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PROBLEM ELLIPTIC ANISOTROPIC NONLINEAR IN $\mathbb{R}^N$ WITH VARIABLE EXPONENT AND LOCALLY INTEGRABLE DATA

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Dedication

I dedicate this humble work:

-To my beloved father, may he rest in peace, and my dear mother, may Allah protect her.

-To my brothers and sisters.

-To all my family members.

-To all my friends and everyone in my department family.

-To all my loved ones (Um Al-Shawashi) whom I have known throughout my life.

summary.pdf

ملخص

تهدف هذه الأطروحة الى دراسة وجود وصقالة الحلول الضعيفة لفئة المعادلات الناقصية غير الخطية و غير المتجانسة في فضاء تابعي ذي أسس متغيرة.
ويعتمد بحثنا في هذا العمل على المسألة التالية:

$$B(u) + F(x,u) = f, \qquad x \in \mathbb{R}^N, N \geq 2$$

حيث

$f \in L^1_{loc}(\mathbb{R}^N)$ و المؤثر $B(u)$ معرف كما يلي

$$B(u) = -\sum_{i=1}^{N} D(d_i(x,u)a_i(x,u,Du))$$

حيث

$$a_i: \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}, d_i: \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$$

هي توابع كاراتيدوري التي تحقق الشرط

$$\frac{\eta}{(1+|u|)^{\gamma_i(x)}} \leq d_i(x,u) \leq \mu$$

و المؤثر $B(u)$ ليس ناقصيا عندما تكون قيم u كبيرة جدا.

تعتمد الخطوات الأساسية للإثبات على الحصول على تقديرات محلية مناسبة للمشكلات التقريبية، ثم المرور الى النهاية. تمثل النتائج المتحصل عليها تعميما للنتائج المعروفة في حالة الأسس الثابتة. بالاضافة الى تعميم بعض النتائج الواردة في المرجع "12".

الكلمات المفتاحية: المسائل الناقصية غير الخطية، الأسس المتغيرة ، معادلات متباينة الخواص، الحلول الضعيفة، توابع قابلة للتكامل محليا.

Abstract

This work is devoted to establishing the existence of weak solutions for a certain class of nonlinear anisotropic elliptic equations, where the involved exponents vary with position and the coercivity condition may degenerate. The equations under consideration take the following general form

$$B(u) + H(x, u) = f, \quad x \in \mathbb{R}^N, \; N \geq 2$$

where f is locally integrable on $\mathbb{R}^N$ and the operator

$$B(u) = -\sum_{i=1}^{N} D_i(e_i(x, u) b_i(x, u, Du))$$

is properly defined between $W_0^{1,p(.)}(\Omega)$, (Ω or $\mathbb{R}^N$)and its dual. Suppose that $b_i : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$, are a Carathéodory functions.
The functions $e_i : \mathbb{R}^N \times \mathbb{R} \longrightarrow \mathbb{R}$ are Carathéodory functions and satisfying the following condition

$$\frac{\eta}{(1 + |u|)^{\gamma_i(x)}} \leq e_i(x, u) \leq \mu,$$

where η, μ are strictly positeve real numbers and $\gamma_i(x) \geq 0, i = 1, ..., N$ are continuous functions on $\mathbb{R}^N$. And $H : \mathbb{R}^N \times \mathbb{R} \longrightarrow \mathbb{R}$ be a Carathéodory functions. The differential operetor B is not coercive if u is large.

The core strategy of the proof involves deriving local estimates for a sequence of appropriately constructed approximate problems, followed by a limiting process. The findings presented here extend known results from the constant exponent framework and also build upon certain results discussed in [12].

KEYWORDS: Anisotropic equations, Variable exponents, Nonlinear elliptic problem, Weak solutions, Locally integrable data.

Résumé

Dans cette thèse, nous prouvons l'existece et la régularité de solutions faibles pour une classe d'équations elliptiques anisotropes non linéaires à exposants variables et à coercivité dégénérée. Nous considérons le problème (P) suivant:

$$B(u) + H(x, u) = f, \quad x \in \mathbb{R}^N, \ N \geq 2$$

où f est intégrable sur tout sous-ensemble compact de $\mathbb{R}^N$ et l'opérateur

$$B(u) = -\sum_{i=1}^{N} D_i(e_i(x, u) b_i(x, u, Du))$$

est bien défini comme une application entre l'espace de Sobolev à exposant variable $W_0^{1,p(.)}(\Omega)$, (Ω or $\mathbb{R}^N$) et son espace dual. $b_i : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$, est une fonetion de Carathéodory, avec $e_i : \mathbb{R}^N \times \mathbb{R} \longrightarrow \mathbb{R}$ est une fonction de Carathéodory satisfait la condtion suivante

$$\frac{\eta}{(1+|u|)^{\gamma_i(x)}} \leq e_i(x, u) \leq \mu,$$

où $\eta > 0, \mu > 0$ et $\gamma_i(x) \geq 0, i = 1, ..., N$ les fonctions continues $\mathbb{R}^N$. et $H : \mathbb{R}^N \times \mathbb{R} \longrightarrow \mathbb{R}$ les fonctions Carathéodory. L'opérateur différentiel B n'est pas coercitif lorsque s est grand. Les principales étapes de la démonstration consistent à établir des estimations locales pour des problémes approchés appropriés, puis à passer à la limite. Nos résultats constituent des généralisations des résultats correspondants obtenus dans le certains résultats présentés dans [12].

MOTS-CLÉS: Équations anisotropes, Exposants variables, Problèmes elliptiques nonlinéaires, Solutions faibles, Données localement intégrables.

Notations

- $\mathbb{N}$: The collection comprising all natural numbers.
- $\mathbb{R}^N$: Real Euclidean space, the N-dimensional.
- Ω: open bounded from $\mathbb{R}^N$.
- $\partial\Omega$: boundary of Ω.
- $x = (x_1, ..., x_n)$ as a general element of the space $\mathbb{R}^N$.
- $U \subset\subset \Omega$: means that the closure of U is compact and $\overline{U} \subset \Omega$.
- $|E|$ or $\mathrm{meas}(E)$: represents the Lebesgue measure associated with the subset E.
- a.e. : stands for almot eveywhere
- V': represents the dual space corresponding to the Banach space V.
- $\langle \cdot, \cdot \rangle$: the duality pairing between V and V'.
- $D_i = \frac{\partial}{\partial x_i}$: corresponds to differentiation with respect to the i-th coordinate of x.
- $Du = (D_1 u, ..., D_N u)$: the gradient of u.
- $\Delta u = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}$: the Laplacian of u.
- $\mathrm{div}\, v = \sum_{i=1}^{N} D_i v_i$: the divergence of the vector $v = (v_1, ..., v_N)$.
- χ_E The function that takes the value 1 on E and 0 elsewhere (characteristic).
- $\mathrm{supp}(u) = \overline{\{x \in \Omega : \; s(x) \neq 0\}}$: the supprt of a function u.

- $C(\Omega)$: represent the collection of real-valued functions that are continuous on the domain Ω.
- $C^k(\Omega)$, $k \in \mathbb{N}$: represent the set of functions on Ω possessing continuous derivatives of all orders up to k.
- $C_0^k(\Omega)$: The space of k times differentiable on Ω with continuity, 0 on $\partial\Omega$.
- $C_0^\infty(\Omega)$ or $\mathcal{D}(\Omega)$: The smooth functions of compact support in Ω. $\mathcal{D}(\Omega)$: the space of smooth functions with compact support in Ω.
- $\mathcal{D}'(\Omega)$: the dual space of $\mathcal{D}(\Omega)$; space of real distributions on Ω.
- $L^\infty(\Omega) = \Big\{u : \Omega \to \mathbb{R}$ mesurable, $\underset{\Omega}{\text{esssup}}(u) < \infty\Big\}$.
- $C_+\left(\overline{\Omega}\right) = \left\{p \in C\left(\overline{\Omega}\right) : \min_{x\in\overline{\Omega}} h(x) > 1\right\}$.
- $p^+ = \max_{x\in\overline{\Omega}} p(x)$ and $p^- = \min_{x\in\overline{\Omega}} p(x)$ for $p \in C_+^0\left(\overline{\Omega}\right)$.
- $p'(\cdot) = \frac{p(\cdot)}{p(\cdot)-1}$: the Hölder conjugate exponent of $p \in C_+\left(\overline{\Omega}\right)$
- $p^*(\cdot) = \begin{cases} \frac{Np(\cdot)}{N-p(\cdot)}, & \text{if } 1 \le p(\cdot) < N \\ \infty, & \text{if } p(\cdot) \ge N, \end{cases}$, the Sobolev critical exponent of $p \in C\left(\overline{\Omega}\right)$.
- $L^{p(\cdot)}(\Omega) = \left\{u : \Omega \to \mathbb{R} \text{ mesurable}, \int_\Omega \left|\frac{u}{\lambda}\right|^{p(x)} dx < \infty \text{ for some } \lambda > 0\right\}$
- $\mathcal{M}^{p(\cdot)}(\Omega) = \Big\{u : \Omega \to \mathbb{R}$ mesurable, $\sup_{\lambda>0} \lambda \|\chi_{\{|u|>\lambda\}}\|_{L^{p(\cdot)}(\Omega)} < \infty\Big\}$.
- $W^{1,p(\cdot)}(\Omega) = \left\{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\right\}$ where $p \in C\left(\overline{\Omega}\right)$ and $p \ge 1$.
- $W_0^{1,p(\cdot)}(\Omega)$: closure of $C_0^\infty(\Omega)$ with respect to $W^{1,p(\cdot)}(\Omega)$ norm.
- $W^{-1,p'(\cdot)}(\Omega)$: the dual space of $W_0^{1,p(\cdot)}(\Omega)$.

- For $\overrightarrow{p}(\cdot) = (p_1(\cdot), ..., p_N(\cdot)) \in C\left(\overline{\Omega}\right)$, we set

$$\frac{1}{\overline{p}(x)} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i(x)}.$$

$$p_+(x) = \max\{p_1(x), ..., p_N(x)\},\ p_-(x) = \min\{p_1(x), ..., p_N(x)\}, x \in \Omega.$$

$$p_+^+ = \max\{p_1^+, \ldots, p_N^+\},\ p_+^- = \max\{p_1^-, \ldots, p_N^-\},\ \text{and } p_-^- = \min\{p_1^-, \ldots, p_N^-\}.$$

- $W^{1,\overrightarrow{p}(\cdot)}(\Omega) = \left\{u \in L^{p_+(\cdot)}(\Omega) :\ D_i u \in L^{p_i(\cdot)}(\Omega),\ \text{for } i = 1, ..., N\right\}$ with $\overrightarrow{p}(\cdot) \in (C_+(\overline{\Omega}))^N$.
- $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$: the closure of $C_0^\infty(\Omega)$ in the norm of $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$.
- $\mathcal{D}_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$: the completion of $C_0^\infty(\Omega)$ with respect to the norm $\sum_{i=1}^{N} \|D_i u\|_{L^{p_i(\cdot)}(\Omega)}$.

Introduction

Nonlinear elliptic and parabolic equations play a crucial role in modern mathematical analysis, especially when studied within the framework of variable exponent Lebesgue Sobolev spaces, denoted by $L^{p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$. These spaces have attracted considerable attention and have been extensively investigated in recent years, as they provide a flexible and effective framework for analyzing such types of equations. They are particularly well-suited for modeling heterogeneous and anisotropic phenomena, making them applicable to a wide range of complex physical and mathematical problems. For further details, we refer to [2]. These function spaces have found applications in various fields, including electro-rheological fluids and image processing (see [4], [3], [20], [27], and [39]). This class of equations emerged from efforts to generalize the classical framework of Laplace and Poisson equations by replacing fixed-exponent spaces such as $L^p(\Omega)$ and $W^{1,p}(\Omega)$ with more flexible spaces of the form $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$, in the case variable exponents; which allow for the modeling of heterogeneous and anisotropic phenomena.

Moreover, these equations have found applications in digital image processing, particularly in image denoising and edge preservation, where it is essential to use coefficients varying pointwise across the image. This naturally leads to PDEs of the $p(.)$-Laplacian type. Key contributions by researchers such as [8], [27], [41] and have played a pivotal role in advancing the analytical theory of these equations, encompassing existence, regularity, and compactness results.

This thesis focuses on investigating the existence and regularity of weak solutions, showing that every weak solution is also a distributional solution for a class of nonlinear in both cases isotropic and anisotropic elliptic equations characterized by variable exponents involving non-regular data. Our approach using the compactness method that involves three steps which are: building the approximate problem, give some priori estimats of the solutions, and passing to the limit via the approximate problem. We employ new techniques that include compactness theorem with data belong to suitable Lebesgue spaces.

Chapter one highlights a comprehensive summary of fundamental results in the field of functional analysis, with an emphasis on key concepts and theories applied to nonlinear partial differential equations. It provides some definitions, facts, and basic properties of generalized Lebesgue-Sobolev spaces $L^{p(\cdot)}(\Omega)$, $W^{1,p(\cdot)}(\Omega)$, and $W_0^{1,p(\cdot)}(\Omega)$, whereΩ denotes an open subset of $\mathbb{R}^N$, as well as some crucial convergence theorems and charactrazation of anisotropic variable exponent Sobolev spaces $W^{1,p_i(\cdot)}(\Omega)$, more details loot at [5, 15, 16, 19, 22, 26, 32, 33].

The second chapter is devoted to establishing the existence and regularity of weak solutions for a class of nonlinear isotropic elliptic equations with variable exponents involving irregular data, within the framework of suitable variable exponent Sobolev spaces. We suppose that the variable exponents $m(\cdot) : \overline{\Omega} \to (1, +\infty)$ and $p(\cdot) : \overline{\Omega} \to (1, +\infty)$ are continuous functions and satisfy the following conditions:

$$1 + \frac{1}{m(x)} - \frac{1}{N} < p(x) < N, \quad \text{for all } x \in \overline{\Omega},$$

and

$$1 < m(x) < \frac{Np(x)}{Np(x) - N + p(x)}, \quad Dm \in L^\infty(\Omega), \quad \text{for all } x \in \overline{\Omega}.$$

Here, we are intersted in studying the following boundary value problem:

$$\begin{cases} -\text{div}\,(\widehat{a}(x, Du)) = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

Where $\widehat{a} : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Leray-Lions type operator. This operator is a Carathéodory function that satisfies, for almost every $x \in \Omega$ and for all $\xi, \xi' \in \mathbb{R}^N$, the following conditions:

$$\widehat{a}(x,\xi) \cdot \xi \geq \alpha |\xi|^{p(x)}, \quad \widehat{a}(x,\xi) = (a_1, \ldots, a_N),$$
$$|\widehat{a}(x,\xi)| \leq \beta \left(h(x) + |\xi|^{p(x)-1}\right),$$
$$(\widehat{a}(x,\xi) - \widehat{a}(x,\xi')) \cdot (\xi - \xi') > 0, \quad \text{for } \xi \neq \xi',$$

Where $\alpha, \beta > 0$ are constants, $h$is a non-negative function in $L^{p'(\cdot)}(\Omega)$ $p'(\cdot)$ is the conjugate exponent of $p(\cdot)$, defined by $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

These conditions ensure that the Leray-Lions type operator is well-defined and suitable for analysis using the variational method; for instance, we refer to [44] for more details about the approach and this work can be found in [1].

The nonlinearity of (1) is more complex than that of the p-Laplacian due to the dependence of the exponent $p(x)$ on the spatial variable x. In the constant case $2 - \frac{1}{N} < p(\cdot) = p$, the existence of a distributional solution u of (1) in the space $W_0^{1,q}(\Omega)$ for all $q \in \left[1, \frac{N(p-1)}{N-1}\right)$ has been established in [18]. Therefore, the study of problem (1) represents a new and interesting direction of research.

Inspired by [13], and [38], we first prove the existence of a weak solution for problem (1) with a right-hand side in $L^{m(\cdot)}(\Omega)$, where $m(\cdot)$ and $p(\cdot)$ satisfy the restrictions given in previous conditions using the approximation method. The main steps of the proof involve obtaining uniform estimates of suitable solutions for an approximate problems and then passing to the limit. Second, we establish the existence of weak solutions for problem (1) using the variational method which different from the results of [**?**]. Furthermore, the strict monotonicity condition of the $p(x)$-Laplacian ensures the uniqueness of the solution. Similar results can be found in [9], [46], and [47].

Third chapter studies the nonlinear anisotropic elliptic equation under Dirichlet boundary conditions with degenerate coercivity in variable exponent Sobolev spaces

$$\begin{cases} Bu + H(x,u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where Ω is a smouth bounded open set of $\mathbb{R}^N$ $(N \geq 2)$ with Lipschi boundary denoted by $\partial\Omega$, the function f belongs to the space $L^\infty(\Omega)$. And B is the oper ator given by

$$Bu = -Div(\hat{b}(x,u,Du)) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}(b_i(x,u,Du)),$$

we suppose that for each $\hat{b}(x,u,Du)$ and $H(x,u)$ are Carathéodory functions satisfying

There exist constant $\alpha > 0$, such that $x \in \Omega, \forall s \in \mathbb{R}, \forall \xi(\xi_1, ..., \xi_N)$ and $\xi'(\xi'_1, ..., \xi'_N) \in \mathbb{R}^N$ for all $i = 1, ..., N$ the function $\hat{b} : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies the conditions:

$$\hat{b}(x, u, \xi).\xi \geq \alpha \sum_{i=1}^{N} |\xi_i|^{\mathfrak{l}_i(x)}, \hat{b}(.) = (b_1(.), b_2(.), ..., b_N(.))$$

$$|b_i(x, u, \xi)| \leq g(x) \left(h(x) + |u|^{\bar{\mathfrak{l}}^-} + \sum_{j=1}^{N} |\xi_j|^{\mathfrak{l}_j(x)} \right)^{1 - \frac{1}{\mathfrak{l}_j(x)}}, \qquad \bar{\mathfrak{l}}^- = \min_{x \in \overline{\Omega}} \bar{\mathfrak{l}}(x),$$

where α is strictly positive real number, and $h(x) \in L^1(\Omega), g(x) \in L^\infty(\Omega)$ are a given positive functions, and the variable exponents $p_i : \mathbb{R}^N \longrightarrow (1, +\infty)$ for all $i = 1, ..., N$ are continuus functios.

For all $(x, u) \in \Omega \times \mathbb{R}$, we have

$$(b_i(x, u, \xi) - b_i(x, u, \xi'))(\xi - \xi') > 0, \qquad \xi \neq \xi'.$$

Let $H : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying the following conditions:

$$\sup_{|u| \leq \tau} |H(x, u)| \in L^1(\Omega), \forall \tau > 0$$

$$H(x, u) sign(u) \geq 0, \qquad \text{a.e}, x \in \Omega, \qquad \text{for all} u \in \mathbb{R}. \tag{3}$$

For solve our problem (2), we emply the monotone theory operator that involves the pseudomonotone techniques. Final chapter deals with the existence and regularity of solutions for certain nonlinear anisotropic elliptic equations whose principal part exhibits degenerate coercivity and whose data are only locally integrable function which is more difficult comparing to the problem of the third chapter. As a prototype, we suppose that f is a locally integrable funtion in $\mathbb{R}^N$ and consider the following problem:

$$-\sum_{i=1}^{N} \left(D_i \left(g(x) \frac{|u|^{\left(\frac{p_i(x)-1}{p_i(x)}\right)\overline{p}^-} + |D_i u|^{p_i(x)-2} D_i u}{(\ln(1 + |u|))^{\gamma_i(x)}} \right) - |u|^{s_i(x)-1} u \right) = f, \quad x \in \mathbb{R}^N, \tag{4}$$

where $s_i(\cdot) \geq p_i(\cdot)$ for all $i = 1, ..., N; D_i u = \frac{\partial u}{\partial u_i}, p_1(\cdot), p_2(\cdot), ..., p_N(\cdot)$.

More general of problem (4), we consider the nonlinear elliptic problem:

$$-\sum_{i=1}^{N} D_i(e_i(x, u) b_i(x, u, Du)) + H(x, u) = f, \quad x \in \mathbb{R}^N, \ N \geq 2 \tag{5}$$

Suppose that $b_i : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}$, are a Carethéodory functions satisfying, a.e $x \in \mathbb{R}^N$, $\forall s \in \mathbb{R}$, $\forall \xi(\xi_1, ..., \xi_N)$ and $\xi'(\xi'_1, ..., \xi'_N) \in \mathbb{R}^N$ for all $i = 1, ..., N$, the follwing conditions:

$$b_i(x, u, \xi).\xi_i \geq \alpha |\xi_i|^{p_i(x)},$$

$$|b_i(x, u, \xi)| \leq g(x) \left(h(x) + |u|^{\overline{p}^-} + \sum_{j=1}^{N} |\xi_j|^{p_j(x)} \right)^{1 - \frac{1}{p_i(x)}}, \qquad \overline{p}^- = \min_{x \in \overline{\Omega}} \overline{p}(x),$$

$$(b_i(x, u, \xi) - b_i(x, u, \xi'))(\xi - \xi') > 0, \qquad \xi \neq \xi', \ \frac{1}{\overline{p}(\cdot)} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i(\cdot)},$$

where α is stritly positeve real numbers and

$$h(x) \in L^1_{loc}(\mathbb{R}^N), g(x) \in L^\infty_{loc}(\mathbb{R}^N)$$

are a given positve functios. The functions $e_i : \mathbb{R}^N \times \mathbb{R} \longrightarrow \mathbb{R}$ are Carathéodory functions and satisfying the following condition

$$\frac{\eta}{(1 + |u|)^{\gamma_i(x)}} \leq e_i(x, u) \leq \mu,$$

where η, μ are strictly positive real numbers and $\gamma_i(x) \geq 0, i = 1, ..., N$ are continous functions on $\mathbb{R}^N$.

The variable exponents $p_i : \mathbb{R}^N \longrightarrow (1, +\infty)$ for all $i = 1, ..., N$ are a continuous functions. And $H : \mathbb{R}^N \times \mathbb{R} \longrightarrow \mathbb{R}$ be a Carathéodory functions satisfing the followng conditions

$$\sup_{|u| \leq \tau} |H(x, u)| \in L^1_{loc}(\mathbb{R}^N), \forall \tau > 0$$

$$H(x, u) sign(u) \geq \sum_{i=1}^{N} |u|^{s_i(x)}, \qquad \text{a.e}, x \in \mathbb{R}^N,$$

for all $u \in \mathbb{R}$ where $s_i(\cdot) > 0, i = 1, ..., N$ are a continuous functions on $\mathbb{R}^N$.

Under the validity of (4.5) the differential operator ceases to be coercive as ss increases, thereby rendering the approach adopted in [36] ineffective.

If $\gamma_i(\cdot) = 0$ the problem (5) has been studied in [36] with different method. In particular, If $p_i(\cdot) = 2$ and $\gamma_i(\cdot) = \gamma \in (0, \frac{1}{N-1})$, the problem (5) has been investigated in ([17], Remark 1.18); remark here that in this case the assumption (4.10) is equivalent to that $\gamma \in (0, \frac{1}{N-1})$. In a bounded domain D and if $p_i(\cdot) = p(\cdot)$ and $\gamma_i(\cdot) = \theta(p-1)$ where $\theta \in [0,1)$, the problem (5) has been treated in [7], where the authors proved that the solution s satisfied

$$Ds \in \mathcal{M}^q(\Omega), q = \frac{N(p-1)(1-\theta)}{N-(1+\theta(p-1))},$$

this result is exactly what we mentioned in (5). In contrast to [11] where the regularity result is established solely in relation to γ^+, our analysis considers both $\gamma_+(\cdot)$ and γ^+ offering a more general framework. Furthermore, the gradient estimates we obtain are novel when compared to those presented in [11, 40]. The regularity result in our work is derived in terms of the more general functions $\gamma_i(\cdot)$ only i.e:

$$D_i u \in \mathcal{M}_{loc}^{q_i(\cdot)}(\mathbb{R}^N)$$

such that

$$q_i(\cdot) = \frac{Np_i(\cdot)(\overline{p}(\cdot) - 1 - \gamma_i(\cdot))}{\overline{p}(\cdot)(N - 1 - \gamma_i(\cdot))},$$

where γ^+, $\gamma_+(\cdot)$ repleced by $\gamma_i(\cdot)$ in (4.10) and in (4.14), this issue remains unresolved, despite related results found in [11, 36, 37, 40, 39].

Our main goal is studying the existence and the regularity of the distributional solutions. For this purpose, we construct an approximate solutions sequence for problem (5) and establish some priori estimates under more restrictive assumptions on $\gamma_i(\cdot)$. Next, we prove the strong convergence of the truncations of the approximate solutions. Finally, we pass to the limit in the approximate equation to obtain the existence of a distributional solution for problem (5).

Considering all aspects, the study of anisotropic problems with nonlinearities and variable exponent coefficients is of great significance due to its wide range of applications in various fields of modern applied sciences. These applications are particularly evident in fluid dynamics within media characterized by directionally varying properties, where the conductivity differs depending on the direction. It also contributes to the field of image processing and restoration, in addition to its use in analyzing the behavior of elastic materials. Readers may refer to specialized references for more detailed information.

Chapter 1

Mathematical preliminaries

This chapter is meant to provide an overview of the real and functional analysis results that will be used afterwards. Moreover, we present some basic facts concerning the necessary function spaces.

Unless otherwise required, in this chapter, $\Omega \subset \mathbb{R}^N$ is a bounded open set equipped with N-dimensional Lebesgue measure. Note that the results in this chapter are not given in full generality, these will be presented as needed in our study.

1.1 Classical Functional Spaces (Lebesgue and Sobolev)

This section provides a brief overview of fundamental concepts related to classical Lebesgue, Marcinkiewicz, and Sobolev spaces, which will serve as essential tools throughout the thesis. For more comprehensive discussions on these topics, the reader is referred to the relevant literature [5, 19, 16, 15, 22, 26, 32, 33].

Let $1 \leq p < \infty$, the Banach space $L^p(\Omega)$ is the space of all measurable functons $u : \Omega \to \mathbb{R}$, with bounded norm

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p dx \right)^{\frac{1}{p}}.$$

When $p = +\infty$, the space $L^\infty(\Omega)$ is defined as the set of all Lebesgue measurable functions on Ω that are essentially bounded,i.e

$$L^\infty(\Omega) = \{u : \Omega \to \mathbb{R} \;\; measurable, \exists M > 0\,; |u(x)| \leq M \;\; a.e.\Omega\}.$$

The norm of u in $L^\infty(\Omega)$ is defined by

$$\|u\|_{L^\infty(\Omega)} = \inf\{M > 0\,;\; |u(x)| \leq M \;\; a.e.\; x \in \Omega\}.$$

The space $L^p(\Omega)$, $1 \leq p \leq \infty$, is a Banach space, is defined by the norm $\|\cdot\|_{L^p(\Omega)}$, the separable for $1 \leq p < \infty$ and reflexive for $1 < p < \infty$. For all $1 \leq p < \infty$, the topological dual space of $L^p(\Omega)$ is isometrically identified with $L^{p'}(\Omega)$ where p' is the Hölder conjugate exponent, i.e., $\frac{1}{p}+\frac{1}{p'} = 1$ (being $(L^1(\Omega))' = L^\infty(\Omega)$). On the other hand, $(L^\infty(\Omega))'$ is strictly bigger than $L^1(\Omega)$.

For $p = 2,$ the space $L^2(\Omega)$ is a Hilbert space with the inner product

$$(u, v)_{L^2(\Omega)} = \int\limits_\Omega u(x)v(x)dx.$$

For every $u \in L^p(\Omega), v \in L^{p'}(\Omega)$ the Hölder inequality holds

$$\int\limits_\Omega uvdx \leq \|u\|_{L^p(\Omega)}\|v\|_{L^{p'}(\Omega)}$$

For $1 \leq p < \infty$, let us present the definition of the Sobolev space:

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) :\; |Du| \in L^p(\Omega)\}\,,$$

which is a Banach space for the norm

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|Du\|_{L^p(\Omega)},$$

or

$$\|u\|_{W^{1,\infty}(\Omega)} = \max\left(\|u\|_{L^\infty(\Omega)} + \|Du\|_{L^\infty(\Omega)}\right) \text{ if } p = \infty,$$

where by $\|u\|_{L^p(\Omega)}$ we understand $\||Du|\|_{L^p(\Omega)}$. The space $W^{1,p}(\Omega)$ is separable for $1 \leq p < \infty$ and reflexive for $1 < p < \infty$. Note that if Ω is sufficiently smooth, then $W^{1,\infty}(\Omega)$ identifies with the space of locally Lipschitz functions.

The space $W_0^{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the norm of $W^{1,p}(\Omega)$. An equivalent norm of $W_0^{1,p}(\Omega)$ is given by

$$\|u\|_{W_0^{1,p}(\Omega)} = \|Du\|_{L^p(\Omega)},$$

For $1 \leq p < \infty$, let us present the Poincaré inequality:

$$\|u\|_{L^p(\Omega)} \leq C\|Du\|_{L^p(\Omega)}, \quad u \in W_0^{1,p}(\Omega)$$

for some $C > 0$ which depends on Ω and p.

Let $1 \leq p \leq \infty$, we define

$$L^p_{loc}(\Omega) = \{u : \Omega \to \mathbb{R}; u \in L^p(U) \text{ for all open } U \subset\subset \Omega\}.$$

Following the same reasoning, we define

$$W^{1,p}_{loc}(\Omega) = \left\{u : \Omega \to \mathbb{R}; u \in W^{1,p}(U) \text{ for all open } U \subset\subset \Omega\right\}.$$

For all non-negative real numbers a, b and every $1 < p < \infty$, the Young inequality holds

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}, \ p' = \frac{p}{p-1},$$

which will be used in the folloing form: for every $\varepsilon > 0$, $1 < p < \infty$ and real non-negative numbers $a, \ b$

$$ab \leq \varepsilon a^p + C_\varepsilon b^{p'} \text{ with } C_\varepsilon = \varepsilon^{\frac{-1}{p-1}}. \tag{1.1}$$

Theorem 1.1 (Stampacchia [33]) *Let $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ be a Lipschitz function, i.e*

$$\forall C > 0, \quad \textit{such that} \quad |\varphi(s) - \varphi(t)| \leq C\,|s-t|\,, \quad \forall s, t \in \mathbb{R},$$

where $\varphi(0) = 0$ Then $\forall u \in W_0^{1,p}(\Omega)$ with $1 \leq p \leq \infty$ we have:

$$\varphi(u) \in W_0^{1,p}(\Omega) \quad \textit{and} \quad D\varphi(u) = \varphi'(u)Du \quad \textit{almost everywhere in} \quad \Omega.$$

Theorem 1.2 (Rellich-Kondrachov [33]) *Let Ω be a bounded open set of $\mathbb{R}^N$ with $1 \leq p < \infty$,*

If $p < N$ then $\forall q \in [1, p^*]$, the injection of $W_0^{1,p}(\Omega)$ in $L^q(\Omega)$ is continuous.

And $\forall q \in [1, p^*[$, the injection is compact, that is bounded of $W_0^{1,p}(\Omega)$ are relatively compact in $L^q(\Omega)$.

1.2 Convergence theorems

Throughout this section, we recell some defintions and results concerning theorems of convergence about sequences of measurable functions. For more dctails, we can refer to [9, 14, 21, 22, 31, 32].

Definition 1.3 Let (u_n) and u be measurable functions in Ω.

1) We say (u_n) converges almost everywhere in Ω to u, and write $u_n \to u$ a.e. in Ω, if

$$\text{meas}\,\{x \in \Omega : \; u_n(x) \text{ does not converge to } u(x)\} = 0,$$

2) We say that the sequence (u_n) converges in measure on Ω to u if for every $\kappa > 0$

$$\lim_{n \to +\infty} \text{meas}\,\{x \in \Omega : \; |u_n(x) - u(x)| > \kappa\} = 0.$$

3) We refer to the sequence as (u_n) is a Cauchy sequene if for every $\varepsilon > 0$ and every $\kappa > 0$ there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, then

$$\text{meas}\,\{x \in \Omega : \; |u_n(x) - u_m(x)| > \kappa\} < \varepsilon.$$

The proposition below establishes a result concerning that for (s_n) being a convergent sequence in measure is a necessary and sufficiant condition of being a Cauchy in measure.

Proposition 1.4 ([32]) *Let (u_n) be a sequence of measurable functions on Ω, then the following statemments are equivalent.*

1) (u_n) is Cauchy in measure.

2) (u_n) converges in measure to a measurable function u.

There is a relationship between the different modes of convergences almost everywhere convergence in measure. This relationship is determined by the next proposition:

Proposition 1.5 ([22]) *Let (u_n) be a sequence of measurable functions on Ω.*

1) If $u_n \to u$ a.e. in Ω then $u_n \to u$ in measure (here Ω is bounded).

2) If $u_n \to u$ in measure, then $\exists (u_{n_k})$ such that $u_{n_k} \to u$ a.e. in Ω as $k \to \infty$.

We proceed to define a Carathéodory function.

Definition 1.6 Let $m \geq 1$. A function $a = a(x, \xi) : D \times \mathbb{R}^m \to \mathbb{R}$ is a Carathéodory function if for all $\xi \in \mathbb{R}^m$ the function

$$f(\cdot, \xi) : \Omega \to \mathbb{R},$$

is measurable and for almost every $x \in \Omega$ the function

$$f(x, \cdot) : \mathbb{R}^m \to \mathbb{R},$$

is continuous.

Proposition 1.7 ([16]) *Let $a = a(x, \xi) : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. Let u_n be a sequence of functions and u be a measuable function such that $u_n \to u$ in measure. Then $a(x, u_n) \to a(x, u)$ in measure.*

We frequently use the following convergence results.

Theorem 1.8 (Monotone convergence theorem [33]) *Let (u_n) be an increaing sequence of non-negative measuable functions on Ω, which converges pointwise to u. Then*

$$\int_\Omega u_n \, dx \longrightarrow \int_\Omega u \, dx \text{ when } n \to \infty.$$

Theorem 1.9 (Fatou's Lemma [33]) *Let (u_n) be a sequence of non-negtive mesurable functions on Ω. Then*

$$\int_\Omega \left(\liminf_{n\to\infty} u_n \right) \, dx \leq \liminf_{n\to\infty} \int_\Omega u_n \, dx.$$

The next result is the analog of Fatou's Lemma.

Theorem 1.10 (Lebesgue's dominated convergence theorem [33]) *Let the sequence (u_n) of $L^p(\Omega)$ with $1 \leq p < \infty$, converges a.e. to u, and be dominated by $v \in L^p(\Omega)$, in the sequense that $|u_n(x)| \leq v(x)$ a.e. in Ω. Then $u_n \to u$ (strongly) in $L^p(\Omega)$, that is, $u \in L^p(\Omega)$ and*

$$\|u_n - u\|_{L^p(\Omega)} \to 0 \text{ as } n \to \infty.$$

Theorem 1.11 (Vitali's convergence theorem [15]) *Let (u_n) be a sequence of functions in $L^p(\Omega)$ with $1 \leq p < \infty$ such that*

- $u_n \to u$ *a.e. on* Ω.
- (u_n) *is equi-integrable, that is, for every* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that*
$$\int_E |u_n(x)|^p \, dx \leq \varepsilon,$$
for all n *and for every measurable set* $E \subset \Omega$ *with* $\mathrm{meas}(E) \leq \delta$.

Then $u_n \to u$ *in* $L^p(\Omega)$.

We remark that when Ω is bounded, the weak-$\star$ convergence of (u_n) in $L^\infty(\Omega)$ to some $u \in L^\infty(\Omega)$ implies the weak convergence of (u_n) to u in any $L^p(\Omega)$, $1 \leq p < \infty$.

It is imprtant to note that the above theorem is false when $p = 1$, since a bounded sequence in $L^1(\Omega)$ has in general no weak convergence property.

The following lemma shows the boundedness of weakly convergent sequences.

Proposition 1.12 ([26]) *Let* (u_n) *be a sequence of functions in* $L^p(\Omega)$ *with* $1 < p < \infty$. *Assume that*

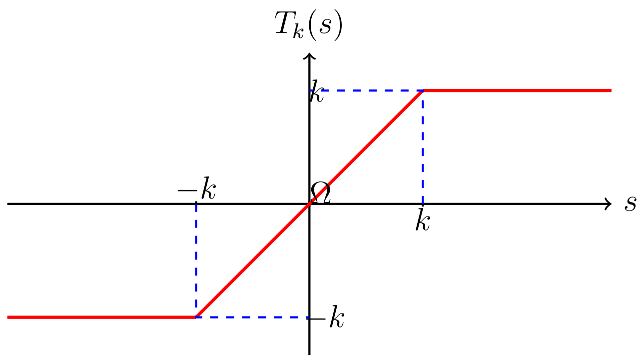
- (u_n) *is bounded in* $L^p(\Omega)$;
- $u_n \to u$ *a.e. in* Ω.

Then $u_n \to u$ *in* $L^q(\Omega)$, *for every* $1 \leq q < p$ *and weakly in* $L^p(\Omega)$, *i.e,*
$$\int_\Omega u_n v dx \longrightarrow \int_\Omega u v dx, \text{ as } n \to \infty,$$
for all $v \in L^{p'}(\Omega)$.

We have the following Charactization of the weak convergence in $W^{1,p}(\Omega)$.

Proposition 1.13 ([33]) *A* (u_n) *weakly converges sequence to* u *in* $W^{1,p}(\Omega)$, *if and only if there exist* $v_i \in L^p(\Omega)$ *such that* $u_n \rightharpoonup u$ *weakly in* $L^p(\Omega)$ *and* $D_i u_n \rightharpoonup v_i$ *weakly in* $L^p(\Omega)$, $i = 1, ..., N$. *In this case,* $v_i = D_i u$.

Throughout this thesis, T_k denotes the truncation function at height k $(k > 0)$, that is
$$T_k(s) = \begin{cases} s, & \text{if } |s| \leq k, \\ k\frac{s}{|s|}, & \text{if } |s| > k. \end{cases}$$



Note that T_k is a Lipschitz continuous functions satisfying $|T_k(s)| \leq k$ and $|T_k(s)| \leq |s|$ and its primitive The superposition operator associated with T_k provides an approximation of the identity in various spaces and this leads us to the next proposition

Proposition 1.14 ([33]) *If $u \in L^p(\Omega)$, then $T_k(u) \to u$ in $L^p(\Omega)$ strongly when $k \to +\infty$. If $u \in W^{1,p}(\Omega)$, then $T_k(u) \to u$ in $W^{1,p}(\Omega)$ strongly.*

The following results concerns the superposition operators.

1.3 Variable exponent Lebesgue spaces

In this section we recall some basic facts on Lebesgue spaces with variable exponent that can be found, for example, in [6, 9, 19, 21, 24, 42].

Let $p(\cdot) : \overline{\Omega} \to [1, +\infty)$ is a continuous function, called the variable exponent. In what follows, we adopt the following notations:

$$C_+\left(\overline{\Omega}\right) = \left\{p \in C\left(\overline{\Omega}\right) : \ 1 < p^- \leq p^+ < \infty\right\},$$

and

$$p^- = \min_{x \in \overline{\Omega}} p(x), \quad p^+ = \max_{x \in \overline{\Omega}} p(x).$$

We define the generalized Lebesgue space $L^{p(\cdot)}(\Omega)$, also called the Lebesgue space with variable exponent, as the set of continuons functions $u : \Omega \longrightarrow (1, +\infty)$ for which the convex modular

$$\rho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)}\, dx < \infty,$$

if $p^+ < \infty$, then the expression

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 \mid \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \le 1\right\},$$

defines a norm in $L^{p(\cdot)}(\Omega)$, called the Luxembourg norm.

The space $(L^{p(\cdot)}(\Omega); \quad \|\cdot\|_{L^{p(\cdot)}(\Omega)})$ is a Banach space and $\mathcal{D}(\Omega)$ is dense in $(L^{p(\cdot)}(\Omega))$.

Proposition 1.15 ([6, 19]) *Let $p \in C_+\left(\overline{\Omega}\right)$. Then for every $u \in L^{p(\cdot)}(\Omega)$, one has $\rho_{p(\cdot)}(u) < 1 (> 1;\ = 1)$ if and only if $\|u\|_{L^{p(\cdot)}(\Omega)} < 1 (> 1;\ = 1)$; further,*

$$\text{if } \|u\|_{L^{p(\cdot)}(\Omega)} < 1 \text{ then } \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)} \le \rho_{p(\cdot)}(u) \le \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}, \tag{1.2}$$

$$\text{if } \|u\|_{L^{p(\cdot)}(\Omega)} > 1 \text{ then } \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)} \le \rho_{p(\cdot)}(u) \le \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}. \tag{1.3}$$

The above proposition states that in qsestions related to convergence, $\rho_{p(\cdot)}(\cdot)$ and $\|\cdot\|_{L^{p(\cdot)}(\Omega)}$ are equivalent, that is to say, if $u_n, u \in L^{p(\cdot)}(\Omega)$, then

$$\|u_n - u\|_{L^{p(\cdot)}(\Omega)} \to 0 \text{ if and only if } \rho_{p(\cdot)}(u_n - u) \to 0.$$

Henceforth, we denote

$$C_+\left(\overline{\Omega}\right) = \left\{p \in C\left(\overline{\Omega}\right) : \min_{x\in\overline{\Omega}} p(x) > 1\right\}.$$

Whenever $p \in C_+\left(\overline{\Omega}\right)$, the space $L^{p(\cdot)}(\Omega)$ is refleive and its dual space can be identified with $L^{p'(\cdot)}(\Omega)$.

Let $p,\ q \in C\left(\overline{\Omega}\right)$ with $p \ge 1$, $q \ge 1$, and $r(\cdot)$ defined by

$$\frac{1}{r(\cdot)} = \frac{1}{p(\cdot)} + \frac{1}{q(\cdot)}.$$

Then for all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{q(\cdot)}(\Omega)$, $fg \in L^{r(\cdot)}(\Omega)$ and the following generalised Hölder inequality holds (see [19, 21])

$$\|u\|_{L^{r(\cdot)}(\Omega)} \le C\|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{q(\cdot)}(\Omega)}, \tag{1.4}$$

with $C = \max_{\overline{\Omega}} \frac{r(x)}{p(x)} + \max_{\overline{\Omega}} \frac{r(x)}{q(x)}$.

Definition 1.16 ([6]) Let $p \in C_+(\overline{\Omega})$. We say that a measurable function $u : \Omega \to \mathbb{R}$ belongs to the Marcinkewicz space $\mathcal{M}^{p(\cdot)}(\Omega)$ if

$$\|u\|_{\mathcal{M}^{p(\cdot)}(\Omega)} = \sup_{\lambda>0} \lambda\|\chi_{\{|u|>\lambda\}}\|_{L^{p(\cdot)}(\Omega)} < \infty, \tag{1.5}$$

where χ_A denotes the charactirestic function of a measurable set A.

The inequalities (1.2)-(1.3) imply that the requirement in Definition 1.16 is equivalent to say that, $\exists M > 0$ such that

$$\int_{\{|u|>\lambda\}} \lambda^{p(x)} dx \leq M, \quad \text{for all } \lambda > 0. \tag{1.6}$$

If $p, q \in C_+\left(\overline{\Omega}\right)$ with $q(\cdot) \leq p(\cdot)$, then we have

$$L^{p(\cdot)}(\Omega) \subset \mathcal{M}^{p(\cdot)}(\Omega) \subset \mathcal{M}^{q(\cdot)}(\Omega).$$

We will need the following property.

Proposition 1.17 ([6]) *If $u \in \mathcal{M}^{q(\cdot)}(\Omega)$ with $q^- > 0$, then*

$$\text{meas}\{|u| > k\} \leq \frac{M + |\Omega|}{k^{q^-}}, \quad \textit{for all } k > 0,$$

where M is the constnt appeared in (1.6). A direct result is that $\text{meas}\{|u| > k\} \to 0$, as $k \to +\infty$.

Proposition 1.18 *Let $p, q \in C_+\left(\overline{\Omega}\right)$. If $(p - q)^- > 0$, then*

$$\mathcal{M}^{p(\cdot)}(\Omega) \subset L^{q(\cdot)}(\Omega).$$

1.4 Variable exponent Sobolev spaces

Based on theories of several variable exponents function spaces have been intensively developed during the last two decades. Our goal in this section is to recall briefly some basic concepts and definitions regarding variable exponent Sobolev Spaces. For an exposition of these concepts, we refer to the books [21, 24] and the references therein.

Everywhere in this section, let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded open domain with Lipschitz boundary.

Let $p(\cdot) : \overline{\Omega} \to [1, +\infty)$ be a continuous function. The (isotropic) Sobolev space with variable exponent $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) : \ D_i u \in L^{p(\cdot)}(\Omega), \ i = 1, ..., N \right\},$$

endowed with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{L^{p(\cdot)}(\Omega)} + \|Du\|_{L^{p(\cdot)}(\Omega)},$$

where $D_i u,\ i \in \{1, ..., N\}$, represent the partial derivatives of u with respect to x_i in the distributions sense.

We define $W_0^{1,p(\cdot)}(\Omega)$ as the completion of $C_0^\infty(\Omega)$ with respect to the above norm.

Whenever $p \in C_+(\overline{\Omega})$, the spaces $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ are separable and reflexive Banach spaces.

Definition 1.19 ([47]) We refer to a function as $k : \overline{\Omega} \to \mathbb{R}$ is log-Hölder continuous on $\overline{\Omega}$ if and only if there $\exists M > 0$ such that

$$|k(x) - k(y)| \leq \frac{M}{-\ln|x-y|}, \text{ for all } x, y \in \overline{\Omega},\ 0 < |x-y| \leq 1/2. \tag{1.7}$$

Nevertheless.

Proposition 1.20 ([8]) *If $p \in C_+(\overline{\Omega})$ satisfies (1.7), then $C_0^\infty(\Omega)$ is dense in $W_0^{1,p(\cdot)}(\Omega)$.*

We recall the famous Poincaré inequality.

Proposition 1.21 ([24]) *Let $p \in C_+(\overline{\Omega})$, then there exists a finite constant $C > 0$ such that for every $u \in W_0^{1,p(\cdot)}(\Omega)$.*

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leq C\|Du\|_{L^{p(\cdot)}(\Omega)}, \tag{1.8}$$

for some constant C which depends on Ω and the function p.

Remark 1.22 ([24]) The following inequity

$$\int_\Omega |u|^{p(x)} dx \leq C \int_\Omega |Du|^{p(x)} dx,$$

generally does not hold (see [29]). But by Proposition 1.15 and (1.10) we have

$$\int_\Omega |u|^{p(x)} dx \leq C \max\{\|Du\|_{L^{p(\cdot)}(\Omega)}^{p^+}, \|Du\|_{L^{p(\cdot)}(\Omega)}^{p^-}\}. \tag{1.9}$$

Let us denote by

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ \text{any number from } [1;\infty[& \text{if } p(x) \geq N. \end{cases}$$

the Sobolev conjugate exponent.

Sobolev-Poincaré inequality [[24]]

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leq C\|Du\|_{L^{p^*(\cdot)}(\Omega)}, \tag{1.10}$$

for some constant C which depends on Ω and the function p.

An important embedding result is as follows:

Lemma 1.23 ([25]) *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary, and let $p : \overline{\Omega} \to (1, N)$ satisfy the logarithm Hölder continuity condition (1.7). Then, we have the following continuous embedding:*

$$W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p^\star(\cdot)}(\Omega)$$

.

Lemma 1.24 ([24]) *Given a set Ω and if $p(\cdot), q(\cdot) \in C_+(\overline{\Omega})$ and $p^*(\cdot) > q(\cdot)$ then there exists a finite constant $C > 0$ such that for every $u \in W_0^{1,p(\cdot)}(\Omega)$.*

$$\|u\|_{L^{q(\cdot)}(\Omega)} \leq C\|Du\|_{L^{p(\cdot)}(\Omega)},$$

The embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{q(\cdot)}(\Omega)$, is continuous and compact. In particular, we have $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{p(\cdot)}(\Omega)$, is continuous and compact.

1.5 Anisotropic variable exponent Sobolev spaces

In this section, we outline some fundamental properties of the anisotropic variable exponent Sobolev spaces to which the solutions of our main problem belong. For more comprehensive discussions, one may refer, for instance, to [28, 29, 30].

Unless otherwise specified, in this section be a bounded open domain with Lipschitz boundary. consider a bounded open domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, with a boundary of Lipschitz type Let $\overrightarrow{p}(\cdot) : \overline{\Omega} \longrightarrow \mathbb{R}^N$ defined by

$$\overrightarrow{p}(\cdot) = (p_1(\cdot), ..., p_N(\cdot)),$$

with $p_i \in C_+(\overline{\Omega})$ for all $i \in \{1, ..., N\}$.

Define

$$p_+(x) = \max\left(p_1(x), ..., p_1(x)\right), \ \forall x \in \overline{\Omega}.$$

The space $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is defined by

$$W^{1,\overrightarrow{p}(\cdot)}(\Omega) = \left\{u \in L^{p_+(\cdot)}(\Omega) : \ D_i u \in L^{p_i(\cdot)}(\Omega), \text{ for } i = 1, ..., N\right\}$$
$$= \left\{u \in L^1_{loc}(\Omega) : \ u \in L^{p_i(\cdot)}(\Omega) \ D_i u \in L^{p_i(\cdot)}(\Omega), \text{ for } i = 1, ..., N\right\}$$

endowed with the norm

$$\|u\|_{W^{1,\overrightarrow{p}(\cdot)}(\Omega)} = \|u\|_{L^{p_+(\cdot)}(\Omega)} + \sum_{i=1}^{N} \|D_i u\|_{L^{p_i(\cdot)}(\Omega)} . \tag{1.11}$$

We denote by $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in the norm of $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$. When equipped with the norm introduced in (1.11) (see [28]), the spaces $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$, $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ exhibit the structure of separable and reflexive Banach spaces. We put for all $x \in \overline{\Omega}$

$$\overline{p}^\star(x) = \begin{cases} \frac{N\overline{p}(x)}{N-\overline{p}(x)}, & \text{if } \overline{p}(x) < N, \\ +\infty, & \text{if } \overline{p}(x) \geq N. \end{cases} \quad \text{where } \overline{p}(x) = \frac{N}{\sum\limits_{i=1}^{N} \frac{1}{p_i(x)}}.$$

Theorem 1.25 ([28]) *Let $D \subset \mathbb{R}^N$ an open bounded domain and $\overrightarrow{p}(\cdot) \in (C_+(\overline{\Omega}))^N$. Suppose that*

$$\forall x \in \overline{\Omega}, \ p_+(x) < \overline{p}^\star(x). \tag{1.12}$$

Then

$$\|u\|_{L^{p_+(\cdot)}(\Omega)} \leq C \sum_{i=1}^{N} \|D_i u\|_{L^{p_i(\cdot)}(\Omega)}, \ \forall u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega), \tag{1.13}$$

where C is a constante positve independante de u. Thus $\sum_{i=1}^{N} \|D_i u\|_{L^{p_i(\cdot)}(\Omega)}$ is an equivalent norm on $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

Proposition 1.26 ([28]) *Let $\overrightarrow{p}(\cdot) \in \left(C_+\left(\overline{D}\right)\right)^N$ and (1.12) hold. Then $\mathcal{D}_0^{1,\overrightarrow{p}(\cdot)}(\Omega) = W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$.*

Let us now recall some anisotropic Sobolev inequalities, proved in [43], that we will use frequently in the sequel.

Theorem 1.27 *Let α_i are constante of $\alpha \geq 1, \ i = 1, \ldots, N$, we put $\overrightarrow{\alpha} = (\alpha_1, \ldots, \alpha_N)$. Suppose that $u \in W_0^{1,\overrightarrow{\alpha}}(\Omega)$, and set*

$$\frac{1}{\overline{\alpha}} = \frac{1}{N}\sum_{i=1}^{N} \frac{1}{\alpha_i}, \quad r = \begin{cases} \overline{\alpha}^* = \frac{N\overline{\alpha}}{N-\overline{\alpha}} & \textit{if } \overline{\alpha} < N, \\ \textit{any number from } [1, +\infty) & \textit{if } \overline{\alpha} \geq N. \end{cases}$$

Then, $\exists C > 0$ dependant de $N, p_1, \ldots, p_N$ if $\overline{\alpha} < N$ and also on r and $\operatorname{meas}(\Omega)$ if $\overline{\alpha} \geq N$, such that

$$\|u\|_{L^r(\Omega)} \leq C \prod_{i=1}^{N} \|D_i u\|_{L^{\alpha_i}(\Omega)}^{\frac{1}{N}} . \tag{1.14}$$

Theorem 1.28 *Let Ω be a cabe of $\mathbb{R}^N$ with faces parallel to the coordinate planes and $\alpha_i \geq 1,\ i = 1, \ldots, N$. Suppose that $u \in W^{1,\overrightarrow{\alpha}}(Q)$, and set*

$$r = \begin{cases} \overline{\alpha}^* & \text{if } \overline{\alpha} < N, \\ \text{any number from } [1, +\infty) & \text{if } \overline{\alpha} \geq N. \end{cases}$$

Then, C depending on $N, \alpha_1, \ldots, \alpha_N$ if $\overline{\alpha} < N$ and also on r and $\operatorname{meas}(Q)$ if $\overline{\alpha} \geq N$, such that

$$\|u\|_{L^r(Q)} \leq C \prod_{i=1}^{N} \left(\|u\|_{L^{\alpha_i}(Q)} + \|D_i u\|_{L^{\alpha_i}(Q)} \right)^{\frac{1}{N}} . \tag{1.15}$$

We finish this brief review by introducing the following space:

$$\mathcal{T}_0^{1,\overrightarrow{p}(\cdot)}(\Omega) = \{u : \Omega \to \mathbb{R} \text{ measurable},\ T_k(u) \in \mathcal{D}_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \text{ for all } k > 0\}.$$

It is worth noticing that $\mathcal{T}_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is not contained in the Sobolev space $W_0^{1,1}(\Omega)$. The next proposition clarifies the meaning of the partial derivatives of $u \in \mathcal{T}_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

Proposition 1.29 ([33]) *Let $u \in \mathcal{T}_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, for $i = 1, \ldots, N$, there exists a unique measurable function $v_i : \Omega \to \mathbb{R}$ such that*

$$D_i T_k(u) = v_i \chi_{\{|u| \leq k\}} \text{ a.e. in } D, \text{ for any } k > 0.$$

The functions v_i are called the weak partial derivatives of u and are still denoted by $D_i u$. Moreover, if u belongs to $W_0^{1;1}(\Omega)$, then v_i coincides with the standard distributional derivatives of u, that is $v_i = D_i u$.

Chapter 2

Nonlinear Elliptic Equation with Variable Exponents

In this chapter, we study a nonlinear anisotropic elliptic equation with variable exponents, non-regular data. We discuss the existence and regularity of weak solutions in appropriate anisotropic variable exponent Sobolev spaces. The results presented here are based on the work in [1].

2.1 Introduction

We will consider the following problem:

$$\begin{cases} -\operatorname{div}\left(|Du|^{p(\cdot)-2}\, Du\right) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open domain with a smooth boundary $\partial\Omega$, f belongs to $L^{m(\cdot)}(\Omega)$, with $m(\cdot)$ satisfying the conditions given in (2.7).

The equation (2.1) generalizes the classical p -Laplace equation, where the constant $p \in (1, +\infty)$ is replaced by a variable exponent $p(\cdot)$. This problem has a variational structure, meaning that weak solutions can be obtained as critical points of the energy functional

$$\mathcal{L}(u) = \int_\Omega \frac{1}{p(x)} |Du|^{p(x)} dx - \int_\Omega f u dx.$$

We consider a more general class of nonlinear elliptic equations with variable exponents of the form

$$\begin{cases} -\text{div}\,(\widehat{a}(x, Du)) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

where $a_i : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}$, is a Carathéodory function that satisfies, for almost every $x \in \Omega$ and for all $\xi, \xi' \in \mathbb{R}^N$, the following conditions:

$$\widehat{a}(x,\xi) \cdot \xi \geq \alpha |\xi|^{p(x)}, \quad \widehat{a}(x,\xi) = (a_1, \ldots, a_N), \tag{2.3}$$

$$|\widehat{a}(x,\xi)| \leq \beta \left(h(x) + |\xi|^{p(x)-1} \right), \tag{2.4}$$

$$(\widehat{a}(x,\xi) - \widehat{a}(x,\xi')) \cdot (\xi - \xi') > 0, \quad \text{for } \xi \neq \xi', \tag{2.5}$$

where α, β are tow constants non-negative, $h > 0$ is a function in $L^{p'(\cdot)}(\Omega)$ and $p'(\cdot)$ is the conjugate exponent of $p(\cdot)$, defined by $1/p(x) + 1/p'(x) = 1$.

We emphasize that the proof of existence of solutions to the problem relies on an abstract surjectivity result.

2.2 Existence result of problem (2.2)

Definition 2.1 *A function u is a weak solution of problem (2.2) if*

$$u \in W_0^{1,1}(\Omega), \quad \widehat{a}(x, Du) \in (L^1(\Omega))^N,$$

and

$$\int_\Omega \widehat{a}(x, Du) D\varphi dx = \int_\Omega f \varphi dx,$$

for all $\varphi \in C_0^\infty(\Omega)$, the C_0^∞ smooth functions of compact support in Ω.

Our main result is the following:

Theorem 2.2 *Let $f \in L^{m(\cdot)}(\Omega)$, the assumptions (2.3)-(2.5) and assume that (1.19) such that*

$$1 + \frac{1}{m(x)} - \frac{1}{N} < p(x) < N, \quad \textit{for all } x \in \overline{\Omega}, \tag{2.6}$$

and

$$1 < m(x) < \frac{Np(x)}{Np(x) - N + p(x)}, \quad Dm \in L^{\infty}(\Omega), \quad \textit{for all } x \in \overline{\Omega}. \tag{2.7}$$

then the problem (2.2) has at least one weak solution $u \in W_0^{1,q(\cdot)}(\Omega)$*, where* $q(\cdot)$ *is a continuous function on* $\overline{\Omega}$ *satisfying*

$$1 \leq q(x) < \frac{Nm(x)(p(x)-1)}{N - m(x)} \quad \textit{for all} \quad x \in \overline{\Omega}. \tag{2.8}$$

2.3 The approximation method for problem (2.2)

In this part, we employ the approximation method to study the existence of weak solutions for the problem (2.2).

Proof of Theorem 2.2 The proof needs three steps.

2.3.1 Approximate problem

Let $(f_n)_n \subset C_0^{\infty}(\Omega)$ be a sequence of bounded functions.

$$f_n \longrightarrow f \quad \text{strongly in } L^{m(\cdot)}(\Omega), \quad \text{as} \quad n \longrightarrow \infty.$$

such that

$$\|f_n\|_{L^{m(\cdot)}(\Omega)} \leq \|f\|_{L^{m(\cdot)}(\Omega)}, \quad \forall n \geq 1. \tag{2.9}$$

The existence of the sequences u_n and f_n smooth functions of compact support in Ω, see for example. We approch the problem (2.2) by following problem

$$\int_{\Omega} \hat{a}(x, Du_n) D\varphi \, dx = \int_{\Omega} f_n \varphi \, dx, \quad \forall \varphi \in W_0^{1,p(\cdot)}(\Omega), \tag{2.10}$$

there exists at least one weak solution

$$u_n \in W_0^{1,p(\cdot)}(\Omega)$$

(cf. J.L. Lions [34] , Theorem 2.7, page 180). Because

For $u_n \in W_0^{1,p(\cdot)}(\Omega)$, we pst

$$Lu_n = -\text{div}\,(\widehat{a}(x, Du_n)).$$

The operator L maps $W_0^{1,p(\cdot)}(\Omega)$ into $\left(W_0^{1,p(\cdot)}(\Omega)\right)'$, thanks (2.5) A is monotone. The growth condition (2.4) implies that A is hemicontinuous.
i.e., for all u_n, v_n, $w_n \in W_0^{1,p(\cdot)}(\Omega)$, the mapping $\mathbb{R} \ni \lambda \mapsto \langle A(u_n + \lambda v_n), w_n\rangle$ is continuous. By (2.3) and Lemma 2.2 [27], we can write

$$\begin{aligned}\frac{\langle Lu_n, u_n\rangle}{\|u_n\|_{W_0^{1,p(\cdot)}(\Omega)}} &\geq \alpha \frac{\rho_{p(\cdot)}(Du_n)}{\|u_n\|_{W_0^{1,p(\cdot)}(\Omega)}} \\ &\geq \alpha \frac{\min\left\{\|u_n\|^{p^+}_{W_0^{1,p(\cdot)}(\Omega)}, \|u_n\|^{p^-}_{W_0^{1,p(\cdot)}(\Omega)}\right\}}{\|u_n\|_{W_0^{1,p(\cdot)}(\Omega)}},\end{aligned}$$

this prove that L is coercive. By (2.4), we get the operator L is bounded.
Thus, we get the desired result.

2.3.2 Uniform estimates

Lemma 2.3 *Let $p(\cdot)$ as in (2.6), and $m(\cdot)$ as in (2.7) with $m^- = \inf_{x\in\overline{\Omega}} m(x) > 1$. Then, for any constant $0 < \delta < 1$, there exists a constant C_δ independent of n such that*

$$\int_\Omega \frac{|Du_n|^{p(x)}}{(1+|u_n|)^\delta} dx \leq C_\delta \left(1 + \Big(\int_\Omega (1+|u_n|)^{(1-\delta)\frac{m^-}{m^- -1}}\, dx\Big)^{1-\frac{1}{m^-}}\right) \tag{2.11}$$

Proof: Let $0 < \delta < 1$, we define the function $\psi_\delta : \mathbb{R} \to \mathbb{R}$ by

$$\psi_\delta(t) = \int_0^t \frac{du}{(1+|u|)^\delta}.$$

It's clear that

$$\psi_\delta(t) = \frac{1}{1-\delta}((1+|t|)^{1-\delta} - 1)sign(t),$$

we have ψ_δ is a continuous function satisfies $\psi_\delta(0) = 0$, $|\psi_\delta'(\cdot)| \leq 1$, taking $\psi_\delta(u_n)$ as a test function in (2.10), we obtain

$$\int_\Omega \hat{a}(x, Du_n) D\psi_\delta(u_n) dx = \int_\Omega f_n \psi_\delta(u_n) dx.$$

Using the from (2.3) and

$$D\psi_\delta(u_n) = Du_n.\psi'_\delta(u_n)$$

we get

$$\alpha \int_\Omega \frac{|Du_n|^{p(x)}}{(1+|u_n|)^\delta} dx \le \frac{1}{1-\delta} \int_\Omega |f||(1+|t|)^{1-\delta} - 1| dx.$$

By application of Holder's inequality, we find

$$\alpha \int_\Omega \frac{|Du_n|^{p(x)}}{(1+|u_n|)^\delta} dx \le \frac{1}{1-\delta} \Big(\int_\Omega |f|^{m^-} dx \Big)^{1/m^-} \Big(\int_\Omega |(1+|t|)^{1-\delta} - 1|^{m^-/(m^- -1)} dx \Big)^{(m^- -1)/m^-}.$$

Since for any $0 < \delta < 1$ and

$$(a_1 + a_2)^r \le \max\{1, 2^{r-1}\} \left(a_1^r + a_2^r\right), \quad a_i \ge 0, \quad r > 0,$$

we obtain

$$\int_\Omega \frac{|Du_n|^{p(x)}}{(1+|u_n|)^\delta} dx \le C_\delta \left(1 + \Big(\int_\Omega (1+|u_n|)^{(1-\delta)\frac{m^-}{m^- -1}} dx \Big)^{1-\frac{1}{m^-}} \right).$$

■

Lemma 2.4 *Let* $p(\cdot)$ *as in* (2.6)*, and* $m(\cdot)$ *as in* (2.7)*, and* $f \in L^{m(\cdot)}(\Omega)$*. Then there exists a constant* C_1 *such that*

$$\|u_n\|_{W_0^{1,q(\cdot)}(\Omega)} \le C_1,$$

for all continuous functions $q(\cdot)$ *as in* (2.8)*.*

Remark 2.5 *Note that the result given in Lemma 2.4 also holds for any measurable function* $q : \overline{\Omega} \to \mathbb{R}$ *such that*

$$ess \inf_{x \in \overline{\Omega}} \left(\frac{Nm(x)(p(x)-1)}{N - m(x)} - q(x) \right) > 0.$$

Indeed, in both cases there exists a continuous function $s : \overline{\Omega} \to \mathbb{R}$ *such that for almost every* $x \in \overline{\Omega}$*:*

$$q(x) \le s(x) \le \frac{Nm(x)(p(x)-1)}{N - m(x)}.$$

From Lemma 2.4, we deduce, in both cases, that $(u_n)_n$ *is bounded in* $W_0^{1,s(\cdot)}(\Omega)$*. Finally, by the continuous embedding*

$$W_0^{1,s(\cdot)}(\Omega) \hookrightarrow W_0^{1,q(\cdot)}(\Omega),$$

we have the desert result.

Proof of Lemma 2.4 Firstly, note that since $m(\cdot) > 1$ and $p(\cdot)$ is defined as in (2.6), we get

$$1 < \frac{Nm(x)(p(x)-1)}{N-m(x)}, \quad \text{for all} \quad x \in \overline{\Omega}.$$

Now, consider the following cases:

Case (a): Let q^+ be a constant satisfying

$$q^+ < \frac{Nm^-(p^- - 1)}{N - m^-}. \tag{2.12}$$

Note that the assumption (2.7) implies that

$$\frac{Nm^-(p^- - 1)}{N - m^-} < p^-. \tag{2.13}$$

Using Hölder's inequality with (2.11), we obtain

$$\begin{aligned} \int_\Omega |Du_n|^{q^+}\, dx &= \int_\Omega \frac{|Du_n|^{q^+}}{(1+|u_n|)^{\delta\frac{q^+}{p^-}}}(1+|u_n|)^{\delta\frac{q^+}{p^-}}\, dx \\ \le C_2 \left(1 + \left(\int_\Omega (1+|u_n|)^{(1-\delta)\frac{m^-}{m^- -1}}\, dx\right)\right)^{(1-\frac{1}{m^-})\frac{q^+}{p^-}} &\cdot \left(1 + \left(\int_\Omega (1+|u_n|)^{\delta\frac{q^+}{p^- - q^+}}\, dx\right)\right)^{1-\frac{q^+}{p^-}}, \end{aligned} \tag{2.14}$$

By (2.12) and (2.13) , we get

$$1 - \left(\frac{Nq^+}{N-q^+}\right)\left(\frac{m^- - 1}{m^-}\right) < \frac{m^-(p^- - q^+)}{(m^- - 1)q^+ + m^-(p^- - q^+)} < 1. \tag{2.15}$$

Now, choose $\delta \in (0,1)$ such that

$$\frac{\delta q^+}{p^- - q^+} < \frac{m^-(1-\delta)}{m^- - 1} < q^{+\star} = \frac{Nq^+}{N-q^+}. \tag{2.16}$$

Notice that (2.15) and (2.16) are respectively equivalent to

$$1 - \left(\frac{Nq^+}{N-q^+}\right)\left(\frac{m^- - 1}{m^-}\right) < \delta < \frac{m^-(p^- - q^+)}{(m^- - 1)q^+ + m^-(p^- - q^+)} < 1. \tag{2.17}$$

Therefore, by (2.14), (2.16) and using Sobolev inequality with $q^{+\star}$, we obtain

$$\begin{aligned} \int_\Omega |Du_n|^{q^+}\, dx &\le C_3 \left(1 + \int_\Omega |u_n|^{\frac{m(1-\delta)}{m^- - 1}}\, dx\right)^{1-\frac{q^+}{m^- p^-}} \\ &\le C_4 \left(1 + \int_\Omega |u_n|^{q^{+\star}}\, dx\right)^{1-\frac{q^+}{m^- p^-}} \\ &\le C_5 \left(1 + \int_\Omega |Du_n|^{q^+}\, dx\right)^{(\frac{N}{N-q^+})(1-\frac{q^+}{m^- p^-})} \\ &\le C_6 + C_6 \left(\int_\Omega |Du_n|^{q^+}\, dx\right)^{(\frac{N}{N-q^+})(1-\frac{q^+}{m^- p^-})}, \end{aligned} \tag{2.18}$$

By the fact that

$$m^- < \frac{Np^-}{Np^- - N + p^-} < \frac{N}{p^-}, \tag{2.19}$$

together with the assumption (2.12), this implies that

$$q^+ < m^-p^- \text{ and } 0 < \left(\frac{N}{N-q^+}\right)\left(1 - \frac{q^+}{m^-p^-}\right) < 1.$$

Hence, the estimate (2.18) imply that (Du_n) is bounded in $L^{q^+}(\Omega)$.
Since $|Du_n|^{q(\cdot)} \leq |Du_n|^{q^+} + 1$, we obtain that (u_n) is bounded in $W_0^{1,q(\cdot)}(\Omega)$. This completes the proof in Case (a).

Case (b): Let q be a continuous functions satisfying (2.8) and

$$q^+ \geq \frac{Nm^-(p^- - 1)}{N - m^-}.$$

By the continuity of $p(\cdot)$ and $q(\cdot)$ on Ω, there exists a constant $\eta > 0$ such that

$$\max_{y \in B(x,\eta)\cap\Omega} q(y) < \min_{y \in B(x,\eta)\cap\Omega} \frac{Nm^-(p(y) - 1)}{N - m^-} \quad \text{for all} \quad x \in \overline{\Omega}.$$

Note that Ω is compact and therefore we can cover it with a finite number of balls $(B_i)_{i=1,\dots,k}$. Moreover, there exists a constant $\rho > 0$ such that

$$|\Omega_i| = \text{meas}(\Omega_i) > \rho, \ \Omega_i := B_i \cap \Omega, \ \text{ for all } i = 1, \dots, k. \tag{2.20}$$

We denote by q_i^+ the local maximum of q on $\overline{\Omega_i}$ (respectively p_i^- the local minimum of p on $\overline{\Omega_i}$), such that

$$q_i^+ < \frac{Nm^-(p_i^- - 1)}{N - m^-} \text{ for all } i = 1, \dots, k. \tag{2.21}$$

Using the same arguments as before locally, we obtain the similar estimate as in (2.18)

$$\int_{\Omega_i} |\Omega u_n|^{q_i^+} dx \leq C_7 \left(1 + \int_{\Omega_i} |u_n|^{q_i^{+\star}} dx\right)^{1 - \frac{q_i^+}{m^- p_i^-}}, \text{ for all } i = 1, \dots, k. \tag{2.22}$$

On the other hand, the Poincaré-Wirtinger inequality gives

$$\|u_n - \widetilde{u_n}\|_{L^{q_i^+ \ star}(\Omega_i) \leq C_8 \|Du_n\|_{L^{q_i^+}(\Omega_i)}}, (2.23)$$

$$\text{where} \qquad \widetilde{u_n} = \frac{1}{|\Omega_i|}\int_{\Omega_i} u_n(x)dx, \quad q_i^{+\star} = \frac{Nq_i^+}{N - q_i^+}.$$

Moreover, note that the sequence $(u_n)_n$ is bounded in $L^1(\Omega)$. So, from (2.20), we have

$$\|\widetilde{u_n}\|_{L^1(\Omega)} \leq C_8,$$

Therefore, by (2.23), we deduce that

$$\begin{aligned}\|u_n\|_{L^{q_i^{+\star}}(\Omega_i)} &\leq \|u_n - \widetilde{u_n}\|_{L^{q_i^{+\star}}(\Omega_i)} + \|\widetilde{u_n}\|_{L^{q_i^{+}}(\Omega_i)} \\ &\leq C_8\|Du_n\|_{L^{q_i^{+}}(\Omega_i)} + C_9, \quad \text{for all} \quad i = 1, ..., k.\end{aligned}$$

Thus, using (2.22), we obtain

$$\int_{\Omega_i} |Du_n|^{q_i^+} dx \leq C_{10} + C_{10} \left(\int_{\Omega_i} |Du_n|^{q_i^+} dx \right)^{\left(\frac{N}{N-q_i^+}\right)\left(1-\frac{q_i^+}{m^+p_i^-}\right)},$$

by (2.21) and arguing locally as in (2.19), we deduce

$$0 < \left(\frac{N}{N-q_i^+}\right)\left(1 - \frac{q_i^+}{m^-p_i^-}\right) < 1,$$

so that

$$\int_{\Omega_i} |Du_n|^{q_i^+} dx \leq C_{11}, \quad \text{for all} \quad i = 1, ..., k.$$

Recall that

$$q(x) \leq q_i^+, \quad \text{for all } x \in \Omega_i \quad \text{and for all } i = 1, ..., k.$$

So, we get

$$\int_{\Omega_i} |Du_n|^{q(x)} dx \leq \int_{\Omega_i} |Du_n|^{q_i^+} dx + |\Omega_i| \leq C_{12}.$$

Since $\Omega \subset \bigcup_{i=1}^{N} \Omega_i,$ for all $i = 1, ..., k.$ we deduce that

$$\int_{\Omega} |Du_n|^{q(x)} dx \leq \sum_{i=1}^{k} \int_{\Omega_i} |Du_n|^{q(x)} dx \leq C_{13}.$$

This finishes the proof of the Case(b). ∎

Remark 2.6 *Remark that in the constant case and $f \in L^{m(\cdot)}(\Omega)$, we choose in* (2.14)

$$\delta = \frac{pN - m^-p - m^-Np + m^-N}{N - m^-p} \in (0, 1),$$

to obtain

$$q = \frac{m^-N(p-1)}{N-m^-} \Longrightarrow (1-\delta)\frac{m^-}{m^- - 1} = \frac{\delta q}{p-q} = \frac{Nq}{N-q},$$

It is easy to check that, instead of the global estimate (2.18)*, we find*

$$\int_\Omega |Du_n|^q dx \leq C + C\left(\int_\Omega |Du_n|^q dx\right)^{\left(\frac{N}{N-q}\right)\left(1-\frac{q}{m^- p}\right)},$$

where $0 < \left(\frac{N}{N-q}\right)\left(1 - \frac{q}{m^- p}\right) < 1$. *Then* (2.2) *has at least one weak solution* u*, possesses the regularity* $u \in W_0^{1,q}(\Omega)$ *far all* $q = \frac{Nm^-(p-1)}{N-m^-}$.

2.3.3 Passage to the limit

From Lemma 2.4 together with the continuous embedding $W_0^{1,q(\cdot)}(\Omega) \hookrightarrow W_0^{1,q^-}(\Omega)$, we have a subsequence (still denoted $(u_n)_n$) such that

$$u_n \rightharpoonup u \quad \text{weakly in} \quad W_0^{1,q^-}(\Omega), \tag{2.24}$$

$$u_n \to u \quad \text{strongly in} \quad L^{q^-}(\Omega) \tag{2.25}$$

$$u_n \to u \quad \text{a.e in} \quad \Omega. \tag{2.26}$$

To complete the proof, we need the following lemmas:

Lemma 2.7 *We have*

$$Du_n \to Du \quad \textit{a.e in} \quad \Omega, \tag{2.27}$$

Proof: In order to prove this lemma it is sufficient to show that:

$$Du_n \to Du \quad \text{in measure.}$$

By (2.25),(2.24),(2.3),(2.4), (2.8) and using Lebesgue's dominated convergence theorem, we get the convergence of (Du_n) to (Du) in measure, which proves the Lemma 2.7. ■

Lemma 2.8 *We have*

$$\widehat{a}(x, Du_n) \to \widehat{a}(x, Du) \quad \textit{strongly in} \quad L^{q(\cdot)}(\Omega), \tag{2.28}$$

for some continuous function $q(\cdot) : \overline{\Omega} \to [1, \frac{Nm(\cdot)}{N-m(\cdot)})$*, where* $m(\cdot)$ *is a defined in* (2.7).

Proof: To prove (2.28), we apply Vitali's theorem with taking in consideration Lemma 2.4, (2.26), (2.27), (2.4) and (2.6).

Finally, for φ the space of smooth functions with compact support in Ω, we know

$$\int_\Omega \hat{a}(x, Du_n) D\varphi \, dx = \int_\Omega f_n \varphi \, dx. \tag{2.29}$$

Using (2.28), we can pass to the limit for $n \to +\infty$ in the weak formulation (2.29), we obtain that u is a weak solution for (2.2). ■

Remark 2.9 *Under the assumption $f \in L^{m(\cdot)}(\Omega)$ in Theorem 2.2, we can deduce that f is never in the dual space $\left(W_0^{1,p(\cdot)}(\Omega)\right)'$, so that the result of this paper deals with irregular data. If $m(\cdot)$ tends to be 1, then $q(\cdot) = \frac{Nm(\cdot)(p(\cdot)-1)}{N-m(\cdot)}$ tends to be $\frac{N(p(\cdot)-1)}{N-1}$.*

2.4 The Variational Method for Problem (2.1)

To prove that the problem (2.1) has a variational structure with $f \in L^{p'(\cdot)}(\Omega)$, meaning that weak solutions can be obtained as critical points of an energy functional, we need to follow these steps:

2.4.1 The Energy Functional

The energy functional $\mathcal{L} : W_0^{1,p(\cdot)}(\Omega) \to \mathbb{R}$ associated with the problem (2.1) is typically defined as:

$$\mathcal{L}(u) = \int_\Omega \frac{1}{p(x)} |Du|^{p(x)} dx - \int_\Omega f u dx.$$

Here

- The first term $\int_\Omega \frac{1}{p(x)} |Du|^{p(x)} dx$ represents the "energy" associated with the gradient of u.
- The second term $\int_\Omega f u dx$ represents the work done by the external force f.

2.4.2 The Gâteaux derivative

To show that weak solutions correspond to critical points of $\mathcal{L}$, we compute the Gâteaux derivative of $\mathcal{L}$ in the direction of a test function $\phi \in W_0^{1,p(\cdot)}(\Omega)$. The Gâteaux derivative is given by

$$\langle \mathcal{L}'(u), \phi \rangle = \lim_{t \to 0} \frac{\mathcal{L}(u + t\phi) - \mathcal{L}(u)}{t}.$$

For the functional $\mathcal{L}$, this derivative can be computed explicitly as

$$\langle \mathcal{L}'(u), \phi \rangle = \int_\Omega |Du|^{p(x)-2} Du \cdot D\phi dx - \int_\Omega f\phi dx.$$

2.4.3 The Weak Formulation

A function $u \in W_0^{1,p(\cdot)}(\Omega)$ is a critical point of $\mathcal{L}$ if $\langle \mathcal{L}'(u), \phi \rangle = 0$ for all $\phi \in W_0^{1,p(\cdot)}(\Omega)$. This condition is equivalent to the weak formulation of the problem:

$$\int_\Omega |Du|^{p(x)-2} Du \cdot D\phi dx = \int_\Omega f\phi dx \quad \text{for all} \quad \phi \in W_0^{1,p(\cdot)}(\Omega).$$

This is precisely the weak form of the equation:

$$-\text{div}\left(|Du|^{p(x)-2} Du\right) = f \quad \text{in} \quad \Omega,$$

with the Dirichlet boundary condition $u = 0$ on $\partial\Omega$.

2.4.4 The Variational Structure

To confirm that the problem (2.1) has a variational structure, we need to ensure that:

1. The energy functional $\mathcal{L}$ is well-defined and differentiable on $W_0^{1,p(\cdot)}(\Omega)$.
2. The critical points of $\mathcal{L}$ correspond to weak solutions of the problem.

These properties follow from the conditions (2.3)-(2.4) of the $p(x)$-Laplacian operator beside this, the continuity and differentiability of the functional $\mathcal{L}$ in the variable exponent Sobolev space setting.

2.4.5 Existence of Critical Points

To prove the existence of critical points (and hence weak solutions) of the problem (**??**), we can use (2.3)-(2.5) to show that $\mathcal{L}$ is bounded below and coercive, and then apply the direct method to find a minimizer.

2.4.6 The Uniqueness of The Solution

We prove the uniqueness of weak solutions of the problem (2.1) as follows:

Proof: Assume there exist two weak solutions $u_1, u_2 \in W_0^{1,q(\cdot)}(\Omega)$ to the problem. Then, for all $\varphi \in W_0^{1,q(\cdot)}(\Omega)$, we have

$$\int_\Omega \widehat{a}(x, Du_1) D\varphi \, dx = \int_\Omega f\varphi dx,$$

$$\int_\Omega \widehat{a}(x, Du_2) D\varphi dx = \int_\Omega f\varphi dx.$$

Subtracting the two equations, we obtain:

$$\int_\Omega \left(\widehat{a}(x, Du_1) - \widehat{a}(x, Du_2)\right) D\varphi dx = 0, \quad \forall \varphi \in W_0^{1,q(\cdot)}(\Omega).$$

Let $\varphi = u_1 - u_2$. Since $u_1, u_2 \in W_0^{1,q(\cdot)}(\Omega)$, it follows that $\varphi \in W_0^{1,q(\cdot)}(\Omega)$. Substituting φ into the equation, we get

$$\int_\Omega \left(\widehat{a}(x, Du_1) - \widehat{a}(x, Du_2)\right) D(u_1 - u_2) dx = 0.$$

By (2.5), we have:

$$\left(\widehat{a}(x, Du_1) - \widehat{a}(x, Du_2)\right) \cdot (Du_1 - Du_2) \geq 0,$$

with equality if and only if $Du_1 = Du_2$ almost everywhere in Ω.

From the integral equation:

$$\int_\Omega \left(\widehat{a}(x, Du_1) - \widehat{a}(x, Du_2)\right) \cdot (Du_1 - Du_2) dx = 0,$$

and (2.5), it follows that

$$Du_1 = Du_2 \quad \text{almost everywhere in} \quad \Omega.$$

Since u_1 and u_2 have the same gradient $Du_1 = Du_2$ and both satisfy the Dirichlet bosndary condition $u_1 = u_2 = 0$ on $\partial\Omega$, we conclude:

$$u_1 = u_2 \quad \text{almost everywhere in} \quad \Omega.$$

Thus, the weak solution $u \in W_0^{1,q(\cdot)}(\Omega)$ of the problem (**??**) is unique. ∎

Chapter 3

Anisotropic nonlinear elliptic equation in bounded domain

3.1 Introduction

This chapter is devoted to the study the Dirichlet problem for the nonlinear anisotropic elliptic equation in variable exponent. We focus particularly on the theory of monotone and pseudo-monotone operators, which we will use later in the approximation framework in Chapter 4. For that purpose, we introduce fundamental notions such as monotonicity, hemicontinuity, coercivity, and pseudo-monotonicity, with simple illustrative examples for each.

3.2 The Operators

3.2.1 Bounded Operators

Definition 3.1 *Let V and V' be two Banach spaces, and let $A : V \longrightarrow V'$ be an operator. We say that A is bounded if it maps every bounded set in V to a bounded set in V' , i.e.,*

$$\forall \rho > 0, \exists C_\rho > 0 : A(B_V(0, \rho)) \subset B_{V'}(0, C_\rho)$$

where $B_V(0, \rho)$ denotes the open ball in V centered at 0 with radius $\rho > 0$, and $B_{V'}(0, C_\rho)$ denotes the open ball in V' centered at 0 with radius $C_\rho > 0$.

Example The operator $Au = -\Delta_p$ is bounded from $W_0^{1,P}(\Omega)$ in $W_0^{-1,P'}(\Omega)$ From the expression of the norm in a dual space, let $\rho > 0$, for $u \in B_V(0,\rho)$, we can write:

$$\|Au\|_{V'} = \sup_{\|\varphi\|\leq 1} |\langle Au, \varphi\rangle| = \sup_{\|\varphi\|\leq 1} |\int_D |Du|^{p-2} Du D\varphi dx|.$$

So,

$$|\int_D |Du|^{p-2} Du D\varphi dx| \leq \int_D |Du|^{p-1}.|D\varphi| dx.$$

$$\leq (\int_D |Du|^p dx)^{\frac{1}{p'}}.(\int_D |D\varphi|^p dx)^{\frac{1}{p}}.$$

$$\leq \|u\|_V^{p-1}.\|\varphi\|_V$$

$$\leq \rho^{p-1}.$$

Hence $\|Au\|_{V'} \leq \rho^{p-1}$ this shows that $A(B_V(0,\rho)) \subset B_{V'}(0, C_\rho)$.

3.2.2 Monotone Operators

Definition 3.2 *Let V be a reflxive Banach space. A single-valued operator $A : V \longrightarrow V'$. We say that:*

A is monotone if:

$$\forall u, v \in V, \langle Au - Av, u - v\rangle \geq 0$$

A is strictly monotone if:

$$\forall u, v \in V, \langle Au - Av, u - v\rangle > 0$$

Example Let $Au = -\Delta u$. The operator A maps $H_0^1(\Omega)$ into its dual $H^{-1}(\Omega)$. It is monotone because for all $u, v \in H_0^1(\Omega)$:

$$\langle Au - Av, u - v\rangle = \int_D D(u-v)D(u-v)dx = \|u-v\|_{H_0^1(\Omega)} \geq 0.$$

3.2.3 Hemicontinuous Operators

Definition 3.3 *Let V be a reflxive Banach space. A single-valued operator $A : V \longrightarrow V'$. We say that A is said to be hemicontinuous if, for any fixed elements*

$$u, v, w \in V, t \longrightarrow \langle A(u + tv), w\rangle_{V' \times V}$$

is continuous with respect to the real parameter.

Example Let $Au = -\Delta u$. The operator A maps $H_0^1(\Omega)$ into its dual $H^{-1}(\Omega)$. It is hemicontinuous. Indeed, for any $u, v \in H_0^1(\Omega)$ and $t \in \mathbb{R}$, we have:

$$\langle A(u + tv), v\rangle = \int_D DA(u + tv)Dvdx.$$

Expanding the integral, we get:

$$\int_D Du + Dvdx + t\int_D |Dv|^2 dx.$$

This shows that $t \longrightarrow \langle A(u + tv), v\rangle$, is a linear function of t and hence continuous.

3.2.4 Coercive Operators

Definition 3.4 *Let V be a reflxive Banach space. An operator $A : V \longrightarrow V'$. is coercive if,*

$$\frac{\langle Av, v\rangle}{\|v\|_V} \longrightarrow +\infty, \quad \text{as} \quad \|v\|_V \longrightarrow +\infty.$$

Example Let $Au = -\Delta u + a(x)u, \quad a(x) \geq 0$. The operator A maps $H_0^1(\Omega)$ into its dual $H^{-1}(\Omega)$. It is coercive. Indeed, we have

$$\frac{\langle Av, v\rangle}{\|v\|_{H_0^1(\Omega)}} = \frac{\int_D |Dv|^2 dx + \int_D a(x)v^2 dx}{\|v\|_{H_0^1(\Omega)}}.$$

Since $a(x) \geq 0$, we obtain:

$$\frac{\langle Av, v\rangle}{\|v\|_{H_0^1(\Omega)}} \geq \alpha\|v\|_{H_0^1(\Omega)} \text{as} \quad \|v\|_{H_0^1(\Omega)} \longrightarrow +\infty.$$

3.2.5 Pseudo-monotone Operators

Definition 3.5 *Let V be a reflexive Banach space. An operator $A : V \to V'$ is pseudomonotone if*

1) A is bounded, that is, the image of a bounded subset of V is a bounded subset of V';

2) if $u_j \rightharpoonup u$ weakly in V and if $\limsup_{j \to +\infty} \langle Au_j, u_j - v\rangle \leq 0$, then

$$\liminf_{j \to +\infty} \langle Au_j, u_j - v\rangle \geq \langle Au, u - v\rangle,$$

for every v in V, where $\langle \cdot, \cdot \rangle$ refers to the duality product between V' and V.

Proposition 3.6 *If $A : V \longrightarrow V'$ be bounded, hemicontinuous and monotonic, then A is pseudo-monotone.*

Proof: Let $(u_j)_{j \geq 0}$ a sequence weakly converging to u in V . Suppose that

$$\limsup_{j \to +\infty} \langle Au_j, u_j - v\rangle \leq 0,$$

we have A is monotone, we obtain

$$\lim_{j \to +\infty} \langle Au_j, u_j - v\rangle = 0 \tag{3.1}$$

Indeed, the monotonicity of A and the weak convergence of u_j to wards u implies that

$$\langle Au_j, u_j - v\rangle \geq \langle Au, u_j - v\rangle \longrightarrow 0 \quad \text{as} \quad j \longrightarrow +\infty.$$

So

$$0 \geq \limsup_{j \to +\infty} \langle Au_j, u_j - v\rangle \geq \liminf_{j \to +\infty} \langle Au_j, u_j - v\rangle \geq \lim_{j \to +\infty} \langle Au, u_j - v\rangle = 0.$$

Hence (3.1).

On the other hand, for $t \in]0, 1[$, let $w = (1 - t)u + tv$. We have $\langle Au_j - Aw, u_j - w\rangle \geq 0$ so that

$$t\langle Au_j, u - v\rangle \geq -\langle Au_j, u_j - u\rangle + \langle Aw, u_j - u\rangle - t\langle Aw, v - u\rangle.$$

From which, thanks to a (3.1):

$$\liminf_{j \to +\infty} t\langle Au_j, u - v\rangle \geq -t\langle Aw, v - u\rangle,$$

from which, dividing by t and taking into account (3.1):

$$\liminf_{j\to+\infty}\langle Au_j, u-v\rangle \geq \langle Aw, u-v\rangle, \tag{3.2}$$

$$w=(1-t)u+tv, \quad \forall t\in]0,1[$$

By making t tend to wards 0 in (3.2), and using hemicontinuity, we deduce

$$\liminf_{j\to+\infty}\langle Au_j, u-v\rangle \geq \langle Au, u-v\rangle, \quad \forall v\in V.$$

Which means that A is pseudo-monotonic. ■

The following theorem stemming form [14], provides a surjectivity result for pseudomonotone operators.

Theorem 3.7 *Let V be a reflexive separable Banach space. Let $A : V \to V'$ be a pseudomonotone coercive operator. Then A is surjective, that is, for every f in V' there exists u in V such that $Au = f$.*

3.3 An application to anisotropic elliptic equation with degenerate coercivity in variable exponent

We consider the following elliptic problem:

$$\begin{cases} -\sum_{i=1}^N D_i(a_i(x,u,Du)) + H(x,u) = f & \text{in } D, \\ u=0 & \text{on } \partial D, \end{cases} \tag{3.3}$$

where D is a smouth bounded open set of $\mathbb{R}^N$ $(N\geq 2)$ with Lipschi boundary denoted by ∂D, the function f belongs to $(W_0^{1,\overrightarrow{p}(\cdot)}(\Omega))'$ the dual space of $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$. And we suppose $a_i : D\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ is Carathéodory function satisfy the condictions:

For all $x\in D, \sigma\in\mathbb{R}$ and $\xi\in\mathbb{R}^N$

$$\hat{a}(x,u,\xi).\xi \geq \alpha\sum_{i=1}^N |\xi_i|^{\mathfrak{p}_i(x)}, \hat{a}(.) = (a_1(.), a_2(.), ..., a_N(.)) \tag{3.4}$$

$$|a_i(x,u,\xi)| \leq g(x)\left(h(x) + |u|^{\overline{\mathfrak{p}}^-} + \sum_{j=1}^N |\xi_j|^{\mathfrak{p}_j(x)}\right)^{1-\frac{1}{\mathfrak{p}_i(x)}}, \qquad \overline{\mathfrak{p}}^- = \min_{x\in\overline{D}}\overline{\mathfrak{p}}(x), \tag{3.5}$$

$$(a_i(x,u,\xi) - a_i(x,u,\xi'))(\xi - \xi') > 0, \qquad \xi \neq \xi'. \tag{3.6}$$

Additionally, we assume that

$$\sup_{|u|\leq\tau} |H(x,u)| \in L^1(\Omega), \quad \forall\tau > 0 \tag{3.7}$$

$$H(x,u)sign(u) \geq 0, \quad \text{a.e}, \quad x \in D, \quad \forall u \in \mathbb{R}. \tag{3.8}$$

where α, η and μ are strictly positive real numbers, and $h(x) \in L^1(\Omega), g(x) \in L^\infty(\Omega)$ are a given positive functions, and the variable exponents $p_i : \mathbb{R}^N \longrightarrow (1, +\infty)$ for all $i = 1, ..., N$ are continuus functios.

We define the following operator:

$$\mathcal{A} : W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \longrightarrow \left(W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)\right)'$$

$$u \mapsto \mathcal{A}u,$$

by

$$\mathcal{A}u = -\sum_{i=1}^{N} D_i(a_i(x,u,Du)) + H(x,u) \tag{3.9}$$

The proof is based on the assertion that the operator $\mathcal{A}$ is pseudo-monotone and coercive.

3.3.1 The coercivity of the operator:

Let $\mathcal{A}$ defined by (3.9), we have

$$\langle \mathcal{A}u, v\rangle = \sum_{i=1}^{N} \int_D a_i(x,u,Du)D_i v dx + \int_D H(x,u)v dx,$$

for any $v \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

For any $u = v$, we get

$$\langle \mathcal{A}v, v\rangle = \sum_{i=1}^{N} \int_D a_i(x,v,Dv)D_i v dx + \int_D H(x,v)v dx,$$

Using the assumptions (3.4) and (3.7), we find

$$\langle \mathcal{A}_i(v), v\rangle \geq \sum_{i=1}^{N}\int_D a_i(x, v, D_i v) D_i v dx \tag{3.10}$$

$$\geq \alpha \sum_{i=1}^{N}\int_D |D_i v|^{p_i(x)} dx. \tag{3.11}$$

In view of Proposition 1.15 , we obtain

$$\langle \mathcal{A}_i(v), v\rangle \geq \alpha \sum_{i=1}^{N} \min\{\|D_i v\|^{p_i^-}_{L^{p_i(\cdot)}(\Omega)}, \|D_i v\|^{p_i^+}_{L^{p_i(\cdot)}(\Omega)}\}$$
$$\geq \alpha \sum_{i=1}^{N} \|D_i v\|^{\lambda_i}_{L^{p_i(\cdot)}(\Omega)},$$

where $\lambda_i = p_i^+$ if $\|D_i v\|_{L^{p_i(\cdot)}(\Omega)} \leq 1$ or $\lambda_i = p_i^-$ if $1 \leq \|D_i v\|_{L^{p_i(\cdot)}(\Omega)} < \infty$.

So

$$\langle \mathcal{A}_i(v), v\rangle \geq \alpha \sum_{i=1}^{N} \|D_i v\|^{\lambda^-}_{L^{p_i(\cdot)}(\Omega)} - \alpha N$$
$$\geq \frac{\alpha}{N^{\lambda^-}} \|v\|^{\lambda^-}_{W^{1,\overrightarrow{p}(\cdot)}(\Omega)} - \alpha N,$$

which implies

$$\frac{\langle \mathcal{A}v, v\rangle}{\|v\|_{W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)}} \longrightarrow +\infty$$

as $\|v\|_{W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)} \longrightarrow +\infty$ since $\lambda^- > 1$.

3.3.2 The pseudomonotonicity of the operator

(a) $\mathcal{A}$ is bounded. Indeed, let u be a bounded function in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ that is : let $\rho > 0$, for $u \in B(o, \rho)$, and for all $v \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, we have

$$\|\mathcal{A}_i(u)\|_{W^{-1,p_i'(\cdot)}(\Omega)} = \sup_{v \in W_0^{1,p_i(x)}(\Omega), \|v\| \leq 1} |\langle A_i u, v\rangle|,$$

then, we have

$$\langle \mathcal{A}u, v\rangle = \sum_{i=1}^{N}\int_D a_i(x, u, Du) D_i v dx + \int_D H(x, u) v dx,$$

which implies

$$|\langle \mathcal{A}u, v\rangle| \leq \sum_{i=1}^{N}\int_D |a_i(x, u, Du) D_i v| dx + \int_D |H(x, u) v| dx,$$

Since (3.5) and the fact that $g \in L^{+\infty}(\Omega)$, we have

$$\begin{aligned}
\|\mathcal{A}_i(u)\|_{W^{-1,p_i'(\cdot)}(\Omega)} &= \sup_{v\in W_0^{1,p_i(x)}(\Omega),\|v\|\leq 1} |\langle A_i u, v\rangle| \\
&\leq \sup_{v\in W_0^{1,p_i(x)}(\Omega),\|v\|\leq 1} \int_D |a_i(x,u,Du)D_i v|dx + \sup_{v\in W_0^{1,p_i(x)}(\Omega),\|v\|\leq 1} \int\limits_D |H(x,u)v|dx,
\end{aligned}$$

which implies

$$\|\mathcal{A}_i(u)\|_{W^{-1,p_i'(\cdot)}(\Omega)} \leq \sup_{v\in W_0^{1,p_i(x)}(\Omega),\|v\|\leq 1} \int_D |a_i(x,u,Du)||D_i v|dx + \sup_{v\in W_0^{1,p_i(x)}(\Omega),\|v\|\leq 1} |H(x,u)| \int\limits_D |v|dx,$$

Using the Holder's inequality (3.6) and (3.8) , we obtain

$$\|\mathcal{A}_i(u)\|_{W^{-1,p_i'(\cdot)}(\Omega)} \leq 2 \sup_{v\in W_0^{1,p_i(x)}(\Omega),\|v\|\leq 1} \|b_i(x,u,D_i u)\|_{L^{p_i'(\cdot)}(\Omega)} \|D_i v\|_{L^{p_i(\cdot)}(\Omega)} + M,$$

$$\begin{aligned}
\|\mathcal{A}_i(u)\|_{W^{-1,p_i'(\cdot)}(\Omega)} &\leq 2 \sup_{v\in W_0^{1,p_i(x)}(\Omega),\|v\|\leq 1} \|a_i(x,u,D_i u)\|_{L^{p_i'(\cdot)}(\Omega)} \|D_i v\|_{L^{p_i(\cdot)}(\Omega)} + M \\
&\leq 2 \sup_{v\in W_0^{1,p_i(x)}(\Omega),\|v\|\leq 1} \|a_i(x,u,D_i u)\|_{L^{p_i'(\cdot)}(\Omega)} (\|D_i v\|_{L^{p_i(\cdot)}(\Omega)} + \|v\|_{L^{p_i^-}(\Omega)}) + M \\
&\leq 2\|a_i(x,u,D_i u)\|_{L^{p_i'(\cdot)}(\Omega)} + M \\
&\leq 2\max\{(\int_D |a_i(x,u,D_i u)|^{p_i'(\cdot)}dx)^{1/p_i^{+\prime}}, (\int_D |a_i(x,u,D_i u)|^{p_i'(\cdot)}dx)^{1/p_i^{-\prime}}\} + M \\
&\leq (\int_D |a_i(x,u,D_i u)|^{p_i'(\cdot)}dx)^{1/p_i^{+\prime}} + (\int_D |a_i(x,u,D_i u)|^{p_i'(\cdot)}dx)^{1/p_i^{-\prime}} + M,
\end{aligned}$$

where $M = \sup_{v\in W_0^{1,p_i(x)}(\Omega),\|v\|\leq 1} |H(x,u)| \int\limits_D |v|dx$, because that $p_i^{-\prime} \geq p_i^{+\prime} \Rightarrow 1/p_i^{-\prime} \leq 1/p_i^{+\prime}$ and we recall that

$$\forall a \geq 0, \alpha \leq \beta \Rightarrow a^{\alpha} \leq a^{\beta} + 1$$

we obtain

$$\begin{aligned}
\|\mathcal{A}_i(u)\|_{W^{-1,p'_i(\cdot)}(\Omega)} &\leq C_1\left(\int_D |a_i(x,u,D_iu)|^{p'_i(\cdot)}dx\right)^{1/p_i^{+\prime}}+1\\
&\leq C_2\left(\int_D g(x)(h(x)+|u|^{\overline{p}^-}+\sum_{j=1}^N |D_ju|^{p_j(x)})dx\right)^{1/p_i^{+\prime}}+1\\
&\leq C_3\left(\|g(x)\|_{L^\infty(\Omega)}(\|h(x)\|_{L^1(\Omega)}+\|u\|^{\overline{p}^-}_{L^1(\Omega)}+2\sum_{i=1}^N\|D_is\|^{p_i^+}_{L^{p_i(\cdot)}(D)}+N)\right)^{1/p_i^{+\prime}}+1\\
&\leq C_4\left(1+\sum_{i=1}^N\|D_iu\|^{p_i^+}_{L^{p_i(\cdot)}(D)}\right)^{1/p_i^{+\prime}}+1,
\end{aligned}$$

where $C_4 = C_3(\max\{\|g(x)\|_{L^\infty(\Omega)}(\|h(x)\|_{L^1(\Omega)} + \|u\|^{\overline{p}^-}_{L^1(\Omega)}, 2\})^{1/p_i^{+\prime}}$. Because $p_i^+ \leq p_+^+ \Rightarrow 1/p_i^{+\prime} \leq 1/p_+^{+\prime}$, we find

$$\|\mathcal{A}_i(u)\|_{W^{-1,p'_i(\cdot)}(\Omega)} \leq C_5\left(1+\|u\|_{W_0^{1,p_i(\cdot)}(D)}\right)^{p_+^+-1}+1.$$

So,

$$\|\mathcal{A}_i(u)\|_{W^{-1,p'_i(\cdot)}(\Omega)} \leq C_5\,(1+r)^{p_+^+-1}+1=r'.$$

Then $\mathcal{A}_i$ is bounded.

(b) If $u_m \rightharpoonup u$ weakly in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, as $m\to+\infty$, and for any $v\in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$

$$\begin{aligned}
0 &\geq \limsup_{m\to+\infty}\langle \mathcal{A}u_m, u_m-v\rangle\\
&= \limsup_{m\to+\infty}\left(\sum_{i=1}^N\int\limits_D a_i(x,u_mD_iu_m)(u_m-v)dx\right.\\
&\qquad\left.+\int\limits_D H(x,u_m)(u_m-v)dx\right).
\end{aligned}\tag{3.12}$$

Then,

$$\liminf_{m\to+\infty}\langle \mathcal{A}u_m, u_m-v\rangle \geq \langle \mathcal{A}u, u-v\rangle.$$

Indeed, the compact embedding yields that $u_m\to u$ in $L^{q(\cdot)}(\Omega)$ for a subsequence still denoted as (u_m). Moreover, we assume that $u_m\to u$ a.e. in D.

Let us first prove that

$$\sum_{i=1}^N\int\limits_D\left[a_i(x,u_m,D_iu_m)-a_i(x,u,D_iu)\right]D_i(u_m-u)dx\to 0,$$

We observe that $\int_D H(x,u_m)(u_m-u)dx \to 0$ since $u_m \to u$ in $L^{q(\cdot)}(\Omega)$ and the sequence $(H(x,u_m))_m$ is bounded in $L^{q'(\cdot)}(\Omega)$. By invoking (3.12) and using the fact that $D_iu_n \rightharpoonup D_iu$ weakly in $L^{p_i'(\cdot)}(\Omega)$, we get

$$\limsup_{m\to+\infty}\sum_{i=1}^{N}\int_D \left[a_i(x,u_m,D_iu_m)-a_i(x,u,D_iu)\right]D_i(u_m-u)dx \le 0.$$

We have for all $i=1,...,N$

$$\int_D \left[a_i(x,u_m,D_iu_m)-a_i(x,u,D_iu)\right]D_i(u_m-u)dx \ge \int_D \left[a_i(x,u_m,D_iu)-a_i(x,u,D_iu)\right]D_i(u_m-u)dx$$

Now, let us prove that

$$\liminf_{m\to+\infty}\langle \mathcal{A}u_m, u_m-v\rangle \ge \langle \mathcal{A}u, u-v\rangle, \quad \forall v\in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega).$$

Because that, from (1.13), we deduce up to a subsequence

$$D_iu_m \to D_iu \quad \text{a.e in } D, \quad i=1,...,N.$$

Therefore, for each $i=1,...,N$

$$a_i(x,u_m,D_iu_m) \rightharpoonup a_i(x,u,D_iu),$$

weakly in $L^{p_i'(\cdot)}(\Omega)$ and a.e in D. Thus

$$\lim_{m\to+\infty}\int_D a_i(x,u_m,D_iu_m)D_ivdx = \int_D a_i(x,u,D_iu)D_ivdx,$$

for all $v\in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$. By virtue of Fatou's lemma and a_i are Carathéodory, we get

$$\liminf_{m\to+\infty}\int_D a_i(x,u_m,D_iu_m)dx \ge \int_D a_i(x,u,D_iu)dx. \tag{3.13}$$

On the other hand, we have

$$\int_D H(x,u_m)(u_m-v)dx \to \int_D H(x,u)(u-v)dx,\ \forall v\in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega). \tag{3.14}$$

Finally, combining , (3.13), and (3.14), we obtain

$$\begin{aligned}\liminf_{m\to+\infty}\langle \mathcal{A}u_m, u_m - v\rangle &\geq \sum_{i=1}^{N}\int_{\Omega} a_i(x, u_m, D_i u_m) D_i(u-v)dx + \int_{\Omega} H(x,u)(u-v)dx \\ &= \langle \mathcal{A}u, u-v\rangle.\end{aligned}$$

Therefore $\mathcal{A}$ is pseudomonotone. Then, according to Theorem 3.7, there exists at least one weak solution $u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ to problem (3.3).

Chapter 4

Nonlinear Anisotropic Degenerate Elliptic Equation in $\mathbb{R}^N$ with L^1_{loc} Data

In this chapter, we establish existence and regularity results for weak solutions to a class of nonlinear anisotropic elliptic equations in $\mathbb{R}^N$, subject to $p_i(x)$ -type growth conditions and locally integrable data, with principal part having degenerate coercivity The results presented here are based on the work in [23].

4.1 Introduction

We consider the following nonlinear anisotropic elliptic equation:

$$-\sum_{i=1}^{N} D_i(e_i(x,u)b_i(x,u,Du)) + H(x,u) = f, \quad x \in \mathbb{R}^N,\ N \geq 2 \tag{4.1}$$

where the function f is locally integrable on $\mathbb{R}^N$. Suppose that $b_i : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$, $e_i : \mathbb{R}^N \times \mathbb{R} \longrightarrow \mathbb{R}$, are Carathéodory functions satisfying, for almost every x in $\mathbb{R}^N$, forall , $u \in \mathbb{R}$,forall $\xi(\xi_1, ..., \xi_N), \xi'(\xi'_1, ..., \xi'_N) \in \mathbb{R}^N$ for all $i = 1, ..., N$, the following:

$$b_i(x,u,\xi).\xi_i \geq \alpha|\xi_i|^{p_i(x)}, \tag{4.2}$$

$$|b_i(x,v,\xi)| \leq g(x)\left(h(x) + |v|^{\overline{p}^-} + \sum_{j=1}^{N}|\xi_j|^{p_j(x)}\right)^{1-\frac{1}{p_i(x)}}, \qquad \overline{p}^- = \min_{x\in\overline{\Omega}} \overline{p}(x), \tag{4.3}$$

$$(b_i(x,u,\xi) - b_i(x,u,\xi'))(\xi - \xi') > 0, \qquad \xi \neq \xi',\ \frac{1}{\overline{p}(\cdot)} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i(\cdot)}, \tag{4.4}$$

$$\frac{\eta}{(1+|u|)^{\gamma_i(x)}} \leq e_i(x,u) \leq \mu, \tag{4.5}$$

where α, η, μ are strictly positive real numbers, $\gamma_i(x) \geq 0, i = 1, ..., N$ are continuous functions $\mathbb{R}^N$, h is locally integrable on $\mathbb{R}^N$, g is locally essentially bounded on $\mathbb{R}^N$ are a given positive functions.

Let $H : \mathbb{R}^N \times \mathbb{R} \longrightarrow \mathbb{R}$ be a Carathéodory function satisfying the conditions:

$$\sup_{|u|\leq\tau} |H(x,u)| \in L^1_{loc}(\mathbb{R}^N), \forall \tau > 0 \tag{4.6}$$

$$H(x,u)sign(u) \geq \sum_{i=1}^{N} |u|^{u_i(x)}, \qquad \text{a.e}, x \in \mathbb{R}^N, \tag{4.7}$$

for all $u \in \mathbb{R}$ where $s_i(\cdot) > 0, i = 1, ..., N$ are continuous functions on $\mathbb{R}^N$.

Example: As a prototype example, we consider the model problem

$$-\sum_{i=1}^{N}\left(D_i\left(g(x)\frac{|u|^{\left(\frac{p_i(x)-1}{p_i(x)}\right)\overline{p}^-} + |D_iu|^{p_i(x)-2}D_iu}{(\ln(1+|u|))^{\gamma_i(x)}}\right) - |u|^{s_i(x)-1}u\right) = f. \tag{4.8}$$

where $s_i(\cdot) \geq p_i(\cdot)$ for all $i = 1, ..., N$.

4.2 Statement of the problem

Definition 4.1 *A function u is a weak solution of problem (P) if*

$$u \in W^{1,1}_{loc}(\mathbb{R}^N) \cap \left(L^{s_+(\cdot)}_{loc}(\mathbb{R}^N)\right), b_i(x,u,Du) \in L^1_{loc}(\mathbb{R}^N), i = 1, ..., N, H(x,u) \in L^1_{loc}(\mathbb{R}^N),$$

$$\sum_{i=1}^{N}\int_{\mathbb{R}^N} e_i(x,u)b_i(x,u,Du)D_i\varphi dx + \int_{\mathbb{R}^N} H(x,u)\varphi dx = \int_{\mathbb{R}^N} f\varphi dx, \tag{4.9}$$

for all $\varphi \in C^1_c(\mathbb{R}^N)$, the C^1_c functions of compect support.

The core contributions of this work are detailed below:

Theorem 4.2 *Let f is locally integrable on $\mathbb{R}^N$ and $p_i(\cdot)$ satisfy (4.21), $i = 1, ..., N$. are continous funtions on $\mathbb{R}^N$ such that $1+\gamma^+_+ < \overline{p}(x) < N$, and for all $i = 1, ..., N$*

$$\frac{\overline{p}(x)(N-1-\gamma_+(x)) - N(\gamma^+_+ - \gamma_+(x))}{N(\overline{p}(x)-1-\gamma^+_+)} < p_i(x) < \frac{\overline{p}(x)(N-1-\gamma_+(x)) - N(\gamma^+_+ - \gamma_+(x))}{(1+\gamma_+(x))(N-\overline{p}(x))}, \tag{4.10}$$

for all $x \in \mathbb{R}^N$ *where*

$$\overline{p}(x) > \frac{N(2+\gamma_+^+ + \gamma_+(x))}{N+1+\gamma_+(x)}, \quad \textit{for all} \quad i = 1, ..., N, \tag{4.11}$$

$$0 \leq \gamma_+(x) < p_i'(x) - 1, \quad \textit{for all} \quad i = 1, ..., N, \tag{4.12}$$

$$s_i(x) \geq p_i(x), \quad \textit{for all} \quad i = 1, ..., N. \tag{4.13}$$

Let b_i *be a functions satisfying* (4.2)-(4.4) *and* H *satisfy* (4.6)-(4.7). *Then the problem* (4.1) *admits at least one weak solution* u *such that*

$$u \in \mathcal{M}_{loc}^{\widetilde{q}(\cdot)}(\mathbb{R}^N), \quad D_i u \in \mathcal{M}_{loc}^{q_i(\cdot)}(\mathbb{R}^N),$$

where

$$q_i(x) = \frac{N p_i(x)(\overline{p}(x) - 1 - \gamma_+^+)}{\overline{p}(x)(N - 1 - \gamma_i(x)) - N(\gamma_+^+ - \gamma_i(x))}, \tag{4.14}$$

$$\widetilde{q}(x) = \frac{N(\overline{p}(x) - 1 - \gamma_+^+)}{N - \overline{p}(x)}. \tag{4.15}$$

Theorem 4.3 *Let* $f \in L_{loc}^1(\mathbb{R}^N)$ *and assume that* $p_i(\cdot) > 1$, (4.21), $s_i(\cdot) > 0$, $i = 1, ..., N$ *are continuous functions on* $\mathbb{R}^N$ *such that*

$$s_i(.) > (1 + \gamma_+(.)) \max_{1 \leq i \leq N} \left(\frac{1}{p_i(.) - 1}; (p_i(.) - 1) \right), \ \forall x \in \mathbb{R}^N. \tag{4.16}$$

$$s_+(x) > \frac{N(\overline{p}(x) - 1 - \gamma_+^+)}{N - \overline{p}(x)}, \ \forall x \in \mathbb{R}^N. \tag{4.17}$$

Let b_i *be a funtions satifying* (4.2)-(4.4) *and* H *satisfy* (4.6)-(4.7). *Then the problem* (4.1) *admits at least one weak solution* u *such that*

$$u \in \mathcal{M}^{r_i(\cdot)}(\mathbb{R}^N), \tag{4.18}$$

$$r_i(x) = \frac{p_i(x) s_+(x)}{1 + s_+(x) + \gamma_i(x)}. \tag{4.19}$$

Remark 4.4 *Let* $0 \leq \gamma_+(x) < p_i'(x) - 1$ *and* $s_i(x) \geq p_i(x)$ *implies that for all* $x \in \mathbb{R}^N$

$$s_i(x) > (1 + \gamma_+(x))(p_i(x) - 1).$$

Under the assumption that $\overline{p}(\cdot) < N$, it is possble to obtain a sharper regularity result for Du provided the contribution of the lower order term $H(x,u)$ when $s_+(\cdot) = \max_{1\leq i\leq N} s_i(\cdot)$ is large enough.

We emphasize that this result remains valid even when $p_i(\cdot) > 1$ as long as $s_+(\cdot) > 1$ is sufficiently large. Indeed, assumption (4.17) ensures that

$$\frac{p_i(\cdot)s_+(\cdot)}{1+s_+(\cdot)+\gamma_i(\cdot)} > \frac{Np_i(\cdot)(\overline{p}(\cdot)-1-\gamma_+^+)}{\overline{p}(\cdot)(N-1-\gamma_i(\cdot))-N(\gamma_+^+-\gamma_i(\cdot))}.$$

for all $i = 1, ..., N$, so Theorem 4.3 improves Theorem 4.2.

Lemma 4.5 ([40]) *Let $p_i(\cdot), s_i(\cdot), i = 1, ..., N$ in $C_+(\overline{\Omega})$ with*

$$s_i(\cdot) \geq p_i(\cdot), \quad \textit{for all} \quad i = 1, ..., N. \tag{4.20}$$

with

$$1+\gamma_+^+ < \overline{p}(\cdot) < N. \tag{4.21}$$

and g be a non-negative function in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$. Suppose that there exists a constant c such that

$$\|g\|_{L^{s_+(\cdot)}(\Omega)} \leq c, \tag{4.22}$$

and

$$\sum_{i=1}^{N}\int_{\{g\leq k\}} |D_i g|^{p_i(x)}dx \leq c(d+1)^{1+\gamma_+^+}, \forall d > 0. \tag{4.23}$$

Thzn there exits a constant C, depending on c such that

$$\int_{\{f>d\}} |d|^{h(x)}dx \leq C, \quad \forall k > 0, \quad h(x) = \frac{N(\overline{p}(x)-1-\gamma_+^+)}{N-\overline{p}(x)}, \quad \forall x \in \overline{D}. \tag{4.24}$$

4.3 Proof of Results:

4.3.1 Proof of Theorem 4.2

Let $R > 0$ and $B_R = \{x \in \mathbb{R}^N / |x| < R\}$ be given. Our objective is to solve the equation (4.1) in domain $\mathbb{R}^N$. Our approach begins by analyzing the case of (4.1) in the bells B_R, for an arbitrary but fixed $R > 0$, given a function f_n which approximated f. If one can derive estimates, which are independent of R and approximate f_n, we can then pass to the limits $R \longrightarrow +\infty$ and $f_n \rightarrow f$ to obtain a solution of the original problem.

4.3.2 Approximate of Problem (4.1)

Let (f_n), $f_n = T_n(f)$ and (u_n) are sequences of bounded defined on a set $B_n = \{x \in \mathbb{R}^N : |x| < n\}$ which convege to f in $L^1_{loc}(\mathbb{R}^N)$, and which verfies the inequalities

$$\begin{cases} \|f_n\|_{L^1_{loc}(\mathbb{R}^N)} \leq \|f\|_{L^1_{loc}(\mathbb{R}^N)}, \\ |f_n| \leq n, \quad \forall n \geq 1 \end{cases}$$

$$\|u_n\|_{L^1_{loc}(\mathbb{R}^N)} \leq \|u\|_{L^1_{loc}(\mathbb{R}^N)}$$

The existence of the sequences u_n and f_n is traditional, see for example. We approch the problem (4.1) by following problem:

$$-\sum_{i=1}^{N} D_i(e_i(x, T_n(u_n))b_i(x, u_n, Du_n)) + H(x, u_n) = f_n, \quad \text{in} \quad \mathbb{R}^N \tag{4.25}$$

There exists at least one weak solution

$$u_n \in \bigcap_{i=1}^{N} \left(W_0^{1,p_i(\cdot)}(B_n) \cap L^{s_i(\cdot)}(B_n) \right), \quad W_0^{1,\overrightarrow{p}(\cdot)}(B_n) = \bigcap_{i=1}^{N} W_0^{1,p_i(\cdot)}(B_n)$$

indeed one has $H(x, u_n) \in L^1(B_n)$ and

$$\sum_{i=1}^{N} \int_{B_n} e_i(x, T_n(u_n))b_i(x, u_n, Du_n)D_i\varphi dx + \int_{B_n} H(x, u_n)\varphi dx = \int_{B_n} f_n\varphi dx, \tag{4.26}$$

for all $\varphi \in \bigcap_{i=1}^{N} W_0^{1,p_i(\cdot)}(B_n) \cap L^\infty(B_n)$. Note that by (4.5) and (4.2) we have

$$e_i(x, T_n(u_n)) \geq \frac{\eta}{(1 + |T_n(u_n)|)^{\gamma_i(x)}} \geq \frac{\eta}{(1 + n)^{\gamma_+^+}}.$$

In such a manner that the operator

$$B : v \to \sum_{i=1}^{N} D_i(e_i(x, T_n(v))b_i(x, v, Dv))$$

is coercive. Thus, chapter 3 provedes a rigorous proof of the existence of the approximate solution u_n.

4.3.3 Uniform estimates

We assume that u_n be a solution of (4.35), (4.2)-(4.4) and (4.6)-(4.7) hold, r such that $0 < 2r < n$, an $B_r = \{x \in \mathbb{R}^N : |x| < r\}$.

Lemma 4.6 *Let $u_+(\cdot) \geq p_i(\cdot)$, $i = 1, ..., N$, consider a redius $R = n$ and let $0 < 2r < n$. Thare exists a constant C, indepzndent of n, such that*

$$||u_n)||_{L^{s_+(\cdot)}(B_r)} \leq C, \quad s_+(\cdot) = \max_{1\leq i\leq N} s_i(\cdot) \tag{4.27}$$

$$||H(x, u_n)||_{L^1(B_r)} \leq C. \tag{4.28}$$

Morever, for every $\delta > 1$ there exists a constant C_δ, depending on δ, such that

$$\sum_{i=1}^{N} \int_{B_r} \frac{|D_i u_n|^{p_i(\cdot)}}{(1 + |u_n|)^{\delta+\gamma_i(x)}} dx \leq C_\delta. \tag{4.29}$$

Proof: We fixed $\delta > 1$ such that

$$\delta \in (1, k), \quad k = \min_{1\leq i\leq N, x\in\overline{B}_{2r}} \left(\frac{u_i(x)}{p_i(x) - 1} - \gamma_+(x) \right), \tag{4.30}$$

we define the function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$\psi(\sigma) = (1 - \delta)\psi_\delta(\sigma) = (1 - \delta) \int_0^\sigma \frac{dt}{(1 + |t|)^\delta} dt, \qquad \forall \delta > 1 \tag{4.31}$$

It's apparent that

$$\psi(\sigma) = ((1 + |\sigma|)^{1-\delta} - 1)sign(\sigma), \tag{4.32}$$

We know $|\psi| \leq 1$ and $|\psi'| \leq \delta - 1$. Taking

$$\psi(u_n)\theta^\alpha \tag{4.33}$$

as test function in (4.35), where α is a number such that

$$\alpha > \max_{1\leq i\leq N, x\in\overline{B}_{2r}} (\frac{s_i(x)p_i(x)}{s_i(x) - (\delta + \gamma_+(x))(p_i(x) - 1)}) > 0, \tag{4.34}$$

θ is a smooth with compact support in B_{2r}, such that $0 \leq \theta \leq 1$ and $\theta \equiv 1$ on B_r and $|D\theta| \leq \frac{2}{r}$. We obtain

$$\sum_{i=1}^{N} \int_{B_R} e_i(x, T_n(u_n))b_i(x, u_n, Du_n)D_i(\psi(u_n)\theta^\alpha)dx + \int_{B_R} H(x, u_n)\psi(u_n)\theta^\alpha dx = \int_{B_R} f_n\psi(u_n)\theta^\alpha dx, \tag{4.35}$$

Using the form (4.2), (4.5) and

$$D_i(\psi(u_n)\theta^\alpha) = D_i u_n \psi'(u_n)\theta^\alpha + \alpha\psi(u_n)\theta^{\alpha-1}D_i\theta, \quad i = 1, ..., N,$$

we obtain

$$\begin{aligned}
\beta\eta\sum_{i=1}^{N}\int_{B_{2r}}\frac{|Du_n|^{p_i(x)}}{(1+|u_n|)^{\gamma_i(x)}}\psi'(u_n)\theta^\alpha dx+\int_{B_{2r}}H(x,u_n)\psi(u_n)\theta^\alpha dx \\
\leq ||f_n||_{L^1(B_{2r})} - \alpha\sum_{i=1}^{N}\int_{B_{2r}} b_i(x,u_n,Du_n)\psi(u_n)\theta^{\alpha-1}D_i\theta dx \\
\leq ||f||_{L^1(B_{2r})} + C_1\sum_{i=1}^{N}\int_{B_{2r}}(h(x)+|u_n|^{\overline{p}^-}+\sum_{j=1}^{N}|D_ju_n|^{p_j(x)})^{1-\frac{1}{p_i(x)}}\psi(u_n)\theta^{\alpha-1}|D_i\theta|dx \\
\leq C_2 + C_3\sum_{i=1}^{N}\int_{B_{2r}}(h(x)+|u_n|^{\overline{p}^-}+\sum_{j=1}^{N}|D_ju_n|^{p_j(x)})^{1-\frac{1}{p_i(x)}}\theta^{\alpha-1}dx.
\end{aligned} \tag{4.36}$$

By Youeng's inequality, we find

$$I = (h(\cdot)+|u_n|^{\overline{p}^-}+\sum_{j=1}^{N}|D_ju_n|^{p_j(\cdot)})^{1-\frac{1}{p_i(\cdot)}}\theta^{\alpha-1}$$

$$= \varepsilon(\cdot)^{\frac{1}{p_i'(\cdot)}}(h(\cdot)+|u_n|^{\overline{p}^-}+\sum_{j=1}^{N}|D_ju_n|^{p_j(\cdot)})^{1-\frac{1}{p_i(\cdot)}}(1+|u_n|)^{\frac{-\gamma_+(\cdot)}{p_i'(\cdot)}}\psi'^{\frac{1}{p_i'(\cdot)}}\theta^{\frac{\alpha}{p_i'(\cdot)}}\varepsilon(\cdot)^{\frac{-1}{p_i'(\cdot)}}(1+|u_n|)^{\frac{\gamma_+(\cdot)}{p_i'(\cdot)}}\psi'^{\frac{-1}{p_i'(\cdot)}}\theta^{\frac{\alpha}{p_i(\cdot)}-1}$$

$$\leq \frac{\varepsilon(\cdot)}{p_i'(\cdot)}(h(\cdot)+|u_n|^{\overline{p}^-}+\sum_{j=1}^{N}|D_ju_n|^{p_j(\cdot)})(1+|u_n|)^{-\gamma_+(\cdot)}\psi'(u_n)\theta^\alpha+\frac{(\delta-1)\theta^{\alpha-p_i(\cdot)}}{p_i(\cdot)\varepsilon(\cdot)^{p_i(\cdot)-1}}(1+|u_n|)^{(\delta+\gamma_+(\cdot))(p_i(\cdot)-1)},$$

then

$$\begin{aligned}
I \leq \frac{\varepsilon(\cdot)}{p_i'(\cdot)}\left(h(\cdot)+\frac{|u_n|^{\overline{p}^-}}{(1+|u_n|)^{\gamma_+(\cdot)}}+\frac{\sum_{j=1}^{N}|D_ju_n|^{p_j(\cdot)}}{(1+|u_n|)^{\gamma_+(\cdot)}}\right)\psi'(u_n)\theta^\alpha \\
+\frac{(\delta-1)\theta^{\alpha-p_i(\cdot)}}{p_i(\cdot)\varepsilon(\cdot)^{p_i(\cdot)-1}}(1+|u_n|)^{(\delta+\gamma_+(\cdot))(p_i(\cdot)-1)},
\end{aligned}$$

so

$$\begin{aligned}
I \leq \frac{\varepsilon(\cdot)}{p_i'(\cdot)}\left(h(\cdot)+\frac{|u_n|^{\overline{p}^-}}{(1+|u_n|)^{\delta+\gamma_+(\cdot)}}+\sum_{j=1}^{N}\frac{|D_ju_n|^{p_j(\cdot)}}{(1+|u_n|)^{\delta+\gamma_j(\cdot)}}\right)\theta^\alpha \\
+ C_\delta\theta^{\alpha-p_i(\cdot)}(1+|u_n|)^{(\delta+\gamma_+(\cdot))(p_i(\cdot)-1)},
\end{aligned} \tag{4.37}$$

where $\varepsilon : \mathbb{R} \to (0, +\infty)$ anyposetive function, and $p_i'(\cdot) = p_i(\cdot)/(p_i(\cdot) - 1), i = 1, ..., N$. Choosing $\varepsilon(\cdot) = \beta/(2C_2 \sum_{i=1}^{N}(1/p_i'(\cdot)))$ and $C_\delta = \frac{(\delta-1)^{1-p_-^-}}{p_-^-}$, the fact that

$$\psi'(u_n) = (\delta - 1)(1 + |u_n|)^{-\delta}.$$

Using (4.36),(4.37), we obtain

$$\frac{\beta\eta(\delta-1)}{2} \sum_{i=1}^{N} \int_{B_{2r}} \frac{|D_i u_n|^{p_i(x)}}{(1+|u_n|)^{\delta+\gamma_i(x)}} \theta^\alpha dx + \int_{B_{2r}} H(x, u_n)\psi(u_n)\theta^\alpha dx$$

$$\leq C_4 + C_5 \sum_{i=1}^{N} \int_{B_{2r}} \left((1 + |u_n|)^{(\delta+\gamma_+(x))(p_i(x)-1)} \theta^{\alpha - p_i(x)} + |u_n|^{\overline{p}^-} \theta^\alpha \right) dx$$

$$\leq C_6 + C_7 \sum_{i=1}^{N} \int_{B_{2r}} \left((1 + |u_n|)^{(\delta+\gamma_+(x))(p_i(x)-1)} \theta^{\alpha - p_i(x)} + |u_n|^{s_+(x)} \theta^\alpha \right) dx. \tag{4.38}$$

Using Young inequality, we can write

$$J = |u_n|^{(\delta+\gamma_+(\cdot))(p_i(\cdot)-1)} \theta^{\alpha - p_i(\cdot)}$$
$$\leq \frac{(\delta + \gamma_+(\cdot))(p_i(\cdot) - 1)}{s_i(\cdot)} \theta^\alpha |u_n|^{s_i(\cdot)} + C_8 \theta^{\alpha - \frac{p_i(\cdot)s_i(\cdot)}{s_i(\cdot) - (\delta+\gamma_+(\cdot))(p_i(\cdot)-1)}}$$

$$\leq \frac{(\delta + \gamma_+(\cdot))(p_i(\cdot) - 1)}{s_i(\cdot)} \theta^\alpha |u_n|^{s_+(\cdot)} + C_9$$

By thisinequality, (4.38) , we have

$$\sum_{i=1}^{N} \int_{B_{2r}} \frac{|D_i u_n|^{p_i(x)}}{(1+|u_n|)^{\delta+\gamma_i(x)}} \theta^\alpha dx + \int_{B_{2r}} H(x, u_n)\psi(u_n)\theta^\alpha dx$$

$$\leq C_{10} + C_{11} \int_{B_{2r}} |u_n|^{s_+(x)} \theta^\alpha dx. \tag{4.39}$$

Now, from the assumption (4.7), we get

$$H(x, \sigma)\psi(\sigma) \geq \alpha_0 \psi(1) \sum_{i=1}^{N} |\sigma|^{s_i(x)}, \quad \forall |\sigma| \geq 1 \quad \text{and a.e.} \quad x \in \mathbb{R}^N,$$

so we have

$$\sum_{i=1}^{N} |\sigma|^{s_i(x)} \leq \frac{1}{\alpha_0 \psi(1)} H(x, \sigma)\psi(\sigma) + N, \quad \forall \sigma \in \mathbb{R} \quad \text{and a.e.} \quad x \in \mathbb{R}^N. \tag{4.40}$$

We combine (4.39) and (4.40), we can write

$$\sum_{i=1}^{N}\int_{B_{2r}}\frac{|D_iu_n|^{p_i(x)}}{(1+|u_n|)^{\delta+\gamma_i(x)}}\theta^\alpha dx+\int_{B_{2r}}H(x,u_n)\psi(u_n)\theta^\alpha dx\leq C_{12}+C_{13}\varsigma\int_{B_{2r}}H(x,u_n)\psi(u_n)\theta^\alpha dx.$$

Using thes inequality and satting $\varsigma=1/(2C_{13})$, weobtain

$$\sum_{i=1}^{N}\int_{B_{2r}}\frac{|D_iu_n|^{p_i(x)}}{(1+|u_n|)^{\delta+\gamma_i(x)}}\theta^\alpha dx+\frac{1}{2}\int_{B_{2r}}H(x,u_n)\psi(u_n)\theta^\alpha dx\leq C_{12}. \tag{4.41}$$

Aftar dropping the nonnegative term, we derive

$$\int_{B_{2r}}H(x,u_n)\psi(u_n)\theta^\alpha dx\leq 2C_{12}, \tag{4.42}$$

estimate (4.27) is then direct consequence of (4.42). By (4.6), (4.42), and the definition of ψ we obtain

$$\int_{B_r}|H(x,u_n)|dx\leq\int_{B_r\bigcap\{|u_n|\leq1\}}|H(x,u_n)|dx+\frac{1}{\psi(1)}\int_{B_r}H(x,u_n)\psi(u_n)dx\leq C.$$

Finally by (4.41), we deduce that

$$\sum_{i=1}^{N}\int_{B_r}\frac{|D_iu_n|^{p_i(x)}}{(1+|u_n|)^{\delta+\gamma_i(x)}}dx\leq C_{12},\qquad\forall\delta\in(1,k). \tag{4.43}$$

so that

$$\sum_{i=1}^{N}\int_{B_r}\frac{|D_iu_n|^{p_i(x)}}{(1+|u_n|)^{\delta+\gamma_i(x)}}dx\leq C,\qquad\forall\delta>1. \tag{4.44}$$

This completes the proof of Lemma 4.6 ■

Lemma 4.7 *There exist a conestant C_k depeendent of k such that*

$$\int_{B_r}|D_iT_k(u_n)|^{p_i(x)}dx\leq C_k, i=1,...,N. \tag{4.45}$$

Proof: Let $\delta>1$ and By the estimmate (4.29), we obtain

$$\begin{aligned}\int_{B_r}|D_iT_k(u_n)|^{p_i(x)}dx &= \int_{B_r}\frac{|D_iT_k(u_n)|^{p_i(x)}}{(1+|u_n|)^{\delta+\gamma_i(x)}}(1+|u_n|)^{\delta+\gamma_i(x)}dx\\ &\leq (1+k)^{\delta+\gamma_+^+}\int_{B_r}\frac{|D_iT_k(u_n)|^{p_i(x)}}{(1+|u_n|)^{\delta+\gamma_i(x)}}dx\\ &\leq C(1+k)^{\delta+\gamma_+^+},\end{aligned}$$

so that

$$\int_{B_r} |D_i T_k(u_n)|^{p_i(x)} dx \leq C_k, i = 1, ..., N.$$

this finishes the proof of Lemma (4.7) ■

Lemma 4.8 *Under the assumptions of Theorem 4.2, there exists tow constants* C_1, C_2 *(independent of* n*) such that*

$$\int_{\{|u_n|>k\}} k^{h(x)} dx \leq C_1, \quad \forall k > 0, \quad h(x) = \frac{N(\overline{p}(x) - 1 - \gamma_+^+)}{N - \overline{p}(x)}, \tag{4.46}$$

and

$$\int_{\{|D_i u_n|>k\}} k^{h_i(x)} dx \leq C_2, \quad \forall k > 0, \quad h_i(x) = \frac{N p_i(x)(\overline{p}(x) - 1 - \gamma_+^+)}{\overline{p}(x)(N - 1 - \gamma_i(x)) - N(\gamma_+^+ - \gamma_i(x))}. \tag{4.47}$$

Proof: The inequality in Lemma (4.6), we have

$$||u_n)||_{L^{s_+(\cdot)}(B_r)} \leq C, \quad s_+(\cdot) = \max_{1 \leq i \leq N} s_i(\cdot),$$

$$s_+(\cdot) \geq p_i(\cdot),$$

and $|D_i|u_n|| \leq |D_i u_n|$ yield

$$\int_{\{|u_n|\leq k\}} |D_i u_n|^{p_i(x)} dx \leq Ck, \qquad i = 1, ..., N,$$

we obtain (4.46). For the estimate (4.47), setting $\alpha_i^\delta(\cdot) = \frac{p_i(\cdot)}{\delta + h(\cdot) + \gamma_i(\cdot)}, i = 1, ..., N,$ then for $k \geq 1$, and from (4.46) we deduce

$$\begin{aligned}
\int_{\{|D_i u_n|^{\alpha_i^\delta(x)}>k\}} k^{h(x)} dx &\leq \int_{\{|D_i u_n|^{\alpha_i^\delta(x)}>k\} \cap \{|u_n|\leq k\}} k^{h(x)} dx + \int_{\{|u_n|>k\}} k^{h(x)} dx \\
&\leq \int_{\{|u_n|\leq k\}} k^{h(x)} \left(\frac{|D_i u_n|^{\alpha_i^\delta(x)}}{k} \right)^{\frac{p_i(x)}{\alpha_i^\delta(x)}} dx + C \\
&\leq \int_{\{|u_n|\leq k\}} k^{-\delta - \gamma_i(x)} |D_i u_n|^{p_i(x)} dx + C. \\
&\leq \int_{\{|u_n|\leq k\}} 2^{\delta + \gamma_+^+} (1 + k)^{-\delta - \gamma_i(x)} |D_i u_n|^{p_i(x)} dx + C. \\
&\leq 2^{\delta + \gamma_+^+} \int_{\{|u_n|\leq k\}} \frac{|D_i u_n|^{p_i(x)}}{(1 + |u_n|)^{\delta + \gamma_i(x)}} dx + C.
\end{aligned}$$

By (4.29), we obtain

$$\int_{\{|D_i u_n|^{\alpha_i^\delta(x)}>k\}} k^{h(x)} dx \le C_\delta, \qquad \forall k \ge 1.$$

If $k \in (0,1)$, we have

$$\int_{\{|D_i u_n|^{\alpha_i^\delta(x)}>k\}} k^{h(x)} dx \le \int_{B_r} k^{h(x)} \le C_\delta.$$

Therefore

$$\int_{\{|D_i u_n|^{\alpha_i^\delta(x)}>k\}} k^{h(x)} dx \le C'_\delta, \qquad \forall k > 0.$$

This shows that, for all $i = 1, ..., N$, $(D_i u_n)$ is bounded in $\mathcal{M}^{h(\cdot)\alpha_i^\delta(\cdot)}(B_r)$ where

$$h(\cdot)\alpha_i^\delta(\cdot) = \frac{p_i(\cdot)h(\cdot)}{\delta + h(\cdot) + \gamma_i(\cdot)} < \frac{p_i(\cdot)h(\cdot)}{1 + h(\cdot) + \gamma_i(\cdot)} = \frac{Np_i(x)(\overline{p}(x) - 1 - \gamma_+^+)}{\overline{p}(x)(N - 1 - \gamma_i(x)) - N(\gamma_+^+ - \gamma_i(x))}.$$

So that $g : \mathbb{R}^N \to \mathbb{R}$ such that

$$g(\cdot) < \frac{p_i(\cdot)h(\cdot)}{1 + h(\cdot) + \gamma_i(\cdot)},$$

we have $(D_i u_n)$ is bounded in $\mathcal{M}^{g(\cdot)}(B_r)$.

Finally suppese that $h_i(x) = \frac{Np_i(x)(\overline{p}(x)-1-\gamma_+^+)}{\overline{p}(x)(N-1-\gamma_i(x))-N(\gamma_+^+-\gamma_i(x))}$ and let $\varepsilon \in (0, h_-^-)$. Then we have

$$\int_{\{|D_i u_n|^{\alpha_i^\delta(x)}>k\}} k^{h_i(x)-\varepsilon} dx \le C, \qquad \forall k > 0.$$

Letting ε go to zero, the proof of lemma 4.8 is completed. ■

Lemma 4.9 *Under the assemptions of Theorem 4.3. Then, sequence* (u_n) *is bounded in* $\mathcal{M}^{r_i(\cdot)}(B_r)$, *such that* $r_i(\cdot) = \frac{p_i(\cdot)s_+(\cdot)}{1+s_+(\cdot)+\gamma_+(\cdot)}$.

Proof: Let $s_+(\cdot) > 0$ and $s_+(\cdot) \ge p_i(\cdot)$, we have

$$\int_{B_r} |u_n|^{s_+(\cdot)} \le C, \quad s_+(\cdot) = \max_{1\le i\le N} s_i(\cdot)$$

we obtain

$$\int_{|u_n|>k} k^{s_+}(x) dx \le C, \qquad \forall k > 0.$$

By applying the same technique used in the proof of Lemma, along with the aforementioned estimate, we deduce that sequence u_n is bounded in $\mathcal{M}^{\tau_i(\cdot)s_+(\cdot)}(B_r)$ such that $\tau_i(\cdot) = \frac{p_i(\cdot)}{1+s_+(\cdot)+\gamma_+(\cdot)}$. ■

4.4 Passage to the limit

In light of the previously obtained estimates, which ensures that for any fixed $r > 0$ the sequence $(u_n)_{n>2r}$ is uniformly bounded in $W^{1,q_-^-}(B_r)$, with $q_-^- = \min_{i=1,...,N} \min\{q_i(x)/x \in \overline{B}_r\}$ and $q_1(\cdot), ..., q_N(\cdot)$ are restricted as in Lemma 4.6 or Lemma 4.8. It is therefore possible to extract a subsequence, still denoted by (u_n), such that

$$u_n \to u \quad \text{a.e. in } D \quad \text{and strongly in} \quad L^{q_-^-}(B_r). \tag{4.48}$$

Lemma 4.10 *Let $f \in L^1_{loc}(\mathbb{R}^N)$ and let e_i, b_i, H be Caratheodory functions, where a_i are satisfying* (4.2)-(4.4) *and H satisfy* (4.6)-(4.7). *Then*

$$H(x,u_n) \to H(x,u) \quad \textit{strongly in} \quad L^1_{loc}(\mathbb{R}^N), \quad \forall r > 0. \tag{4.49}$$

Proof: Let $\lambda > 0$, $r > 0$ and We shallfirst obtain local-integrability of $(H(x,u_n))$ on B_r. We define $\phi_\lambda : \mathbb{R} \to \mathbb{R}$ such that

$$\phi_\lambda(\sigma) = \begin{cases} \phi(\sigma - \lambda) & , \sigma \geq \lambda, \\ 0 & ,|\sigma| < \lambda, \\ \phi(\sigma + \lambda) & , \sigma \leq -\lambda, \end{cases}$$

Let $\alpha > 0$. We choose $\phi_\gamma(u_n)\theta^\alpha$ as test function, where θ be a cutoff function as in (4.35), we have

$$\sum_{i=1}^N \int_D e_i(x, T_n(u_n)) b_i(x, Du_n) D_i u_n \phi'_\lambda(u_n)\theta^\alpha dx + \int_D H_n(x,u_n)\phi_\lambda(u_n)\theta^\alpha dx$$
$$= \int_D f\phi_\lambda(u_n)\theta^\alpha dx,$$

working as in (4.36), we obtain

$$\sum_{i=1}^N \int_D e_i(x, T_n(u_n)) b_i(x, Du_n) D_i u_n \phi'_\lambda(u_n)\theta^\alpha dx + \int_D H_n(x,u_n)\phi_\lambda(u_n)\theta^\alpha dx$$
$$\leq C_2 + C_3 \sum_{i=1}^N \int_{B_{2r}} (h(x) + |u_n|^{\overline{p}^-} + \sum_{j=1}^N |D_j u_n|^{p_j(x)})^{1-\frac{1}{p_i(x)}} \theta^{\alpha-1} dx.$$

Then

$$\sum_{i=1}^N \int_{B_{2r}} |D_i u_n|^{p_i(x)} \phi'_\gamma(u_n)\theta^\alpha dx + \frac{1}{2}\int_{B_{2r}} H(x,u_n)\phi_\gamma(u_n)\theta^\alpha dx$$

$$\leq C_{17} \int_{B_{2r}\cap\{|u_n|\geq\gamma\}} (|f| + |h| + |u_n|^{\overline{p}^-})dx + C_{17}|B_{2r} \cap \{|u_n| \geq \gamma\}|.$$

Sincethe sequence (u_n) is bounded in $L^1(B_{2r})$ for any $n > 2r$. and $f, h \in L^1_{loc}(\mathbb{R}^N)$, we deduce form the above inequality that

Using the properties of the function θ, ϕ_λ, and the fact that $H(x,\sigma)\sigma \geq 0$, we get

$$\int_{B_{2r}\bigcap\{|u_n|\geq 2\gamma\}} |H(x,u_n)|dx \leq \frac{1}{\phi_\gamma(2\gamma)} \int_{B_{2r}} H(x,u_n)\phi_\lambda(u_n)\theta^\alpha dx \to 0 \qquad \text{as} \qquad \lambda \to +\infty.$$

This inequality and the assumption (6) give equi-integrable of $(H(x,u_n))$ on (B_r). Form this (54), and Vitali's theorem we obtain the result.

$$H(x,u_n) \to H(x,u) \quad \text{a.e. in} \quad \mathbb{R}^N. \tag{4.50}$$

By the assumption (4.6), (4.28), (4.50), and with the help of techniques used in [37], we get (4.49). ■

Taking $u_n^k = T_k(u_n)$ and $u^k = T_k(u)$, The result obtained is as follows.

Lemma 4.11 *For all $k > 0$, we have there exists a function θ_k such that for all ε, we have*

$$\limsup_n \int_{\{|u_n-u^k|\leq\varepsilon\}} \theta(x)e_i(x,T_n(u_n))b_i(x,u_n,D_iu_n)(D_iu_n - D_iu^k)dx \leq l_k(\varepsilon),$$

with for all $i = 1,...,N$ and $\lim_{\varepsilon\to 0} l_k(\varepsilon) = 0$.

Proof: The proof of Lemma 4.11 is similar to that of Lemma 2.16 of [35]. ■

Proposition 4.12 *Let b_i be a function satisfying (4.2)-(4.4). Then*

$$\theta(x)e_i(x,T_n(u_n))b_i(x,u_n,Du^k) \longrightarrow \theta(x)e_i(x,u)a_i(x,u,Du^k) \quad \textit{strongly in} \quad L^{p_i'(\cdot)}(B_{2r}) \tag{4.51}$$

for all $i = 1,...,N$ and $l_i'(\cdot) = \dfrac{p_i(\cdot)}{p_i(\cdot)-1}$.

Proof: we have (4.48), implies that

$$\theta(x)e_i(x,T_n(u_n))b_i(x,u_n,Du^k) \longrightarrow \theta(x)e_i(x,u)b_i(x,u,Du^k) \quad \text{a.e. in} \quad B_{2r}. \tag{4.52}$$

and in the fact

$$|\theta(x)e_i(x,T_n(u_n))b_i(x,u_n,Du^k)| \leq C|b_i(x,u,Du^k)| \in L^{p_i'(\cdot)}(B_{2r}) \tag{4.53}$$

according to Lebesgue's dominoated convergence theorem, we have (4.51). ■

Lemma 4.13 *There exists a subsequence (still denoted (u_n)) such that*

$$Du_n \to Du, \qquad a.e.\ in \qquad \mathbb{R}^N. \tag{4.54}$$

Proof: We write for all $\varepsilon \in (0,1)$

$$\begin{aligned} A_{ni}(\varepsilon) &= \int_{\{|u_n-u^k|\le\varepsilon\}} \theta(x)e_i(x,T_n(u_n))(b_i(x,u_n,Du_n)-b_i(x,u_n,Du^k))(D_iu_n-D_iu^k)dx \\ &= A^1_{ni}(\varepsilon)-A^2_{ni}(\varepsilon), \end{aligned}$$

with

$$A^1_{ni}(\varepsilon) = \int_{\{|u_n-u^k|\le\varepsilon\}} \theta(x)e_i(x,T_n(u_n))b_i(x,u_n,Du_n)(D_iu_n-D_iu^k)dx$$

and

$$\begin{aligned} A^2_{ni}(\varepsilon) &= \int_{\{|u_n-u^k|\le\varepsilon\}} \theta(x)e_i(x,T_n(u_n))b_i(x,u_n,Du^k)(D_iu_n-D_iu^k)dx \\ &= \int_{\{|u_n-u^k|\le\varepsilon\}} \theta(x)e_i(x,T_n(u_n))b_i(x,u_n^{k+1},Du^k)(D_iu_n^{k+1}-D_iu^k)dx. \end{aligned}$$

By (4.49) and Lebesgue's dominated convergence theorem we have

$$\theta(x)e_i(x,T_n(u_n))b_i(x,u_n,Du^k) \to \theta(x)e_i(x,u)b_i(x,u,Du^k) \quad \text{strongly in} \quad L^{p_i'(x)}_{loc}(\mathbb{R}^N).$$

Therefore, by Lemma 4.7, we can write

$$\lim_{n\to+\infty} A^2_{ni}(\varepsilon) = \int_{\{|u|>k,|u-u^k|\le\varepsilon\}} \theta(x)e_i(x,u)b_i(x,u,Du^k)(D_iu^{k+1}-D_iu^k)dx.$$

Consequently

$$\lim_{\varepsilon\to0}\lim_{n\to+\infty} A^2_{ni}(\varepsilon)=0.$$

By Lemma 4.11, we get

$$\lim_{\varepsilon\to0}\lim_{n\to+\infty} A_{ni}(\varepsilon)=0. \tag{4.55}$$

We put for all $i=1,\dots,N$

$$L_i(u_n,u) = e_i(x,T_n(u_n))\{(b_i(x,u_n,Du_n)-b_i(x,u_n,Du)).D_i(u_n-u)\}^{1/p_+^+} \ge 0.$$

From (4.3) and Young inequality's, we derive

$$L_i(u_n,u) \le C\left(h(x)+|u|+\sum_{j=1}^N |D_ju_n| + \sum_{j=1}^N |D_ju|\right). \tag{4.56}$$

Let us write

$$\int_{B_{2r}} \theta(x)L_i(u_n,u)dx = \int_{\{|u|\le k\}} \theta(x)L_i(u_n,u)dx + \int_{\{|u|>k\}} \theta(x)L_i(u_n,u)dx. \tag{4.57}$$

By (4.56), the $L^1(B_{2r})$ -bound on $h(x)$ and $L^{q_i(\cdot)}(B_r)$ -bound on D_iu_n for $q_i(\cdot)$ satisfy (4.14), we have for $i=1,...,N$

$$\int_{B_{2r}} |L_i(u_n,u)|^{q_-^-}dx \le C, \quad q_-^- = \min_{1\le i\le N}\min_{x\in\overline{B}_{2r}} q_i(x). \tag{4.58}$$

So, we have

$$\int_{\{|u|>k\}} \theta(x)L_i(u_n,u)dx \le C|\{|u|>k\}|^{1-1/q_-^-} = o(1)(\text{as} \quad k\to+\infty). \tag{4.59}$$

For the first integral in (4.57), we decompose it as

$$\int_{\{|u|\le k\}} \theta(x)L_i(u_n,u)dx = \int_{\{|u_n-u^k|\le\varepsilon,|u|\le k\}} \theta(x)L_i(u_n,u)dx$$
$$+\int_{\{|u_n-u^k|>\varepsilon,|u|\le k\}} \theta(x)L_i(u_n,u)dx. \tag{4.60}$$

By (4.55), we get

$$\lim_{n\to+\infty}\int_{\{|u_n-u^k|\le\varepsilon,|u|\le k\}} \theta(x)L_i(u_n,u)dx \le o(1)(\text{as} \quad \varepsilon\to 0)+o(1)(\text{as} \quad n\to+\infty). \tag{4.61}$$

Arguing as in (4.59), we obtain

$$\int_{\{|u_n-u^k|>\varepsilon,|u|\le k\}} \theta(x)L_i(u_n,u)dx \le C|\{|u_n-u|>\varepsilon\}|^{1-1/q_-^-} = o(1)(\text{as} \quad n\to+\infty). \tag{4.62}$$

We combine (4.57), (4.59), (4.61) and (4.62) to obtain

$$\limsup_{n\to+\infty}\int_{B_{2r}} \theta(x)L_i(u_n,u)dx = 0. \tag{4.63}$$

Since the integralfunction in (4.63) is nonnegative and $\theta=1$ on B_r, this implies that

$$e_i(x,T_n(u_n))(b_i(x,u_n,Du_n)-b_i(x,u,Du))(D_iu_n-D_iu)dx \to 0, \quad \text{strongly in} \quad L^1(B_r).$$

Thus, up to subsequence still denoted by u_n

$$e_i(x,T_n(u_n))(b_i(x,u_n,Du_n)-b_i(x,u_n,Du))(D_iu_n-D_iu)dx \to 0, \quad \text{a.e} \quad x\in B_r. \tag{4.64}$$

Let $x\in B_r$ be such that $u_n(x)$ converges to $u(x)$, that $|s(x)|<+\infty$, and that (4.25) hol Du true. Due to (4.49), the set of $x\in B_r$ such that at least one of the above properties does not hold has zero measure. Since $|s(x)| < +\infty$, one has $|s_n(x)| \leq |s(x)| + 1 \leq n$ for n large enough, so that (4.64) becomes

$$e_i(x, u_n)(b_i(x, u_n, Du_n) - b_i(x, u_n, Du))(D_i u_n - D_i u)dx \to 0, \quad \text{a.e} \quad x \in B_r. \tag{4.65}$$

Adopting the approach of [34], we obtain the desert result. ∎

Lemma 4.14 *Let b_i be a function satisfying (4.2)-(4.4) and H let satisfy (4.6)-(4.7), for all $i = 1, ..., N$, Then*

$$e_i(x, T_n(u_n))b_i(x, u_n, Du_n) \to e_i(x, u)b_i(x, u, Du), \quad \textit{strongly in} \quad L^1_{loc}(\mathbb{R}^N). \tag{4.66}$$

Proof: Let $m_i(\cdot) > 1$ are continuous functions on $\mathbb{R}^N$ such that

$$\frac{1}{p_i(\cdot) - 1} \leq m_i(\cdot) < \frac{Np_i(\cdot)}{p_i(\cdot) - 1}\left(\frac{\overline{p}(\cdot) - 1 - \gamma_+^+}{\overline{p}(\cdot)(N - 1 - \gamma_+(\cdot)) - N(\gamma_+^+ - \gamma_+(\cdot))}\right),$$

this is possible since we have (4.10). Let $\sigma : \mathbb{R}^N \to (0, 1)$ be a continuous function such that

$$m_i(\cdot)\frac{p_i(\cdot) - 1}{p_i(\cdot)} < \sigma(\cdot) < N\left(\frac{\overline{p}(\cdot) - 1 - \gamma_+^+}{\overline{p}(\cdot)(N - 1 - \gamma_+(\cdot)) - N(\gamma_+^+ - \gamma_+(\cdot))}\right) < 1,$$

$$m_i(\cdot)(p_i(\cdot) - 1) \leq \sigma(\cdot)p_i(\cdot) < Np_i(\cdot)\left(\frac{\overline{p}(\cdot) - 1 - \gamma_+^+}{\overline{p}(\cdot)(N - 1 - \gamma_+(\cdot)) - N(\gamma_+^+ - \gamma_+(\cdot))}\right). \tag{4.67}$$

Using the fact that

$$|b_i(., u_n, D_i u_n)|^{m_i(\cdot)} \leq \left(h + |u_n|^{\overline{p}^-} + \sum_{j=1}^{N} |D_j u_n|^{p_j(\cdot)}\right)^{m_i(\cdot)\frac{p_i(\cdot)-1}{p_i(\cdot)}},$$

$$\leq \left(h^{\sigma(\cdot)} + |u_n|^{\sigma(\cdot)\overline{p}^-} + \sum_{j=1}^{N} |D_j u_n|^{\sigma(\cdot)p_j(\cdot)}\right)^{m_i(\cdot)\frac{p_i(\cdot)-1}{\sigma(\cdot)p_i(\cdot)}}. \tag{4.68}$$

By (4.54), (4.67), (4.68), (4.47), and Vitali's Theorem, we derive

$$e_i(x, T_n(u_n))b_i(x, u_n, Du_n) \to e_i(x, u)b_i(x, u, Du), \quad \text{strongly in} \quad L^{m_i(\cdot)}(B_r). \tag{4.69}$$

This finishes the proof of Lemma 4.14. ∎

Using the convergence (4.49) and (4.66), we conclude that the function u is weak solution of equation (4.1). The Theorem 4.2 is so proved.

Remark 4.15 *If we replace the hypothesis* (4.3) *by*

$$|a_i(x,u,\xi)| \leq g(x)\left(h(x)+|u|^{\overline{p}^-}+|\xi_i|^{p_i(x)}\right)^{1-\frac{1}{p_i(x)}},$$

we can prove the same regularity reported in Theorem 4.2 but the exponent $p_i(\cdot)$ *satisfies a better condition*

$$\frac{\overline{p}(x)(N-1-\gamma_+(x))-N(\gamma_+^+-\gamma_i(x))}{N(\overline{p}(x)-1-\gamma_+^+)} < p_i(x) < \frac{\overline{p}(x)(N-1-\gamma_i(x))-N(\gamma_+^+-\gamma_i(x))}{(1+\gamma_i(x))(N-\overline{p}(x))},$$

compared to (4.10). *Indeed, it suffices to substitute* $\sigma(\cdot)$ *in* (4.67) *by* $\sigma_i(\cdot)$ *such that*

$$m_i(\cdot)\frac{p_i(\cdot)-1}{p_i(\cdot)} < \sigma_i(\cdot) < N\left(\frac{\overline{p}(\cdot)-1-\gamma_+^+}{\overline{p}(\cdot)(N-1-\gamma_i(\cdot))-N(\gamma_+^+-\gamma_i(\cdot))}\right) < 1.$$

4.4.1 Proof of Theorem (4.3)

By applying Lemma (4.14) where $r_i(\cdot)>0$ such that

$$1 < r_i(\cdot) < \frac{p_i(x)s_+(x)}{1+s_+(x)+\gamma_i(x)(p_i(x)-1)}.$$

Hence, by

$$H(x,u_n) \to H(x,u) \quad \text{strongly in} \quad L^1_{loc}(\mathbb{R}^N), \quad \forall r>0.$$

and

$$e_i(x,T_n(u_n))b_i(x,u_n,Du_n) \to e_i(x,u)b_i(x,u,Du), \quad \text{strongly in} \quad L^1_{loc}(\mathbb{R}^N).$$

we conclude that the limit function u is a weak solution of equation possessing the regularity stated in (4.18) this finishes the proof of Theorem 4.3.

Conclusions and Future Research

This work focuses on the study of nonlinear elliptic equations with variable exponents, which represent a natural generalization of classical elliptic problems with constant exponents. The main objective is to prove the existence of weak solutions, even in cases where standard assumptions, such as coercivity, may fail. These equations arise in various physical and engineering models, particularly when the properties of the medium change from point to point. The proof strategy relies on constructing approximate problems, obtaining a priori estimates, and applying a limiting process, which allows us to extend classical results to a more general framework.

In future research, several directions can be explored:

Regularity: studying the smoothness and uniqueness of weak solutions.

Parabolic Extensions: extending the analysis to time-dependent problems with variable exponents.

Applications: applying the theoretical results to real-world models such as heat conduction and fluid flow in non-homogeneous materials.

Numerical Approaches: developing efficient numerical methods to approximate weak solutions and analyze their convergence.

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