On $p(\cdot)$-Laplacian equations with variable exponents and L^m data

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Abstract

In this work, we prove existence and regularity of weak solutions for a class of $p(\cdot)$-Laplacian equations with variable exponents and L^m data, with m being small. The functional setting involves Lebesgue-Sobolev spaces with variable exponents.

Keywords

$p(\cdot)-$Laplacian equations, weak solution, variable exponents, L^m data.

Introduction

In this paper we prove existence and regularity of weak solutions for a class of nonlinear elliptic equations with variable exponents. A prototype example is

$$(P)\qquad \begin{cases} -\mathrm{div}\left(|Du|^{p(\cdot)-2}Du\right)=f, & \text{in } \Omega \\ u=0, & \text{on } \partial\Omega \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$ with Lipchitz boundary $\partial\Omega$, the right-hand side $f \in L^m(\Omega)$, m as in (6).
The equation (P) can be viewed as a generalization of the classical p-Laplace equation where the constant $p \in (1, +\infty)$.
Instead of (P) we will consider more general nonlinear elliptic equations with variable exponents of the form

$$\begin{cases} -\mathrm{div}\left(\widehat{a}(x, Du)\right)=f, & \text{in } \Omega \\ u=0, & \text{on } \partial\Omega \end{cases} \tag{1}$$

The vector field $\widehat{a} : \Omega\times\mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function and satisfying, a.e $x \in \Omega$ and for all $\xi, \xi' \in \mathbb{R}^N$, the following:

$$\widehat{a}(x,\xi)\xi \geq \alpha|\xi|^{p(.)}, \quad \widehat{a}(x,\xi)=(a_1,\dots,a_N) \tag{2}$$
$$|\widehat{a}(x,\xi)| \leq \beta\left(h+|\xi|^{p(\cdot)-1}\right) \tag{3}$$
$$(\widehat{a}(x,\xi)-\widehat{a}(x,\xi'))(\xi-\xi')>0,\ \xi\neq\xi', \tag{4}$$

where α, β are strictly positive real numbers, h is a given positive function in $L^{p'(\cdot)}(\Omega)$ where $\frac{1}{p(\cdot)}+\frac{1}{p'(\cdot)}=1$, and the variable exponent $p(\cdot) : \overline{\Omega} \to (1, +\infty)$ is continuous function such that:

$$1+\frac{1}{m}-\frac{1}{N}<p(x)<N, \quad \text{for all } x \in \overline{\Omega}. \tag{5}$$

where

$$1<m<\frac{Np(x)}{Np(x)-N+p(x)}, \quad \text{for all } x \in \overline{\Omega}. \tag{6}$$

As another prototype example we consider the model problem

$$\begin{cases} -\mathrm{div}\left(|Du|^{p(\cdot)-2}Du\right)=\delta & \text{in } B \\ u=0 & \text{on } \partial B \end{cases} \tag{7}$$

where δ is the Dirac measure at the origin, $p(\cdot)$ as in (5) , and

$$B=\{x\in\mathbb{R}^N \mid |x|<1\}.$$

The study of various mathematical problems with variable exponent has received considerable attention in recent years. These problems concern applications and raise many difficult mathematical problems. In some cases, they provide realistic models for the study of natural phenomena in electro-rheological fluids and an important applications are related to image processing. We refer the reader to [3] and the references therein. Clearly, the nonlinearity of (7) is more complicated than nonlinearity of the p-Laplacian. As the exponent which appear in (7) depends on the variable x, the functional setting involves Lebesgue and Sobolev spaces with variable exponent $L^{p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$, we refer to [4], [5] for further properties of variable exponent LebesgueSobolev spaces. In the constant case $2-\frac{1}{N}<p(\cdot)=p$ with $Au=-\mathrm{div}\left(\widehat{a}(x,u,Du)\right)$, the existence of a distributional solution u of (1) in the space $W_0^{1,q}(\Omega)$ for all $q \in \left[1; \frac{N(p-1)}{(N-1)}\right)$ has been proved in [2]. Therefore, the study of problem (1) is a new and interesting topic. Inspired by [1] and [7], we prove the existence of weak solution for the problem (1) with right-hand side in $L^m(\Omega)$ where m and the variable exponent $p(\cdot)$ are restricted as in (5)-(6),(we refer to [7] for more details), similar results can be found in [8], [9].
The main steps of the proof consist of obtaining uniform estimates for suitable approximate problems and then passing to the limit. There are essentially two difficulties introduced in treating nonlinear elliptic operators $Au=-\mathrm{div}\left(\widehat{a}(x,Du)\right)$ instead of the Laplacian with variable exponents. The first one is to obtain uniform estimates on the solution u and the gradients Du. The second difficulty is to pass to the limit when the nonlinearity of A depends on Du.

Main Objectives

1. The objective of this work is investigate the existence of solutions for a class of equations elliptic with variable exponents and irregular data.
2. know the application of partial differential equation with variable exponents in image processing.

Materials and Methods

Some basic notations and properties of Lebesgue and Sobolev spaces with variable exponents
The method used in [7] was the following: Compactness method

Mathematical Preliminaries

Sobolev Spaces with variable exponents

We recall some definitions and basic properties of the generalized Lebesgue-Sobolev spaces $L^{p(\cdot)}(\Omega)$, $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$, where Ω is an open subset of $\mathbb{R}^N$. We refer to [4], [5] for further properties of variable exponent Lebesgue-Sobolev spaces.

Statement of results

Definition 0.1 ([2]). *A function u is a weak solution of problem* (1) *if*

$$u\in W_0^{1,1}(\Omega), \quad \widehat{a}(x,Du)\in (L^1(\Omega))^N,$$

and

$$\int_\Omega \widehat{a}(x,Du)D\varphi\,dx=\int_\Omega f\varphi\,dx, \quad \forall\varphi\in C_0^\infty(\Omega).$$

Our main results are the following:

Theorem 0.1 ([7]). *Let $f \in L^m(\Omega)$ and assume that $p(\cdot)$ and m are restricted as in* (5)-(6)*. Let $\hat{a}$ be a Carathéodory function satisfying* (2)-(4)*. Then the problem* (1) *has at least one weak solution $u \in W_0^{1,q(\cdot)}(\Omega)$ where $q(\cdot)$ is a continuous function on $\overline{\Omega}$ satisfying*

$$1\leq q(x)<\frac{Nm(p(x)-1)}{N-m} \quad \textit{for all} \quad x\in\overline{\Omega}. \tag{8}$$

Proof of Theorem 0.1. The proof needs three steps. (we refer to [7] for more details)
Firstly approximation, second uniform estimates and finally passage to the limit. □

Conclusions

- Under the assumption $f \in L^m(\Omega)$ in Theorem 0.1, we can deduce that f is never in the dual space $\left(W_0^{1,p(\cdot)}(\Omega)\right)'$, so that the result of this paper deals with irregular data. If m tends to be 1, then $q(\cdot)=\frac{Nm(p(\cdot)-1)}{N-m}$ tends to be $\frac{N(p(\cdot)-1)}{N-1}$.
- The result given in Lemma 2 [7] also holds for any measurable function $q : \overline{\Omega} \to \mathbb{R}$ such that

$$ess\ \inf_{x\in\overline{\Omega}}\left(\frac{Nm(p(x)-1)}{N-m}-q(x)\right)>0.$$

Indeed, in both cases there exists a continuous function $s : \overline{\Omega} \to \mathbb{R}$ such that for almost every $x \in \overline{\Omega}$:

$$q(x)\leq s(x)\leq\frac{Nm(p(x)-1)}{N-m}.$$

From Lemma 2 [7], we deduce, in both cases, that $(u_n)_n$ is bounded in $W_0^{1,s(\cdot)}(\Omega)$. Finally, by the continuous embedding $W_0^{1,s(\cdot)}(\Omega) \hookrightarrow W_0^{1,q(\cdot)}(\Omega)$, we have the desert result.

Forthcoming Research

We study the existence, uniqueness and qualitative properties of solutions of nonlinear anisotropic elliptic and parabolic equations with variable exponents and irregular data.

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