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Maximum principle for BSDEs with locally Lipschitz and logarithmic growth

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Abstract

This paper tackles a stochastic control problem involving a backward stochastic differential equation (BSDE) with a local Lipschitz coefficient and logarithmic growth. We derive the necessary and sufficient conditions for optimality that hold for all optimal controls, even without convexity assumptions on the control domain. These conditions involve a local Lipschitz stochastic differential equation and a minimized Hamiltonian. We begin by demonstrating the existence and uniqueness of the solution to the associated adjoint equation under suitable conditions. Next, we introduce a series of control problems with global Lipschitz coefficients using an approximation approach. This framework allows us to derive a stochastic maximum principle, facilitating the analysis of near-optimal controls within these approximated systems. Finally, we seamlessly transition back to the initial control problem through a well-defined limit process.

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1 Introduction

The domain of stochastic optimal control problems is commonly traversed through two primary avenues: the maximum principle introduced by Pontryagin and the method of dynamic programming. Each of these methodologies necessitates distinct mathematical treatments. Dynamic programming, for instance, aims to derive a second-order partial differential equation, commonly known as the Hamilton-Jacobi-Bellman (HJB) equation, serving as a characterization of the value function.

However, a significant drawback arises when employing this approach-classical solutions to the HJB equation are only guaranteed for sufficiently smooth value functions, a condition often unmet in practical scenarios. Crandall and Lions [9] addressed this limitation by introducing viscosity solutions, wherein (set-valued) sub-derivatives replace conventional derivatives. This innovation empowers dynamic programming with enhanced applicability in real-world situations.

While the maximum principle is extensively employed for solving optimal control problems in deterministic systems, translating theoretical results into practical solutions encounters numerous obstacles. The inherent difficulty lies in explicitly solving the resultant adjoint systems. Some scholars (e.g. [12, 13]) have proposed numerical methods to address such challenges, expanding the applicability of Pontryagin's maximum principle into fields like mathematical finance and economics. Several attempts have been made to loosen the constraints on coefficients, facilitating the extension of the stochastic maximum principle to irregular cases.

Mezerdi [7] pioneered this direction by deriving a maximum principle for a controlled stochastic differential equation (SDE) that accounts for a non-smooth drift, leveraging Clarke's generalized gradients, and stable convergence of probability measures. Building on this, Bahlali et al. [5] extended the principle to SDEs characterized by Lipschitz continuity and a non-degenerate diffusion matrix, employing Krylov's inequality with uniform ellipticity. In a broader context, Bahlali et al. [2] developed a stochastic maximum principle for optimizing control over a broad category of diffusion processes that exhibit degeneracy, assuming only Lipschitz continuity in state equation coefficient and continuous differentiability in cost functional coefficients. Chighoub et al. [8] further expanded these results to cases where both state equation and cost functional coefficients lack differentiability.

Recent advancements include Xu and Wu's [17] work, where they established the well-posedness of mild solutions for a class of mean-field BSDEs within Hilbert spaces with less restrictive conditions than Lipschitz continuity. They then demonstrated a maximum principle for optimal control problems involving mean-field type backward stochastic partial differential equations. Additionally, Orrieri [14] introduced a version of the maximum principle for optimal control in SDEs influenced by multidimensional Brownian processes. Dokuchaev and Zhou [10] derived conditions that are both required and sufficient for optimality in cases where the control domain lacks convexity.

Consider $T > 0$ and let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$ be a probability space with completeness, equipped with a filtration that satisfies the usual conditions. Given this probability space, we introduce a one-dimensional Brownian motion $W = (W_t)_{0\leq t\leq T}$. We make the assumption that $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ represents the $\mathbb{P}$-augmentation of the natural filtration associated with $(W_t)_{0\leq t\leq T}$. For our subsequent analysis, we define the following spaces for $p \geq 1$:

- $S^p([0,T],\mathbb{R})$: the space of continuous, $\mathbb{F}$-adapted stochastic processes $\{Y_t : t \in [0,T]\}$, such that

$$\mathbb{E}[\sup_{0\leq t\leq T} |Y_t|^p] < \infty.$$

- $\mathcal{M}^2([0,T],\mathbb{R})$: the space of $\mathbb{F}$-predictable, $\mathbb{R}$-valued processes $\{Z_t : t \in [0,T]\}$, satisfying

$$\int_0^T \mathbb{E}[|Z_r|^2]\mathrm{d}r < \infty.$$

- $\mathbb{L}^p_{loc}(\mathbb{R}_+,\mathbb{R})$: the set of $\mathbb{F}$-adapted processes taking values in $\mathbb{R}$, denoted by $\{X_t : t \geq 0\}$, such that

$$\int_0^T |X_r|^p \,\mathrm{d}r < \infty \ \mathbb{P}\text{-a.s for every } T.$$

We examine the following controlled backward stochastic differential equation (BSDE):

$$\begin{cases} \mathrm{d}Y_t &= f(t, Y_t, Z_t, v_t)\mathrm{d}t + Z_t \mathrm{d}W_t, \\ Y_T &= \zeta. \end{cases} \tag{1}$$

Here, f is a function defined on $f : [0, T] \times \mathbb{R} \times \mathbb{R} \times U \longrightarrow \mathbb{R}$. The terminal condition ζ is a random variable adapted to $\mathcal{F}_T$. The control variable $(v_t)_{t\geq 0}$ is represented by the process v_t, considered as an $\mathbb{F}$-adapted process that takes values within a non-empty subset U within $\mathbb{R}$. The collection of all admissible controls is represented by $\mathcal{U}_{ad}$.

Given a mapping g from $\mathbb{R}$ to $\mathbb{R}$, we describe the cost functional associated with the stochastic control issue as:

$$\mathcal{J}(v_\cdot) = \mathbb{E}\left[g(Y_0^v)\right]. \tag{2}$$

The aim is to minimize the cost functional (2) among all admissible controls. The control problem can now be stated as follows:

Problem (A) Given the cost functional (2) and the constraint (1), the objective is to identify an optimal control, denoted as u from the set $\mathcal{U}_{ad}$, that minimizes the specified cost functional.

There exists an extensive body of literature addressing stochastic optimal control issues related to BSDEs and Forward-BSDEs within the global Lipschitz framework. Azizi and Khelfallah [1] were the first to investigate a stochastic control problem involving BSDEs with local Lipschitz continuity in y and global Lipschitz continuity in z under the initial assumption. In their study, they demonstrated that the generator satisfies specific conditions, which include:

- There exist three constants, $M_0, M_1 > 0$, $\kappa \in (0, 1)$ and a non-negative function h defined on $\mathbb{R}_+$. For all y and z,

$$\begin{aligned} \left\langle y, f(t, y, z, v)\right\rangle &\leq M_0(1 + \left|y\right|^2 + \left|y\right| |z|) \text{ a.e. } t \in [0, T], \\ \left|f(t, y, z, v)\right| &\leq M_1(1 + h\left(\left|y\right|\right) + |z|^\kappa) \text{ a.e. } t \in [0, T]. \end{aligned}$$

Moreover, they present results under another assumption where the generator exhibits local Lipschitz continuity in both (y, z), along with linear growth. They establish necessary and sufficient optimality conditions for non-convex control domains, characterized by a linear SDE with local Lipschitz continuity and a maximum condition on the Hamiltonian.

In our context, we loosen the standard Lipschitz condition on the generator of the BSDEs, imposing a logarithmic growth condition with respect to y and linear growth with respect to z in the first assumption. In the second assumption, we require the generator to satisfy the logarithmic growth condition for both y and z, and we employ the Malliavin approach.

For ease of notation, we denote $h_\theta = \frac{\partial h}{\partial \theta}$ for a given function h and parameter θ. The primary challenge we face is with the coefficients in the resulting local Lipschitz linear adjoint equation,

$$\begin{cases} -\mathrm{d}x_t &= f_y(t, Y_t, Z_t, u_t)x_t \mathrm{d}t + f_z(t, Y_t, Z_t, u_t)x_t \mathrm{d}W_t, \\ x_0 &= g_y(Y_0), \end{cases} \tag{3}$$

which are only locally bounded. Consequently, they are locally Lipschitz on x but do not satisfy the linear growth condition.

Given the existing results in the literature, confirmation regarding whether the adjoint Equation (3) admits a unique solution remains elusive.

The organization of the paper is as follows: Sect. 2 introduces the foundational concepts of our study, including the existence and uniqueness of both BSDE (1) and SDE (3), as well as the control problem framework and preliminary lemmas. Section 3 establishes the necessary and sufficient conditions for optimality (maximum principle). Finally, Sect. 4 relaxes the linear condition on z by extending the logarithmic growth condition to both (y,z), with the analysis grounded in Malliavin differentiability.

2 Foundational concepts and existence findings

In this section, we will state some basic results related to BSDEs theory and prove the existence and uniqueness results for one kind of linear SDEs with local Lipschitz coefficients.

Assumption 1

(A.1.1) f and g are continuously differentiable with respect to (y,z) and there exists a positive constant L such that: $|g(y)| \leq L(1+|y|)$.

(A.1.2) We posit the existence of a positive constant λ, large enough where the expected value of $|\zeta|^{1+e^{\lambda T}}$ is finite.

(A.1.3) (i) f is continuous in (y,z).

(ii) There exist constants η, c_0, c_1: for any $t \geq 0$, y, $z, u \in U$:

$$|f(t,y,z,u)| \leq \eta + c_0|y||\ln|y|| + c_1|z|, \ a.e.$$

(A.1.4) There exist a real-valued sequence $(A_N)_{N>1}$ and constants $M_2 \in \mathbb{R}_+$, $r > 0$ such that:

(i) $\forall\, N > 1$, $1 < A_N \leq N^r$.

(ii) $\lim_{N\to\infty} A_N = \infty$.

(iii) For every $N \in \mathbb{N}$, $u \in U$ and every y, y', z, z' such that $|y|$, $|y'|$, $|z|$, $|z'| \leq N$, we have:

$$\begin{aligned} &\big(y-y'\big)\big(f(t,y,z,u) - f(t,y',z',u)\big) \\ &\leq M_2\big(\,|\,y-y'\,|^2 \ln(A_N) + |y-y'||z-z'|\sqrt{\ln(A_N)}\big). \end{aligned}$$

Remark 1 If f satisfies (A.1.1), then it satisfies a local Lipschitz condition, i.e., for all $N \in \mathbb{N}$, there exist two constants $L_{1,N}$, $L_{2,N} > 0$ such that for any $u \in U$ and for those $y, y', z, z' \in \mathbb{R}$ with $\max\{|y|,|y'|,|z|,|z'|\} \leq N$, the following condition holds:

$$\begin{aligned} &|f(t,y,z,u) - f(t,y',z,u)| \leq L_{1,N}|y-y'|, \\ &|f(t,y,z,u) - f(t,y,z',u)| \leq L_{2,N}|z-z'|. \end{aligned}$$

Remark 2 Assume that f satisfies (A.1.1) and (A.1.4). Consequently, $L_{1,N} = M_2 \ln(A_N)$, $L_{2,N} = M_2\sqrt{\ln(A_N)}$.

Remark 3 If f satisfies (A.1.3), then for every t, y, z and $u \in U$:

$$|f(t,y,z,u)| \leq \widetilde{\eta} + c_0|y||\ln|y|| + c_1|z|\sqrt{|\ln(|z|)|},$$

where $\widetilde{\eta} = \eta + c_1 e$.

The following lemmas establish estimates, guaranteeing the boundedness of both the generator and the solutions. The first two lemmas are thoroughly detailed and proven in [4], with further details provided in [6].

Lemma 1 *Assuming that conditions* (*A*.1.2) *and* (*A*.1.3) *hold, there exists a positive constant* $C(T,\alpha,\eta,c_0,c_1)$ *such that,*

$$\int_0^T \mathbb{E}\left[\left|f(s,Y_s,Z_s,u_s)\right|^{\alpha}\right] \mathrm{d}s \leq C(T,\alpha,\eta,c_0,c_1)\left(1 + \int_0^T \mathbb{E}\left[\,|Y_s|^{\mu_s+1} + |Z_s|^2\,\right]\mathrm{d}s\right),$$

where $1 < \alpha < 2$.

Lemma 2 *Let* $(Y_t, Z_t)_{t\geq 0}$ *represent the unique solutions to Equation* (1). *Then, there are constants* $C_{T,\eta}$ *and* $C(T,c_0,c_1)$, *both positive, such that, under Assumption* 1, *the following hold*:

$$\mathbb{E}\Big[\sup_{0\leq t\leq T}|Y_t|^{1+e^{\lambda T}}\Big] \leq C_{T,\eta}\mathbb{E}\Big[1+|\zeta|^{e^{\lambda T}+1}\Big].$$

$$\int_0^T \mathbb{E}[|Z_s|^2]\mathrm{d}s \leq C(T,\eta,c_0,c_1)\mathbb{E}\Big[1+|\zeta|^2+\sup_{0\leq t\leq T}|Y_t|^{1+e^{\lambda T}}\Big].$$

Lemma 3 *If the assumption of the previous Lemma* 2 *holds and if* ζ *is bounded, we can find constants* $C_{1,T}$, $C_{2,T}$ *and* $C_{3,T}$, *which depend on* η, *such that*:

(i) $\sup_{0\leq t\leq T}|Y_t|^{1+e^{\lambda T}} \leq C_{1,T}$, $\int_0^T \mathbb{E}[|Z_s|^2]\mathrm{d}s \leq C_{2,T}$.
(ii) $\int_0^T \mathbb{E}[|f(s,Y_s,Z_s,u_s)|^2]\mathrm{d}s \leq C_{3,T}$.

Proof We derive the following insight based on the work of Bahlali *et al.* [4].

$$|\,Y_t\,|^{1+e^{\lambda T}} \leq \ell(\eta)\Big(1+|\,\zeta\,|^{1+e^{\lambda T}} - \int_t^T (e^{\lambda s}+1)\,|\,Y_s\,|^{e^{\lambda s}}\,\mathrm{sgn}(Y_s)Z_s\mathrm{d}W_s\Big),$$

and

$$\int_t^T |Z_s|^2\,\mathrm{d}s \leq \ell(\eta)\left(1+|\zeta|^2+\sup_{s\in[0,T]}|Y_s|^{1+e^{\lambda T}} + \int_t^T Y_s Z_s\,\mathrm{d}W_s\right),$$

where $\ell(\eta)$ is a universal positive constant. We get the assertion (i) by taking the conditional expectation for Y and the expectation for the rest.

By (A.1.3) and assertion (i), and since $|Y_t| \leq 1 + |Y_t|^{1+e^{\lambda T}} \leq 1 + C_{1,T}$, we get:

- $$\int_0^T \mathbb{E}[|f(s,Y_s,Z_s,u_s)|^2]\mathrm{d}s \leq \ell(\eta)\left(1+\int_0^T \mathbb{E}\left[|Y_s|^{\mu_s+1}+|Z_s|^2\right]\mathrm{d}s\right) \leq C_{3,T}.$$ □

Theorem 4 *Let Assumptions* $(A.1.2)$–$(A.1.4)$ *hold, then the BSDE* (1) *admits a unique solution* (Y,Z) *in* $S^{1+e^{\lambda T}}([0,T],\mathbb{R})\times\mathcal{M}^2([0,T],\mathbb{R})$.

Under Assumptions (A.1.2)–(A.1.4), the conditions in [4] that guarantee the existence and uniqueness of the BSDE solution are satisfied. Therefore, the preceding theorem is applicable.

It is important to observe that, for every $v\in\mathcal{U}_{ad}$, the functions $f_y(t,\cdot,\cdot,v_t)$ and $f_z(t,\cdot,\cdot,v_t)$ are generally unbounded.

In the subsequent theorem, we establish the existence and uniqueness outcomes for the SDE given by (3) up to a potential explosion time.

Theorem 5 *Assuming that Assumption* 1 *is satisfied, we can assert that for any* $v\in\mathcal{U}_{ad}$, *the SDE* (3) *possesses a unique solution*.

Remark 4 The previous theorem cannot guarantee the existence of a global solution but rather only up to an 'explosion time' denoted as

$$\tau_N^{ex} := \inf\{t\in[0,T];\ |f_y(t,y,z,u)|\wedge|f_z(t,y,z,u)|\geq N\}.$$

We require the following additional assumptions to ensure the existence of a global solution.

- $\mathbf{H}_{loc}$: $f_y\in\mathbb{L}^1_{loc}(\mathbb{R}_+,\mathbb{R}), f_z\in\mathbb{L}^2_{loc}(\mathbb{R}_+,\mathbb{R})$.
- $\mathbf{H}_{lin}$: There exists a positive constant $L>0$, such that $\forall\,(y,z,u)\in\mathbb{R}\times\mathbb{R}\times U$:

$$|f_y(t,y,z,u)|\leq L(1+|y|)+\epsilon\ln(|z|+1),\ \text{a.e. } t\in[0,T],$$
$$|f_z(t,y,z,u)|\leq L(1+|y|)+\epsilon\sqrt{\ln(|z|+1)}\ \text{a.e. } t\in[0,T],$$

where ϵ is a positive constant that is small enough.

Remark 5 The assumption $\mathbf{H}_{loc}$ ensures that for any $(Y_t,Z_t)_{t\geq0}$ $\mathbb{F}$-adapted stochastic processes, the SDE (3) has a global solution, while the assumption $\mathbf{H}_{lin}$ guarantees the global solution under square-integrable $\mathbb{F}$-adapted stochastic processes (i.e., $(Y_t,Z_t)_{t\geq0}\in\mathbb{L}^2_{loc}(\mathbb{R}_+,\mathbb{R})$).

2.1 Control problem framework

This paper aims to address the control problem outlined in Equation (1) and the associated cost functional (2). Our objective is to determine both necessary and sufficient conditions for optimality. It is important to note that due to the unbounded nature of the derivatives of f, standard duality methods are not directly applicable to this context.

For any $p\geq1$ and $v\in\mathcal{U}_{ad}$, we introduce a family of semi-norms $(\rho^v_{N,p}(f))_{N\in\mathbb{N}}$ defined as follows:

$$\rho^v_{N,p}(f)=\left(\mathbb{E}\int_0^T\sup_{|y|,|z|\leq N}\left|f(r,y,z,v_r)\right|^p\mathrm{d}r\right)^{\frac{1}{p}}.$$

Lemma 6 *Let f be a function that satisfies Assumption 1 and $\mathbf{H}_{loc}$ or $\mathbf{H}_{lin}$. Then, there exists a sequence of functions f^n such that:*

(i) *For each n, f^n is globally Lipschitz in (y,z) -a.e. $t \in [0,T]$.*
(ii) *For each n, f^n satisfies Assumption 1.*
(iii) *For every n, $\rho^{v}_{n,p}(f^n - f) \to 0$ as $n \to \infty$.*
(iv) *For every n, $|f^n_y| \leq |f_y| + \frac{c}{n}|f|$, $|f^n_z| \leq |f_z| + \frac{c}{n}|f|$ and $\lim\limits_{n\to+\infty} g^n$ (resp. g^n_y) $\to g$ (resp. g_y).*

The following paragraphs are dedicated to transforming the original Problem *(A)* into a sequence of control issues characterized by globally Lipschitz continuous functions. To achieve this, take any specific $n \in \mathbb{N}^*$ and a control $v \in \mathcal{U}_{ad}$. Let $(\bar{Y}^n_t, \bar{Z}^n_t)_{t\geq 0}$ represent the solution of the corresponding controlled BSDE:

$$\begin{cases} \mathrm{d}\bar{Y}^n_t &= f^n(t, \bar{Y}^n_t, \bar{Z}^n_t, v_t)\mathrm{d}t + \bar{Z}^n_t \mathrm{d}W_t, \\ \bar{Y}^n_T &= \zeta. \end{cases} \tag{4}$$

Furthermore, define

$$\mathcal{J}^n(v_\cdot) = \mathbb{E}\left[g^n(\bar{Y}^n_0)\right]. \tag{5}$$

The subsequent lemma provides estimates that will be employed to establish a relationship between the control problem (4), (5) and Problem *(A)*.

Lemma 7 *Let $(Y_t)_{t\geq 0}$ and $(\bar{Y}^n_t)_{t\geq 0}$ be the solutions of BSDE (1) and (4), respectively, corresponding to the control $v \in \mathcal{U}_{ad}$. Then, for any $\alpha \in (1,2)$, $q \in (0,2)$ and any $\beta \in (1, 3 - \frac{2}{\alpha})$, the following hold:*

(i) $\sup_{t\in[0,T]} \mathbb{E}[|\bar{Y}^n_t - Y_t|^\beta] \leq K_{n,N}$, *and* $\mathbb{E}[\int_0^T |\bar{Z}^n_r - Z_r|^q \mathrm{d}r] \leq K_{n,N}$.
(ii) $|\mathcal{J}^n(v) - \mathcal{J}(v)| \leq C\varepsilon_{n,N}$,

where $K_{n,N}$ and $\varepsilon_{n,N}$ approach 0 as n goes to $+\infty$ and N approaches $+\infty$ subsequently, here N denotes the radius of the ball $B(0,N)$.

The proof of assertion (i) follows a similar methodology to that of Theorem 2.1 in [4], while assertion (ii) is derived using the approach outlined in [1].

Consider an optimal control u defined as the solution to:

$$\mathcal{J}(u_\cdot) = \inf_{v\in\mathcal{U}_{ad}} \mathcal{J}(v_\cdot),$$

under to the constraint (1). It is crucial to observe that u might not remain optimal for the modified control problem. By Lemma 7, there is a positive sequence (δ_n) approaching to 0 where:

$$\mathcal{J}^n(u_\cdot) \leq \inf_{v\in\mathcal{U}_{ad}} \mathcal{J}^n(v_\cdot) + \delta_{n,N},$$

where $\delta_{n,N} = 2C\varepsilon_{n,N}$. To facilitate the application of Ekeland's lemma, we need to introduce a metric d on $\mathcal{U}_{ad}$. For any two controls $u, v \in \mathcal{U}_{ad}$, the metric d is given by:

$$d(u_\cdot, v_\cdot) = \mathbb{P} \otimes \mathrm{d}t\,\{(\omega,t) \in \Omega \times [0,T] : u(\omega,t) \neq v(\omega,t)\},$$

here, $\mathbb{P}\otimes \mathrm{d}t$ denotes the product measure combining $\mathbb{P}$ and the Lebesgue measure over the interval $[0,T]$. Utilizing Ekeland's lemma on $\mathcal{J}^n(u_\cdot)$ allows us to find $u^n \in \mathcal{U}_{ad}$ satisfying:

$$d(u^n_\cdot, u_\cdot) \leq (\delta_{n,N})^{\frac{1}{2}}$$

and

$$\tilde{\mathcal{J}}^n(u^n_\cdot) \leq \tilde{\mathcal{J}}^n(v_\cdot) \quad \forall\, v \in \mathcal{U}_{ad},$$

given that

$$\tilde{\mathcal{J}}^n(v_\cdot) = \mathcal{J}^n(v_\cdot) + (\delta_{n,N})^{\frac{1}{2}} d(v_\cdot, u^n_\cdot).$$

From the preceding arguments, we can deduce that u^n solves the optimal control problem given by Equations (4) and (5), but with $\tilde{\mathcal{J}}^n$. For every $n \in \mathbb{N}^*$, consider the pair $(Y^n_t, Z^n_t)_{t\geq 0}$, representing the distinctive solution to the subsequent BSDE under the influence of u^n:

$$\begin{cases} \mathrm{d}Y^n_t &= f^n(t, Y^n_t, Z^n_t, u^n_t)\mathrm{d}t + Z^n_t \mathrm{d}W_t, \\ Y^n_T &= \zeta. \end{cases} \tag{6}$$

Associated with this control problem is the following cost function:

$$\mathcal{J}^n(u^n_\cdot) = \mathbb{E}\left[g^n(Y^n_0)\right]. \tag{7}$$

Now, we pose the subsequent optimization task, denoted as Problem *(B)*: For any integer n, find $u^n \in \mathcal{U}_{ad}$ that minimizes (7) subject to (6).

In concluding this subsection, we introduce a set of adjoint equations. For every integer n, examine the SDE given by:

$$\begin{cases} -\mathrm{d}x^n_t &= f^n_y(t, Y^n_t, Z^n_t, u^n_t)x^n_t \mathrm{d}t + f^n_z(t, Y^n_t, Z^n_t, u^n_t)x^n_t \mathrm{d}W_t, \\ x^n_0 &= g^n_y(Y^n_0). \end{cases} \tag{8}$$

Given that f^n is a globally Lipschitz, implying boundedness of f^n_y and f^n_z. Consequently, the coefficients of Equation (8) satisfy a global Lipschitz condition and exhibit linear growth behavior. This implies that for any integer n, Equation (8) possesses a unique solution.

Furthermore, we introduce a collection of Hamiltonian functions $\mathcal{H}^n : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times U \to \mathbb{R}$ as follows:

$$\mathcal{H}^n(t, y, z, x, u) = x f^n(t, y, z, u) \text{ for each } n \in \mathbb{N}.$$

2.2 Preliminary lemmas

In the following part of this subsection, we aim to consolidate and establish several helpful lemmas. These lemmas are instrumental in achieving our primary findings within the subsequent section.

Lemma 8 *Define* (f^n) *as the sequence of functions linked to* f *according to Lemma 6. Let* $(Y_t^n, Z_t^n)_{t\geq 0}$ *stand for the solution of Equation (6). Consequently, there exist constants* K_1, K_2 *and* K_3 *satisfying*:

(i) $\sup_n \mathbb{E}[\sup_{0\leq t\leq T} |Y_t^n|^{e^{\lambda T}+1}] \leq K_1$.

(ii) $\sup_n \mathbb{E}[\int_0^T |Z_s^n|^2 \mathrm{d}s] \leq K_2$.

(iii) $\sup_n \mathbb{E}[\int_0^T |f^n(s, Y_s^n, Z_s^n, u_s^n)|^\alpha \mathrm{d}s] \leq K_3$,

where $\alpha \in (1,2)$.

The demonstration of the following Lemma is outlined in [4].

Lemma 9 *Under Assumption 1, we have*:

$$\lim_{n\to\infty} \mathbb{E}\Big[\sup_{t\in[0,T]} |Y_t^n - Y_t|^\beta\Big] = 0. \tag{9}$$

$$\lim_{n\to\infty} \mathbb{E}\int_0^T \left|Z_t^n - Z_t\right|^q \mathrm{d}t = 0. \tag{10}$$

Lemma 10 *Under Assumption 1 and* $\mathbf{H}_{lin}$, *the following estimates hold*:

$$\lim_{n\to\infty} \mathbb{E}\int_0^T \left|f^n(r, Y_r^n, Z_r^n, u_r^n) - f(r, Y_r, Z_r, u_r)\right|^{\bar{\alpha}} \mathrm{d}r = 0. \tag{11}$$

$$\lim_{n\to\infty} \mathbb{E}\int_0^T \left|f_y^n(r, Y_r^n, Z_r^n, u_r^n) - f_y(r, Y_r, Z_r, u_r)\right|^q \mathrm{d}r = 0. \tag{12}$$

$$\lim_{n\to\infty} \mathbb{E}\int_0^T \left|f_z^n(r, Y_r^n, Z_r^n, u_r^n) - f_z(r, Y_r, Z_r, u_r)\right|^q \mathrm{d}r = 0, \tag{13}$$

where $q \in (0,2)$ *and* $\bar{\alpha} \in (1,\alpha)$.

Remark 6 To demonstrate the convergence of a sequence X_n of random variables in $\mathbb{L}^p$, where $p \geq 1$, it suffices to establish convergence in probability and ensure that $\{|X_n|^p,\ n \in \mathbb{N}^*\}$ is uniformly integrable.

Proof Assuming Assumption 1 and $\mathbf{H}_{lin}$ hold. Drawing from our knowledge and the preceding remark, it is essential to demonstrate the convergence in $\mathbb{L}^1$.

$$\begin{aligned}
&\mathbb{E}\int_0^T |f^n(r, Y_r^n, Z_r^n, u_r^n) - f(r, Y_r, Z_r, u_r)|\mathrm{d}r \\
\leq\ &\mathbb{E}\int_0^T |f^n(r, Y_r^n, Z_r^n, u_r) - f(r, Y_r, Z_r, u_r)|\mathrm{d}r \\
&+\mathbb{E}\int_0^T |f^n(r, Y_r^n, Z_r^n, u_r^n) - f^n(r, Y_r^n, Z_r^n, u_r)|\mathbb{1}_{\{u_r^n \neq u_r\}}\mathrm{d}r.
\end{aligned}$$

Considering the previous derivation in [4], we have:

$$\lim_{n\to\infty} \mathbb{E}\int_0^T |f^n(r, Y_r^n, Z_r^n, u_r) - f(r, Y_r, Z_r, u_r)|\mathrm{d}r = 0.$$

Holder's inequality yields to:

$$
\begin{aligned}
&\mathbb{E}\int_0^T |f^n(r,Y_r^n,Z_r^n,u_r^n)-f^n(r,Y_r^n,Z_r^n,u_r)|\mathbb{1}_{\{u_r^n\neq u_r\}}\mathrm{d}r\\
&\leq \left(\mathbb{E}\int_0^T |f^n(r,Y_r^n,Z_r^n,u_r^n)-f^n(r,Y_r^n,Z_r^n,u_r)|^\alpha \mathrm{d}r\right)^{\frac{1}{\alpha}}\left(\mathbb{E}\int_0^T \mathbb{1}_{\{u_r^n\neq u_r\}}\mathrm{d}r\right)^{1-\frac{1}{\alpha}}\\
&\leq (4K_3)^{\frac{1}{\alpha}}\left(d(u_\cdot^n,u_\cdot)\right)^{1-\frac{1}{\alpha}}.
\end{aligned}
$$

$d(u_\cdot^n,u_\cdot)$ approaches 0 as n tends to infinity, thus (11) is satisfied.

We give the proof of (12). The proof of (13) can be performed similarly. Since $|y||\ln|y|| \leq e^{-1}+|y|^2$ and for any $n\in\mathbb{N}^*$, $t\in[0,T]$, we have $|Y_t^n|,\ |Z_t^n|\leq n$. Thus by (A.1.3), we have for any $v\in\mathcal{U}_{ad}$ that,

$$
\begin{aligned}
\frac{1}{n^2}|f^n(r,Y_r^n,Z_r^n,v_r)|^2 &\leq \frac{C}{n^2}(1+\eta^2+|Y_r^n|^4+|Z_r^n|^2)\\
&\leq C(1+\frac{1}{n^2}+\frac{\eta^2}{n^2}+|Y_r^n|^2).
\end{aligned}
$$

By (i) of Lemma 8, we get:

$$
\sup_n \mathbb{E}\int_0^T \frac{1}{n^2}|f^n(r,Y_r^n,Z_r^n,v_r)|^2\mathrm{d}r \leq C, \tag{14}
$$

where C is a universal constant. Using assertion (iv) of Lemma 6, along with $\mathbf{H}_{lin}$ and (14), we obtain:

$$
\sup_n \mathbb{E}\int_0^T (|f_y^n(r,Y_r^n,Z_r^n,v_r)|^2+|f_z^n(r,Y_r^n,Z_r^n,v_r)|^2)\mathrm{d}r \leq K_4. \tag{15}
$$

Let $N>1$, we put $\Lambda_n^N := \left\{(r,\omega),\ |Y_r^n|+|Z_r^n|>N\right\}$ and $\bar{\Lambda}_n^N=\Omega\backslash\Lambda_n^N$, then we have:

$$
\begin{aligned}
&\mathbb{E}\int_0^T \left|f_y^n(r,Y_r^n,Z_r^n,u_r^n)-f_y(r,Y_r,Z_r,u_r)\right|\mathrm{d}r\\
&\leq \mathbb{E}\int_0^T \left|f_y^n(r,Y_r^n,Z_r^n,u_r^n)-f_y^n(r,Y_r^n,Z_r^n,u_r)\right|\mathbb{1}_{\{u_r^n\neq u_r\}}\mathrm{d}r\\
&+\mathbb{E}\int_0^T \left|f_y^n(r,Y_r^n,Z_r^n,u_r)-f_y(r,Y_r^n,Z_r^n,u_r)\right|\mathrm{d}r\\
&+\mathbb{E}\int_0^T \left|f_y(r,Y_r^n,Z_r^n,u_r)-f_y(r,Y_r,Z_r,u_r)\right|\mathrm{d}r.
\end{aligned}
$$

By Schwarz's inequality and $\mathbf{H}_{lin}$, we have:

$$
\begin{aligned}
&\mathbb{E}\int_0^T |f_y^n(r,Y_r^n,Z_r^n,u_r^n)-f_y^n(r,Y_r^n,Z_r^n,u_r)|\mathbb{1}_{\{u_r^n\neq u_r\}}\mathrm{d}r\\
&\leq 2\mathbb{E}\int_0^T \left(L(1+|Y_r^n|)+\epsilon\ln(|Z_r^n|+1)\right)\mathbb{1}_{\{u_r^n\neq u_r\}}\mathrm{d}r\\
&\leq 2L\mathbb{E}\int_0^T \left(2+|Y_r^n|+|Z_r^n|\right)\mathbb{1}_{\{u_r^n\neq u_r\}}\mathrm{d}r
\end{aligned}
$$

$$\leq 2L\left(8T+4\mathbb{E}\int_0^T(|Y_r^n|^2+|Z_r^n|^2)\mathrm{d}r\right)^{\frac{1}{2}}\left(\mathbb{E}\int_0^T 1\!\!1_{\{u_r^n\neq u_r\}}\mathrm{d}r\right)^{\frac{1}{2}}$$
$$\leq 4L\,(2T+TK_1+K_2)^{\frac{1}{2}}\left(d(u_.^n,u_.)\right)^{\frac{1}{2}}.$$

Therefore,

$$\lim_{n\to\infty}\mathbb{E}\int_0^T|f_y^n(r,Y_r^n,Z_r^n,u_r^n)-f_y^n(r,Y_r^n,Z_r^n,u_r)|1\!\!1_{\{u_r^n\neq u_r\}}\mathrm{d}r=0.$$

Due to the fact that $1\!\!1_{A^N}<\frac{|Y_r^n|+|Z_r^n|}{N}1\!\!1_{A^N}$, and by using Schwarz's inequality, we obtain:

$$\mathbb{E}\int_0^T\left|(f_y^n-f_y)(r,Y_r^n,Z_r^n,u_r)\right|\mathrm{d}r$$
$$\leq\rho_{N,1}^u(f_y^n-f_y)+\frac{2(TK_1+K_2)^{\frac{1}{2}}}{N}\left(\mathbb{E}\int_0^T\left|(f_y^n-f_y)(r,Y_r^n,Z_r^n,u_r)\right|^2\mathrm{d}r\right)^{\frac{1}{2}}.$$

By (15), we can assert the presence of a constant $\ell>0$ for which:

$$\mathbb{E}\int_0^T\left|(f_y^n-f_y)(r,Y_r^n,Z_r^n,u_r)\right|\mathrm{d}r\leq\rho_{N,1}^u(f_y^n-f_y)+\ell\Big(\frac{2(TK_1+K_2)}{N}(K_4)^{\frac{1}{2}}\Big).$$

Taking the limit initially with respect to n followed by N, we obtain,

$$\lim_{n\to\infty}\mathbb{E}\int_0^T\left|(f_y^n-f_y)(r,Y_r^n,Z_r^n,u_r)\right|\mathrm{d}r=0.$$

Assumption $\mathbf{H}_{lin}$ and Lemma 8 enable the use of the Lebesgue Dominated Convergence Theorem, which facilitates the demonstration that:

$$\lim_{n\to\infty}\mathbb{E}\int_0^T\left|f_y(r,Y_r^n,Z_r^n,u_r)-f_y(r,Y_r,Z_r,u_r)\right|\mathrm{d}r=0.$$

Hence, (12) is established. □

Assumption 2 The validity of Assumption 1 in conjunction with $\mathbf{H}_{lin}$, along with the constraint that ζ is bounded.

Lemma 11 *Assume that Assumption 2 holds. Let $(Y_t,Z_t)_{t\geq 0}$ (resp. $(Y_t^n,Z_t^n)_{t\geq 0}$) denote the unique solutions of the BSDE (1) (resp. (6)). Then, for any $v\in\mathcal{U}_{ad}$ and $p\geq 2$ there exists a universal constant C, for which:*

$$\mathbb{E}\int_0^T(|f|^2+|f_y|^p+|f_z|^p)(r,Y_r,Z_r,v_r)\mathrm{d}r\leq C,$$
$$\sup_n\mathbb{E}\int_0^T(|f^n|^2+|f_y^n|^p+|f_z^n|^p)(r,Y_r^n,Z_r^n,v_r)\mathrm{d}r\leq C.$$

Proof By assertion (i) of Lemma 3, we have Y is bounded. Moreover,

$$\ln(|z|+1)=\frac{p}{2}\ln(|z|+1)^{\frac{2}{p}}\ \leq\ \frac{p}{2}(|z|+1)^{\frac{2}{p}}.$$

Thus $(\ln(|z|+1))^p \le C(|z|^2+1)$. By $\mathbf{H}_{lin}$ and Lemma 3 we get:

$$\mathbb{E}\int_0^T (|f|^2+|f_y|^p+|f_z|^p)(r,Y_r,Z_r,v_r)\mathrm{d}r \le C.$$

For any $n\in\mathbb{N}^*$ and $t\in[0,T]$, we have $|Y_t^n|\le C_{1,T}$. Since $|Z_t^n|\le n$, Assumption (A.1.3) yields,

$$\frac{1}{n^p}|f^n(r,Y_r^n,Z_r^n,v_r)|^p \;\le\; C \text{ and } |f^n(r,Y_r^n,Z_r^n,v_r)|^2 \;\le\; C(1+|Z_r^n|^2).$$

Thus, by assertion (iv) of Lemma 6, assertion (ii) of Lemma 8, and the previous result, we have:

$$\sup_n \mathbb{E}\int_0^T (|f^n|^2+|f_y^n|^p+|f_z^n|^p)(r,Y_r^n,Z_r^n,v_r)\mathrm{d}r \le C. \qquad \square$$

Remark 7 If Assumption 2 holds, then for any $\alpha\in(1,2)$ and $p\ge 2$, Lemma 10 and Lemma 11 guarantee the following convergence:

$$\lim_{n\to\infty}\mathbb{E}\int_0^T \left|f^n(r,Y_r^n,Z_r^n,u_r^n)-f(r,Y_r,Z_r,u_r)\right|^\alpha \mathrm{d}r=0. \tag{16}$$

$$\lim_{n\to\infty}\mathbb{E}\int_0^T \left|f_y^n(r,Y_r^n,Z_r^n,u_r^n)-f_y(r,Y_r,Z_r,u_r)\right|^p \mathrm{d}r=0. \tag{17}$$

$$\lim_{n\to\infty}\mathbb{E}\int_0^T \left|f_z^n(r,Y_r^n,Z_r^n,u_r^n)-f_z(r,Y_r,Z_r,u_r)\right|^p \mathrm{d}r=0. \tag{18}$$

Lemma 12 *Under the fulfillment of Assumptions 2, the solutions x and x^n to Equations (3) and (8), respectively, are bounded in the space $S^p([0,T],\mathbb{R})$ for all $p\ge 2$. More specifically, two positive constants ℓ_T and $\overline{\ell}_T$ can be found, ensuring that*:

$$\mathbb{E}\left[\sup_{0\le t\le T}|x_t|^p\right]\le \ell_T,$$
$$\mathbb{E}\left[\sup_{0\le t\le T}|x_t^n|^p\right]\le \overline{\ell}_T,\ \forall n\in\mathbb{N}.$$

Proof Let $p\ge 2$. By applying Itô's formula, we obtain ($\mathrm{sgn}(x_t)x_t=|x_t|$):

$$\begin{aligned}
|x_t|^p &\le |g_y(Y_0)|^p + p\int_0^T |x_s|^p\big(|f_y|+\frac{p-1}{2}|f_z|^2\big)(s,Y_s,Z_s,u_s)\mathrm{d}s \\
&\quad + \left|\int_0^t |x_s|^p f_z(s,Y_s,Z_s,u_s)\mathrm{d}W_s\right| \\
&\le |g_y(Y_0)|^p + p\int_0^T \sup_{0\le r\le s}\{|x_r|^p\}\big(|f_y|+\frac{p-1}{2}|f_z|^2\big)(s,Y_s,Z_s,u_s)\mathrm{d}s \\
&\quad + \left|\int_0^t |x_s|^p f_z(s,Y_s,Z_s,u_s)\mathrm{d}W_s\right|.
\end{aligned} \tag{19}$$

By BDG's inequality

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T}\left|\int_0^t |x_s|^p f_z(s,Y_s,Z_s,u_s)\mathrm{d}W_s\right|\right]\\
&\le 3\mathbb{E}\left[\left(\int_0^T |x_s|^{2p}|f_z(s,Y_s,Z_s,u_s)|^2\mathrm{d}s\right)^{\frac{1}{2}}\right]\\
&\le 3\mathbb{E}\left[\left(\int_0^T \sup_{0\le r\le s}|x_r|^{2p}|f_z(s,Y_s,Z_s,u_s)|^2\mathrm{d}s\right)^{\frac{1}{2}}\right]\\
&\le 3\mathbb{E}\left[\sup_{0\le t\le T}|x_t|^{\frac{p}{2}}\left(\int_0^T \sup_{0\le r\le s}|x_r|^{p}|f_z(s,Y_s,Z_s,u_s)|^2\mathrm{d}s\right)^{\frac{1}{2}}\right]\\
&\le \mathbb{E}\left[\frac{1}{2}\sup_{0\le t\le T}|x_t|^p+\frac{9}{2}\int_0^T \sup_{0\le r\le s}|x_r|^{p}|f_z(s,Y_s,Z_s,u_s)|^2\mathrm{d}s\right],
\end{aligned}$$

the last inequality is obtained using Young's ($ab\le \frac{1}{6}a^2+\frac{3}{2}b^2$). Therefore, by taking the supremum and then the expectation of (19), and applying the previous inequality, we obtain:

$$\begin{aligned}
\mathbb{E}\Big[\sup_{0\le t\le T}|x_t|^p\Big] \le \mathbb{E}\Big[2|g_y(Y_0)|^p+\int_0^T \sup_{0\le r\le s}|x_r|^p\big(2p|f_y(s,Y_s,Z_s,u_s)|\\
+(p(p-1)+9)|f_z(s,Y_s,Z_s,u_s)|^2\big)\mathrm{d}s\Big].
\end{aligned}$$

Gronwall's lemma, yields,

$$\mathbb{E}\Big[\sup_{0\le t\le T}|x_t|^p\Big]\le 2\mathbb{E}\left[|g_y(Y_0)|^p\exp\left(\int_0^T\big(2p|f_y|+(p(p-1)+9)|f_z|^2\big)(s,Y_s,Z_s,u_s)\mathrm{d}s\right)\right].$$

Since g_y is locally bounded and $Y_0, Y_0^n\le C_{1,T}$ (where $C_{1,T}$ does not depend on n), $g_y(Y_0)$ and $g_y(Y_0^n)$ are bounded. Moreover, by $\mathbf{H}_{lin}$, we have:

$$\mathbb{E}\Big[\sup_{0\le t\le T}|x_t|^p\Big]\le C\mathbb{E}\left[\exp\left(\int_0^T\big(2p\epsilon\ln(|Z_s|+1)+(p(p-1)+9)\epsilon^2\ln(|Z_s|+1)\big)\mathrm{d}s\right)\right],$$

where C is a constant that may change from line to line. Since ϵ is sufficiently small, therefore $2p\epsilon+(p(p-1)+9)\epsilon^2\le 2$. Thus, by Jensen's inequality, we get:

$$\begin{aligned}
\mathbb{E}\Big[\sup_{0\le t\le T}|x_t|^p\Big]&\le C\mathbb{E}\left[\exp\left(\int_0^T\ln(|Z_s|+1)^2\mathrm{d}s\right)\right]\\
&\le C\Big(1+\int_0^T\mathbb{E}[|Z_s|^2]\mathrm{d}s\Big)\ =:\ell_T.
\end{aligned}$$

Following the same arguments as previously, and since $\frac{1}{n^p}|f^n(r,Y_r^n,Z_r^n,v_r)|^p\le C$, we have:

$$\sup_n\mathbb{E}\Big[\sup_{0\le t\le T}|x_t^n|^p\Big]\le \overline{\ell}_T.$$

□

Lemma 13 *Let* $(x_t)_{t\geq 0}$ *and* $(x^n_t)_{t\geq 0}$ *be respectively the solution of* (3) *and* (8), *then under Assumption* 2, *we have*:

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\mathbb{E}\left[\left|x^n_t - x_t\right|^p\right] = 0, \quad \forall\, p\geq 2. \tag{20}$$

Proof Lemma 12 implies that $\{|x^n_t|^p,\ t\in[0,T],\ n\in\mathbb{N}^*,\ p\geq 2\}$ is uniformly integrable. Based on Equations (3) and (8), applying Itô's formula, we get:

$$\begin{aligned}
|x^n_t - x_t|^2 \leq\, & |g^n_y(Y^n_0) - g_y(Y_0)|^2 \\
& +2\int_0^T |x^n_r - x_r||x^n_r f^n_y(r,Y^n_r,Z^n_r,u^n_r) - x_r f_y(r,Y_r,Z_r,u_r)|\mathrm{d}r \\
& +\int_0^T |x^n_r f^n_z(r,Y^n_r,Z^n_r,u^n_r) - x_r f_z(r,Y_r,Z_r,u_r)|^2\mathrm{d}r \\
& -2\int_0^t (x^n_r - x_r)(x^n_r f^n_z(r,Y^n_r,Z^n_r,u^n_r) - x_r f_z(r,Y_r,Z_r,u_r))\mathrm{d}W_r.
\end{aligned}$$

By using Young's inequality and taking the expectation, we arrive at:

$$\begin{aligned}
\mathbb{E}\big[|x^n_t - x_t|^2\big] \leq\, & \mathbb{E}\left[|g^n_y(Y^n_0) - g_y(Y_0)|^2\right] \\
& +2\mathbb{E}\left[\int_0^T |x^n_r - x_r|^2(|f^n_y| + |f^n_z|(r,Y^n_r,Z^n_r,u^n_r)\mathrm{d}r\right] \\
& +2\mathbb{E}\left[\int_0^T |x^n_r - x_r||x_r||f^n_y(r,Y^n_r,Z^n_r,u^n_r) - f_y(r,Y_r,Z_r,u_r)|\mathrm{d}r\right] \\
& +2\mathbb{E}\left[\int_0^T |x_r|^2|f^n_z(r,Y^n_r,Z^n_r,u^n_r) - f_z(r,Y_r,Z_r,u_r)|^2\mathrm{d}r\right].
\end{aligned}$$

Since for any $n\in\mathbb{N}^*$ and $p\geq 2$, $\mathbb{E}[\sup_{0\leq t\leq T}(|x_t|^p + |x^n_t|^p)]\leq \ell_T + \overline{\ell}_T$. By Hölder's inequality, we get a universal constant C, such that:

$$\begin{aligned}
\mathbb{E}\left[|x^n_t - x_t|^2\right] \leq\, & \mathbb{E}\left[|g^n_y(Y^n_0) - g_y(Y_0)|^2\right] + C\gamma^n \\
& +2\mathbb{E}\left[\int_0^T |x^n_r - x_r|^2(|f^n_y| + |f^n_z|^2)(r,Y^n_r,Z^n_r,u^n_r)\mathrm{d}r\right],
\end{aligned}$$

where,

$$\begin{aligned}
\gamma^n := \mathbb{E}\Big[\int_0^T & (|f^n_y(r,Y^n_r,Z^n_r,u^n_r) - f_y(r,Y_r,Z_r,u_r)|^2 \\
& +|f^n_z(r,Y^n_r,Z^n_r,u^n_r) - f_z(r,Y_r,Z_r,u_r)|^4)\mathrm{d}r\Big].
\end{aligned}$$

The sequence γ^n tends to zero as n approaches infinity, as indicated by (17) and (18). Moreover, with the same steps as in the proof of Lemma 12, we can obtain:

$$\sup_n \mathbb{E}\Big[\exp\Big(2\int_0^T (|f^n_y| + |f^n_z|^2)(r,Y^n_r,Z^n_r,u^n_r)\mathrm{d}r\Big)\Big]\leq C.$$

Establishing the desired result is facilitated by demonstrating the convergence of the initial terms to zero and applying Gronwall's lemma. Since $g_y(Y_0), g_y^n(Y_0)$ and $g_y(Y_0^n)$ are bounded, allowing us to use the Dominated Convergence Theorem. Furthermore, by (iv) of Lemma 6 and Equation (9), we obtain:

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}\left|g_y^n(Y_0^n) - g_y(Y_0)\right|^2 &\leq 2 \lim_{n\to\infty} \mathbb{E}\left[|g_y^n(Y_0^n) - g_y(Y_0^n)|^2 + |g_y(Y_0^n) - g_y(Y_0)|^2\right] \\ &= 0. \end{aligned}$$

□

3 Necessary and sufficient conditions for optimality

This section is dedicated to establishing the necessary optimality condition for the optimization Problem (A).

3.1 Necessary optimality condition

We rely on the following lemma to establish the necessary condition for optimality, which forms the foundation for our further investigation.

Lemma 14 *Under the fulfillment of Assumption 2, we can establish the following*:

$$\lim_{n\to\infty} \mathbb{E}\int_0^T \left|\Phi^n(r) - \Phi(r)\right| \mathrm{d}r = 0,$$

where

$$\Phi^n(r) = \left[\mathcal{H}^n(r, Y_r^n, Z_r^n, x_r^n, u_r^n) - \mathcal{H}^n(r, Y_r^n, Z_r^n, x_r^n, v_r)\right],$$

and

$$\Phi(r) = \left[\mathcal{H}(r, Y_r, Z_r, x_r, u_r) - \mathcal{H}(r, Y_r, Z_r, x_r, v_r)\right].$$

Proof A straightforward computation demonstrates that:

$$\begin{aligned}\mathbb{E}\int_0^T \left|\Phi^n(r) - \Phi(r)\right| \mathrm{d}r &\leq \mathbb{E}\int_0^T \left|f^n(r, Y_r^n, Z_r^n, u_r^n)x_r^n - f(r, Y_r, Z_r, u_r)x_r\right| \mathrm{d}r \\ &+ \mathbb{E}\int_0^T \left|f^n(r, Y_r^n, Z_r^n, v_r)x_r^n - f(r, Y_r, Z_r, v_r)x_r\right| \mathrm{d}r.\end{aligned}$$

To simplify matters, we represent the first and second integrals by I_1^n and I_2^n, respectively, and demonstrate their convergence to 0 as n goes to $+\infty$.

By applying Hölder's inequality (for α, $\bar{\alpha} = \frac{\alpha}{\alpha-1}$) and utilizing both 12 and property *(iii)* from Lemma 8, we obtain:

$$\begin{aligned} I_1^n &\leq \left(\int_0^T \mathbb{E}|f^n(r, Y_r^n, Z_r^n, u_r^n)|^\alpha \mathrm{d}r\right)^{\frac{1}{\alpha}} \left(\int_0^T \mathbb{E}|x_r^n - x_r|^{\bar{\alpha}} \mathrm{d}r\right)^{\frac{1}{\bar{\alpha}}} \\ &+ \left(\int_0^T \mathbb{E}|x_r|^{\bar{\alpha}} \mathrm{d}r\right)^{\frac{1}{\bar{\alpha}}} \left(\int_0^T \mathbb{E}|f^n(r, Y_r^n, Z_r^n, u_r^n) - f^n(r, Y_r, Z_r, u_r|^\alpha \mathrm{d}r\right)^{\frac{1}{\alpha}} \\ &\leq K_3^{\frac{1}{\alpha}} \left(\int_0^T \mathbb{E}|x_r^n - x_r|^{\bar{\alpha}} \mathrm{d}r\right)^{\frac{1}{\bar{\alpha}}} \end{aligned}$$

$$+T\ell_T\left(\int_0^T \mathbb{E}|f^n(r,Y_r^n,Z_r^n,u_r^n)-f(r,Y_r,Z_r,u_r)|^\alpha \mathrm{d}r\right)^{\frac{1}{\alpha}}.$$

By (16) and (20), I_1^n converges to 0 as $n\to\infty$. On the flip side, utilizing similar arguments as presented earlier, it becomes apparent that the limit of I_2^n tends to 0 as n approaches $+\infty$. This concludes the proof. □

Primary result of this paper

Theorem 15 *Let u denote the optimal control for the problem (A), and (Y,Z) represent the unique solution of BSDE (1) corresponding to u. There exists a unique adapted process $(x_t)_{t\geq 0}$ in $\mathcal{S}^2([0,T],\mathbb{R})$, which is the solution to the associated forward SDE (3). This process $(x_t)_{t\geq 0}$ is uniquely characterized by ensuring that the Hamiltonian $\mathcal{H}$ is minimized at the control $(u_t)_{t\geq 0}$, such that*

$$\mathcal{H}(t,Y_t,Z_t,x_t,u_t)=\min_{v\in\mathcal{U}_{ad}}\mathcal{H}(t,Y_t,Z_t,x_t,v_t)\,\mathrm{d}t\text{-}a.e.,\ \mathbb{P}\text{-}a.s. \tag{21}$$

Proof To elucidate the key steps in our proof, we begin by transforming Problem *(A)* into a more manageable Problem *(B)*.

Next, we employ the spike variation approach to establish a necessary condition for approximate optimality while addressing Problem *(B)*. Finally, leveraging Lemma 14 and taking appropriate limits, we culminate the desired optimality condition (21).

For any integer n, let u^n be a control that is optimal for Problem *(B)*, satisfying $\mathcal{J}^n(u^n_\cdot)\leq \inf_{v\in\mathcal{U}_{ad}}\mathcal{J}^n(v_\cdot)$. Let the solution of BSDE (6) be denoted by $\left(Y_t^n,Z_t^n\right)_{t\geq 0}$ associated with u^n. Define the spike variation as:

$$u_t^{n,\theta}=\begin{cases} v_t \text{ if } t\in[t_0,t_0+\theta),\\ u_t^n \text{ otherwise.}\end{cases}$$

where t_0 is a fixed time within the interval $[0,\ T]$, $\theta>0$ is a small positive constant, and v represents any $\mathcal{F}_{t_0}$-measurable random variable.

Consider the following:

$$\tilde{\mathcal{J}}^n(u^n_\cdot)\leq\tilde{\mathcal{J}}^n(u^{n,\theta}_\cdot),$$

and

$$d(u^{n,\theta}_\cdot,u^n_\cdot)\leq\theta.$$

These lead to:

$$\mathcal{J}^n(u^{n,\theta}_\cdot)-\mathcal{J}^n(u^n_\cdot)\geq-(\delta_{n,N})^{\frac{1}{2}}\theta. \tag{22}$$

Utilizing standard arguments (see, for instance, [18]), we can show that the expression on the left side of (22) is:

$$\mathbb{E}\int_{t_0}^{t_0+\theta}\left[\mathcal{H}^n(r,Y_r^n,Z_r^n,x_r^n,v_r)-\mathcal{H}^n(r,Y_r^n,Z_r^n,x_r^n,u_r^n)\right]dr+o\left(\theta\right).$$

By dividing each side of (22) by θ, we obtain:

$$-(\delta_{n,N})^{\frac{1}{2}} \leq \frac{1}{\theta}\mathbb{E}\int_{t_0}^{t_0+\theta}\left[\mathcal{H}^n(r,Y_r^n,Z_r^n,x_r^n,v_r)-\mathcal{H}^n(r,Y_r^n,Z_r^n,x_r^n,u_r^n)\right]\mathrm{d}r+\frac{o\,(\theta)}{\theta}.$$

Applying Lemma 14 and successively taking limits on n, N, and θ, while considering the arbitrary nature of t_0 in $[0,T]$, yields:

$$\mathbb{E}\left[\mathcal{H}(t,Y_t,Z_t,x_t,v_t)-\mathcal{H}(t,Y_t,Z_t,x_t,u_t)\right]\geq 0.$$

Consider a fixed element a within the set U, and let B be any set belonging to the σ-algebra $\mathcal{F}_t$. Define:

$$w_t = a\mathbb{1}_B + u_t\mathbb{1}_{\Omega\setminus B}.$$

The control w satisfies the admissibility criteria. Utilizing the aforementioned inequality with w, we infer:

$$\mathbb{E}\left[\mathbb{1}_B\left(\mathcal{H}(t,Y_t,Z_t,x_t,a)-\mathcal{H}(t,Y_t,Z_t,x_t,u_t)\right)\right]\geq 0,\quad \forall\, B\in\mathcal{F}_t,$$

which leads to:

$$\mathbb{E}^{\mathcal{F}_t}\left[\mathcal{H}(t,Y_t,Z_t,x_t,a)-\mathcal{H}(t,Y_t,Z_t,x_t,u_t)\right]\geq 0.$$

Since the expression inside the conditional expectation is measurable with respect to $\mathcal{F}_t$, the desired result follows directly. □

Example 1 Let $f(t,y,z,u) := \eta + y\ln|y| + z + u$. Clearly, f satisfies the conditions of existence and uniqueness, i.e., (*A.2*)-(*A.4*), as outlined in the first example in [3]. Therefore, it remains to verify that (*A.1*) holds.

The partial derivatives of f are given by:

$$f_y(t,y,z,u) = 1+\ln|y|;\;\; f_z(t,y,z,u)=1.$$

This shows that f is continuously differentiable with respect to z and is differentiable almost everywhere in y. Moreover, it is evident that for any $L\geq 1$ and $\epsilon>0$:

$$|f_y(t,y,z,u)|\leq L(1+|y|)+\epsilon\ln(|z|+1),\;\text{ a.e. } t\in[0,T],$$
$$|f_z(t,y,z,u)|\leq L(1+|y|)+\epsilon\sqrt{\ln(|z|+1)}\;\text{ a.e. } t\in[0,T].$$

Thus, assumption $\mathbf{H}_{lin}$ is satisfied.

Next, consider the function $g(y) := y^2$, which is continuously differentiable and locally Lipschitz. As a result, Assumption 2 is also fulfilled. According to Theorem 15, we then obtain:

$$\mathcal{H}(t,Y_t,Z_t,x_t,u_t) = \min_{v\in\mathcal{U}_{ad}}\mathcal{H}(t,Y_t,Z_t,x_t,v_t)\;\mathrm{d}t\text{-a.e., }\mathbb{P}\text{-a.s.},$$

where $\mathcal{H}(t,Y_t,Z_t,x_t,v_t) = x_t(\eta+Y_t\ln|Y_t|+Z_t+v_t)$.

3.2 The inverse problem

This section investigates the extension of a previously established necessary optimality condition (21) to serve as a sufficient condition under additional assumptions.

Theorem 16 *Assume that the mapping* $(y,z,u) \mapsto \mathcal{H}(t,y,z,x,u)$ *is convex for almost everywhere* $t \in [0,T]$, *and that* f *satisfies the Lipschitz condition with respect to* u. *Furthermore, assume that* g *is convex. Provided that condition* (21) *is satisfied, then* $(u_t)_{t\geq 0}$ *is optimal for Problem* (A).

Proof For any $t \in [0,\ T]$, let $\mathcal{V}_t$ denote the set of all $\mathcal{F}_t$-measurable, U-valued random variables. Consider any $B \in \mathcal{F}_t$. For $v \in \mathcal{V}_t$, define

$$\mathcal{I}_n(v) = \mathbb{E}[\mathcal{H}_n(t, \hat{X}_t^n, \hat{Y}_t^n, v)1\!\!1_B].$$

Let u satisfy the condition in Equation (21). Note that u does not necessarily satisfy the necessary condition for optimality for the perturbed control problem (6) and (7).

Using convergence results, a simple computation shows that:

$$\mathcal{I}_n(u_t) = \inf_{v\in\mathcal{V}_t} \mathcal{I}_n(v) + \delta_n.$$

where δ_n is a positive sequence that approaches 0.

Applying Ekeland's variational principle to $\mathcal{I}_n$, we can find $u^n \in \mathcal{U}_{ad}$ for which:

$$\mathcal{I}_{n,\delta}(v_t) = \mathcal{I}_n(v_t) + \sqrt{\delta_n} d(v_\cdot, u_\cdot^n), \qquad \text{for any } v \in \mathcal{U}_{ad}.$$

We want to show that u is an optimal control for the original cost function $\mathcal{J}$.

(i) u^n minimizes $\mathcal{I}_{n,\delta}$:

$$\mathcal{I}_{n,\delta}\left(u_t^n\right) \leq \mathcal{I}_{n,\delta}(v_t), \quad \text{for any } v \in \mathcal{U}_{ad}.$$

(ii) The distance between u^n and u is bounded by:

$$d\left(u_\cdot^n, u_\cdot\right) \leq \sqrt{\delta_n}.$$

(iii) Given that B is an arbitrary element of the σ-algebra $\mathcal{F}_t$ and since u^n minimizes $\mathcal{I}_{n,\delta}$ by definition, we have:

$$\begin{aligned}\sqrt{\delta_n} d(v_\cdot, u_\cdot^n) &\geq \mathbb{E}\left[\mathcal{H}^n(t, Y_t^n, Z_t^n, x_t^n, u_t^n) - \mathcal{H}^n(t, Y_t^n, Z_t^n, x_t^n, v_t)\middle|\mathcal{F}_t\right] \\ &= \mathcal{H}^n(t, Y_t^n, Z_t^n, x_t^n, u_t^n) - \mathcal{H}^n(t, Y_t^n, Z_t^n, x_t^n, v_t).\end{aligned}$$

Therefore, it follows that:

$$\mathcal{H}_\delta^n(t, Y_t^n, Z_t^n, x_t^n, u_t^n) - \mathcal{H}_\delta^n(t, Y_t^n, Z_t^n, x_t^n, v_t) \leq 0, \tag{23}$$

where $\mathcal{H}_\delta^n(t, Y_t^n, Z_t^n, x_t^n, v_t) = \mathcal{H}^n(t, Y_t^n, Z_t^n, x_t^n, v_t) + \sqrt{\delta_n} d(v_\cdot, u_\cdot^n)$.

Since f^n is globally Lipschitz and x^n is bounded, $\mathcal{H}^n$ is also globally Lipschitz. Thus, by applying the result from [10], the necessary condition (23) becomes sufficient. Therefore, we obtain:

$$\mathcal{J}_\delta^n\left(u_.^n\right) = \inf_{v\in\mathcal{U}_{ad}} \mathcal{J}_\delta^n(v_.).$$

This definition of the modified cost-functional:

$$\mathcal{J}_\delta^n(v_.) := \mathcal{J}^n(v_.) + \sqrt{\delta_n} d\left(v_., u_.^n\right),$$

allows us to conclude that for each admissible control $v \in \mathcal{U}_{ad}$,

$$\mathcal{J}^n\left(u_.^n\right) \le \mathcal{J}^n(v_.) + O\left(\delta_n\right),$$

where $O\left(\delta_n\right)$ represents terms that vanish as δ_n approaches zero.

According to assertion (ii) in Lemma 7, $\mathcal{J}^n(v_.)$ approaches $\mathcal{J}(v_.)$ as n goes to infinity. Moreover, we have:

$$\begin{aligned}
|\mathcal{J}^n\left(u_.^n\right) - \mathcal{J}\left(u_.\right)| &\le \mathbb{E}[|g^n(Y_0^n) - g(Y_0)|] \\
&\le \mathbb{E}[|g^n(Y_0^n) - g^n(Y_0)|] + \mathbb{E}[|g^n(Y_0) - g(Y_0)|] \\
&\le C\mathbb{E}[|Y_0^n - Y_0|] + \mathbb{E}[|g^n(Y_0) - g(Y_0)|].
\end{aligned}$$

Since g (respectively, g^n) has linear growth and Y_0 (respectively, Y_0^n) is bounded, this enables the application of the Dominated Convergence Theorem. By assertion (iv) of Lemma 6 and Lemma 9, we obtain: $\lim_{n\to+\infty} \mathcal{J}^n\left(u_.^n\right) \to \mathcal{J}(u_.)$. Thus,

$$\mathcal{J}(u_.) = \inf_{u\in\mathcal{U}_{ad}} \mathcal{J}(v_.),$$

which implies that u is an optimal control for the cost function $\mathcal{J}$. □

4 A generalized logarithmic growth condition in the context of Malliavin differentiability

In this section, we address a significant extension of the previous part by relaxing the linear condition on z. Specifically, we generalize the logarithmic growth condition to include both y and z, thereby broadening the applicability of the results beyond the restrictive linear case. This relaxation is achieved through the use of Malliavin differentiability.

Assumption 3

(A.3.1) f and g are continuously differentiable with respect to (y, z) and f is globally Lipschitz with respect to v.

(A.3.2) Assume that ζ is bounded and an element of $\mathbb{D}^{1,2}$, and there are constants M_3 and M_4 for which, for all $v \in \mathcal{U}_{ad}$, we have:

$$\int_0^T |D_r v_s| \mathrm{d}s \le M_3, \text{ and } |D_r \zeta| \le M_4, \forall\, r \le T.$$

(A.3.3) There exists a positive constant c such that, for every $t, y, z, v \in U$:

$$|f(t,y,z,v)| \leq c(1 + |y||\ln|y|| + |z|\sqrt{|\ln(|z|)|}).$$

(A.3.4) There is a positive constant $L > 0$ such that for all $(y,z,v) \in \mathbb{R} \times \mathbb{R} \times U$:

$$|f_y(t,y,z,v)| \leq L(1 + |y|) + \ln(|z| + 1), \text{ -a.e. } t \in [0,T].$$

Theorem 17 *Assuming conditions* $(A.3.2)$ *and* $(A.3.3)$ *hold, the BSDE* (1) *possesses at least one solution* (Y,Z) *in* $S^{1+e^{\lambda T}}([0,T],\mathbb{R}) \times \mathcal{M}^2([0,T],\mathbb{R})$.

Indeed, if Assumptions (A.3.2) and (A.3.3) hold, then conditions (H1) and (H2) in [4] are fulfilled. Therefore, according to Theorem 2.2 in [4], the BSDE (1) is guaranteed to have a solution.

Lemma 18 *If Assumption* 3 *holds, we can get constants* $C_{1,T}$, $C_{2,T}$ *and* $C_{3,T}$ *such that*:

(i) $\sup_{0\leq t\leq T} |Y_t| \leq C_{1,T}$.
(ii) $\sup_{0\leq t\leq T} |Z_t| \leq C_{2,T}$.
(iii) $\sup_{0\leq t\leq T} |f(t,Y_t,Z_t,v_t)| \leq C_{3,T}$.

Proof By following the identical procedure used in the proof of Lemma 3, we can show that assertion (i) also holds.

We aim to substantiate assertion (ii).

Let $N \in \mathbb{N}^*$ and $f^N(t,y,z,v) = f(t,y,z,v)\psi(\frac{z}{N})$, where $\psi(x) = 1$ if $|x| \leq 1$ and $\psi(x) = 0$ if $|x| \geq 2$. Clearly f^N satisfies Assumption 3, thus:

$$\begin{cases} dY_t = f^N(t,Y_t,Z_t,v_t)dt + Z_t dW_t, \\ Y_T = \zeta, \end{cases}$$

admits a solution $(Y,Z) \in S^{1+e^{\lambda T}}([0,T],\mathbb{R}) \times \mathcal{M}^2([0,T],\mathbb{R})$. Moreover, $\sup_{0\leq t\leq T}|Y_t| \leq C_{1,T}$. According to Proposition 2.2 in [15], we deduce that for all $t \leq T$, Y and Z are elements of $\mathbb{D}^{1,2}$. Furthermore, for all $r \in [0,T]$ the pair $(D_rY_t, D_rZ_t)_{t\leq T}$ satisfies linear BSDE:

$$\begin{aligned} D_rY_t &= D_r\zeta - \int_t^T \big(f_y^N(s,Y_s,Z_s,v_s)D_rY_s + f_z^N(s,Y_s,Z_s,v_s)D_rZ_s\big)ds \\ &\quad - \int_t^T A_sD_rv_s ds - \int_t^T D_rZ_s dW_s, \\ D_tY_t &= Z_t, \end{aligned}$$

where $(A_s)_{s\geq 0}$ is a bounded process, with the bound denoted by a constant M_5 [11]. Consider the process $\gamma^{f_z} = (\gamma_t^{f_z})_{0\leq t\leq T}$ given by:

$$\gamma_t^{f_z} := \mathcal{E}\left(-\int_0^t f_z^N(s,Y_s,Z_s,v_s)dW_s\right), \; t \in [0,T], \; \mathbb{P} \; a.s.,$$

where $\mathcal{E}$ denotes the stochastic exponential. Since f_z^N is uniformly bounded it follows that, the process $(\gamma_t^{f_z})_{0\leq t\leq T}$ is a martingale process. Moreover, $\mathbb{E}[|\gamma_t^{f_z}|^2]$ is finite. Let

$\gamma_t^{f_z} := \frac{d\mathbb{P}^{f_z}}{d\mathbb{P}}|\mathcal{F}_t$, this implies absolute continuity of $\mathbb{P}^{f_z}$ with respect to $\mathbb{P}$ under Girsanov's theorem.

Girsanov's theorem further establishes that:

$$W_t^{f_z} = W_t + \int_0^t f_z^N(s, Y_s, Z_s, \nu_s)\mathrm{d}s, \text{ for } t \in [0, T],$$

is a Brownian motion under $\mathbb{P}^{f_z}$. Therefore, under $\mathbb{P}^{f_z}$ we have

$$\begin{aligned} D_r Y_t &= D_r\zeta - \int_t^T \left(f_y^N(s, Y_s, Z_s, \nu_s) D_r Y_s + A_s D_r \nu_s\right) \mathrm{d}s \\ &\quad - \int_t^T D_r Z_s \mathrm{d}W_s^{f_z} \quad t \le T, \\ D_r Y_t &= 0 \quad r > t. \end{aligned} \tag{24}$$

Moreover,

$$\begin{aligned} \mathbb{E}^{\mathbb{P}^{f_z}}\left[\left(\int_0^T |D_r Z_s|^2 \mathrm{d}s\right)^{\frac{1}{2}}\right] &= \mathbb{E}\left[\gamma_T^{f_z}\left(\int_0^T |D_r Z_s|^2 \mathrm{d}s\right)^{\frac{1}{2}}\right] \\ &\le \mathbb{E}[|\gamma_T^{f_z}|^2] + \mathbb{E}\left[\int_0^T |D_r Z_s|^2 \mathrm{d}s\right] < \infty. \end{aligned}$$

By taking the conditional expectation of (24) and applying Jensen's inequality, we obtain:

$$\begin{aligned} |D_r Y_t| &\le M_4 + \mathbb{E}^{\mathbb{P}^{f_z}}\Big[\int_0^T |A_s D_r \nu_s|\mathrm{d}s + \int_t^T |f_y^N(s, Y_s, Z_s, \nu_s) D_r Y_s|\mathrm{d}s \Big| \mathcal{F}_t\Big] \\ &\le M_4 + M_3 M_5 + \mathbb{E}^{\mathbb{P}^{f_z}}\Big[\int_t^T |f_y^N(s, Y_s, Z_s, \nu_s) D_r Y_s|\mathrm{d}s \Big| \mathcal{F}_t\Big]. \end{aligned} \tag{25}$$

Since $\sup_{t\in[0,T]} |Y_t| \le C_{1,T}$ and ψ guarantees that $|Z_t| \le N$, there exists a constant $C_{T,N}$ such that $|f_y^N(s, Y_s, Z_s, \nu_s)| \le C_{T,N}$. For any $\iota \le t$, we have:

$$\mathbb{E}^{\mathbb{P}^{f_z}}\Big[|D_r Y_t|\Big|\mathcal{F}_\iota\Big] \le M_4 + M_3 M_5 + C_{T,N}\int_t^T \mathbb{E}^{\mathbb{P}^{f_z}}\Big[|D_r Y_s|\Big|\mathcal{F}_\iota\Big]\mathrm{d}s.$$

Gronwall's Lemma yields to,

$$\mathbb{E}^{\mathbb{P}^{f_z}}\Big[|D_r Y_t|\Big|\mathcal{F}_\iota\Big] \le (M_4 + M_3 M_5)e^{TC_{T,N}} := M_6 e^{TC_{T,N}}.$$

For $\iota = t$, we get $|D_r Y_t| \le M_6 e^{TC_{T,N}}$; thus, $(D_r Y_t)_{t\ge 0}$ is uniformly bounded. Therefore, we apply Gronwall's Lemma to (25) (Theorem 1 in [16]), and obtain:

$$|D_r Y_t| \le M_6 \mathbb{E}^{\mathbb{P}^{f_z}}\Big[\exp\Big(\int_t^T |f_y^N(s, Y_s, Z_s, \nu_s)|\mathrm{d}s\Big)\Big|\mathcal{F}_t\Big].$$

Using (A.3.4) and the boundedness of Y and for $r = t$,

$$|Z_t| \le M_6 \exp\left(L(1 + C_{1,T})\right)\mathbb{E}^{\mathbb{P}^{f_z}}\left[\exp\left(\int_t^T \ln\left(|Z_s| + 1\right)\mathrm{d}s\right)\Big|\mathcal{F}_t\right]$$

$$\leq M_6 \exp(L(1+C_{1,T}))\mathbb{E}^{\mathbb{P}^{fz}}\left[\int_t^T (|Z_s|+1)\mathrm{d}s|\mathcal{F}_t\right]$$
$$\leq M_6 \exp(L(1+C_{1,T}))\left(T+\mathbb{E}^{\mathbb{P}^{fz}}\left[\int_t^T |Z_s|\mathrm{d}s|\mathcal{F}_t\right]\right).$$

By computing the conditional expectation given $\mathcal{F}_\iota$, where $\iota \leq t$, we obtain:

$$\mathbb{E}^{\mathbb{P}^{fz}}[|Z_t||\mathcal{F}_\iota] \leq M_6 \exp(L(1+C_{1,T}))\left(T+\mathbb{E}^{\mathbb{P}^{fz}}\left[\int_t^T |Z_s|\mathrm{d}s|\mathcal{F}_\iota\right]\right).$$

By applying Gronwall's Lemma and then setting $\iota = t$, we obtain:

$$\sup_{0\leq t\leq T}|Z_t| \leq M_6 T\exp\left(L(1+C_{1,T})\right)\exp\left(M_6 T\exp\left(L(1+C_{1,T})\right)\right)$$
$$= M_6 T\exp\left(L(1+C_{1,T})+M_6 T\exp\left(L(1+C_{1,T})\right)\right) =: C_{2,T}.$$

Alternatively, we can use Theorem 1 from [16], as $Z_t = D_t Y_t$, and thus it is uniformly bounded.

Thus, for any $N \geq C_{2,T}$, $f^N = f$ and $\sup_{0\leq t\leq T}|Z_t| \leq C_{2,T}$. The assertion (iii) follows directly from (A.3.3) and the preceding assertions. □

Theorem 19 *Under Assumption 3, the BSDE (1) has one solution.*

Proof According to Theorem 17, the BSDE (1) has a solution. To prove uniqueness, we consider two solutions (Y,Z), (Y',Z') of (1) with the same terminal condition. It follows that:

$$Y_t - Y'_t = -\int_t^T \left(f(s,Y_s,Z_s,\nu_s) - f\left(s,Y'_s,Z'_s,\nu_s\right)\right)\mathrm{d}s - \int_t^T \left(Z_s - Z'_s\right)\mathrm{d}W_s$$
$$= -\int_t^T \left(f(s,Y_s,Z_s,\nu_s) - f\left(s,Y'_s,Z_s,\nu_s\right)\right)\mathrm{d}s$$
$$-\int_t^T \left(f\left(s,Y'_s,Z_s,\nu_s\right) - f\left(s,Y'_s,Z'_s,\nu_s\right)\right)\mathrm{d}s - \int_t^T \left(Z_s - Z'_s\right)\mathrm{d}W_s.$$

Since f is locally Lipschitz and according to Lemma 18 the solutions are bounded, thus there is a positive constant C_T that is determined by $C_{1,T}$ and $C_{2,T}$, such that $\forall s \in [0,T]$:

$$|f(s,Y_s,Z_s,\nu_s) - f(s,Y'_s,Z'_s,\nu_s)| \leq C_T(|Y_s - Y'_s| + |Z_s - Z'_s|).$$

By taking similar steps as the proof of Lemma 18, we get:

$$Y_t - Y'_t = -\int_t^T \left(f(s,Y_s,Z_s,\nu_s) - f\left(s,Y'_s,Z_s,\nu_s\right)\right)\mathrm{d}s$$
$$-\int_t^T \left(Z_s - Z'_s\right)d\widetilde{W}_s,$$

where

$$\widetilde{W}_s = W_s + \int_0^t \left(f\left(s,Y'_s,Z_s,\nu_s\right) - f\left(s,Y'_s,Z'_s,\nu_s\right)\right)\left(Z_s - Z'_s\right)^{-1}\mathbb{1}_{\{Z_s\neq Z'_s\}}\mathrm{d}s.$$

Moreover, the same arguments yield that for all $t \in [0, T]$: $|Y_t - Y'_t| = 0$. This implies Y and Y' coincide. Intuitively, this should also imply, $Z_t = Z'_t$ for all t. Thus, the uniqueness is satisfied. □

These results ensure that the control problem is well-posed. Additionally, the boundedness of f and f_y allows us to leverage the previous control result under Assumption 3.

5 Conclusion

In this study, we explored a stochastic optimal control problem for a specific type of controlled BSDE characterized by a locally Lipschitz coefficient and a generator with logarithmic growth. The main challenges stemmed from the local Lipschitz nature of the BSDE generator and the adjoint equation, which is described by a linear SDE, complicating the application of standard duality techniques for solving the control problem. To address these challenges, we introduced certain assumptions to ensure the existence and uniqueness of the associated adjoint process. By applying the variational principle of Ekeland in conjunction with approximation techniques and taking limits, we established the required conditions for both necessity and sufficiency in optimality.

6 Perspective

We plan to extend our results to backward stochastic differential equations driven by a Poisson process and a Brownian motion, aiming to establish the necessary and sufficient conditions for optimality in this broader context [6]. Additionally, we intend to relax the bounded condition on the terminal variable ζ and to illustrate our findings through a finance application supported by a numerical study.

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