \times \mathcal{U} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$$

by

$$\begin{aligned} H_i(x, z, \theta, u, p_i, q_i) \\ = f_i(x, z, \theta, u) + p_i b(x, z, \theta, u) + q_i \sigma(x, z, \theta, u) \end{aligned} \quad (0.672)$$

The quantities p, q are called the *adjoint variables*. We define the *adjoint processes* $p_i(t, x), q_i(t, x)$ as the solution of the backward stochastic partial differential equation (BSPDE)

$$\begin{cases} dp_i(t, x) = -[L_{u, \theta(t, x)}^* p_i(t, x) + \frac{\partial H_i}{\partial y}(t, x)] dt + q_i(t, x) dB(t); & (t, x) \in (0, T) \times D \\ p_i(T, x) = \frac{\partial g_i}{\partial y}(x, Y(T, x)); & x \in D \\ p_i(t, x) = 0; & (t, x) \in [0, T] \times \partial D, \end{cases} \quad (0.673)$$



Zero-sum game

In the zero-sum case we have

$$J_1(u, \theta) + J_2(u, \theta) = 0.$$

Then the Nash equilibrium $(\hat{u}, \hat{\theta}) \in \mathcal{A}_1 \times \mathcal{A}_2$ satisfying (0.669)-(0.670) becomes a saddle point for $J(u, \theta) = J_1(u, \theta)$. To see this, note that (0.669)-(0.670) imply that

$$J_1(u, \hat{\theta}) \leq J_2(\hat{u}, \hat{\theta}) = -J_2(\hat{u}, \hat{\theta}) \leq -J_2(\hat{u}, \theta)$$

Since we always have $\inf \sup \geq \sup \inf$, we conclude that

$$\begin{aligned} \inf_{\theta \in \mathcal{A}_2} \sup_{u \in \mathcal{A}_1} J(u, \theta) &= \sup_{u \in \mathcal{A}_1} J(u, \hat{\theta}) = J(\hat{u}, \hat{\theta}) \\ &= \inf_{\theta \in \mathcal{A}_2} J(\hat{u}, \theta) = \sup_{u \in \mathcal{A}_1} \left(\inf_{\theta \in \mathcal{A}_2} J(u, \theta) \right). \end{aligned}$$

i.e. $(\hat{u}, \hat{\theta}) \in \mathcal{A}$ is a saddle point for $J(u, \theta)$.

Thus we want to find $(\hat{u}, \hat{\theta}) \in \mathcal{A}$ such that

$$\sup_{u \in \mathcal{A}_1} \left\{ \inf_{\theta \in \mathcal{A}_2} J(u, \theta) \right\} = \inf_{\theta \in \mathcal{A}_2} \left\{ \sup_{u \in \mathcal{A}_1} J(u, \theta) \right\} = J(\hat{u}, \hat{\theta}),$$

where

$$\begin{aligned} J(u, \theta) &= E \left[\int_0^T \left[\int_D f(t, x, Y(t, x), u(t, x), \theta(t, x)) dx \right] dt \right. \\ &\quad \left. + \int_D g(x, Y(T, x)) dx \right]. \end{aligned}$$

In this case we see that only one Hamiltonian H is needed, namely

$$\begin{aligned} H(t, x, y, \varphi, u, \theta, p, q) &= H(t, x, y, \varphi, u, \theta, p, q, \omega) \\ &= f(t, x, y, u, \theta) + [L_{u, \theta}(\varphi) + b(t, x, y, u, \theta)]p + \sigma(t, x, y, u, \theta)q, \end{aligned}$$

and we have put $g_i = g$; $i = 1, 2$ and $f_1 = f = -f_2$. Moreover, there is only one triple (p, q) of adjoint processes, given by the BSPDE

$$\begin{cases} dp(t, x) = -[L_{u, \theta}^*(t, x)p(t, x) + \frac{\partial H}{\partial y}(t, x)]dt + q(t, x)dB(t); & (t, x) \in (0, T) \times D \\ p(T, x) = \frac{\partial g}{\partial y}(x, Y(T, x)); & x \in D \\ p(t, x) = 0; & (t, x) \in [0, T] \times \partial D. \end{cases}$$

Theorem 0.230 (Sufficient zero-sum maximum principle). *Let $(\hat{\theta}, \hat{u}) \in \mathcal{A}_1 \times \mathcal{A}_2$ with corresponding solutions $\hat{Y}(t, x), \hat{p}(t, x), \hat{q}(t, x)$. Suppose the following hold:*

$$\begin{aligned} &y \rightarrow g(x, y) \text{ is concave for all } x \\ &(\varphi, u, \theta) \rightarrow H(t, x, \varphi(x), \varphi, u, \theta, \hat{p}(t, x, z), \hat{q}(t, x, z)) \\ &\text{is concave for all } t, x, \\ &\sup_{u \in \mathcal{A}_1} H(t, x, \hat{Y}(t, x), \hat{Y}(t, \cdot)(x), w, \hat{p}(t, x), \hat{q}(t, x)) \\ &= H(t, x, \hat{Y}(t, x), \hat{Y}(t, \cdot)(x), \hat{u}(t, x), \hat{\theta}(t, x), \hat{p}(t, x), \hat{q}(t, x)) \\ &\text{and } \inf_{\theta \in \mathcal{A}_2} H(t, x, \hat{Y}(t, x), \hat{Y}(t, \cdot)(x), w, \hat{p}(t, x), \hat{q}(t, x)) \\ &= H(t, x, \hat{Y}(t, x), \hat{Y}(t, \cdot)(x), \hat{u}(t, x), \hat{\theta}(t, x), \hat{p}(t, x), \hat{q}(t, x)). \end{aligned}$$

$\mathbb{P}$.a.s for each $t \in [0, T]$. Then $(\hat{u}, \hat{\theta})$ is a saddle point for $J(\theta, u)$.



Theorem 0.231 (Necessary maximum principle). *Let $(\hat{\theta}, \hat{u}) \in \mathcal{A}_1 \times \mathcal{A}_2$. Then the following are equivalent:*

- $$\frac{d}{da} J((u + a\beta_1, \theta))|_{a=0} = \frac{d}{da} J((u, \theta + a\beta_2))|_{a=0} = 0.$$
- for all bounded $\beta_1 \in \mathcal{A}_1, \beta_2 \in \mathcal{A}_2$.

$$\frac{\partial H}{\partial u}(t, x)_{u, \theta = \hat{u}, \hat{\theta}} = 0, \quad \text{for all } (t, x) \in [0, T] \times D. \quad (0.674)$$

Conclusion

In this work, we have considered the problem of optimal control of stochastic partial differential equations (SPDEs) under model uncertainty. We represented this model uncertainty control problem as a stochastic differential game of stochastic partial differential equations with two players. The control of one of the players, representing the uncertainty, is a measure-valued stochastic process $\theta(t)$ and the control of the other player is a classical real-valued stochastic process $u(t)$. We proved a sufficient and necessary maximum principle for Nash equilibria for such games in the general nonzero-sum case and for saddle points in zero-sum games.

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Asymptotic Normality of the Conditional Hazard Rate Function Estimator

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Abstract. In this paper, a smooth estimator of the conditional hazard function is studied in the censorship model when the data exhibit some dependence structure. It is shown, under some regularity conditions, that, when suitably normalized, the kernel estimator of the conditional hazard rate function is asymptotically normally distributed.

Keywords: Association, Asymptotic normality, Censored data, Conditional hazard rate function, Kaplan-Meier estimator.

Introduction

In many life testing studies, such as medical follow-ups and engineering, the variable of interest, known as lifetime, is not always fully observable. We consider survival times $T_1, \dots, T_n$ of n individuals, which are assumed to be strictly stationary and have a common unknown distribution function F . Due to censoring, we observe pairs (Y_i, δ_i) where $Y_i = \min(T_i, C_i)$ and $\delta_i = \mathbb{I}(T_i \leq C_i)$. Additionally, we include a stationary covariate X . The concept of association, introduced by Esary et al. (1967), indicates that lifetimes may be positively associated rather than independent. In this paper, the focus is on non-parametric estimation of the conditional hazard rate function using observed data. The aim is to establish the asymptotic normality of a kernel estimator for the conditional hazard rate function under the assumption of positive association among the variables. The paper is structured as follows: section* 2 outlines the notations and estimators, section* 3 presents the main results and assumptions, and section* 4 contains the proofs.



Main results

The estimator of the conditional hazard function is obtained by estimating the conditional density $f(\cdot|\cdot)$ and the conditional survival function $1-F(\cdot|\cdot)$. We consider the following smooth estimator of $F(\cdot|x)$:

$$\tilde{F}_n(t|x) = \frac{\sum_{i=1}^n \frac{\delta_i}{\bar{G}(Y_i)} K\left(\frac{x-X_i}{h}\right) H\left(\frac{t-Y_i}{h}\right)}{\sum_{i=1}^n K\left(\frac{x-X_i}{h}\right)},$$

where $\bar{G} = 1 - G$, K is a probability density function (called the kernel function), h is a sequence of positive real numbers (called bandwidth) tending to zero as n goes to infinity, and H is a distribution function.

Recall that (2.1) can be rewritten as: $\tilde{F}_n(t|x) = \frac{\tilde{F}_{1,n}(t,x)}{\ln(x)}$.

Therefore, the feasible estimator of the conditional distribution function $F(\cdot|x)$ is given by

$$F_n(t|x) = \frac{\sum_{i=1}^n \frac{\delta_i}{\bar{G}(Y_i)} K\left(\frac{x-X_i}{h}\right) H\left(\frac{t-Y_i}{h}\right)}{\sum_{i=1}^n K\left(\frac{x-X_i}{h}\right)},$$

The conditional density estimator is given by: $f_n(t|x) = \frac{F'_{1,n}(t,x)}{\ln(x)}$.

Then the natural estimator of the hazard rate function, $h(\cdot|x)$, is given by: $h_n(t|x) = \frac{f_n(t|x)}{1-F_n(t|x)}$.

The infeasible estimator $\tilde{h}_n$ is the same but it involves infeasible estimators $\tilde{f}_n$ and $1 - \tilde{F}_n$ of the conditional density and survival functions.

Let Ω be a compact set such that $\Omega \subset \Omega_0 = \{x \in \mathbb{R} \mid l(x) > 0\}$, and let $\mathcal{C}$ be a compact set included in $(-\infty, \tau_F)$ where τ is a positive real number such that $\tau < \tau_F$.

A1. The Kernel K satisfies:

K is strictly positive and bounded with compact support. Moreover, there exist constants M^* and m^* such that $0 < M^* < \infty$, $0 < m^* < \infty$, $\sup K(u) = M^*$, and $\inf K(u) = m^*$.

K is Lipschitzian and $\int K^2(u) du = \kappa < +\infty$.

$\int K(u) du = 0$, $\int |u|K(u) du < +\infty$ and $\int u^2 K(u) du < +\infty$.

A2. The bandwidth h satisfies:

$$\sqrt{\frac{\log \log n}{n}} = \mathcal{O}(h^3).$$

A4. The distribution function H has a first derivative H' which is positive and bounded such that:

There exist two constants $0 < M < \infty$ and $0 < m < \infty$ such that $\sup H'(t) = M$ and $\inf_{\mathbb{R}} H'(t) = m$.

$\int H'(t) dt = 1$ and $\int |t|H'(t) dt < +\infty$. and $\int (H'(s))^2 ds = \sigma < +\infty$.

A8. For the large p -blocks and small q -blocks, we suppose that: $\frac{pk}{n} \rightarrow 1$, $\frac{qk}{n} \rightarrow 0$, $\frac{p^2}{nh^2} \rightarrow 0$, as $n \rightarrow \infty$, $\frac{1}{h^4} \sum_{j=q}^{\infty} |\text{cov}(X_1, X_{j+1})| \rightarrow 0$.

Theorem 0.232. Under assumptions A1-A8, and for any $x \in \Omega_0$ such that $l(x) > 0$, and for n large enough, we have:

$$(nh^2)^{\frac{1}{2}} (h_n(t|x) - h(t|x)) \xrightarrow{D} N(0, \sigma^2(t, x)),$$

where $\xrightarrow{D}$ denotes convergence in distribution, and

$$\sigma^2(t, x) = \frac{1}{l^2(x)[1 - F(t|x)]^2} \left[\frac{\kappa \sigma F'_1(t, x)}{\bar{G}(t)} \right].$$



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Problem of Control for Stationary Marguerre-von Kármán Equations

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Abstract. The classical Marguerre-von kármán equations are two-dimensional equations for a nonlinearly elastic shallow shell subjected to boundary conditions analogous to those of a von kármán plate. They were initially proposed by Marguerre and von kármán and Tsien, and later justified by Ciarlet and Paumier by means of a formal asymptotic analysis. In this paper we prove the existence of an optimal control for the cost. Then, by proving the *Fréchet* differentiability of the state we establish necessary optimality condition for the optimal control corresponding to the cost. Further, we study the uniqueness of the optimal controls, referring our study to I.Bock and B.Rao.

Keywords: Non-linear elasticity; stationary Marguerre-von kármán's equations; optimal control.



Introduction

Let $\omega \in \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial\omega$. We consider the Von Kármán equations for a thin elastic plate by means the right hand side of the equilibrium equation is considered, where the second side of the first equation depends to $\lambda\Delta$ operator.

In this paper, our system may be described as follows: Let $\omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial\omega$. We consider the Marguerre-Von Kármán of the shalow shell model with Dirichlet boundary conditions

$$\begin{cases} \Delta^2 \zeta = [\varphi, \zeta] + [\varphi, \theta] + f & x \in \omega \\ \Delta^2 \varphi = -[\zeta, \zeta] - 2[\zeta, \theta] & x \in \omega \end{cases} \quad (0.675)$$

where

$$\begin{cases} \zeta = \frac{\partial \zeta}{\partial \nu} = 0 & x \in \partial\omega \\ \varphi = \varphi_0, \quad \frac{\partial \varphi}{\partial \nu} = \varphi_1 & x \in \partial\omega \end{cases} \quad (0.676)$$

Next, we use the reduced cubic form for the Marguerre-von Kármán equations, following the recent definitions with an unknown changes $\tilde{\zeta} = u = \zeta + \theta$ and $\tilde{F} = F + \theta$. Finally, we define the energy given by Ciarlet $\mathcal{I}(v) = \mathcal{J}(u(v)) + j(v)$ to study the optimal control. Moreover, in this paper, we discuss the existence and uniqueness of optimal control.

We show *Frèchet* differentiability of $\mathcal{J}$, j and of the state $x(\cdot)$. As a consequence, we prove uniqueness of optimal control, and necessary condition of optimality. This is another novelty of the paper.

Reduced Cubic Canonical Problem

Definition 0.114. The cubic operator is C . First, we let the bilinear and linear continuous operators are respectively B , and L_2, L_1

$$B : H^2(\omega) \times H^2(\omega) \longrightarrow H_0^2(\omega)$$

The canonical Marguerre-von Kármán equations are reduced to a cubic operator equation, such that the pair $(\tilde{\zeta}, \lambda) \in (H_0^2(\omega), \mathbb{R})$ satisfies the continuous operator equation:

$$\tilde{\zeta} - \lambda L_2(\tilde{\zeta}) - L_1(\tilde{\zeta}) + C(\tilde{\zeta}) - \tilde{F} = 0 \quad (0.677)$$

B be a bilinear operator, unique solution of $\Delta^2 B(\xi, \eta) = [\xi, \eta]$. Then define the operator $C : H_0^2(\omega) \longrightarrow H_0^2(\omega)$, written by $C(\eta) = B(B(\eta, \eta), \eta)$, and we define the linear operators $L_2, L_1 : H_0^2(\omega) \longrightarrow H_0^2(\omega)$ by letting $\lambda L_2(\eta) = B(\theta_0, \eta)$, and $L_1(\eta) = B(\chi, \eta) = B(B(\theta, \eta), \theta)$. Let $\chi \in H^2(\omega)$ is the unique solution of the boundary value problem

$$\begin{aligned} \Delta^2 \chi &= 0, \quad \chi = \varphi_0, \quad \frac{\partial \chi}{\partial \nu} = \varphi_1 \quad \text{on } \partial\omega, \\ \chi &= B(\theta, \theta), \\ \varphi &= \chi - [\tilde{\zeta}, \tilde{\zeta}]. \end{aligned}$$

The Variational Problem Equation

Definition 0.115. The canonical Marguerre-von Kármán equations are reduced to a cubic operator equation, such that the pair $(\tilde{\zeta}, \lambda) \in (H_0^2(\omega), \mathbb{R})$ satisfies the variational equation:

$$\begin{cases} \text{Find } (\tilde{\zeta}, \lambda) \in (H_0^2(\omega), \mathbb{R}) & \text{such that,} \\ (\tilde{\zeta} - \lambda L_2(\tilde{\zeta}) - L_1(\tilde{\zeta}) + C(\tilde{\zeta}) - \tilde{F}, \eta)_\Delta = 0 & \forall \eta \in H_0^2(\omega) \end{cases} \quad (0.678)$$

Then $\forall \eta \in H_0^2(\omega)$, $\tilde{\zeta}$ it is a weak solution of problem (0.675)



Spectral theory for operators: L_1 and L_2

Theorem 0.233. *The linear operators L_2 and L_1 are compact, self-adjoint and positive definite, then L_2 and L_1 have an infinite number of distinct positive eigenvalues, such that*

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \dots < \lambda_k < \dots \rightarrow +\infty \quad \text{for } L_2,$$

$$0 < \alpha_1 < \alpha_2 < \alpha_3 < \dots < \alpha_k < \dots \rightarrow +\infty \quad \text{for } L_1.$$

where

$$\lambda_1 = \inf_{u \in H_0^2(\omega), \|u\|_{2,\omega} \neq 0} \frac{\|u\|_{2,\omega}^2}{(L_2 u, u)_\Delta},$$

$$\mu_1 = \frac{1}{\lambda_1} = \sup_{u \in H_0^2(\omega), \|u\|_{2,\omega} = 1} (L_2 u, u)_\Delta,$$

$$\alpha_1 = \inf_{u \in H_0^2(\omega), \|u\|_{2,\omega} \neq 0} \frac{\|u\|_{2,\omega}^2}{(L_1 u, u)_\Delta},$$

$$\beta_1 = \frac{1}{\alpha_1} = \sup_{u \in H_0^2(\omega), \|u\|_{2,\omega} = 1} (L_1 u, u)_\Delta.$$

Existence and Uniqueness of a Solution of the State Problem

Theorem 0.234. *Let*

$$\|Mv + \theta\|_{2,\omega} < \|B\|^{-1}, \quad (0.679)$$

$$\lambda < \lambda^* (\|Mv + \theta\|_{2,\omega}). \quad (0.680)$$

Where

$$\lambda^* (\|Mv + \theta\|_{2,\omega}) = \frac{\lambda_1}{\alpha_1} + \lambda_1 \left(1 - \|B\|^{\frac{2}{3}} \|Mv + \theta\|_{2,\omega}^{\frac{2}{3}} \right).$$

Then there exists a unique solution $u(v) \in H_0^2(\omega)$ of the equation (0.677). Moreover $u(v)$ satisfies the estimate

$$\|u(v)\|_{2,\omega} \leq \tilde{K}_1(\lambda)^{-1} \|Mv + \theta\|_{2,\omega}, \quad (0.681)$$

where

$$\tilde{K}_1(\lambda) = 1 - \left(\frac{\lambda}{\lambda_1} + \frac{1}{\alpha_1} \right) \quad (0.682)$$

The Cost function and the minimization problem

Let the functional

$$\forall v \in H_0^2(\omega), \mathcal{I}(v) := \mathcal{J}(u(v)) + j(v), \quad (0.683)$$

$$\begin{aligned} \mathcal{I}(u) := & \frac{1}{4} (C(u), u)_\Delta + \frac{1}{2} ((I - \lambda L_2 - L_1)u, u)_\Delta \\ & + (Mv + \theta, u)_\Delta. \end{aligned}$$

Theorem 0.235. *solving the reduced Marguerre-Von Karman equation is equivalent to finding all the stationary points of the functional $\mathcal{I}$, that is, those ζ satisfy*

$$\zeta \in H_0^2(\omega),$$

$$\mathcal{I}'(\zeta) = 0.$$

Then, there exists at least one minimizer for the energy functional $\mathcal{I}$:

$$\zeta \in H_0^2(\omega), \quad (0.684)$$

$$\mathcal{I}(\zeta) = \inf_{u \in H_0^2(\omega)} \mathcal{I}(u). \quad (0.685)$$



Existence of the optimal control

Assuming that

$$G(v) := \|v + \Delta^2 \theta\|_{0,\omega} - \frac{\alpha}{c} \|B\|^{-1} \tilde{K}_1(\lambda)^{\frac{3}{2}},$$

$$\alpha \in (0, 1).$$

Let us define the admissible set of controls

$$\mathcal{U}_{ad} = \{v \in L^2(\omega) : G(v) \leq 0\} = G^{-1}(-\infty, 0],$$

$$\text{when } \lambda < \lambda_1 \left(1 - \frac{1}{\alpha_1}\right).$$

Where $\mathcal{U}_{ad}$ is a closed and convex set.

Theorem 0.236. Let $\mathcal{J} : H_0^2(\omega) \rightarrow \mathbb{R}$, $j : L^2(\omega) \rightarrow \mathbb{R}$ be weakly lower semicontinuous. Then there exists a solution $u \in \mathcal{U}_{ad}$ of problem

$$\mathcal{I}(u) = \min_{v \in \mathcal{U}_{ad}} \mathcal{I}(v) \quad (0.686)$$

Uniqueness of the Control

Assuming that $\mathcal{J}$ and j are Fréchet differentiable and satisfying conditions

$$\langle \mathcal{J}'(u_1) - \mathcal{J}'(u_2), u_1 - u_2 \rangle_* \geq m \|u_1 - u_2\|_{2,\omega}, \forall m > 0, \forall u_1, u_2 \in H_0^2(\omega). \quad (0.687)$$

$$\langle j'(v_1) - j'(v_2), v_1 - v_2 \rangle_0 \geq N \|v_1 - v_2\|_{0,\omega}, \forall N > 0, \forall v_1, v_2 \in L^2(\omega). \quad (0.688)$$

If $v \in \mathcal{U}_{ad}$ is a solution of problem (0.686), then $\langle \mathcal{I}'(v), w - v \rangle_0 \geq 0, \forall w \in \mathcal{U}_{ad}$.

$$\|\mathcal{J}'(x)\|_* \leq a_0 \|x\|_{2,\omega} + a_1, \forall x \in H_0^2(\omega); a_0, a_1 > 0, \quad (0.689)$$

where $\|\cdot\|_* = \|\cdot\|_{\mathcal{L}(H_0^2(\omega), H_0^2(\omega))}$,

Theorem 0.237. Let the functionals $\mathcal{J}$, j be lower bounded, weakly lower semi-continuous with Fréchet derivatives satisfying the conditions (0.687), (0.688) and (0.689), if

$$\left[-m + \left(a_0 \alpha \|B\|^{-1} \tilde{K}_1(\lambda)^{\frac{1}{2}} + a_1 \right) \cdot \frac{6\alpha}{1 - \alpha^2} \tilde{K}_1(\lambda)^{-\frac{1}{2}} \|B\| \right],$$

$$c^2 [\tilde{K}_1(\lambda)(1 - \alpha^2)]^{-2} < N.$$

where $\tilde{K}_1(\lambda) = 1 - \left(\frac{\lambda}{\lambda_1} + \frac{1}{\alpha_1}\right)$, then there exists a unique solution u of the problem (0.684).

Necessary Condition of Optimality

$$\langle p, v - u \rangle + \langle R_2 j'(u), v - u \rangle \geq 0, \quad \forall v \in \mathcal{U}_{ad}, \quad (0.690)$$

$$[I - \lambda L_2 - L_1 + C'(x(u))]p = R_1 \mathcal{J}'(x(u)), \quad (0.691)$$

$$x(u) - \lambda L_2 x(u) - L_1 x(u) + C(x(u)) = Mu + \theta. \quad (0.692)$$

The system (0.690), (0.691) and (0.692) is the optimality system of the problem (0.686). The problem (0.691) is the adjoint problem to the problem (0.692)

Conclusion

This study is concerned with optimal control problem for Marguerre-von Kármán equations. First, we establish the existence and uniqueness of non trivial solutions. Then we define the admissible set of control to study the existence and by the Fréchet differentiability of the state we get the uniqueness of the control. At the end by Riesz theorem, and the adjoint problem we establish the system of optimality to define the necessary condition of optimality. Note that, in the case $\theta \equiv 0$ in $\bar{\omega}$, we recover the optimal control problem for von Kármán equations.





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Weak law of large numbers for dependent random variables with infinite means

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Abstract. We provide necessary and sufficient conditions for the convergence in probability of weighted averages of random variables with infinite means. Our results extend and improve the corresponding theorems obtained in the independent setup by Adler (2012) and Nakata (2016).

Keywords: Weak laws, weighted exact laws, Rosenthal-type maximal inequalities.

Introduction

Consider a sequence $\mathcal{X} = \{X_n, n \geq 1\}$ of real-valued random variables (rv's) defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and satisfying

$$\mathbb{P}(|X_j| > x) \asymp x^{-\alpha} \quad \text{for } j \geq 1 \text{ and some } 0 < \alpha \leq 1 \quad (0.693)$$

and

$$\limsup_{x \rightarrow \infty} \sup_{j \geq 1} x^\alpha \mathbb{P}(|X_j| > x) < \infty, \quad \text{for some } 0 < \alpha \leq 1 \quad (0.694)$$



The strong law of large numbers fails for these rv's since they have infinite means. Herein, we establish necessary and sufficient conditions for the convergence in probability of

$$W_n := \frac{1}{b_n} \max_{1 \leq k \leq n} \left| \sum_{j=1}^k a_j (X_j - c_{nj}) \right|,$$

for a suitable sequence $\{c_{nj}, 1 \leq j \leq n\}$.

Main results

Theorem 0.238. Let $0 < \alpha \leq 1$ and consider two sequences of positive constants $\tilde{a} = \{a_n, n \geq 1\}$ and $\tilde{b} = \{b_n, n \geq 1\}$ such that $\sum_{j=1}^n a_j^\alpha = o(b_n^\alpha)$. If $\mathcal{X} = \{X_n, n \geq 1\}$ is a sequence of rv's satisfying (0.693), (0.694) and a Rosenthal-type maximal inequality, then $\mathcal{W} = \{W_n, n \geq 1\}$ converges in probability.

A necessary condition for the convergence in probability of $\mathcal{W}$ is also derived.

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Investigation of full information availability in a Markovian queue with differentiated vacations

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Abstract. We investigate the availability of complete information in a Markovian queue featuring two distinct vacation types. Customers decide whether to join the queue based on a reward-cost structure. We derive the equilibrium thresholds and analyze the system stationary behavior. Through numerical simulations, we demonstrate how various system parameters influence the equilibrium thresholds and overall social benefit.

Keywords: Queuing systems, Thresholds, Equilibrium, Social benefit.

Introduction

Queueing behavior has attracted substantial research interest due to its critical role in optimizing service quality across industries. By analyzing customer interactions within these systems, valuable insights can be gained to improve operational efficiency while balancing service quality and cost. As a result, many studies now focus on understanding the equilibrium behavior of customers in queueing systems.

Hassin and Haviv's (2003) and Hassin's (2016) comprehensive monographs have traced the development of this field, showcasing the vast research and diverse models that have emerged. The literature covers numerous models of continuous and discrete Markovian queues, both single and multi-server, which examine customer equilibrium behavior. These models have been key in enhancing our understanding of queueing dynamics and guiding the design of more efficient and fair service systems. Recent contributions from Sun and Wang (2022), Zhang and Wang (2023), He et al. (2024), and others continue to expand this knowledge, offering fresh insights and methods for analyzing queueing behavior.

This study focuses on a specific type of queueing system known as queues with server vacations, where managing the server's time-off periods plays a critical role in system performance and customer satisfaction. We analyze the strategic behavior of customers in a Markovian queue with two types of server vacations. In this setup, the server takes a first vacation after serving customers, followed by a second vacation if no customers are present. The second vacation repeats until a customer arrives. We investigate how customer strategies are shaped



in the fully observable cases. For each, we derive the Nash equilibrium threshold and present a numerical example to illustrate its effects on social welfare. Our findings shed light on the interplay between vacation policies and information availability, offering both theoretical insights and practical guidelines for improving customer satisfaction and operational efficiency in such systems.

The paper presents in section* 2 the studied model. section* 3 and 4 focuses on fully and almost observable cases. section* 5 contains a numerical example.

Fully Observable Case

In the fully observable $M/M/1$ queue with two types of vacations, customers are aware of both the server's state, $I(t)$, and the number of customers present, $N(t)$, at the time of their arrival, t . A pure threshold strategy is defined as $(n_e(0), n_e(1), n_e(2))$, where a customer joins the system if $N(t) \leq n_e(I(t))$ and chooses to balk otherwise.

Theorem 0.239. *In the fully observable $M/M/1$ queue with differentiated vacations, the optimal thresholds are:*

$$(n_e(0), n_e(1), n_e(2)) = \left(\left\lfloor \frac{R\mu}{c} \right\rfloor - 1, \left\lfloor \frac{R\mu}{c} - \frac{\mu}{\gamma_1} \right\rfloor - 1, \left\lfloor \frac{R\mu}{c} - \frac{\mu}{\gamma_2} \right\rfloor - 1 \right). \quad (0.695)$$

This threshold strategy constitutes a weakly dominant equilibrium.

The state space of the resulting Markov process is given by:

$$\Omega_{ob} = \{(n, 1) \mid 0 \leq n \leq n_e(1) + 1\} \cup \{(n, 2) \mid 0 \leq n \leq n_e(2) + 1\} \cup \{(n, 0) \mid 0 \leq n \leq n_e(0) + 1\}. \quad (0.696)$$

For the case where $n_e(1) > n_e(2)$, the stationary queue length distribution is:

$$\pi_{n,1} = \beta_1^n \alpha_1 \pi_{1,0}, \quad 0 \leq n \leq n_e(1), \quad (0.697)$$

$$\pi_{n_e(1)+1,1} = \frac{\lambda}{\gamma_1} \beta^{n_e(1)} \alpha_1 \pi_{1,0}, \quad (0.698)$$

$$\pi_{n,2} = \beta_2^n \alpha_2 \pi_{1,0}, \quad 0 \leq n \leq n_e(2), \quad (0.699)$$

$$\pi_{n_e(2)+1,2} = \frac{\lambda}{\gamma_2} \beta^{n_e(2)} \alpha_2 \pi_{1,0}, \quad (0.700)$$

$$\pi_{n,0} = A + B\rho^n + C_1\beta_1^n + D_1\beta_2^n, \quad (0.701)$$

where A , B , C_1 , and D_1 are given by the following relations:

$$\begin{aligned} C_1 &= \frac{\alpha_1 \rho}{\beta_1 - \rho} \pi_{1,0}, \\ D_1 &= \frac{\alpha_2 \rho}{\beta_2 - \rho} \pi_{1,0}, \\ A &= \frac{\pi_{1,0}}{1 - \rho} - C_1 \beta_1 \left(1 - \frac{\gamma_1}{\mu(1 - \rho)} \right) - D_1 \beta_2 \left(1 - \frac{\gamma_2}{\mu(1 - \rho)} \right), \\ B &= -\frac{\pi_{1,0}}{1 - \rho} - C_1 \beta_1 \frac{\gamma_1}{\lambda(1 - \rho)} - D_1 \beta_2 \frac{\gamma_2}{\lambda(1 - \rho)}. \end{aligned} \quad (0.702)$$

The social benefit per unit of time is calculated as:

$$\begin{aligned} B_e(n_e(0), n_e(1), n_e(2)) &= \lambda R (1 - \pi_{n_e(2)+1,2} - \pi_{n_e(1)+1,1} - \pi_{n_e(0)+1,0}) \\ &\quad - c \left(\sum_{n=1}^{n_e(0)+1} n \pi_{n,0} + \sum_{n=0}^{n_e(1)+1} n \pi_{n,1} + \sum_{n=0}^{n_e(2)+1} n \pi_{n,2} \right). \end{aligned} \quad (0.703)$$



Numerical Results

In this section*, we analyze the influence of the system's service rate μ on the equilibrium thresholds and social benefit. The goal is to gain insights into how changes in the service rate affect customer decision-making and overall system performance.

First, we investigate how an increase in the service rate μ impacts the equilibrium threshold, which determines the point at which customers choose to join or balk. As shown in Figure. 1, higher service rates correspond to an increase in the equilibrium threshold. This result is intuitive: when μ increases, the expected waiting time decreases, making the queue more attractive to customers. Consequently, more customers are willing to join the system, raising the threshold at which they decide to enter the queue.

Next, we assess the effect of the service rate μ on social benefit, a measure that reflects the overall welfare of the system's customers. Figure. 2 shows that as μ increases, the social benefit improves. This positive trend indicates that a higher service rate not only encourages greater participation by reducing waiting times but also leads to a more efficient utilization of resources within the system.

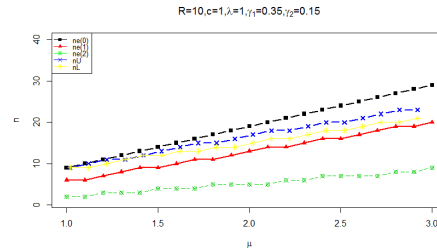


Figure 43: Thresholds vs μ .

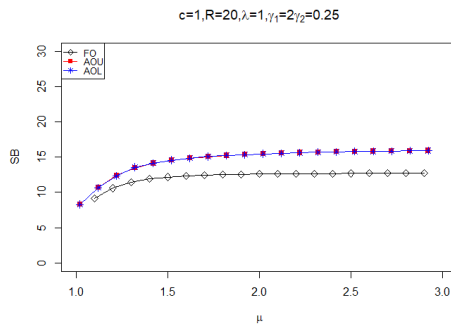


Figure 44: Social Benefit vs μ .

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Conclusion

This study examined customer decision-making in a single-server queue with two types of vacations under fully observable conditions. We derived the equilibrium threshold strategies and provided a numerical example to demonstrate the findings. Understanding customer behavior in these queueing scenarios allows for optimizing system configurations to align more



closely with customer preferences, ultimately leading to enhanced satisfaction and improved overall performance.

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On the study of Ψ –caputo fractional impulsive differential equations involving variational approach.

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Abstract.

This paper is devoted to contribute sufficient conditions for the existence of at least one solution of a Ψ –caputo fractional initial value problem by using critical point theory and variational approach.

Keywords: Fractional impulsive differential equations; Ψ -Caputo fractional derivative type; IVP problem.

Introduction

Variational methods and critical point theory are one of the most useful technique in the recent years. In this work we contribute by presenting the Ψ caputo spaces $W_p^{\alpha, \Psi}([0, \tau])$ using help from critical point theory then we reduce it into an equivalent form. As an application, we discuss the existence of solutions of the nonlinear fractional problem.



Main results

Theorem 0.240. [1] Let H be a reflexive complete metric space. if the mapping $\Lambda : H \rightarrow \mathbb{R}^n$ is weakly lower semi-continuous and coercive i.e $\Lambda(\Upsilon) = +\infty$ as $\|\Upsilon\| \rightarrow \infty$, Λ is frechet differentiable on H , then there exist $\Upsilon_0 \in H$ such that $\Lambda(\Lambda_0) = \inf_{\Lambda \in H} \Lambda(\Upsilon)$.

Definition 0.116. [2],[3],[4] For $h, \Psi \in C^n(J)$, the right ψ -caputo fractional derivatives of h are defined as

$$\begin{aligned} {}^{\Psi, C}_z \mathbf{D}_{d-}^{\beta} h(z) &= \tau I_k^{n-\beta, \Psi} \left(-\frac{1}{\Psi'(z)} \frac{d}{d\tau} \right)^n h(z), \\ &= \frac{1}{\Gamma(n-\beta)} \int_z^d (\Psi(s) - \Psi(z))^{n-\beta-1}, \\ &\quad \Psi'(s) \left(-\frac{1}{\Psi'(s)} \frac{d}{ds} \right)^n h(s) ds. \end{aligned}$$

Define the left ψ -caputo fractional derivatives of f for $f, \psi \in C^n(J)$ as follows:

$$\begin{aligned} {}^{\Psi, C}_z \mathbf{D}_{c+}^{\beta} h(z) &= {}_c I_k^{n-\beta, \Psi} \left(-\frac{1}{\psi'(z)} \frac{d}{d\tau} \right)^n h(z), \\ &= \frac{1}{\Gamma(n-\beta)} \int_a^{\tau} (\Psi(z) - \Psi(s))^{n-\beta-1}, \\ &\quad \Psi'(s) \left(\frac{1}{\Psi'(s)} \frac{d}{ds} \right)^n h(s) ds. \end{aligned}$$

Theorem 0.241. Let $C_0^\infty([0, \tau], \mathbb{R}^n)$ the space of the function $\mathcal{K}$ such that $\mathcal{K} \in C_0^\infty([0, \tau])$ and $\mathcal{K}(0) = \mathcal{K}(\tau) = 0$. The Ψ caputo fractional derivative space $W_p^{\alpha, \psi}$, where $p \in (1, \infty)$ and $\alpha \in (0, 1]$ define by the closure of $C_0^\infty([0, \tau], \mathbb{R}^n)$ with respect to the norme

$$\|\mathcal{K}\|_{\alpha, \Psi}^p = \left(\int_0^\tau |\mathcal{K}(\tau)|^p + \int_0^\tau |\Psi'(\tau) {}^{\Psi, C}_z \mathbf{D}^\alpha \mathcal{K}(\tau)|^p \right)^{\frac{1}{p}}.$$

Application

We prove the existence of solution for the following nonlinear Ψ caputo fractional impulsive differential equation

$$(p) \begin{cases} {}^{\Psi, C}_{\tau-} \mathbf{D}^\alpha (\Psi'(\zeta) {}^{\Psi} \mathbf{D}^\alpha) v(\zeta, x) + \Delta v(\zeta, x) = f(\zeta, x, v(\zeta, x), {}^{\Psi, C}_{\tau-} \mathbf{D}^\alpha v(\zeta, x)), \\ \zeta \in J \setminus \{\zeta_1, \dots, \zeta_m\}, J = [0, T] \ x \in \mathbb{R}^n, \\ \Delta({}^{\Psi, C}_{\tau-} \mathbf{D}^{\alpha-1} ({}^{\Psi} \mathbf{D}^\alpha_\zeta) v)(\zeta_k) = I_k(v(\zeta_k)), \\ v(0, x) = v(T, x) = 0, k \in \{1, \dots, m\}, x \in \mathbb{R}^n \end{cases}$$

where ${}^{\Psi, C}_{\tau-} \mathbf{D}^\alpha$, ${}^{\Psi}_t \mathbf{D}^{\alpha+}$ are the left-sided Ψ -Caputo and Riemann-Liouville fractional derivatives with respect to Ψ_1 and $p > 1$, $0 < \alpha \leq 1$. Let Ψ be strictly increasing and n times differentiable function on J , then $AC_\Psi^n(J) = \left\{ u : J \rightarrow \mathbb{R} \text{ and } \delta_\Psi^{[n-1]} u \in AC(J), \delta_\Psi^{[n-1]} u = \left(\frac{1}{\Psi'(\tau)} \frac{d}{d\tau} \right) u \right\}$ denotes the Banch space of n times absolutely continuous with respect to the strictly $0 = \zeta_0 < \zeta_1 < \dots < \zeta_k < \dots < \zeta_m < \zeta_{m+1} = T$.

$$\Delta({}^{\Psi, C}_{\tau-} \mathbf{D}^{\alpha-1} ({}^{\Psi} \mathbf{D}^\alpha_\zeta) v)(\zeta_k) = {}^{\Psi, C}_{\tau-} \mathbf{D}^{\alpha-1} ({}^{\Psi} \mathbf{D}^\alpha_\zeta) v(\zeta_k^+) - {}^{\Psi, C}_{\tau-} \mathbf{D}^{\alpha-1} ({}^{\Psi} \mathbf{D}^\alpha_\zeta) v(\zeta_k^-).$$

denote the right and the left limits of ${}^{\Psi, C}_{\tau-} \mathbf{D}^{\alpha-1} ({}^{\Psi} \mathbf{D}^\alpha_\zeta) v(\zeta_k)$ at $\zeta = \zeta_k$ ($k = 1, 2, \dots, n$) respectively.

Assume that $\alpha(\frac{1}{2}, 1]$ and $L \in L^{\frac{2}{\theta}}([0, \tau], \mathbb{R})$, $\theta \in (0, 2)$, $a \in [0, \frac{\Gamma^2(\alpha+1)}{2(M_{\Psi'})^3[\Psi(\tau)]^2})$ if $|f(\zeta, x, v(\zeta, x), {}^{\Psi, C}_{\tau-} \mathbf{D}^\alpha v(\zeta, x)) -$



$\Delta v(\zeta, x) \leq a |v|^2 + L |v|^{2-\theta} + c(x)$, for $v \in \mathbb{R}$.

Then the problem (p) has at least one weak solution which also minimizes Λ on ${}_0W_p^{\alpha, \psi}$.

$$\begin{aligned} \Lambda(v) &= \frac{1}{2} \int_0^\tau |\Psi'(\zeta)^{\Psi, C}_{\tau-} \mathbf{D}^\alpha v(\zeta, x)|^2 d\tau, \\ &- \frac{1}{2} \int_0^\tau \Psi'(\zeta) |v(\zeta, x)|^2 dt, \\ &- \int_0^\tau \Psi'(\zeta) f(\zeta, x, v(\zeta, x))^{\Psi, C}_{\tau-} \mathbf{D}^\alpha v(\zeta, x) \end{aligned}$$

Conclusion

In the survey paper, we present some background material for our problems and by variational approach, we will prove the existence of at least one solutions for the problem.

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