People's Democratic Republic of Algeria
Ministry of higher education and scientific research
University of Algiers Ben Youssef Ben Khedda
Faculty of Sciences - Department of Mathematics

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CERTIFICATE OF PARTICIPATION

THIS IS TO CERTIFY THAT

Abdelaziz Hellal

Has successfully participated in the First National Colloquium on Mathematics : **Trends and Actual Applications (CNMT2A).**

held on February 22 and 23, 2025

By delivering an online oral presentation titled:

Neumann boundary value problems in anisotropic Sobolev spaces with variable exponents

Colloquium Chair

Président du Colloque
Dr . BENAISSA Lakhdar

Dean of Faculty

Colloque National sur les Mathématiques :
Tendances et Applications Actuelles
CNMT2A

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MATHEMATICS : TRENDS AND ACTUAL APPLICATIONS

NMT2A

1 ST NATIONAL COLLOQUIUM

Organized by the Laboratory of Mathematical Analysis and Applications (LAMA-University of Algiers 1)

22.02.2025 | 8:30-17:50

PROGRAM

8 :30-9 :00	Registration
9 :00-9 :30	Opening ceremony
9 :30-10 :30	Session Chairperson : **Pr. Hamid Haddadou** **Plenary talk 1 :** De la variole au COVID-19 : l'apport des mathématiques dans l'analyse des épidémies - **Pr. A. Moussaoui**
10:30-10:45	Coffee break
10:45-11:00	Poster Session • Variational and Asymptotic Analysis of a Dynamic Viscoelastic Problem with Wear in a 3D-Thin Domain - **Mme. Benzatat Rim** • Existence Solutions for a Multidimensional Nonlinear Reaction-Diffusion Equations - **M. Djemiat Rabah** • Initial value problems using Euler,Taylor and Volterra integral equations - **M. Mohamed Raid Nadir** • Numerical Approximation of Optimal Control for Wave Equation with Missing Data - **Mme. Abdelli Mouna** • Numerical Solution of Coupled Parabolic PDEs for NO Vibrational States in the Upper Atmosphere - **M. Bouziane Abdelaaziz** • Periodicity Detection in Irregularly Sampled data by Robust Regression and Outlier Detection - **M. Mansouri Charef Eddine**
11:00-12:00	Session Chairperson : Pr. **Benyattou Benabderrahmane** Plenary talk 2 : "Industrial mathematics" in the Wilaya of Bejaia: analysis of 160 problems posed by 60 industrial companies and socio-economic organizations - **Pr. Aissani**
12:00-13:00	Session Chairperson : **Pr. Benyattou Benabderrahmane** **Plenary talk 3 :** Les modèles à base d'agents et quelques applications - **Pr. Nadjia El Saadi**
13:00-14:00	Lunch Break

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PROGRAM

14:00-16:00

Parallel Sessions

Session 1 : Functional analysis and partial differential equations (in-person)
Session chairperson : **Dr. Belkacem Chaouchi**
Session 2 : Numerical analysis, optimization and optimal control (online)
Session chairperson : **Dr. Djedaidi Noura**
Session 3 : Dynamic systems, differential equations (online)
Session chairperson : **Pr. Bachouche Kamel**

16:00-16:15

Coffee break

16:15-16:30

Poster Session

- On the well-posedness of a spatial fractional nonlinear diffusion equation - **Mme. Boudrissa Imane**
- Asymptotic behaviours of solution to generalized fractional integro-differential diffusion equation - **M. Benaissi Brahim**
- Numerical Analysis for Image Processing in Antifungal Testing of Cobalt-Doped SiO2 - **Mme. Hadjila Nadia**
- Semi-analyticle approach for solving the Fractional Differential Equations - **Mme. Chita Fouzia**
- Un résultat de multiplicité pour un problème aux limites via la théorie du point fixe - **Mme. Zahar Samira**
- AutoML Study on the Mortality of COVID-19 Vaccine in the Maghreb Region - **Mme. Zemouli Doua**
- Model of Formalization of Binary Relations by Multi-Functions - M. **Zehani Oussama**

16:30-17:50

Parallel Sessions

Session 4 : Algebra and number theory (in-person)
Session chairperson : **Pr. Abdelmoumene Zekiri**
Session 5 : Statistical and stochastic analysis (online)
Session chairperson : **Dr. Ikhlef Lyes**
Session 6 : Discrete mathematics and optimization (online)
Session chairperson : **Dr. Zerfa Lamia**
Session 7 : Differential equations (online)
Session chairperson : **Dr. Ayadi Hocine**

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PROGRAM

08:30-10:30

Parallel Sessions

Session 8 : Data analysis and AI (in-person)
Session chairperson : **Pr. Djamil Aissani**
Session 9 : Statistical and stochastic analysis (online)
Session chairperson : **Dr. Atik Touazi**
Session 10 : Numerical analysis, optimization and optimal control (online)
Session chairperson : **Dr. Amine Zouatini**

10:30-10:45

Coffee break

10:45-11:00

Poster Session

- A Study of the Relationships Between Various Notions of Semi-Convexity - **M. Merabet Ibrahim**
- Force de blocage de l'écoulement stationnaire du fluide herschel-bulkley - **M. Hibaterrahmane Benmessaoud**
- Nontrivial solution for non-degenerate fractional Kirchhoff-Schrödinger-Poisson system with concave-convex nonlinearities - **M. Boutebba Hamza**
- Un problème d'évolution revisité - **Mme. Boukrouk Wafia**
- A note on double Roman domination in graphs - **M. Omar Abdelhak**
- Optimizing Linear Discriminant Analysis for Imbalanced Data : Outliers Handling and Oversampling Techniques - **Mme. Chettih Nadjet**
- Prévision d'un Processus Autorégressif Hilbertien d'ordre 1 - **Mme. Bensmain Nawel**

11:00-13:00

Parallel Sessions

Session 11 : Dynamic systems, differential equations (in-person)
Session chairperson : **Pr. Benmazai Abdelhamid**
Session 12 : Algebra and number theory (online)
Session chairperson : **Pr. Abdelmoumene Zekiri**
Session 13 : Functional analysis and partial differential equations (online)
Session chairperson : **Dr. Abderachide Saadi**

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SESSION 1

- Stability and Regularity Analysis of Nonlinear Wave Equations with Localised Internal and Ventcel Boundary Conditions - **Mme. Rayan Ikram ADDOUN**
- Problems of diffraction by planar obstacles covered with thin dielectric multilayers - **M. Alliti Bachir**
- Nonlocal Neumann boundary conditions for the fractional p(.,.)-Laplacian operator - **Mme. Nawal IRZI**
- Survey of Energy Dissipation in Wave Equations with Kelvin-Voigt Damping on Star-Shaped Networks - **Mme. Naima MEHENAOUI**
- Existence result for a non-convex Lagrange optimal control problem - **Mme. Sara Attab**
- Coupled Cooperative Differential System : Example of two patches - **M. bilel elbetch**

SESSION 2

- Osmosis image editing - **Mme. Sabira BEN ALIA**
- Quadratic convex lower bound function for univariate nonconvex functions - **M. Djamel Zerrouki**
- Robust Control for the Stokes-Darcy system with pointwise tracking - **M. Merwan Abdelbari**
- Sur le processus de la rafledégénéré d'ordre deux - **Mme. Amira Aibeche**
- Construction of DNA codes from skew cyclic codes over R - **Mme. Amina Delhoum**

SESSION 3

- A family of polynomial differential systems with three hyperbolic algebraic limit cycles - **Mme. Rima Chouader**
- On boundary value problems of fractional differential equations involving Stieltjes derivatives - **M. Said Baghdad**
- A Vector Approach to the Study of Integro-Differential Systems - **M. Abdelhamid Bensalem**
- Mathematical Analysis and Operating Diagram of the Competition Model in a Chemostat - **M. Nabil Hamidi**
- Mathematical Insights into Infectious Disease Transmission - **Mme. Maroua Amel Boubekeur**
- New criterion for the global practical exponential stability of nonlinear time-varying systems - **M. Abir Kicha**

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PROGRAM

SESSION 4

- Almost Repdigits in k-Pell sequences - **Mme. Safia Seffah**
- Formulesexplicites pour les polynômes et nombres de Genocchi et d'Euler - **M. FARID BENCHERIF**
- Pell numbers with property D(4) - Mme. Khadidja MOUSSAOUI
- Study of Hull Dimensions and LCD Codes over Finite Fields- **Mme. Fatima Zohra KHELLOUL**

SESSION 5

- Analysis of Exponential-Two parameters Lindley distribution and its modeling - **Mme. Ahlem Ghouar**
- Asymptotic Normality of the Kernel Estimate of the Conditional Distribution Function for functional data - **Mme. Oum elkheir Benaouda**
- Asymptotic normality of the MLq estimator based on an extreme values distribution - **Mme. Nesrine IDIOU**
- Availability density function estimation based on Gamma and Inverse Gaussian kernels - **Mme. Yasmina Zitout**

SESSION 6

- A new type of Kannan's fixed point theorem - **Mme. Safia Bazine**
- Almost cycle extendability of a class of graphs generalizing claw free graphs - **Mme. Zineb Benmezianedaimellah**
- An new hybrid genetic algorithm for maximizing the coverage area in wireless sensor networks - **M. Nabil boumedine**
- Une énumération intelligente des solutions non dominées du problème du sac à dos bi-objectif en 0/1 - **Mme. Nadia Lachemi**

SESSION 7

- A study on a class of sixth-order fractional differential equations that involve Caputo derivatives - **M. Rakah Mahdi**
- An nonlinear elliptic problem involves two types of terms : degenerate coercivity and singular nonlinearity - **M. Zouatini Amine**
- Degenerate Elliptic Equations with Variable Exponents and Multiple Types of Terms - **Mme. Teyar Radjia**
- Sliding Mode Control of a chaotic System: Analysis and Dynamic Regulation - **Mme. Manal Mechekef**
- On HARDY-STEKLOV OPERATORS where $0 < p < 1$ - **M.Abdelaziz Gherdaoui**
- Neumann boundary value problems in anisotropic Sobolev spaces with variable exponents - **M. Hellal Abdelaziz**

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PROGRAM

SESSION 8

- Enhancing KPCA Performance with Bivariate Copulas: The BCFS-KPCA Approach - **Mme. Karima Femmam**
- Asymptotic propreties of a kernel estimator of regression function under mixing condition - **Mme. Sabrina BENZAMOUCHE**
- Kernel estimation of the trimmed regression for functional and truncated data - **Mme. Sarra Leulmi**
- Stochastic Modeling of Enzymatic Functioning: A Non-reliable Server Queueing System with Retrials and Orbital Search Approach - **M. Haroun Bouaouaou**
- Statistical analysis of the M/G/1/N/N queue with retrial and vacation - **M. Lyes Ikhlef**
- Local Bayesian approach in the context of estimation of the probability mass function using associated kernel method - **Mme. Lynda Harfouche**

SESSION 9

- Bayesian alternative in the estimation of the conditional density - **Mme. Ladaouri Nour El-hayet**
- Dynamics of Fractional Stochastic Systems with Set-Valued Conditions - **Mme. Fatima Zahra ARIOUI**
- MISE of the copula estimators under censoring - Mme. Samia Toumi
- Partial Functional Mean Characterization based tests for the Bivariate Skew-Normal Distribution - **Mme. Samia MAZOUZ**
- Quadratic Fit Search Optimizer To The Machine Repair Problem With Two Removable Servers Operating Under The Triadic (0, Q, N, M) Policy And Multiple Vacation - **M. Abir Kadi**
- Risk-Sensitive SMP for Fractional BSDE via Malliavin calculus - **M. Tayeb BOUAZIZ**

SESSION 10

- A characteristic discontinuous Galerkin method for optimal control problems - **Mme. Souheyla Zelmat**
- An efficient interior-point algorithm for semidefinite optimization - **M. Billel Zaoui**
- An improved full-Newton step feasible interior-point algorithm for convex quadratic optimization - **M. Welid Grimes**
- Analysis of nonlinear mixed integral equations - Mme. Hanane Belhireche
- First and second Chebyshev polynomials to Volterra-Fredholm integral equations - **M. Mohamed Nasseh NADIR**

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SESSION 11

- Cryptage d'image basé sur des transformations chaotiques – **M. Ahmed Sahnoune**
- Un système couplé d'inclusion différentielle fractionnaire avec des multi-applications pseudo-Lipschitz – **Mme. Somia Tamouza**
- Système dynamique du premier ordre avec perturbation – **Mme. Nadjiba Abdi**
- Existence and Uniqueness de Solution pour Problème A Termo-Electro Viscoelastic Fractional Contact – **Mme. Roufaida Ketfi**
- Existence of positive solutions for a Fractional boundary value problem – **Mme. Lydia Bouchal**
- Systèmes MIMO Utilisant les polynômes orthogonaux – **M. Djilali Naar**

SESSION 12

- Secure Encryption Scheme Using General Linear Groups over Groupring – **M. Sassia Makhlouf**
- Polynômes d-orthogonaux de type Meixner – Mme. Benamira Wissem
- The study of the D(1)-extensibility of the D(-1)-triple {1,5,c} and its associated elliptic curve – **M. Yacine Briedj**
- On Hyperlattice Ordered Commutative Semigroups and Some of Their Properties – **Mme. Imane Douadi**
- Examples of Lie groups in Artificial intelligence – Mme. Noura Djellali
- A New Algebraic Construction of Some LCD Linear Codes Over the Ring – **Mme. Haddouche Ouarda**

SESSION 13

- Singular anisotropic degenerate elliptic problems with Lm data – **Mme. Khaoula Mazouz**
- Global and Non-Global Existence Result in a Class of Viscoelastic Wave Equation With Non-linear Damping and Source Terms – **Mme. Zakia Tebba**
- Compactness of weighted Riemann-Liouville fractional operator in Morrey spaces – **M. Khirani Mohammed**
- Dynamical behavior of piezoelectric system with Fourier's law and delay – **M. Sami Loucif**
- Fixed point results for generalized contractions in dislocated quasi b-metric space – **Mme.Nadjet Hala**
- Global existence of a dynamic contact problem with normal damped response – **Mme. Imane Ouakil**

Neumann boundary value problems in anisotropic Sobolev spaces with variable exponents

H. Abdelaziz

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Mots-clés : *Anisotropic Sobolev space, variable exponent, Neumann elliptic problem, weak solutions, existence, uniqueness, trace theorem.*

1 Introduction

One of our work's novelty is that we examine a problem with a nonlinear term on the boundary, for which a trace theorem must be introduced and proven. More specifically, we examine the problem :

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \partial_{x_i} a_i\left(x, \partial_{x_i} u\right) + b(x)|u|^{p_M(x)-2}u = f(x,u) & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ \displaystyle\sum_{i=1}^{N} a_i\left(x, \partial_{x_i} u\right)\nu_i = g(x,u) & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here $\Omega \subset \mathbb{R}^N$ is a bounded open set with smooth boundary, ν_i, $i \in \{1,\ldots,N\}$, are the components of the outer normal unit vector, and $a_i : \Omega \times \mathbb{R} \to \mathbb{R}$, $i \in \{1,\ldots,N\}$, are the Carathéodory functions characterized by :

(A1) There exists a positive constant $\bar{c}_i$ such that a_i satisfies the growth condition

$$|a_i(x,s)| \leq \bar{c}_i(d_i(x) + |s|^{p_i(x)-1}),$$

for all $x \in \Omega$ and $s \in \mathbb{R}$, where $d_i \in L^{p_i'(\cdot)}(\Omega)$ (with $1/p_i(x)+1/p_i'(x) = 1$), is a nonnegative function.

(A2) If we denote by $A_i : \Omega \times \mathbb{R} \to \mathbb{R}$,

$$A_i(x,s) = \int_0^s a_i(x,t)dt,$$

then the following inequalities hold :

$$|s|^{p_i(x)} \leq a_i(x,s)s \leq p_i(x)\ A_i(x,s),$$

for all $x \in \Omega$ and $s \in \mathbb{R}$.

(A3) a_i is fulfilling

$$(a_i(x,s) - a_i(x,t))(s-t) > 0,$$

for all $x \in \Omega$ and s, $t \in \mathbb{R}$ with $s \neq t$;.

The operator presented above is the anisotropic variable mean curvature operator

$$\sum_{i=1}^{N} \partial_{x_i} \left[\left(1 + |\partial_{x_i} u|^2\right)^{(p_i(x)-2)/2} \partial_{x_i} u \right].$$

In our case, $\overrightarrow{p} : \overline{\Omega} \to \mathbb{R}^N$, $\overrightarrow{p}(x) = (p_1(x), p_2(x), ...p_N(x))$ with $p_i \in C_+(\overline{\Omega})$, $i \in \{1, ..., N\}$, and for all $x \in \overline{\Omega}$ we put

$$p_M(x) = \max\{p_1(x), \dots, p_N(x)\} \quad \text{and} \quad p_m(x) = \min\{p_1(x), \dots, p_N(x)\}.$$

In addition, for the Carathéodory functions $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$, we consider the antiderivatives $F : \Omega \times \mathbb{R} \to \mathbb{R}$

$$F(x, s) = \int_0^s f(x, t)dt,$$

respectively $G : \partial\Omega \times \mathbb{R} \to \mathbb{R}$

$$G(x, s) = \int_0^s g(x, t)dt.$$

Under the previous notation, the functions $\overrightarrow{p}$, b, f and g satisfy the conditions :

(B) $b \in L^\infty(\Omega)$ and there exists $b_0 > 0$ such that $b(x) \geq b_0$ for all $x \in \Omega$.

(F) There exist a positive constant k_1 and $q \in L^\infty_+(\Omega)$ with $q^+ < p_m^-$, such that

$$|f(x, s)| \leq k_1 \left(1 + |s|^{q(x)-1}\right),$$

for all $x \in \Omega$ and $s \in \mathbb{R}$.

(G) There exist a positive constant k_2 and $r \in C(\overline{\Omega})$ with $r^+ < \min_{x\in\partial\Omega}\{p_1^\partial(x), \dots, p_N^\partial(x)\}$ and $r^+ < p_m^-$, such that

$$|g(x, s)| \leq k_2 \left(1 + |s|^{r(x)-1}\right),$$

for all $x \in \partial\Omega$ and $s \in \mathbb{R}$.

Note that by adding the following assumptions

(F0) f is fulfilling the monotonicity condition

$$(f(x, s) - f(x, t))(s - t) < 0,$$

for all $x \in \Omega$ and s, $t \in \mathbb{R}$ with $s \neq t$.

(G0) g is fulfilling the monotonicity condition

$$(g(x, s) - g(x, t))(s - t) < 0,$$

for all $x \in \partial\Omega$ and s, $t \in \mathbb{R}$ with $s \neq t$.

We can infer that the solution is unique.

2 Main results

We present a trace theorem prior to examining the existence of solutions to issue (1).

Theorem 1 *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded open set with smooth boundary and let $\overrightarrow{p} \in \left(C_+(\overline{\Omega})\right)^N$, $r \in C(\overline{\Omega})$ satisfy the condition*

$$1 \leq r(x) < \min_{x\in\partial\Omega}\{p_1^\partial(x), \dots, p_N^\partial(x)\}, \qquad \forall\, x \in \partial\Omega.$$

Then there is a compact boundary trace embedding

$$W^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\partial\Omega).$$

Hypotheses (H). *We consider* $\Omega \subset \mathbb{R}^N$, $N \geq 2$, *to be a bounded open set with smooth boundary and* $\overrightarrow{p} \in \left(C_+(\overline{\Omega})\right)^N$. *We assume that for all* $i \in \{1, \ldots, N\}$, *the applications* a_i, $f : \Omega \times \mathbb{R} \to \mathbb{R}$, $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ *are Carathéodory functions satisfying (A1) - (A3), (F), respectively (G), and* $b : \Omega \to \mathbb{R}$ *satisfies (B).*

Theorem 2 *([12, 1.2 Theorem]) Suppose* X *is a reflexive Banach space with norm* $\|\cdot\|_X$ *and let* $M \subset X$ *be a weakly closed subset of* X. *Suppose* $\Phi : M \to \mathbb{R} \cup \{\infty\}$ *is coercive and (sequentially) weakly lower semi-continuous on* M *with respect to* X, *that is, suppose the following conditions are fulfilled : (i)* $\Phi(u) \to \infty$ *as* $\|u\|_X \to \infty$, $u \in M$. *(ii) For any* $u \in M$, *any subsequence* $(u_m)_m$ *in* M *such that* $u_m \rightharpoonup u$ *weakly in* X *there holds*

$$\Phi(u) \leq \liminf_{m\to\infty} \Phi(u_m).$$

Then Φ *is bounded from below on* M *and attains its infimum in* M.

Here, we give the notion of weak solution.

Définition 1 *By a weak solution to problem (1) we understand a function* $u \in W^{1,\overrightarrow{p}(\cdot)}(\Omega)$ *such that*

$$\int_\Omega \sum_{i=1}^N a_i\left(x, \partial_{x_i} u\right) \partial_{x_i}\varphi \, dx + \int_\Omega b(x)|u|^{p_M(x)-2} u\varphi \, dx - \int_\Omega f(x,u)\varphi \, dx - \int_{\partial\Omega} g(x,u)\varphi \, dS = 0, \quad (2)$$

for all $\varphi \in W^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

We use an energetic functional $I : W^{1,\overrightarrow{p}(\cdot)}(\Omega) \to \mathbb{R}$ to problem (1), which is described by :

$$I(u) = \int_\Omega \sum_{i=1}^N A_i\left(x, \partial_{x_i} u\right) dx + \int_\Omega \frac{b(x)}{p_M(x)}|u|^{p_M(x)} \, dx - \int_\Omega F(x, u_+) \, dx - \int_{\partial\Omega} G(x, u_+) \, dx,$$

where $u_+(x) = \max\{u(x), 0\}$. We denote by Λ, $J : W^{1,\overrightarrow{p}(\cdot)}(\Omega) \to \mathbb{R}$ the functionals

$$\Lambda(u) = \int_\Omega \sum_{i=1}^N A_i\left(x, \partial_{x_i} u\right) dx$$

and

$$J(u) = \int_\Omega \sum_{i=1}^N A_i\left(x, \partial_{x_i} u\right) dx + \int_\Omega \frac{b(x)}{p_M(x)}|u|^{p_M(x)} \, dx = \Lambda(u) + \int_\Omega \frac{b(x)}{p_M(x)}|u|^{p_M(x)} \, dx.$$

We remind the following result.

Lemma 1 *(see [8, Lemma 3.4]) The functional* Λ *is well-defined on* : $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$. *In addition, the functional* Λ *is of class* $C^1(W^{1,\overrightarrow{p}(\cdot)}(\Omega), \mathbb{R})$ *and*

$$\langle \Lambda'(u), \varphi \rangle = \int_\Omega \sum_{i=1}^N a_i(x, \partial_{x_i} u)\partial_{x_i}\varphi \, dx,$$

for all $u, \varphi \in: W^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

Due to Lemma 1, a standard calculus leads to the fact that I is well-defined on : $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and $I \in C^1(W^{1,\overrightarrow{p}(\cdot)}(\Omega), \mathbb{R})$ with the derivative given by

$$\langle I'(u), \varphi \rangle = \int_\Omega \sum_{i=1}^N a_i\left(x, \partial_{x_i} u\right) \partial_{x_i}\varphi \, dx + \int_\Omega b(x)|u|^{p_M(x)-2} u\varphi \, dx - \int_\Omega f(x,u)\varphi \, dx - \int_{\partial\Omega} g(x,u)\varphi \, dS,$$

for all $u, \varphi \in W^{1,\overrightarrow{p}(\cdot)}(\Omega)$. Obviously, the critical points of I are weak solutions to (1), so, by means of Theorem 2, we intend to establish the existence of critical points in order to deduce the existence of weak solutions. Our second main result is the following.

Theorem 3 *If hypotheses (H) are fulfilled, then there exists a weak solution to problem (1).*

For the proof of Theorem 3 we show that our energetic functional I fulfils the hypotheses of Theorem 2. The uniqueness of the solution is established below.

Theorem 4 *The weak solution to problem* (1) *is unique if the conditions (F0), (G0), and the hypothesis (H) are fulfilled.*

Références

[1] H. Abdelaziz, Singular Elliptic Equations with Variable Exponents. *Int. J. Math. And Appl.*, 11(**4**) : (2023) 141-168.

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Neumann boundary value problems in anisotropic Sobolev spaces with variable exponents

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Outline:

Introduction:

- Problem with a nonlinear term on the boundary

$$
\begin{cases}
-\sum_{i=1}^{N} \partial_{x_i} a_i\left(x, \partial_{x_i} u\right) + b(x)|u|^{p_M(x)-2}u = f(x,u) & \text{in } \Omega, \\
u \geq 0 & \text{in } \Omega, \\
\sum_{i=1}^{N} a_i\left(x, \partial_{x_i} u\right)\nu_i = g(x,u) & \text{on } \partial\Omega.
\end{cases} \tag{1}
$$

Here $\Omega \subset \mathbb{R}^N$ is a bounded open set with smooth boundary, ν_i, $i \in \{1, \dots, N\}$, are the components of the outer normal unit vector.

$a_i : \Omega \times \mathbb{R} \to \mathbb{R}$, $i \in \{1, \ldots, N\}$, are the Carathéodory functions characterized by:

$$\exists \bar{c}_i > 0, \ |a_i(x,s)| \leq \bar{c}_i(d_i(x) + |s|^{p_i(x)-1}), \tag{2}$$

for all $x \in \Omega$ and $s \in \mathbb{R}$, where $d_i \in L^{p_i'(\cdot)}(\Omega)$ (with $1/p_i(x) + 1/p_i'(x) = 1$), is a nonnegative function.

If we denote by $A_i : \Omega \times \mathbb{R} \to \mathbb{R}$,

$$A_i(x,s) = \int_0^s a_i(x,t)dt, \text{ then } |s|^{p_i(x)} \leq a_i(x,s)s \leq p_i(x)\, A_i(x,s), \tag{3}$$

for all $x \in \Omega$ and $s \in \mathbb{R}$.

a_i is fulfilling:

$$(a_i(x,s) - a_i(x,t))(s-t) > 0, \ \forall x \in \Omega, \text{ and } s, \ t \in \mathbb{R}, \ s \neq t. \tag{4}$$

The operator presented above is the anisotropic variable mean curvature operator

$$\sum_{i=1}^{N} \partial_{x_i} \left[\left(1 + |\partial_{x_i} u|^2\right)^{(p_i(x)-2)/2} \partial_{x_i} u \right]$$

In our case, $\overrightarrow{p} : \overline{\Omega} \to \mathbb{R}^N$, $\overrightarrow{p}(x) = (p_1(x), p_2(x), ...p_N(x))$ with $p_i \in C_+(\overline{\Omega})$, $i \in \{1, ..., N\}$, and for all $x \in \overline{\Omega}$ we put

$$p_M(x) = \max\{p_1(x), \ldots, p_N(x)\} \quad \text{and} \quad p_m(x) = \min\{p_1(x), \ldots, p_N(x)\}.$$

In addition, for the Carathéodory functions $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$, we consider the antiderivatives $F : \Omega \times \mathbb{R} \to \mathbb{R}$

$$F(x, s) = \int_0^s f(x, t)dt,$$

respectively $G : \partial\Omega \times \mathbb{R} \to \mathbb{R}$

$$G(x, s) = \int_0^s g(x, t)dt.$$

where $C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) \ : \ \inf_{x\in\Omega} h(x) > 1\}$
$h^+ = \sup_{x\in\Omega} h(x)$ and $h^- = \inf_{x\in\Omega} h(x)$.

$$h^{\partial}(x) = \begin{cases} (N-1)h(x)/[N-h(x)] & \text{if} \quad h(x) < N, \\ \infty & \text{if} \quad h(x) \geq N. \end{cases}$$

Under the previous notation, the functions $\overrightarrow{p}$, b, f and g satisfy the conditions:

$$b \in L^{\infty}(\Omega),\ \exists b_0 > 0 :\ b(x) \geq b_0,\ \forall x \in \Omega. \tag{5}$$

$$\exists k_1 > 0,\ q \in L^{\infty}_{+}(\Omega),\ q^{+} < p_m^{-}\ : |f(x,s)| \leq k_1 \left(1 + |s|^{q(x)-1}\right), \tag{6}$$

for all $x \in \Omega$ and $s \in \mathbb{R}$.

$\exists k_2 > 0$ and $r \in C(\overline{\Omega})$, $r^{+} < \min_{x\in\partial\Omega}\{p_1^{\partial}(x), \ldots, p_N^{\partial}(x)\},\ r^{+} < p_m^{-}$:

$$|g(x,s)| \leq k_2 \left(1 + |s|^{r(x)-1}\right), \tag{7}$$

for all $x \in \partial\Omega$ and $s \in \mathbb{R}$.

Note that we can deduce the uniqueness of the solution by adding the following assumptions:

f is the monotonicity condition:

$$(f(x, s) - f(x, t))(s - t) < 0, \tag{8}$$

for all $x \in \Omega$ and $s,\ t \in \mathbb{R}$ with $s \neq t$.

g is fulfilling the monotonicity condition

$$,\,(g(x, s) - g(x, t))(s - t) < 0, \tag{9}$$

for all $x \in \partial\Omega$ and $s,\ t \in \mathbb{R}$ with $s \neq t$.

Variable exponent spaces:

For any measurable subset $\Omega \subset \mathbb{R}^N$, $N \geq 2$, with $0 < |\Omega| < \infty$, we consider $p \in C_+(\overline{\Omega})$.

The continuous embedding: [1]

If $0 < |\Omega| < \infty$ and $p_1,\ p_2 \in C(\overline{\Omega};\mathbb{R})$, $1 < p_i^- \leq p_i^+ < \infty$ $(i = 1, 2)$, are such that $p_1 \leq p_2$ in Ω, then the embedding $L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$ is continuous.

Hölder-type inequality: [1]

$$\left| \int_\Omega u(x)v(x)\ dx \right| \leq 2\, \|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{p'(\cdot)}(\Omega)}$$

for all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$.

The connection between the $p(\cdot)$-modular and $\|u\|_{L^{p(\cdot)}(\Omega)}$: [3]

If $u \in L^{p(\cdot)}(\Omega)$ and $p < \infty$ then,

$$\|u\|_{L^{p(\cdot)}(\Omega)} < 1 \ (= 1; > 1) \quad \Leftrightarrow \quad \rho_{\Omega, p(\cdot)}(u) < 1 \ (= 1; > 1)$$

$$\|u\|_{L^{p(\cdot)}(\Omega)} > 1 \quad \Rightarrow \quad \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)} \leq \rho_{\Omega, p(\cdot)}(u) \leq \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}$$

$$\|u\|_{L^{p(\cdot)}(\Omega)} < 1 \quad \Rightarrow \quad \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)} \leq \rho_{\Omega, p(\cdot)}(u) \leq \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}$$

$$\|u\|_{L^{p(\cdot)}(\Omega)} \to 0 \ (\to \infty) \quad \Leftrightarrow \quad \rho_{\Omega, p(\cdot)}(u) \to 0 \ (\to \infty).$$

If, in addition, $(u_n)_n \subset L^{p(\cdot)}(\Omega)$, then

$$\lim_{n\to\infty} \|u_n - u\|_{L^{p(\cdot)}(\Omega)} = 0 \quad \Leftrightarrow \quad \lim_{n\to\infty} \rho_{\Omega, p(\cdot)}(u_n - u) = 0 \quad \Leftrightarrow$$

$\Leftrightarrow$ $(u_n)_n$ converges to u in measure and $\lim_{n\to\infty} \rho_{\Omega, p(\cdot)}(u_n) = \rho_{\Omega, p(\cdot)}(u)$.

The trace theorem for the isotropic case: [2]

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded open set with smooth boundary. Suppose that $p \in C_+(\overline{\Omega})$ and $r \in C(\overline{\Omega})$ satisfy the condition

$$1 \leq r(x) < p^{\partial}(x), \qquad \forall\, x \in \partial\Omega.$$

Then there is a compact boundary trace embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\partial\Omega)$.

Next, we consider the space $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$, where $\overrightarrow{p}:\overline{\Omega}\to\mathbb{R}^N$ is the vectorial function

$$\overrightarrow{p}(\cdot)=(p_1(\cdot),...,p_N(\cdot))$$

and $p_i\in C_+(\overline{\Omega})$ for all $i\in\{1,\ldots,N\}$. The anisotropic variable exponent Sobolev space is defined by

$$\begin{aligned} W^{1,\overrightarrow{p}(\cdot)}(\Omega) &= \{u\in L^{p_M(\cdot)}(\Omega): \quad \partial_{x_i}u\in L^{p_i(\cdot)}(\Omega),\ i\in\{1,\ldots,N\}\} \\ &= \{u\in L^1_{loc}(\Omega): \quad u\in L^{p_i(\cdot)}(\Omega),\ \partial_{x_i}u\in L^{p_i(\cdot)}(\Omega)\} \end{aligned}$$

endowed with the norm

$$\|u\|_{W^{1,\overrightarrow{p}(\cdot)}(\Omega)}=\|u\|_{L^{p_M(\cdot)}(\Omega)}+\sum_{i=1}^{N}\|\partial_{x_i}u\|_{L^{p_i(\cdot)}(\Omega)}.$$

The space $\left(W^{1,\overrightarrow{p}(\cdot)}(\Omega), \|\cdot\|_{W^{1,\overrightarrow{p}(\cdot)}(\Omega)}\right)$ is a reflexive Banach space and the following embedding theorem takes place.

The compact embedding for the anisotropic case:[1]

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and for all $i \in \{1, \ldots, N\}$, $p_i \in L^\infty(\Omega)$, let $p_i(x) \geq 1$ a.e. in Ω. Then for any $q \in L^\infty(\Omega)$ with $q(x) \geq 1$ a.e. in Ω such that

$$\operatorname{ess}\inf_{x\in\Omega}(p_M(x) - q(x)) > 0$$

we have the compact embedding

$$W^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega).$$

Note that since $p_i^- > 1$, $W^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow W^{1,1}(\Omega)$ continously and by the Gagliardo trace theorem $\mathrm{W}^{1,1}(\Omega) \hookrightarrow L^1(\partial\Omega)$ compactly, with $\Omega \subset \mathbb{R}^N$ being a bounded open set with smooth boundary. Hence for $u \in W^{1,\overrightarrow{p}(\cdot)}(\Omega)$ the trace has definite meaning.

Result 1

Theorem 1 [2]

Let X be a reflexive Banach space, and let $f : M \subseteq X \to \mathbb{R}$ be Gâteaux differentiable over the closed, convex set M. Then the following conditions are equivalent:

(i) f is convex over M.

(ii) We have

$$f(u) - f(v) \geq \langle f'(v), u - v \rangle_{X^\star \times X} \quad \forall\, u,\, v \in M,$$

where $X^\star$ denotes the dual of the space X.

(iii) The first Gâteaux derivative is monotone, that is,

$$\langle f'(u) - f'(v), u - v \rangle_{X^\star \times X} \geq 0, \quad \forall\, u,\, v \in M.$$

(iv) The second Gâteaux derivative of f exists and it is positive, that is,

$$\langle f''(u) \circ v, v \rangle_{X^\star \times X} \geq 0, \quad \forall\, v \in M.$$

Result 2

Theorem 2 [4]

Suppose X is a reflexive Banach space with norm $\|\cdot\|_X$ and let $M \subset X$ be a weakly closed subset of X. Suppose $\Phi : M \to \mathbb{R} \cup \{\infty\}$ is coercive and (sequentially) weakly lower semi-continuous on M with respect to X, that is, suppose the following conditions are fulfilled:

(i) $\Phi(u) \to \infty$ as $\|u\|_X \to \infty$, $u \in M$.

(ii) For any $u \in M$, any subsequence $(u_m)_m$ in M such that $u_m \rightharpoonup u$ weakly in X there holds

$$\Phi(u) \leq \liminf_{m\to\infty} \Phi(u_m).$$

Then Φ is bounded from below on M and attains its infimum in M.

The trace theorem for the anisotropic case: [2]

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded open set with smooth boundary and let $\overrightarrow{p} \in \left(C_+(\overline{\Omega})\right)^N$, $r \in C(\overline{\Omega})$ satisfy the condition

$$1 \leq r(x) < \min_{x \in \partial\Omega} \{p_1^{\partial}(x), \dots, p_N^{\partial}(x)\}, \qquad \forall\, x \in \partial\Omega.$$

Then there is a compact boundary trace embedding

$$W^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\partial\Omega).$$

The proof of this theorem is based on the trace theorem for the isotropic case, the Gagliardo trace theorem, and the previous continuous embedding.

Main results:

Hypotheses (H)

We consider $\Omega \subset \mathbb{R}^N$, $N \geq 2$, to be a bounded open set with smooth boundary and $\overrightarrow{p} \in \left(C_+(\overline{\Omega})\right)^N$. We assume that for all $i \in \{1, \dots, N\}$, the applications a_i, $f : \Omega \times \mathbb{R} \to \mathbb{R}$, $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions satisfying (2)-(4), (6), respectively (7), and $b : \Omega \to \mathbb{R}$ satisfies (5).

The notion of weak solution

A function $u \in W^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is a weak solution to problem 1 if

$$\int_\Omega \sum_{i=1}^N a_i\left(x, \partial_{x_i} u\right) \partial_{x_i}\varphi \, dx + \int_\Omega b(x)|u|^{p_M(x)-2} u\varphi \, dx$$

$$-\int_\Omega f(x,u)\varphi \, dx - \int_{\partial\Omega} g(x,u)\varphi \, dS = 0, \text{ for all } \varphi \in W^{1,\overrightarrow{p}(\cdot)}(\Omega).$$

Existence Theorem

If hypotheses (H) are fulfilled, then there exists a weak solution to the problem 1.

Uniqueness Theorem

If, in addition to the hypotheses (H), the conditions (8), (9) are fulfilled, then the weak solution to the problem 1 is unique.

Strategy of the proof of the existence theorem:

- We determine the energetic functional Φ associated with the problem 1.
- We show that the energetic functional Φ fulfills the hypotheses of Result 2 (Theorem 2).

For the problem 1 we associate an energetic functional $\Phi : W^{1,\overrightarrow{p}(\cdot)}(\Omega) \to \mathbb{R}$, defined by

$$\Phi(u) = \int_\Omega \sum_{i=1}^{N} A_i\left(x, \partial_{x_i} u\right) \, dx + \int_\Omega \frac{b(x)}{p_M(x)} |u|^{p_M(x)} \, dx - \int_\Omega F(x, u_+) \, dx - \int_{\partial\Omega} G(x, u_+) \, dx,$$

where $u_+(x) = \max\{u(x), 0\}$.

We denote by T, $J : W^{1,\overrightarrow{p}(\cdot)}(\Omega) \to \mathbb{R}$ the functionals

$$T(u) = \int_\Omega \sum_{i=1}^{N} A_i\left(x, \partial_{x_i} u\right) dx$$

and

$$J(u) = \int_\Omega \sum_{i=1}^{N} A_i\left(x, \partial_{x_i} u\right)\, dx + \int_\Omega \frac{b(x)}{p_M(x)} |u|^{p_M(x)}\, dx = T(u)$$

$$+ \int_\Omega \frac{b(x)}{p_M(x)} |u|^{p_M(x)}\, dx.$$

Lemma 1 [4]

The functional T is well-defined on $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$. In addition, the functional T is of class $C^1(W^{1,\overrightarrow{p}(\cdot)}(\Omega),\mathbb{R})$ and

$$\langle T'(u),\varphi\rangle = \int_\Omega \sum_{i=1}^{N} a_i(x,\partial_{x_i}u)\partial_{x_i}\varphi\, dx,$$

for all $u,\varphi \in W^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

Due to this Lemma 1, a standard calculus leads to the fact that Φ is well-defined on $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and $\Phi \in C^1(W^{1,\overrightarrow{p}(\cdot)}(\Omega), \mathbb{R})$ with the derivative given by

$$\langle \Phi'(u), \varphi \rangle = \int_\Omega \sum_{i=1}^{N} a_i\left(x, \partial_{x_i} u\right) \partial_{x_i} \varphi \, dx + \int_\Omega b(x)|u|^{p_M(x)-2} u\varphi \, dx$$

$$- \int_\Omega f(x, u)\varphi \, dx - \int_{\partial\Omega} g(x, u)\varphi \, dS,$$

for all $u, \varphi \in W^{1,\overrightarrow{p}(\cdot)}(\Omega)$. Obviously, the critical points of Φ are weak solutions to the problem 1, so, by means of Result 2 (Theorem 2), we establish the existence of critical points in order to deduce the existence of weak solutions.

To this end, we proceed with the following lemma.

Lemma 2

- If hypotheses (H) are fulfilled, then the functional Φ is coercive.
- If hypotheses (H) are fulfilled, then the functional Φ is weakly lower semi-continuous.

For the coercivity of Φ, we use (3) to get

$$T(u) \geq \frac{1}{p_M^+} \sum_{i=1}^{N} \int_\Omega |\partial_{x_i} u|^{p_i(x)}.$$

By the connection between the $p_i(\cdot)$-modular and $\|u\|_{L^{p(i^\cdot)}(\Omega)}$, (5), (6), and the compact embedding for the anisotropic case, we deduce that

$$T(u) \geq \frac{1}{p_M^+} \left[\frac{1}{N^{p_m^- - 1}} \left(\sum_{i=1}^{N} \|\partial_{x_i} u\|_{L^{p_i(\cdot)}} \right)^{p_m^-} - N \right].$$

It follows from the two cases corresponding to the values of $\|u\|_{L^{p_M(\cdot)}}$ that there exists $\tilde{k}_0, \tilde{k}_3 > 0$ such that

$$J(u) \geq \tilde{k}_0 \|u\|^{p_m^-} - \tilde{k}_3.$$

and there exists $\tilde{k}_1 > 0$ such that

$$\int_\Omega F(x, u_+)\, dx \leq \tilde{k}_1 \|u\|^{q^+}.$$

Using hypothesis (7) and the trace theorem for the anisotropic case, by similar arguments we obtain the existence of a positive constant $\tilde{k}_2$ such that

$$\int_\Omega G(x, u_+)\, dx \leq \tilde{k}_2 \|u\|^{r^+},$$

Putting together the previous inequalities, we arrive at

$$\Phi(u) \geq \tilde{k}_0 \|u\|^{p_m^-} - \tilde{k}_1 \|u\|^{q^+} - \tilde{k}_2 \|u\|^{r^+} - \tilde{k}_3.$$

Knowing that $q^+, r^+ < p_m^-$, we find that $\Phi(u) \to \infty$ when $\|u\| \to \infty$, hence Φ is coercive.

Now, we show that Φ is weakly lower semi-continuous. According to Brezis [1], to show that J is weakly lower semi-continuous, it is enough to prove that J is lower semi-continuous. By (4) and Result 1 (Theorem 1) with $\epsilon > 0$, we deduce that

$$J(v) \geq J(u)+\sum_{i=1}^{N}\int_{\Omega} a_i(x,\partial_{x_i}u)(\partial_{x_i}v-\partial_{x_i}u)\,dx+\int_{\Omega} b(x)|u|^{p_M(x)-2}u(v-u)\,dx.$$

Using (2), (5), the Hölder-type inequality, and the connection between the $p_i(\cdot)$-modular and $\|u\|_{L^{p(_i\cdot)}(\Omega)}$, we deduce that there exists $C > 0$ such that

$$J(v) \geq J(u) - C\|v-u\| \geq J(u) - \epsilon,$$

for all $v \in W^{1,\overrightarrow{p}(\cdot)}(\Omega)$ with $\|v-u\| < \delta = \epsilon/C$. Therefore, J is weakly lower semi-continuous.

Next, we denote

$$w_1(u) = \int_\Omega F(x,u)\,dx \quad \text{and} \quad w_2(u) = \int_{\partial\Omega} G(x,u)\,dx.$$

Then $w_1^{'}, w_2^{'} : W^{1,\vec{p}(\cdot)}(\Omega) \to (W^{1,\vec{p}(\cdot)}(\Omega))^{\star}$ are completely continuous, that is, if $u_n \rightharpoonup u$, then $w_1^{'}(u_n) \to w_1^{'}(u)$ and $w_2^{'}(u_n) \to w_2^{'}(u)$. Hence the functionals $w_1^{'}, w_2^{'}$ are weakly continuous and, since J is weakly lower semi-continuous, we conclude that Φ is weakly lower semi-continuous.
Finally, the proof of the existence theorem follows directly from Lemma 2 and Result 2 (Theorem 2).

Proof of the uniqueness theorem

We suppose there exist two weak solutions to problem 1, that is, u_1 and u_2. We replace the solution u by u_1 in the definition of weak solution and we choose $\varphi = u_1 - u_2$. Then

$$\int_\Omega \sum_{i=1}^{N} a_i\left(x, \partial_{x_i} u_1\right) \partial_{x_i}(u_1 - u_2)\, dx + \int_\Omega b(x)|u_1|^{p_M(x)-2} u_1 (u_1 - u_2)\, dx$$

$$- \int_\Omega f(x, u_1)(u_1 - u_2)\, dx - \int_{\partial\Omega} g(x, u_1)(u_1 - u_2)\, dx = 0.$$

Next, we replace the solution u by u_2 in the definition of weak solution and we choose $\varphi = u_2 - u_1$. We have

$$\int_\Omega \sum_{i=1}^{N} a_i\left(x, \partial_{x_i} u_2\right) \partial_{x_i}(u_2 - u_1)\, dx + \int_\Omega b(x)|u_2|^{p_M(x)-2} u_2 (u_2 - u_1)\, dx$$

$$- \int_\Omega f(x, u_2)(u_2 - u_1)\, dx - \int_{\partial\Omega} g(x, u_2)(u_2 - u_1)\, dx = 0.$$

By combining the previous two equalities we obtain

$$\int_{\Omega}\left\{\sum_{i=1}^{N}\left[a_i\left(x,\partial_{x_i}u_1\right)-a_i\left(x,\partial_{x_i}u_2\right)\right]\left(\partial_{x_i}u_1-\partial_{x_i}u_2\right)\right\}dx$$
$$+\int_{\Omega}b(x)\left[\left|u_1\right|^{p_M(x)-2}u_1-\left|u_2\right|^{p_M(x)-2}u_2\right](u_1-u_2)\,dx$$
$$-\int_{\Omega}\left[f(x,u_1)-f(x,u_2)\right](u_1-u_2)\,dx$$
$$-\int_{\partial\Omega}\left[g(x,u_1)-g(x,u_2)\right](u_1-u_2)\,dx=0.$$

By (4), (8) and (9), all the terms in the above equality are positive unless $u_1=u_2$, and this yields the uniqueness of the solution.

Conclusion and perspectives

$$\begin{cases} -\sum_{i=1}^{N} \partial_{x_i} a_i\left(x, \partial_{x_i} u\right) + b_1(x)|u|^{p_M(x)-2}u = f(x,u) + b_2(x) & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ \sum_{i=1}^{N} a_i\left(x, \partial_{x_i} u\right)\nu_i = g(x,u) & \text{on } \partial\Omega, \end{cases}$$

where Ω, a_i, b_1, $\overrightarrow{p}$, f, g satisfy hypotheses (H) and $b_2 \in L^\infty(\Omega)$.

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Thank you for your attention

Questions are welcome