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The well-posedness of an inverse source problem for a time-fractional telegraph equation

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Abstract *In this paper, we consider an inverse time-dependent source problem for a time-fractional telegraph equation with mixed boundary conditions and an additional measurement at a fixed point. The fractional derivative is described in the conformable sense. Under some assumptions on the input data, the well-posedness of this inverse source problem is shown by using Fourier's method and Banach's contraction mapping principle.*

1 Introduction

Let $\ell, T > 0$ *be some fixed numbers and let* Ω_T *be a rectangular region defined by:*

$$\Omega_T := \{(x,t) : 0 < x < \ell,\ 0 < t < T\}.$$

We consider the one-dimensional time-fractional telegraph equation

$$\mathcal{D}_t^{(2\alpha)} u(x,t) + 2a\mathcal{D}_t^{(\alpha)} u(x,t) + b^2 u(x,t) = \omega u_{xx}(x,t) + F(x,t),\ (x,t) \in \Omega_T, \tag{1.1}$$

where $\mathcal{D}_t^{(\alpha)}$ *represent the left-conformable fractional derivative of order* $0 < \alpha \le 1$ *with respect to* t *such that* $\mathcal{D}_t^{(2\alpha)} = \mathcal{D}_t^{(\alpha)}\left(\mathcal{D}_t^{(\alpha)}\right)$, $F(x,t)$ *is the source term and* $u(x,t)$ *represent the voltage or the current inside a piece of telegraph or transmission wire, whose electrical properties per unit length are: resistance* R, *inductance* L, *capacitance* C, *and conductance of leakage current* G *where* $\omega = \frac{1}{LC}$, $2a = \frac{G}{C} + \frac{R}{L}$, *and* $b^2 = \frac{GR}{LC}$.

For $\alpha = 1$, *equation* (1.1) *is the classical telegraph equation developed by Oliver Heaviside in last decades of* 19^{th} *century [9]. This equation is a second-order linear hyperbolic equation and it models several phenomena in many different fields such as signal analysis [13], wave propagation [19], random walk theory [7].*

Suppose the unknown function u *satisfy the following initial conditions*

$$u(x,0) = \varphi(x),\ \mathcal{D}_t^{(\alpha)} u(x,0) = \psi(x),\ 0 \le x \le \ell, \tag{1.2}$$

and the homogeneous mixed boundary conditions

$$u_x(0,t) = u(\ell,t) = 0,\ 0 \le t \le T, \tag{1.3}$$

where φ *and* ψ *are given functions. If all functions* $F(x,t), \varphi(t), \psi(t)$ *are given appropriately, the problem* (1.1)-(1.3) *is a direct problem. It should be noted that the direct problem* (1.1)-(1.3) *has been investigated in the works [3, 4, 2], and the references therein. When the source term* $F(x,t) = r(t) f(x,t)$ *with* $f(x,t)$ *is a given function. The problem of finding the solution pair* $\{u(x,t), r(t)\}$ *of the problem* (1.1)-(1.3) *with additional measurement condition*

$$u(x_0,t) = h(t),\ 0 \le t \le T, \tag{1.4}$$

is called the inverse problem where $x_0 \in [0, \ell[$ *is a fixed point and* $h(t)$ *is a given function. We note that inverse source problems for fractional diffusion and wave equations were investigated in [5, 10, 20, 11, 17, 18] and inverse coefficient problems for a semilinear time fractional telegraph equation in [16, 15]. It should also be noted that all the articles on the inverse problems mentioned used the fractional derivative of the Caputo sense. However, the inverse source problems for fractional telegraph equations have not yet been studied.*

Our aim in this paper is to study the existence and uniqueness of the solution as well as the continuous dependence of the solution upon data of the inverse time-dependent source problem (1.1)-(1.4)*. As far as we know, the study of this inverse source problem will be discussed in this paper for the first time.*

The rest of this paper is structured as follows: in Section 2, we give some definitions and properties of the conformable fractional calculus. In Section 3, under some natural regularity and consistency conditions on the input data, the well-posedness of inverse problem (1.1)-(1.4) *is shown by using eigenfunction expansion of a self-adjoint spectral problem along the Fourier's method and Banach's contraction mapping principle.*

2 Preliminaries

We recall some definitions and properties of the conformable fractional calculus theory.

Definition 2.1 ([14]). Given a function $f : [0, \infty[\longrightarrow \mathbb{R}$. Then, the conformable fractional derivative of f of order α is defined by

$$\mathcal{D}^{(\alpha)} f(t) = \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon t^{1-\alpha}\right) - f(t)}{\varepsilon}$$

for all $t > 0$, $\alpha \in]0, 1]$. If $\mathcal{D}^{(\alpha)} f(t)$ exists in some $]0, a[$, $a > 0$, and $\lim_{t \to 0^+} \mathcal{D}^{(\alpha)} f(t)$ exists, then define

$$\mathcal{D}^{(\alpha)} f(0) = \lim_{t \to 0^+} \mathcal{D}^{(\alpha)} f(t).$$

If the conformable fractional derivative of f *of order* α *exists, then we simply say* f *is* α*-differentiable.*

Definition 2.2 (**[14]**). The conformable fractional integral of a function f starting from $a \geq 0$ of order α is defined by

$$I_\alpha^a(f)(t) = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx,$$

where $\alpha \in]0, 1]$.

Theorem 2.3 ([14]). *Let* $\alpha \in]0, 1]$ *and* f, g *be* α*-differentiable at a point* $t > 0$. *Then*

(1) $\mathcal{D}^{(\alpha)}(af + bg) = a\mathcal{D}^{(\alpha)}(f) + b\mathcal{D}^{(\alpha)}(g)$, *for all* $a, b \in \mathbb{R}$.

(2) $\mathcal{D}^{(\alpha)}(\lambda) = 0$, *for all constant functions* $f(x) = \lambda$.

(3) $\mathcal{D}^{(\alpha)}(fg) = f\mathcal{D}^{(\alpha)}(g) + g\mathcal{D}^{(\alpha)}(f)$.

(4) *If* f *is differentiable, then* $\mathcal{D}^{(\alpha)}(f)(t) = t^{1-\alpha}\frac{df}{dt}(t)$.

Theorem 2.4 ([14]). *Let* $\alpha \in]0, 1]$, $a \geq 0$ *and* $f : [a, \infty[\longrightarrow \mathbb{R}$ *be continuous function. Then, for all* $t > a$ *we have*

$$\mathcal{D}^{(\alpha)} I_\alpha^a(f)(t) = f(t).$$

Definition 2.5 ([1]). Let $0 < \alpha \leq 1$ and $f : [0, \infty[\longrightarrow \mathbb{R}$ be function. Then the **fractional Laplace transform** of order α of f is defined by

$$\mathcal{L}_\alpha \{f(t)\}(s) = \mathcal{F}_\alpha(s) = \int_0^\infty e^{-s\frac{t^\alpha}{\alpha}} f(t)\, t^{\alpha-1} dt.$$

Theorem 2.6 ([1]). *Let* $0 < \alpha \leq 1$ *and* $f :]0,\infty[\longrightarrow \mathbb{R}$ *be differentiable function. Then*

$$\mathcal{L}_\alpha \left\{ \mathcal{D}^{(\alpha)} f(t) \right\}(s) = s\mathcal{F}_\alpha(s) - f(0).$$

Property 2.7 ([1, 6, 12]). Let $0 < \alpha \leq 1$, $a, b \in \mathbb{R}$ and $f, g : [0,\infty[\longrightarrow \mathbb{R}$ be a functions such that $\mathcal{L}_\alpha\{f(t)\}(s) = \mathcal{F}_\alpha(s)$ and $\mathcal{L}_\alpha\{g(t)\}(s) = \mathcal{G}_\alpha(s)$ exists. Then

(i) The fractional Laplace transform is linear operator:

$$\mathcal{L}_\alpha\{af(t) + bg(t)\}(s) = a\mathcal{F}_\alpha(s) + b\mathcal{G}_\alpha(s). \tag{2.1}$$

(ii) We have:

$$\mathcal{L}_\alpha\{f(t)\}(s) = \mathcal{L}\left\{ f\left((\alpha t)^{\frac{1}{\alpha}} \right) \right\}(s), \tag{2.2}$$

where $\mathcal{L}$ is the usual Laplace transform such that $\mathcal{L}\{g(t)\}(s) = \int_0^\infty e^{-st} g(t)\, dt$.

(iii) We have $\mathcal{L}_\alpha \left\{ e^{-a\frac{t^\alpha}{\alpha}} f(t) \right\}(s) = \mathcal{L}\left\{ e^{-at} f\left((\alpha t)^{\frac{1}{\alpha}} \right) \right\}(s) = \mathcal{L}\left\{ f\left((\alpha t)^{\frac{1}{\alpha}} \right) \right\}(s+a)$. For example:

$$\mathcal{L}_\alpha \left\{ e^{-a\frac{t^\alpha}{\alpha}} \cosh\left(b\frac{t^\alpha}{\alpha} \right) \right\}(s) = \mathcal{L}\left\{ e^{-at}\cosh(bt) \right\}(s) = \frac{s+a}{(s+a)^2 - b^2}, \tag{2.3}$$

$$\mathcal{L}_\alpha \left\{ e^{-a\frac{t^\alpha}{\alpha}} \sinh\left(b\frac{t^\alpha}{\alpha} \right) \right\}(s) = \mathcal{L}\left\{ e^{-at}\sinh(bt) \right\}(s) = \frac{b}{(s+a)^2 - b^2}. \tag{2.4}$$

(iv) The derivative of the fractional Laplace transform satisfy:

$$\frac{d\mathcal{L}_\alpha\{f(t)\}(s)}{ds} = -\mathcal{L}_\alpha \left\{ \frac{t^\alpha}{\alpha} f(t) \right\}(s). \tag{2.5}$$

(v) The fractional Laplace transform of the α-convolution of $f(t)$ and $g(t)$ is:

$$\mathcal{L}_\alpha\{(f * g)(t)\}(s) = \mathcal{F}_\alpha(s) \cdot \mathcal{G}_\alpha(s), \tag{2.6}$$

where $(f * g)(t) = \int_0^t f\left((t^\alpha - \tau^\alpha)^{1/\alpha} \right) g(\tau)\, \tau^{\alpha-1} d\tau$.

Theorem 2.8. *Let* $\eta, \gamma > 0$ *and* $g : [0,\infty[\longrightarrow \mathbb{R}$ *be continuous function. For all* $0 < \alpha \leq 1$, *the following Cauchy problem:*

$$\begin{cases} \mathcal{D}^{(2\alpha)} y(t) + 2\eta \mathcal{D}^{(\alpha)} y(t) + \gamma^2 y(t) = g(t), & 0 < t, \\ y(0) = y_1,\ \mathcal{D}^{(\alpha)} y(0) = y_2, \end{cases} \tag{2.7}$$

admits a unique solution given in the following three cases:

(i) If $\eta < \gamma$ *(by Theorem 2.1 in [2])*

$$\begin{aligned} y(t) &= y_1 e^{-\eta\frac{t^\alpha}{\alpha}} \cos\left(\sqrt{\gamma^2 - \eta^2}\frac{t^\alpha}{\alpha} \right) + \frac{\eta y_1 + y_2}{\sqrt{\gamma^2 - \eta^2}} e^{-\eta\frac{t^\alpha}{\alpha}} \sin\left(\sqrt{\gamma^2 - \eta^2}\frac{t^\alpha}{\alpha} \right) \\ &+ \frac{1}{\sqrt{\gamma^2 - \eta^2}} \int_0^t g(\tau)\, e^{-\eta\frac{t^\alpha - \tau^\alpha}{\alpha}} \sin\left(\sqrt{\gamma^2 - \eta^2}\frac{t^\alpha - \tau^\alpha}{\alpha} \right) \tau^{\alpha-1} d\tau. \end{aligned} \tag{2.8}$$

(ii) If $\eta = \gamma$

$$y(t) = y_1 e^{-\eta\frac{t^\alpha}{\alpha}} + (\eta y_1 + y_2)\frac{t^\alpha}{\alpha} e^{-\eta\frac{t^\alpha}{\alpha}} + \int_0^t g(\tau)\frac{t^\alpha - \tau^\alpha}{\alpha} e^{-\eta\frac{t^\alpha - \tau^\alpha}{\alpha}} \tau^{\alpha-1} d\tau. \tag{2.9}$$

(iii) If $\eta > \gamma$

$$\begin{aligned} y(t) &= y_1 e^{-\eta\frac{t^\alpha}{\alpha}} \cosh\left(\sqrt{\eta^2-\gamma^2}\frac{t^\alpha}{\alpha}\right) + \frac{\eta y_1 + y_2}{\sqrt{\eta^2-\gamma^2}} e^{-\eta\frac{t^\alpha}{\alpha}} \sinh\left(\sqrt{\eta^2-\gamma^2}\frac{t^\alpha}{\alpha}\right) \\ &+ \frac{1}{\sqrt{\eta^2-\gamma^2}} \int_0^t g(\tau) e^{-\eta\frac{t^\alpha-\tau^\alpha}{\alpha}} \sinh\left(\sqrt{\eta^2-\gamma^2}\frac{t^\alpha-\tau^\alpha}{\alpha}\right) \tau^{\alpha-1} d\tau. \end{aligned} \tag{2.10}$$

Proof. According to Theorem 2.6, and from (2.1) and (2.7) we get:

$$\begin{aligned} \mathcal{L}_\alpha\{y(t)\}(s) &= \frac{y_1(s+\eta)}{(s+\eta)^2-(\eta^2-\gamma^2)} \\ &+ \frac{y_1\eta + y_2}{(s+\eta)^2-(\eta^2-\gamma^2)} + \frac{1}{(s+\eta)^2-(\eta^2-\gamma^2)} \cdot \mathcal{L}_\alpha\{g(t)\}(s). \end{aligned} \tag{2.11}$$

(i) If $\eta < \gamma$, the proof in [2, page 30].

(ii) If $\eta = \gamma$, using (2.2) we have

$$\mathcal{L}_\alpha\left\{e^{-\eta\frac{t^\alpha}{\alpha}}\right\}(s) = \mathcal{L}\left\{e^{-\eta t}\right\}(s) = \frac{1}{s+\eta}, \tag{2.12}$$

using (2.5), from (2.12) we obtain:

$$\frac{d\mathcal{L}_\alpha\left\{-e^{-\eta\frac{t^\alpha}{\alpha}}\right\}(s)}{ds} = \mathcal{L}_\alpha\left\{\frac{t^\alpha}{\alpha}e^{-\eta\frac{t^\alpha}{\alpha}}\right\}(s) = \frac{1}{(s+\eta)^2}, \tag{2.13}$$

by using (2.8), from (2.13) we obtain:

$$\mathcal{L}_\alpha\left\{\int_0^t g(\tau)\frac{t^\alpha-\tau^\alpha}{\alpha}e^{-\eta\frac{t^\alpha-\tau^\alpha}{\alpha}}\tau^{\alpha-1}d\tau\right\}(s) = \mathcal{L}_\alpha\left\{\frac{t^\alpha}{\alpha}e^{-\eta\frac{t^\alpha}{\alpha}}\right\}(s)\cdot\mathcal{L}_\alpha\{g(t)\}(s), \tag{2.14}$$

after substituting (2.12)-(2.14) in equation (2.11), we find

$$\mathcal{L}_\alpha\{y(t)\}(s) = \mathcal{L}_\alpha\left\{y_1e^{-\eta\frac{t^\alpha}{\alpha}} + (\eta y_1+y_2)\frac{t^\alpha}{\alpha}e^{-\eta\frac{t^\alpha}{\alpha}} + \int_0^t g(\tau)\frac{t^\alpha-\tau^\alpha}{\alpha}e^{-\eta\frac{t^\alpha-\tau^\alpha}{\alpha}}\tau^{\alpha-1}d\tau\right\}(s),$$

hence, by using the inverse fractional Laplace transform, we get (2.9).

(iii) The last case (if $\eta > \gamma$): using (2.4) and (2.6) with putting $b = \sqrt{\eta^2-\gamma^2}$ in equation (2.4), we obtain:

$$\begin{aligned} &\mathcal{L}_\alpha\left\{\int_0^t g(\tau)\frac{e^{-\eta\frac{t^\alpha-\tau^\alpha}{\alpha}}}{\sqrt{\eta^2-\gamma^2}}\sinh\left(\sqrt{\eta^2-\gamma^2}\frac{t^\alpha-\tau^\alpha}{\alpha}\right)\tau^{\alpha-1}d\tau\right\}(s) \\ &\qquad = \mathcal{L}_\alpha\left\{\frac{e^{-\eta\frac{t^\alpha}{\alpha}}}{\sqrt{\eta^2-\gamma^2}}\sinh\left(\sqrt{\eta^2-\gamma^2}\frac{t^\alpha}{\alpha}\right)\right\}(s)\cdot\mathcal{L}_\alpha\{g(t)\}(s), \end{aligned} \tag{2.15}$$

after rearranging (2.3), (2.4) and (2.15) in equation (2.11), we find

$$\begin{aligned} \mathcal{L}_\alpha\{y(t)\}(s) &= \mathcal{L}_\alpha\Bigg\{y_1e^{-\eta\frac{t^\alpha}{\alpha}}\cosh\left(\sqrt{\eta^2-\gamma^2}\frac{t^\alpha}{\alpha}\right) + \frac{\eta y_1+y_2}{\sqrt{\eta^2-\gamma^2}}e^{-\eta\frac{t^\alpha}{\alpha}}\sinh\left(\sqrt{\eta^2-\gamma^2}\frac{t^\alpha}{\alpha}\right) \\ &+ \frac{1}{\sqrt{\eta^2-\gamma^2}}\int_0^t g(\tau)e^{-\eta\frac{t^\alpha-\tau^\alpha}{\alpha}}\sinh\left(\sqrt{\eta^2-\gamma^2}\frac{t^\alpha-\tau^\alpha}{\alpha}\right)\tau^{\alpha-1}d\tau\Bigg\}(s), \end{aligned}$$

thus, by using the inverse fractional Laplace transform, we get (2.10).

□

Definition 2.9 ([8]). Let $\alpha \in \,]\,0,1\,]$. Define a function space

$$C^{\alpha}[0,1] = \left\{u : u(t) = I_{\alpha}^{0}x(t) + c, c \in \mathbb{R}, x \in C[0,1]\right\}.$$

Define

$$\|u\|_{\alpha} = \|u\|_{0} + \left\|\mathcal{D}^{(\alpha)}u\right\|_{0},$$

where $\|u\|_{0} = \max_{t\in[0,1]} |u(t)|$.

Theorem 2.10 ([8]). *$(C^{\alpha}[0,1], \|\cdot\|_{\alpha})$ is a Banach space.*

Definition 2.11. Let $\alpha \in \,]\,0,1\,]$ and $T > 0$. We define the set of functions as:

$$C^{2\alpha}[0,T] = \left\{u : \mathcal{D}^{(\alpha)}u \in C^{\alpha}[0,T]\right\} = \left\{u : \mathcal{D}^{(\alpha)}u(t) = I_{\alpha}^{0}x(t) + c, c \in \mathbb{R}, x \in C[0,T]\right\}.$$

Define

$$\|u\|_{C^{2\alpha}[0,T]} = \|u\|_{C[0,T]} + \left\|\mathcal{D}^{(\alpha)}u\right\|_{C[0,T]} + \left\|\mathcal{D}^{(2\alpha)}u\right\|_{C[0,T]},$$

where $\|u\|_{C[0,T]} = \max_{t\in[0,T]} |u(t)|$.

Theorem 2.12. *$\left(C^{2\alpha}[0,T], \|\cdot\|_{C^{2\alpha}[0,T]}\right)$ is a Banach space.*

Proof. To prove this theorem, we follow the same steps as the proof of Theorem 2.10. It is easy to verify that $\|\cdot\|_{C^{2\alpha}[0,T]}$ satisfies the norm axioms.

The following proof is the completeness of $C^{2\alpha}[0,T]$. Let $\{u_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $C^{2\alpha}[0,T]$:

$$\mathcal{D}^{(\alpha)}u_n(t) = I_{\alpha}^{0}x_n(t) + c_n,$$

where $x_n \in C[0,T]$, and $c_n \in \mathbb{R}$. Then

$$\mathcal{D}^{(\alpha)}u_n(t) - \mathcal{D}^{(\alpha)}u_m(t) = I_{\alpha}^{0}(x_n(t) - x_m(t)) + c_n - c_m,$$

by using parts **(1)** and **(2)** of Theorem 2.4 and using Theorem 2.6, we find

$$\mathcal{D}^{(2\alpha)}(u_n(t) - u_m(t)) = x_n(t) - x_m(t).$$

Because $\{u_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $C^{2\alpha}[0,T]$, we have

$$\|u_n - u_m\|_{C^{2\alpha}[0,T]} = \|u_n - u_m\|_{C[0,T]} + \left\|\mathcal{D}^{(\alpha)}(u_n - u_m)\right\|_{C[0,T]} + \left\|\mathcal{D}^{(2\alpha)}(u_n - u_m)\right\|_{C[0,T]} \xrightarrow[n,m\to+\infty]{} 0.$$

Thus every term of the above formula converges to 0. By

$$\|u_n - u_m\|_{C[0,T]} \xrightarrow[n,m\to+\infty]{} 0 \text{ and } \left\|\mathcal{D}^{(2\alpha)}(u_n - u_m)\right\|_{C[0,T]} = \|x_n - x_m\|_{C[0,T]} \xrightarrow[n,m\to+\infty]{} 0$$

we know $\{u_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ are a Cauchy sequences in $C[0,T]$. By the completeness of $C[0,T]$, there exists $u, x \in C[0,T]$ such that $u_n \xrightarrow[n\to+\infty]{} u$ and $x_n \xrightarrow[n\to+\infty]{} x$. The second term is

$$\left\|\mathcal{D}^{(\alpha)}(u_n - u_m)\right\|_{C[0,T]} = \left\|I_{\alpha}^{0}(x_n - x_m) + (c_n - c_m)\right\|_{C[0,T]} \xrightarrow[n,m\to+\infty]{} 0.$$

We have

$$\|c_n - c_m\|_{C[0,T]} \leq \left\|I_{\alpha}^{0}(x_n - x_m) + (c_n - c_m)\right\|_{C[0,T]} + \left\|I_{\alpha}^{0}(x_n - x_m)\right\|_{C[0,T]} \xrightarrow[n,m\to+\infty]{} 0,$$

that is to say, $\{c_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$. By the completeness of $\mathbb{R}$, there exists $c \in \mathbb{R}$ such that $c_n \xrightarrow[n\to+\infty]{} c$. Let $\mathcal{D}^{(\alpha)}u(t) = I_{\alpha}^{0}x(t) + c$, then $u \in C^{2\alpha}[0,T]$ and

$$\|u_n - u\|_{C^{2\alpha}[0,T]} \xrightarrow[n\to+\infty]{} 0.$$

The completeness of $C^{2\alpha}[0,T]$ is proved. □

3 Main results

The following lemma is obtained with the help of integration by parts and the Cauchy–Schwarz inequality.

Lemma 3.1. *Let* $\mu_n = \frac{\pi(2n+1)}{2\ell}$ *with* $n \in \mathbb{N}$*. We have:*

(i) If $g \in \mathcal{C}^3[0,\ell]$ *satisfies the conditions* $g'(0) = g(\ell) = g''(\ell) = 0$*, then the inequality*

$$\left|\frac{2}{\ell}\int_0^\ell g(x)\cos(\mu_n x)\,dx\right| \le \frac{\sqrt{2}}{\mu_n^3}\|g\|_{\mathcal{C}^3[0,\ell]}, \text{ hold.}$$

(ii) If $g \in \mathcal{C}^4[0,\ell]$ *satisfies the conditions* $g'(0) = g(\ell) = g''(\ell) = g'''(0) = 0$*, then the inequality*

$$\left|\frac{2}{\ell}\int_0^\ell g(x)\cos(\mu_n x)\,dx\right| \le \frac{\sqrt{2}}{\mu_n^4}\|g\|_{\mathcal{C}^4[0,\ell]}, \text{ hold.}$$

Lemma 3.2. *The numerical series* $\sum_{n=0}^{+\infty}\frac{1}{\mu_n\sqrt{\omega\mu_n^2+b^2-a^2}}$ *converges to* $S > 0$*, where* $\mu_n = \frac{\pi}{2\ell}(2n+1)$.

Proof. Using Riemann's rule, we get:

$$\lim_{n\to+\infty}\frac{n^2}{\mu_n\sqrt{\omega\mu_n^2+b^2-a^2}} = \frac{\ell^2}{\pi^2\sqrt{\omega}},$$

then, the numerical series $\sum_{n=0}^{+\infty}\frac{1}{\mu_n\sqrt{\omega\mu_n^2+b^2-a^2}}$ converges to $S > 0$. □

Let $\alpha \in\ ]0,1]$*,* $T > 0$ *and* $L > 0$*. For* $f \in \mathcal{C}[0,T]$*, we define in* $\mathcal{C}[0,T]$ *the norm*

$$\|f\|_{L,\alpha} := \max_{0\le t\le T} e^{-Lt^\alpha/\alpha}|f(t)|.$$

Lemma 3.3. *The norms* $\|\cdot\|_{L,\alpha}$ *and* $\|\cdot\|_{\mathcal{C}[0,T]}$ *are equivalent.*

Proof. For $f \in \mathcal{C}[0,T]$, we have

$$e^{-LT^\alpha/\alpha}|f(t)| \le e^{-Lt^\alpha/\alpha}|f(t)| \le |f(t)|.$$

For all $t \in [0,T]$, we obtain

$$e^{-LT^\alpha/\alpha}\|f\|_{\mathcal{C}[0,T]} \le \|f\|_{L,\alpha} \le \|f\|_{\mathcal{C}[0,T]},$$

where, $\|f\|_{\mathcal{C}[0,T]} = \max_{0\le t\le T}|f(t)|$. □

3.1 Existence and uniqueness of the solution

The first main result on existence and uniqueness of the solution of the inverse time-dependent source problem (1.1)-(1.4) *is presented as follows.*

Theorem 3.4. *Suppose that the following assumptions hold:*

$$(A_1): \varphi \in \mathcal{C}^4[0,\ell];\ \varphi'(0) = \varphi(\ell) = \varphi''(0) = \varphi'''(\ell) = 0;$$

$$(A_2): \psi \in \mathcal{C}^3[0,\ell];\ \psi'(0) = \psi(\ell) = \psi''(\ell) = 0;$$

$$(A_3): f(\cdot,t) \in \mathcal{C}^3[0,\ell];\ f(x_0,\cdot) \in \mathcal{C}[0,T] \text{ with }, f(x_0,t) \ne 0 \text{ for all } t \in [0,T];$$

$$\frac{\partial f}{\partial x}(0,t) = f(\ell,t) = \frac{\partial^2 f}{\partial x^2}(\ell,t) = 0, \text{ for all } t \in [0,T];$$

$$(A_4): h \in \mathcal{C}^{2\alpha}[0,T];\ \varphi(x_0) = h(0) \text{ and } \psi(x_0) = \mathcal{D}_t^{(\alpha)}h(0).$$

Then, the inverse problem (1.1)-(1.4) *has a unique solution* $\{u(x,t), r(t)\}$.

Proof. The proof of this theorem takes place in three steps:
Step 1: Construction of solution. By using the Fourier's method (separation of variables), the associated spectral problem of the direct problem (1.1)-(1.3) is given by:

$$\begin{cases} X''(x) + \lambda X(x) = 0, \ 0 < x < \ell, \\ X'(0) = X(\ell) = 0. \end{cases} \tag{3.1}$$

Eigenvalues and eigenfunctions of the spectral problem (3.1) are

$$\lambda_n = \mu_n^2 \text{ where } \mu_n = \frac{\pi(2n+1)}{2\ell} \text{ and } X_n(x) = \cos(\mu_n x), \ n \in \mathbb{N}. \tag{3.2}$$

We can easily show that problem (3.1) is self-adjoint, then the system of functions (3.2) forms an orthogonal basis in the space $L^2[0,\ell]$.

By applying the standard procedure of the Fourier method, we obtain the following representation for the solution of the direct problem(1.1)-(1.3) for arbitrary $r \in \mathcal{C}[0,T]$,

$$u(x,t) = \sum_{n=0}^{\infty} u_n(t) X_n(x).$$

where the functions $u_n(t)$ with $n \in \mathbb{N}$ satisfy the following sequence of Cauchy problems:

$$\begin{cases} \mathcal{D}^{(2\alpha)} u_n(t) + 2a\mathcal{D}^{(\alpha)} u_n(t) + \left(b^2 + c\mu_n^2\right) u_n(t) = r(t) f_n(t), \ 0 < t \le T, \\ u_n(0) = \varphi_n, \ \mathcal{D}^{(\alpha)} u_n(0) = \psi_n, \end{cases} \tag{3.3}$$

where

$$f_n(t) = \frac{2}{\ell}\int_0^\ell f(x,t)\cos(\mu_n x)\,dx,$$

$$\varphi_n = \frac{2}{\ell}\int_0^\ell \varphi(x)\cos(\mu_n x)\,dx,$$

$$\psi_n = \frac{2}{\ell}\int_0^\ell \psi(x)\cos(\mu_n x)\,dx.$$

According to Theorem 2.8, the solutions of (3.3) are given in the following three cases:
Case 1: If $\frac{a^2-b^2}{\omega} < \mu_0^2$, then the solutions of (3.3) are:

$$\begin{aligned} u_n(t) &= e^{-a\frac{t^\alpha}{\alpha}}\left(\varphi_n \cos\left(\delta_n \frac{t^\alpha}{\alpha}\right) + \frac{\psi_n + a\varphi_n}{\delta_n}\sin\left(\delta_n\frac{t^\alpha}{\alpha}\right)\right) \\ &+ \frac{e^{-a\frac{t^\alpha}{\alpha}}}{\delta_n}\int_0^t \sin\left(\delta_n \frac{t^\alpha - s^\alpha}{\alpha}\right) s^{\alpha-1} e^{a\frac{s^\alpha}{\alpha}} r(s) f_n(s)\,ds, \end{aligned} \tag{3.4}$$

where $\delta_n = \sqrt{\omega\mu_n^2 + b^2 - a^2}$.
Case 2: If exist $n_0 \in \mathbb{N}$ such that $\mu_{n_0}^2 < \frac{a^2-b^2}{\omega} < \mu_{n_0+1}^2$, then the solutions of (3.2) are:

$$\begin{aligned} \text{for } n \le n_0,\ u_n(t) &= e^{-a\frac{t^\alpha}{\alpha}}\left(\varphi_n \cosh\left(\Delta_n \frac{t^\alpha}{\alpha}\right) + \frac{\psi_n + a\varphi_n}{\Delta_n}\sinh\left(\Delta_n\frac{t^\alpha}{\alpha}\right)\right) \\ &+ \frac{e^{-a\frac{t^\alpha}{\alpha}}}{\Delta_n}\int_0^t \sinh\left(\Delta_n \frac{t^\alpha - s^\alpha}{\alpha}\right) s^{\alpha-1} e^{a\frac{s^\alpha}{\alpha}} r(s) f_n(s)\,ds \\ \text{for } n > n_0,\ u_n(t) &= e^{-a\frac{t^\alpha}{\alpha}}\left(\varphi_n \cos\left(\delta_n \frac{t^\alpha}{\alpha}\right) + \frac{\psi_n + a\varphi_n}{\delta_n}\sin\left(\delta_n\frac{t^\alpha}{\alpha}\right)\right) \\ &+ \frac{e^{-a\frac{t^\alpha}{\alpha}}}{\delta_n}\int_0^t \sin\left(\delta_n \frac{t^\alpha - s^\alpha}{\alpha}\right) s^{\alpha-1} e^{a\frac{s^\alpha}{\alpha}} r(s) f_n(s)\,ds, \end{aligned} \tag{3.5}$$

where $\Delta_n = \sqrt{a^2 - b^2 - \omega\mu_n^2}$.

Case 3: If exist $n_0 \in \mathbb{N}$ such that $\frac{a^2 - b^2}{\omega} = \mu_{n_0}^2$, then the solutions of (3.3) are:

$$\begin{aligned}
&\text{for } n < n_0, \;\; u_n(t) = e^{-a\frac{t^\alpha}{\alpha}}\left(\varphi_n \cosh\left(\Delta_n \frac{t^\alpha}{\alpha}\right) + \frac{\psi_n + a\varphi_n}{\Delta_n}\sinh\left(\Delta_n \frac{t^\alpha}{\alpha}\right)\right) \\
&\qquad + \frac{e^{-a\frac{t^\alpha}{\alpha}}}{\Delta_n}\int_0^t \sinh\left(\Delta_n \frac{t^\alpha - s^\alpha}{\alpha}\right) s^{\alpha-1} e^{a\frac{s^\alpha}{\alpha}} r(s) f_n(s)\, ds \\
&\text{if } n = n_0, \; u_{n_0}(t) = e^{-a\frac{t^\alpha}{\alpha}}\left(\varphi_{n_0} + (\psi_{n_0} + a\varphi_{n_0})\frac{t^\alpha}{\alpha} + \int_0^t \frac{t^\alpha - s^\alpha}{\alpha} e^{a\frac{s^\alpha}{\alpha}} s^{\alpha-1} r(s) f_{n_0}(s)\, ds\right) \\
&\text{for } n > n_0, \; u_n(t) = e^{-a\frac{t^\alpha}{\alpha}}\left(\varphi_n \cos\left(\delta_n \frac{t^\alpha}{\alpha}\right) + \frac{\psi_n + a\varphi_k}{\delta_n}\sin\left(\delta_n \frac{t^\alpha}{\alpha}\right)\right) \\
&\qquad + \frac{e^{-a\frac{t^\alpha}{\alpha}}}{\delta_n}\int_0^t \sin\left(\delta_n \frac{t^\alpha - s^\alpha}{\alpha}\right) s^{\alpha-1} e^{a\frac{s^\alpha}{\alpha}} r(s) f_n(s)\, ds.
\end{aligned} \tag{3.6}$$

Hence, the representation of the first component of solution pair $\{u(x,t), r(r)\}$ is given in the following three cases:

☞ If $\frac{a^2 - b^2}{\omega} < \mu_0^2$, then

$$\begin{aligned}
u(x,t) = \sum_{n=0}^{\infty} \Bigg[& e^{-a\frac{t^\alpha}{\alpha}}\left(\varphi_n \cos\left(\delta_n \frac{t^\alpha}{\alpha}\right) + \frac{\psi_n + a\varphi_n}{\delta_n}\sin\left(\delta_n \frac{t^\alpha}{\alpha}\right)\right) \\
&+ \frac{e^{-a\frac{t^\alpha}{\alpha}}}{\delta_n}\int_0^t \sin\left(\delta_n \frac{t^\alpha - s^\alpha}{\alpha}\right) s^{\alpha-1} e^{a\frac{s^\alpha}{\alpha}} r(s) f_n(s)\, ds\Bigg] \cos(\mu_n x).
\end{aligned} \tag{3.7}$$

☞ If $\mu_{n_0}^2 < \frac{a^2 - b^2}{\omega} < \mu_{n_0+1}^2$, then

$$u(x,t) = \sum_{n=0}^{n_0} u_n(t)\cos(\mu_n x) + \sum_{n=n_0+1}^{\infty} u_n(t)\cos(\mu_n x), \tag{3.8}$$

where $u_n(t)$ is defined by (3.5).

☞ If $\frac{a^2 - b^2}{\omega} = \mu_{n_0}^2$, then

$$u(x,t) = \sum_{n=0}^{n_0-1} u_n(t)\cos(\mu_n x) + u_{n_0}(t)\cos(\mu_{n_0} x) + \sum_{n=n_0+1}^{\infty} u_n(t)\cos(\mu_n x), \tag{3.9}$$

where $u_n(t)$ is defined by (3.5). In this case, if $n_0 = 0$, we delete the first series from the representation (3.9).

Now we construction the second component of solution pair $\{u(x,t), r(t)\}$. Under the condition (A_4) and from (1.1), (1.4) and (3.6)-(3.9), we obtain the following Volterra integral equation of the second kind for $r(t)$:

$$r(t) = B(t) + \int_0^t Q(t,s)\, r(s)\, ds, \; t \in [0,T], \tag{3.10}$$

where

✓ If $\dfrac{a^2-b^2}{\omega} < \mu_0^2$, then

$$\begin{aligned} B(t) &= \frac{\mathcal{D}^{(2\alpha)}h(t) + 2a\mathcal{D}^{(\alpha)}h(t) + b^2h(t)}{f(x_0,t)} \\ &+ \frac{\omega e^{-a\frac{t^\alpha}{\alpha}}}{f(x_0,t)} \sum_{n=0}^{\infty} \left[\varphi_n \cos\left(\delta_n \frac{t^\alpha}{\alpha}\right) + \frac{\psi_n + a\varphi_n}{\delta_n} \sin\left(\delta_n \frac{t^\alpha}{\alpha}\right)\right] \mu_n^2 \cos(\mu_n x_0), \end{aligned} \tag{3.11}$$

and

$$Q(t,s) = \frac{\omega e^{a\frac{s^\alpha - t^\alpha}{\alpha}} s^{\alpha-1}}{f(x_0,t)} \sum_{n=0}^{\infty} \frac{\mu_n^2 \cos(\mu_n x_0)}{\delta_n} \sin\left(\delta_n \frac{t^\alpha - s^\alpha}{\alpha}\right) f_n(s). \tag{3.12}$$

✓ If $\mu_{n_0}^2 < \dfrac{a^2-b^2}{\omega} < \mu_{n_0+1}^2$, then

$$\begin{aligned} B(t) &= \frac{\mathcal{D}^{(2\alpha)}h(t) + 2a\mathcal{D}^{(\alpha)}h(t) + b^2h(t)}{f(x_0,t)} \\ &+ \frac{\omega e^{-a\frac{t^\alpha}{\alpha}}}{f(x_0,t)} \left(\sum_{n=0}^{n_0} \left[\varphi_n \cosh\left(\Delta_n \frac{t^\alpha}{\alpha}\right) + \frac{\psi_n + a\varphi_n}{\Delta_n} \sinh\left(\Delta_n \frac{t^\alpha}{\alpha}\right)\right] \mu_n^2 \cos(\mu_n x_0)\right. \\ &+ \left.\sum_{n=n_0+1}^{\infty} \left[\varphi_n \cos\left(\delta_n \frac{t^\alpha}{\alpha}\right) + \frac{\psi_n + a\varphi_n}{\delta_n} \sin\left(\delta_n \frac{t^\alpha}{\alpha}\right)\right] \mu_n^2 \cos(\mu_n x_0)\right), \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} Q(t,s) &= \frac{\omega e^{a\frac{s^\alpha - t^\alpha}{\alpha}} s^{\alpha-1}}{f(x_0,t)} \left(\sum_{n=0}^{n_0} \frac{\mu_n^2 \cos(\mu_n x_0)}{\Delta_n} \sinh\left(\Delta_n \frac{t^\alpha - s^\alpha}{\alpha}\right) f_n(s)\right. \\ &+ \left.\sum_{n=n_0+1}^{\infty} \frac{\mu_n^2 \cos(\mu_n x_0)}{\delta_n} \sin\left(\delta_n \frac{t^\alpha - s^\alpha}{\alpha}\right) f_n(s)\right). \end{aligned} \tag{3.14}$$

✓ If $\dfrac{a^2-b^2}{\omega} = \mu_{n_0}^2$, then

$$\begin{aligned} B(t) &= \frac{\mathcal{D}^{(2\alpha)}h(t) + 2a\mathcal{D}^{(\alpha)}h(t) + b^2h(t)}{f(x_0,t)} \\ &+ \frac{\omega e^{-a\frac{t^\alpha}{\alpha}}}{f(x_0,t)} \left(\sum_{n=0}^{n_0-1} \left[\varphi_n \cosh\left(\Delta_n \frac{t^\alpha}{\alpha}\right) + \frac{\psi_n + a\varphi_n}{\Delta_n} \sinh\left(\Delta_n \frac{t^\alpha}{\alpha}\right)\right] \mu_n^2 \cos(\mu_n x_0)\right. \\ &+ \left[\varphi_{n_0} + (\psi_{n_0} + a\varphi_{n_0}) \frac{t^\alpha}{\alpha}\right] \mu_{n_0}^2 \cos(\mu_{n_0} x_0) \\ &+ \left.\sum_{n=n_0+1}^{\infty} \left[\varphi_n \cos\left(\delta_n \frac{t^\alpha}{\alpha}\right) + \frac{\psi_n + a\varphi_n}{\delta_n} \sin\left(\delta_n \frac{t^\alpha}{\alpha}\right)\right] \mu_n^2 \cos(\mu_n x_0)\right), \end{aligned} \tag{3.15}$$

and

$$Q(t,s)=\frac{\omega e^{a\frac{s^\alpha-t^\alpha}{\alpha}}s^{\alpha-1}}{f(x_0,t)}\left(\sum_{n=0}^{n_0-1}\frac{\mu_n^2\cos(\mu_n x_0)}{\Delta_n}\sinh\left(\Delta_n\frac{t^\alpha-s^\alpha}{\alpha}\right)f_n(s)\right.$$
$$+\mu_{n_0}^2\cos(\mu_{n_0}x_0)f_{n_0}(s)\frac{t^\alpha-s^\alpha}{\alpha} \tag{3.16}$$
$$\left.+\sum_{n=n_0+1}^{\infty}\frac{\mu_n^2X_n(x_0)}{\delta_n}\sin\left(\delta_n\frac{t^\alpha-s^\alpha}{\alpha}\right)f_n(s)\right).$$

In this case, if $n_0=0$, we delete the first series from the two representations (3.15) and (3.16).

Step 2: Existence of the solution

To establish the regularity of the first component $u(x,t)$, we need to show $u(x,t)$, $u_x(x,t)$, $u_{xx}(x,t)$, $\mathcal{D}_t^{(\alpha)}u(x,t)$ and $\mathcal{D}_t^{(2\alpha)}u(x,t)$ are continuous functions in $\bar{D}_T$.

Under the conditions $(A_1)-(A_3)$ and Lemma 3.1, by using the series (3.8), the following inequalities hold for any $(x,t)\in\bar{D}_T$ such that

$$|u(x,t)|\le\sum_{n=0}^{\infty}\left[\frac{\sqrt{2}\|\varphi\|_{\mathcal{C}^4[0,\ell]}}{\mu_n^4}+\frac{\sqrt{2}\|\psi\|_{\mathcal{C}^3[0,\ell]}}{\delta_n\mu_n^3}+\frac{\sqrt{2}a\|\varphi\|_{\mathcal{C}^4[0,\ell]}}{\delta_n\mu_n^4}\right.$$
$$\left.+\frac{\sqrt{2}e^{aT^\alpha/\alpha}\|r\|_{\mathcal{C}[0,T]}\|f\|_{\mathcal{C}^3[0,\ell]\times\mathcal{C}[0,T]}}{a\alpha\delta_n\mu_n^3}\right]. \tag{3.17}$$

$$|u_x(x,t)|\le\sum_{n=0}^{\infty}\left[\frac{\sqrt{2}\|\varphi\|_{\mathcal{C}^4[0,\ell]}}{\mu_n^3}+\frac{\sqrt{2}\|\psi\|_{\mathcal{C}^3[0,\ell]}}{\delta_n\mu_n^2}+\frac{\sqrt{2}a\|\varphi\|_{\mathcal{C}^4[0,\ell]}}{\delta_n\mu_n^3}\right.$$
$$\left.+\frac{\sqrt{2}e^{aT^\alpha/\alpha}\|r\|_{\mathcal{C}[0,T]}\|f\|_{\mathcal{C}^3[0,\ell]\times\mathcal{C}[0,T]}}{a\alpha\delta_n\mu_n^2}\right]. \tag{3.18}$$

$$|u_{xx}(x,t)|\le\sum_{n=0}^{\infty}\left[\frac{\sqrt{2}\|\varphi\|_{\mathcal{C}^4[0,\ell]}}{\mu_n^2}+\frac{\sqrt{2}\|\psi\|_{\mathcal{C}^3[0,\ell]}}{\delta_n\mu_n}+\frac{\sqrt{2}a\|\varphi\|_{\mathcal{C}^4[0,\ell]}}{\delta_n\mu_n^2}\right.$$
$$\left.+\frac{\sqrt{2}e^{aT^\alpha/\alpha}\|r\|_{\mathcal{C}[0,T]}\|f\|_{\mathcal{C}^3[0,\ell]\times\mathcal{C}[0,T]}}{a\alpha\delta_n\mu_n}\right]. \tag{3.19}$$

$$\left|\mathcal{D}_t^{(\alpha)}u(x,t)\right|\le\sum_{n=0}^{\infty}\left[\sqrt{2}\left(2a+\delta_n+\frac{a^2}{\delta_n}\right)\frac{\|\varphi\|_{\mathcal{C}^4[0,\ell]}}{\mu_n^4}+\left(\sqrt{2}+\frac{a\sqrt{2}}{\delta_n}\right)\frac{\|\psi\|_{\mathcal{C}^3[0,\ell]}}{\mu_n^3}\right.$$
$$\left.+\left(\frac{\sqrt{2}}{a}+\frac{\sqrt{2}}{\delta_n}\right)e^{aT^\alpha/\alpha}\|r\|_{\mathcal{C}[0,\ell]}\frac{\|f\|_{\mathcal{C}^3[0,\ell]\times\mathcal{C}[0,T]}}{\mu_n^3}\right]. \tag{3.20}$$

$$\left|\mathcal{D}_t^{(2\alpha)}u(x,t)\right|\le\sum_{n=0}^{\infty}\left[\left(\delta_n^2+\frac{2a}{\delta_n}+2\right)\left(\sqrt{2}\left(1+\frac{a}{\delta_n}\right)\frac{\|\varphi\|_{\mathcal{C}^4[0,\ell]}}{\mu_n^4}+\frac{\sqrt{2}\|\psi\|_{\mathcal{C}^3[0,\ell]}}{\delta_n\mu_n^3}\right)\right.$$
$$\left.+\sqrt{2}\left(\frac{a}{\delta_n}+\frac{\delta_n}{a}+2\right)e^{aT^\alpha/\alpha}\|r\|_{\mathcal{C}[0,T]}\frac{\|f\|_{\mathcal{C}^3[0,\ell]\times\mathcal{C}[0,T]}}{\mu_n^3}\right] \tag{3.21}$$

From (3.17)-(3.21) and by Weierstrass M-test, the series corresponding to $u(x,t)$, $u_x(x,t)$, $u_{xx}(x,t)$, $\mathcal{D}_t^{(\alpha)}u(x,t)$ and $\mathcal{D}_t^{(2\alpha)}u(x,t)$ are uniformly convergent on $\bar{D}_T$. Hence, $u(x,t)$, $u_x(x,t)$, $u_{xx}(x,t)$, $\mathcal{D}_t^{(\alpha)}u(x,t)$ and $\mathcal{D}_t^{(2\alpha)}u(x,t)$ are continuous functions on $\bar{D}_T$.

Similarly, we show that the two series (3.8) and (3.9) are uniformly convergent in $\bar{D}_T$. Therefore, their sums $u(x,t)$, $u_x(x,t)$, $u_{xx}(x,t)$, $\mathcal{D}_t^{(\alpha)}u(x,t)$ and $\mathcal{D}_t^{(2\alpha)}u(x,t)$ are continuous in $\bar{D}_T$. Thus, $u(x,t)$ satisfies the conditions (1.1)-(1.3) for arbitrary $r \in \mathcal{C}[0,T]$.

We define the following operator:

$$\mathcal{P}(r(t)) := B(t) + \int_0^t Q(t,s)\, r(s)\, ds,$$

on the space $\mathcal{C}[0,T]$ with $\|\phi\|_{\mathcal{C}[0,T]} := \max\limits_{0\le t\le T} |\phi(t)|$. To show $\mathcal{P}$ is well defined.

Under assumption (A_4), the function $t \mapsto \frac{\mathcal{D}^{(2\alpha)}h(t)+2a\mathcal{D}^{(\alpha)}h(t)+b^2h(t)}{f(x_0,t)}$ is continuous on $[0,T]$. Under assumptions $(A_1)-(A_2)$, using Lemma 3.1 and (3.11)-(3.12), we have:

$$\begin{aligned}\sum_{n=0}^{\infty}\left[\varphi_n \cos\left(\delta_n \frac{t^\alpha}{\alpha}\right) + \frac{\psi_n + a\varphi_n}{\delta_n}\sin\left(\delta_n\frac{t^\alpha}{\alpha}\right)\right]\mu_n^2\cos(\mu_n x_0) \\ \le \sum_{n=0}^{+\infty}\sqrt{2}\left[\left(1+\frac{a}{\delta_n}\right)\frac{\|\varphi\|_{\mathcal{C}^4[0,\ell]}}{\mu_n^2} + \frac{\|\psi\|_{\mathcal{C}^3[0,\ell]}}{\delta_n\mu_n}\right]\end{aligned} \tag{3.22}$$

and

$$\sum_{n=0}^{\infty}\frac{\mu_n^2\cos(\mu_n x_0)}{\delta_n}\sin\left(\delta_n\frac{t^\alpha - s^\alpha}{\alpha}\right) f_n(s) \le \sum_{n=0}^{+\infty}\frac{\sqrt{2}\,\|f\|_{\mathcal{C}^3[0,\ell]\times\mathcal{C}[0,T]}}{\delta_n\mu_n} \tag{3.23}$$

From (3.22) and (3.23), the series functions

$$\sum_{n=0}^{\infty}\left[\varphi_n\cos\left(\delta_n\frac{t^\alpha}{\alpha}\right) + \frac{\psi_n + a\varphi_n}{\delta_n}\sin\left(\delta_n\frac{t^\alpha}{\alpha}\right)\right]\mu_n^2\cos(\mu_n x_0),$$

$$\sum_{n=0}^{\infty}\frac{\mu_n^2\cos(\mu_n x_0)}{\delta_n}\sin\left(\delta_n\frac{t^\alpha - s^\alpha}{\alpha}\right) f_n(s)$$

are uniformly convergent. Then, $B(t)$ and $Q(t,s)$ defined by (3.11) and (3.12) are continuous functions on $[0,T]$ and $[0,T]\times[0,T]$, respectively. Hence, the operator $\mathcal{P}$ is well defined.

For the rest of this proof, we take only the first case (if $\frac{a^2-b^2}{\omega} < \mu_0^2$). Now we prove that $\mathcal{P}$ is a contraction operator in the space $\mathcal{C}[0,T]$. We choose $L > 0$ and let $r_1, r_2 \in \mathcal{C}[0,T]$. Under assumption (A_3) and using (3.12), we have the following estimates:

$$\begin{aligned} e^{-Lt^\alpha/\alpha}\left|\mathcal{P}(r_1(t)) - \mathcal{P}(r_2(t))\right| &\le e^{-Lt^\alpha/\alpha}\int_0^t e^{Ls^\alpha/\alpha}|Q(t,s)|\, e^{-Ls^\alpha/\alpha}|r_1(s) - r_2(s)|\, ds \\ &\le \|r_1 - r_2\|_{L,\alpha}\int_0^t e^{L(s^\alpha - t^\alpha)/\alpha}|Q(t,s)|\, ds \\ &\le \frac{\sqrt{2}\omega\,\|f\|_{\mathcal{C}^3[0,\ell]\times\mathcal{C}[0,T]}\sum\limits_{n=0}^{+\infty}\frac{1}{\delta_n\mu_n}}{(L+a)\min\limits_{0\le t\le T}|f(x_0,t)|}\|r_1 - r_2\|_{L,\alpha}. \end{aligned}$$

Consequently, we obtain:

$$\|\mathcal{P}(r_1) - \mathcal{P}(r_2)\|_{L,\alpha} \le \frac{\sqrt{2}\omega S\,\|f\|_{\mathcal{C}^3[0,\ell]\times\mathcal{C}[0,T]}}{(L+a)\min\limits_{0\le t\le T}|f(x_0,t)|}\|r_1 - r_2\|_{L,\alpha}, \tag{3.24}$$

where $S = \sum\limits_{n=0}^{+\infty}\frac{1}{\mu_n\sqrt{\omega\mu_n^2 + b^2 - a^2}}$. It is easy to choose the real $L > 0$ such that,

$$\frac{\sqrt{2}\omega S\,\|f\|_{\mathcal{C}^3[0,\ell]\times\mathcal{C}[0,T]}}{(L+a)\min\limits_{0\le t\le T}|f(x_0,t)|} < 1. \tag{3.25}$$

Then the operator $\mathcal{P}$ is a contraction. Consequently, by Banach's contraction mapping principle, $\mathcal{P}$ has a unique fixed point $r \in \mathcal{C}[0,T]$.

Step 3: Uniqueness of the solution

Let $\{u(x,t), r(t)\}$ and $\{\tilde{u}(x,t), \tilde{r}(t)\}$ be two solution sets of the inverse problem (1.1)-(1.4). From (3.7) and (3.9) , we have

$$u(x,t) - \tilde{u}(x,t) = e^{-a\frac{t^\alpha}{\alpha}} \sum_{n=0}^{\infty} \frac{\cos(\mu_n x)}{\delta_n} \int_0^t \sin\left(\delta_n \frac{t^\alpha - s^\alpha}{\alpha}\right) s^{\alpha-1} e^{a\frac{s^\alpha}{\alpha}} (r(s) - \tilde{r}(s)) f_k(s)\, ds, \tag{3.26}$$

and

$$r(t) - \tilde{r}(t) = \mathcal{P}(r(t)) - \mathcal{P}(\tilde{r}(t)). \tag{3.27}$$

From (3.24) and (3.27) we get:

$$\|r - \tilde{r}\|_{L,\alpha} \leq \frac{\sqrt{2}\omega S \|f\|_{\mathcal{C}^3[0,\ell]\times\mathcal{C}[0,T]}}{(L+a)\min\limits_{0\leq t\leq T}|f(x_0,t)|} \|r - \tilde{r}\|_{L,\alpha}, \tag{3.28}$$

which implies that $r = \tilde{r}$. After inserting $r = \tilde{r}$ in (3.26), we have $u = \tilde{u}$. □

Remark 3.5. We can prove Theorem 3.4 so that the function $Q(t,s)$ given by (3.14) and (3.16). In both cases, the contraction constant in (3.25) will be given as follows:

✓ If $\mu_{n_0}^2 < \dfrac{a^2 - b^2}{\omega} < \mu_{n_0+1}^2$, the contraction constant in (3.25) is replaced by

$$\frac{\sqrt{2}\omega S' \|f\|_{\mathcal{C}^3[0,\ell]\times\mathcal{C}[0,T]}}{(L+a)\min\limits_{0\leq t\leq T}|f(x_0,t)|} \text{ where } S' = \sum_{n=0}^{n_0} \frac{1}{\mu_n\sqrt{a^2 - b^2 - \omega\mu_n^2}} + \sum_{n=n_0+1}^{+\infty} \frac{1}{\mu_n\sqrt{\omega\mu_n^2 + b^2 - a^2}}. \tag{3.29}$$

✓ If $\dfrac{a^2 - b^2}{\omega} = \mu_{n_0}^2$, the contraction constant in (3.25) is replaced by

$$\frac{\sqrt{2}\omega S'' \|f\|_{\mathcal{C}^3[0,\ell]\times\mathcal{C}[0,T]}}{(L+a)\min\limits_{0\leq t\leq T}|f(x_0,t)|} \text{ where } S'' = \frac{T^\alpha}{\alpha\mu_{n_0}^3} + \sum_{n=0}^{n_0-1} \frac{1}{\mu_n\sqrt{a^2 - b^2 - \omega\mu_n^2}} + \sum_{n=n_0+1}^{+\infty} \frac{1}{\mu_n\sqrt{\omega\mu_n^2 + b^2 - a^2}}. \tag{3.30}$$

3.2 Continuous dependence upon the data of the solution

In this subsection, we give the second main result on continuous dependence upon the data of the solution pair $\{u(x,t), r(t)\}$ of the inverse problem (1.1)-(1.4).

Let $\Im$ be the set of quartiles $\{\varphi, \psi, f, h\}$ where the functions φ, ψ, f and h satisfy the conditions $(A_1) - (A_4)$ of Theorem 3.4 and

$$\|\varphi\|_{\mathcal{C}^4[0,\ell]} \leq M_1,\ \|\psi\|_{\mathcal{C}^3[0,\ell]} \leq M_2,\ \|f\|_{\mathcal{C}^3[0,\ell]\times\mathcal{C}[0,T]} \leq M_3,\ \|h\|_{\mathcal{C}^{2\alpha}[0,T]} \leq M_4, \\ M_5 = \min_{0\leq t\leq T}|f(x_0,t)|,\ M_6 = \max\left\{1, 2a, b^2\right\}. \tag{3.31}$$

For $\phi \in \Im$, we define the norm

$$\|\phi\|_{\Im} := \|\varphi\|_{\mathcal{C}^4[0,\ell]} + \|\psi\|_{\mathcal{C}^3[0,\ell]} + \|f\|_{\mathcal{C}^3[0,\ell]\times\mathcal{C}[0,T]} + \|h\|_{\mathcal{C}^{2\alpha}[0,T]}. \tag{3.32}$$

Theorem 3.6. *The solution* $\{u(x,t), r(t)\}$ *of the inverse problem* (1.1)-(1.4) *under the assumptions of Theorem 3.4, depends continuously upon the data.*

Proof. Let $\{u(x,t), r(t)\}$ and $\{\tilde{u}(x,t), \tilde{r}(t)\}$ be two solution sets of the inverse problem (1.1)-(1.4), corresponding to the data $\phi = \{\varphi, \psi, f, h\}$ and $\tilde{\phi} = \{\tilde{\varphi}, \tilde{\psi}, \tilde{f}, \tilde{h}\}$, respectively.

From (3.11), we have

$$\begin{aligned} B(t) - \tilde{B}(t) &= \frac{1}{f(x_0,t)} \left[\mathcal{D}^{(2\alpha)}\left(h-\tilde{h}\right) + 2a\mathcal{D}^{(\alpha)}\left(h-\tilde{h}\right) + b^2\left(h-\tilde{h}\right)\right] \\ &+ \frac{\tilde{f}(x_0,t) - f(x_0,t)}{f(x_0,t)\tilde{f}(x_0,t)}\left[\mathcal{D}^{(2\alpha)}\tilde{h}(t) + 2a\mathcal{D}^{(\alpha)}\tilde{h}(t) + b^2\tilde{h}(t)\right] \\ &+ \frac{\omega e^{-at^\alpha/\alpha}}{f(x_0,t)}\sum_{n=0}^{+\infty}\left[(\varphi_n - \tilde{\varphi}_n)\cos(\delta_n t^\alpha/\alpha) + \frac{1}{\delta_n}\left(\psi_n - \tilde{\psi}_n + a(\varphi_n - \tilde{\varphi}_n)\right)\sin(\delta_n t^\alpha/\alpha)\right]\mu_n^2\cos(\mu_n x_0) \\ &+ \frac{\omega e^{-at^\alpha/\alpha}\left(\tilde{f}(x_0,t) - f(x_0,t)\right)}{f(x_0,t)\tilde{f}(x_0,t)}\sum_{n=0}^{+\infty}\left(\tilde{\varphi}_n\cos(\delta_n t^\alpha/\alpha) + \frac{\tilde{\psi}_n + a\tilde{\varphi}_n}{\delta_n}\sin(\delta_n t^\alpha/\alpha)\right)\mu_n^2\cos(\mu_n x_0). \end{aligned} \tag{3.33}$$

Under conditions $(A_1) - (A_4)$ and using Lemme 3.1, (3.30) and (3.32) we obtain:

$$\begin{aligned} \left\|B - \tilde{B}\right\|_{\mathcal{C}[0,T]} \leq M_7\left\|\varphi - \tilde{\varphi}\right\|_{\mathcal{C}^4[0,\ell]} + M_8\left\|\psi - \tilde{\psi}\right\|_{\mathcal{C}^3[0,\ell]} + M_9\left\|f - \tilde{f}\right\|_{\mathcal{C}^3\times\mathcal{C}[0,T]} \\ + M_{10}\left\|h - \tilde{h}\right\|_{\mathcal{C}^{2\alpha}[0,T]}, \end{aligned} \tag{3.34}$$

where

$$\begin{aligned} M_7 &= \frac{\omega}{M_5}\sum_{n=0}^{+\infty}\left(\frac{1}{\mu_n^2} + \frac{a}{\delta_n\mu_n^2}\right), \\ M_8 &= \frac{\omega}{M_5}\sum_{n=0}^{+\infty}\frac{1}{\delta_n\mu_n}, \\ M_9 &= \frac{M_4M_6}{M_5^2} + \frac{\omega}{M_5^2}\sum_{n=0}^{+\infty}\left(1 + \frac{a}{\delta_n}\frac{M_1}{\mu_n^2} + \frac{M_2}{\delta_n\mu_n}\right), \\ M_{10} &= M_6/M_5. \end{aligned}$$

From (3.10) and for all $L > 0$, we have

$$\begin{aligned} e^{-Lt^\alpha/\alpha}\left[r(t) - \tilde{r}(t)\right] &= e^{-Lt^\alpha/\alpha}\left[B(t) - \tilde{B}(t)\right] \\ &+ \int_0^t e^{-Lt^\alpha/\alpha}Q(t,s)\left[r(s) - \tilde{r}(s)\right]ds \\ &+ \int_0^t e^{-Lt^\alpha/\alpha}\left[Q(t,s) - \tilde{Q}(t,s)\right]\tilde{r}(s)\,ds. \end{aligned} \tag{3.35}$$

Under condition (A_3), using Lemma 3.1, Lemma 3.2 and (3.12), we obtain

$$\int_0^t e^{-Lt^\alpha/\alpha}Q(t,s)\left[r(s) - \tilde{r}(s)\right]ds \leq \frac{\sqrt{2}\omega S\left\|f\right\|_{\mathcal{C}^3[0,\ell]\times\mathcal{C}[0,T]}}{(L+a)M_5}\left\|r - \tilde{r}\right\|_{L,\alpha}, \tag{3.36}$$

and

$$\int_0^t e^{-Lt^\alpha/\alpha}\left[Q(t,s) - \tilde{Q}(t,s)\right]\tilde{r}(s)\,ds \leq \frac{2\sqrt{2}d\omega M_3 S}{(L+a)M_5^2}\left\|f - \tilde{f}\right\|_{\mathcal{C}^3[0,\ell]\times\mathcal{C}[0,T]}, \tag{3.37}$$

where $d = \left\|\tilde{r}\right\|_{L,\alpha}$.

From (3.34)-(3.37), we have the estimate

$$\left(1-\frac{\sqrt{2}\omega S\,\|f\|_{\mathcal{C}^3[0,\ell]\times\mathcal{C}[0,T]}}{(L+a)\,M_5}\right)\|r-\tilde{r}\|_{L,\alpha}\le M_7\,\|\varphi-\tilde{\varphi}\|_{\mathcal{C}^4[0,\ell]}+M_8\left\|\psi-\tilde{\psi}\right\|_{\mathcal{C}^3[0,\ell]} + \left[M_9+\frac{2\sqrt{2}d\omega M_3 S}{(L+a)\,M_5^2}\right]\left\|f-\tilde{f}\right\|_{\mathcal{C}^3\times\mathcal{C}[0,T]} + M_{10}\left\|h-\tilde{h}\right\|_{\mathcal{C}^{2\alpha}[0,T]}. \tag{3.38}$$

From (3.38) and (3.32), we get

$$\left(1-\frac{\sqrt{2}\omega S\,\|f\|_{\mathcal{C}^3[0,\ell]\times\mathcal{C}[0,T]}}{(L+a)\,M_5}\right)\|r-\tilde{r}\|_{L,\alpha}\le M_{11}\,\|\phi-\phi\|_{\Im}, \tag{3.39}$$

where

$$M_{11}=\max\left\{M_7,M_8,M_9+\frac{2\sqrt{2}d\omega M_3 S}{(L+a)\,M_5^2},M_{10}\right\}.$$

From (3.25) and (3.39), we have

$$\|r-\tilde{r}\|_{L,\alpha}\le M_{12}\,\|\phi-\phi\|_{\Im}, \tag{3.40}$$

where $M_{12}=\frac{M_{11}}{1-\frac{\sqrt{2}\omega S\|f\|_{\mathcal{C}^3[0,\ell]\times\mathcal{C}[0,T]}}{(L+a)M_5}}$.

Under conditions $(A_1)-(A_3)$ and from (3.4), we obtain

$$\|u-\tilde{u}\|_{\mathcal{C}(\bar{D}_T)}\le M_{13}\,\|\varphi-\tilde{\varphi}\|_{\mathcal{C}^4[0,\ell]}+M_{14}\left\|\psi-\tilde{\psi}\right\|_{\mathcal{C}^3[0,\ell]}+M_{15}\left\|f-\tilde{f}\right\|_{\mathcal{C}^3[0,\ell]\times\mathcal{C}[0,T]} + M_{16}\,\|r-\tilde{r}\|_{L,\alpha}, \tag{3.41}$$

where

$$M_{13}=\sum_{n=0}^{+\infty}\left(\frac{\sqrt{2}}{\mu_n^4}+\frac{\sqrt{2}}{\delta_n\mu_n^4}\right),\; M_{14}=\sum_{n=0}^{+\infty}\frac{\sqrt{2}}{\delta_n\mu_n^3},\; M_{15}=\sum_{n=0}^{+\infty}\frac{d\sqrt{2}}{(L+\alpha)\,\mu_n^3},\; M_{16}=\sum_{n=0}^{+\infty}\frac{M_3\sqrt{2}}{(L+\alpha)\,\mu_n^3}$$

From (3.40) and (3.41), we get

$$\|u-\tilde{u}\|_{\mathcal{C}(\bar{D}_T)}+\|r-\tilde{r}\|_{L,\alpha}\le M_{17}\left\|\phi-\tilde{\phi}\right\|_{\Im},$$

where $M_{17}=\max\{M_{13},M_{14},M_{15},M_{12}M_{16}\}$. Then, the solution of the inverse problem (1.1)-(1.4) is depends continuously upon the data. □

4 Conclusion

An inverse time-dependent source problem of finding the time-dependent source term in a time-fractional telegraph equation with mixed boundary conditions and an additional measurement at a fixed point, have been investigated. Under some assumptions on the input data, the well-posedness of this inverse source problem is shown by the Fourier's method and Banach's contraction mapping principle.

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